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Høyrup, Jens

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Several algebraic unknowns – the road from Pacioli to Descartes

Jens Høyrup

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Wolfgang Köhler's Sultan
in memoriam

Abstract

At the Annual conference of the Indian Society for History of Mathematics in 2020 I spoke about the scattered use of several algebraic unknowns in Italian algebra from Fibonacci to Pacioli, and in 2021 about Benedetto da Firenze's introduction of symbolic algebraic calculations with up to five unknowns in 1463 – the latter having no impact whatsoever on future developments. Here I shall complete what was not originally planned to become a triptych, looking at the development of the technique from Pacioli onward in the writings of Rudolff, Stifel and Mennher. In the end I shall consider the likely influence on Viète's and Descartes' algebras, together with the reasons for their unprecedented introduction of abstract coefficients.

Contents

At the Annual conference of the Indian Society for History of Mathematics in 2020 I spoke about the appearance of two algebraic unknowns in Italian material from Leonardo Fibonacci until Luca Pacioli; in that of 2021 I spoke about Benedetto da Firenze's use of up to five unknowns in symbolic first-degree algebra, in a manuscript from 1463, and about how this seeming revolutionary innovation found absolutely no echo.[1]

I shall now return from this blind alley to the main course of the development until Viète and Descartes – still staying within the European environment. As emphasized in the first leaf of the triptych, this is no search for priority; nor is it a cross-culturally comparative endeavour (note 23 being a minimal exception). It is, stricly, a piece of *local history*.

It is evident and generally recognized that Viète and Descartes provided the starting point for a wholly new kind of algebra.[2] It can be argued that their (rather different) algebras presupposed new ontologies.[3] More important for the *practising* of algebra (and thus for its future unfolding) were probably various technical changes: (1) the abstract representation of coefficients by letters; (2) the general parenthesis function; (3) the stabilization of symbolization; and (4) the free use of many unknowns (becoming variables, an aspect of the ontology).

I have dealt with aspects of (2) and (3) elsewhere.[4] Here I shall look at the gradually increasing use of several unknowns from Pacioli onward, and (in the end) on the connection to the question of abstract coefficients. For this, like another Köhler chimpanzee, I suddenly realized at a late moment how to combine a banana outside my reach to the left with a stick to the right. I leave the eating to my readers. *Buon appetito!*

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[1] The two contributions have since then been published – evidently in revised form – as [Høyrup 2019] and [Høyrup 2020].

[2] Since Thomas Harriot not only refers to Viète but also adopts his characteristic terminology (*logistica speciosa* and *numeroca*, *zetetices* etc.) in the posthumous *Artis analyticae praxis* [1631: 1–2 and *passim*], there is no reason to include him in the present discussion; cf. also [Stedall 2008]. Admittedly, his notation innovates with regard to Viète – but in a way that points toward Stifel (cf. [Seltman & Goulding 2007: 7]), probably as meditated by Borrel or Mennher (se below).

For similar reasons, we may leave Pierre de Fermat out of the picture.

[3] See, for instance, [Oaks 2018], dealing with Viète.

[4] With (2) in [Høyrup 2015] and with (3) in [Høyrup 2010] – both investigating the development from the Italian abbacus school onward.

Status after Pacioli

Pacioli presents the use of a second unknown called *quantità* (abbreviated q^a) in his *Summa* [1494] as a minor topic, first on fol. 148^v, where the *cosa* as well as the *quantità* are spoken of as *doi quantita sorde*, "two deaf quantities".[5] Returning to the topic in earnest on fol. $191^{v,}$ Pacioli states to explain it "only to show how one operates with a deaf quantity which the ancients called second things so as to distinguish them from the first position". He then operates with *three* unknowns, but eliminates the second one before introducing the third, which allows him to recycle the name *quantità*. In the end Pacioli explains that

> by means of these deaf quantities which the ancients called second *things* a great many strong problems can be solved by the one who handles the equations well.

The identity of Pacioli's "ancients" is a mystery, but at least Pacioli's reference to them shows that two unknowns had been practised in abbacus algebra more broadly than can be derived from the surviving sources (those of them where a second or further unknowns occur never give a name to the category). This is confirmed by the repeated appearance of a recycled second unknown in Nicolas Chuquet's *Triparty*, written in Lyons[6] in 1484 – see [Heeffer 2012: 134*f*]. Since the *Triparty* was a manuscript and apparently did not circulate (apart from Étienne de la Roche's selective use of it in [1520]), Chuquet was certainly not Pacioli's source. On the other hand, the shared principle supports Pacioli's claim that he presents existing ideas.

We shall have to speak about de la Roche again, but first we turn our attention to German lands ("Germany", like "Italy" and pre-colonial "India", was a cultural area but rarely a single state.

Around 1450, abbacus-type mathematics started to spread into southern Germany – the area involved in trade with and through Italy, in particular but not only Venice. Our sources show that quite a few of those who participated in the process had received a Latin university education, and also that a number of the latter had discovered the existence of a "new" mathematical discipline – new with respect to the ancient heritage and the mathematics that was taught in universities. This new discipline was *algebra*. Apart from minimal influence of al-Khwārizmī as translated by Gerard of Cremona and Robert of Chester, the algebra they took over was that of the Italian abbacus school. Not its most advanced level, and not what could be found in systematic treatises. The algebra that resulted was at first quite eclectic. Even when systematic treatises were

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[5] My translation, as all translations in the following where nothing different is stated.

[6] Lyons was the financial capital of France; much more abbacus-style teaching must have taken place there than we know about from extant manuscripts.

written in the outgoing 15th and initial 16th centuries they never dealt with more than one unknown, for which reason there is no reason to go into details.[7]

Rudolff

That was to change with Christoff Rudolff's *Coss* from [1525], the book that (in Thomas Kuhn's original sense, that is, the book everybody in a field reads and learns to emulate) became the paradigm for German algebra, known as *coß* – a term coming from the standard name for the unknown, borrowed from the north Italian orthography *cossa*. Rudolff not only makes use of a second unknown, he introduces it twice. On fol. I vi^{v}, early in the section dealing with "the first *coß*" (first-degree equations) he uses the *regula quantitatis* – the use of a second unknown called *quantitet*[8] – and speaks of it there as "a completion of the *coß*, indeed a completion without which it would not be worth much more than a trifle". Then, on fol. P vi^{v}, before the next time he uses it, he explains that the *regula quantitatis* teaches how to "avoid confusion and mistake" when extra positions beyond the usual one are needed.

In French, the corresponding name *Regle de la quantite* had been used by de la Roche a few years before [1520: 42^{r}, 61^{r}]. It is also used in a marginal note in Chuquet's *Triparty* (*Ceste regle est appelee La Regle de la quantite*, "this rule is called the rule of the quantity"), according to Albrecht Heeffer [2012: 134] in the hand of de la Roche – almost certainly an indication that the term was already circulating (and then used by de la Roche when he published his own work, dependent on the *Triparty* though deprived of its more innovative aspects). This is no proof that Rudolff took the expression from de la Roche – Rudolff drew on many sources which we cannot identify. On the other hand, de la Roche's book may well have been one of them.[9]

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7 Details and documentation will be found in [Høyrup, forthcoming, chapter V].

8 Often but no always abbreviated (inside symbolic equations as well as outside) *quant*, *quanti*, *quantit* or *q*.

9 De la Roche [1520: 61^{r}] speaks of the "rule of the quantity" as "accomplishment and perfection" of the "rule of the thing". Immediately afterwards, he speaks about the occasional need to make "two, three or more positions" and about how to avoid the confusion that would result if the same name were used for all. Both of Rudolff's presentations might contain echoes of this. On the other hand, de la Roche uses ρ for the *thing*, a notation neither in Pacioli's *Summa* nor in Chuquet's *Triparty*, which shows him to build also on the Italian manuscript tradition. All in all I tend to consider shared inspiration not only for the technique but also for the characterization to be as likely than direct borrowing.

Fog is added by de la Roche's abbreviations for the algebraic powers (wholly different from what he had seen in the *Triparty*). The first power, as just said, is ρ, which had been used by many Florentine writers since the late 14th century. The second power is , seemingly identical

Rudolff uses the technique copiously for problems of the first degree, more systematically and much more extensively than Pacioli, but not in different ways. His main step “forward” (i.e., forward toward us) is the emphasis on the importance of the rule. This he shares with de la Roche, but Rudolff’s book came to define the discipline while that of de la Roche had a limited impact only.[10]

Stifel

From our point of view, it seems strange that Chuquet, Pacioli, de la Roche and Rudolff all restricted themselves to one extra name for an unknown, which creates the need for a certain amount of dexterity when more than two unknowns are involved. After all, Benedetto had developed a technique of naming up to five unknowns when he needed it (and I cannot imagine how the intricate problems where he used this technique would have allowed him to eliminate the extra unknowns one by one).

In any case, the first algebra writer after Benedetto to create a system for naming more than two unknowns (and the first to have explained it *as a system*, which Benedetto did not do) appears to have been Michael Stifel in the *Arithmetica integra* from [1544]. This is done from fol. 252^{v} onward. Stifel still thinks of the *res* (“thing”) *r*[11] as a primary unknown (his headline is *De secundis radicibus*, “on second roots”). For these second

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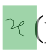
with an abbreviation (*ce* in ligature) used in southern Germany in the second half of the 15th century in the same function but also for the metrological unit *centenarius* (Munich, Clm 14908 fol. 90$^{r–v}$; Clm 14783 fol. 433^{r}, 439^{v}; Clm 14111, fol. 306^{v}). The third power is □, which seems to be wholly idiosyncratic.

[10] Petrus Ramus mentions no names in his succinct *Algebra* published anonymously in [1560] (probably anonymous because of a printer’s omission, many copies carry Ramus’s name inserted in ink in a way that emulates print). In any case, nothing in the contents of the booklet points to de la Roche; Ramus’s logorrhoeic *Schola mathematica* [1569] mentions many names, but in the vicinity of practical arithmetic and algebra only Johann Scheubel, Caspar Peucer and Christian Wurstisen (all three university professors writing in Latin). That is certainly no proof that Ramus did not know about de la Roche, but still evidence that he could easily be neglected.

Jacques Peletier’s *L’algebre* [1554] also mentions many names, in his case pertinent names. He even tells (p. 2) not to have seen the books of Pedro Nuñez, Rudolff and Adam Ries, but has nothing to say about de la Roche; it seems he had not heard about him at the moment (in his Latin *De occulta parte numerorum*, *quam algebram vocant* de la Roche does turn up [1560: * iiii^{r}]).

Jean Borrel [Buteo 1559: 189] may have learned the name *regula quantitatis* from de la Roche: he says explicitly to follow neither him nor Pacioli, and Pacioli, as we know, did not use the expression. As we shall see, his source for the technique itself is different.

[11] *r* here stands for what is actually written  (probably meant as *r*^{e}). Its second and third powers (*zensus* and *cubus*), written  and  by Stifel, I shall render *z* and *k*.

roots he uses the sequence of letters of the alphabet, “1*A* (that is, 1*Ar*), 1*B* (that is, 1*Br* 1*C* (that is, 1*Cr*), 1*D* etc.”; for their second powers he uses 1*Az* etc. For the product of *r* and *A* he suggests *rA*, while that of *A* and *B* will be written *AB*.[12] Stifel also shows what results from divisions of such products of powers of the unknowns (to use modern terms), though only such that do not lead to negative powers.

A first example (fol. 252v) is borrowed from Rudolff and only involves two unknowns. It is uninteresting on both accounts, involving nothing but a substitution of *A* for *q* (for this problem, admittedly, Rudolff uses only one unknown).

The next example (fol. 253v) teaches us much more. It is mathematically simple, belonging to a type which we may speak of as “all except each”:

> Seven men owe me money in this way. The first and second, third, fourth, fifth and sixth owe 142 florins. (Here observe, that only the debt of the seventh debtor is excluded from this amount of florins.) I posit therefore that the amount of the seventh is 1*r*, and thus that the amount of all the debts will be 142+*r*. The second, third, fourth, fifth, sixth and seventh owe 126 florins. (Here the debt of the first is excluded.) I posit therefore for the amount of the first 1*A* florins. And thus again the amount of all results, making 126+1*A*. [...].

The problem formulation continues cyclically. The following numbers are therefore equal (*r*, *A*, *B*, *C*, *D*, *E* and *F* being Stifel’s names for the respective debts):

$$142+1r$$
$$126+1A$$
$$136+1B$$
$$128+1C$$
$$130+1D$$
$$120+1E$$
$$148+1F$$

In Fibonacci’s *Liber abbaci* as well as a in number of abbacus treatises, such problems are solved without recourse to algebra. In the present case their authors would have observed that the sum 142+126+136+...+148 = 930 contains the sum of the debts seven times, less the same sum once. Dividing the sum by 6 therefore shows that the sum of the debts is 155.

But Stifel’s primary interest is not to solve problems: he wants to illustrate a technique, and therefore the possibility to eschew algebra does not interest him. Nor is there any reason he should point out that the problem allows easy recycling of the

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[12] That 1*A* is explained to stand for 1*Ar* shows that *A*, *B*, etc. are thought of as markings. So, all the first powers are *r*, but they are distinguished as ^{A}r (“the *A*-kind of *r*”), ^{B}r, etc. *r* standing to the left, on the other hand, is meant as a factor. This system seems somewhat heavy and prone to produce mistakes; as we shall see, Stifel would soon give it up,

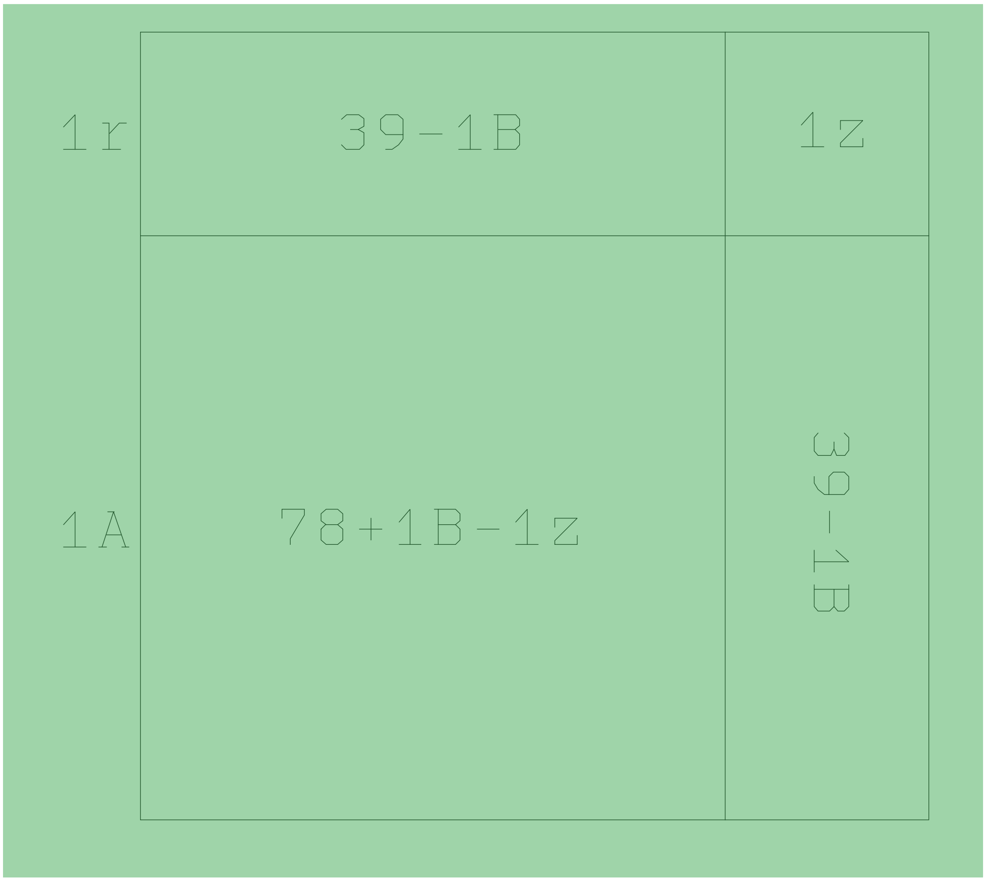
secondary unknown. He therefore proceeds as follows:

At first, from the equality of the first two amounts follows A = 16+1r. Similarly, B = 6+1r, C = 14+1r, D = 12+1r, E = 22+1r, F = 1r–6. Therefore A+B+C+D+E+F+r = 7r+64, which must therefore equal 142+1r, and further r = 13. From this the remaining debts can be found.

The last problem in this section making use of several unknowns is a reducible quartic (fol. 254v). It asks for two numbers (say, P and Q) fulfilling the condition

$$P^2+Q^2-(P+Q) = 78\ ,\quad PQ+(P+Q) = 39\ .$$

Stifel posits the first number to be r and the second to be A, and for convenience represents their sum by B. He proceeds in a way that has more to do with *Elements* II or with square-grid geometry than with algebra, using this diagram:



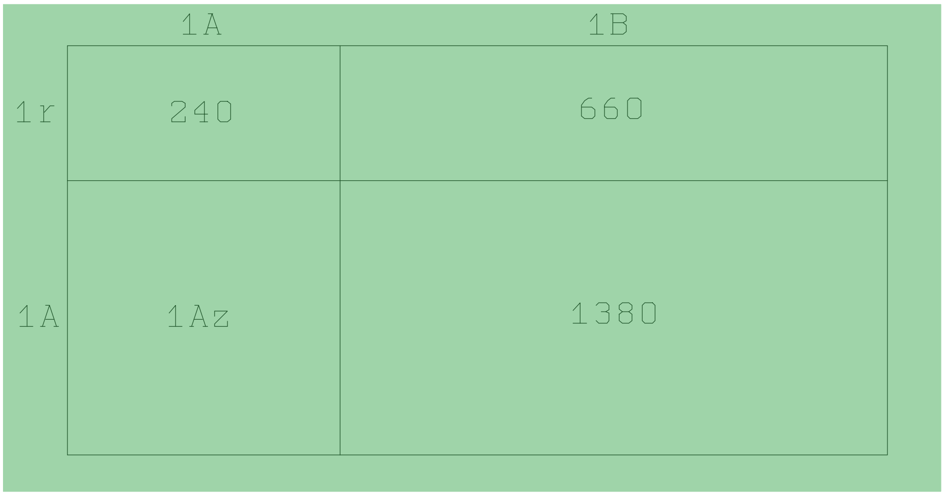
E

From the second condition he gets that rA = 39–1B. That allows him to complete the square, etc.

Two things are to be observed here. Firstly, that Stifel avoids using his new formalism in non-linear *algebra*. And secondly, that the geometric embedding allows him to take over from geometry the habit of naming more than a minimal set of unknowns by letters. In a lettered geometric diagram all occurring entities may indeed be treated on an equal footing.

Several algebraic unknowns return on fols 292^{r}–301^{r}. Most problems there are linear; they only go beyond the tradition in the choice of concrete topics dealt with in some of them. One (fol. 293^{v}) is said to be borrowed from the problem which causes Rudolff to make use of the rule of quantity for the first time; Rudolff's problem, however, deals simply with proportions between line segments, while Stifel speaks of sections of the monochord and harmonic intervals, thus linking the problem (rather gratuitously) to music theory. Another one (fol. 293^{v}) borrows its topic from Regiomontanus, who has it from Ptolemy. The others are traditional problems about "men having money" or partnership, mostly borrowed from or inspired by Adam Ries or Rudolff.

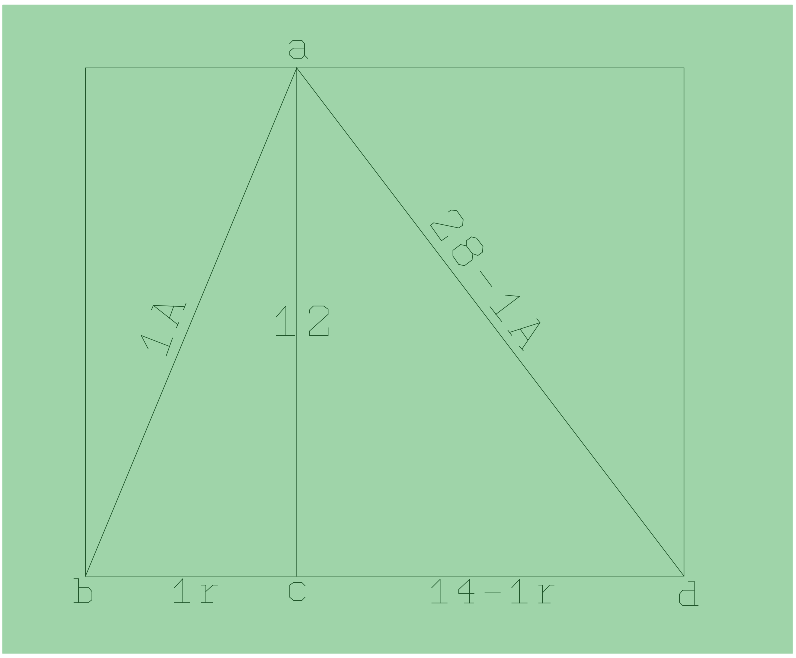
Only two are non-linear. The first of these (fol. 292^{r}) asks for three line segments, given the areas of the three rectangles they contain pairwise. They are posited as *r*, *A* and *B*. The solution follows from another square-grid diagram:



F

No algebra is used, and no equation formulated. But as we see, the second power of *A* is written as *Az*, cf. note ?.

The last problem (fol. 300^{v}) – according to Stifel more difficult than the preceding ones and chosen so as to show the potential of the technique – is inherently geometric but solved by means of algebra. It treats of a rectangle with sides 12 and 14, subdivided into two rectangles, the sum of whose diagonals is 28:



G

By the Pythagorean theorem ab^2 is found to be $12^2+(1r)^2 = 1z+144$, which is identified with $1Az$ (i.e., $(1A)^2$). ad^2 is found in a similar way to be $340–28r+24z$, but also to be $(28–1A)^2 = 784–56A+1Az$. This leads to $1Az = 56A+1z–28r–444$; but since $1Az = 1z+144$, A can be eliminated; the problem is thus of the second degree, but only in r.

In [1553], Stifel prepared an "improved and much augmented" edition of Rudolff's *Coss*. Here he replaces Rudolff's notation for the second unknown by his own, and also uses it in a number of problems that Rudolff had solved without using a second quantity. More interesting, however, is an *Anhang* containing 24 new problems. 12 of these (all of higher degree) make use of the new technique. One may have been borrowed from Pacioli [1494: 148^v], a question for two numbers fulfilling the conditions $mn = 96$, $m^2+n^2 =$ 292, where Stifel makes the positions $m := r+A$, $n := r–A$. Pacioli's parameters are different but his positions the same. The others I do not remember to have seen elsewhere (certainly no proof of whatever it might be). Two are solved by means of geometric diagrams similar to what we have already seen, the others algebraically. They are quite sophisticated.

There is one noteworthy change with respect to the *Arithmetica integra:* the potentially ambiguous notation for higher powers of the extra unknowns is left behind, instead of *Az* Stifel now writes *AA* (etc.); in one problem (fol. 469^r), powers until *k*, *zA*, *rAA* and *AAA* appear.

Immediate influence in Germany

Two works written in the German area show influence from Stifel's new technique.[13] One is Mattheus Nefe's *Zwey newe Rechenbuecher* from [1565]. Nefe does not treat algebra in general (very few *Rechenbücher* do), but he gives a single example of *Regula quantitatis* (fol. P iii^r). The accompanying words might be inspired by Rudolff – "without understanding this rule one can do little that is useful in the *coß*, and in others much less" – but he must also have known Stifel (directly or indirectly). His example is of type "all except each" and has four participants. Mathematically it is thus not innovative; but Nefe has seen that there is no reason to regard one unknown as primary; his names are therefore *A*, *B*, *C* and *D*.

Caspar Peucer's Latin *Logistice Regulae Arithmeticae, quam Cossam & Algebram quadratam vocant*, the second part of [Peucer 1556], was written and printed in Wittenberg and thus apparently meant for the higher Lutheran educational system; it does not seem to have been a conspicuous commercial success. Towards the end (fol. T vi^r) it contains a section *De radicibus secundis*, "on second roots", which initially refers to Rudolff, Cardano and Stifel. The notation is that of the *Arithmetica integra* (that is, "1 *A*, id est 1 *A z*", etc.). In agreement with the Humanist program, some examples are drawn from Greek arithmetical epigrams (mathematical riddles originally meant to be solved without algebra, and thus by necessity elementary). One is of type "all except each". None are spectacular.

Valentin Mennher

Much more interesting is Valentin Mennher's *Arithmetique seconde* from [1556]. Mennher was born in Kempten in Schwaben in 1520 and when young he worked as an accountant in the Fugger firm.[14] He became a citizen of Antwerpen in 1549, where he had come in Fugger service, and already in 1550 he had promised this "second arithmetic" (thus stated in the preface in 1556 – the promise must have been made in a first *Arithmetic* that is now lost).

This *Arithmetique seconde* consists of three parts. The first is a regular and well-structured *Rechenbuch*, the second an algebra, the third a geometry going well beyond the traditional geometries of the abbacus tradition, containing both a Euclidean *proof*

[13] There may be more – I have looked at around one hundred *Rechenbücher* from the 16th century, but there are many more (nor have I read all in detail).

Rechenbuch, "book about calculation", was the standard title for German commercial arithmetics of the time (and for centuries to come).

[14] See [Meskens 2013: 14*f* and *passim*], where Mennher is discussed copiously.

of the Pythagorean theorem (fol. S i^r) and an Archimedean determination of the ratio between the perimeter and the diameter of a circle.[15]

The second part, about algebra, explains on fol. F $iiii^r$ that understanding of "the high and liberal art of arithmetic is infinite"; several questions, moreover, "cannot be solved except by the very ingenious rule of algebra, or *cos*; as also commanded by the very subtle and liberal art of geometry". The "style and manner" of the very renowned Christoph Rudolff has been of great help, and Mennher has found him very competent. Therefore, he says, he has not deviated much from him, knowing well that the very renowned Michael Stifel has renewed and augmented him much in the same High German language with several beautiful examples – from which, however, "I have extracted a fair part of the best only, adding other matters needed by merchants".

First in the second part comes a very orderly exposition of traditional *coß* with a single unknown. Toward the end (fol. O i^v) we find the *regle de la quantité ou seconde radix*. Here Mennher is inspired by Stifel, yet without copying; his notation is the one used by Stifel in [1553]. Since not many books were around that presented the technique and apparently none treating it in depth we may safely assume that Mennher was working on his own. As we see, he is also not shy of speaking about his sources.

First (fol. O i^v) comes an "all less each" problem with four participants, similar to one in [Stifel 1553: 312^r], but with different parameters and a different choice of the unknowns. The third problem (fol. O ii^r) is similar to Stifel's two-number problem from [1544: 254^v]; the parameters are different, but the argument similar (including the use of a somewhat superfluous third unknown). Other two-number problems are similar to what we can find in [Stifel 1553] but solved differently – either by a better choice of the algebraic unknowns or by using a diagram instead of algebra; still others are without counterpart in Stifel.

Channels to France

The "new algebra" was produced in France: If it received some inspiration from the *coß* (as we shall see it did), in particular from its use of several unknowns, what were the channels?

Mennher, writing in French and publishing in Antwerpen, an internationally well

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[15] Traditional abbacus- and *Rechenmeister* geometries had always taken the Archimedean approximation 22 : 7 as a quasi-axiom (as does Mennher himself in the preceding pages).

We should take note that in [1564] Mennher published the "practice of spherical triangles, the distances on the globe, clocks, shadows, and other ingenious and new mathematical questions" – adding, thus the preamble, to what had been done by the very learned Regiomontanus (*viz*, in *De triangulis*) the labour of calculation.

connected city, was certainly one such channel. He was to present the *regle de la quantité* once again in [1565: Ffi^{r}–Ggii^{v}], by then expanding the treatment, though not much. He was not forgotten. Descartes' friend and mentor Isaac Beeckman (on whom more below) possessed one of Mennher's arithmetics at his death – see [Beeckmann 1637x: Biv^{v}]; in a letter from around 1666, moreover, John Collins [ed. Beeley & Scriba 2005] mentions Mennher together with Viète and Viète's translator Jean-Louis Vaulezard (etc.) as good introductions to algebra.

Mennher was not alone, however. Petrus Ramus, it is true, makes no use of several unknowns, nor does Johann Scheubel, whose *Algebrae compendiosa facilisque descriptio* was published in Paris in [1551] (and republished the next year, showing that it sold well). In his *L'algebre* from [1554], Peletier does.[16] On pp. 95–117, under the heading *Des racines secondes*, he presents Stifel's system from the *Arithmetica integra* together with a number of examples – all but one of the first degree. The one which is not coincides (parameters apart) from Stifel's first higher-degree problem; Peletier solves it by means of a geometric diagram, as does Stifel, but in contrast to Stifel he refers explicitly to *Elements* II (more precisely to *Elements* II.4).

Though promising *the foundations* for algebra in [1556], Pierre Forcadel does not deal with the topic itself, and *a fortiori* not with several unknowns. Jean Borrel does, however [Buteo 1559]. On p. 189, when introducing the *regula quantitatis*, he refers to Pacioli and de la Roche, leaving no doubt that he knew them. As can be seen throughout the book, Borrel only mentions precedessors by name when he can castigate them, so the absence of Stifel is no proof that he did not learn from him. His unknowns are 1*A*, 1*B*, etc., apparently the same reduction of Stifel's system as effectuated by Nefe (in his notation for a single unknown, Borrel has also changed what he had found in his model, still using ρ for the first power as de la Roche had done, but  for the second power and  for the third.[17]

Since Borrel presents only 5 problems using the technique, all of the first degree (pp. 189–196, 357) he has no occasion to introduce higher power or products of the unknowns. Nor would there be much to learn from him for anybody who had studied Mennher's *Arithmetique seconde* (or even Peletier's *L'algebre*).

Nor would there have been anything to learn *on this account* from Guillaume Gosselin's *De arte magna* from [1577]. Toward the end of that book (fols 80^{r}–84^{r}) Gosselin deals with the topic of several unknowns under the heading *De quantitate*

[16] He also does in his Latin algebra from [1560: 27^{r}–33^{v}].

[17] Curiously, Pacioli uses  for the *cosa* and  for the *censo* in *De divina proportione* [1509: 3^{v} and *passim*].

absoluta.[18] His names are the same as those of Borrel (that is, the same as those of Stifel apart from elimination of the *thing* as a primary unknown), and all his problems are of the first degree.

The transition

I have found no more French writers before Viète who trade in several algebraic unknowns. Gosselin in [1577: 80r] also refers to Pacioli, de la Roche, Cardano, Borrel and unspecified "others" only; these latter may well be the Germans. We shall therefore turn our attention to Viète and Descartes – but in reverse order, since more can be said about Descartes' inspiration than about that of Viète

First, however, some generalities will be adequate. As generally recognized and as said here in the beginning, Viète and Descartes did not just use several unknowns freely, each in his particular way. They also introduced the use of *abstract* coefficients, thereby creating the foundation for Modern algebra. Descartes does not seem to have been inspired by Viète – the undertakings as well as the notations are radically different, only *post festum* to be seen as belonging together, and Descartes appears to have learned about Viète's work only when his own was close to maturation. We must therefore ask which new, and shared, conditions could call for parallel inventions.

Benedetto's creation of a technique for symbolic operations with five unknowns was an outgrowth from a particular aspect of the Italian abbacus culture: a culture of agonistic challenges. This culture was not new in the mid-15th century; it had already characterized the "proto-abbacus culture" spanning the Mediterranean coastal cities in Fibonacci's time: in the *Liber abbaci* the latter refers repeatedly to question he had been asked in Constantinople, obviously as challenges; his *Liber quadratorum* and much of his *Flos* were also answers to such challenges, this time posed by Frederick II's court philosophers.

Rechenmeister culture was different. The *Rechenmeister* were certainly commercial competitors no less than the abbacus masters had been, and their struggles might even end up in court. But theirs was a print culture, and once a mathematical invention had been published in a *Rechenbuch*, everybody sufficiently competent might learn. So, their intellectual competition was located in the book market; by far the larger part of the *Rechenbücher* published in the 16th century claim on their title page to be *new, never seen before* (which was rarely true, and never completely true).

The beginning of the *coß* was somewhat different. With the exception of Heinrich

[18] This name may have been taken over from Pedro Nuñez [1567: 224v], a book referred to by Gosselin. Nuñez operates with *cosa* and *quantidad*, apparently borrowing from Pacioli, to whom Nuñez refers (fol. 225v).

Schreyber, Rudolff's predecessors wrote manuscripts that were only printed in the 20th century (if ever), and by then only as interesting historical sources. As a rule (though we do not know for sure about everybody) they were university scholars. Once Rudolff's *Coss* had been printed, nobody tried to outcompete this paradigm (nor to deny its existence). But even Rudolff was not competing, he taught *methods* (at times with a shade of arrogance), not clever particular problem solutions. Stifel's *Arithmetica integra* went further in this direction, still teaching theory (doing it well) and methods. That, as said, is why his illustrations of the use of several unknowns could use simple examples, at times examples that could have been solved without any use at all of algebra. This has always been the way of mathematicians presenting a new technique: taking some well-known problem and illustrating how a new invention works better than what was done before.[19]

Viète and Descartes, on their part, participated in still another mathematical culture – once again agonistic, intellectually competitive (not commercially competitive – as mathematicians they were amateurs who might perhaps gain protection from the dedicatees of their books but no money). As an illustration we may quote a familiar story [Busard 1976: 22]:

> Viete's mathematical reputation was already considerable when the ambassador from the Netherlands remarked to Henry IV that France did not possess any geometricians capable of solving a problem propounded in 1593 by Adrian Romanus [van Roomen] to all mathematicians and that required the solution of a forty-fifth-degree equation. The king thereupon summoned Viete and informed him of the challenge. Viete saw that the equation was satisfied by the chord of a circle (of unit radius) that subtends an angle $2\pi/45$ at the center. In a few minutes he gave the king one solution of the problem written in pencil and, the next day, twenty-two more.

In the Italian Hight Renaissance, good Latin style had been not only a matter of prestige but also a diplomatic weapon for Florence and other Italian Renaissance city states. Now, two centuries later, mathematical prowess had thus become an *affaire d'état*. However, the battlefield was no longer defined by sophisticated versions of recreational classics – "purchase of a horse", "men finding a purse", etc. In the van-Roomen–Viète case, as we see, it had to do with new developments of trigonometry, but more broadly it grew out of late Humanism.

14th- and early 15th-century Humanism – that of Francesco Petrarca, Giovanni Boccaccio, etc. – had been totally ignorant of mathematics and not been interested in learning. A slight change set in (with Bessarion, Leon Battista Alberti and others) after

[19] "Always" – that is, at least since al-Khwārizmī demonstrated 1200 years ago how "divided 10" problems could be solved by means of algebra.

1450, but editions of the Greek mathematical classics and translations directly from the Greek were only made systematically from *ca* 1500 onward.[20]

In parallel with this, we see that 16th-century French mathematical writers such as Borrel and Gosselin,[21] even when writing about abbacus-Rechenmeister mathematics, played the Humanist card, on one hand by the dress of the problems they presented (borrowed, for instance, from Greek epigrams, or speaking of Alexander the Great), on the other by using a Grecizing terminology.

The mathematical level of a Viète and a Descartes was much higher. If they were to Grecise, it was not enough to show Humanist allegiance in their terminology (although Viète's do-it-yourself Greek certainly did so too). In the age of printing, as already observed, abbacus-style riddles were no longer fit as intellectual challenges, at least not at their level. The kind of problems by which late-16th-century mathematicians challenged each other is reflected in Viète's *Variorum de rebus mathematicis responsorum liber VIII*, "Book 8 of various responses about mathematical matters" [1593]:

- two intermediate proportionals;
- squaring and rectification of the circle and of circular segments, using Archimedean spirals and the quadratrix;
- construction of a regular heptagon;
- lunules; etc.

In the end comes much spherical trigonometry, a topic that had Viète's special interest and the only topic pointing to broader practices (astronomy and navigation). Soon, Fermat, Roberval and course Descartes were to extend the field of interest, within as well as beyond the Greek horizon.

Traditional recreational problems leading to the use of several algebraic unknowns would be formulated as paradigmatic examples. For instance – I quote the problem that first causes Benedetto to introduce five algebraic unknowns,[22]

> Four have *denari*, and walking on a road they found a purse with *denari*. <u>The first and</u>

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[20] Bartolomeo Zamberti's translation of the *Elements* in 1505, and Grynaeus's complete edition of Euclid with Proclos's commentary in 1533. Pappos's *Collection* appeared in Greek in Basel in 1538, and Commandino's Latin translation in 1588. The *editio princeps* of Archimedes was printed in 1544, and Memmo's Latin translation of Apollonios's *Conics* I–IV appeared in 1537 (that of Commandino in 1566); Xylander's Latin translation of Diophantos was published in 1575 (the Greek text only in 1621).

[21] They were not alone. Peucer, Cuthbert Tunstall and Scheubel did as much, but these parallels do not concern us here except for the fact that Tunstall as well as Scheubel were re-published in Paris and were thus expected by local printers to find a receptive public there.

[22] Siena, Biblioteca degl'Intronati L.IV.21, fol. 270^{v}.

second say to the third, if you give us the purse we shall have 2 times as much as you. The second and third men say to the fourth, if we had the *denari* of the purse we should have 3 times as much as you. The third and fourth say to the first, if we had the *denari* of the purse we should have 4 times as much as you. The fourth and the first say to the second, if you give us the *denari* of the purse we shall have 5 times as much as you. It is asked how much each one had, and how many *denari* there were in the purse.

It would be difficult to formulate this (and even more difficult to follow it) without numerically fixed factors (I already felt the need to clarify the structure by means of underlinings that have no counterpart in the manuscript). Just try to formulate this in terms of “the first factor”, “the second factor”, etc., and then to go through the calculations. (Fibonacci does go through the calculations by means of rhetorical quasi-algebra in the *Liber abbaci* [ed. Giusti 2020: 372] but evidently with identified factors.)

It would be just as difficult, on the other hand, to follow this geometric proof [trans. Heath 1926: I, 349] if we had no diagram and were not allowed to draw one:

> Let *ABC* be a right-angled triangle having the angle *BAC* right; I say that the square on *BC* is equal to the squares on *BA*, *AC*. For let there be described on *BC* the square *BDEC*, and on *BA*, *AC* the squares *GB*, *HC*; through *A* let *AL* be drawn parallel to either *BD* or *CE*, and let *AD*, *FC* be joined. Then, since each of the angles *BAC*, *BAG* is right, it follows that with a straight line *BA*, and at the point *A* on it the two straight lines *AC*, *AG* not lying on the same side make the adjacent angles equal to two right angles; therefore *CA* is in a straight line with *AG*. [...].

As can be guessed from the first period of the quotation, this is (the first half of) Euclid’s proof of the Pythagorean theorem. The letters of the (here missing) diagram serve to replace the abstract line segments with something just as manifest as the numerical coefficients in the purse problem.

The formulation of the theorem itself,

> In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle

is properly abstract. The contrast with the ensuing demonstration illustrates that the latter is not abstract but a paradigmatic example.

We may also look at what Euclid does when an undetermined *number* is involved in a theorem, for instance in *Elements* V.1 [trans. Heath 1926: 138]:

> If there be any number of magnitudes whatever which are, respectively, equimultiples of any magnitude equal in multitude, then, whatever multiple one of the magnitude is of one, that multiple also will all be of all.

In *the proof*, Euclid takes the “any number of magnitudes” to be 2, and similarly, the “equimultiples ... equal in multitude” to be 2. Even here the proof is performed on a paradigmatic example – neither more nor less general than the solutions offered by

Fibonacci and Benedetto for their horse problems.

When 17th-century geometers were confronted with a problem, they looked at or drew a lettered diagram, that is, a paradigmatic example showing its structure. Then, what did they do if they wanted to solve it by means of algebra? Descartes [trans. Smith & Latham 1925: 6–9] tells us:

> If, then, we wish to solve any problem, we first suppose the solution already effected, and give names to all the lines that seem needful for its construction,– to those that are unknown as well as to those that are known. Then, making no distinction between known and unknown lines, we must unravel the difficulty in any way that shows most naturally the relations between these lines.

That is, just as said above in connection with Stifel, Descartes does not look for a minimal set of unknowns – everything that seems to play a role gets a name. Once the idea to apply algebra with several unknowns to geometric problems, pseudo-abstract names (that is, not numbers but names linked to specific entities appearing in a diagram) would be used for everything pertinent with no distinction between what would turn up as coefficients in the equations and what would turn up as unknowns.

Only later, Descartes introduces (but does not explain) the principle to use letters from the end of the alphabet (*z*, *y*, *x*) for the unknown magnitudes and letters from its beginning for those that are known.

Even Viète's distinction between known and unknown entities is secondary, introduced when problems are to be *solved*. In the beginning of *In artem analyticem isagoge* [1591], Viète also used the letters *A*, *B*, *Z*, *G*, *D* indiscriminately. Then, when coming (fol. 7ʳ) to the *De legibus Zetetis* ("the laws of equation formulation"), he explains to use vowels for the magnitudes that are asked for and consonants for the given elements "because it will somehow be useful for the art".

Once Descartes' and Viète's equations were there, of course, the magnitudes that were supposed to be known would be *coefficients*, and these (being in the equation and no longer in the diagram) would be genuinely abstract. We may claim that they did not *invent* abstract coefficients – they were forced upon them. Thereby – but that is peripheral to the present discussion – the *unknowns* were transformed into *variables*, changing with the variations of the coefficients.

So, in spite of the decisive difference made by the introduction of abstract coefficients, their *introduction* (rather than "invention") was a "natural" consequence of the new kind of questions that were asked (namely geometric questions) – but natural *only* because of the idea to apply to these an algebra making use of several unknowns.[23]

How did they learn?

In order to establish whether Viète and Descartes reinvented this algebraic gunpowder or learned about the use of several unknowns from their predecessors we shall round off by asking from where they learned their algebra. Since Descartes is once again the easy case, we shall look at him first.

In the *Discours de la methode* [ed. Adam & Tannery 1902: 17] we find that, as a youngster, he had read some

> logic and, among the mathematics, the analysis of the geometers and algebra, three arts or sciences which seemed to promise something for my purpose.

The combination leaves no doubt that these topics he had studied at the Jesuit college of La Flèche, from where he graduated in 1614. There he had been taught algebra on the basis of Clavius's textbook [1608], a very pedagogical introduction to *coß* in German style. His first algebra was thus in debt to the cossic tradition, not least to Stifel. From this book (p. 72 onward) he may also already have learned about the use of several unknowns, in Stifel's notation from 1544 (not 1553), but expanded also to negative powers.

Thinking back at a moment when he had created something better, Descartes [ed.

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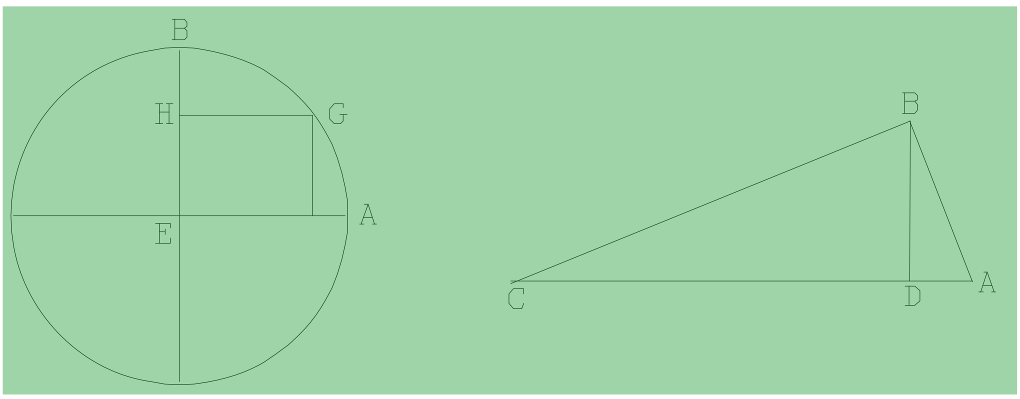
[23] In contrast we may look at al-Khayyāmī, who operated with a single unknown only, but certainly not inferior to Viète and Descartes as a mathematician. A small treatise of his [ed. trans. Rashed 1981: 73–90] deals with a particular partition of a circular arc. The arc AB (left in the below diagram) is to be divided at G in such a way that $AE : GH = H : HB$. A long analysis reduces this to the finding of a right-angled triangle ABC (right in the diagram), with height BD, in which $AB+BD = AC$. In order to apply *his* algebra with only one unknown, al-Khayyāmī needs to posit that $AD = 10$; that leads him to an equation whose coefficients are numerically fixed. Viète and Descartes would have posited AD to be, for instance, b, which would automatically (though obviously after as much calculation as made by al-Khayyāmī) have given then an equation with abstract coefficients.



Adam & Tannery 1897: VI, 17] thought of what he had been taught on the basis of Clavius's book as "a confused and obscure art that puts the mind in difficulty instead of a science that cultivates it". Yet in a letter to Beeckman from 1619, Descartes had still used Clavius's notation; it is the same letter that expresses the ambition to solve all problems "dealing with any kind of quantities, discrete as well as continuous", by means of curves corresponding to higher-degree equations [ed. Adam & Tannery 1897: 154–160, quotation p. 157] – the project that eventually came to fruition in the *Geometrie*.

In 1628 Descartes met Beeckman again; by then the project had grown to encompass something like the future *Geometrie*, and by then Beeckman may have shown him Mennher, or Descartes may have discovered Mennher in Beeckman's library.

In the *Geometrie* Descartes also knows Cardano [ed. Adam & Tannery 1897: VI, 471–474], but from him he cannot have learned anything better than from Clavius concerning several unknowns (nor from Borrel or Gosselin, should he have read them).

As often observed, in the *Geometrie* Descartes prefers *aa* (and similarly) for the second power even though he had now created the symbolism allowing him to write a^3 for the third. This proves nothing but might reflect influence from Mennher.

Viète says little about his algebraic inspiration. He speaks of algebra as "a new art, or rather so old and so defiled and polluted by barbarians that I have found it necessary to bring it into, and invent, a completely new form" [1591a: 2^v]. We may discard as a fancy any hypothesis that he should have read Arabic, medieval Latin or Italian manuscripts. His "old" art *could* be that of Mennher – after all, they shared an interest in spherical geometry; for the same reason, he may have read Nuñez. It seems most likely, however, that Viète took his inspiration from French writers. As pointed out by Frédéric Ritter [1895: 11], he appears to have had among his personal acquaintances Ramus, Forcadel, and Gosselin.[24] Moreover, Viète's terminology for the algebraic powers [1591a: 4^v–5^r] – *latus*, *quadratum*, *cubus*, *quadrato-quadratum*, *quadrato-cubus*, *cubo-cubus*, *quadrato-cubo-cubus*, *cubo-cubo-cubus*, close to what he could have found in Xylander's translation [1575: 1] of Diophantos, is not *close to* but *identical with* that of Gosselin as far as it goes (Gosselin, like Viète, uses hyphens within the composite names, Xylander doesn't). There can be no doubt that Viète borrowed from Gosselin.

Evidently, the borrowing of a ready-made Latin *terminology* from Gosselin is no proof that Viète had no better sources for his *mathematics*. Gosselin, after all, makes nothing advanced. However, when we go beyond the terminology for the powers, Viète transmutes what he may have found in predecessors radically – "I have found it necessary to bring it into, and invent, a completely new form", we remember. That prevents us

[24] Cf. also [Witmer 1983: 4 n. 7].

from tracing in his texts whom he read (or with whom he discussed). We may just take note that the use of several algebraic unknowns was no secret at the time, Viète could have found it at least in rudimentary form in most of the works he can be conjectured to have read more or less thoroughly (Ramus and his favourite Forcadel being the exceptions).[25] So, even Viète had access to the tool which made the shift to abstract coefficients an obvious move.

All in all we may conclude that Viète's and Descartes' fundamental innovation in algebra was *not* the introduction of abstract coefficients, however much that would turn out in the long run to be their most important contribution to the transformation of mathematics (together with Descartes' introduction of the general parenthesis function). It was *the application of algebra to geometric problems*, often of ancient Greek descent but first of all too complex to be dealt with by means of a single unknown – that is, a *change in algebraic practice suggesting new tools*.

References

Adam, Charles, & Paul Tannery (eds), 1897. *Oeuvres* de Descartes. 12 vols. Paris: Léopold Cerf, 1897–1910.

Beeckman, Isaac, 1637x. *Catalogus Variorum & insignium Librorum Clarissimi Doctissimique viri D. Isaaci Catalogus ... Quorum Auctio habebitur in aedibus defuncti ad diem 14 Iulij M DC XXXVII.* Dordrecht: Isaac Andreae.

Beeley, Philip, & Christoph J. Scriba (eds), 2005. *The Correspondence of John Wallis*. II. *(1660–September 1668)*. Oxford & New York: Oxford University Press.

Busard, Hubert L. L., 1976. "Viète, François", pp. 18–25 *in Dictionary of Scientific Biography*, vol. XIV. New York: Scribner.

Buteo, Joannes, 1559. *Logistica, quae et arithmética vulgò dicitur in libros quinque digesta*. Lyon: Guillaume Rouillé.

Cardano, Girolamo, 1539. *Practica arithmetice, et mensurandi singularis*. Milano: Bernardini Calusco.

Clavius, Cristopher, 1608. *Algebra.* Roma: Bartolomeo Zanetti.

de la Roche, Etienne, 1520. *Larismethique novellement composee*. Lyon: Constantin Fradin.

Forcadel, Pierre, 1556. *L'arithmeticque*. Paris: Guillaume Cavellat.

Giusti, Enrico (ed.), 2020. Leonardi Bigolli Pisano *vulgo* Fibonacci *Liber Abbaci*. Firenze: Leo

[25] One work which Viète does refer to a few times (because it gives him the occasion for censure) in [1591b: 9^{v}, 10^{r}] is Cardano's *Practica arithmetice et mensurandi singularis* from [1539] (thus *not* the *Ars magna*). Even in this work (L viii^{r}) the use of a second unknown is presented as *modus pulcher operandi*, "a beautiful way to operate"), in a language that points to Pacioli (the second unknown is *quantitas surda*, "a deaf quantity"). Just like many of the other books which Viète *may* have read or perused, this work uses the method for linear problems only.

S. Olschki.
Gosselin, Guillaume, 1577. *De arte magna, seu de occulta parte numerorum quae et Almucabala vulgo dicitur*. Paris: Egide Beys.
Harriot, Thomas, 1631. *Artis analyticae praxis*. London: Robert Barker.
Heath, Thomas L. (ed., trans.), 1926. *The Thirteen Books of Euclid's Elements*. 2nd revised edition. 3 vols. Cambridge: Cambridge University Press / New York: Macmillan.
Heeffer, Albrecht, 2012. "The Rule of Quantity by Chuquet and de la Roche and its Influence on German Cossic Algebra", pp. 127–147 *in* Sabine Rommevaux, Maryvonne Spiesser & Maria Rosa Massa Esteve (eds), 2012. *Pluralité de l'algèbre à la Renaissance*. Paris: Honoré Champion.
Høyrup, Jens, 2010. "Hesitating progress – the slow development toward algebraic symbolization in abbacus- and related manuscripts, c. 1300 to c. 1550", pp. 3–56 *in* Albrecht Heeffer & Maarten Van Dyck (eds), *Philosophical Aspects of Symbolic Reasoning in Early Modern Mathematics*. London: College Publications.
Høyrup, Jens, 2015. "Embedding: Another Case of stumbling progress". *Physis* **50**, 1–38.
Høyrup, Jens, 2019. "Reinventing or Borrowing Hot Water? Early Latin and Tuscan Algebraic Operations with Two Unknowns". *Gaṇita Bhāratī* **41**, 115-159.
Høyrup, Jens, 2020. "Fifteenth-Century Italian Symbolic Algebraic Calculation with Four or Five Unknowns". *Gaṇita Bhāratī* **42**, 161–192.
Høyrup, forthcoming. *The World of the Abbaco: Abbacus Mathematics Analyzed and Situated Historically between Fibonacci and Stifel*. Cham etc.: Springer.
Mennher, Valentin, 1556. *Arithmetique seconde*. Antwerpen: Ian Loë.
Mennher, Valentin, 1564. *Praticque des triangles sphériques, des distances sur les globes, et autres ingenieuses et nouvelles questions mathematique*. Antwerpen: Gilles Coppenius.
Mennher, Valentin, 1565. *Praticque pour brievement apprendre à ciffrer, et tenir livre de compte, avec la regle de coss, et geometrie*. Antwerpen.
Meskens, Ad, 2013. *Practical Mathematics in a Commercial Metropolis: Mathematical Life in Late 16th Century Antwerp*. Dordrecht etc.: Springer.
Nefe, Mattheus, 1565. *Arithmetica: Zwey newe Rechenbuecher, das erste auff der Linien und Federn, darinn die gebreuchlichsten Regeln mit gruendlichem Unterricht erklert... das ander ist die kuenstliche Rechnung, wie man rechnet, wie weit von einer Stadt zur andern ist*. Breslau.
Nuñez, Pedro, 1567. *Libro de Algebra en Arithmetica y Geometria*. Antwerpen: Herederos d'Arnaldo Birckman.
Oaks, Jeffrey A., 2018. "François Viète's Revolution in Algebra". *Archive for History of Exact Sciences* **72**, 245–302.
Pacioli, Luca, 1494. *Summa de Arithmetica Geometria Proportioni et Proportionalita*. Venezia: Paganino de Paganini.
Pacioli, Luca, 1509. *Divina proportione*. Venezia: Paganius Paganinus.
Peletier, Jacques, 1554. *L'algebre*. Lyon: Ian de Tournes.
Peletier, Jacques, 1560. *De occulta parte numerorum, quam algebram vocant, libri duo*. Paris:

Guillaume Cavellat.
[Ramus, Petrus], 1560. *Algebra*. Paris: Andreas Wechelum.
Ritter, Frédéric, 1895. *François Viète, inventeur de l'algèbre moderne, 1540–1603: Notice sur so vie et son œuvre*. Paris: La Revue occidentale.
Rudolff, Christoff, 1525. *Behend unnd hübsch Rechnung durch die kunstreichen Regeln Algebra, so gemeincklich die Coss genennt werden*. Straßburg.
Scheubel, Johann, 1551. *Algebrae compendiosa facilísque descriptio, qua depromuntur magna Arithmetices miracula*. Parisiis: Guillaume Cavellat.
Seltman, Muriel, & Robert Goulding (ed., trans.), 2007. Thomas Harriot's *Artis analyticae praxis*. An English Translation with Commentary. New York: Springer.
Smith, David Eugene, & Marcia L. Latham (eds, trans.), 1925. The *Geometry* of René Descartes, translated from the French and Latin. Chicago: Open Court.
Stedall, Jacqueline, 2008. "Notes made by Thomas Harriot on the Treatises of François Viète". *Archive for History of Exact Sciences* **62**, 179–200.
Stifel, Michael, 1544. *Arithmetica integra*. Nürnberg: Johan Petreius, 1544.
Stifel, Michael, 1553. *Die Coss* Christoffs Rudolffs. Die schönen Exemplen der Coss gebessert und gemehrt. Königsberg in Preussen: Alexander Lutomyslensis, 1553.
Viète, François, 1591a. *In artem analyticem isagoge*. Tours: Jamet Mettayer.
Viète, François, 1591b. *Zeteticorum libri quinque*. Tours: Jamet Mettayer.
Viète, François, 1593. *Variorum de rebus mathematicis responsorum, liber VIII*. Tours: Jamet Mettayer.
Witmer, T. Richard (ed., trans.), 1983. François Viète, *The Analytic Art: Nine Studies in Algebra, Geometry, and Trigonometry from the Opus restitutae mathematicae analyseos, seu algebrâ novâ*. Kent, Ohio: Kent State University Press.
Xylander, Wilhelm (ed., trans.), 1575. Diophanti Alexandrini *Rerum arithmeticarum libri sex*. Item *Liber de polygonis*. Basel: Eusebius Episcopius.