Hybrid logic with propositional quantiers

Natural deduction style (Work in progress)

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Two-layered Belnapian logics for uncertainty

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When it comes to information, its potential incompleteness, uncertainty, and contradictoriness needs to be dealt with adequately. Separately, these characteristics have been taken into account by various appropriate logical formalisms and (classical) probability theory. While incompleteness and uncertainty are typically accommodated within one formalism, e.g. within various models of imprecise probability, contradictoriness and uncertainty less so — conflict or contradictoriness of information is rather chosen to be resolved than to be reasoned with. To reason with conflicting information, positive and negative support—evidence in favour and evidence against—a statement are quantified separately in the semantics. This two-dimensionality gives rise to logics interpreted over twisted-product algebras or bi-lattices, the well known Belnap-Dunn logic of First Degree Entailment being a prominent example [2, 6].

In a spirit similar to Belnap-Dunn logic, one can introduce many-valued logics for uncertainty which are interpreted over twisted-product algebras based on the $[0, 1]$ real interval. They can be seen to account for the two-dimensionality of positive and negative component of (the degree of) belief based on potentially contradictory information. The logics presented in this talk include extensions of Łukasiewicz or Gödel logic with a de-Morgan negation which swaps between the positive and negative semantical component. The extensions of Gödel logic in particular relate to the extensions of Nelson's paraconsistent logic $N4$ [10, 11], or Wansing's paraconsistent logic I_4C_4 [12], with the prelinearity axiom. The resulting logics inherit (finite) standard completeness and decidability properties of Łukasiewicz or Gödel logic respectively, and allow for an efficient reasoning using the constraint tableaux calculi formalism [3].

Many-valued logics with such a two-dimensional semantics can be applied to reason about belief based on evidence within a two-layer logical framework. Two-layer logics for reasoning under uncertainty were introduced in [7, 8], and developed further within an abstract algebraic framework by [5] and [1]. They separate two layers of reasoning: the lower layer consists of a logic chosen to reason about events (often classical propositional logic interpreted over sets of possible worlds), the connecting modalities are interpreted by a chosen uncertainty measure on propositions of the lower layer (typically a probability or a belief function), and the upper layer consists of a logical framework to reason about probabilities or beliefs. The modalities apply to lower level formulas only, to produce upper level atomic formulas, and they never nest. Logics introduced in [7] use classical propositional logic on the lower level, and reasoning with linear inequalities on the upper level. [8] on the other hand uses Łukasiewicz logic on the upper level, to capture the quantitative reasoning about probabilities within a propositional logical language.

Building on that idea and having in mind the two-dimensionality of information, another two-layer modal logic has been introduced to reason with non-standard probabilities [9] in our recent conference paper [4]. It presents a logical framework in which belief is based on potentially contradictory information obtained from multiple, possibly conflicting, sources and is of a probabilistic nature. It uses an extension of Łukasiewicz logic on the upper layer, Belnap-Dunn logic on the lower layer to model evidence, and its probabilistic extension [9] to give rise to a belief modality. The many-valued logics with two-dimensional semantics mentioned above can all be used on the upper layer in this framework to reason about agent's beliefs, producing two-layer logics suitable for various scenarios: extensions of Łukasiewicz logic are adequate in cases when aggregated evidence yields a non-standard probability measure or a belief function (on a De Morgan algebra), while extensions of Gödel logic are useful to reason about comparative uncertainty in cases where it is not so.

(This talk is rooted in joint work with S. Frittella, D. Kozhemiachenko, O. Majer and S. Nazari.)

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Is Intuitionistic Mathematics Compatible with Classical Logic?

Liron Cohen

Intuitionistic mathematics, as conceived by Brouwer, extends the standard Curch-Turing notion of effective algorithmic constructions by also admitting constructions based on temporal intuitions. In particular, the key notion of infinitely proceeding sequences of freely chosen objects, known as free choice sequences, regards functions as being constructed over time.

In this talk, we will describe how, despite this stronger computational power, free choice sequences can be embedded in an implemented formal framework, namely the constructive type theory of the Nuprl proof assistant. This implementation requires a major overhaul to all components of the Nuprl proof assistant, and in particular an extension of the notion of truth to a multi-world truth. We will present two possible models and show that while one is inconsistent with classical reasoning, the other is compatible with it. We will also discuss broader implications of supporting such an extended notion of computability in a formal system, focusing on formal verification and constructive mathematics.

Potentialism and Critical Plural Logic

Øystein Linnebo

Potentialism is the view that certain types of entity are successively generated, in such a way that it is impossible to complete the process of generation. What is the correct logic for reasoning about all entities of some such type? Under some plausible assumptions, classical first-order logic has been shown to remain valid, whereas the traditional logic of plurals needs to be restricted. Here I seek to answer the open question of what is the correct plural logic for reasoning about such domains. The answer takes the form of a critical plural logic. An unexpected benefit of this new logic is that it paves the way for an alternative analysis of potentialism, which is simpler and more user-friendly than the extant modal analysis.

From definability in finitely-valued modal logic

Carles Noguera

This talk will present some new results (obtained in cooperation with Guillermo Badia and Xavier Caicedo) in the area of many-valued modal logics. After introducing the topic, we will establish two main facts: (1) in finitely-valued modal logics we cannot define more classes of frames than are already definable in classical modal logic, and (2) a large family of finitely-valued modal logics define exactly the same classes of frames as classical modal logic (including modal logics based on finite Heyting and MV-algebras). In particular, we will have that the celebrated Goldblatt-Thomason theorem applies immediately to these logics. Therefore, we obtain the central result of a previous work by Teheux (a generalization of Goldblatt-Thomason theorem for finitely-valued modal Lukasiewicz logics) with a much simpler proof and for a much wider class of logics, and answer one of the open questions left in that paper.

Normative Dilemmas, Dialetheias, and their Modal Logic

Graham Priest

Systems of norms deliver (at least arguably) both dilemmas and dialetheias. In the first part of the talk I will illustrate this with examples concerning norms of law, morality, and rationality. In the second part I will discuss the appropriate modal logic of obligation. Unsurprisingly, a paraconsistent logic is required to do justice to matters.

New results on Kripke completeness and incompleteness in modal predicative logic

Valentin Shehtman

As noticed some 50 years ago, Kripke frame semantics does not fit well for modal predicate logics. This contrasts to the propositional case, when we can expect the logics to be complete.

The talk gives an overview of some well-known and some recent results in this field. We are mainly interested in minimal predicate extensions of propositional logics and their Kripke completions. In particular we consider a certain operation of 'boxing', show that it preserves strong Kripke completeness and find axiomatization for 'boxed' logics. We also axiomatize logics of some predicate Kripke frames based on finite trees.

Revision without revision? Two case studies in inconsistent mathematics

Zach Weber

Inconsistent mathematics (IM) is a branch of non-classical mathematics that allows some contradiction, by using a background paraconsistent logic. Since IM allows inconsistent theories, but classical mathematics does not, there is a natural sense in which IM is revisionary, challenging or rejecting some aspects of conventionally accepted mathematics. However, most of the (few) people who have worked in IM have not advocated for revisionism; they suggest various ways it is more conservative after all, and seek instead reassurance that under IM no classical mathematics is lost. In this talk I explore whether IM is best thought of as revisionist, by looking at two case studies: first, the expressive completeness of the logic LP (studied by Omori and Weber); and second, the cardinality of the set of all computable functions (studied by Sylvan and Copeland). Both of these questions have well-established classical answers. But when approached from IM we see a sense in which those answers can be challenged—yet without simply 'overturning' received wisdom. In light of this I ask whether IM offers revision without revision, and, appropriately enough, whether this provides reassurance without reassurance.

Abstracts of contributed talks

Maximality of logic without identity

Guillermo Badia, Xavier Caicedo and Carles Noguera

In the 1960s, Lindström [11] showed that first-order logic is the maximal logic (in terms of expressive power) satisfying certain combinations of model-theoretic results. The best known of these combinations are:

Löwenheim–Skolem theorem + Compactness

Löwenheim–Skolem theorem + Recursively enumerable set of validities

These are by no means exhaustive though (the reader can consult the encyclopaedic monograph [2] for a thorough treatment of this topic). Philosophically, these results have been interpreted as providing a case for first-order logic being the "right" logic in contrast to higher order, infinitary or logics with generalized quantifiers, which can be argued to be more mathematical beasts (see [12]). An implicit assumption of Lindström's work is that identity (=) is a most basic notion and belongs in the base logic.

Lindström's theorems clearly fails for first-order logic without identity ($\mathcal{L}^-_{\omega\omega}$) since first-order logic with identity ($\mathcal{L}_{\omega\omega}$) is a proper extension of $\mathcal{L}^-_{\omega\omega}$. In fact, there are continuum-many logics between the former and the latter obtained by adding finite cardinality quantifiers of the form $\exists^{\geq n}$.

In this talk we aim at finding a way to amend Lindström's two central theorems so they would apply in the identity-free context. We use the notion of an *abstract logic* from [2, Def. II.1.1.1]. Recall that this notion presents logics as model-theoretic languages [6] (see also [1, 11]), not as consequence relations or collections of theorems. Furthermore, we assume logics to have the basic closure properties from [2, Def. II.1.2.1] except that in the atom property we use $\mathcal{L}^-_{\omega\omega}$ as the base logic. We use an essential definition from [5]: $\mathcal{A} \sim \mathcal{B}$ means that there is a *relativeness correspondence* between the models [5, Def. 2.5], i.e. an isomorphism relation without functionality or injectivity requirements (we prefer to call this a *weak isomorphism*). We will use the following property of logics:

- *Weak isomorphism property*: for any structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \sim \mathcal{B}$ only if $\mathcal{A} \equiv_{\mathcal{L}} \mathcal{B}$.

As usual, if $\mathcal{L}$ and $\mathcal{L}'$ are logics, we write $\mathcal{L} \leq \mathcal{L}'$ if, for any vocabulary τ and any formula $\varphi \in \mathcal{L}(\tau)$, we can find an equivalent formula $\varphi' \in \mathcal{L}'(\tau)$.

$\mathcal{L}^-_{\omega\omega}$ is, properly speaking, a fragment of $\mathcal{L}_{\omega\omega}$ containing the guarded fragments corresponding to basic modal logics. In the modal setting, the most fruitful approach has been to use bisimulations as modal analogues of potential isomorphisms in first-order logic to obtain Lindström-style characterizations [3]. In the present context all we require is the notion of weak isomorphism introduced in [5], which is stronger than bisimulation.

We are now ready to state the central maximality result of this talk:

Theorem 1. *Let $\mathcal{L}$ be an abstract logic such that $\mathcal{L}^-_{\omega\omega} \leq \mathcal{L}$. If $\mathcal{L}$ has the weak isomorphism, compactness, and Löwenheim–Skolem properties, then $\mathcal{L} \leq \mathcal{L}^-_{\omega\omega}$.*

By $\mathcal{L}^{1-}_{\omega\omega}$ we denote the logic obtained from $\mathcal{L}^-_{\omega\omega}$ by allowing just vocabularies where all the relation symbols are unary. If only this kind of vocabulary is admitted, we call the logic *monadic*. In this case, we obtain the same result from compactness alone:

Theorem 2. *Let $\mathcal{L}$ be a monadic logic such that $\mathcal{L}^{1-}_{\omega\omega} \leq \mathcal{L}$. If $\mathcal{L}$ satisfies the compactness and weak isomorphism properties, then $\mathcal{L} \leq \mathcal{L}^{1-}_{\omega\omega}$.*

Corollary 3. *$\mathcal{L}^-_{\omega\omega}$ (resp. $\mathcal{L}^{1-}_{\omega\omega}$) is the fragment of $\mathcal{L}_{\omega\omega}$ (resp. $\mathcal{L}^1_{\omega\omega}$) preserved under weak isomorphisms.*

Theorem 4. *Let $\mathcal{L}$ be an effectively regular abstract logic [2, Def. II.1.2.4] such that $\mathcal{L}^-_{\omega\omega} \leq \mathcal{L}$. If $\mathcal{L}$ has the weak isomorphism property, is recursively enumerable for validity, and has the Löwenheim–Skolem property, then $\mathcal{L} \leq \mathcal{L}^-_{\omega\omega}$.*

Adding a Lindström quantifier to $\mathcal{L}^-_{\omega\omega}$ usually destroys the weak isomorphism property, as is the case with cardinality and cofinality quantifiers. However, each quantifier has a natural version closed under weak isomorphisms. This technique provides interesting examples.

Example 5 (The logic $\mathcal{L}^-_{\omega\omega}(Q^-_\alpha)$)**.** Consider the Lindström quantifier Q^-_α defined as:

$$\{\langle A, M, E\rangle \mid M \subseteq A, E \text{ equivalence relation on } A \text{ congruent with } M, \left|{}^{M}\!/_{E}\right| \geq \omega_\alpha\}.$$

The satisfaction condition for this operator then is

$$\begin{aligned}\mathcal{A} \models Q^-_\alpha xyz[\varphi(x), \theta(y,z)] \text{ iff } & \{\langle a,b\rangle \in A^2 \mid \mathcal{A} \models \theta[a,b]\} \text{ is an equivalence relation on } A,\\ & \mathcal{A} \models \forall xy(\theta(x,y) \to (\varphi(x) \to \varphi(y))), \text{ and}\\ & \left|{}^{\{a \in A \mid \mathcal{A} \models \varphi[a]\}}\!/_{\{\langle a,b\rangle \in A^2 \mid \mathcal{A} \models \theta[a,b]\}}\right| \geq \omega_\alpha.\end{aligned}$$

It is easy to verify that this logic has the weak isomorphism property. In the case of $\mathcal{L}^-_{\omega\omega}(Q^-_1)$ it inherits countable compactness and the recursive axiomatizability of validity from $\mathcal{L}_{\omega\omega}(Q_1)$ (these facts are not immediate).

Example 6 (The logic $\mathcal{L}^{-}_{\omega\omega}(Q^{\mathrm{cf}\omega-})$)**.** Consider now the following Lindström quantifier:

$$Q^{\mathrm{cf}\omega-} = \{\langle A, M, E\rangle \mid M \subseteq A^2, E \text{ is an equivalence relation on } A \text{ congruent with } M,$$
$$\langle A, M\rangle\!\big/_{E} \text{ is a linear order with cofinality } \omega\}.$$

Then, we have that $\mathcal{A} \models Q^{\mathrm{cf}\omega-}xyzw[\varphi(x,y), \theta(z,w)]$ iff

- $\theta^{\mathfrak{A}} = \{\langle a, b\rangle \in A^2 \mid \mathcal{A} \models \theta[a,b]\}$ is an equivalence relation on A,
- $\mathcal{A} \models \forall xy((\theta(x,y) \wedge \theta(z,w)) \to (\varphi(x,z) \to \varphi(y,w)))$,
- $\mathcal{A} \models$ "$\varphi(x,y)$ is an irreflexive transitive relation",
- $\mathcal{A} \models (\forall xy)(\varphi(x,y) \vee \varphi(y,x) \vee \theta(x,y))$, and
- $\langle A, \theta^{\mathfrak{A}}\rangle\!\big/_{\{\langle a,b\rangle \in A^2 \mid \mathcal{A} \models \theta[a,b]\}}$ has cofinality ω.

$\mathcal{L}^{-}_{\omega\omega}(Q^{\mathrm{cf}\omega-})$ is closed under weak isomorphisms and inherits the compactness and recursive axiomatizability of full $\mathcal{L}_{\omega\omega}(Q^{\mathrm{cf}\omega})$. This logic also shows that the Löwenheim–Skolem property is needed in Thm. 1, in contrast to Thm. 2 where only compactness is required.

A remarkable example of a Lindström quantifier which respects weak isomorphims is the Henkin quantifier Q^H.

Example 7 (The logic $\mathcal{L}^{-}_{\omega\omega}(Q^H)$)**.** Recall the Henkin quantifier Q^H which is defined as follows:

$$Q^H = \{\langle A, M\rangle \mid M \subseteq A^4, M \supseteq f \times g \text{ for some } f, g\colon A \longrightarrow A\}.$$

It is known that when Q^H is added to $\mathcal{L}^{1-}_{\omega\omega}$ it collapses into $\mathcal{L}^{1-}_{\omega\omega}$ itself [10] so that the resulting logic is compact. However, $\mathcal{L}^{-}_{\omega\omega}(Q^H)$ may be shown to be incompact.

The results in the present paper may also provide a foundation for a philosophical discussion on whether $\mathcal{L}^{-}_{\omega\omega}$ is suitable as a contender for the title of the "right logic" against $\mathcal{L}_{\omega\omega}$. After all, the logicality of the $=$ predicate is not obvious (cf. [7]). Another work where $\mathcal{L}^{-}_{\omega\omega}$ has attracted mathematical attention is [9], where the problem of categoricity of theories in that logic is studied.

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Logic	Compactness	LöwSko Property	Weak Iso Property
$\mathcal{L}_{\omega\omega}$	$+$	$+$	$-$
$\mathcal{L}^-_{\omega\omega}$	$+$	$+$	$+$
$\mathcal{L}^-_{\omega\omega}(\{\exists^{\geq n} \mid n \in X\})$	$+$	$+$	$-$
$\mathcal{L}^-_{\omega\omega}(Q_1^-)$	$+$ (for countable theories)	$-$	$+$
$\mathcal{L}^-_{\omega\omega}(Q_1)$	$+$ (for countable theories)	$-$	$-$
$\mathcal{L}^-_{\omega\omega}(Q^{\mathrm{cf}\omega-})$	$+$	$-$	$+$
$\mathcal{L}^-_{\omega\omega}(Q^{\mathrm{cf}\omega})$	$+$	$-$	$-$
$\mathcal{L}^-_{\omega\omega}(Q^{H})$	$-$	$-$	$+$

Table 1: Summary of properties of some logics.

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On the connexivity of fuzzy counterfactuals

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The validity of the axioms of connexivity in various semantics of counterfactual conditionals has been a widely discussed topic (e.g., [4, 6, 3]). Recently, a fuzzy semantics of counterfactuals [1] has been proposed, which accommodates graded antecedents and consequents and reflects the vagueness of the similarity ordering of possible worlds. This brings up the natural question of whether some form of the connexive axioms is valid in this semantics, or perhaps in some variant thereof. In this contribution, I will briefly introduce the semantics in question and show that it indeed validates some restricted forms of the connexive axioms, especially if the original definition of the semantics is strengthened in an arguably natural way.

The mentioned gradual semantics for the propositional logic of counterfactuals can be based on a broad class of first-order fuzzy logics; for simplicity, though, we will restrict ourselves to the standard semantics of the logic Ł$\forall_\Delta$, or the well-known infinite-valued logic of Łukasiewicz with the additional connective Δ (see, e.g., [2]). The standard semantics of Ł$\forall_\Delta$ evaluates propositions in the real interval $[0, 1]$, using the following Tarski conditions:

$$\|\neg\varphi\| = 1 - x \qquad \|\Delta\varphi\| = 1 - \operatorname{sgn}(x)$$
$$\|\varphi \mathbin{\&} \psi\| = \max(0, x + y - 1) \qquad \|\varphi \wedge \psi\| = \min(x, y)$$
$$\|\varphi \oplus \psi\| = \min(1, x + y) \qquad \|\varphi \vee \psi\| = \max(x, y)$$
$$\|\varphi \to \psi\| = \min(1, 1 - x + y) \qquad \|\varphi \leftrightarrow \psi\| = 1 - |x - y|$$
$$\|(\forall x)\varphi(x)\| = \inf_{a\in D}\big(\|\varphi\|(a)\big) \qquad \|(\exists x)\varphi(x)\| = \sup_{a\in D}\big(\|\varphi\|(a)\big)$$

The graded semantics for counterfactuals is based on Lewis' Analysis 2 of [5], according to which a counterfactual $A \mathrel{\Box\!\!\to} C$ is true in a world w if all the closest (in terms of similarity to w) A-worlds (i.e., worlds satisfying A) are C-worlds. We will see that, unlike Lewis' bivalent Analysis 2, the gradual variant does not rely on the implausible Limit Assumption. In its most basic form, the semantics is defined as follows:

A *fuzzy counterfactual frame* is a non-empty set W of possible worlds equipped with a system of fuzzy relations $\preceq_w \colon W^2 \to [0, 1]$, for all $w \in W$, satisfying the conditions:

(i) $(v \preceq_w w) = 1$ only if $v = w$ (strict minimality of w)

(ii) $(u \preceq_w v) = 1$ or $(v \preceq_w u) = 1$ (linearity)
(iii) $(v \preceq_w v') \,\&\, (v' \preceq_w v'') \leq (v \preceq_w v'')$ (fuzzy transitivity)

An evaluation in a fuzzy counterfactual frame is a mapping $e\colon \mathrm{PropVar} \times W \to [0,1]$, extended to all propositional formulae of $Ł_\Delta$ as usual; the value of the formula A in the world w is denoted by $\|A\|_w$. The semantics of the counterfactual conditional $\Box\!\!\rightarrow$ is defined as follows:

$$\|A \Box\!\!\rightarrow C\|_w =_{\mathrm{df}} \big(\mathrm{Min}_{\preceq_w} \|A\| \subseteq \|C\|\big),$$

where $X \subseteq Y \equiv_{\mathrm{df}} (\forall v)(Xv \to Yv)$ and $(\mathrm{Min}_{\preceq_w} \|A\|)(v) \equiv_{\mathrm{df}} \|A\|_v \wedge (\forall v')(\|A\|_{v'} \to v \preceq_w v')$, both interpreted in the standard semantics of $Ł\forall_\Delta$. It can be observed that the definition uses the same Tarski conditions as that of Lewis' Analysis 2, only reinterpreted in $Ł\forall_\Delta$. The consequence relation $\models$ is defined as the global preservation of the designated truth degree 1.

The semantics can readily be extended by fuzzy S5-modalities (e.g., [2, §8.3]), defined as $\|\Box A\|_w =_{\mathrm{df}} \inf_{v\in W} \|A\|_w$ and $\|\Diamond A\|_w =_{\mathrm{df}} \sup_{v\in W} \|A\|_w$, and the usual definition of strict implication, $A \multimap C \equiv_{\mathrm{df}} \Box(A \to C)$. Then, similarly as in the bivalent case, the counterfactual implication is intermediate between the material and strict implications: $\|A \multimap C\|_w \leq \|A \Box\!\!\rightarrow C\|_w \leq \|A \to C\|_w$.

It can be shown that the gradual semantics validates various desirable properties of counterfactuals (such as $A \Box\!\!\rightarrow A \vee B$) and refutes various undesirable properties (e.g., the transitivity of $\Box\!\!\rightarrow$). Although the axiomatization of this semantics in the propositional language of counterfactual logic has not yet been attempted, a syntactic method for verifying counterfactual laws is obtained by the standard translation tr_x into first-order Łukasiewicz logic $Ł\forall_\Delta$ with additional axioms formalizing the properties (i)–(iii) of the ternary predicate $\preceq$, where the atoms, the connectives of $Ł_\Delta$, and the modalities $\Box, \Diamond$ are translated in the usual manner, and the counterfactual implication $\Box\!\!\rightarrow$ is translated as follows:

$$\mathrm{tr}_x(A \Box\!\!\rightarrow C) = (\forall y)\big(\mathrm{tr}_y(A) \wedge (\forall z)(\mathrm{tr}_z(A) \to y \preceq_x z) \to \mathrm{tr}_y(C)\big).$$

As mentioned, Lewis' bivalent Analysis 2 of [5] employs the implausible Limit Assumption to ensure the existence of the closest A-worlds, provided any A-worlds exist. In the described gradual semantics of [1], however, a considerably weaker and arguably more plausible condition of the *right-connectedness* of the fuzzy counterfactual frame is sufficient to ensure the non-emptiness of $\mathrm{Min}_{\preceq_w} \|A\|$, and hence the non-triviality of $A \Box\!\!\rightarrow C$ whenever A is possible to a non-zero degree. This condition says that for every strictly decreasing (w.r.t. $\preceq_w$) set of worlds $X \subseteq W$, there are distinct worlds $u, v \in X$ whose closeness to w is mutually indistinguishable to a non-zero degree, i.e., $(u \approx_w v) > 0$, where $(u \approx_w v) =_{\mathrm{df}} (u \preceq_w v) \wedge (v \preceq_w u)$; or equivalently,

$$\sup_{\substack{u,v\in X\\ u\neq v}} (u \approx_w v) > 0. \tag{1}$$

As shown in [1], this condition already ensures that $(\mathrm{Min}_{\preceq_w} \|A\|)(v) > 0$ for some $v \in W$ if $\|A\|_u > 0$ for some $u \in W$—and so the non-triviality of the counterfactual $A \boxright C$ whenever $\Diamond A$ is true to a non-zero degree. (In [1], the condition is equivalently formulated in terms of a fuzzy indistinguishability relation between abstract, possibly uninhabited distances from w. The condition of right-connectedness is then a plausible property of fuzzy indistinguishability on ordered sets, saying that the infimum of a set is not fully distinguishable from all of its elements.)

In the context of discussing the validity of the connexive axioms in various semantics for counterfactuals (e.g., [4, 6, 3]), a natural question is whether these axioms are valid in (right-connected) fuzzy counterfactual frames. First, it is clear that fuzzy counterfactuals can only validate them for bivalent (and possible, due to classical counterexamples) antecedents: since $A \leftrightarrow \neg A$ is satisfiable in Ł$_\Delta$, fuzzy counterfactual frames can validate $\Box(A \leftrightarrow \neg A)$, and thus also $A \boxright \neg A$ and $\neg A \boxright A$. Easy calculations moreover show that even assuming $A \vee \neg A$ and $\Diamond A$, right-connected fuzzy counterfactual frames only validate a weak form of the connexive axioms—namely, holding only to a non-zero, rather than full, degree; e.g., Aristotle's theses only hold in these forms: $A \vee \neg A, \Diamond A \models \neg\Delta(A \boxright \neg A)$ and $A \vee \neg A, \Diamond A \models \neg\Delta(\neg A \boxright A)$.

Counterexamples to the stronger variants of the connexive axioms (not weakened by the connective Δ) are easy to find. The condition of right-connectedness, while sufficient for the non-triviality of counterfactuals with possible antecedents, is thus too weak to ensure the stronger forms of connexivity. Nevertheless, the stronger connexive axioms turn out to be valid in fuzzy counterfactual frames that satisfy a stronger form of the right-connectedness condition, namely: For any strictly decreasing (w.r.t. $\preceq_w$) set of worlds $X \subseteq W$,

$$\sup_{\substack{u,v \in X \\ u \neq v}} (u \approx_w v) = 1. \tag{2}$$

In terms of abstract distances of [1], condition (2) says that the infimum of a set of distances is indistinguishable from some of its elements up to the full degree 1, which is still a plausible property of fuzzy indistinguishability on an ordered set. It is not difficult to show that condition (2) ensures the existence of $\preceq_w$-minimal worlds for bivalent possible antecedents not only to a non-zero degree as does condition (1), but even to the full degree 1. In consequence of this, fuzzy counterfactual frames satisfying (2) validate the connexive axioms as follows:

$$A \vee \neg A, \Diamond A \models \neg(A \boxright \neg A) \tag{3}$$
$$A \vee \neg A, \Diamond\neg A \models \neg(\neg A \boxright A) \tag{4}$$
$$A \vee \neg A, \Diamond A \models (A \boxright B) \Rightarrow \neg(A \boxright \neg B) \tag{5}$$
$$A \vee \neg A, \Diamond A \models (A \boxright \neg B) \Rightarrow \neg(A \boxright B) \tag{6}$$

for $\Rightarrow \in \{\rightarrow, \multimap, \boxright\}$. In the terminology of [3], connexivity restricted to possible antecedents is called *humble modal connexivity*. In our case, the bivalence of antecedents (which in classical logic is automatic) must additionally be required. On the other hand, the premises $\Diamond(A \rightarrow B)$ and $\Diamond(A \rightarrow \neg B)$, assumed in the definition of humble modal Boëthius' theses in [3], are already ensured in fuzzy counterfactual frames by the premises of (5)–(6) and condition (2). Fuzzy counterfactual frames satisfying (2) thus validate, and can even be shown to be characterized by, the humble modal connexive axioms for fuzzy counterfactuals with bivalent antecedents.

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Paraconsistent modal logic of comparative uncertainty

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General project. This work is a part of the project introduced in [1]. We are developing a modular logical framework for reasoning based on uncertain, incomplete or inconsistent information. In this framework, an agent is constructing their belief using probabilistic incomplete and/or conflicting information aggregated from multiple sources. We formalise such probabilistic reasoning using the framework of many-valued crisp modal logics akin to those presented in [2, 3, 7]. These logics allow statements of the form $\Box\phi$ construed as 'the agent believes that ϕ' or 'the agent is certain in ϕ' to have not only the classical values but also the intermediate ones.

Two-dimensional treatment of uncertainty. For the purpose of our talk, we consider agents who although not being always able to give an exact level of their certainty in some proposition, can compare their certainty in one proposition to the certainty in the other. Thus, we are interested in the expansions of Gödel logic which can be treated as the logic of comparative truth (or comparative certainty).

Two-dimensionality comes from the definition of the logics using expansions of the product bilattice $[0,1] \odot [0,1]$. Here, the left coordinate is interpreted as the agent's certainty in truth of a statement ('positive support') and the right coordinate — as certainty in falsity ('negative support'). Since agents can collect their data from different (and possibly conflicting) sources, we treat positive and negative supports independently. The usual truth order of $[0,1]$ becomes 'truth-and-falsity' order and is defined as follows

$$(x,y) \leq (x',y') \text{ iff } x \leq x' \text{ and } y \geq y'$$

While $\wedge$ and $\vee$ are defined in the same way as on twisted structures (cf., e.g. [5]), there are several ways to construe the negative support of implication. We interpret the negative support of $\phi \rightarrow \chi$ as $\neg\chi \prec \neg\phi$ with $\prec$ being the coimplication of Gödel logic. This treatment of $\rightarrow$ goes back to one of Wansing's logic of [8], namely $\mathsf{I_4C_4}$, which, in its turn is derived from bi-intuitionistic logic [4, 6]. We call the propositional fragment $\mathsf{G}^2(\rightarrow,\prec)$.

Definition 1 (Propositional connectives)**.** For all $a, b \in [0,1]$, we set $a \wedge b := \min(a,b)$, $a \vee b := \max(a,b)$ as well as

$$a \rightarrow_G b := \begin{cases} 1, & \text{if} \quad a \leq b \\ b & \text{else} \end{cases} \qquad\qquad b \prec_G a := \begin{cases} 0, & \text{if} \quad b \leq a \\ b & \text{else} \end{cases}$$

Negation and 1 are defined as $\sim_{\mathsf{G}} a := a \rightarrow_{\mathsf{G}} 0$, and $1 :=\sim_{\mathsf{G}} 0$, respectively.

Now fix a countable set Prop of propositional letters and consider the following language:

$$\phi \coloneqq \mathbf{0} \mid \mathbf{1} \mid p \mid \neg\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi) \mid (\phi \prec \phi)$$

where $p \in \mathsf{Prop}$. We define $\sim\phi \coloneqq \phi \to \mathbf{0}$.

Let $v : \mathsf{Prop} \to [0,1] \times [0,1]$, and denote v_1 and v_2 its left and right coordinates, respectively. We extend v as follows.

$$\begin{array}{rclrcl} v(\mathbf{0}) & = & (0,1) & v(\phi_1 \wedge \phi_2) & = & (v_1(\phi_1) \wedge v_1(\phi_2), v_2(\phi_1) \vee v_2(\phi_2)) \\ v(\mathbf{1}) & = & (1,0) & v(\phi_1 \vee \phi_2) & = & (v_1(\phi_1) \vee v_1(\phi_2), v_2(\phi_1) \wedge v_2(\phi_2)) \\ v(\neg\phi) & = & (v_2(\phi), v_1(\phi)) & v(\phi_1 \to \phi_2) & = & (v_1(\phi_1) \to_{\mathsf{G}} v_1(\phi_2), v_2(\phi_2) \prec_{\mathsf{G}} v_2(\phi_1)) \\ & & & v(\phi_1 \prec \phi_2) & = & (v_1(\phi_1) \prec_{\mathsf{G}} v_1(\phi_2), v_2(\phi_2) \to_{\mathsf{G}} v_2(\phi_1)) \end{array}$$

Modalities and frame semantics. We expand the language with $\Box$ — a modal operator that we treat as 'the agent is certain that…', as well as its dual — $\Diamond$ and denote the resulting language with $\mathcal{L}_{\Box,\Diamond}$. Modalities are interpreted in an expected fashion on crisp models.

Definition 2 (Frames)**.** A crisp frame is a tuple $\mathfrak{F} = \langle W, R \rangle$ with with $W \neq \varnothing$ and $R \subseteq W \times W$.

Definition 3 (Models)**.** A crisp model on a frame $\mathfrak{F}$ is a tuple $\mathfrak{M} = \langle \mathfrak{F}, v \rangle$ with $\mathfrak{F}$ being a frame and $v : \mathsf{Var} \times W \to [0,1] \odot [0,1]$ is a valuation which is uniquely extended to a map with domain $\mathcal{L}_{\Box,\Diamond} \times W$ in such a way that it is a propositional G^2 homomorphism (for the propositional connectives) and where the modal operators are interpreted as infima and suprema w.r.t. the order on $[0,1] \odot [0,1]$.

$$\begin{array}{rr} v(\mathbf{0}, w) = (0,1) & v(\phi \circ \phi', w) = v(\phi) \circ v(\phi') \\ & (\text{with } \circ \in \{\wedge, \vee, \to, \prec\}) \\ v(\mathbf{1}, w) = (1,0) & v(\Box\phi, w) = \inf\{v(\phi, w') : wRw'\} \\ v(\neg\phi, w) = (v_2(\phi), v_1(\phi)) & v(\Diamond\phi, w) = \sup\{v(\phi, w') : wRw'\} \end{array}$$

The definitions of validity and entailment are also as expected.

Definition 4 (Truth, falsity, and entailment)**.**

- ϕ is true at $w \in \mathfrak{M}$ (denote, $\mathfrak{M}, w \vDash^+ \phi$) iff $v_1(\phi, w) = 1$. ϕ is false at $w \in \mathfrak{M}$ (denote, $\mathfrak{M}, w \vDash^- \phi$) iff $v_2(\phi, w) = 1$.
- ϕ is true in $\mathfrak{M}$ (denote, $\mathfrak{M} \vDash^+ \phi$) iff $v_1(\phi, w) = 1$ for any $w \in \mathfrak{M}$. ϕ is false at $w \in \mathfrak{M}$ (denote, $\mathfrak{M} \vDash^- \phi$) iff $v_2(\phi, w) = 1$ for any $w \in \mathfrak{M}$.[1]
- ϕ is valid on $\mathfrak{F}$ (denote, $\mathfrak{F} \models \phi$) iff for any valuation v on $\mathfrak{F}$ and for any state $w \in \mathfrak{F}$, it holds that $v(\phi, w) = (1,0)$.

[1] As expected, $\mathfrak{M}, w \vDash^+ \Gamma$ stands for $\forall\gamma \in \Gamma : \mathfrak{M}, w \vDash^+ \gamma$ and likewise for $\vDash^-$.

- ϕ is (universally) valid iff it is valid on every frame.
- $\Gamma \subseteq \mathcal{L}_{\Box,\Diamond}$ locally entails ϕ iff for any $\mathfrak{M}$, it holds that

$$\forall w \in \mathfrak{M} : \inf \{v(\gamma) : \gamma \in \Gamma\} \leq v(\phi, w)$$

We denote the set of valid formulas with $\mathbf{K}\mathsf{G}^2$.

Results. We provide a sound and complete Hilbert-style axiomatisation of $\mathbf{K}\mathsf{G}^2$. We also develop its model theory. In particular, we show that Scott — Lemmon correspondence holds and obtain complete axiomatisations for the canonical extensions of $\mathbf{K}\mathsf{G}^2$. Furthermore, we prove the finite model property for $\mathbf{K}\mathsf{G}^2$ using a modification of the filtration technique and provide decidability and complexity results.

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Hybrid logic with propositional quantifiers: Natural deduction style
(Work in progress)

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1 Introduction

Hybrid logic extends standard modal logic so it can refer to worlds. It does so using *nominals*, a second kind of propositional symbol, usually written i, j, and k, to distinguish them from the p, q, and r used for ordinary propositional symbols. Nominals are true at one and only one world in any model, so a nominal is an atomic 'propositional term' that names a world (or time, or state, or...); here we call such symbols *standard nominals*. Arthur Prior introduced early forms of hybrid logic in the 1950s and 60s; see [5, 2] for background.

Sometimes, rather than introducing nominals as a second kind of propositional symbol, Arthur Prior would create them using the Q operator: Qp, the result of prefixing the ordinary propositional symbol by Q operator, converted p to a nominal. As Prior put it in [7], page 237:

> For 'p is an individual' (or an instant, or a possible total world-state) we write Qp. If we have propositional quantifiers, we can define Qp thus:
> $$Qp = \Diamond p \wedge \forall q(\Box(p \to q) \vee \Box(p \to \neg q))$$

Here the $\Box$ and $\Diamond$ are the box and diamond forms of the *universal modality*, that is, $\Box$ means *true at all worlds* and $\Diamond$ means *true at some world*. Thus, if the quantifier is read as ranging across all sets of worlds, then Qp says that the denotation of p is a singleton, that is, p is a standard nominal.

But Prior's definition is ambiguous: there are *two* mathematically well understood ways of interpreting a propositional quantifier like $\forall q$. The first is to interpret it as quantifying across all subsets of the set of possible worlds. This is called the *standard* interpretation, and it is the standard interpretation that gives rise to standard nominals. But we can also interpret propositional quantifiers as ranging over a pre-selected set of subsets of worlds called the *admissible subsets*. This interpretation traces back to Leon Henkin's pioneering work on higher-order logic in the 1950s, and is often called the *general* semantics. Read this alternative way, Prior's operator Qp says that the denotation of p is an *atom*, that is, a minimal non-empty admissible set.

The distinction between the standard and the general semantics is of direct relevance to Prior's definition of the Q operator: when interpreted standardly

Figure 1: Natural deduction rules for propositional quantifiers

$$\frac{@_i\phi[q/p]}{@_i\forall p\phi}\,(\forall I)^* \qquad\qquad \frac{@_i\forall p\phi}{@_i\phi[\psi/p]}\,(\forall E)^\dagger$$

$*$ The propositional variable q does not occur free in $@_i\forall p\phi$ or in any undischarged assumptions.
$\dagger$ The formula ψ is free for p in φ and ψ does not contain any standard nominal in formula position.

we get standard nominals, but when interpreted according to the general semantics, we get something interestingly different; here we call them *non-standard nominals*. In this note we explore this distinction by working with a basic hybrid language enriched with propositional quantifiers. Thus we will have standard nominals and — because of the propositional quantifiers — we will be able to define Prior's Q operator and hence non-standard nominals too.

That is, we work with a language which contains ordinary propositional symbols, a universal modality $\Box$ (and its dual $\Diamond$), standard nominals i, j, k and so on, together with a *satisfaction operator* $@_i$, $@_j$, $@_k$ for each nominal. A formula of the form $@_i\varphi$ says that φ is true at one particular world, namely the world the standard nominal i refers to, and similarly for j, k and so on. Note that standard nominals are used in two syntactically distinct ways: if i appears as a subscript to @, then we say it occurs in *operator position* and if it occurs as an atomic symbol, then we say it occurs in *formula position*. The above are the syntactic elements of what nowadays is called the *basic hybrid language*. To this we add propositional quantifiers.

2 Natural deduction rules for propositional quantifiers

So we have two species of nominals — standard nominals and non-standard nominals, the latter being defined via the Q operator interpreted by the general semantics. To make the differences between our two species of nominals concrete, it will help to have a proof system. In the present note we consider the natural deduction system for hybrid logic obtained by adding the two rules in Figure 1 to the system for propositional hybrid logic given in Chapter 2 of the book [4]. The rules in Figure 1 are translations into natural deduction style of the tableau rules given in [1].

A characteristic feature of natural deduction style proof-systems is that there are two different kinds of rules for each logical connective: one to introduce it, the other to eliminate it. The rules $(\forall I)$ and $(\forall E)$ in Figure 1 are the introduction and elimination rules for the propositional quatifier. The $(\forall I)$ rule is standard, but the $(\forall E)$ rule is not: Whereas the first part of the $\dagger$

side-condition simply prevents accidental symbol binding (defined in the usual way), the second part of the † side-condition has a clear-cut model-theoretic meaning: If the second part of the side-condition is included, we get a proof-system wrt. the general semantics, but on the other hand, if the side-condition is not included, we get a system which is sound (but not complete) wrt. the standard semantics (see, for example, Chapter 4 of [6]).

We are currently investigating how to make sense of the above distinction from a more proof-theoretic point of view, which is the reason why we are here working with a proof-system in natural deduction style.

Our first remark is in connection with the branch of logic called proof-theoretic semantics, which is based on the idea of explaining the meaning of a logical connective in terms of derivation rules, see [10]. If we take that idea for granted, then the side-condition tells something about the (proof-theoretic) semantics of propositional quantifiers, namely that if nominals are allowed in formula position, in particular, if nominals can be substituted for propositional symbols, then the range of the quantifiers includes the denotations of nominals, which is in line with the model-theoretic observation that even though the unrestricted ($\forall E$) rule is not sound wrt. the general semantics, it *is* sound wrt. what are called *discrete* general semantics, based on frames having the property that all singleton sets are admissible, cf. [1].

This raises a more general and somewhat speculative question: What would Prior have said to the contemporary issue of proof-theoretic semantics versus model-theory?

- Would he endorse a proof-theoretic semantics for temporal and modal logics? Probably not: In his paper [9] he uses his famous "tonk" argument to raise doubt as to whether the meaning of logical connectives can be explained in terms of derivation rules.
- Would he prefer a model-theoretic semantics? Definitely not when it comes to temporal logics. This is clear from many places in his works where he objects to the abstract character of instants, which are reified in the usual Kripke models of time.

We remark that it is less clear whether Prior had a problem with the metalinguistic nature of model-theoretic semantics, at least, it seems that he accepted a specific metalinguistic semantics which is not set-theoretic, namely what is called a *homophonic*[2] semantic theory of tenses, cf. [8] pages 8–9, as described in the paper [3].

Beside these philosophical and historical issues, there are a number of more technical issues that calls for investigation: the inversion principle and maximum

[2] A semantic theory is homophonic if the constructions of the object language are interpreted in terms of analogous constructions of the metalanguage, for example, the standard truth-table semantics for propositional logic is homophonic since $\neg$ is interpreted i terms of negation and $\wedge$ is interpreted in terms of conjunction, etc. This is contrary to the Kripke semantics for tense logic where the tense operators P and F are interpreted in terms of quantification—not in terms of tenses.

formulas, normalisation and the subformula property of normal derivations, analyticity, conservativity, etc.

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Connexive arithmetic formulated relevantly

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In [2], Ferguson shows that Richard Angell's connexive logics **PA1** and **PA2**, and Graham Priest's $\mathbf{P}_N$, are questionable bases for connexive arithmetic. Ferguson shows these theories to be quite limited, if not unserviceable. Some "pathologies", as he calls them, of these systems include the following:

1. For any theory T in any universal-identity extension of arithmetics based on **PA1**, there exists at least one formula $(\forall x \leq n)(\varphi(x))$ that is true for natural even n and false for natural odd n. (This is an arithmetic expression of a well-known feature —or defect, as Woods and Routley and Montgomery considered it— of Angell-McCall's extension of **PA1**, **CC1**, to wit: a proposition p connexively implies only odd-numbered conjunctions of occurrences of itself, and never even-numbered ones.)

2. The arithmetics based on **PA2** solve the above problem of bounded quantification. However, for any theory T in any universal-identity extension of arithmetics based on **PA2**, denoted 'PA2$^+$', every complete protoarithmetical theory is either literal or illiterate, i.e. either $T \models_{\mathbf{PA2}^+} (t = t) \leftrightarrow \mathbf{T}(t = t)$ or $T \models_{\mathbf{PA2}^+} {\sim}((t = t) \leftrightarrow \mathbf{T}(t = t))$ for any term t, where $\mathbf{T}$ is a truth predicate.

3. Finally, there are no numerically inductive theories of arithmetic in any universal-identity extension of Priest's $\mathbf{P}_N$. These theories are decidible but only due to the fact that the Peano axioms have no consequences in these systems.

In view of the results above, Ferguson conjectures that further attempts may consider using Wansing's **C**, since there is an embedding of its quantified version into positive intuitionistic logic, from which Heyting arithmetic is available. In [3], Ferguson borrows techniques from relevant logic to prove that $\mathbf{C}^{\#}$, the connexive arithmetic based on Wansing's **C**, is Post-consistent *simpliciter*, guaranteeing that it has models. Notably, it seems that all its models are inconsistent.

However, we believe Ferguson attempted the task using considerably weak connexive logics. We approach connexive arithmetic in a rather distinct, but simpler, manner. We obtain **cRM3** by replacing the conditional of **RM3**, namely

$A \to B$	$\{1\}$	$\{1,0\}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{0\}$	$\{0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{0\}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$

with the Belikov-Loginov conditional [1],

$A \to_{BL} B$	$\{1\}$	$\{1,0\}$	$\{0\}$
$\{1\}$	$\{1\}$	$\{0\}$	$\{0\}$
$\{1,0\}$	$\{1\}$	$\{1,0\}$	$\{0\}$
$\{0\}$	$\{1,0\}$	$\{1,0\}$	$\{1,0\}$

thus validating Aristotle's and Boethius' theses (and their variants) and invalidating Symmetry for the conditional, which suffice for connexivity. **cRMn** are obtained following the strategy to obtain **RMn**. Then, following the methods of [4], we formulate the connexive arithmetics $\mathbf{cRM3}^{\#}$, $\mathbf{cRM3}^{\#\#}$, $\mathbf{cRMn}^{i}$ and $\mathbf{cRM}^{\omega}$ through the underlying connexive logics **cRMQ**, **cRM3Q** and **cRMnQ** which we obtain by adapting their relevant counterparts.

Among the results we prove in the connexive arithmetics considered, which are essentially the same as those available in their relevant counterparts, the following are noteworthy:

1. Every $\mathbf{cRM3}^{\#}$ is inconsistent and ω-inconsistent but non-trivial.
2. $\mathbf{cRM3}^{\#}$ is ω-complete.
3. $\mathbf{cRM3}^{\#}$ is prime and complete.
4. $\mathbf{cRM3}^{\#}$ is decidible.
5. The theorems of $\mathbf{cRM3}^{i\#}$ are exactly the truths of $\mathbf{cRM3}^{i}$.
6. $\mathbf{cRM}^{\omega}$ is inconsistent, ω-inconsistent but non-trivial.
7. $\mathbf{cRM}^{\omega}$ is ω-complete.

Finally, we comment on some differences between the relevant systems and the connexive versions we offer. Foreseeable, some theorems require new proofs, as some principles of **R** are lacking in **cRM**. Moreover, we do not have analogues for other theorems. For instance, though the relevant arithmetics $\mathbf{RM3}^{i}$ contain $\mathbf{R}^{\#}$, $\mathbf{R}^{\#\#}$, $\mathbf{RM}^{\#}$ and $\mathbf{RM}^{\#\#}$, a connexive analogue of this result is not available without knowing **cR**: the largest fragment of **R** that does not trivialize when the connexive theses are added to it.

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Defeasible Linear Temporal Logic

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1 Defeasible LTL

Linear temporal logic (LTL) was introduced by Pnueli [5] as a formal tool for reasoning about programs execution. Many properties that an execution should have can be expressed elegantly using this formalism. The logic LTL is used for systems verification [6]. With advances in technologies, systems became more and more complex, displaying new features and behaviours. One of these behaviours is tolerating exceptions. In more general terms, if an error occurs within an execution of a program at certain points of time where it is tolerated, the program can still function properly.

Defeasible linear temporal logic ($LTL^{\sim}$) is a defeasible temporal formalism for representing and verifying exception-tolerant systems. It is based on linear temporal logic (LTL) and builds on the preferential approach of Kraus et al. [4] (a.k.a. the KLM approach) for non-monotonic reasoning, which allows us to formalize and reason with exceptions. We want a formalism for verifying properties of executions that can, on one hand, be strictly required at some points of time, and on the other hand, be missing in other points of time deemed to be exceptional or anodyne. The defeasible aspect of $LTL^{\sim}$ adds a new dimension to the verification of a program's execution. We can order time points from the important ones, which we call *normal*, to the lesser and lesser ones. *Normality* in LTL indicates the importance of a time point within an execution compared to others.

1.1 Introducing defeasible temporal operators

Britz & Varzinczak [1] introduced new modal operators called defeasible modalities. In their setting, defeasible operators, unlike their classical counterparts, are able to single out normal worlds from those that are less normal or exceptional. Here we extend the vocabulary of LTL with the *defeasible temporal operators* $\widetilde{\Box}$ and $\widetilde{\Diamond}$. Sentences of the resulting logic $LTL^{\sim}$ are built up according to the following grammar:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Box\alpha \mid \Diamond\alpha \mid \bigcirc\alpha \mid \alpha\mathcal{U}\alpha \mid \widetilde{\Box}\alpha \mid \widetilde{\Diamond}\alpha$$

Standard Boolean operators are a part of the syntax. The symbol $\top$ is an abbreviation of $p \vee \neg p$, $\bot$ is an abbreviation of $p \wedge \neg p$, the implication operator $\alpha \rightarrow \beta \stackrel{\text{def}}{=} \neg\alpha \vee \beta$ and the equivalence operator $\alpha \leftrightarrow \beta \stackrel{\text{def}}{=} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$. The classical temporal operators are: $\Box$ always, $\Diamond$ eventually, $\bigcirc$ next and $\mathcal{U}$ until.

The intuition behind defeasible operators is the following: $\widetilde{\Box}$ reads as *defeasible always* and $\widetilde{\Diamond}$ reads as *defeasible eventuality.*

The set of all well-formed $LTL^{\sim}$ sentences is denoted by $\mathcal{L}^{\sim}$. Using defeasible operators, we can express *defeasible properties* of an execution, targeting current and future states that are *normal* on one hand, and leaving states that are *exceptional* on the other. Here are some defeasible properties that can be expressed in $LTL^{\sim}$.

- Defeasible safety: $\widetilde{\Box}\alpha$ means that the property α for all normal future time points of the execution.
- Pertinent liveness: $\widetilde{\Diamond}\alpha$ means that the the property α will hold in a normal future time point of the execution.
- Defeasible response: $\widetilde{\Box}\widetilde{\Diamond}\alpha$ means that for all normal time points of the execution, there is a later normal time point where α holds.
- Defeasible persistence: $\widetilde{\Diamond}\widetilde{\Box}\alpha$ means that there exists a normal point in the execution such that from then and onward all the normal points are points where α holds.

Next we shall discuss how to interpret statements that have this defeasible aspect and how to determine the truth values of each well-formed sentence in $\mathcal{L}^{\sim}$.

1.2 Preferential semantics

In order to interpret the sentences of $\mathcal{L}^{\sim}$, we define two components. First, we consider $(\mathbb{N}, <)$ to be a temporal structure. The temporal structure represents the chronological sequence of time points. Let $\mathcal{P}$ be a finite set of propositional atoms, the first component of the interpretation is a mapping function $V : \mathbb{N} \longrightarrow 2^{\mathcal{P}}$ that associates each time point $t \in \mathbb{N}$ to a set of atoms $V(t)$ that are true in t.

For the second component, the preferential aspect of the interpretation is directly inspired by the preferential semantics proposed by Shoham [7] and used in the KLM approach [4]. The ordering relation $\prec$ is a strict partial order on our points of time. Following Kraus et al. [4], $t \prec t'$ means that t is more preferred than t'.

Definition (Minimality w.r.t. $\prec$)**.** Let $\prec$ be a strict partial order on a set $\mathbb{N}$ and $N \subseteq \mathbb{N}$. The set of the minimal elements of N w.r.t. $\prec$, denoted by $\min_{\prec}(N)$, is defined by $\min_{\prec}(N) \stackrel{\text{def}}{=} \{t \in N \mid \text{there is no } t' \in N \text{ such that } t' \prec t\}$.

Definition (Well-founded set)**.** Let $\prec$ be a strict partial order on a set $\mathbb{N}$. We say $\mathbb{N}$ is *well-founded w.r.t.* $\prec$ if $\min_{\prec}(N) \neq \emptyset$ for every $\emptyset \neq N \subseteq \mathbb{N}$.

Definition (Preferential temporal interpretation)**.** An $LTL^{\sim}$ interpretation on a set of propositional atoms $\mathcal{P}$, also called preferential temporal interpretation on $\mathcal{P}$, is a pair $I \stackrel{\text{def}}{=} (V, \prec)$ where $V : \mathbb{N} \longrightarrow 2^{\mathcal{P}}$ is a valuation function on time points, and $\prec\ \subseteq \mathbb{N} \times \mathbb{N}$ is a strict partial order on $\mathbb{N}$ such that $\mathbb{N}$ is well-founded w.r.t. $\prec$. We denote the set of preferential temporal interpretations by $\mathfrak{I}$.

In what follows, given a preference relation $\prec$ and a time point $t \in \mathbb{N}$, the set of *preferred time points relative to* t is the set $\min_{\prec}([t, +\infty[)$ which is denoted in short by $\min_{\prec}(t)$. It is also worth to point out that given a preferential interpretation $I = (V, \prec)$ and $\mathbb{N}$, the set $\min_{\prec}(t)$ is always a non-empty subset of $[t, +\infty[$ at any time point $t \in \mathbb{N}$.

Preferential temporal interpretations provide us with an intuitive way of interpreting sentences of $\mathcal{L}^{\sim}$. Let $\alpha \in \mathcal{L}^{\sim}$, let $I = (V, \prec)$ be a preferential interpretation, and let t be a time point in I in $\mathbb{N}$. Satisfaction of α at t in I, denoted $I, t \models \alpha$, is defined as follows:

- $I, t \models p$ if $p \in V(t)$;
- $I, t \models \neg\alpha$ if $I, t \not\models \alpha$;
- $I, t \models \alpha \wedge \alpha'$ if $I, t \models \alpha$ and $I, t \models \alpha'$;
- $I, t \models \alpha \vee \alpha'$ if $I, t \models \alpha$ or $I, t \models \alpha'$;
- $I, t \models \Box\alpha$ if $I, t' \models \alpha$ for all $t' \in \mathbb{N}$ s.t. $t' \geq t$;
- $I, t \models \Diamond\alpha$ if $I, t' \models \alpha$ for some $t' \in \mathbb{N}$ s.t. $t' \geq t$;
- $I, t \models \widetilde{\Box}\alpha$ if $I, t' \models \alpha$ for *all* $t' \in \min_{\prec}(t)$;
- $I, t \models \widetilde{\Diamond}\alpha$ if $I, t' \models \alpha$ for *some* $t' \in \min_{\prec}(t)$.

The truth values of Boolean connectives and classical modalities are defined as in LTL. The intuition behind a sentence like $\widetilde{\Box}\alpha$ is that α holds in *all* preferred time points that come after t. $\widetilde{\Diamond}\alpha$ intuitively means that α holds on at least one preferred time point relative in the future of t.

We say $\alpha \in \mathcal{L}^{\sim}$ is *preferentially satisfiable* if there is a preferential temporal interpretation I and a time point t in $\mathbb{N}$ such that $I, t \models \alpha$. We can show that $\alpha \in \mathcal{L}^{\sim}$ is *preferentially satisfiable* if there is a preferential temporal interpretation I s.t. $I, 0 \models \alpha$. A sentence $\alpha \in \mathcal{L}^{\sim}$ is *valid* (denoted by $\models \alpha$) iff for all temporal interpretation I and time points t in $\mathbb{N}$, we have $I, t \models \alpha$.

We can see that the addition of $\prec$ relation preserves the truth values of all classical temporal sentences. Moreover, for every $\alpha \in \mathcal{L}_{LTL}$, we have that α is satisfiable in LTL if and only if α is preferentially satisfiable in $LTL^{\sim}$.

Let α, β be well-formed sentences in $\mathcal{L}^{\sim}$. We have a duality between our defeasible operators: $\models \widetilde{\Box}\alpha \leftrightarrow \neg\widetilde{\Diamond}\neg\alpha$. We also have $\models \Box\alpha \rightarrow \widetilde{\Box}\alpha$ and $\models \widetilde{\Diamond}\alpha \rightarrow \Diamond\alpha$. Intuitively, this property states that if a statement holds in all of future time points of any given t, it holds on all our *future preferred* time points of t. As intended, this property establishes the defeasible always as "weaker" than the classical always. It can commonly be accepted since the set of all preferred future states are in the future. This is why we named $\widetilde{\Box}$ as *defeasible always*. On the other hand, we see that $\widetilde{\Diamond}$ is "stronger" than classical eventually, the statement within $\widetilde{\Diamond}$ holds at a preferable future.

We established the decidability of the satisfiability problem of the sub-language that has $\Box, \Diamond, \widetilde{\Box}, \widetilde{\Diamond}$ as modalities [3]. We also defined a semantic tableau for another fragment [2] that can serve as the basis for further exploring tableau methods for this logic.

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Logic with Two-Layered Modal Syntax: Abstract, Abstracter, Abstractest

Petr Cintula and Carles Noguera

Two-layered modal syntax is given by three propositional languages (collections of connectives together with their arities): the *lower* one (also known as language of events), the *modal* one (whose connectives are actually called modalities), and the *upper* one (also known as language of statements). Using these three languages and a fixed set of event variables, we construct three disjoint sets of formulas:

- *non-modal* (or event) formulas are built from event variables using the lower language,
- *atomic modal* formulas (or atomic statements) are built by applying the modalities to non-modal formulas, and
- *complex* modal formulas (or statements) are built from the atomic ones using the upper language.

Note that, by construction, the modalities cannot be nested and each event variable has to be in the scope of some modality.

Early examples of logics with two-layered syntax were modal logics of uncertainty stemming from Hamblin's seminal idea of reading the atomic statement $P\varphi$ as 'probably φ' [16] and semantically interpreting it (in a given Kripke frame equipped with a finitely additive probability measure) as *true* iff the probability of the set of worlds where φ is true is bigger than a given threshold. This idea was later elaborated and extended by Fagin, Halpern and many others; see e.g. [5, 15].

These initial examples used classical logic to govern the behavior of formulas on both the modal and the non-modal layers. A departure from this classical paradigm was proposed by Hájek and Harmancová in [13] and later developed by them in collaboration with Godo and Esteva in [12]. They kept classical logic as the interpretation of the lower syntactical layer of events, but proposed Łukasiewicz logic to govern the upper layer of statements on probabilities of these events, so that the truth degree of the atomic statement $P\varphi$ could be directly identified with the probability of the set of worlds where φ is true. Later, numerous other authors changed even the logic governing the lower layer (e.g., another fuzzy logic in order to allow for the treatment of uncertainty of vague events) or considered additional possibly non-unary modalities (e.g. for conditional probability), see e.g. [10, 9, 17, 11, 7, 6, 8, 14].

This research thus gave rise to an interesting way of combining logics which allows to use one logic to reason about formulas (or rules) of another one with numerous examples described and developed in the literature. The existing bulk of literature constitutes an area of logic screaming for systematization through the development and application of uniform, general, and abstract methods. In our previous work [3] we took the first steps towards such a theory by providing an abstract notion of two-layered syntax and logic, a general semantics of *measured* Kripke frames and proved, in a rather general setting, two forms of completeness theorem most commonly appearing in the literature. Although the level of generality seemed quite sufficient back then (*finitary weakly implicative logics with unit and lattice conjunction*, see [4]), the recent development in the field shows the need for more: e.g., the lower logic in [2] and the upper logic in [1] are not weakly implicative.

The aim of this talk is to present the state of the art of the abstract theory of two-layered logics which is now much more extensive, streamlined and mature than in the original paper [3]. In particular, we will show how we can obtain the completeness results for an arbitrary equivalential or degree-preserving lower logic and an arbitrary equivalential upper logic.

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Towards First-Order Partial Fuzzy Modal Logic

Antonín Dvořák

In recent decades, modal logics have been studied not just in the classical bivalent setting, but also in non-classical settings including that of fuzzy logic (e.g., [9, §8.3], [6, 12]). In this contribution, we present our research program aiming at the introduction and investigation of first-order partial fuzzy modal logic with varying domains. To the best of our knowledge, such logic has not been yet proposed. To this end, we already studied some necessary building blocks of such logic, namely fuzzy relational modalities admitting truth-valueless propositions [3] and free fuzzy logic [2]. These systems are based on partial fuzzy logic [4], that is, a variant of fuzzy logic that admits truth-valueless propositions. For first-order modal logic in the classical setting, see, e.g., [7, 8].

Partial fuzzy logic L^* proposed in [4] represents truth-value gaps by an additional truth value $*$, added to an algebra of truth degrees of an underlying $\triangle$-core [10] fuzzy logic L. The intended L^*-algebras are thus defined as expansions $\boldsymbol{L}_* = \boldsymbol{L} \cup \{*\}$ of L-algebras $\boldsymbol{L}$, where $* \notin \boldsymbol{L}$. The connectives of $\boldsymbol{L}$ are extended to $\boldsymbol{L}_*$ in several parallel ways, including the following prominent families of $\boldsymbol{L}_*$-connectives:

- The *Bochvar-style* connectives c_{B} for each connective c of L, which treat $*$ as the absorbing element.
- The *Sobociński-style* connectives c_{S}, which treat $*$ as the neutral element.
- The *Kleene-style* connectives c_{K}, which preserve the neutral and absorbing elements of the corresponding connectives of L and otherwise are evaluated Bochvar-style.

The first-order variant $\mathrm{L}\forall^*$ of L^*, introduced in [1], is defined as usual in fuzzy [9, 10] logics, with predicates evaluated in L^*-algebras. Like the connectives of L^*, the quantifiers of $\mathrm{L}\forall^*$ also come in several families. For example, the *Bochvar-style* quantifiers $\exists_{\mathrm{B}}, \forall_{\mathrm{B}}$ yield $*$ whenever an instance is $*$-valued, etc.

Partial fuzzy relational modalities. As a next step towards first-order partial fuzzy modal logic, we studied partial fuzzy relational modalities [3]. We aimed at extending the propositional language of L^* by meaningful Kripke-style modalities in **K**. Above, we already met three important families of propositional connectives of partial fuzzy logic L^*, namely Bochvar, Kleene and Sobociński, and also the corresponding families of quantifiers of $\mathrm{L}\forall^*$. Naturally, we searched for the corresponding partial fuzzy modalities with a behavior in some sense analogous to the behaviour of these families of connectives and quantifiers. First, we characterized these modalities by means of their semantic behavior. Later, we showed how these modalities can be defined using propositional connectives and quantifiers of Ł$\forall^*$. Kripke modalities can be understood as monadic quantifiers over possible worlds restricted by the accessibility relation, hence it is not surprising that a similar situation occurs in partial fuzzy modal logic. In particular, since the semantic definitions of, e.g., $\Box$ involve a universal quantifier and a restricting connective, namely implication, partial fuzzy modalities can be generally indexed by two indices that specify the quantifier and the connective.

Free fuzzy logic. For our system of first-order fuzzy modal logic, we expect that in some possible worlds, a term has a referent, but that it may not be the case in other ones. Standard predicate (fuzzy) logic does not admit non-denoting terms. In [2], we introduced two variants of first-order fuzzy logic that can deal with non-denoting terms, or terms that lack existing referents. Logics designed for this purpose in the classical setting are known as free logics [5, 11]. We discussed the features of free logics and selected the options best suited for fuzzification, deciding on the so-called dual-domain semantics for positive free logic with truth-value gaps and outer quantifiers. We fuzzified the latter semantics in two levels of generality, first with a crisp and subsequently with a fuzzy predicate of existence.

First-order partial fuzzy modal logic with varying domains. Now we have at our disposal the ingredients for our system of first-order fuzzy modal logic. In the talk, we will introduce the structure of a Kripke model for this logic with one common domain and domains of individual possible worlds (subsets of the common domain). Then, we will propose Tarski conditions for terms and formulae including also the conditions for modalities and quantifiers. As above, modalities will come in various families differing in the modes of propagation of the undefined value. The same holds for quantifiers. Moreover, there will be two kinds of quantifiers, similarly as in our proposed system of free logic, namely "outer" quantifiers that quantify over the common domain, and "inner" quantifiers that quantify over domains of possible worlds. These "inner" quantifiers can be defined as relativizations of "outer" quantifiers using the predicate of existence. We will finish by presenting initial observations about this system.

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Glivenko classes and constructive cut elimination in infinitary logic

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Notable parts of algebra and geometry can be formalised as *coherent theories* over first-order classical or intuitionistic logic. Their axioms are *coherent implications*, i.e., universal closures of implications $D_1 \supset D_2$, where both D_1 and D_2 are built up from atoms using conjunction, disjunction and existential quantification. Examples include all algebraic theories, such as group theory and ring theory, all essentially algebraic theories, such as category theory [4], the theory of fields, the theory of local rings, lattice theory [13], projective and affine geometry [13, 10], the theory of separably closed local rings (aka "strictly Henselian local rings") [5, 10, 15].

Although wide, the class of coherent theories leaves out certain axioms in algebra such as the axioms of torsion abelian groups or of Archimedean ordered fields, or in the theory of connected graphs, as well as in the modelling of epistemic social notions such as common knowledge. All the latter examples can however be axiomatised by means of *geometric axioms*, a generalisation of coherent axioms that allows infinitary disjunctions.

Orevkov [11] has established some well-known conservativity results of classical logic over intuitionistic and minimal predicate logics with equality. In particular, [11] isolates seven classes of sequents – the so-called *Glivenko sequent classes* – having this property and it shows that these classes are optimal: any class of sequents for which classical derivability implies intuitionisitc derivability is contained in one of these seven classes. The interest of such conservativity results is twofold. First, since proofs in intuitionistic logic obtain a computational meaning via the Curry-Howard correspondence, such results identify some classical theories having a computational content. Second, since it may be easier to prove theorems in classical than in intuitionistic logic and since there are more well-developed automated theorem provers for classical than for intuitionistic logic, such results simplify the search for theorems in intuitionistic theories.

Coherent and geometric implications form sequents that give a Glivenko class [11], as shown by Barr's Theorem.

Theorem 1 (Barr's Theorem [2])**.** *If $\mathcal{T}$ is a coherent (geometric) theory and A is a sentence provable from $\mathcal{T}$ with (infinitary) classical logic, then A is provable from $\mathcal{T}$ with (infinitary) intuitionistic logic.*

If we limit our attention to first-order coherent theories $\mathcal{T}$, an extremely simple and purely logical proof of Barr's Theorem has been given in [7] by means of **G3**-style sequent calculi. [7] shows how to express coherent implications by means of rules that preserve the admissibility of the structural rules of inference. As a consequence, Barr's theorem is proved by simply noticing that a proof in **G3cT** is also a proof in the intuitionistic multisuccedent calculus **G3iT**. This simple and purely logical proof of Barr's Theorem has been extended to cover all other first-order Glivenko classes in [8].

A purely logical proof of Barr's Theorem for infinitary geometric theories has been given [9]. This work considers the **G3**-style calculi for classical and intuitionistic infinitary logic $\mathbf{G3[ci]}_\omega$ (with finite sequents instead of countably infinite sequents) and their extension with rules expressing geometric implications $\mathbf{G3[ci]}_\omega\mathbf{T}$. The main results in [9] are that in $\mathbf{G3[ci]}_\omega\mathbf{T}$ all rules are height-preserving invertible, the structural rules of weakening and contraction are height-preserving admissible, and cut is admissible. Hence, Barr's Theorem for geometric theories is proved by showing that a proof in $\mathbf{G3c}_\omega\mathbf{T}$ is also a proof in the intuitionistic multisuccedent calculus $\mathbf{G3i}_\omega\mathbf{T}$.

In this paper we extend this purely logical proof of the infinitary Barr's Theorem to cover all other infinitary Glivenko sequent classes: for each class we give a purely constructive proof of conservativity of classical infinitary logic over intuitionistic and minimal infinitary logics.

One weakness of the results in [9] is that the cut-elimination procedure given in Sect. 4.1 is not constructive. This is a typical limitation of cut eliminations in infinitary logics that are based on ordinal numbers [3, 6, 14]. The main problem is that the proof makes use of the 'natural' (or Hessenberg) commutative sum of ordinals which is not available in **CZF** nor in **IZF** [12, p.369]. We constructivise the proof of (height-preserving) admissibility of the structural rules for $\mathbf{G3[cim]}_\omega\mathbf{T}$ by giving procedures that avoid completely the need for ordinal numbers: inductions on (sums of) ordinals are replaced by inductions on well-founded trees and by Brouwer's principle of bar induction.[3]

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Defusing Small Explosions in Topic-Sensitive Intentional Modals

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Topic-Sensitive Intentional Modals: The framework of *topic-sensitive intentional modal operators* (TSIMs) described by Berto in *e.g.* [2] and [3] provides a general platform for representing agents' intentional states of various kinds. In the general case, a TSIM is a two-place modal operator $\Box^{\varphi}\,\psi$ with an intended reading: "given φ, one stands in such-and-such a relation to ψ."

The core Kripke-style semantic interpretation of these operators has two components, one *truth-theoretic* and the other *content-theoretic*. The truth-theoretic component employs a variably strict conditional in the sense of [10], reflected by the presence of an accessibility relation R_{φ} for each formula φ. This is complemented by a content inclusion filter reminiscent of Fine's semantics for analytic implication in [8], in which a function c maps each formula to an element of a semilattice of *topics* with ordering $\preccurlyeq$. The general truth conditions for a TSIM are

- $w \Vdash \Box^{\varphi}\,\psi$ if $\begin{cases} \forall w' \text{ s.t. } wR_{\varphi}w',\ w' \Vdash \psi, \text{ and} \\ c(\psi) \preccurlyeq c(\varphi) \end{cases}$

Varying semantic conditions allow one to model a number of intentional relations. *E.g.*, among those documented in [3] is a *doxastic* reading in which $\Box^{\varphi}\,\psi$ is understood as "after revising one's beliefs with φ, one believes ψ."

Calling it a "small explosion principle," [3] identifies a potentially counterintuitive feature: In any interpretation in which $\Box^{\varphi}\,\varphi$—the **(Success)** axiom—holds, we will have the following:

(S-EXP) $\models \Box^{\varphi\wedge\neg\varphi\wedge\psi}\,\neg\psi$

Given **(Success)**, the classicality of the stock TSIM framework ensures vacuous satisfaction of the first clause while the second clause follows from the guarantee that $c(\neg\psi) \preccurlyeq c(\varphi \wedge \neg\varphi \wedge \psi)$.

(S-EXP) is clearly a special case of a phenomenon identified in [5] as the paradox of "making too much of one small, if nasty, mistake"—the theoremhood of $((\varphi \wedge {\sim}\varphi) \wedge \psi) \to (\psi \wedge {\sim}\psi)$. Given a *doxastic* reading, *e.g.*, **(S-EXP)** encapsulates a picture in which the presence of *any* item of inconsistent information is sufficient to *explicitly undermine* any *occurrent* belief of an agent.

Deutsch's remarks in [6] on strategies for resolving the relevantly objectionable $\varphi \to (\psi \to \psi)$ are equally applicable to the paradox of small-if-nasty-mistakes. Broadly, the two strategies correspond to *truth-theoretic* and *content-theoretic* considerations, respectively; applied to $\Box^{\varphi\wedge\neg\varphi\wedge\psi}\,\neg\psi$, these may be made precise as

- positing inconsistent situations satisfying $\varphi \wedge \neg\varphi \wedge \psi$ and not $\neg\psi$, and
- relaxing the condition that the subject-matter of $\neg\psi$ is included in that of $\varphi \wedge \neg\varphi \wedge \psi$

The Content-Theoretic Strategy: The inclusion of the topic of $\neg\psi$ within that of $\varphi \wedge \neg\varphi \wedge \psi$ is a consequence of the condition of *negation transparency*, according to which $c(\varphi) = c(\neg\varphi)$, *i.e.*, the subject-matters of a formula and its negation are identical. But in a system like Angell's AC of [1], negation transparency need not hold, further motivated by the "fact-based" account of subject-matter in Fine's interpretation of AC in [9].

We define a modified semantics for TSIMs by replacing the content semilattice with the *signed* semilattice defined in [7] for the containment logic PAC and redefining the conditions on c accordingly. Importantly, the models allow for cases in which $c(\varphi)$ and $c(\neg\varphi)$ are incommensurable.

Observation 1. **(S-EXP)** fails in the logic of signed TSIMs, even in case **(SUCCESS)** is assumed.

Thus, adopting a content-theoretic strategy *à la* Angell is strong enough to resolve the paradoxical **(S-EXP)**.

The limitations of the strategy are revealed by similar cases of small-if-nasty-mistakes. Thus, it remains too hasty to trumpet Observation 1 as a solution to the *general* form of the paradox. Consider the following:

(S-EXP*) $\vDash \Box^{\varphi\wedge\neg\varphi\wedge\psi\wedge(\psi\vee\neg\psi)}\neg\psi$

In natural language, **(S-EXP*)** reflects that the entry of inconsistent information in an intentional state undermines an unrelated explicit belief ψ *in the presence of explicit information that* ψ *satisfies excluded middle.*

Observation 2. Assuming **(SUCCESS)**, **(S-EXP*)** is valid in the logic of signed TSIMs.

The Truth-Theoretic Strategy: The truth-theoretic strategy posits a device by which a state can verify $\varphi \wedge \neg\varphi \wedge \psi$ without verifying $\neg\psi$. This strategy is formalized by describing a modification to the TSIM model theory in which an atom p is given a positive interpretation $v^+(p) \subseteq W$ and negative interpretation $v^-(p) \subseteq W$ inducing verification ($\Vdash^+$) and falsification ($\Vdash^-$) relations given standard FDE evaluations.

Observation 3. **(S-EXP)** and **(S-EXP*)** fail in the logic of TSIMs with FDE states, even in case **(SUCCESS)** is assumed.

The truth-theoretic strategy thus resolves both **(S-EXP)** and **(S-EXP*)**.

Although going paraconsistent may work for a *doxastic* interpretation, the *epistemic* interpretation reveals its limitations. In [4], the following is identified as a desirable property when TSIMs are given a knowledge reading:

(COOKIT) $\{\Box^{\varphi}\,\psi, \Box^{\varphi}\,\psi \supset \xi\} \vDash \Box^{\varphi}\,\xi$

Given the definition of $\supset$, the following should also hold:

(COOKIT*) $\{\Box^{\varphi}\,\psi, \Box^{\varphi}\,\neg\psi \vee \xi\} \vDash \Box^{\varphi}\,\xi$

If **(COOKIT*)** is to be respected, devices like FDE—for which disjunctive syllogism fails—are inappropriate tools.

(COOKIT*) thus presents a dilemma. **(S-EXP)** and **(S-EXP*)** seem equally paradoxical on the knowledge reading. (Consider a *textbook*—a paradigmatic aggregate of *knowledge-constitutive* propositional information. Clearly, a student's knowledge of a *truth* ψ explained in a physics textbook should not be undermined by some *irrelevant* inconsistency $\varphi \wedge \neg\varphi$ in its concluding remarks.) Straightforward, paraconsistent truth-theoretic strategies contradict **(COOKIT)** while content-theoretic strategies do not dispel **(S-EXP*)**. We conclude by considering what is necessary to resolve this dilemma.

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On the role of Dunn and Fisher Servi axioms in relational frames for Gödel modal logics

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Extending modal logics to a non-classical propositional ground has been, and still is, a fruitful research line that encompasses several approaches, ideas and methods. In the last years, this topic has significantly impacted on the community of many-valued and mathematical fuzzy logic that have proposed ways to expand fuzzy logics (t-norm based fuzzy logics, in the terminology of Hájek [8]) by modal operators so as to capture modes of truth that can be faithfully described as "graded".

In this line, one of the fuzzy logics that has been an object of major interest without any doubt is the so called *Gödel logic*, i.e., the axiomatic extension of intuitionistic propositional calculus given by the *prelinearity axiom*: $(\varphi \to \psi) \vee (\psi \to \varphi)$. As first observed by Horn in [9], prelinearity implies completeness of Gödel logic with respect to totally ordered Heyting algebras, i.e., *Gödel chains*. Indeed, prelinear Heyting algebras form a proper subvariety of that of Heyting algebras, usually called the variety of Gödel algebras and denoted $\mathbb{G}$ whose subdirectly irreducible elements are totally ordered. Furthermore, in contrast with the intuitionistic case, $\mathbb{G}$ is locally finite, whence the finitely generated free Gödel algebras are finite.

Modal extensions of Gödel logic have been intensively discussed in the literature [2, 3, 10]. Following the usual methodological and philosophical approach to fuzzy logic, they have been mainly approached semantically by generalizing the classical definition of Kripke model $\langle W, R, e\rangle$ by allowing both the evaluation of (modal) formulas and the accessibility relation R to range over a Gödel algebra, rather than the classical two-valued set $\{0, 1\}$ (see [1] for a general approach). More precisely, a model of this kind, besides evaluating formulas in a more general structure than the classical two-element boolean algebra, regards the accessibility relation R as a function from the cartesian product $W \times W$ to a Gödel algebra $\mathbf{A}$ so that, for all $w, w' \in W$, $R(w, w') = a \in A$ means that a is the *degree of accessibility* of w' from w.

Here, we put forward a novel approach to Gödel modal logic that leverages on the duality between finite Gödel algebras and finite forests. This line, that was previously presented in [7], is deepened and extended by the present approach. In particular, we ground our investigation on finite Gödel modal algebras and their dual structures, that is, the prime spectra of finite Gödel algebras ordered by reverse-inclusion. These ordered structures can be regarded as the prelinear version of posets and they are known in the literature as *finite forests*: finite posets whose principal downsets are totally ordered. In general, Gödel algebras with modal operators form a variety denoted by $\mathbb{GAO}$ for *Gödel algebras with operators*. Hence, the algebras we are concerned with are those belonging to the finite slice of $\mathbb{GAO}$. The associated relational structures based on forests, as we briefly recalled above, might hence be regarded as the prelinear version of the usual relational semantics of intuitionistic modal logic. Accessibility relations $R_\Box$ and $R_\Diamond$ on finite forests are defined, in our frames, by ad hoc properties that we express in terms of (anti)monotonicity on the first argument of the relations themselves. These relational frames will be called *forest frames*.

Furthermore, we put forward a comparison between our approach to the ones that have been proposed for intuitionistic modal logic and, in particular, those developed by Palmigiano in [12] and Orłowska and Rewitzky in [11]. By analyzing the role that these different relational frames (namely, those presented by Palmigiano, Orłowska and Rewitzky, and ours) have in proving a Jónsson-Tarski like representation theorem for Gödel algebras with modal operators, we realized that forest frames situate in a middle level of generality between those of Palmigiano and those of Orłowska and Rewitzky. The former being the less and the latter being the more general ones.

More in details, we observe that, if we start from any Gödel algebra with operators $(\mathbf{A}, \Box, \Diamond)$, its associated forest frame $(\mathbf{F_A}, R_\Box, R_\Diamond)$ allows to construct another algebraic structure $(\mathbf{S_{F_A}}, \beta_\Box, \delta_\Diamond)$ isomorphic to the starting one. Interestingly, the forest frame $(\mathbf{F_A}, R_\Box, R_\Diamond)$ is not the unique one that reconstructs $(\mathbf{A}, \Box, \Diamond)$ up to isomorphisms. Indeed, for every Gödel algebra with operators $(\mathbf{A}, \Box, \Diamond)$, there are non-isomorphic forest frames, Palmigiano-like, and Orłowska and Rewitzky-like frames that determine the same original modal algebra $(\mathbf{A}, \Box, \Diamond)$ up to isomorphism.

We start by considering the most general way to define the operators $\Box$ and $\Diamond$ on Gödel algebras and investigating the relational structures corresponding to the resulting algebraic structures. Later on, we focus on particular and well-known extensions. Precisely we consider two main extensions of Gödel algebras with operators: (1) the first one is obtained by adding the Dunn axioms, typically studied in the fragment of positive classical (and intuitionistic) logic [5, 4]; (2) the second one is determined by adding the Fischer-Servi axioms [6]. From the algebraic perspective, adding these identities to Gödel algebras with operators identifies two proper subvarieties of $\mathbb{GAO}$ that we respectively denoted by $\mathbb{DGAO}$ and $\mathbb{FSGAO}$.

In contrast with the case of general Gödel algebras with operators whose relational structures need two independent relations to treat the modal operators, the structures belonging to $\mathbb{DGAO}$ and $\mathbb{FSGAO}$ only need, for their Jónnson-Tarski like representation, frames with only one accessibility relation. In addition, we study in detail the relational structures corresponding to two further subvarieties of $\mathbb{GAO}$. The first one is the variety obtained as the intersection $\mathbb{DGAO} \cap \mathbb{FSGAO}$. The algebras belonging to such variety have been called *bi-modal Gödel algebras* in [3] and a modal algebra $(\mathbf{A}, \Box, \Diamond) \in \mathbb{DGAO} \cap \mathbb{FSGAO}$ is characterized by the property stating that, for every boolean element $b \in A$, both $\Box b$ and $\Diamond b$ are boolean as well. The second subvariety that we consider refines $\mathbb{DGAO}$. Indeed, any algebra $(\mathbf{A}, \Box, \Diamond)$ belongs to this class iff it satisfies

Dunn axioms, plus the requirement that $\Box a$ and $\Diamond a$ are boolean for all $a \in A$.

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Semantical investigations on non-classical logics with recovery operators (using the Isabelle proof assistant)

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What & Why

Non-classical negations have an important application in computer science by enabling knowledge representation and reasoning in presence of partial (incomplete) and excessive (contradictory) information. A logic is called paraconsistent if it 'tolerates contradictions' and paracomplete if it does not 'enforce completeness/exhaustiveness'. Thus, in paraconsistent logics, the so-called "principle of explosion" or *ex contradictione (sequitur) quodlibet* (`ECQ`) is not valid: from a contradiction $A \wedge \neg A$ not everything follows. Dually, in paracomplete logics the "law of excluded middle" or *tertium non datur* (`TND`) is not valid: the formula $A \vee \neg A$ is not a tautology. Recovery operators are unary connectives employed in these logics to selectively recover properties of classical logic in a sentence-wise fashion (cf. [12] for a discussion).

This presentation gives an overview of formal reconstruction work using the Isabelle proof assistant (available at the *Archive of Formal Proofs* [17] and summarised in [18]). The present semantical investigations employ a formal metalanguage: Church's simple theory of types [3], also known as classical higher-order logic (HOL) and instantiated here as Isabelle's logic: Isabelle/HOL [23]. This work introduces a *shallow semantical embedding* [2, 6] for a family of non-classical logics featuring a semantics based upon (Stone-type representations of) Boolean algebras extended with additional unary operations by drawing upon early works on topological Boolean algebras (by Kuratowski [19, 20], Zarycki [24, 25], McKinsey & Tarski [22]). The range of such logics is, in fact, very wide and includes *quantified* paraconsistent and paracomplete logics. The present work also aims at reformulating (and extending) the neighbourhood semantics for paraconsistent Logics of Formal Inconsistency (LFIs) *with replacement* as proposed in previous work [11, §5–§7]. On the more practical side, this approach allows for harnessing theorem provers, model generators and so-called 'hammers' [8] for reasoning with combinations of quantified non-classical logics.

This work has its roots in ongoing efforts towards embedding symbolic automated ethical reasoning capabilities in new-generation AI [5, 4], where the ability of reasoning under real-world partial and inconsistent information (e.g. avoiding 'explosion') is essential. The work is also of interest to the working logician, who might appreciate tedious pen-and-paper calculations being replaced by invocations of (counter-)model generators (such as *Nitpick* [9]) or automated theorem provers (via *Sledgehammer* [8]) integrated into modern mathematical proof assistants.

How

Boolean algebras of propositions

Two notions play a fundamental role in this work: *propositions* and *propositional functions*. Propositions are intended to act as sentence denotations. In this respect, we will follow the "propositions as sets of worlds" paradigm as traditionally employed in the semantics of modal logics. For this, we introduce a type w for the domain of points (interpreted as 'worlds', 'states', etc.), where σ, the type for propositions, is an alias for type $w{\Rightarrow}bool$ (i.e. the type for characteristic functions of sets of points). Propositional functions are, as the name suggests, functions assigning propositions to objects in a given domain; formally, they are basically anything with a (parametric) type $'t{\Rightarrow}\sigma$ (where $'t$ acts as a type variable representing an arbitrary but fixed type; cf. *type polymorphism* in Isabelle/HOL [23]).

Below we show how to encode in Isabelle/HOL Boolean algebras (BAs) by drawing upon their set-based Stone representations. They will later be extended with topological operators used to define negations and recovery operators.

The relations of equality and lattice-ordering are encoded as terms of type $\sigma{\Rightarrow}\sigma{\Rightarrow}bool$ in the following manner:

$$A \approx B := \forall w.\ (A\ w) \longleftrightarrow (B\ w) \qquad A \preceq B := \forall w.\ (A\ w) \longrightarrow (B\ w)$$

Following the *shallow semantical embeddings* approach [2, 6], we reuse HOL's meta-logical classical connectives ($\wedge$, $\vee$, $\longrightarrow$, $\neg$, $\texttt{True}$, $\texttt{False}$) in order to define object-logical connectives for BAs ($\boldsymbol{\wedge}$, $\boldsymbol{\vee}$, $\boldsymbol{\rightarrow}$, $\boldsymbol{-}$, $\boldsymbol{\top}$, $\boldsymbol{\bot}$; using **boldface** symbols) as HOL-terms of types $\sigma{\Rightarrow}\sigma{\Rightarrow}\sigma$, $\sigma{\Rightarrow}\sigma$, and σ for binary, unary, and zero-ary connectives respectively; this gives:

$$A \boldsymbol{\wedge} B := \lambda w.\ (A\ w) \wedge (B\ w) \qquad A \boldsymbol{\vee} B := \lambda w.\ (A\ w) \vee (B\ w) \qquad A \boldsymbol{\rightarrow} B := \lambda w.\ (A\ w) \longrightarrow (B\ w)$$

$$\boldsymbol{-}A := \lambda w.\ \neg(A\ w) \qquad \boldsymbol{\top} := \lambda w.\ \texttt{True} \qquad \boldsymbol{\bot} := \lambda w.\ \texttt{False}$$

The aim is to encode a complete BA that can be used to interpret quantified formulas in the spirit of Boolean-valued models for first-order theories (e.g. LFIs with replacement [11, §9] or set theory [1]). For this we encode (infinitary) infima and suprema as terms of type $(\sigma{\Rightarrow}bool){\Rightarrow}\sigma$ by reusing the (*polymorphic* [23]) meta-logical quantifiers $\forall$ and $\exists$.

$$\bigwedge S := \lambda w.\ \forall X.\ S(X) \longrightarrow (X\ w) \qquad \bigvee S := \lambda w.\ \exists X.\ S(X) \wedge (X\ w)$$

After verifying that our encoded BAs are complete, we proceed to define object-logical quantification: $\boldsymbol{\mathcal{Q}}_\forall$, $\boldsymbol{\mathcal{Q}}_\exists$, by taking suprema and infima over the range $\texttt{Ra}(\pi) := \lambda Y.\ \exists x.\ \pi(x) = Y$ of a propositional function π (of type $'t{\Rightarrow}\sigma$). Observe that these quantifiers can be given an alternative, equivalent formulation that corresponds, in fact, to the one introduced for the *shallow semantical embedding* of quantified modal logics in HOL; cf. [2, 6]. Moreover, we can introduce a more familiar variable-binder notation: $\forall$, $\exists$ (with **boldface** symbols) by exploiting λ-abstraction; cf. [3].

$$\mathcal{Q}_\forall\,\pi := \bigwedge \mathtt{Ra}(\pi) \quad \mathcal{Q}_\forall\,\pi :=^{\mathtt{alt}} \lambda w.\, \forall X.\, \pi(X)\ w \quad \boldsymbol{\forall} x.\ \pi := \mathcal{Q}_\forall\,(\lambda x.\ \pi)$$

$$\mathcal{Q}_\exists\,\pi := \bigvee \mathtt{Ra}(\pi) \quad \mathcal{Q}_\exists\,\pi :=^{\mathtt{alt}} \lambda w.\, \exists X.\, \pi(X)\ w \quad \boldsymbol{\exists} x.\ \pi := \mathcal{Q}_\exists\,(\lambda x.\ \pi)$$

Exploiting *type polymorphism* (cf. [23]) we observe that by taking the type variable 't above as σ we obtain propositional quantifiers, whereas by taking 't as some further type i (for 'individuals') we obtain predicate quantification.

In fact, the expressivity of HOL allows us to seamlessly generalise the alternative definition above by explicitly passing a domain of quantification (δ). We call these variants *restricted*, aka. 'free', quantifiers.[4]

$$\mathcal{Q}^R_\forall(\delta)\,\pi := \lambda w.\, \forall X.\, \delta(X)\ w \longrightarrow \pi(X)\ w \quad \big(\text{i.e.}\quad \mathcal{Q}^R_\forall(\!|D|\!)\,\pi = \bigwedge \mathtt{Ra}[\pi|D]\,\big)$$

$$\mathcal{Q}^R_\exists(\delta)\,\pi := \lambda w.\, \exists X.\, \delta(X)\ w \wedge \pi(X)\ w \quad \big(\text{i.e.}\quad \mathcal{Q}^R_\exists(\!|D|\!)\,\pi = \bigvee \mathtt{Ra}[\pi|D]\,\big)$$

Note in the right-side correspondence above that $\mathtt{Ra}[\pi|D] := \lambda Y.\ \exists x.\ D(x) \wedge \pi(x) = Y$, and $(\!|\cdot|\!)$ converts (characteristic functions of) sets into their corresponding 'rigid' propositional functions, i.e. $(\!|S|\!) := \lambda X.\ \lambda w.\ S(X)$.

Finally, we introduce a suitable *fixed point* notion to allow for the encoding of recovery operators. We speak of propositions (of type σ) being fixed points of an operation φ (of type $\sigma{\Rightarrow}\sigma$). For this we define in the usual way a fixed point predicate on propositions $\mathtt{fp}(\varphi) := \lambda X.\ \varphi(X) \approx X$. Moreover, this predicate can become 'operationalised' by defining a fixed-point construction $\varphi^{\mathtt{fp}} := \lambda X.\ \varphi(X) \leftrightarrow X$ such that $\mathtt{fp}(\varphi)\ X \longleftrightarrow \varphi^{\mathtt{fp}}(X) \approx \top$ always holds.

Negations

It is well known that Boolean algebras extended with *closure* $\mathcal{C}$ (resp. *interior* $\mathcal{I}$) operators (aka. closure, resp. interior algebras) and topologies are two sides of the same coin. This fact has been exploited, at least since the seminal work by McKinsey & Tarski in the 1940's [22], to provide topological semantics for intuitionistic and modal logics (cf. [14] for a partial survey). In the present approach, the key to reusing HOL's classical connectives for the encoding of paraconsistent (resp. paracomplete) negations lies in the interplay between the classical complement operation and the closure $\mathcal{C}$ (resp. interior $\mathcal{I}$) operators. We generalise McKinsey & Tarski's approach in two respects: (i) by defining unary operations satisfying some (not necessarily all!) of the corresponding topological conditions (e.g. Kuratowski [19] axioms); and (ii) by introducing

[4]These *restricted* quantifiers take a propositional 'domain' function $\delta(\cdot)$ as an additional parameter. This function can be interpreted as mapping elements X to the proposition "X exists". Readers acquainted with quantified modal logics may observe that $\delta(\cdot)$ generalises the 'existence' meta-predicate employed to restrict the domains of quantification in the varying-domains semantics for first-order modal logics [15] (cf. [7] for the use of restricted quantification in the encoding of category theory in Isabelle/HOL).

some alternative, yet related, sets of conditions of a topological nature which characterise the *derivative* $\mathcal{D}$ [19, 25], and the so-called *frontier* $\mathcal{F}$ (aka. *boundary*) and *border* $\mathcal{B}$ [24] operations.[5]

Our recipe for the encoding of a non-classical negation is thus as follows:

1. We choose a particular topological BA as the base algebra to work with, observing that we may leave its corresponding primitive topological operator ($\mathcal{I}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{B}$, $\mathcal{F}$) initially unconstrained. In this regard, different choices give us different points of departure for defining (possibly distinct) families of non-classical negations.[6]

2. We define non-classical negations as composite algebraic operations employing the (primitive or derived) topological operators, leaving them initially unconstrained. We have opted in [17] for composing the operators $\mathcal{I}$ and $\mathcal{C}$ with the complement $-$.[7] This gives us (a family of) paracomplete and paraconsistent negations, defined respectively as: $\neg^I A := \mathcal{I}(-A)$ and $\neg^C A := \mathcal{C}(-A)$.

3. We start 'customising' our negation by adding semantic restrictions (e.g. Kuratowski closure axioms or their analogues) for the topological operators *on-demand* in order to *minimally* satisfy the desired 'negation-like' properties (`ECQ`, `TND`, double-negation introduction/elimination, De Morgan laws, different versions of contraposition and *modus tollens*, etc.). This give-and-take process is supported by automated tools like *Nitpick* and *Sledgehammer*.[8]

4. Alternatively (or rather complementarily) we can employ fixed-point predicates $\texttt{fp}(\cdot)$ and operators $(\cdot)^{\texttt{fp}}$ in order to recover particular 'negation-like' properties as above, many of which cannot be readily recovered by only adding the semantic (e.g. Kuratowski) conditions from the previous step. This is further discussed in the next section.

[5]Recalling our previously defined fixed-point predicate $\texttt{fp}(\cdot)$, we note that if an element A satisfies $(\texttt{fp}\ \mathcal{I})(A)$ (resp. $(\texttt{fp}\ \mathcal{C})(A)$), in other words, if A is a fixed point of $\mathcal{I}$ (resp. $\mathcal{C}$), then A is called *open* (resp. *closed*). In a similar vein, we have considered in [17] the fixed points of other operators $(\mathcal{F}, \mathcal{B}, \mathcal{D})$ and verified several interesting properties.

[6]For instance, by taking a BA extended with a (minimally constrained) frontier $\mathcal{F}$ operator as a base, we obtain a semantics for the logic **Rmbc(ciw)**, which is the least extension of the basic logic **RmbC** that features a definable consistency connective; cf. [11].

[7]Other, more creative choices are surely possible. Their plausibility is easy to verify with our approach using theorem provers and model finders. Which properties we thereby obtain and which limits exist are intriguing topics that deserve further investigation.

[8]It is worth remarking that we have leveraged automated provers (via Sledgehammer) together with model finder *Nitpick* to uncover minimal semantic conditions (modulo existence of 'not-too-big' finite countermodels) under which the above relationships hold. There were actually only very few cases in which the question of minimality remained undecided: either is the conjecture a theorem but the provers can not verify it (...yet), or there exist indeed infinite or too-big-for-Nitpick countermodels. We found it rather surprising that the great majority of non-theorems have indeed finite countermodels of 'small' cardinalities (most of them algebras with 2 or 4 elements, in rare cases 8 were required).

Recovery operators

We elaborate on the last item (4) above, observing that fixed-point predicates and operators can serve as a mechanism for 'recovering' the classical properties of negation in a *sentence-wise* fashion.

It is clear that, for the paraconsistent negation $\neg^C$, `ECQ` does not hold in general. As it happens, we can 'switch on' `ECQ` for some proposition A by further assuming A open. Thus, by defining a *degree-preserving* consequence relation [16] such that $A \vdash B := A \preceq B$, we have that $(\texttt{fp}\ \mathcal{I})\ A \longrightarrow A \wedge \neg^C A \vdash \bot$. Analogously, we can recover `TND` for the paracomplete negation $\neg^I$ relative to some A, by assuming A closed, such that $(\texttt{fp}\ \mathcal{C})\ A \longrightarrow \top \vdash A \vee \neg^I A$.

Interestingly, other 'negation-like' properties can analogously be recovered in a sentence-wise fashion. For instance, we can recover, for the paracomplete negation $\neg^I$, the properties of double-negation introduction, and (a variant of) contraposition, for some given proposition A, by assuming A open. In fact, by restricting set-quantification over only open A's (assuming all interior conditions for $\mathcal{I}$) we obtain a quite special paracomplete logic, namely, intuitionistic logic. (Observing that this construction actually corresponds to the Stone-type representation of a Heyting algebra.)

Moreover, we can employ the 'operationalised' fixed-point construction $(\cdot)^{\texttt{fp}}$ in order to recover `ECQ` (among other properties) for $\neg^C$ by purely object-logical means, for instance, by noting that $\mathcal{I}^{\texttt{fp}}(A) \wedge A \wedge \neg^C A \vdash \bot$ (for A arbitrary). Let us now introduce a corresponding algebraic operation $\circ A := \mathcal{I}^{\texttt{fp}}(A)$, which we shall call a *consistency* operator, drawing upon the paraconsistent *Logics of Formal Inconsistency* (LFIs; introduced in [13], cf. also [10, 12]). Moreover, we can do the same exercise for the dual property `TND` for $\neg^I$, thus obtaining an operator $\boldsymbol{P}A := \mathcal{C}^{\texttt{fp}}(A)$, such that $\boldsymbol{P}A \rightarrow A \vee \neg^I A \approx \top$, or, more suggestively, by defining $\bigstar A := -\boldsymbol{P}A$, thus obtaining $\top \vdash \bigstar A \vee A \vee \neg^I A$ (for A arbitrary). We call $\boldsymbol{P}$ a *determinedness* operator, drawing upon the (dual) *Logics of Formal Undeterminedness* (LFUs; introduced in [21], cf. also [12]).

Last but not least, we can also recover several properties of negation by applying these fixed-point constructions to topological operators other than $\mathcal{I}$ and $\mathcal{C}$ such as $\mathcal{D}$, $\mathcal{B}$, $\mathcal{F}$, thus obtaining other, hitherto unnamed recovery operators. For instance, the operator $\mathcal{B}^{\texttt{fp}}(\cdot)$, while not able to recover `TND` or `ECQ`, can, under some conditions, recover some contraposition, De Morgan, and double-negation rules. We refer the reader to [17] and its summary [18] for details.

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Truth Tables for Modal Logics: The Forgotten Papers

Lukas Grätz

1 Introduction

The *truth table method* is used to test validity or satisfiability of a propositional formula by listing all combinations of truth values in what we call a *truth table*. The discovery of this brute force method is often attributed to Charles S. Peirce. Modern investigations present a multi-layered picture around the utilization of truth table device and truth table technique by Emil Post, Bertrand Russel, Ludwig Wittgenstein, Paul Bernays and others [1]. Notably, truth tables influenced the early development of modal logic as well—logics proposed by Jan Łukasiewicz have a many-valued semantics.

It was soon shown that the classical many-valued semantics is not sufficient for the modern modal logics developed by C.I. Lewis. Kurt Gödel's proof to the contrary was adapted to modal logics by James Dugundji [2, 3]. We will see that this problem is solved as soon as the notions of truth functions and truth tables are decoupled. Modal logic might not be truth functional, but truth tables can be defined nevertheless.

2 Henry S. Leonard

In 1941, Henry S. Leonard gave a talk "Modal Propositions and Truth Tables" at the annual meeting of the eastern division of the American Philosophical Association [4].[9] He presented three methods of truth tables for modal propositions. The first is restricted "accidental" modal forms. The second one seems to be the one he elaborates in his later paper on two-valued truth tables [5]. The third, unpublished method involves the use of four truth values.

In [5], Leonard defines truth tables with propositional formulas as headings and rows with truth values "T" or "F" below each heading. Using this general notion of a truth table, he defines *valid* tables in §§12–20 by semantical constraints. For modal functions, he defines prescriptions D, D_1, D_2 and D_3. As Leonard concludes, the resulting modal systems seem to be distinct from standard modal logics. Notably, D corresponds to Ivlev's S_{min}.

3 Anderson and Poliferno

Until the beginning of the 1960th there was a great lack of semantic tools for modal logic. Alan Ross Anderson and his disciple M. J. Poliferno wanted to

[9] Some notes and manuscripts of Leonard are preserved under UA.17.361 by *Michigan State University Archives & Historical Collections, East Lansing, Michigan.*

close this gap by developing decision procedures based on truth tables for important modal logics [6, 8]. Unfortunately, these procedures contained several mistakes [7, 9, 10]. A final corrected version was never provided by the authors. Nevertheless, it is worth looking into these papers as they present an early construction of truth tables modal logics M and $S4$.

4 Arnould Bayart

According to google scholar, Arnould Bayart's paper "On truth-tables for M, B, S4 and S5" [11] is cited only by Luis Fariñas-del-Cerro [12], in fact in the same Belgian journal "Logique et Analyse". Bayart's paper is from 1970 and contains 41 pages. Historical background on Bayart's work can be found in [13]. It seems that Bayart's work disappeared invisibly in the shadow of the just emerging Kripke semantics. You may notice that [11] does not include the word "modal" in its title, which makes it hard to find.

Nevertheless, Bayart's truth tables are unrelated to possible worlds semantic. He fixes and extends Anderson's approach to modal logics M, B, $S4$ and $S5$. Starting in Section 2, he gives relational constraints on truth table rows for these logics. These relations he calls C0 and C1. Basically, they are defined for truth table rows r and v as follows:

$$r \ \mathrm{C0} \ v \quad \text{iff} \quad r(\Box\alpha) = \mathrm{T} \text{ implies } v(\alpha) = \mathrm{T} \text{ for every formula } \Box\alpha \text{ on the headings of the truth table.}$$
$$r \ \mathrm{C1} \ v \quad \text{iff} \quad r(\Box\alpha) = \mathrm{T} \text{ implies } v(\Box\alpha) = \mathrm{T} \text{ for every formula } \Box\alpha \text{ on the headings of the truth table.}$$

Bayart shows soundness, which he calls *consistency*, and completeness of his truth table method.

5 Kripke and Massey

In [14], Saul Kripke makes a suggestion to construct truth tables based on the possible worlds semantics for $S5$. This approach is further developed by Gerald J. Massey [15], constructing several truth tables for a single formula to reflect different assignments in each possible world. In contrast to the papers we mentioned before, Massey's paper seems to be widely known [1]. Notably, Massey refers to Leonard but not to Anderson.

6 Conclusion and Outlook

We have seen a number of approaches to constructing truth tables for modal logic. Some of these are very little known. Noteworthy are the truth tables of Arnould Bayart for well-known modal logics.

There is another tradition using many-valued, non-deterministic semantics with valuation levels: This was proposed by Yuri Ivlev [16, 17] and John T. Kearns [18] and then developed in [19, 20, 21, 22].

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Nonclassical first-order logics: Semantics and proof theory

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The existential and universal quantifiers in first-order logic have a clear intuitive meaning and a very well understood behaviour from their interpretation in classical models. However, many challenges arise when trying to interpret quantifiers in a non-classical setting. Already in intuitionistic logic the semantics of the universal quantifier cannot be defined locally (i.e. its intuitionistic interpretation requires to look across possible worlds in models and across the individuals inhabiting those worlds) [7], and unless the constant domain axiom $\forall x(A \vee B(x)) \leftrightarrow A \vee \forall x B(x)$ is assumed, the domains of the models might vary. Likewise for modal logic, even within a classical framework, different semantics have led to different axiomatizations [8, 11]. For weaker logics than the intuitionistic, it is unclear how to properly axiomatize the quantifiers and how to interpret them. In [12, 14, 13], a general approach on quantification is given based on the theory of hyperdoctrines. In [18], semantics with a constant domain are given for distributive modal logic. In [17, 3, 15], an algebraic approach is explored which covers a class of logics, based on the algebraic interpretation of quantifiers as suprema and infima [16]. For classical modal logic, a very general complete axiomatization is given in [4].

Our proposal builds on [19, 1], where a proper display calculus is introduced for classical first-order logic, based on a well known semantic analysis that represents classical first-order models as hyperdoctrines. As was the case of other logical frameworks (cf. e.g. [5, 6, 9]), this semantic analysis makes it possible to define a suitable multi-type calculus for first-order logic in which the side conditions of introduction rules for the quantifiers are encoded into analytic structural rules involving different types. Wansing's insight [21, 20, 2] that quantifiers can be treated proof-theoretically as modal operators is incorporated into this approach by simply regarding $(\forall x)$ and $(\exists x)$ as modal operators bridging different types (i.e. as *heterogeneous* operators). Following Lawvere [12, 14, 13] and Halmos [10], this requires to consider as many types as there are finite sets of free variables; that is, two first-order formulas have the same type if and only if they have exactly the same free variables.

In this environment, both substitutions and quantifiers are explicitly represented as logical and structural connectives, which allows to encode the axioms capturing their interaction as analytic (hererogeneous) modal reduction principles, and hence as analytic structural rules. Thanks to this, we are now in a position to explore systematically the space of properties of substitutions and quantifiers and their possible interactions, and more importantly, to conduct a finer-grained analysis of fundamental interactions between quantifiers and intensional connectives. For instance, certain rules encode the fact that the cylindrification maps are Boolean algebra homomorphisms, which in turn captures the fact that classical propositional connectives are all extensional.

This algebraic analysis, performed in [19, 1] in order to define the multi-type display proof calculus, allows us to isolate a multi-sorted algebraic representation of classical first-order logic, a directed system of Boolean algebras connected via the cylindrification maps.

Building on this algebraic reformulation of classical first-order logic, we introduce multi-type algebraic semantics and proper multi-type display calculi for first-order logics based on varieties of lattice expansions (normal LE-logics). The algebraic semantics are based on directed systems of lattices with normal operators, connected via embeddings which correspond to cylindrifications and lattice homomorphisms which correspond to substitutions. The algebraic inequalities that describe the interactions between these embeddings and homomorphisms and the operators, belong to the class of analytic inductive inequalities and can therefore be transformed to analytic rules in the multi-type display calculus.

We show that these semantics, in the cases of intuitionistic and classical modal logic, correspond to existing logical frameworks by recasting completeness proofs. Having obtained modular algebraic semantics we use duality theory and the theory of canonical extensions to investigate the corresponding relational models. We discuss the challenges that arise in the interpretation of quantifiers when the propositional base is weaker than intuitionistic and highlight the connection of these challenges with the possible constructions of filters and ideals in different algebraic settings. We present as case studies the relational semantics for first-order co-intuitionistic logic, where information can disappear in models whose domains shrink (as opposed to intuitionistic logic where new information arises as models' domains expand), and the relational semantics for first-order distributive logic where we have models whose domains can both expand and shrink. Finally we present general semantics for first-order logic on non-distributive propositional base. We will present both polarity-based and graph-based semantics and discuss possible interpretations for each construction.

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The Simple Model and the Deduction System for Dynamic Epistemic Quantum Logic

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Quantum logic (QL) has been studied to handle strange propositions of quantum physics. In particular, logic based on *orthomodular lattices*, namely, *orthomodular logic* (OML), has been studied since 1936, proposed by Birkhoff and Von Neumann [11]. An orthomodular lattice is related to the closed subspaces of a Hilbert space, which is a state space of a particle in quantum physics. Instead of these lattices, the *Kripke model* (possible world model) of OML can be used, which is called the *orthomodular-model* (OM-model) [14].

OML contains only the propositions for states and does not include the propositions expressing the agent's *knowledge*. To treat an agent's knowledge in quantum mechanics, some studies combine *epistemic logic* (EL) with QL. Ref [9] and [10] can be cited as one of the studies of logic that deal with the concept of knowledge with quantum physics. In these studies, the models which incorporate specific *quantum information* concepts were used. Ref [1] and [3] can be cited as the studies of knowledge with more general concepts of quantum physics. In these studies, similar to EL, knowledge was expressed using the indistinguishability of states. These studies mainly focus on the analysis of *static* knowledge.

To discuss the general *transitions* of knowledges due to the procurement of informations, other concepts, such as *dynamic EL* (DEL), have to be introduced and the field of *dynamic epistemic QL* (DEQL) has to be developed. In [4], *quantum test frame* is introduced as a part of the study of the *dynamic logic of test* (DLT). DLT is a logic for dealing with general changes in knowledge due to information obtained by testing. Quantum test frame is based on the frame for DLT and the frame for *dynamic QL* (DQL) [5] [6] [7]. DQL uses modal symbols for several types of transitions of quantum states, such as *unitary evolutions* and *projections*. An important aspect of quantum physics is the change of state due to measurement. In quantum physics, when a physical quantity is observed, the state is projected to an eigenstate of the physical quantity. That is, when information is obtained from a particle, the state of the particle itself changes depending on the obtained information. This is an important difference between classical and quantum information, and it is reflected in a quantum test frame.

Although the transition of knowledge in quantum mechanics has been analyzed in some directions, some problems remain.

1. These models in previous studies are little complicated because these models introduce almost every element related to knowledge of quantum mechanics. Such a model is also needed, but a somewhat simple model that leaves only the important notions is also useful.

2. As the models and symbols are complicated, constructing a *deduction system* for this types of logic is somewhat complicated task because we have to deal with the mutual consistency of many conditions. Actually, deduction systems for these logics have not been well analyzed.

3. One of the reasons for the complexity of the model is that the conditions used for the models are often unique. Therefore, it is difficult to compare the logic on these models with other logics.

In this study, as a basis for solving these problems, we construct new logic and models for the transition of knowledge in quantum mechanics that is simpler than the studies in [3] and [4], while retaining the essence of these studies. Furthermore, we construct a deduction system that holds soundness and completeness for those new models. In general, *public announcement logic* (PAL) is treated as the most basic and simple logic in DEL. Therefore, this logic fits the purpose of this study. We construct *dynamic epistemic orthomodular logic* (DEOML) by combining the frames and models of OML and PAL, and we simply use a combination of logical symbols for OML and PAL. OML is adopted instead of DQL for the foundation of logic because of the following advantages.

Although OML is not a modal logic, OM-models *implicitly* include the concept of the modality of projection as binary relations that satisfies some important conditions [16]. Therefore, OML can handle the concept of projection while being a simpler model than DQL. This is a completely different feature from previous studies that added all the movement of projections to models *expressly*.

OML does not include the other dynamic concepts of quantum mechanics, such as unitary evolutions. However, the most important strange properties of the agent's knowledge that appear in quantum mechanics are related to projective observations. Therefore, the important properties can be analyzed as long as the concept of projection is included in the logic.

Different from DQL, deduction systems for OML are well argued in previous studies [17] [21] [22], and we can use them directly to construct a deduction system for DEOML. Furthermore, the modal symbols used in DEOML are the same that used in PAL, which is not the case for logics based on DQL [4]. This nature is useful to prove some theorems.

We construct a *sequent calculus* type deduction system for DEOML and prove the soundness and completeness theorem. Sequent calculus is suitable for this study because it is compatible with OML [17] [21] [22]. Furthermore, by using properties similar to PAL, the deduction rules for knowledge-related parts can be easily constructed.

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Two-Dimensional Rigidity

Jonathan Mai

Multi-dimensional modal logics are modal logics interpreted over relational structures, whose points are tuples or sequences over some base set. These logics, and here especially two-dimensional modal logic, have many interesting applications in reverse correspondence theory, natural language semantics and philosophy [5] [2] [4]. However, two-dimensional modal logic can also be applied to address conceptual issues concerning the semantics of first-order modal languages. This important application has not yet received the attention it deserves. Due to the influential [1] the opinion is widespread that a varying domain semantics for first-order modal languages with rigid designators must be based on a free logic. In my talk I want to show that this belief is wrong. I do this by developing a two-dimensional varying domain semantics for first-order modal languages with rigid designators that preserves classical quantification theory.

My talk has three parts. The first part reconstructs and criticizes an argument due to Garson, which aims to show that varying domain semantics for first-order modal languages with rigid designators should not be combined with classical quantification principles [1]. The argument starts with the observation that for any rigid constant c the sentence $\exists x(c = x)$ is true at a point in a varying domain model if and only if c's denotation is in the domain of the point. Since classical quantification theory implies that $\exists x(c = x)$ is valid, it follows according to Garson that every rigid constant c of the language denotes an object that is in the domain of every point of the model. This is supposed to lead to severe technical as well as conceptual problems. However, Garson himself does not provide any convincing reason for thinking that under a varying domain semantics the validity of $\exists x(c = x)$ implies that rigid constants denote objects that exist at every point of a model. Garson's argument is sound only if the implication mentioned holds.

The second part of my talk shows that the implication does not hold and so that Garson's argument fails. I do that by defining a varying domain two-dimensional semantics for a language with rigid constants, where quantifiers behave classically. I confine myself to first-order modal languages with relation symbols (plus identity), individual constants and without any function symbols. A two-dimensional varying domain model on a non-empty set D is a triple $M = (W, \delta, I)$, where $W \neq \emptyset$ and $\delta : W \to \mathcal{P}(D) \setminus \{\emptyset\}$. I is an interpretation function with $I(c)(w) \in \delta(w)$, for any constant c and $I(P)(w) \subseteq D^n$, for any n-place relation symbol P. Assignments are functions g from variables to D. The definitions of denotation and satisfaction are given with respect to pairs (w, v) of points. For denotation, note that the denotation of a constant with respect to (w, v) is $I(c)(v)$. The satisfaction clauses for identity and the boolean connectives are the standard ones. The clauses for atoms, quantifications and boxed formulas go as follows:

- $M, (w, v), g \models Pt_1 \ldots t_n \quad :\Leftrightarrow \quad (t_1^{M,(w,v),g}, \ldots, t_n^{M,(w,v),g}) \in I(P)(v)$;
- $M, (w, v), g \models \forall x\varphi \quad :\Leftrightarrow \quad M, (w, v), g\frac{a}{x} \models \varphi$, for all $a \in \delta(v)$;
- $M, (w, v), g \models \Box\varphi \quad :\Leftrightarrow \quad M, (w, u), g \models \varphi$, for every $u \in W$.

Let a formula φ be true in a model M ($M \models \varphi$), if for any $w \in W$ and any assignment g it holds that $M, (w, w), g \models \varphi$. A formula φ is a logical consequence of a set of formulas Σ ($\Sigma \models \varphi$), if $M \models \varphi$, for any model M with $M \models \psi$, for any $\psi \in \Sigma$. Validity is logical consequence from the empty set.

Call a term t (two-dimensionally) rigid with respect to a model $M = (W, \delta, I)$, a point $w \in W$ and an assignment g, if for all $u, v \in W$ we have that $t^{M,(w,v),g} = t^{M,(w,u),g}$. I assume for any model M and any $w \in W$ a (maybe empty) collection C_w of exactly those constants which are rigid w.r.t M, w, g, for some g. Furthermore, let $C_M = \bigcup_{w \in W} C_w$. I call a model M fully rigid, if for all $w \in W$: $C_w \neq \emptyset \Rightarrow C_w \subseteq \bigcap_{v \in W} C_v$. Note that t is rigid w.r.t M, w, g only if $t^{M,(w,v),g} = t^{M,(w,w),g}$, for all $v \in W$, even if M is not fully rigid. Therefore, for every fully rigid M it follows that $M \models c = c'$ only if $M \models \Box(c = c')$, for any constants $c, c' \in C_M$. Let fully rigid semantics be defined as above but where the admissible models are restricted to be fully rigid models. Thus the results from above show that fully rigid semantics captures the notion of rigidity and is varying domain. Finally, fully rigid semantics preserves classical quantification theory: For any fully rigid model M, $M \models \exists x(c = x)$, where c is a constant from C_M. This is because, for any $w \in W$, $c^{M,(w,w,),g} \in \delta(w)$. For the same reason, for any fully rigid model M, $M \models Pc$ only if $M \models \exists xPx$. At the same time, rigid constants need not denote objects that exist at every point of a fully rigid model M, since for $w \neq u$ we may have $c^{M.(w,v),g} \neq c^{M,(u,s),g}$.

In the third part of my talk I outline how fully rigid semantics can be used to give a promising semantic treatment of natural language descriptive names, i.e. proper names whose reference is determined by a definite description. I argue that the meaning of descriptive names is of a two-dimensional nature. It can be represented as a partial function from centered worlds to constant functions from possible worlds to individuals.This partial function I call the name's deep intension. The idea is the following: Centered worlds represent reference fixing situations. Assume the individual a centered world centers on uniquely satisfies the description that is supposed to fix the reference of a name n. Then the deep intension of n is defined for the centered world and outputs the constant function which sends every possible world to that individual. In case the individual at the center of the world does not uniquely satisfy the associated reference fixing description, n's deep intension is undefined for the centered world in question. I apply this semantics to the notorious case of 'Newman-1', a descriptive name introduced in [3]. The semantics predicts that prior to the first instant of the 21st century, the name did not denote anything, since its deep intension was not defined for any actual centered world with a time parameter earlier than that first instant. However, since the beginning of the 21st century the name rigidly denotes the individual that uniquely satisfies the condition of being the first to be born in the 21st century.

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Completing most quantified modal logics

Eugenio Orlandelli

This work, which is a revised version of [15, 14], is a proof-theoretic study of quantified modal logics (QMLs) in the context of indexed modalities and transition semantics. We hope in this way to dispel some 'locus communis' on the intractability of quantified modal logics. The core of our work is the introduction and the systematic study of labelled sequent calculi for indexed modal logics (IMLs).

An IML is defined as the set of indexed modal formulas that are valid on a class of transition frames, see [3, 4]. In general IMLs determine a very general and expressive family of QMLs which allows to simulate most other families of QMLs, such as all those considered in [1]. IMLs are obtained by indexing modal operators with sets of terms and by considering a counterpart-theoretic semantics. This allows us to distinguish between *de re* and *de dicto* modalities and to obtain a better control of quantifiers, substitutions, and identity in modal contexts. Axiomatic systems for IMLs have been introduced in [3] and studied in [16]. Nevertheless, save for some of the simpler calculi, there exists no proof that such systems are complete with respect to the intended class of transition frames. This is an instance of a general problem for QMLs: completeness results for QMLs are extremely difficult to find. Moreover, in most cases the quantified extensions of a complete propositional modal logic are not complete with respect to the intended semantics [19, 8, 5, 6, 2, 9].

In modal logics the incompleteness phenomenon is widespread at the propositional level and it becomes even worst at the predicative level, see [18]. Even if we limit our attention to the quantified extensions of those propositional modal logics (PMLs) which are complete and canonical, we find that most of them are incomplete. Of course the analysis needs to be more precise because 'quantified extensions' can mean different things – e.g., there are logics with or without classical quantification, with or without the Barcan formulas, with or without the Ghilardi formula and with or without the necessity of identity. In general, given a complete PML **S** two kinds of incompleteness may arise for its quantified extensions: there are *incompletable* logics such as the non recursively axiomatizable **Q.GL** [10] and *completable* logics such as **Q.S4M** where, in order to obtain a complete system, we need to add some *de re* axiom that regulates the interaction between modalities and the first-order machinery [6].

We are interested in completable logics: we aim at extending to the predicate level a wide range of completeness results established at the propositional level. In doing so, we want to solve the following open problem for IMLs:

> how should we define a modular family of proof systems that characterizes quantified extensions of a wide class of complete PMLs?

This will be done by introducing labelled sequent calculi for IMLs which behave extremely well from a proof-theoretic point of view – all the structural rules of inference are admissible and all rules are invertible – and characterize quantified extensions of any first-order definable PML.

The labelled calculi introduced in [12] allow to internalize the semantics of a propositional modal logics into the syntax of the calculi by extending the language and to express the semantic conditions defining classes of modal frames via rules expressing coherent (aka geometric) implications [11]. In [7] it has been shown that each first-order formula can be transformed into a coherent implication and, hence, in a rule preserving the good structural properties of the underlying calculus. In [13, 17] it has been shown that the same holds for quantified modal logics and, as it shown in [14], the same holds for some indexed epistemic logics. In this work we show that this holds holds for all indexed extension of each first-order definable propositional modal logic.

This proves that labelled sequents are better behaved than axiomatic systems with respects to QMLs: for any completable – but incomplete – axiomatic system for a first-order definable QML, we can define a labelled sequent calculus that proves all theorems of the completion of the axiomatic system. To illustrate, in [5] it has been shown that the formula:

$$\Diamond(\forall x(A \to \Box A) \land \Box\neg\forall x A) \ \land \ \Diamond\forall x(A \lor \Box A) \ \land \ \forall x(\Diamond A \to \Box A) \tag{7}$$

is consistent in the axiomatic system **Q.2.BF** (i.e., Cresswell's **KG1+BF**), but it is unsatisfiable in the class $\mathcal{C}^{2,BF}$ of all Kripke frames for that logic. This proves the incompleteness of **Q.2.BF**, which is completable by adding some presently unknown axiom. On the other hand, our modular completeness theorem entails that the labelled sequent calculus for $\mathcal{C}^{2,BF}$ is complete with respect to it.

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Hyperintensional models for non-congruential modal logics

Matteo Pascucci and Igor Sedlár

Overview In this work we illustrate applications of a semantics for non-congruential modal logic introduced in [9] and based on *hyperintensional models*. We start by discussing some philosophical ideas behind the approach; in particular, the difference between the proposition expressed by a formula (its intension) and the semantic content of a formula (its hyperintension), which is captured in a rigorous way in hyperintesional models. Next, we rigorously specify the approach and provide a fundamental completeness theorem. Moreover, we analyse examples of non-congruential systems that can be semantically characterized within this framework in an elegant and modular way. In the light of the results obtained, we argue that hyperintensional models constitute a *basic*, *general* and *unifying* framework for the interpretation of modal logic.

Non-congruential modal logic. The use of non-congruential systems of modal logic is crucial to represent contexts of reasoning that can be named "logically hyperintensional" [3]. These are contexts in which two formulas that express the same proposition (i.e., which are logically equivalent) cannot be always substituted *salva veritate* in the scope of a modal operator. Epistemic reasoning provides well-known examples of hyperintensional contexts; for instance, the set of formulas representing what is explicitly known or believed by a subject endowed with bounded rationality is not closed under logical equivalence.

Alternative proposals for the semantic analysis of non-congruential modal systems have been formulated over the years. Some are tailored to specific classes of systems (see, e.g., [2], [1] and [6]); others aim at constituting a general framework (see, e.g., [7], [8] and [4]). Here we focus on the semantics formulated in [9], which employs hyperintensional models.

Formal setting. Let a *propositional language* $\mathcal{P}$ contain a countable set of propositional variables Pr and the set of connectives $Con_{\mathcal{P}} = \{\wedge, \vee, \rightarrow, \bar{0}\}$, where $\wedge, \vee, \rightarrow$ are binary and $\bar{0}$ is zero-ary. The language $\mathsf{Mod}(\mathcal{P})$, which is a modal extension of $\mathcal{P}$, contains Pr and $Con_{\mathsf{Mod}(\mathcal{P})} = Con_{\mathcal{P}} \cup \{\Box\}$, where $\Box$ is unary. The set of formulas of $\mathcal{X}$, for $\mathcal{X} \in \{\mathcal{P}, \mathsf{Mod}(\mathcal{P})\}$, is denoted as $Fm_{\mathcal{X}}$. We define $\neg\varphi := \varphi \rightarrow \bar{0}$ and $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

An $\mathcal{X}$*-type algebra* is any algebra $\mathbf{A} = (A, \{c^{\mathbf{A}} \mid c \in Con_{\mathcal{X}}\})$. $Fm_{\mathcal{X}}$ can be seen as an $\mathcal{X}$-type algebra. Given two algebras $\mathbf{A}$ and $\mathbf{B}$, an $\mathcal{X}$*-homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ is a mapping that commutes with all elements of $Con_{\mathcal{X}}$.

Definition 1. A *hyperintensional model* is a tuple $\mathfrak{M} = (W, \mathbf{C}, O, N, I)$, where

- W is a non-empty set;
- $\mathbf{C}$ is a $\mathcal{P}$-type algebra;
- O is a $\mathcal{P}$-homomorphism from $Fm_{\mathsf{Mod}(\mathcal{P})}$ to $\mathbf{C}$;
- N is a function from W to subsets of (the universe of) $\mathbf{C}$;
- I is a $\mathcal{P}$-homomorphism from $\mathbf{C}$ to the power-set algebra over W such that, for all $\varphi \in Fm_{\mathsf{Mod}(\mathcal{P})}$ and $w \in W$

$$w \in I(O(\Box\varphi)) \iff O(\varphi) \in N(w) \tag{8}$$

We define E as the composition of O and I. A formula φ is *valid in* $\mathfrak{M}$ iff $E(\varphi) = W$. We will sometimes write $(\mathfrak{M}, w) \models \varphi$ instead of $w \in E(\varphi)$, where E is defined over $\mathfrak{M}$. Informally, $\mathbf{C}$ is a set of "semantic contents" of declarative sentences endowed with some algebraic structure. $O(\varphi)$ is the semantic content assigned to φ.

General completeness theorem. A logic is here regarded either as an axiomatic system or as a set of theorems. Given a logic L based on a language $\mathcal{X}$, φ^L is the set of all maximal consistent L-theories Γ s.t. $\varphi \in \Gamma$. Moreover, $Fm^L_{\mathcal{X}} = \{\varphi^L \mid \varphi \in Fm_{\mathcal{X}}\}$.

Theorem 1. *For each logic L over* $\mathsf{Mod}(\mathcal{P})$*, there is a hyperintensional model $\mathfrak{M}^L$ s.t., for all ϕ, $\phi \in L$ iff ϕ is valid in $\mathfrak{M}^L$.*

Proof. An adaptation of the usual canonical model construction, where the model $\mathfrak{M}^L = (W^L, \mathbf{C}^L, O^L, N^L, I^L)$ is such that: W^L is the set of all maximal consistent L-theories, $\mathbf{C}^L = Fm_{\mathsf{Mod}(\mathcal{P})}$, $N^L(\Gamma) = \{\varphi \mid \Box\varphi \in \Gamma\}$, $O^L(\varphi) = \varphi$ and $I^L(\varphi) = \varphi^L$.

□

Modular semantic characterization. Properties of hyperintensional models can be added in a modular way to obtain a semantic characterization for specific non-congruential modal systems. Some examples are illustrated below.

Definition 2. A *Boolean-content model* is a hyperintensional model where $\mathbf{C}$ is a Boolean algebra.

Let x, y range over the universe of $\mathbf{C}$. We define $x \leq y$ as $x \vee^{\mathbf{C}} y = y$. A Boolean-content model is

- *monotonic* iff, for all $w \in W$, $x \leq y$ only if $x \in N(w) \implies y \in N(w)$;
- *regular* iff it is monotonic and, for all $w \in W$, $x \in N(w)$ and $y \in N(w)$ only if $x \wedge y \in N(w)$;
- *N-consistent* iff, for all $w \in W$, $x \in N(w)$ implies $\neg x \notin N(w)$.

We analyse the following non-congruential systems over $\mathsf{Mod}(\mathcal{P})$, all of which contain the Propositional Calculus (PC) and are closed under Modus Ponens:

- $C0.1$ is the weakest system that is closed under the rule (RE$_{PC}$) $\dfrac{\varphi \leftrightarrow \psi \in PC}{\Box\varphi \leftrightarrow \Box\psi}$
- $C0.5$ is the weakest system that is closed under the rule (RM$_{PC}$) $\dfrac{\varphi \to \psi \in PC}{\Box\varphi \to \Box\psi}$
- $C1$ is the weakest system including the axiom (K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ that is closed under (RM$_{PC}$)
- $D1$ is the weakest system including the axioms (K) and (D) $\Box\varphi \to \neg\Box\neg\varphi$ that is closed under (RM$_{PC}$).

Definition 3. Let L be a logic over $\mathsf{Mod}(\mathcal{P})$. The *$PC$-content L-model* is

$$\mathfrak{M}^{PC/L} = (W^{PC/L}, \mathbf{C}^{PC/L}, O^{PC/L}, N^{PC/L}, I^{PC/L})$$

such that

- $W^{PC/L}$ is the set of all maximal consistent L-theories;
- $\mathbf{C}^{PC/L} = (Fm^{PC}, \{c^{PC} \mid c \in Con_{\mathcal{P}}\})$ where $c^{PC}(\varphi_1^{PC}, \ldots, \varphi_n^{PC}) = \big(c(\varphi_1, \ldots, \varphi_n)\big)^{PC}$;
- $O^{PC/L}(\varphi) = \varphi^{PC}$;
- $N^{PC/L}(\Gamma) = \{\varphi^{PC} \mid \Box\varphi \in \Gamma\}$;
- $I^{PC/L}(\varphi^{PC}) = \varphi^{L}$.

Lemma 1. *Let L be a logic over $\mathsf{Mod}(\mathcal{P})$; then, $\mathfrak{M}^{PC/L}$ is a Boolean-content model.*

Proof. $O^{PC/L}$ is a $\mathcal{P}$-homomorphism by definition of c^{PC}. $\mathbf{C}^{PC/L}$ is a Boolean algebra thanks to the definition of PC; in fact, it is a set algebra. Next, $I^{PC/L}$ is a $\mathcal{P}$-homomorphism thanks to the properties of maximal consistent L-theories. $\Box$

Lemma 2. *For each logic L over $\mathsf{Mod}(\mathcal{P})$, each $\Gamma \in W^{PC/L}$ and each φ:*

$$\varphi \in \Gamma \iff (\mathfrak{M}^{PC/L}, \Gamma) \models \varphi \tag{9}$$

Proof. $E^{PC/L}(\varphi) = I^{PC/L}(O^{PC/L}(\varphi)) = I^{PC/L}(\varphi^{PC}) = \varphi^{L}$. $\Box$

Theorem 2 (Semantic Characterization)**.**

1. *$\varphi \in C0.1$ iff φ is valid in all Boolean-content models.*
2. *$\varphi \in C0.5$ iff φ is valid in all monotonic Boolean-content models.*
3. *$\varphi \in C1$ iff φ is valid in all regular Boolean-content models.*
4. *$\varphi \in D1$ iff φ is valid in all regular N-consistent Boolean-content models.*

Final remarks. The results proven have important consequences with respect to applications of the framework, which can be described adopting the terminology in [11]. First, the approach at issue is *basic* since, in its more general form, can be used to semantically characterize the Propositional Calculus formulated over $\mathsf{Mod}(\mathcal{P})$. Second, the approach is *general*, since properties can be added to classes of models in a modular way, thus characterizing modal systems with a different deductive power. Moreover, as it is shown in [9], within this framework one can simulate related approaches developed in the literature; most importantly, hyperintensional models embed Rantala models [7, 8], which, in turn, embed models of many other frameworks available [11, 10]. For these reasons, hyperintensional models constitute also a *unifying* framework for the analysis of non-congruential modal logics.

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Yes, Fellows, Well-known Modal Logics are at Most 8-valued

Pawel Pawlowski and Elio La Rosa

In this paper, we show that *well-known* modal logics have a *non-deterministic* characterization that is at most *8-valued*. By well-known modal logics, we mean logics depicted in the modal cube of the *Stanford Encyclopedia of Philosophy* entry on modal logic [7]. These systems are defined by the modal logic **K** extended with an arbitrary combination of axioms: D, T, B, 4, 5. While some of these results are already known, we complete the overall picture and simplify it by providing *reductions* that minimize the number of values. We also emphasize on the use of more economical axiomatizations that do not incorporate axioms that become redundant in the presence of the *necessitation rule*. In this process, some logics are coupled with *non-normal* (in the sense of lacking the necessitation rule) companions.

Consider the following 2-valued (V, V') table for the connective $\circ$:

$\circ$	V	V'
V	V	V, V'
V'	V'	V, V'

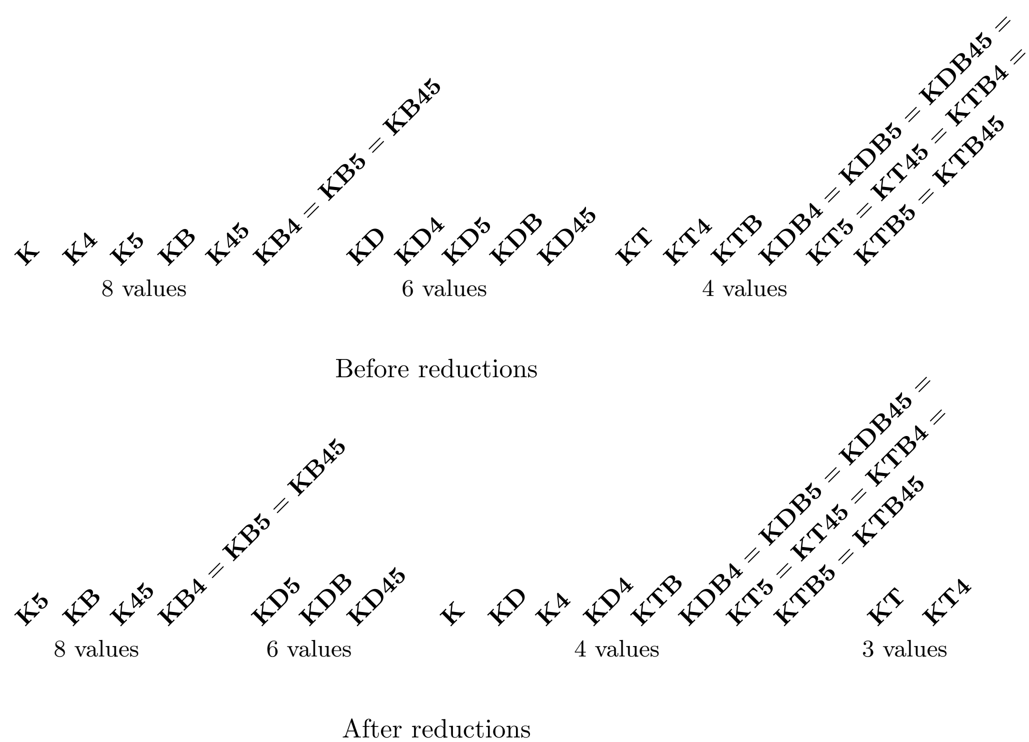
Notice how, for the value associated with the V' column, the table does not single out a single value, but a set $\{V, V'\}$ of them, indicated in the abbreviated form by V, V' in our table. We call these tables *non-deterministic*. Non-deterministic *semantics* are based on such a generalization of (many-valued) tables. Since the interpretation of connectives can give a non-empty set of truth-values instead of a single one, the valuation function singles out one of the possible values given by the set.

This allows for introducing new interpretations for an otherwise extensional reading of a connective, making it possible to semantically characterize logics that cannot be characterized by finitely many valued (deterministic) approaches. Examples of this can be found in [10, 1, 13] and [4]. In this paper, we deal with finitely many-valued characterizations of modal logics that cannot be captured by deterministic tables, as those studied in [6], possibly strengthened by the method on mth-level valuations in order to validate necessitation.[10] So far, approaches of this kind were developed for capturing modal logics weaker than **K** [9], for **K**, **T**, **S4**, **S5**, **KD**, **KB** and **KTB** [11, 2, 12].

[10] [5], extend this study providing more general results of incompleteness, while [4] and [8] show that not even non-deterministic many-valued matrices alone (i.e., not characterized by the mth-level valuations introduced below) are not enough to characterize common modal logics, in particular those systems comprised between **K** and **S5**. Moreover, [8] shows that no *deterministic* many-valued matrix can characterize any of the weakening of modal logics such as the ones above that do not validate the necessitation rule.

No systematic study of non-deterministic semantics has hence been conducted for all *well-known* modal logic, that is, logics of the modal cube portrayed in the *Stanford Encyclopedia of Philosophy* entry on modal logic [7].[11] In this paper, we fill this gap and rework the logics found in the works cited above in order to guarantee a more economical axiomatization while preserving modularity over the presence of the necessitation rule and, in some cases, a reduction of the number of values will be given. As a consequence, axiomatizations which do not include necessitation are not closed under the rule of *substitutivity of equivalents*.[12]

Figure 2: *Table of results*



[11]For reasons of clarity, notation will slightly deviate from that of [7], being more explicit on the axiom added to the base system **K**, as Figure 2 shows.

[12]This was first noticed by [12], where the authors spotted a mistake in the work of [9]. The proposed axiomatic system provided there is not sound with respect to the substitution of a subformula φ for $\neg\neg\varphi$ and vice versa in a formula ψ. This mistake carries over to [2], but was later acknowledged in [3].

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Information Types in Intuitionistic Predicate Logic

Vít Punčochář and Carles Noguera

Following [1] we can distinguish between a concrete piece of information and an information type. While the sentence *the white queen is on b5* (when uttered in a suitable context of a chess game) provides a piece of information, the expression *the position of the white queen* represents an information type under which some concrete pieces of information fall, namely the pieces expressed by the sentences *the white queen is on b5*, *the white queen is on a3*, and so on.

Even though expressions representing information types are not declarative sentences, one can observe that it makes sense to combine them by logical operators. For instance, the type *the position of the white queen* when combined via conjunction with the type *the position of the black king* forms a new type, namely *the position of the white queen and the position of the black king*. For example, the piece of information *the white queen is on b5 and the black king is on a1* falls under this complex type. In a similar sense, one can form complex information types by means of universal quantifier, obtaining, for example, the type *the positions of all chess pieces*. A piece of information falls under this type if it specifies for every chessmen what is its position.

One can generalize the notion of entailment to be applicable not only to pieces of information but also to information types. For example, the type *the positions of all chess pieces* "entails" in this generalized sense the type *the position of the white queen*, meaning that every piece of information falling under the former type entails (in the usual sense) some piece of information falling under the latter type.

One can codify a logic of information types. This is done by first-order inquisitive logic [2] where information types are identified with questions. However, the standard inquisitive logic is based on classical logic of declarative sentences. In other words, the standard framework allows us to express information types only in the context of classical predicate logic. The goal of this paper is to generalize the framework so that we can express information types also in the context of non-classical predicate logics. We will focus especially on the case of first-order intuitionistic logic. In particular, we will formulate a non-standard semantic framework for intuitionistic predicate logic and prove completeness with respect to this semantics. Then we extend the first-order language with new expressive means that allow as to formulate information types. The non-standard framework enables us to equip these new expressions with a suitable semantics. We define the notion of entailment and generalize the ideas from [2] to obtain an axiomatization. In the final part of our talk we will formulate also an algebraic semantics for the first-order intuitionsitic logic of information types. A possiblility of further generalization to the substructural setting will also be briefly discussed.

In the rest of this abstract we explain the main technical points in more detail. We start with the following first-order language $\mathcal{L}$. Terms are just variables and names. Complex formulas are defined as follows:

$$\alpha ::= \bot \mid Pt_1 \dots t_n \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \alpha \to \alpha \mid \forall x\alpha \mid \exists x\alpha$$

A complete infinitely distributive lattice (CID-lattice, for short) is a structure $\langle S, \bigsqcup, \bigsqcap \rangle$ where S is a nonempty set, $\bigsqcup, \bigsqcap$ respectively assign to every subset of S its join and meet ($\sqcup, \sqcap$ will denote the finitary versions of $\bigsqcup, \bigsqcap$) and the infinitary distributive laws are satisfied: $s \sqcup \bigsqcap_{i\in I} t_i = \bigsqcap_{i\in I}(s \sqcup t_i)$, $s \sqcap \bigsqcup_{i\in I} t_i = \bigsqcup_{i\in I}(s \sqcap t_i)$.

A signature is just a set of predicates and names. Given a signature σ, we define a σ-model as a tuple $\mathcal{M} = \langle \mathcal{A}, U, V \rangle$, where $\mathcal{A}$ is a CID-lattice (of information states), U is the nonempty set (the universe of discourse), and V is a σ-valuation, i.e. a function that assigns to each name from σ an element of U, and to every n-ary predicate from σ a function that assigns to each n-tuple of elements from U an element of $\mathcal{A}$. Note that $\bigsqcup \emptyset$ is the least element of $\mathcal{A}$. It represents the inconsistent state (the state of maximal information) and it will be denoted as 0.

An evaluation is a function that assigns to each variable of the language an element of U. If e is an evaluation, x a variable and $m \in U$, then $e(m/x)$ is the evaluation that assigns m to x and $e(y)$ to every other variable y. For any term t, $V^e(t)$ is identical with $V(t)$ if t is a name, and with $e(t)$ if t is a variable.

A $\sigma\mathcal{L}$-formula is a formula of $\mathcal{L}$ in the signature σ. Given a σ-model $\mathcal{M} = \langle \mathcal{A}, U, V \rangle$ and an evaluation e, we define a relation of support $\Vdash_e$ between the states of $\mathcal{M}$ and $\sigma\mathcal{L}$-formulas by the following clauses:

$s \Vdash_e \bot$	iff	$s = 0$,
$s \Vdash_e Pt_1 \dots t_n$	iff	$s \leq V(P)(V^e(t_1), \dots, V^e(t_n))$,
$s \Vdash_e \alpha \wedge \beta$	iff	$s \Vdash_e \alpha$ and $s \Vdash_e \beta$,
$s \Vdash_e \alpha \vee \beta$	iff	there are $t, u \in S$ such that $s \leq t \sqcup u$ and $t \Vdash_e \alpha$, $u \Vdash_e \beta$,
$s \Vdash_e \alpha \to \beta$	iff	for every $t \leq s$, if $t \Vdash_e \alpha$, then $t \Vdash_e \beta$,
$s \Vdash_e \forall x\alpha$	iff	for every $m \in U$, $s \Vdash_{e(m/x)} \alpha$,
$s \Vdash_e \exists x\alpha$	iff	there is a $g \colon U \to S$ such that $s \leq \bigsqcup_{m\in U} g(m)$ and for all $m \in U$, $g(m) \Vdash_{e(m/x)} \alpha$.

If $s \Vdash_e \alpha$, we will also say that s e-supports α. We define a consequence relation $\vDash$ as a preservation of support, i.e. if $\Phi \cup \{\alpha\}$ is a set of $\sigma\mathcal{L}$-formulas, then $\Phi \vDash \alpha$ iff for every σ-model $\mathcal{M}$, every state s of $\mathcal{M}$ and every evaluation e in $\mathcal{M}$, if s e-supports all formulas from Φ then s e-supports α. Let $\vdash$ be the derivability relation of intuitionistic predicate logic with constant domains.

Theorem 1. *Let $\Phi \cup \{\alpha\}$ be a set of $\mathcal{L}$-sentences. Then, $\Phi \vDash \alpha$ iff $\Phi \vdash \alpha$.*

Information types are expressed using inquisitive disjunction and inquisitive existential quantifier, so we extend the language in the following way:

$$\varphi ::= \bot \mid Pt_1 \ldots t_n \mid \varphi \wedge \varphi \mid \alpha \vee \alpha \mid \alpha \rightarrow \varphi \mid \forall x \varphi \mid \exists x \alpha \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \exists] x \varphi$$

The resulting language will be called $\mathcal{L}_{typ}$. Non-inquisitive existential quantifier and disjunction are applicable only to formulas from $\mathcal{L}$. Also antecedents of implications are always from $\mathcal{L}$. (It is an open problem whether or not one can recursively axiomatize the resulting logic for the full language without these restrictions, especially the restriction concerning implication.) Within this language, the semantic clauses for the operators from $\mathcal{L}$ are as before and for the new operators they are defined as follows:

$s \Vdash_e \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $s \Vdash_e \varphi$ or $s \Vdash_e \psi$,
$s \Vdash_e \exists] x \varphi$ iff for some $m \in U$, $s \Vdash_{e(m/x)} \varphi$.

The definition of entailment can now be extended to the full language $\mathcal{L}_{typ}$. A sound and complete system of the resulting logic is obtained by a system for intuitionistic logic (extrapolated to the language $\mathcal{L}_{typ}$) and enriched with the usual introduction and elimination rules for inquisitive disjunction and inquisitive existential quantifier and the following rules:

R1 $\alpha \rightarrow (\varphi \mathbin{\vee\!\!\!\vee} \psi) \;/\; (\alpha \rightarrow \varphi) \mathbin{\vee\!\!\!\vee} (\alpha \rightarrow \psi)$, if $\alpha \in \mathcal{L}$
R2 $\alpha \rightarrow \exists] x \varphi \;/\; \exists] x(\alpha \rightarrow \varphi)$, if $\alpha \in \mathcal{L}$ and x is not free in φ,
R3 $\forall x(\varphi \mathbin{\vee\!\!\!\vee} \psi) \;/\; \varphi \mathbin{\vee\!\!\!\vee} \forall x \psi$, if x is not free in φ.

Theorem 2. *Let $\Phi \cup \{\varphi\}$ be a set of $\mathcal{L}_{typ}$-sentences. Then, $\Phi \vDash \varphi$ iff $\Phi \vdash \varphi$.*

For a better understanding of this untypical logic, it is illuminating to see it also from a more algebraic perspective. We will introduce a class of structures that are isomorphic to lattices of nonempty downsets of CID-lattices. We will provide an alternative characterization of these structures and show that these structures are exactly what is needed for an algebraic semantics of intuitionistic logic with information types.

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Modal QUARC

Jonas Raab

The purpose of this paper is to showcase a modal extension of the Quantified Argument Calculus (QUARC)—a novel logical system developed by Hanoch Ben-Yami (2014). To indicate its potential, we consider the Barcan and Converse Barcan formulas in modal QUARC (M-QUARC). In particular, I show that they are not valid even if we employ the usual restrictions capable of validating them. For that reason, I develop a model-theoretic semantics with variable domains for M-QUARC. Moreover, I introduce new conditions—conditions not on the *domain* but on the *extension* of predicates—which, in turn, validate the Barcan and Converse Barcan formulas.

As most readers won't be familiar with QUARC, let me start by outlining its main features. QUARC is motivated by Ben-Yami's claim that classical logic is not the logic of natural language—which QUARC is supposed to capture. There are several differences between classical logic and QUARC. Most tellingly, as suggested by its name, QUARC treats *quantification* differently from how classical logic does.

Consider, for example, the sentence 'All animals are mortal'. The formal rendering of such a sentence based on classical logic introduces several differences to the sentence's surface-form. In particular, classical logic introduces a quantifier $\forall x$ acting on an open formula $\varphi(x)$. Moreover, the open formulas has a conditional $\rightarrow$ as its main connective. Overall, classical logic renders the sentence as '$\forall x(A(x) \rightarrow M(x))$'. QUARC, on the other hand, has *quantified arguments* capturing the expression 'all animals' ($\forall A$). It is called quantified *argument* because it can be the argument of another predicate such as 'mortal' ($(\cdot)M$). Overall, QUARC formalizes the sentences as '$(\forall A)M$'—keeping the surface form intact. As quantified arguments are plural referring expressions, QUARC is, in contrast to classical logic, a *plural* logic.

Moreover, QUARC distinguishes between sentence and predicate negation. Sentence negation is the negation as known from classical logic, i.e., an operator on formulas. Predicate negation, on the other hand, allows us to introduce new predicates. For example, classical logic renders a sentence like 'Pegasus does not fly' as '$\neg(F(p))$'. QUARC, on the other hand, can render this sentence as '$(p)\neg F$', where '$(\cdot)\neg F$' is the negated form of 'F'. For quantifier-free QUARC-formulas, sentence negation and predicate negation are equivalent ($\neg((\cdot)F) \Leftrightarrow (\cdot)\neg F$), but the position of the negation makes a difference for quantified formulas ($\neg((\forall A)M) \not\Leftrightarrow (\forall A)\neg M$). For example, 'It is not the case that all animals are mortal' is *not* equivalent to 'All animals are not mortal'.

This has also consequences for M-QUARC. We can treat not only negation as an operator on predicates, but also other modalities such as necessity ($\Box$) and possibility ($\Diamond$). As in the case of negation, the position of the modality in quantifier-free formulas does not make a difference; however, in formulas involving quantifiers, the formulas do come apart. For example, the sentence 'All horses possibly fly' ($(\forall H)\Diamond F$) is (arguably) not equivalent to the sentence 'It is possible that all horses fly' ($\Diamond((\forall S)F)$) (cf. Ben-Yami 2020).

Further differences to classical logic are the incorporation of *anaphora* and what Ben-Yami calls *reordered predicates* which capture, for example, active/passive constructions. For example, a sentence like 'Bellerophon rides Pegasus' concerns a two-place predicate '$(\cdot, -)F$' whose positions can be reordered as '$(-, \cdot)F^{2,1}$' where the '$2, 1$' superscript indicates the reordering. This reordered predicate can be used to turn the active sentence into the passive sentence 'Pegasus is ridden by Bellerophon'.

Anaphora, on the other hand, play the role of variables in classical logic, i.e., they allow for cross-reference. For example, it allows us to formalize a sentence like 'Bellerophon captured Pegasus and he rides it' as '$(b_\alpha, p_\beta)C \wedge (\alpha, \beta)R$' where α and β are the anaphora and the subscripts indicate their *sources*. Anaphora can also attach to quantifiers. As the introduction of anaphora is *substitutional*, we have to treat quantification substitutionally, too.

Having seen the main differences between QUARC and classical logic, let me somewhat informally introduce the language of M-QUARC and indicate the main ingredients for the model-theoretic semantic with variable domains.

The language $\mathcal{L}_Q$ of QUARC consists of the following countably infinite sets: a set of anaphora $\mathsf{Ana}_{\mathcal{L}_Q}$, of singular arguments $\mathsf{SA}_{\mathcal{L}_Q}$, of n-place predicates $\mathsf{Pred}^n_{\mathcal{L}_Q}$ ($n \geq 1$), and of n-place reordered predicates $\mathsf{Reord}^n_{\mathcal{L}_Q}$ (where every $P \in \mathsf{Reord}^n_{\mathcal{L}_Q}$ is the reorder of a $Q \in \mathsf{Pred}^n_{\mathcal{L}_Q}$). Moreover, it includes the usual logical symbols and some auxiliary symbols.

The language $\mathcal{L}_{MQ}$ of M-QUARC is $\mathcal{L}_Q$ augmented by the logical symbols $\Box$ and $\Diamond$. Moreover, we recursively define a set $\mathsf{Mod}_{\mathcal{L}_{MQ}}$ of *modalities* as follows: $\neg, \Box, \Diamond \in \mathsf{Mod}_{\mathcal{L}_{MQ}}$; if $\pi \in \mathsf{Mod}_{\mathcal{L}_{MQ}}$, then $\neg\pi, \Box\pi, \Diamond\pi \in \mathsf{Mod}_{\mathcal{L}_{MQ}}$. The reason is that we allow for modalities $\pi \in \mathsf{Mod}_{\mathcal{L}_{MQ}}$ to be operators on predicates; for every $P \in \mathsf{Reord}^n_{\mathcal{L}_{MQ}}$ and $\pi \in \mathsf{Mod}_{\mathcal{L}_{MQ}}$, we have a predicate πP. For example, for a predicate 'is a horse', we also have the predicates 'is possibly a horse', 'is necessarily a horse', 'is not possibly a horse', etc.

The set of $\mathcal{L}_{MQ}$-formulas ($\mathsf{Form}_{\mathcal{L}_{MQ}}$) is recursively defined. The quantifier clauses are slightly more complex than usual. I introduce a *variable domain semantics*; the models are tuples $\mathfrak{M} = \langle W, R, D, \mathcal{D}, V\rangle$ consisting of a set of possible worlds (W), an accessibility relation (R), a domain (D), a domain map ($\mathcal{D}$), and a valuation function (V). The domain map $\mathcal{D}$: $W \to \mathcal{P}(D)$ assigns every possible world $w \in W$ its domain $D_w \coloneqq \mathcal{D}(w) \subseteq D$.

As quantification is *substitutional*, we have to consider *language expansions* in order to get the truth conditions right. I introduce $\mathcal{L}_{MQ}$-w-A-expansions: for a model $\mathfrak{M}$, $w \in W$, and $A \subseteq D_w$, we define the language $\mathcal{L}'_{MQ} \coloneqq \mathcal{L}_{MQ} \cup \{c_a | a \in A\}$ where the c_as are new symbols ($c_a \in \mathsf{SA}_{\mathcal{L}'_{MQ}}$). To interpret these language expansions, we have to consider *model expansions*. A model $\mathfrak{M}'$ is a w-A-expansion of a model $\mathfrak{M}$ to the language $\mathcal{L}'_{MQ}$ if $W' = W$, $R' = R$, $D' = D$, $\mathcal{D}' = \mathcal{D}$, $V \subseteq V'$, and, if $a \in D_u$, $V'(c_a)(u) = a$, and if $a \notin D_u$, $V'(c_a)(u) \in D_u$.

Given these definitions, we can define the relation $\mathfrak{M}, w \models \varphi$ in the usual way. In particular, a model $\mathfrak{M}$ satisfies a formula $\psi(\forall P)$ at a world w iff. every w-a-expansion $\mathfrak{M}'$ of $\mathfrak{M}$ such that $a \in V(P)(w)$, $\mathfrak{M}', w \models \psi(c_a/\forall P)$, i.e., if every expansion for which an element of the respective domain is a P satisfies the formula ψ.

This brings us to the Barcan and Converse Barcan formulas. M-QUARC-analogues of the Barcan formulas are of the form '$(\forall S)\Box P \to \Box((\forall S)P)$' and '$\Diamond((\exists S)P) \to (\exists S)\Diamond P$'. Moreover, M-QUARC-analogues of the Converse Barcan formulas are formulas of the form '$\Box((\forall S)P) \to (\forall S)\Box P$' and '$(\exists S)\Diamond P \to \Diamond((\exists S)P)$'.

The usual conditions necessary to validate Barcan and Converse Barcan formulas are conditions on the *domain*: we call $\mathfrak{M}$ a *constant domain model* iff. $\mathcal{D}$ is constant, i.e., $D_w = D_u$ $(u, w \in W)$; we call it an *isotone domain model* iff. for wRu, $D_w \subseteq D_u$; we call it an *antitone domain model* iff. for wRu, $D_w \supseteq D_u$; and we call it an *absolute and constant domain model* iff. $\mathcal{D}$ is absolute and constant, i.e., $D_w = D_u = D$.

Clearly, the last condition implies the other three. What we can show is that in absolute and constant domain models of M-QUARC, analogues of the Barcan and Converse Barcan formulas have countermodels. The reason is that we evaluate predicates $P \in \mathsf{Reord}^n_{\mathcal{L}_{MQ}}$ relative to domains, i.e., the extension of P varies across different domains. Since quantification always involves a restricting predicate, this means that we cannot just swap modalities and quantifiers.

However, we can impose conditions on the extensions of predicate. We call $\mathfrak{M}$ an *isotone extension model* iff. for $P \in \mathsf{Reord}^1_{\mathcal{L}_{MQ}}$, if wRu, then $V(P)(w) \subseteq V(P)(u)$; an *antitone extension model* iff. for $P \in \mathsf{Reord}^1_{\mathcal{L}_{MQ}}$, if wRu, then $V(P)(w) \supseteq V(P)(u)$; and a *constant extension model* iff. for $P \in \mathsf{Reord}^1_{\mathcal{L}_{MQ}}$, if wRu, then $V(P)(w) = V(P)(u)$.

With these conditions we can formulate new results: antitone extension models validate M-QUARC-analogues of Barcan formulas, and isotone extension models validate M-QUARC-analogues of Converse Barcan formulas. Thus, M-QUARC shows that Barcan and Converse Barcan formulas are connected to the predicates extensions more than the behaviour of the domains. This contrasts it with "classical" modal logic.

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Completeness in Partial Type Logic

Jiří Raclavský and Petr Kuchyňka

Background. If equipped with logical operators, *simple type theory* (STT) (Church [5], Andrews [1]) is a comprehensive variant of *higher-order logic* (HOL):[13] its variables are tied to 'ranges' called *types* τ, which are symbols interpreted not only by the *domain* D_ι of individuals as in FOL but also by various sets of functions over the type *basis* $\mathscr{B} = \{D_\iota, D_o\}$, where $D_o = \{\mathtt{T}, \mathtt{F}\}$ (truth values). Definition of *types* $\tau \in \mathcal{T}$: i. $\iota, o \in \mathcal{T}$; ii. $\tau_0 \mapsto \tau_1 \in \mathcal{T}$, whenever each $\tau_i \in \mathcal{T}$. $D_{\tau_0 \mapsto \tau 1}$ is the set of all functions-as-graphs from D_{τ_0} to D_{τ_1}. Then, a *frame* is an indexed family of sets $\mathscr{F} = \{D_\tau\}_{\tau \in \mathcal{T}}$. A *model* is $\mathscr{M} = \langle \mathscr{F}, \mathscr{I} \rangle$, where $\mathscr{I}$ is an *interpretation function* (it interprets constants of $\mathcal{L}$).

STT is known for its i. great expressive power – a feature which is however accompanied by STT's ii. incompleteness (as famously proved by Gödel). See e.g. van Bentham and Doets [18], Farmer [8] for discussion and defence of HOL. STT, whose language $\mathcal{L}$ is a *typed* λ*-calculus*, is substantially justified by its iii. broad range of applications in computer science (esp. functional programming languages, proof assistants). Henkin [10] remarkably accompanied the fact ii. by the (proven) fact that STT is *complete w.r.t. general* (or *Henkin*, *non-standard*) *models*.

The crucial ideas of his proof are: (a) domains of a *general model* contain not all, but *some* of all possible functions, (b) these functions are *named* by λ-abstracts (and also by constants, if available in $\mathcal{L}$), (c) the hierarchy of terms denoting the objects in $\mathscr{F}$ are closed under β-conversion rules and few operational rules, (d) each D_τ only consists of values of a certain function Φ at equivalence classes of closed λ-terms. So the questions of semantics became the questions of derivation rules, as stressed by Henkin [11]. Henkin's proof became an 'industry standard' in various versions of HOL, see e.g. Andrews [1], Brown [4], Areces et al. [2].[14]

Various writers recently proposed a needed (cf. e.g. Feferman [9]) extension of STT by embracing, in addition to total functions, *partial functions*(-as-graphs): a partial function f is *undefined* (i.e. has no value at all) for at least one argument x of its domain, i.e. $f(x) =$ (which we will write $f(x) = \_$ for its convenience in our further notation). See the *partial TTs* by e.g. Tichý [16], Farmer [7], Muskens [14].

[13] We can speak about a *pure STT*, if it is equipped only with λ-rules (α-, β-, η-), while *STT* as logic has even rules for logical operators (which are usually defined in the style of e.g. $\forall(s) \equiv (s = \lambda x.\mathtt{T})$, where s is a variable for sets as characteristic functions, and x ranges over the (interpretation of the) type of its members).

[14] Henkin's famous 1949 completeness proof for FOL, often adapted for modal logic, arose as a simplification of his proof for STT. For its recent application for various non-classical logics, see e.g. Cintula and Noguera [6] (and their forthcoming book).

Unlike Tichý, both Farmer and Muskens offered Henkin-style completeness proofs for their systems. However, Muskens' TT only contains relations and the logic is four-valued (with 'strong Kleene' basis), hence his proof cannot be adapted for the other systems. Farmer's proof is based on the proof by Andrews, for his partial TT is an extension of Andrews' TT. But Farmer's STT only contains total functions over D_o and partial functions over D_ι. Again, his proof cannot be adapted for the other systems. In other words, Tichý's partial TT lacks a completeness proof.[15]

Goal of the paper. In our paper, we *provide a Henkin-style completeness proof for* a (significant) adjustment of Tichý's *partial TT*, which we call TT^*. There are three main *reasons* why it is important: (1) Tichý's partial TT, and thus even TT^*, is the only TT which systematically uses *all* total and partial multiargument functions (so it has the greatest expressivity if compared with the other system mentioned in this abstract), (2) it has an extension enabling to capture fine-grained hyperintensionality (needed e.g. for the analysis of natural language), (3) its natural deduction system, which we adjust as $\mathsf{ND}_{\mathsf{TT}^*}$, allows a plausible expression of inference with partial functions (while no criticism in the style of Blamey [3] is applicable).

Syntax and semantics of TT^*. In the first phase of our whole project (cf. e.g. [15]), we reformulated and adjusted Tichý's system to obtain its clear and unambiguous *syntax* and (Henkin-style model-theoretic) *semantics*:[16]

a. $\mathcal{L}_{\mathsf{TT}^*}$ $\quad C ::= x \mid \mathrm{c} \mid C_0(\bar{C}_m) \mid \lambda\tilde{x}_m.C_0 \mid \ulcorner C_0 \urcorner \mid [\![C_0]\!]_\tau$

b. For definitions of *type* over the (type) *basis* $\mathscr{B}_{\mathsf{TT}^*}$ (which significantly extend definitions e.g. in [5] and [16]), see [15].

c. A standard or general *model* is $\mathscr{M} = \langle \mathscr{F}, \mathscr{I} \rangle$ with appropriate $\mathscr{F}$ and $\mathscr{I}$.

TT^* conforms to Tichý's [17] neo-Fregean algorithmic approach: $\mathcal{L}_{\mathsf{TT}^*}$ consists of *terms* t that express (abstract) acyclic, not necessarily effective algorithmic computations called *constructions* C. Each C either v-constructs (dependently on assignment v) an object O *denoted* by t, or it v-constructs nothing at all, being v*-improper* (so t is *non-denoting*).

An *application* $C_0(\bar{C}_m)$ applies a function F v-constructed by C_0 to an m-ary argument A v-constructed by $\bar{C}_m$, provided F is defined for A (it is v-improper otherwise); a λ*-abstraction* $\lambda\tilde{x}_m.C_0$ v-constructs a function. *Acquisitions* $\ulcorner C_0 \urcorner$ and *immersions* $[\![C_0]\!]_\tau$ are adjusted from Tichý [17]. $\ulcorner C_0 \urcorner$ v-constructs C_0 as such (this is useful for capturing *Sinn* of expressions in *hyperintensional contexts*). $[\![C_0]\!]_\tau$ v-constructs, if v-proper, what is v-constructed by what is v-constructed by C_0; our *future work* will embed $[\![C_0]\!]_\tau$ into our proof.

In our proof, $\mathcal{L}_{\mathsf{TT}^*}$ contains constants for Bochvar-like $\neg$ and $\rightarrow$ and strong-Kleene-like $\exists$ and $\forall$ (these are not interdefinable in partial TT).

[15] Moreover, Tichý never employed model-theoretic notions such as model, frame or interpretation.

[16] Let $\tilde{X}_m$ be short for $X_1X_2...X_m$ and $\bar{X}_m$ be short for $X_1, X_2, ..., X_m$.

Natural deduction $\mathsf{ND}_{\mathsf{TT}^*}$. We adjust *natural deduction in sequent style* from [16] (most of this our work was presented in [15, 12]), we call our system $\mathsf{ND}_{\mathsf{TT}^*}$. Since it utilises sequents, it has i. *structural rules*. It also has ii. *constructional rules*. Further, it has iii. *operational rules* for logical constants of (an extended) $\mathcal{L}_{\mathsf{TT}^*}$ (and even iv. *rules for extralogical constants*). Instantiation rules, necessitated by partiality, complement introductory and elimination rules. $\mathsf{ND}_{\mathsf{TT}^*}$ has 27 basic rules.

Sequents $\mathcal{S}$ – from which *rules* $\mathcal{R}$ are build – are made from '*labelled* (or *signed*) *terms*', which are congruence statements called *matches* $\mathcal{M}$. Matches are either i. of the form $C{:}^{\tau}\mathbf{a}$, which says that C is v-proper (and its value is v-constructed by a. the variable a ranging over τ, b. the constant a of type τ, or c. the acquisition $\ulcorner a \urcorner$, where a is a construction), or ii. of the form $C{:}^{\tau}\_$, which says that C is v-improper. Matches are definitely (counter)*satisfiable* w.r.t. $\mathscr{M}$ and bring also further benefits for our Henkin-style proof: C of $C{:}\_$ is v-improper in $\mathscr{M}$ either for the fact that C v-constructs no object in any model whatsoever, or it v-constructs no object in $\mathscr{M}$ under consideration.

Main results presented in the paper consist in proofs that establish the following theorems:

Theorem 1. *If Δ is any consistent set of matches over $\mathscr{B}_{\mathsf{TT}^*}$, there is a general model $\mathscr{M}$ in which Δ is satisfiable.*

Theorem 2. *For any sequent $\mathcal{S}$ over $\mathscr{B}_{\mathsf{TT}^*}$, $\vdash_{\mathsf{ND}_{\mathsf{TT}^*}} \mathcal{S}$ iff $\vDash_g \mathcal{S}$ (i.e. valid in the general sense).*

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Modal Intuitionistic Algebras and Twist Representations

Umberto Rivieccio

A *modal Heyting algebra* (henceforth, MHA) is obtained by enriching a Heyting algebra $\langle H; \wedge, \vee, \to, 0, 1\rangle$ with a unary modal operator that satisfies $x \to \Box y = \Box x \to \Box y$. Such an operator is also known in the literature as a *nucleus.*

MHAs (and subreducts thereof) have been studied since the 1970s, usually within the framework of topology and sheaf theory [7, 8, 2]. A more recent paper [4] proposed a logic (called *Lax Logic*) based on MHAs as a tool in the formal verification of computer hardware. Even more recently, another connection between MHAs and logic emerged within the study of the algebraic semantics of *quasi-Nelson logic* [15, 14]. The latter may be viewed as a common generalization of both intuitionistic logic and *Nelson's constructive logic with strong negation* [9] obtained by deleting the double negation law.

As shown in [14, 11, 10], there exists a formal relation between the algebraic counterpart of quasi-Nelson logic (*quasi-Nelson algebras*) and MHAs which parallels the well-known connection between *Nelson algebras* and Heyting algebras (see e.g. [16]). This relation – which concerns the algebras in the full language as well as some of their subreducts – provides further motivation for an investigation of MHAs from a logical as well as an algebraic point of view. Studies of this kind, perhaps owing to the mainly topological interest in this class of algebras, can hardly be found in the existing literature, with the notable exception of [1]. The purpose of the my contribution is to (at least partly) fill in this gap and at the same time to draw attention to certain subreducts of MHAs whose interest is motivated by the recent developments in the theory of quasi-Nelson logic.

Given that a MHA is usually presented in the language $\{\wedge, \vee, \to, \Box, 0, 1\}$, fragments that appear to be of natural interest (from a logico-algebraic-topological perspective) are, for instance, the implication-free one $\{\wedge, \vee, \Box\}$ and the $\{\to, \Box\}$-fragment. The former, whose models are distributive lattices enriched with a modal operator, is the main object of [1], while the latter was studied (mainly from a topological perspective) as far back as in [7] and as recently as in [3]. Other less obvious but equally interesting "fragments" have emerged in the course of recent investigations on quasi-Nelson logic and quasi-Nelson algebras [11, 10, 12]. An interest in these classes of algebras, however, can also be motivated within the limits of the traditional framework of MHAs.

A well-known fact within the theory MHAs [7, Thm. 2.12] is that, for every such algebra $\mathbf{H} = \langle H; \wedge, \vee, \to, \Box, 0, 1\rangle$, the set $H^{\Box}$ of the fixpoints of the $\Box$

operator can itself be endowed with a MHA structure by defining, for every n-ary algebraic operation $f \in \{\wedge, \vee, \rightarrow, \Box, 0, 1\}$, the operation $f^{\Box}$ given, for all $a_1, \ldots, a_n \in H^{\Box}$, by $f^{\Box}(a_1, \ldots, a_n) := \Box f(a_1, \ldots, a_n)$. This algebra, denoted $\mathbf{H}^{\Box}$, is indeed a MHA, and a very special one[17], for the $\Box$ operator is the identity map on $H^{\Box}$. This very fact, in turn, is essential in ensuring that $\mathbf{H}^{\Box}$ has a Heyting algebra reduct; for instance we have

$$a \wedge^{\Box} b = \Box(a \wedge b) = \Box a \wedge \Box b = a \wedge b$$

for all $a, b \in H^{\Box}$, guaranteeing that $\wedge^{\Box}$ is a meet semilattice operation on $H^{\Box}$. A similar reasoning applies to the other operations as well, although the join $\vee^{\Box}$ (computed in $\mathbf{H}^{\Box}$) does not coincide with the join $\vee$ (computed in $\mathbf{H}$), i.e. $\mathbf{H}^{\Box}$ is not a Heyting subalgebra of $\mathbf{H}$.

Thus, although nothing prevents us from considering each operation $f^{\Box}$ as defined on the whole universe H, in general $\wedge^{\Box}$ and $\vee^{\Box}$ will not be semilattice operations on H, and likewise $\rightarrow^{\Box}$ will not be the Heyting implication. By definition, all these operations generalize the intuitionistic ones, which can be retrieved by requiring $\Box$ to be the identity map on H. Indeed, a natural question to ask is what properties each generalized operation $f^{\Box}$ retains and, more generally, whether $f^{\Box}$ has any independent interest deserving further study.

An answer to the latter question may come from the theory of quasi-Nelson logic. Indeed, as shown in the papers [14, 11, 10, 12], some of the above-defined operations of type $f^{\Box}$ naturally arise (via the so-called *twist* representation) within the study of fragments of the quasi-Nelson language. From this standpoint it is interesting to observe that, on these new algebras, the original Heyting operations coexist with the new ones. Thus, for instance, one of the classes of algebras arising in this way retains the original meet semilattice operation (and the lattice bounds) while replacing the Heyting implication with its generalized counterpart: that is, we are dealing with the $\{\wedge, \rightarrow^{\Box}, 0, 1\}$-subreducts of modal Heyting algebras (on which the modal operator is in fact definable by $\Box x := 1 \rightarrow^{\Box} x$). Note that these new algebras are not the result of an arbitrary choice of operations, but arise as factors in the twist representation of subreducts of quasi-Nelson algebras, as explained below.

A *quasi-Nelson algebra* may be defined as a commutative integral bounded residuated lattice $\mathbf{A} = \langle A; \sqcap, \sqcup, *, \Rightarrow, \bot \rangle$ that (upon letting ${\sim} x := x \Rightarrow \bot$) satisfies the *Nelson identity*:

$$(x \Rightarrow (x \Rightarrow y)) \sqcap ({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x)) = x \Rightarrow y.$$

Quasi-Nelson algebras arise as the algebraic counterpart of quasi-Nelson logic, which can be viewed either as a generalization (i.e. a weakening) common to Nelson's constructive logic with strong negation and to intuitionistic logic or as the extension (i.e. strengthening) of the well-known substructural logic FL_{ew} (the *Full Lambek Calculus with Exchange and Weakening* [5]) by the *Nelson axiom*:

$$((x \Rightarrow (x \Rightarrow y)) \sqcap ({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x))) \Rightarrow (x \Rightarrow y).$$

[17]This construction is easily seen to be a generalization of Glivenko's result relating Heyting and Boolean algebras (corresponding to the case where $\Box x = (x \rightarrow 0) \rightarrow 0$).

We refer to [15] for further details on quasi-Nelson logic as well as for other equivalent characterization of quasi-Nelson algebras (these can also be obtained as the class of $(0,1)$-*congruence orderable* commutative integral bounded residuated lattices).

Each Heyting algebra $\mathbf{H} = \langle H; \wedge, \vee, \rightarrow, 0, 1 \rangle$ may be viewed as a quasi-Nelson algebra on which $\wedge = * = \sqcap$, $\vee = \sqcup$, $\rightarrow = \Rightarrow$ and $0 = \bot$, and we know that on $\mathbf{H}$ the double negation is a modal operator. On the other hand, if we replace $\mathbf{H}$ by a quasi-Nelson algebra $\mathbf{A}$, then the double negation need not define a modal operator on $\mathbf{A}$, but can be used to obtain one on a special quotient $H(\mathbf{A})$, which is the Heyting algebra canonically associated to each quasi-Nelson algebra $\mathbf{A}$ via the twist construction.

Given a quasi-Nelson algebra $\mathbf{A}$, consider the map given, for all $a \in A$, by $a \mapsto a * a$. The kernel θ of this map is a congruence of the reduct $\langle A; \sqcap, \sqcup, \bot \rangle$ which is also compatible with the double negation operation and with the *weak implication* $\Rightarrow^2$ given by $x \Rightarrow^2 y := x \Rightarrow (x \Rightarrow y)$. Thus, letting $\Box(x/\theta) := {\sim}{\sim} x/\theta$, we have a quotient algebra $H(\mathbf{A}) = \langle A/\theta; \sqcap, \sqcup, \Rightarrow^2, \Box, \bot \rangle$, which is a MHA. Moreover, $\mathbf{A}$ embeds into a *twist-algebra* over $H(\mathbf{A})$, defined as follows.

Let $\mathbf{H} = \langle H; \wedge, \vee, \rightarrow, \Box, 0, 1 \rangle$ be a modal Heyting algebra. Define the algebra $\mathbf{H}^{\bowtie} = \langle H^{\bowtie}; \sqcap, \sqcup, *, \Rightarrow, \bot \rangle$ with universe $H^{\bowtie} := \{\langle a_1, a_2 \rangle \in H \times H : a_2 = \Box a_2,\ a_1 \wedge a_2 = 0\}$ and operations given, for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in H \times H$, by:

$$\begin{aligned}
\bot &:= \langle 0, 1 \rangle \\
\langle a_1, a_2 \rangle * \langle b_1, b_2 \rangle &= \langle a_1 \wedge b_1, \Box((a_1 \rightarrow b_2) \wedge (b_1 \rightarrow a_2)) \rangle \\
\langle a_1, a_2 \rangle \sqcap \langle b_1, b_2 \rangle &:= \langle a_1 \wedge b_1, \Box(a_2 \vee b_2) \rangle \\
\langle a_1, a_2 \rangle \sqcup \langle b_1, b_2 \rangle &:= \langle a_1 \vee b_1, \Box(a_2 \wedge b_2) \rangle \\
\langle a_1, a_2 \rangle \Rightarrow \langle b_1, b_2 \rangle &:= \langle (a_1 \rightarrow b_1) \wedge (b_2 \rightarrow a_2), \Box(a_1 \wedge b_2) \rangle.
\end{aligned}$$

A *quasi-Nelson twist-algebra over* $\mathbf{H}$ is any subalgebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$ satisfying $\pi_1[A] = H$.

The *twist representation theorem* [15] states that every quasi-Nelson algebra $\mathbf{A}$ embeds into the twist-algebra $(H(\mathbf{A}))^{\bowtie}$ through the map given by $a \mapsto \langle a/\theta, {\sim} a/\theta \rangle$ for all $a \in A$. This result is obviously an extension to the non-involutive setting of the well-known twist representation of Nelson algebras (see e.g. [16]), which can be retrieved as the special case where $\Box$ is the identity map on $H(\mathbf{A})$.

The twist-algebra definition suggests that certain term operations of the language of MHAs are of particular interest in the study of fragments of quasi-Nelson logic. Consider, for instance, the monoid operation. In order to define $*$ on a quasi-Nelson algebra $\mathbf{A} \leq \mathbf{H}^{\bowtie}$, we need two operations on $\mathbf{H}$: the semilattice operation $\wedge$ (for the first component) and, for the second component, an implication-like operation (denote it by $\rightharpoonup$) which can be given by $x \rightharpoonup y := x \rightarrow \Box y$. The latter claim may not be obvious, but using the properties of the twist construction and of the $\Box$ operator it is not hard to verify the following equalities:

$$\begin{aligned}
\Box((a_1 \rightarrow b_2) \wedge (b_1 \rightarrow a_2)) &= \Box((a_1 \rightarrow \Box b_2) \wedge (b_1 \rightarrow \Box a_2)) = \Box(a_1 \rightarrow \Box b_2) \wedge \Box(b_1 \rightarrow \Box a_2) \\
&= (a_1 \rightarrow \Box b_2) \wedge (b_1 \rightarrow \Box a_2) = (a_1 \rightharpoonup b_2) \wedge (b_1 \rightharpoonup a_2).
\end{aligned}$$

These observations led to the introduction of the class of algebras dubbed $\rightharpoonup$-

semilattices in [12], where we show that the $\{*, \sim\}$-subreducts of quasi-Nelson algebras are precisely the twist-algebras over $\rightharpoonup$-semilattices. Similar considerations motivate the introduction of other term operations of the language of MHAs, for instance $x \odot y := \Box(x \wedge y)$ and $x \oplus y := \Box(x \vee y)$. The classes of modal algebras thus obtained allow us to establish twist representations for (respectively) the $\{\Rightarrow^2, \sim\}$- and the $\{\wedge, \Rightarrow, \sim\}$-subreducts of quasi-Nelson algebras (studied, respectively, in [10] and in [12]). Other subreducts may be obtained by adding a modal operator to more traditional classes of intuitionistic algebras, such as implicative semilattices (giving us the $\{*, \Rightarrow, \sim\}$-subreducts of quasi-Nelson algebras), distributive lattices (giving us the $\{\wedge, \vee, \sim\}$-subreducts studied in [13]) and pseudo-complemented lattices (corresponding to the "two-negations" subreducts studied in [11]). It is worth mentioning that all these results specialize straightforwardly to the involutive case, yielding previously unknown characterizations of the corresponding fragments of Nelson's constructive logic with strong negation.

The preceding considerations indicate the above-mentioned classes of modal intuitionistic algebras as mathematical objects which may be interesting both in themselves and in relation to the study of non-classical logics (Nelson's logics in particular[18]). In this contribution I will give an overview of the results that have been achieved thanks to the introduction of these new classes of "modal intuitionistic algebras", and above all I will report on ongoing research aimed at obtaining a better understanding of these algebras from a logical, an algebraic and a topological point of view.

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Changing the World Constructively

Igor Sedlár

The finite model property of intuitionistic logic entails completeness with respect to posets where each element is under a maximal element. Maximal elements can be seen as complete and consistent possible worlds and elements strictly under them as situations containing incomplete yet consistent information about the world. This suggests a natural semantics for intuitionistic modal logic based on posets with maximal elements and an accessibility relation on the set of maximal elements. In models for a modal language containing $\Box$ and $\Diamond$ based on such posets, a situation x is seen as containing the information that φ is necessary, $\Box\varphi$, if φ is true in each world u accessible from any world w in which x is contained; similarly, x contains the information that φ is possible, $\Diamond\varphi$, if φ is true in each world u accessible from any w in which x is contained. In this paper, we study the intuitionistic modal logic **AK** arising from this semantics.

1 The logic AK

The *modal language* $\mathcal{M}$ contains a countable set of propositional variables Pr, propositional connectives $\wedge, \vee, \rightarrow, \bot$ and modal operators $\Box, \Diamond$. The set of formulas of $\mathcal{M}$ is defined in the usual inductive way and denoted $Fm_{\mathcal{M}}$.

A *w-frame* is $\mathfrak{F} = (S, \leq, W, R)$, where (i) $(S, \leq)$ is a partially ordered set; (ii) (W, R) is a directed graph; (iii) W is the set of maximal elements of $(S, \leq)$; and (iv) for all $s \in S$ there is $w \in W$ such that $s \leq w$. A *w-model* based on $\mathfrak{F}$ is $\mathfrak{M} = (\mathfrak{F}, V)$, were V is a function from Pr to subsets of S closed upwards under $\leq$. For each $\mathfrak{M}$, the *satisfaction relation* induced by $\mathfrak{M}$, $\models_{\mathfrak{M}}$, is a relation between states $s \in S$ of $\mathfrak{M}$ and formulas $\varphi \in Fm_{\mathcal{M}}$ defined by induction on the complexity of formulas in such a way that the clauses for propositional variables and propositional connectives are as in the semantics of intuitionistic propositional logic and

- $s \models_{\mathfrak{M}} \Box\varphi$ iff, for all $w, u \in W$, $s \leq w$ and Rwu only if $u \models_{\mathfrak{M}}$;
- $s \models_{\mathfrak{M}} \Diamond\varphi$ iff, for all $w \in W$ such that $s \leq w$, there is $u \in W$ where Rwu and $u \models_{\mathfrak{M}} \varphi$.

A formula φ is *valid in* $\mathfrak{M}$ iff $s \models_{\mathfrak{M}} \varphi$ for all s in $\mathfrak{M}$; φ is *valid in* $\mathfrak{F}$ iff φ is valid in all $\mathfrak{M}$ based on $\mathfrak{M}$. **AK** is the set of formulas valid in all $\mathfrak{F}$.

The logic **AK** is "almost" the basic classical normal modal logic **K** in the sense that $(\varphi_1 \wedge \ldots \wedge \varphi_n) \rightarrow \psi$ is in **K** iff $(\Box\varphi_1 \wedge \ldots \wedge \Box\varphi_n) \rightarrow \Box\psi$ is in **AK**. However, (DD) $\Diamond(p \vee q) \rightarrow \Diamond p \vee \Diamond q$ is not in **AK**, which means that it is a non-normal intuitionistic modal logic. It is also noteworthy that (DD) is valid in all $\mathfrak{F}$ where R is a universal relation, but not in all $\mathfrak{F}$ where R is an equivalence relation. **AK** also enjoys the disjunction property and the Glivenko property with respect to **K** (i.e. $\varphi \in \mathbf{K}$ iff $\neg\neg\varphi \in \mathbf{AK}$).

2 Completeness results

Fix an axiom system *Int* for intuitionistic propositional logic with Modus Ponens and Uniform Substitution as rules of inference. Using the canonical model construction, we obtain the following result.

Theorem 3. $\varphi \in \mathbf{AK}$ *iff* φ *is a theorem of the system AK, which results by extending Int with the Necessitation rule* $\dfrac{\varphi}{\Box\varphi}$ *and the axioms*

(A1) $\Box(p \to q) \to (\Box p \to \Box q)$

(A2) $\Diamond p \leftrightarrow \neg\Box\neg p$

(A3) $\Box(p \vee \neg p)$

(A4) $\neg\neg\Box p \to \Box p$

For each directed graph (classical Kripke frame) (W, R), let $Po(W, R)$ be the class of w-frames $(S, \leq, W, R)$, and similarly for $Po(\mathsf{F})$ where F is a class of directed graphs. Let K be the proof system for the basic normal modal logic **K**. Let $\neg\neg\Sigma := \{\neg\neg\varphi \mid \varphi \in \Sigma\}$.

Theorem 4. *If* $K + \Sigma$ *is a normal modal logic that is canonical for the first-order property of directed graphs* Φ *and* $\mathsf{F}(\Phi)$ *is the class of directed graphs satisfying* Φ*, then* $AK + (\neg\neg\Sigma)$ *is sound and complete with respect to* $Po(\mathsf{F}(\Phi))$*.*

3 Comparisons to other intuitionistic modal logics

The satisfaction clauses for $\Box, \Diamond$ in our semantics are a variant of the clauses used by Wijesekera [5]. **AK** extends Božić and Došen's logic $\mathbf{HK}_\Box$ (over $\mathcal{M}$), see [1], and the propositional fragment **WK** of Wijesekera's logic, studied also in [3]. **AK** does not extend Božić and Došen's bimodal logic $\mathbf{HK}_{\Box\Diamond}$ (since $\Diamond p \vee \neg\Box p$ fails) or **IK** of [2, 4], which contains (DD).

AK is the logic of all "standard" intuitionistic modal frames $(S, \leq, Q)$ where Q is a binary relation on S satisfying the "Božić–Došen condition" $(\leq \circ Q \circ \leq) \subseteq Q$ and

(W1) $\forall xy\,(Qxy \to \forall z(y \leq z \to z \leq y))$

(W2) $\forall xy\,(Qxy \to \exists x'\,(x \leq x' \wedge \forall z(x' \leq z \to Qzy)))$

4 Informal interpretation and possible applications

AK might turn out to be a convenient formalism capturing constructive reasoning in settings where the modal accessibility relation is interpreted in terms of *actions* that modify the state of the *world*. Building on this intuition, we will also briefly outline a version of Propositional Dynamic Logic extending **AK**.

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Carnap's Problem for Generalised Quantifiers

Sebastian G. W. Speitel

Carnap's Problem concerns the question of how much of the semantics of an expression one can 'read off' of its inferential behaviour. More precisely, it asks what model-theoretic value of an expression is determined by a given consequence relation in the context of a particular semantic framework. Carnap [4] showed that even at the level of the propositional connectives the standard (single-conclusion) consequence relation of classical propositional logic is incapable of determining their standard truth-conditional semantics in any but the simplest cases. The underdetermination of the model-theoretic value of logical constants by consequence relations extends and deepens at the level of quantification. Recently, Bonnay & Westerståhl [3] characterised the extent to which the standard universal and existential quantifiers are underdetermined by the consequence relation of classical first-order logic (FOL). Their treatment of these expressions as quantifiers in the sense of generalised quantifier theory invites an extension of the investigation of the determination and underdetermination of quantifiers in the context of first-order consequence relations in general. To map out the framework for such an investigation, and to present some initial results, is the purpose of this talk.

A *generalised* or *Lindström-quantifier* $\mathcal{Q}$ is a class of structures of the same signature [6]. Every quantifier $\mathcal{Q}$ determines a unique *quantifier-on-a-domain* M, $\mathcal{Q}(M)$. Where $\mathscr{L}$ is the language of FOL, we designate by $\mathscr{L}(Q)$ the language of FOL extended by a quantifier symbol Q, and by $\mathscr{L}(Q_1, \dots, Q_n)$ the language of FOL extended by quantifier symbols $Q_1, \dots, Q_n$. Given the standard interpretations of the logical constants of FOL and an interpretation of Q (interpretations of $Q_1, \dots, Q_n$) of appropriate type(s) we denote the model-theoretic consequence relation over the relevant logics by $\models_{\mathcal{Q}}$ ($\models_{\mathcal{Q}_1,\dots,\mathcal{Q}_n}$). An interpretation $\mathcal{Q}'$ is *consistent with a consequence relation* $\models_{\mathcal{Q}}$ if $\models_{\mathcal{Q}} \subseteq \models_{\mathcal{Q}'}$. We then ask the following question:

> *for what values $\mathcal{Q}$, and under what conditions, is it the case that $\mathcal{Q}$ is the unique interpretation of Q that is consistent with* $\models_{\mathcal{Q}}$?

In other words: under what conditions is the consequence relation $\models_{\mathcal{Q}}$ 'strong enough' to uniquely 'pin down' or determine the intended interpretation $\mathcal{Q}$ of Q? When a quantifier is such that it is the unique $\mathcal{Q}$ (satisfying certain conditions) consistent with $\models_{\mathcal{Q}}$, we say that it is *uniquely determined by* $\models_{\mathcal{Q}}$ (with respect to these conditions).

Bonnay and Westerståhl showed in [3] that the demand that quantifiers be *isomorphism-invariant* suffices to uniquely determine the standard interpretation of the universal (and thus also the existential) quantifier in the context of the standard consequence relation of FOL. We show that the condition of isomorphism-invariance, a very natural constraint for values interpreting quantifier-expressions, also renders several non-first-order definable quantifiers unique with respect to their associated consequence relation $\models_{\mathcal{Q}}$. In the class of type $\langle 1 \rangle$ cardinality quantifiers this is the case for, e.g.,

(i) $\mathcal{Q} = \mathcal{Q}_0$, where $\mathcal{Q}_0(M) = \{A \subseteq M \mid \omega \leq |A|\}$ is the quantifier *there exist infinitely many*

(ii) $\mathcal{Q} = \mathcal{Q}_{fin}$, where $\mathcal{Q}_{fin}(M) = \{A \subseteq M \mid |A| < \omega\}$ is the quantifier *there exist finitely many*

These results are, despite their elementary nature, philosophically interesting: both uniqueness and isomorphism-invariance have, in different traditions, been considered essential components of the *logicality* of an expression.[19] Unique determination of model-theoretic value by inference, in the sense outlined above, and under the assumption of further semantic constraints, thus delineates a class of logical constants far extending the usual class of first-order logical expressions. A criterion of logicality based on unique determination by inference (*categoricity*) and isomorphism-invariance (*formality*) was formulated and defended in [2].

In the class of type $\langle 1 \rangle$ cardinality quantifiers the ability to be uniquely determined by a consequence relation over a language of the form $\mathscr{L}(Q)$ appears to abruptly stop at $\aleph_1$. Based on old results by Keisler and others[20] we show that the quantifier *there exist uncountably many* ($\mathcal{Q}_1$), given by $\mathcal{Q}_1(M) = \{A \subseteq M \,|\, \aleph_1 \leq |A|\}$, fails to be uniquely determinable over any consequence relation of the form $\models_{\mathcal{Q}}$. This result generalises, in a strengthened formulation, to various other classes of cardinality quantifiers of the form $\mathcal{Q}_\alpha(M) = \{A \subseteq M \,|\, \aleph_\alpha \leq |A|\}$. If there is time, we will present further examples of quantifiers of various types that are not uniquely determined by their associated consequence relations.

Returning to a more abstract perspective and taking isomorphism-invariance to be a desirable constraint on potential interpretations of quantifier-expressions we further study the extent and limits of unique determinability of generalised quantifiers by appropriate consequence relations. In particular, we show that the EC_Δ-definability of a class in FOL is sufficient for a quantifier $\mathcal{Q}$ identified with that class to be uniquely determined by the consequence relation $\models_{\mathcal{Q}}$. We explore further relationships between notions of definability (e.g., projective definability) and unique determination by a consequence relation, and investigate various closure conditions of unique determination under, for example, Boolean combinations of quantifiers.

We conclude this talk by briefly looking at analogous questions for the case in which more than one generalised quantifier-expression is present in the language, i.e., at unique determinability with respect to consequence relations $\models_{\mathcal{Q}_1,\ldots,\mathcal{Q}_n}$ over languages $\mathscr{L}(Q_1, \ldots, Q_n)$, advancing some questions and conjectures, and reflecting on the philosophical significance of the results presented.

This talk is based on joint work with D. Bonnay and D. Westerståhl.

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Modals and Quantifiers in Neighbourhood Semantics for Relevant Logics

Andrew Tedder and Nicholas Ferenz

The Mares-Goldblatt analysis of quantifiers in relevant logic, introduced in [9], presents a particularly elegant and transferable solution to the well-known problem that the most plausible axiomatisations of first-order relevant logics are incomplete with respect to their constant-domain semantics [4]. The solution revolves around two innovations:

1. The use of *general* frame semantics, incorporating a set of admissible propositions.
2. The use of a *non-Tarskian* truth condition for the quantifiers, exploiting the difference between sets of points which are, and those which are not, admissible propositions.

The resulting semantics is simple, elegant, and allows for completeness proofs for the resulting logics which straightforwardly adapt the canonical model methods usually employed for the frame semantics of relevant logics. It has been developed in the direction of first-order classical modal logics by Goldblatt [7] and Mares [6], and for use in propositionally quantified relevant logic in further work by Goldblatt and Kane [8]. More recently, Ferenz has employed the Mares-Goldblatt analysis of the quantifiers to provide adequate frame semantics for a range of quantified relevant modal logics [2, 3]. Furthermore, Ferenz and Tedder [13] have developed the Mares-Goldblatt semantics to accommodate a range of logics weaker than those complete w.r.t. classes of *relational* frames, by investigating first-order relevant logics in a *neighbourhood* setting, following on work by Sylvan (née Routley) and Meyer [11, 12] and Goble [5]. The resulting semantics, in essence, treats of Mares-Goldblatt style extensions of *algebraic* semantics for weak relevant logics, as the resulting semantics is a *general neighbourhood* semantics, as related structures are called by Pacuit [10] in the classical modal case, and so provides for a semantic characterisation of a wide range of logics.

In this paper, we shall extend our previous work by investigating general neighbourhood semantics for quantified modal relevant logics. We shall give a general completeness argument, along the lines of those in [3, 13], investigate the various forms of *augmentation* for the intensional connectives (both relevant and modal), and how these interact with the quantifiers, and discuss upshots for debates surrounding constant-domain modal logics and metaphysics.

Neighbourhood Mares-Goldblatt frames can be defined as follows:

Definition. An NQMRL frame is a tuple $F = \langle W, N, R, *, S_\Box, S_\Diamond, Prop, D, PropFun\rangle$ where:

- $\varnothing \neq N \subseteq W$
- $R \subseteq W \times \wp(W)^2$
- $* : W \longrightarrow W$
- $S_\Box, S_\Diamond \subseteq W \times \wp(W)$
- $Prop \subseteq \wp(W)$
- $D \neq \varnothing$
- $PropFun \subseteq \{\varphi \mid \varphi : D^\omega \longrightarrow Prop\}$

Given F, we define the following operations on $X, Y \subseteq W$:

- $\neg X = \{\alpha \in W \mid \alpha^* \notin X\}$
- $X \to Y = \{\alpha \in W \mid R\alpha XY\}$
- $\Box X = \{\alpha \in W \mid S_\Box \alpha X\}$
- $\Diamond X = \{\alpha \in W \mid S_\Diamond \alpha X\}$

Then F is required to satisfied the following constraints:

(c0) $N \in Prop$ and $Prop$ closed w.r.t. $\cap, \cup, \neg, \to, \Box, \Diamond$

(c0.0) There is a $\varphi_N \in PropFun$ s.t. for any $f \in D^\omega$, $\varphi_N f = N$

(c0.1) $(\oplus\varphi)f = \oplus(\varphi f)$ for any $f \in D^\omega, \oplus \in \{\neg, \Box, \Diamond\}$

(c0.2) $(\varphi \otimes \psi)f = \varphi f \otimes \psi f$ for any $f \in D^\omega, \otimes \in \{\cap, \cup, \to\}$

(c0.3) For any $\varphi \in PropFun, n \in \omega, f \in D^\omega$, the following is an element of $PropFun$:

$$(\forall_n \varphi)f = \bigsqcap_{f' \sim_x f} \varphi f' = \bigcup\{X \in Prop \mid X \subseteq \bigcap_{f' \sim_x f} \varphi f'\}$$

(c0.4) For any $\varphi \in PropFun, n \in \omega, f \in D^\omega$, the following is an element of $PropFun$:

$$(\exists_n \varphi)f = \bigsqcup_{f' \sim_x f} \varphi f = \bigcap\{X \in Prop \mid \bigcup_{f' \sim_x f} \varphi f' \subseteq X\}$$

(c1) $X \subseteq Y \iff N \subseteq X \to Y$ for any $X, Y \in Prop$

We can then define models on such frames straightforwardly:

Definition. A model M on F consists of a multi-type function M, on constants Con and predicates $Pred$ s.t. $M : Con \longrightarrow D$ and $M : Pred^n \times D^n \longrightarrow Prop$. Then, given $f : \omega \longrightarrow D$, we fix:

- $M_f(c) = M(c)$
- $M_f(x_n) = fn$

With these, we extend M to a valuation $[\![\cdot]\!]^M : \mathcal{L} \longrightarrow PropFun$ s.t., for any $f \in D^\omega$ – we'll write $[\![A]\!]^M_f$ for $([\![A]\!]^M)f$:

- $[\![P(\tau_0, \ldots, \tau_n)]\!]^M_f$
  $= M(P)(M_f(\tau_0), \ldots, M_f(\tau_n))$
- $[\![\oplus A]\!]^M_f = \oplus [\![A]\!]^M_f$
  for $\oplus \in \{\neg, \Box, \Diamond\}$
- $[\![A \wedge B]\!]^M_f = [\![A]\!]^M_f \cap [\![B]\!]^M_f$
- $[\![A \vee B]\!]^M_f = [\![A]\!]^M_f \cup [\![B]\!]^M_f$
- $[\![A \to B]\!]^M_f = [\![A]\!]^M_f \to [\![B]\!]^M_f$
- $[\![\forall x_n A]\!]^M_f = (\forall_n [\![A]\!]^M)f$
- $[\![\exists x_n A]\!]^M_f = (\exists_n [\![A]\!]^M)f$

A formula A is true in M, written $\models_M A$ iff $N \subseteq [\![A]\!]^M_f$ holds for every $f \in D^\omega$. A is true on a frame F, $\models_F A$ iff $\models_M A$ holds for every F on M. Finally, A is valid for a class $\mathcal{F}$ of frames just in case it is true on every $F \in \mathcal{F}$.

The range of logics for which we can obtain completeness w.r.t. classes of such frames is very large – including, at least, any logic obtainable from the basic system **F** of [13] (called “Min” in [5]) by any of the quantifier or modal axioms/rules mentioned in [2, 3], and a range of others. Using the very general tools of gaggle theory, for which, see [1], we can obtain equivalence results between neighbourhood models and relational models when the former obey so-called *augmentation* principles. These, in effect, enforce the requirement that the operations defined by the neighbourhood relations ($\rightarrow, \Box, \Diamond$) have *distribution types* and, where appropriate, stand in relations of *abstract residuation* with each other. Consideration of the quantifiers against this background adds some additional complexities, but allows for new results and insights. The aim of this talk is to set out some of these insights, and discuss philosophical upshots.

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On Three-Valued Modal Logics: from a Four-Valued Perspective

Xinyu Wang, Yang Song and Satoshi Tojo

0. Preliminaries

Various non-classical logics thrive, the most successful ones including three-valued logic and modal logic. Here immediately arises a natural question: how can we combine three-valued logic with modal logic into so-called "three-valued modal logic"? We provide an answer by taking an uncommon detour approach of interpreting three-valued weak Kleene logic through auxiliary four-valued logic, so as to obtain a deeper and clearer philosophical insight, which then guides us to evolve three-valued propositional logic into three-valued modal logic in a systematical way and spontaneously generates very natural three-valued semantics suitable for modality $\Box$. To demonstrate our method, two practical example cases are presented and analyzed in detail, with sound and strongly complete natural deduction proof systems. One case is deontic and another one is epistemic, both of which are quite interesting and popular topics in study of modal logic as well as philosophy, and our technique of three-valued modal logic provides a clean and elegant way to combine deontic or epistemic notion into temporal logic, without too much complexity to make use of multiple modalities.

Definition 1. Given a non-empty countable set of propositions $\mathbf{P}$, formula A in Language $\mathsf{3VL}$ is recursively defined as the following BNF, where $p \in \mathbf{P}$:

$$A ::= p \mid \neg A \mid A \wedge A \mid A \vee A$$

A three-valued valuation model is a function $V : \mathbf{P} \to \{T, U, F\}$.

Song et al. in [1] devises a novel methodology of interpreting three-valued propositional logic with the assistance of four-valued propositional logic. Define a four-valued propositional model as a four-valued valuation function $V_4 : \mathbf{P} \to \{T_1, F_1\} \times \{T_2, F_2\}$. The core philosophical idea is that in the finest-grained view, everything is ultimately two-valued, for example, we can ably pick any one out of arbitrarily finite many possible values by just asking a series of *yes/no* questions. Hence the pair of truth values $V_4(p) = (\mathrm{Val}_1^{V_4}(p), \mathrm{Val}_2^{V_4}(p)) \in \{T_1, F_1\} \times \{T_2, F_2\}$ just represent two different *yes/no* properties of the "bundled" propositional letter p, but when we zoom out to a courser-grained view, resolution decreases and p blurs so as to look like one solitary three-valued propositional letter. Thus, the heart of the whole story settles on semantics of these two two-valued truth values, as well as a "compression" function $f_C : \{T_1, F_1\} \times \{T_2, F_2\} \to \{T, U, F\}$.

As for the case of weak Kleene logic, we let $\mathrm{Val}_1^{V_4}$ behave classically:

$$\begin{aligned}
\mathrm{Val}_1^{V_4}(\neg A) = T_1 &\iff \mathrm{Val}_1^{V_4}(A) = F_1 \\
\mathrm{Val}_1^{V_4}(A \wedge B) = T_1 &\iff \mathrm{Val}_1^{V_4}(A) = T_1 \text{ and } \mathrm{Val}_1^{V_4}(B) = T_1 \\
\mathrm{Val}_1^{V_4}(A \vee B) = T_1 &\iff \mathrm{Val}_1^{V_4}(A) = T_1 \text{ or } \mathrm{Val}_1^{V_4}(B) = T_1
\end{aligned}$$

We let $\mathrm{Val}_2^{V_4}$ behave *False*-infectiously:

$$\begin{aligned} \mathrm{Val}_2^{V_4}(\neg A) = F_2 &\iff \mathrm{Val}_2^{V_4}(A) = F_2 \\ \mathrm{Val}_2^{V_4}(A \wedge B) = F_2 &\iff \mathrm{Val}_2^{V_4}(A) = F_2 \text{ or } \mathrm{Val}_2^{V_4}(B) = F_2 \\ \mathrm{Val}_2^{V_4}(A \vee B) = F_2 &\iff \mathrm{Val}_2^{V_4}(A) = F_2 \text{ or } \mathrm{Val}_2^{V_4}(B) = F_2 \end{aligned}$$

We let $f_C(T_1, T_2) = T$, $f_C(F_1, T_2) = F$, and $f_C(T_1, F_2) = f_C(F_1, F_2) = U$.

The advantage of this four-valued interpretation is straightforward: with its assistance, we can easily expand three-valued propositional logic onto three-valued modal logic, with ample confidence to philosophically justify our choice of definition for modality $\Box$'s three-valued semantics, since we already have a good intuition about how $\Box$ may act upon a two-valued truth value.

1. Case I Deontic Three-Valued Modal Logic

Definition 2. Given a non-empty countable set of propositions $\mathbf{P}$, formula A in Language 3VML is recursively defined as the following BNF, where $p \in \mathbf{P}$:

$$A ::= p \mid \neg A \mid A \wedge A \mid A \vee A \mid \Box A$$

A three-valued Kripke model $\mathfrak{M}$ is a triple (S, R, V) where:

- S is a non-empty set of possible worlds.
- $R \subseteq S \times S$ is a binary relation on S.
- $V : S \times \mathbf{P} \to \{T, U, F\}$ is a three-valued valuation function.

Suppose A is any 3VL-formula, for its first two-valued truth value $\mathrm{Val}_1^{V_4}(A)$, T_1 means the agent is obligated to do A, and so F_1 means the agent does not have to do A; for its second two-valued truth value $\mathrm{Val}_2^{V_4}(A)$, T_2 means the agent is allowed to do A, and so F_2 means the agent is forbidden to do A. Thus $f_C(T_1, T_2) = T$ means the agent must do A, $f_C(F_1, T_2) = F$ means the agent can either do A or not do A, and $f_C(T_1, F_2) = f_C(F_1, F_2) = U$ means the agent must not do A since ethically speaking, an immoral deed is afterall immoral even if it is also an obligation, for example, a soldier kills an enemy on the battlefield. Further we designate a temporal interpretation to modality $\Box$ in Language 3VML, then semantics of $\Box$ can be assigned as the following:

1. At any possible world, $\mathrm{Val}_1^{V_4}(\Box A) = T_1$ iff on all successors $\mathrm{Val}_1^{V_4}(A) = T_1$, because that the agent must keep doing A all the time in the future is the same as that at any time in the future the agent must be doing A.
2. At any possible world, $\mathrm{Val}_2^{V_4}(\Box A) = T_2$ iff on all successors $\mathrm{Val}_2^{V_4}(A) = T_2$, because that the agent is allowed to keep doing A all the time in the future is the same as that at any time in the future the agent is allowed to do A.

The above four-valued semantics can be mapped down to three-valued semantics:

$$\mathrm{Val}_{\mathrm{I}}^{\mathfrak{M}}(s, \Box A) = \begin{cases} T, & \text{if } \forall sRt, \mathrm{Val}_{\mathrm{I}}^{\mathfrak{M}}(t, A) = T \\ U, & \text{if } \exists sRt, \mathrm{Val}_{\mathrm{I}}^{\mathfrak{M}}(t, A) = U \\ F, & \text{otherwise} \end{cases}$$

2. Case II Epistemic Three-Valued Modal Logic

Suppose A is any 3VML-formula, for its first two-valued truth value $\mathrm{Val}_1^{V_4}(A)$, T_1 means objectively A is true and so F_1 means A is false, just as classical two-valued logic; for its second two-valued truth value $\mathrm{Val}_2^{V_4}(A)$, T_2 means the agent understands A, and so F_2 means the agent does not understand A. Thus $f_C(T_1, T_2) = T$ means the agent understands A is true, $f_C(F_1, T_2) = F$ means the agent understands A is false, and $f_C(T_1, F_2) = f_C(F_1, F_2) = U$ means the agent does not understand A since under such a circumstance, it is sheer nonsense for the agent to talk about truth value of some statement that he does not even understand at all. Further we designate a temporal interpretation to modality $\Box$ in Language 3VML, then semantics of $\Box$ can be assigned as the following:

1. At any possible world, $\mathrm{Val}_1^{V_4}(\Box A) = T_1$ iff on all successors $\mathrm{Val}_1^{V_4}(A) = T_1$.
2. At any possible world, $\mathrm{Val}_2^{V_4}(\Box A) = T_2$ iff on the very same possible world $\mathrm{Val}_2^{V_4}(A) = T_2$, because understanding a sentence depends solely on status quo, regardless whether the sentence itself talks about past, present or future.

Moreover, it can be reasonably assumed that the agent never forgets his knowledge [2], namely, the following restriction should be put onto the Kripke model:

3. For any propositional letter $p \in \mathbf{P}$, at any possible world if $\mathrm{Val}_2^{V_4}(p) = T_2$, then on all successors $\mathrm{Val}_2^{V_4}(p) = T_2$.

This restriction can be mapped down as: a three-value Kripke model is a three-valued Kripke model–II, iff for any $s \in S$ and any $p \in \mathbf{P}$, $V(s, p) = U \Rightarrow \forall tRs, V(t, p) = U$. Within the class of three-valued Kripke models–II, the above four-valued semantics can be mapped down to three-valued semantics:

$$\mathrm{Val}_{\mathrm{II}}^{\mathfrak{M}_{\mathrm{II}}}(s, \Box A) = \begin{cases} T, & \text{if } \mathrm{Val}_{\mathrm{II}}^{\mathfrak{M}_{\mathrm{II}}}(s, A) \neq U \text{ and } \forall sRt, \mathrm{Val}_{\mathrm{II}}^{\mathfrak{M}_{\mathrm{II}}}(t, A) = T \\ U, & \text{if } \mathrm{Val}_{\mathrm{II}}^{\mathfrak{M}_{\mathrm{II}}}(s, A) = U \\ F, & \text{otherwise} \end{cases}$$

References

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[2] Wang, Y., Cao, Q.: On axiomatizations of public announcement logic. Synthese **190**, 103–134. Springer (2013)