Two algebraic unknowns in Latin and Italian mathematics 1200-1500: known but not considered anything special

Jens Høyrup

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The use of several algebraic unknowns was familiar in India at least since Brahmagupta, and in Arabic algebra we know it in a wholly different form. So, my discussion of the appearance of two unknowns (never more) in Latin and post-Latin Europe until 1500 is not about priorities.

In Arabic, several algebraic unknowns were used regularly, for example by Abū Kāmil, the first unknown being a *thing* (*šay'*), the following names of coins:[1] *dinar*, *fals*, and *khātam* (on a single occasion Abū Kāmil uses "large thing" and "small thing"[2]). Coin names for algebraic unknowns are used a couple of times in the *Liber mahameleth*,[3] which was translated into Latin around 1160 in the Toledo environment[4]. They are obviously also used in the Latin translation of Abū Kāmil's algebra.[5] However, the idea did not spread beyond that in the Latin world.

1 Ed. trans. [Rashed 2012: 370, 396, 400–408, 736–755].

2 Ed. trans. [Rashed 2012: 410].

3 Ed. [Vlasschaert 2010: 209*f*]; ed. [Sesiano 2014: 258–260].

4 Some colleagues hold it to be an original Latin composition, but there are strong arguments against that – see [Høyrup 2015: 13–15].

5 Ed. [Sesiano 1993].

Fibonacci

So, I shall start with a look at the occasional use of *two algebraic* unknowns by Leonardo Fibonacci, mainly in the *Liber abbaci* and, in particular, at how Fibonacci saw this technique.

> "Algebraic unknowns" are new entities that are introduced – "posited" – as *representatives* of the magnitudes about which a problem is formulated, and submitted to arithmetical and algebraic operations.

It is known that Fibonacci uses two algebraic unknowns in one problem in the *Flos*, "The Flower", a pure-number version of an unusual variant of the "purchase of a horse", asking (emphasis added in the interest of intelligibility) for[6]

> five numbers, of which the *first with the half* of the second and third and fourth makes as much as the *second with the third part* of the third and fourth and fifth numbers, and as much as the *third with the fourth part* of the fourth and the fifth and the first numbers, and also as much as the *fourth with the fifth part* of the fifth and the first and the second numbers, and besides as much as the *fifth number with the sixth part* of the first and the second and the third numbers.

Here, the first number is posited to be a *causa*, and the fifth a *res* – both standing for *thing*, the former probably borrowed from Catalan, Castilian or Provençal, the latter evidently Latin. This (and, since the problem is indeterminate, the assumption that the shared sum is 17), allows Fibonacci to perform a *bona fide* rhetorical algebraic calculation.

At closer inspection, there are also a number of instances in the *Liber abbaci* – some of them observed by Heinz Lüneburg and Laurence Sigler,[7] who however do not distinguish them from the many other problems that

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[6] All English translations from Latin and Italian are mine.

[7] [Lüneburg 1993: 181*f*], [Sigler 2002: 626].

deal simply with several unknown magnitudes (the money possessed by three, four or five men, etc.) but which are solved without algebraic representation.

A first version of the *Liber abbaci* was written in 1202. Around 1228, Fibonacci created a new version, using a master copy of the original version, adding and deleting, but faithfully conserving a number of obvious miswritings etc. from the original version.[8] With one exception, all surviving manuscripts descend from the 1228 version. The exception, discovered a few years ago by Enrico Giusti, is one manuscript that contains chapter 12 of the 1202 version, a collection of mixed problems, constituting a third of the whole work.

Since two algebraic unknowns occur in the context of *regula recta* or "direct rule", first a few words about this "rule", that is, *technique* or *method* – namely the technique of positing some magnitude occurring in the question to be a *thing*, and then to solve or reduce the problem by means of rhetorical algebra.

In the 1228 version, the *regula recta* is said when it first occurs to be "much used by the Arabs" and "immensely praiseworthy"; here it serves in an alternative solution to a "give-and-take" problem.[9] If we look at the same problem in the 1202-version of the chapter,[10] no alternative solution is offered, and in consequence the *regula recta* goes unmentioned there. That does not mean, however, that Fibonacci did not use the *regula recta* in 1202. An alternative solution to a different problem by *regula recta* is announced in both versions as something "which you can also find by

[8] This is what demonstrates the use of a master copy. Discussed in [Høyrup 2020].

[9] [Ed. Giusti 2020: 324]. *A* asks *B* for 7 of his *denari*, and will then have 7 times as much as *B*. *B* on his part asks 5 *denari* from *A* and will then have 7 times as much as *A*.

[10] [Ed. Giusti 2017: 59].

regula recta").[11] That is, in 1202 Fibonacci used this technique and referred to it as something familiar; in 1228 he then discovered that it was one of the things that needed to be introduced explicitly to his audience.

Fibonacci's appeals to two algebraic unknowns follow a similar pattern. The first instance in 1228 is in an alternative solution to a problem in chapter 12 about two men finding a purse, a *bursa*;[12] the alternative solution is absent from the 1202 edition, although the problem itself is there. The possession of the first man is posited to be a *thing* (which makes us understand that we are within a *regula recta* calculation, even if it is not said); thereby the *bursa* too becomes an algebraic unknown, and algebraic manipulation of the resulting rhetorical equations yields the ratio between these two unknowns.

This being an alternative solution (as a matter of fact a third possibility), the two unknowns are not a *necessary* tool for the problem in question. In their next appearance they are, however. There they turn up in a problem dealing with gains and expenses in repeated commercial travels. The first problem in the sequence to which it belongs runs:[13]

> Somebody proceeding to Lucca made double there, and disbursed 12 δ. Going out from there he went on to Florence; and made double there, and disbursed 12 δ. As he got back to Pisa, and doubled there, and disbursed 12 δ, nothing is said to remain for him. It is asked how much he had in the beginning.

Instead of calculating backwards, Fibonacci employs a hidden single false position, showing that an initial capital of 1 δ after three doublings will

[11] [Ed. Giusti 2020: 334; Giusti 2017: 70]. Similarly [ed. Giusti 2020:341; Giusti 2017: 78].

[12] [ed. Giusti 2020: 355; Giusti 2017: 92]. *A* if getting the purse will have thrice as much as *B*. *B* if getting the purse will have four times as much as *A*.

[13] [Ed. Giusti 2020: 417; Giusti 2017: 127].

have a "Pisa value" of $2{\cdot}2{\cdot}2$ δ = 8 δ. The total Pisa value of the expenses is $(1+2+4){\cdot}12$ δ = 84 δ. Since the Pisa value of the initial capital should equal the Pisa value of the expenses, the initial capital itself must be $(7{\cdot}12\ \delta)/8 = 10\frac{1}{2}\ \delta$. The principles involved are those of composite interest and discounting, familiar mercantile techniques.

In the problem that concerns us here[14] that will not do:

> Again, in a first travel somebody made double; in the second, of two he made three; in the third, of three he made 4; in the fourth, of 4 he made 5. And in the first travel he expended I do not know how much; in the second, he expended 3 more than in the first; in the third, 2 more than in the second; in the fourth, 2 more than in the third; and it is said that in the end nothing remained for him.

Therefore, Fibonacci posits, this time with explicit reference to the *regula recta*, that the initial capital is an *amount* (*summa*), and the first expenditure a *thing*. This problem is also in the 1202 version of chapter 12,[15] solved in the same way and with the same reference to the *regula recta*.

The third instance is an alternative solution to a problem found in chapter 13, part 2.[16] Here, three men ask each other for money, and it is supposed that the second and third together have a *thing*, and the third alone *part of a thing*, afterwards reduced to *part* and actually an independent unknown. Whether this was already in the 1202 version we cannot know. What we can observe is that Fibonacci presents the solution simply as one "according to an investigation of proportions" (namely the proportions between the three possessions).

Together, these three problems demonstrate that Fibonacci saw nothing particular in the introduction of a second algebraic unknown; just as with

[14] [Ed. Giusti 2020: 426].

[15] [Ed. Giusti 2020: 134].

[16] [Ed. Giusti 2020: 530].

the *regula recta* (where we know it from his own words) we may conclude with confidence that he drew on a source or source environment where the technique was considered obvious – and that he took over that perspective. Since one of the instances classifies the solution as being according to *regula recta*, we can safely assume that the two techniques were borrowed from the same place – obviously located in the Arabic world, and almost certainly in the Maghreb or al-Andalus.

A fourth and final instance belongs in chapter 15, part 3, the presentation of algebra (under which Fibonacci did not count the *regula recta*). 10 is divided into two parts (say, *a* and *b*), and $^{a}/_{b}+^{b}/_{a} = \sqrt{5}\,\delta$ (Fibonacci, following Arabic algebra, regularly provides pure numbers with the unit *denarius* – Arabic *dirham*). An alternative solution[17] explains that

> you posit one of the two parts a *thing*, and the other certainly 10 less a *thing*. And let from the division of 10 less a *thing* in a thing a *denarius* result.

Obviously, Fibonacci here borrows the Arabic use of a coin name as second algebraic unknown. Unfortunately, he does not stop using the same name as a unit for pure numbers, and so shows not to be aware of what goes on in the source he copies;[18] if he had understood, he would almost certainly have called the new unknown a *dragma* or used some third coin name. Other mistakes confirm that Fibonacci did not understand his source. This instance therefore tells us nothing certain about how Fibonacci himself thought about the use of two algebraic unknowns. On the other hand, there is no reason to doubt his full understanding of what he was doing in the

[17] [Ed. Giusti 2020: 660].

[18] Abū Kāmil would have used *dinar* as the unknown and *dirham* as the unit for pure numbers; Antonio de' Mazzinghi, in problem solutions where he adopts *quantità* as a second unknown along with *cosa* in the *Fioretti* (from problem 18 onward [ed. Arrighi 1967: 41 onward], abstains from using it as a synonym for number, as he does elsewhere.

three previous instances (and in the *Flos*), and in these he did not treat it as something special. We may say that he agreed with Lüneburg and Sigler, however much *we* discern something that was to become important centuries later.

Antonio de' Mazzinghi

After Fibonacci, we have to wait until 1380–1390 and until Antonio de' Mazzinghi's *Fioretti* ("Small Flowers")[19] before known sources make use of two algebraic unknowns.

Antonio did not learn his use of two unknowns from anybody else. In the *Fioretti* we can follow the gradual development of the idea. What we possess is indeed a work in progress copied in 1463 by Benedetto da Firenze, in which Antonio does not hide the traces of his progress.

Problem 9[20] deals with two numbers, which for brevity we may designate A and B, fulfilling the conditions that

$$AB = 8\ ,\quad A^2+B^2 = 27\ .$$

A first solution makes use of *Elements* II.4. Next (*census* is the square on the *thing)*:

> we can also make it by the equations of algebra; and that is that we posit that the first quantity is a thing less the root of some quantity, and the other is a thing plus the root of some quantity. Now you will multiply the first quantity [A] by itself and the second quantity [B] by itself, and you will join together, and you will have 2 *censi* and an unknown quantity, which unknown quantity is that which there is from 2 *censi* until 27, which is 27 less 2 *censi*, where the product of these quantities is $13^1/_2$ less a *censo*. The smaller part is thus a *thing* minus the root of $13^1/_2$ less a *censo*, and the other is a thing plus the root of $13^1/_2$ less 1 *censo*. [...].

This is quite opaque. The "unknown quantity" has the potentiality to

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[19] [Ed. Arrighi 1967].

[20] [Ed. Arrighi 1967: 28].

become a second unknown, but that the "unknown quantity ... that ... there is from 2 *censi* until 27" is twice the square on the "some quantity" is a secret that stays in Antonio's mind.

The rest of the calculation is impeccable and algebraic, but with only one unknown.

The following problem 10 is very similar. But in the more intricate problem 18[21] the idea comes to fruition:

> Find two numbers which, one multiplied with the other, make as much as the difference squared, and then, when one is divided by the other and the other by the one and these are joined together make as much as these numbers joined together. Posit the first number to be a *quantity* less a *thing*, and posit that the second be the same *quantity* plus a *thing*. Now it is up to us to find what this *quantity* may be, which we will do in this way. [...]

What follows is a perfect algebraic solution. From here onward, Antonio uses the technique in no less than 8 problems, all of which *could* have been solved with the more intuitive technique of problems 9 and 10. So, Antonio has created something new (disregarding the one he copied without understanding and from which nothing could be learned, Fibonacci's problems were all of the first degree), and he almost certainly knew it.

Nobody else discovered, even though Antonio's problems were copied is two more "abbacus encyclopedias" around 1460.

Florentine anonymous, *ca* 1390

Use of two algebraic unknowns is sometimes also ascribed to an anonymous Florentine abbacus writer.[22] The claim is based on problem solutions like this variant of the "purchase of a horse":

[21] [ed. Arrighi 1967: 41].

[22] Florence, Bibl. Naz. Centr., fondo princ. II.V.152. [Franci & Pancanti 1988] is an edition of its extensive algebra section.

> Three have *denari* and they want to buy a goose, and none of them has so many *denari* that he is able to buy it on his own. Now the first says to the other two, if each of you would give me $^1/_3$ of his *denari*, I shall buy the goose. The second says [and so on cyclically]. Then they joined together the *denari* all three had together and put on top the worth of the goose, and the sum will make 176, it is asked how much each one had for himself, and how much the goose was worth. [...]. I have made it in such way that in this one and those that follow it will have to be shown that the question examined by the *thing* will lead to new questions that cannot be decided without false position. [...]. I shall make this beginning, let us make the position that the first man alone had a *thing*, whence, when the position is made, you shall say thus, if the first who has a *thing* asks the other two so many of their *denari* that he says to be able to buy the goose, these two must give to the first that which a *goose* is worth less what a *thing* is worth, which the first has on his own. So that the first can say to ask from the other two a *goose* less a *thing* [...]

After longwinded arguments it is concluded that

$$7geese = 13things + 4 \ .$$

and that

$$4geese - 2things = 176 \ .$$

Now, for instance, the *thing* might be found from the latter equation to be 2 *geese* less 88 and be inserted in the former, which would easily lead to the goal. Instead the author goes on, using a complicated double-false position, first that the *goose* is worth 40, next that it is 80.

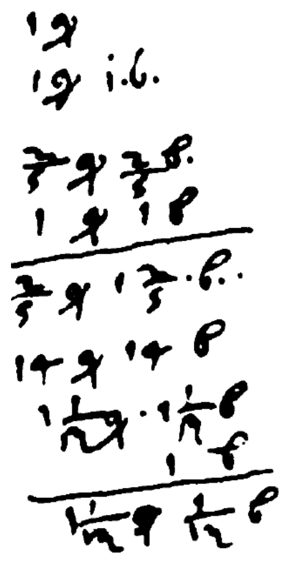
So, not only did the author not speak about using two algebraic unknowns. Even though he was almost certainly influenced by something like Fibonacci's *regula-recta* problems with two unknowns (hardly Fibonacci's texts with their full explanations), he evidently did not really see the two unknowns as algebraic entities, and therefore did not eliminate one by means of the two equations, as Fibonacci had done in the *Flos*, and as Antonio had done repeatedly. Instead, as he has announced, our anonymous makes use of the non-algebraic double false position, a familiar but opaque technique – more opaque in the present context than normally.

Benedetto da Firenze

Benedetto da Firenze copied Antonio, but did not learn from him. He does use two algebraic unknowns, but in first-degree problems, in the same way as Fibonacci had done (apparently under no direct influence, as shown by his terminology).

He does so both in his monumental encyclopedic *Trattato de praticha d'arismetrica* and in the more elementary *Tractato d'abbacho*. In the latter he also introduces the *regula recta* (as *modo recto*), in a way that suggest he has it from the teaching tradition. For example, in a problem about five men finding a purse (the other instances are similar)

> Five men have *denari*, and going on a road they find a purse with *denari*. The first says to the others, if I got the *denari* of the purse, then I would have $2^1/_2$ times as much as you. The second says, if I got the *denari* of the purse, then I would have $3^1/_3$ times as much as you [and so on cyclically]. It is asked how much each one had, and how many *denari* there were in the purse. You will make the position that the first had a *quantity*, and having got the purse he had a *quantity* and a *purse*, and he says to have $2^1/_2$ of the others. [...]



after which Benedetto makes a perfect algebraic argument. Most interesting is that the manuscript page of his autograph shows that after formulating the question he has first made all calculations in algebraic letter symbolism in the margin (*q* standing for *quantity*, *b* for *borsa*, "purse"), and only then written the text.

Pacioli

In Pacioli's *Summa de arithmetica*[23] from 1494 we find evidence that

[23] [Pacioli 1494: fol. 191^{v}].

the use of two unknowns must have been more widely known that we would believe from surviving sources. Pacioli proposes a three-participant give-and-take problem "merely to show how one operates with a deaf quantity, which the ancients call second thing"; this is the first time we see that the technique, though considered of minor importance, was recognized as being something specific and discover that it was even provided with a kind of name. Since Pacioli himself uses *cosa* as the first and *quantità* as the second unknown, Benedetto can be excluded as his source (being less than one generation older, it would also be strange if he were considered "ancient"). Antonio uses *cosa* and *quantità* but in second-degree problems; even though Pacioli knew the *Fioretti* directly or indirectly, Pacioli could not have taken the use of two algebraic unknowns in this type of first-degree problems from the *Fioretti*.

Why no take-off?

From Viète onward, the use of several algebraic unknowns became essential when algebra was applied to high-level questions. Why were almost four centuries needed before the promises of the idea were explored?

The answer is inherent in the notion of "high-level questions" submitted to algebraic treatment. From Fibonacci's *Flos* to Pacioli's *Summa*, these were represented by problems belonging to the family comprising the purchase of a horse, the finding of a purse, etc. When these became too intricate to be solved by verbalized arithmetic, line diagrams served just as well as – often better than – rhetorical algebra with several unknowns, and such line diagrams indeed occur profusely in the *Liber abbaci*. In this context, the idea of two unknowns therefore did not promise anything, and Pacioli was not mistaken when he "merely" wanted to show how it worked.

In the time of Viète, Descartes etc., high-level questions were those inspired by Archimedes and Pappos but going beyond what could conveniently be done by classical geometry. Then not only two but more unknowns became a necessity.

So, the use of two algebraic unknowns, a minor technique barely kept alive for centuries, unfolded only within a mathematical practice where it became an indispensable tool.

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