Roskilde University

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Standing Wave Patterns

a study on the effect of stiffness on the vibration of a free square plate

BP2

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Abstract

Despite the approximation of the vibrating thin plate model which reduces its three dimensional problems into a two dimensional one, bending stiffness of the plate creates complexities that have limited the scope of many studies. The Calculation of modes of a vibrating rectangular free plate was described by Rayleigh as a problem "of great difficulty". However, it is suggested that the behavior of a vibrating plate can be likened to a stiff membrane or a two-dimensional rod (14) . Based on this, this study aims to analogize the Kirschoff thin plate model to other systems with the objective of experimentally investigating the effect of stiffness properties of a free square isotropic plate under mechanical vibration.

This theoretical and experimental study attempts to answer this question: **How does varying the stiffness properties of a free square plate affect the configuration of its nodal lines under mechanical vibration?**, by using two plates (stainless steel and aluminum) of different stiffness parameters, namely Young's modulus and mass density, and investigating their behavior at their natural frequencies. Our experimental method of investigation is based from the famous Chladni experiment, in which he was able to visualize patterns of nodal lines formed on a plate by placing fine granular media on its surface.

Our experimental results confirm that the above mentioned parameters lead to a disparity of patterns between the two plates above f=2600 Hz, while the exact patterns of the same natural frequencies are formed on both plates at frequencies within the range of f=540-2600 Hz. This led to the conclusion that influence of these parameters increases at higher frequencies. Overall, this research found that for the stiffer plate (stainless steel), more patterns were formed within smaller ranges of frequency at at lower frequencies, which led to new deductions about the relationship between a frequency's wavelength and a plate's stiffness.

1 Preface

The semester constraint for the second semester project is the 'Interplay between Theory, Model, Experiment and Simulation in Science.'
In this project we first approached our research question through theoretical research and analysis. From this theory, we were better able to understand what our experiment should be and what model we should choose. However, for the entirety of the process, the interplay was anything but linear. The theory both informed the experimental process, and vice versa. For each step we took within each methodological facet of the project, we had to consider the elements of the others. For example, for each new discovery we made in our theoretical research, we had to consider how we might need to change our model and experiments so that they could be more aligned with our new knowledge. Conversely, we had to consider what we should research and include as our theoretical concepts based off of our experimental setup, goals, and limitations.
We additionally researched and successfully ran a simulation of our model, and considered how it would relate to our experiments and theory. However, we did not conduct experiments with the simulation. Our results are taken from our experimental results and our analysis of how the said results compare with our model and theory. In this way, we fulfilled the semester constraint.

Contents

2 Introduction

2.1 A Brief History of Cymatics

Towards the end of the 18th century, Physicist, Musician, and instrument maker Ernst Chladni devised of an experiment that would influence Acoustics for the centuries to come. He discovered that by striking a plate with a violin bow, when a resonant frequency was reached, one could see different geometrical patterns on the plate. These patterns were visualized by placing some material medium such as sand on the surface. From this Chladni was able to surmise that frequency has a direct effect on the nodal lines of a plate. It was from here that he was able to derive *Chladni's Law*. Chladni's Law approximates the number and positions of the nodal lines that lie on a membrane (2). This discovery led to the birth of the study known as Cymatics. Cymatics refers to the phenomena of modal vibration. It is generally associated with making resonance visible in different material mediums.

2.2 Aim of Research

Centuries later the phenomena is still being studied, however the understanding has been expanded to a much broader range than just tones struck onto a plate. While the system may seem relatively simple from an introductory level, the underlying phenomena is much more complex than just looking at resonant frequencies. The different patterns formed for each modal frequency depends on several factors, among these being the boundary condition, thickness, stiffness and shape of the plate. of the plate. This has an influence on the behavior of its vibration.

Initially, our interest purely stemmed from the cymatic patterns formed on plates. We became curious to see if there was any quantifiable way of understanding how and why these patterns form. Further research then led us to the question; what about the material medium which is experiencing mechanical vibration affects these patterns, if at all? How might it affect these patterns, and can we explain these changes in a quantitative way? We thus began to research which parameters in the system might be most influential on what patterns appear. There are many factors which contribute to how a pattern is formed, the most influential of which include; the shape of the plate, the material of the plate, the driving frequency, and the proportions of the plate (i.e rectangularity). After looking more into classical plate theory, specifically using the aid of the work done by NASA in "Vibration of Plates" (11), we were able to determine what might be most useful to examine in terms of influential material parameters. The controlled variable of the model in this research is a square plate which is free along all four edges. This is the most simple case we could have in terms of both the mathematics behind the equations and the physical set up. Our experimentation takes the form of vibrating two free square plates of different materials, specifically with very different Young's moduli and mass densities. These parameters will be discussed in more detail further on. We do not vary and the plates' thickness and surface area. This research aims to investigate how varying material properties of plate effects the patterns formed when an external force is applied. The scope of this research is limited to a theoretical and experimental analysis of the vibration of solid shapes and a study of the Classical Plate Theory when applied to square plates of different materials. The most concise formation of what we aim to investigate is the question:

How does varying the stiffness properties of a free square plate affect the configuration of its nodal lines under mechanical vibration?

this work aims to answer this by investigating the phenomenon of patterns formation due to standing waves propagating within elastic solids by likening the vibration of a plate to that of a rod. Theoretical and experimental analysis of the vibration of square plates will allow for a better understanding of effect of a vibrating plate's stiffness.

There is a substantial amount of theory that we must take into account when looking to decipher how standing wave patterns form on a plate. To have a basic understanding of classical mechanics and calculus is necessary for investigating this problem and understanding this work.

3 Theoretical Concepts

One of the purposes of this project is to look into structures of the standing waves on a plate. It is thus necessary to investigate and understand the physics that govern these properties and phenomenon. Since the physics which it stands on are based on Newtons laws, the general equations that connect Newton laws with standing waves are considered. Starting with the most basic case, which is small displacement as stretch or compression for a material, it is crucial to introduce Hooke's law and Newton's Second Law:

$$F_y = m\frac{d^2y}{dt^2} = -ky \tag{3.1}$$

where y is position and k is *the spring constant*. In which there is a standard coordinate system, where y is the vertical direction and is the direction of the amplitude of the waves (see figue3.1)

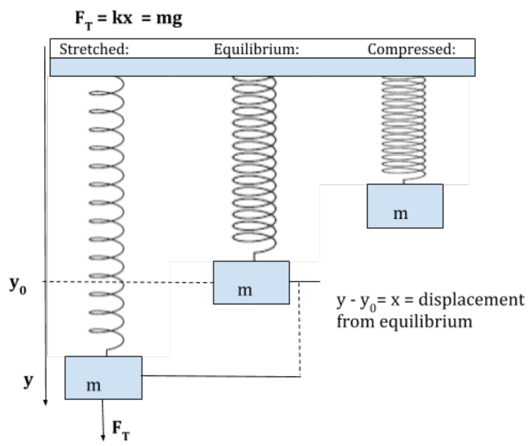

Figure 3.1: *a classical interpretation of Hooke's law*

3.1 Hooke's Law

Hooke's law states that if a mass m attached to a spring is influenced by some external force F_y, it is displaced some distance y from its equilibrium position (proportional to the force of influence)[1]. The reaction force will act, due to Newtons third law, with an equal but opposite direction,that will balance the external force. As the vertical force is dealt with, the gravitational force should also be taken into account($F_t = mg$). Considering the resulting force on the mass in figure 3.3:

$$F_{res} = -ky + mg(9) \tag{3.2}$$

[1]For small displacements, Hookes law is an example of deformations, which can go back and forth without changing any properties of the material. If a material has this property, it is said to be elastic

(this is illustrated in figure 3.1). The equilibrium point of the mass is not at $y = 0$, but actually where $F(y = 0) = mg$. Therefore the equilibrium position is determined by which value of y is in $F(y) = 0$. Correspondingly, this gives:

$$0 = -ky_0 + mg \Leftrightarrow y_0 = \frac{mg}{k} \tag{3.3}$$

Taking equation 3.1 into eq. 3.3, we get:

$$\frac{d^2y}{dt^2} + \frac{k}{m}(y - y_0) = 0 \tag{3.4}$$

If one takes a new variable x, such that: $x = y - y_0$, such that:

$$\frac{d^2y}{dt^2} = \frac{d^2x}{dt^2} \tag{3.5}$$

A unique attribute of this new variable is that it turns eq. 3.4 into:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{3.6}$$

Which gives:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{3.7}$$

This all describes Hooke's law for a mass m and a spring with spring constant k in horizontal movement on a friction-less surface, as in figure 3.2. Here the Normal force from the surface balances the force of gravitation. This is rather important, because knowing that:

$$\frac{d^2y}{dt^2} = \frac{d^2x}{dt^2}$$

gravitation can be neglected, and we can thus say we do not have any *preferred directions.* The notion of no preferred directions is paramount, as the theories used in later sections require that both medium and reference frame is *isotropic*[2].

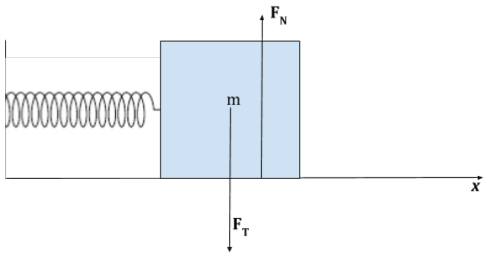

Figure 3.2: *Hooke's law with a mass m sliding along a frictionless surface, where the normal force balances the gravitational force.*

[2]isotropic reference frame: what occurs in the frame is independent of direction. If a force affects a segment in the plate, the impact would be the same if the same force had affected the same segment from another direction.

3.1.1 Harmonic Oscillation

The purpose of this section is to discuss the relation between a harmonic oscillator and Hooke's law. A harmonic oscillator is normally described with a sine or cosine function. It will be described in more detail in a later sections, but for now it will be considered as a statement and assumed to be true.A function which will fulfill Hooke's law and the demand of a periodic function is the cosine function. Consequently, the displacement in Hooke's law can be written as:

$$x = Acos(\omega t) \tag{3.8}$$

Where ω is the *angular acceleration* and is related to the frequency f by $\omega = 2\pi f$. A is the amplitude. Substituting eq. (3.8) into Hooke's law (eq.3.1):

$$m\frac{d^2x}{dt^2} = -mA\omega^2 cos(\omega t) \tag{3.9}$$

$$-kx = -kAcos(\omega t) \tag{3.10}$$

This gives:

$$m\frac{d^2x}{dt^2} = -kx \Leftrightarrow -mA\omega^2 cos(\omega t) = -kAcos(\omega t) \Leftrightarrow m\omega^2 = k \Leftrightarrow \omega = \sqrt{\frac{k}{m}} \tag{3.11}$$

Which defines the frequency as:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{3.12}$$

This shows that the frequency depends solely on the mass of the object which is oscillating and the spring constant. This material parameter depends on the type of metal (or any material) the spring is made of. This fact supports this research by considering to what extent a springs can be compared with plates and how their material properties influence their vibration .

To understand the vibrations of plates in the context of Hooke's law, the plate can be considered as a lot of independent masses, attached to each other with springs in a coupled oscillation which is illustrated in figure 3.3.

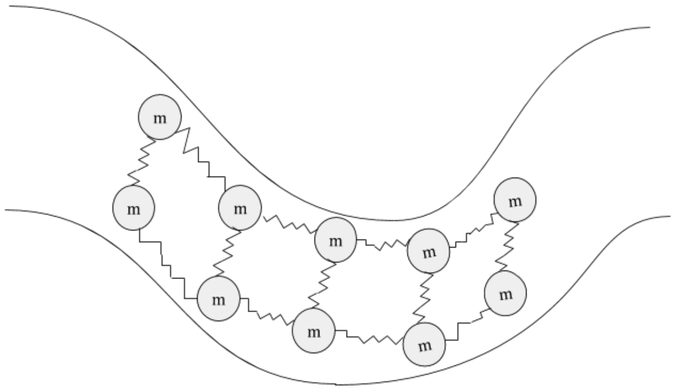

Figure 3.3: *Hooke's law applied to the context of a bending sheet or rod*

Every small segment dm is correlated to the neighboring segments with springs. For example: if the segment is inside the plate, the segment is attached with six springs (figure 3.4). From this it could look like, the plate equation can be derived only by using Hooke's law (and coupled differential equations). But there is another problem this comes from the fact that the spring forces will stretch or compress the segment, and Hooke's law only works with masses that are well defined in space and does not change shape. This is because, if a segment is stretched or compressed, there will be some energy change of the system, and this is not taken into account in Hooke's law, therefore it can not be used to make a general model for the plate.

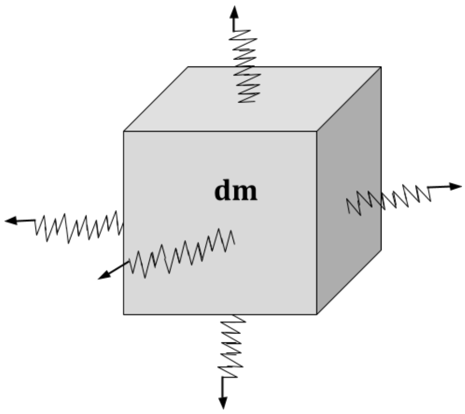

Figure 3.4: *The 3D case for coupled oscilation of a mass. Sometimes reffered to as a tensor. To model the system of a vibrating plate, we say every small segment dm (pictured) is correlated to the neighboring segments with springs*

3.2 Standing Waves

A classical system of a standing wave is a string with fixed ends. When a wave is forced to travel from one end to another, it will reach the other end without oscillating it. The disturbance is reflected to meet the primary wave and interfere with it. These two interfering waves have the same wavelength and amplitude and are travelling at the same speeds. This is because the speed of a wave on a string depends only on the length (L) of the string and the density (μ) of the string. This is written as (9):

$$v = \sqrt{\frac{L}{\mu}} \tag{3.13}$$

There is also a simple mathematical relation between wave velocity, frequency f and wavelength λ that can be written as:

$$v = f\lambda \tag{3.14}$$

It can be shown with Newtonian physics that when the wave meets an end point, it will be reflected in a way such that the amplitude of the reflected wave has opposite direction of the incoming wave (see fig. 3.5). This means that the two waves will have a phase difference of 90^o (or $\pi/2$). The amplitude in a standing wave formed as a result of this superposition varies, the points at which there is no vibration hence the amplitude is equal to zero is called a node. The portion that is displaced from its equilibrium position is called a loop, and the maximum point of the loop (i.e where maximum displacement occurs) is called the antinode. Antinodes occur at the crests and troughs of a standing wave. These portions have different but stationary positions. These are the general properties of a standing wave on a string.

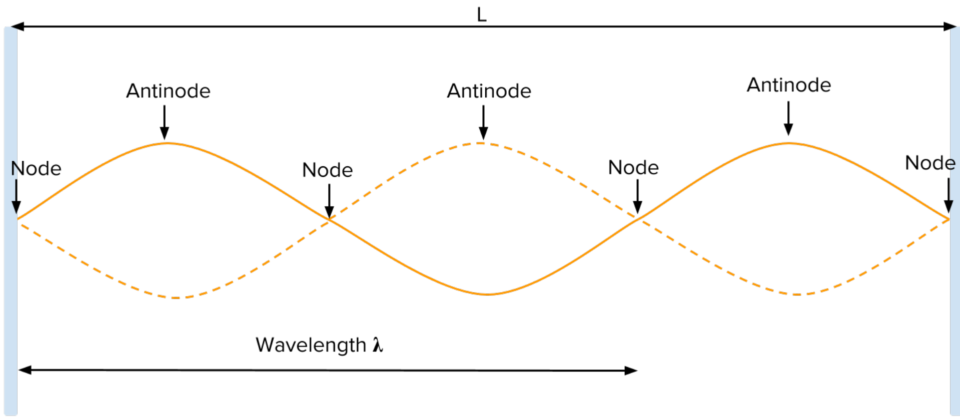

Figure 3.5: *Nodes and antinodes positioned on a standing wave. (1)*

The superposition of the primary wave and the reflected wave can be shown mathematically by the sum of the harmonic functions. If the primary wave travelling to the right

$$y1 = Acos(kx - \omega t) \tag{3.15}$$

interferes with the wave travelling to the left

$$y2 = Acos(kx + \omega t) \tag{3.16}$$

their sum will be

$$y1 + y2 = Acos(kx + \omega t) + Acos(kx - \omega t) \tag{3.17}$$

This sum is evaluated using trigonometric identities to give

$$y = 2Acos(\omega t)cos(kx) \tag{3.18}$$

this equation shows the dependence of the standing wave displacement on both time (represented by the factor $cos(\omega t)$), and its dependence on the x, (represented by the factor $cos(kx)$) which also describes the nodes and antinodes. The antinodes occur at the maximum value of $cos(kx)$ where $kx = 0,\pi,2\pi,\ 3\pi,...$ The nodes occur when the factor $cos(kx)$ is zero for $kx = \pi/2,\ 3\pi/2,\ 5\pi/2,...$ These values match with phase difference for constructive and destructive interference and are illustrated in figure3.5.

If the standing wave on a **plate** is considered, the nodes become a series of nodes on a line called the *nodal line*, or nodal lines, if there are more than one as in 3.6.

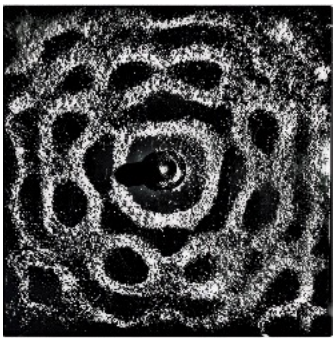

Figure 3.6: *One of the modes for an Aluminum plate representing nodal lines where salt is accumulated and displacement in empty regions*

The nodal lines on a vibrating plate can not be visualized the same way they would be on a string because the deflection distance is so small that it cannot be seen with the naked eye. However, it can be observed by sprinkling salt or any fine substance on the plate inspecting where the salt accumulates and eventually remains static. Antinodes and loops are the portions where displacement occurs, therefore they are the region where there is continuous movement and hence no accumulation of salt (see 3.6). The accumulation and continuous movement of the salt on the plate forms the patterns for each specific frequency. The frequencies at which patterns are formed is the natural frequency of the plate. One of the parameters investigated in this research that influence these pattern formations is the plate's material stiffness.

3.3 Vibrations in a Plate

The phenomenon of vibrations in a plate is simply a specific case within the general field of mechanical vibration. In its essence, it is applying the theory behind bending and vibrations to a plate. Mechanical vibrations in a plate are of great interest to many due to their cymatic behavior. As a general overview, When a plate is subject to a load (i.e mechanical vibration), it is displaced from its equilibrium state and responses with its vibration. When the plate vibrates at its natural frequencies, patterns can be observed. The patterns that form on the surface of a plate are a product of many parameters. The following section will introduce which parameters are deemed relevant in the scope of vibrations in a plate. The conditions that determine the behavior of waves and thus nodal/anti-nodal lines in a plate that include:

- Thickness of the plate
- Geometry
- Boundary conditions
- Properties of stiffness

Thickness of the plate: Considering the theory regarding *thin plates*[3]. The plate has to be of a specific thickness as a plate can either behave as a thick plate, a thin plate, or as a membrane. The governing equations for a plate get more complicated if its third dimension is considered which is the case in thick plates. However, the governing equations for vibrating membranes (the 2-D case) can give excellent approximations for the case of a plate. This is because the magnitude of third dimension of the plate (i.e the thickness, z axis) can be assumed negligible when compared to the other two dimensions (lengths a,b, for the x,y axes) based on thin plate theory. However, a membrane model does not take into account the material parameters that this research aims to investigate. Therefore, the thickness of plate is taken into account in the governing equation for a thin plate but not as a third dimension.

[3]Discussed in later sections

Geometry: The vibrating plate can be of any shape and any geometry which is related to the shapes of patterns formed. One of the most commonly used plate shapes is a circular plate. For a square plate, which is the shape chosen in this study, the conditions are somewhat complicated, since the waves are starting out as a circular wave, much like a the wave from a drop of water into a basin or sink. The circular wave will be reflected in a way that is, and will make much more complex wave-fronts when reflected.

Boundary Conditions: The boundary conditions restrict what happens at the end points of a standing wave, and is satisfied by describing the amplitude of the wave at the end. For example, for a string with fixed ends, the boundary conditions are satisfied if the endpoints are node of zero amplitude. In the case of free boundary condition, the endpoints are anti-nodes. Boundary conditions are restricted in any system, in a plate there is generally 4 different boundary conditions:

- free plate
- Clamped plate
- Supported
- Combination of condition

Different boundary conditions affect the behavior of the plate due to different reflection directions of the waves. This means that the plate's resonant frequency will be different for different boundary conditions. Despite the complexity of free boundary condition when compared to the frequently used clamped plate due to its simplicity (16), it is assumed for the plates in this study because it is more experimentally feasible to apply.

Properties of Stiffness This parameter, which is investigated, is dependant on several properties of the metal of the plate. A stiff material is defined as a material that requires high loads to be elastically deformed or as the resistance to deform property. Besides its mechanical properties, stiffness depends on many characteristics of the object that is subjected to deformation, however in this context, *stiffness* means stiffness properties related to the mechanical properties of the material, namely, mass density, Poission's ratio, and Young's modulus.

3.3.1 Material Properties of a Plate

This section discusses the material properties of a material related to stiffness.

Mass Density: The relationship between mass, stiffness, and vibration was already defined by Hooke's Law.

Poisson's Ratio

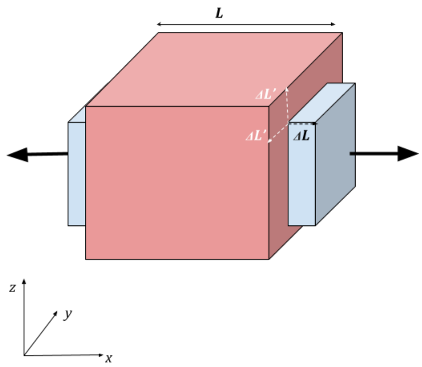

Figure 3.7: *Visualization of Poisson's Ratio in a standard ductile cube.* $\nu = -\frac{u_{yy}}{u_{xx}}$ *indicates that the strain in the y direction is divided by the strain in the x direction to yield a result of either compression or expansion, depending on the sign of the values.*

The compression or expansion of a material in one direction causes its expansion or compression in the direction parallel to the direction of applied force. If a force is applied to extend or compress a (ductile) material in the x direction, then length in the y direction will decrease or increase. This is to say that lateral strain[4] is in the direction perpendicular to the pull.

Normal materials will always contract in directions transverse to their direction of extension. If the transverse side, which we can call the "diameter" D, of a rod or spring changes by amount y , then the transverse strain becomes of order $u_{yy} = \frac{y}{D}(4)$. In a linear elastic material the transverse strain u_{xx} must be proportional to F,so that the ratio $\frac{u_{yy}}{u_{xx}}$ is independent of F. The ratio is called *Poisson's ratio* and is represented by:

$$\nu = -\frac{u_{yy}}{u_{xx}} \tag{3.19}$$

Young's Modulus

Solid bodies made of homogeneous and isotropic materials will act as springs when stretched. However, when using a rod instead of a spring, one must to take other parameters into consideration if analyzing said stretching. For example, when stretching a spring with spring constant k, a one dimensional problem is only considered. However, when stretching a rod, one must also take the rod's cross section into consideration. This could be likened to constructing a spring

[4]Strain: The magnitude of elongation over the compression with respect to the original length

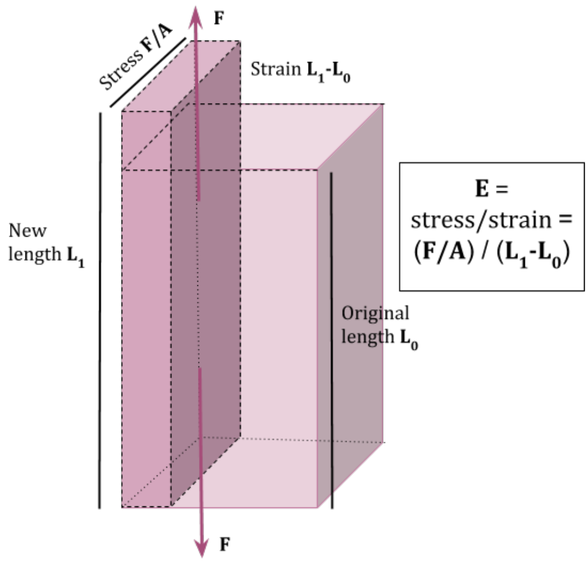

Figure 3.8: *visualization of Young's Modulus in a standard, rectangular cube.*

that has a spring constant N times stronger than one dimensional spring. Thus N of springs is needed of the same type. The effect of Young's modulus is illustrated in figure 3.8

If a spring has a cross section A, then the total cross section of the springs will be $N \cdot A$. If the force from a single spring is $F = kx$ then the force from N springs will have a resulting force of $N \cdot F$. This gives rise to a ratio(4):

$$\frac{N \cdot F}{N \cdot A} = \frac{kx}{A} = \sigma_{xx} \tag{3.20}$$

which is the *normal stress* applied per unit area. Another important property is the *Normal Strain* (u_{xx}). It comes from the fact, that the force on any given segment on a stretched spring (or rod) is the same every where. So if one have a spring, with length L and extension x and look at piece of spring with length $L' < L$, then the amount of extension for the piece must be x'. Normal strain can be represented by the ratio:

$$\frac{x}{L} = \frac{x'}{L'} = u_{xx} \tag{3.21}$$

The ratio of the normal stress to the normal strain defines *Young's modulus* also called *modulus of extension* :

$$E = \frac{\sigma_{xx}}{u_{xx}} = \frac{F/A}{x/L} \tag{3.22}$$

This is a material parameter and it is independent of x, A and L, from a unit perspective it can be defined as

$$[E] = \frac{\frac{N}{m^2}}{\frac{m}{m}} = \frac{N}{m^2} \tag{3.23}$$

As seen Young's Modulus has the unit of force per square meter or Pascals [Pa].

3.4 Wave Systems

In order to understand the vibration in a plate and interpret its wave equation, the vibration of a string and a membrane system are examined in this section. The most common case for a one-dimensional system is the vibration of a string. The most common two-dimensional systems are the vibration of a membrane. However, a string and a membrane can be categorized under non-stiff systems unlike a plate. Another system that is likened to the plate is the one dimensional rod. Despite its one dimensional property, its behavior is likened to a plate since it has stiffness, and can vibrate with fixed or free ends and with or without any external tension as their elastic forces supply the necessary restoring force, unlike a string and a membrane whose oscillation is caused by external tension application. A rod is the simplest continuous mechanical vibrator (10), and both rods and plates can transmit longitudinal, shear, torsional, or bending waves. Stiff Solid materials, which have both shear and compressed elasticity such as a rod, allow the propagation of both longitudinal and transverse waves (14).Likewise, the vibration of a plate can be likened to the vibration of membrane that has stiffness. Therefore, the vibration of a thin plate can be considered as a three-dimensional bar vibration or a non-ideal stiff membrane. The equations of motion for a string and a membrane are explained in the following section. In this paper, the vibration of a thin plate is likened to these systems and their equations are physically interpreted.

The equations of motion of the vibration of these non-stiff systems both differ and correlate with the plate equation to certain extent. The simplest example of the hyperbolic differential equation is the wave equation, which is a second order partial differential equation that describes the wave motion. These equations' dependent variable is the displacement in a one or more space dimensions denoted by w. The independent variables besides time t can vary from one space dimension variable to more. In the one dimensional string case which is located in the $x - y$ plane, the displacement of the wave W is in the y direction which depends on both time and the x space dimension. A higher dimension equation can describe a membrane. For example, for a membrane in a rest position in the $x-y$ plane, the dependent variable w, is the wave displacement in the z direction, this variable depends both time t and the x and y plane dimensions. The acceleration of the membrane arises from the unbalanced tensions coming from both x and y directions. In the same way, the thin plate equation is described by two-dimensional differential equations.

3.4.1 Waves on a String

The fundamental principles and equations for waves in materials are easier to understand in an ideal string. The string example is non-stiff and is one-dimensional, which of course will exclude important parameters this research aims to investigate. The stretching of a string between two points along the x-axis, means that the string has a tension T, and that every segment dm has a tension T from both sides 3.9.

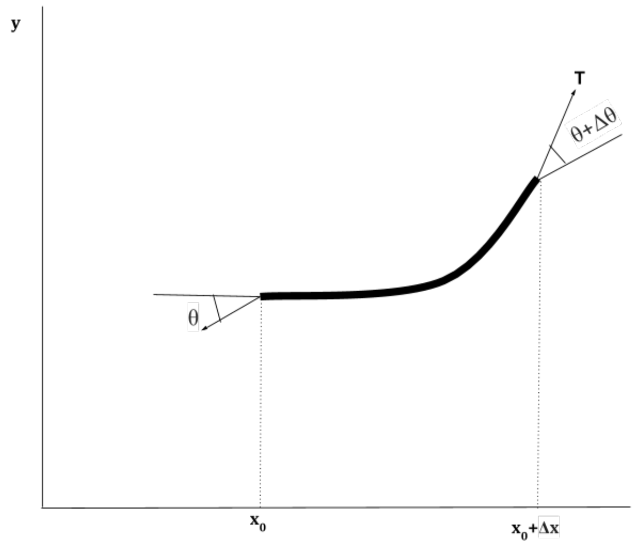

Figure 3.9: *line segment on a stretched string*

We now let a force F_y, which is perpendicular to the string, make a small displacement of the string. The force will make a change on the tension on the string, which can be written as:

$$F_y = -Tsin(\theta) + Tsin(\theta + \Delta\theta) \approx -T\theta + T\theta + T\Delta\theta = T\Delta\theta \tag{3.24}$$

By approximating for small angles: $sin(\theta) \approx \theta$.

Going back to the force, it is defined as (9):

$$F_y = dm\frac{d^2y}{dt^2} \tag{3.25}$$

and using the substitution os $dm = \rho\Delta x$,where ρ is the density of the string. equation 3.25 is written as:

$$\rho\Delta x\frac{d^2y}{dt^2} = T\Delta\theta \tag{3.26}$$

The angle θ has the relation $tan(\theta) = \frac{dy}{dx}$. For small angles, $sin(\theta) \approx \theta$ and $cos(\theta) \approx 1$, so

$$tan(\theta) = \frac{sin(\theta)}{cos(\theta)} \approx \frac{\theta}{1} = \theta$$

This is used to rewrite the relation between the angle θ and the displacement the y-direction:

$$\theta \approx \frac{dy}{dx} \Rightarrow \frac{d\theta}{dx} \approx \frac{d^2y}{dx^2} \tag{3.27}$$

For small angles:

$$\Delta\theta \approx \Delta x \frac{d\theta}{dx} = \Delta x \frac{d^2y}{dx^2} \tag{3.28}$$

And putting equation 3.28 into 3.26 we now have:

$$\rho \Delta x \frac{d^2y}{dt^2} = T \Delta x \frac{d^2y}{d}^{x^2} \Leftrightarrow \rho \frac{d^2y}{dt^2} = T \frac{d^2y}{d}^{x^2} \Leftrightarrow \frac{\rho}{T} \frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} \tag{3.29}$$

Taking a new constant c to write the relation:

$$\frac{1}{c^2} = \frac{\rho}{T} \tag{3.30}$$

The one dimensional wave equation is derived:

$$\frac{1}{c^2} \frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} \tag{3.31}$$

Where c is to be the velocity of the wave, where a relation between the tension T the material parameter (the density) ρ and the velocity c is given by (3):

$$c = \sqrt{\frac{T}{\rho}} \tag{3.32}$$

Information received this from this derivation is that the mass density of the string is a factor of the speed of the wave and that there is direct relation between Newtons second law ($F = m\frac{d^2y}{dx^2}$) and the wave equation. The patterns observed for a string as previosly mentioned can be visualized by its harmonics modes as in figure 3.10

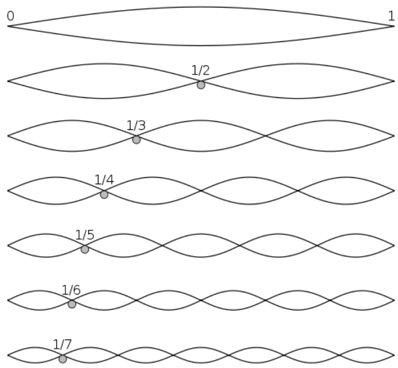

Figure 3.10: *different modes in a one dimension system of standing waves with clamped ends. In acoustics and music theory, this known as the classical model for the 'harmonic overtone series'(5). The algebraic representation of the number modes that exist, also known as the harmonic series, is*

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$$

3.4.2 Standing Waves in a Vibrating Membrane

The vibrating string is the basic one-dimensional problem, the two dimensional problem is described by an elastic membrane vibration. The ideal membrane case is considered to be a simple system the plate since it has no stiffness. Its equation of motion is derived from considering the components of forces acting on the plate and by applying Newtons' Law as in the string case. Consider a two dimensional, clamped, elastic, and isotropic membrane under tension. A simple everyday example of this system is [an ideal] drum-head. We can thus say that this system supports transverse vibrations (8).

The governing equation describing the vibration of a membrane is the same as the general equation for a string which is

$$\nabla^2 w = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \tag{3.33}$$

in a membrane system which is two dimensional

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \tag{3.34}$$

which gives

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \tag{3.35}$$

solutions to the wave equation 3.35 are discussed in the next sections.

As seen in the last section, we can say that the speed of the propagating wave is proportional to the tension(T) of the system as as well as the mass density (ρ). The same idea applies to a membrane, where constant c has a unit of velocity defined as in the one-dimensional case.

As the natural frequencies of a string relate to the normal modes of a plate, the normal modes of a membrane also relate to the normal modes of a plate, in fact, they correlate to a certain degree. At this material's resonant frequencies, the membrane will move in a pattern of standing waves of the same frequency, this is said to be the normal mode. There exists a fundamental mode as seen in figure 3.11, and thereafter an infinite amount of modes. The fundamental mode corresponds to the lowest input frequency, and as the frequency increases, so does the number of modes. It is important to note that a normal modes' continuous form is a standing wave. For a mode to exist in this system, there must be a medium under vibration (the rope), a stress on said medium, and a modal variable.

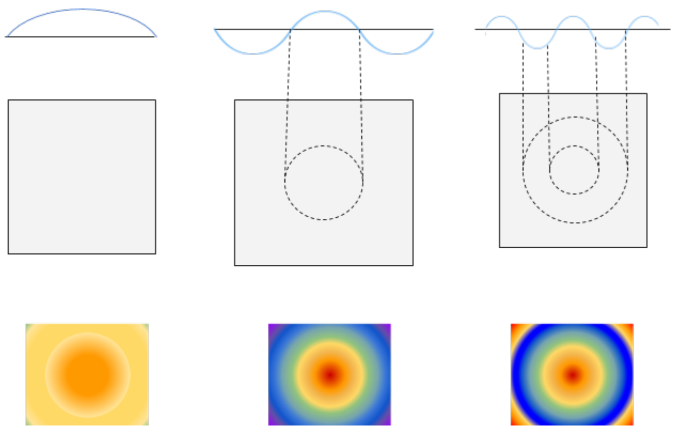

Figure 3.11: *visualization of the wave equation for a square membrane. warm colors represent positive displacement along the zaxis, cool in negative, and yellow/green means minimal to 0 displacement.*

As in the fixed string case, the natural frequencies for a rectangular or square membrane with clamped boundary conditions and lengths (a) and (b), can be found by the following equation (3),

$$w(x, y, t) = sin(\frac{\pi n x}{a})sin(\frac{\pi m x}{b})cos(\omega t) \tag{3.36}$$

where $w(x, y, t)$ is the amplitude of the membrane's displacement over time. These equations which are unique solutions of the general wave equation are found by implementing boundary conditions as in any system.

4 Wave Equation in a plate

This section describes and introduces the governing wave equations for a vibrating thin plate. When an external force is applied to a plate, an internal stress occurs and the plate resists the applied load by bending and twisting forces. among other factors, the response of the plate depends on its bending stiffness which is defined by the Flexural Rigidity. Flexural Rigidity is the resistance offered by the plate when it undergoes bending which is similar to the definition given to material properties of stiffness except that flexural rigidty does not take into account the mass density of the material. Plates are continuous elastic systems, unlike the systems discussed earlier, plates have stiffness and can be considered as 2 dimensional systems when thin. The advantage of a thin plate, in which there is a typical thickness to width ratio, is utilized to reduce the three dimensional problem into two according to the thin plate theory approximation. In order to study the formation of patterns for certain natural frequencies, the effect of vibration on the performance of the plate and the internal forces are analyzed (13), this is done by the study of structure dynamics of a plate which covers the behavior of it under time-dependent loading (which this study does not investigate). A plate dynamics can be mathematically modelled by partial differentiation based on the application of Newton's Law or by integration equations of stresses and strains. There is various types of motions of plates, when a time-dependent load is applied the vibration response is called a forced vibration which could be either harmonic or not. The differential equation of a forced vibration of a plate is:

$$D\nabla^2\nabla^2 w(x,y,t) = p(x,y,t) - h\rho\frac{\partial^2 w}{\partial t^2}(x,y,t). \tag{4.1}$$

(15)

where ρ is the mass density of the plate; h is the thickness of the plate; p is the time dependant forced vibration; w is the plate deflection in the transverse direction;∇^4 which denotes the bi-harmonic operator, is the square of the Laplacian operator where ∇^4= $\nabla^2(\nabla^2)$ defined as:

$$\nabla^2\nabla^2 = \nabla^4 = \partial^4 x + 2\partial^2 xy + \partial^4 y \tag{4.2}$$

and D is the *Flexural Rigidity*, for a plate defined by:

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{4.3}$$

where E is Young's modulus, h is the plates thickness, and v is Poisson's ratio.

A plate responses with a free vibration when there is no applied load, therefore $p(x, y, z)$ which causes a dynamic response is zero, the equation becomes as the following linear homogeneous partial differential equation when damping is ignored:

$$D\nabla^2\nabla^2 w(x, y, t) = -h\rho \frac{\partial^2 w}{\partial t^2}(x, y, t) \tag{4.4}$$

(15) The free vibration is only dependent on the characteristics of the plate and is initiated by the application of initial conditions where w satisfies both the boundary conditions and the following initial conditions.

at t=0: w=w$_o(x, y)$, $\frac{\partial w}{\partial t} = v_o(x, y)$ (15)

For simplicity purposes, the equation of the free vibration of a plate is written in the following form:

$$D\nabla^4 w + h\rho \frac{\partial^2 w}{\partial t^2} = 0 \tag{4.5}$$

Plates are not classified only based on their shapes, mechanical properties, or boundary conditions but also their thickness. The most extensive group of plates, when classified according to their ratio of dimension to thickness, is the intermediate type of plate which is not thick, in which there is not a third dimension considered, nor too thin, in which it acts like a membrane that has no bending stiffness. This ratio is within the range of 10...80 $\leq \frac{a}{h} \leq$ 80...100 (15), where a is one of the plate's dimensions and h is its thickness as seen in the following figure.

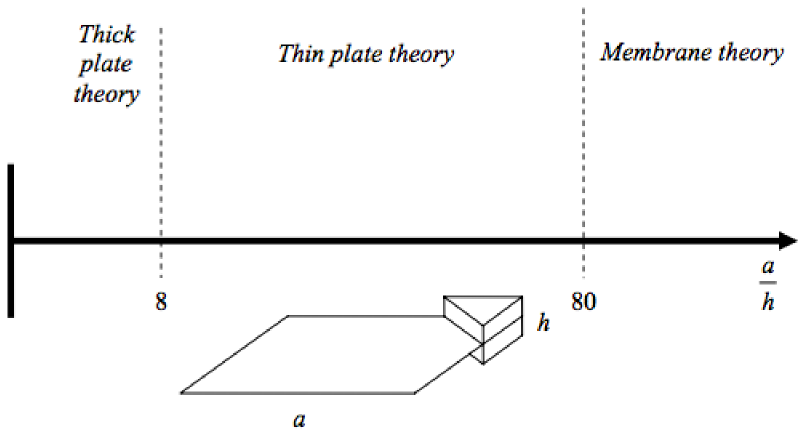

Figure 4.1: *Dimension by thickness ratio limits distinguishing thick, thin plate and membrane theory (15)*

The plate bending theory is represented by both The Classical Plate Theory and Kirchhoff's Plate Theory which are based on the same fundamental assumptions [5] stated below adapted from (15).

- The plate material is isotropic, homogeneous and linear; their mechanical properties are the same throughout the plate directions and points, and the plate is initially flat.
- Deflection is small compared to thickness of plate
- The hypothesis if straight normals: vertical shear strains and the normal strain are negligible.
- The stress component normal to the middle plane is small compared with the other stress components and is neglected.

Under these assumptions, of which Kirchhoff has proposed a few, the three-dimensional plate problem is reduced to a two-dimensional one. The common plate theories that are based on these assumptions are the Kirchhoff's plate theory and the Classical plate theory. Another plate theory that does consider the third dimension and often referred to as the thick plate theory is the Mindlin-Reissner Plate Theory. The main difference the between Kirschoff's and Mindlin's plate theory is that Kirschoff's assumptions ignore any shearing of cross section and rotational inertia. This means that the Mindlin theory will yeild more complicated yet more accurate results.

4.1 Physical Interpretation of the Wave equations

This section aims to interpret the difference between the general wave equation and the flexural wave equation for a freely vibrating plate, for the purpose of comprehending the significance of its terms.
The general wave equation of a membrane:

$$\nabla^2 w = \frac{1}{c^2} \frac{\partial w}{\partial t^2} \tag{4.6}$$

The flexural wave equation for a freely vibrating plate:

$$D\nabla^4 w + \rho \frac{\partial^2 w}{\partial t^2} = 0 \tag{4.7}$$

there are both differences and similarities between the two equations. Both equations have a second-order dependence on time. In Newtonian terms, this is the expression of acceleration. The plate equation has the constants D and ρ, in which both are material parameters, while the general wave equation has the constant c, defined earlier by tension and mass density.

[5]The consequences of these assumptions are not examined s they are beyond the scope of this research.

The General wave equation clearly does not take into account properties like Young's modulus and Poisson's ratio. Another major difference between the two equations are in their Laplacian: ∇^2 versus ∇^4 (or$\nabla^2 \cdot \nabla^2$) for the membrane versus the plate, respectively. The general wave equation is in its ideal form, describing a wave which is propagating through material without disturbing it. The wave equation for the plate describes the vibration of the material itself.

Another system that should be introduced in order to interpret stiffness and deformation is simpler system of plate in which it is a rod. This is a valid comparison, as a rod describes the general idea and physical interpretation in the same way and its behavior is likened to the plate behavior as suggested by (14). The rod used for this description is not necessarily an ordinary rod. It has length and height, but it has a significantly small width. It is essentially a thin sheet. If we look at a small segment (filament) of the rod (see illustration 4.2) located a distance z under the natural axis of the rod. This filament is compressed an amount $d\theta$. This is defined as $z\frac{d\theta}{dx}$ where z is perpendicular to the length of the rod, and is related to the width of the rod. By using the relation $tan(\theta) = \frac{dy}{x}$,where θ is the bending angle and x is in the direction of the length of the rod, and y is the direction of vibration. For small angles, $sin(\theta) \approx \theta$ and $cos(\theta) \approx 1$, so

$$tan(\theta) = \frac{sin(\theta)}{cos(\theta)} \approx \frac{\theta}{1} = \theta$$

This we use to rewrite the strain equation:

$$\theta \approx \frac{dy}{dx} \Rightarrow \frac{d\theta}{dx} \approx \frac{d^2y}{dx^2} \tag{4.8}$$

And the strain can be written as:

$$z\frac{d\theta}{dx} \approx z\frac{d^2y}{dx^2} \tag{4.9}$$

-where the term $\frac{d^2y}{dx^2}$ tells us how rapid the amplitude changes over a range dx .
From this we can get the amount of force to produce the strain:

$$dF = EdSz\frac{d^2y}{dx^2} \tag{4.10}$$

- where dS is the cross section area of the filament and E is Youngs modulus. So an amount of force is required to bend the filament. The amount of force can be described as(14):

$$dM = dFz = (EdSz\frac{d\theta}{z})z \tag{4.11}$$

If we integrate this, we will get the *total moment* that is required to compress the filament:

$$M = \int dM = E\frac{d\theta}{dx}\int z^2 dS \tag{4.12}$$

It is customary to define a constant K called the *radius of gyration* of the cross section:

$$K^2 = \frac{1}{S}\int z^2 dS$$

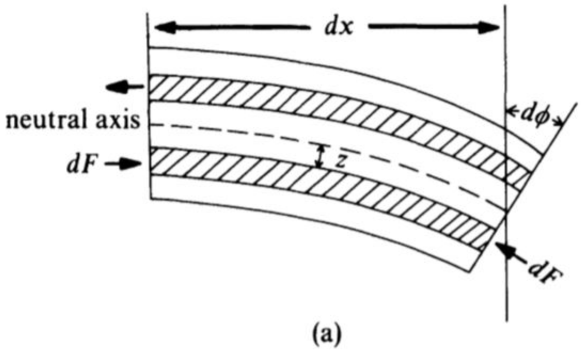

Figure 4.2: *The bending of a filament. figure taken from "Principles of Vibration and Sound",T. D. Rossing and N. H. Fletcher,2004.*

This gives us the total moment:

$$M = E\frac{d\theta}{dx}SK^2 = -ESK^2\frac{\partial^2 y}{\partial x^2} \tag{4.13}$$

But the bending moment gives rise to a *shear force*:

$$F = \frac{\partial M}{\partial x} = -ESK^2\frac{\partial^3 y}{\partial x^3} \tag{4.14}$$

Further on, the shearing force is not constant, because of a net force which is perpendicular to the axis of bar:

$$dF = (\frac{\partial F}{\partial x})dx = -ESK^2 dx\frac{\partial^4 y}{\partial x^4} \tag{4.15}$$

A line segment dm, as mentioned in the last section, can be written as: $dm = \rho S dx$ This gives us a displacement force:

$$dF = (\frac{\partial F}{\partial x})dx = \rho S dx\frac{\partial^2 y}{\partial t^2} \tag{4.16}$$

This gives us the wave equation we are looking for:

$$\rho S dx\frac{\partial^2 y}{\partial t^2} = -ESK^2 dx\frac{\partial^4 y}{\partial x^4} \Leftrightarrow \frac{\partial^2 y}{\partial t^2} = \frac{-EK^2}{\rho}\frac{\partial^4 y}{\partial x^4} \tag{4.17}$$

- and if we establish that $\frac{D}{\rho} = \frac{-EK^2}{\rho}$ we are where we want to be, back at equation 4.7

This shows that the difference between the two wave equations comes from the shearing force and a net force perpendicular to the axis of the bar.

4.2 Wave Equation Solutions

In order to answer the research question from a theoretical point of view, the wave equation can be utilized to find the natural frequencies since solving the wave equations gives the vibration of the system at any frequency. Analytical and numerical solutions for the modal behavior can be obtained, which are key indicators of what the patterns formed look like. These solutions do not only show the influence of stiffness on the patterns formed but are also used to evaluate the observations found from the experimental results. The analysis of the solutions to the plate equation is described as a complicated problem especially that for a free plate (14), and involves analysis and mathematical methods that are beyond the scope of this research. The same applies for a rod, therefore, this study does not utilize the plate wave equation to investigate the effect of stiffness on these solutions, which is an excellent method for checking the reliability of any experimental results. However, this section introduces an example solution of the membrane wave equation by the separation of variables method. The wave equation of a membrane is very similar to the simple one dimensional equation of a string with an second dimension which will not result in any complications while solving the equation.

In a membrane $w = w(x, y, t)$ obeys the wave equation:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{c_o^2}\frac{\partial w}{\partial t^2} \tag{4.18}$$

Because the x and y can be treated separately, the wave function can be factorized in a product form where $w(x, y, t) = X(x)Y(y)T(t)$ or more simply $w = XYT$. This is called *separation of variables* and is the technique used to mathematically solve the membrane wave equation.

The wave function can now be expressed in a new form when partially differentiated (12)

$$\frac{\partial^2 X}{\partial x^2}YT + X\frac{\partial Y}{\partial y^2}T = \frac{1}{c_o^2}XY\frac{\partial T}{\partial t^2} \tag{4.19}$$

Which can be expressed more simply as:

$$X''YT + XY''T = \frac{1}{c_o^2}XYT'' \tag{4.20}$$

dividing both sides by XYZ to get:

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c_o^2}\frac{T''}{T} \tag{4.21}$$

From this, it can be seen that the left-hand side depends on both x and y, and the right-hand side only depends on t.
Since x,y and t are independent, the only solution to equation is that all terms must be equal to a constant. This can be expressed as:

$$\frac{X''}{X} = -\alpha^2, \frac{Y''}{Y} = -\beta^2, \frac{1}{c_o^2}\frac{T''}{T} = -\mu^2 \tag{4.22}$$

Substituting these constants in equation 4.21.

$$-\mu^2 = -(\alpha^2 + \beta^2) \tag{4.23}$$

The important task isn't the simplicity of the function, but the fact that it can be divided up into separate solutions for each of the independent variables. The solutions can be looked up in most mathematics books that deal with PDE's ,and for that reason we will not use space here to describe them but merely write the general solutions where X depends on x and not y or t; Y depends on y and not x or t; T depends on t but not x or y :

$$X(x) = Acos(\alpha x) + Bsin(\alpha x) \tag{4.24}$$

$$Y(y) = Acos(\beta y) + Bsin(\beta x) \tag{4.25}$$

$$T(t) = Ecos(c\mu t) + Fsin(c\mu t) \tag{4.26}$$

Where *A*,*B*,..,*F* are constants which are related to and can be found from the boundary conditions. These functions can also be written in complex form.
The general solutions obtained above can be limited to unique solutions by applying the initial and boundary conditions. A membrane compared to the plate studied in this research will have free boundary condition as in the experimental setup. Setting the correct boundary conditions is crucial for developing the right equations. The boundary conditions for a free membrane or plate are not geometrical but rather are forces and moments. These are conditions which are not covered in this investigation.

4.2.1 Solutions to wave equation and normal modes

This section relates the solutions to the membrane equation to the patterns formed. The nodal lines on a membrane are represented by m and n in the x and the y direction. This applies for both a membrane and a plate. The fundamental mode is when m=n=1. A visual representation of simulated patterns can be seen in figure 4.3.

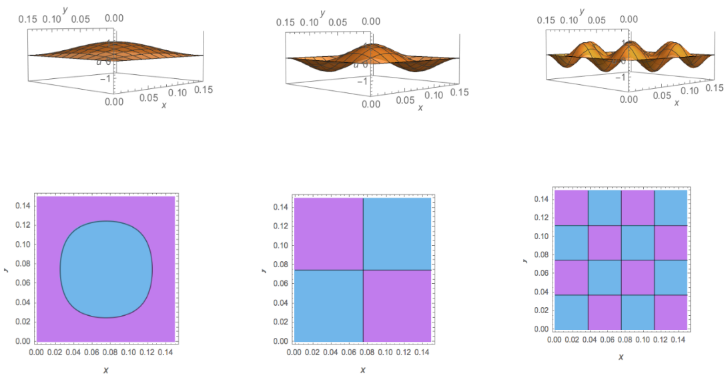

Figure 4.3: *modes 1-3 in a square membrane (a=b=150mm),where m=n. for the modes 1-3, visualized in a mesh plot and 2-D plot.from left to right: m=n=1, m=n=2, m=n=3. Code taken from (3)*

In order to solve the system for finding the modes, we solve for the angular frequency ω of the wave, with respect to m and n. This can be described as:

$$\omega = \pi c \sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}} = \pi c \sqrt{\frac{n^2 + m^2}{a^2}} \tag{4.27}$$

The lengths a and b are equal in this case as we are studying square membranes, not rectangular. The solution for ω_{mn} is crucial for this project. Since it relates to the frequency (Hz), a known variable in our experiments, we can consequentially see where our modes should appear according to our frequency. This solution is obtained from solving for unique solutions by restricting boundary conditions for a clamped membrane from the general wave solutions obtained by separation of variables previously.

In this same way the solutions to the wave equation can analytically and numerically support the prediction of patterns on a plate. For example, in figure 4.4 the first 5 modes of frequency on a fully clamped plate (a) and a fully clamped membrane in (b), show the positions of the nodal lines which are m and n in the solution.

As seen in figure 4.4, the first 3 modes for a plate and a membrane look the same number of nodal lines and displacement regions. However, the behavior of the plate and membrane

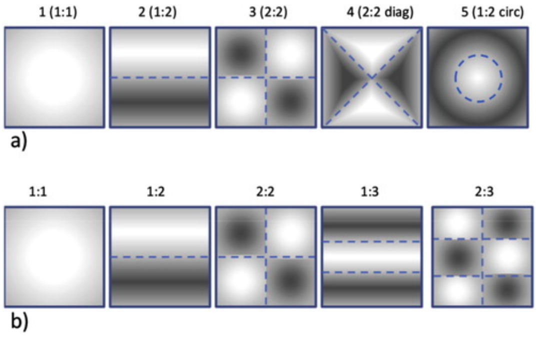

Figure 4.4: *The first resonance mode of a fully clamped square plate (a) and a fully clamped square membrane (b) (the dashed lines represent nodal lines and the the dark and light regions vibrate away from and towards the page plane respectively(6)*

after the third mode, varies significantly not only by the number of modes but also their shapes being curved or straight. Considering these data found in literature, a plate when compared to a membrane tends to form more curved shapes. However, this is true for square and clamped boundary conditions and not necessarily for other conditions. The modes of a plate which could be derived from the modes for a rod can be of different types for the same plate, for example, some of the mode could be a twisting or bending ones (14)

5 Experiment on Vibration of a thin plate

To answer the research experimentally, an experiment of mechanical vibration on a plate with specific characteristics is conducted. In order to study the influence of some mechanical parameters, which are flexural rigidity and mass density on the plate vibration, which can be observed by the patterns formed when salt is used for visualizing them, two plates of different mechanical properties are used and the frequency is varied until the natural frequencies are reached, the patterns formed are interpreted by description only. The plates stiffness can be varied by varying the type of material for a plate, in this study aluminium and stainless steel metal plates are used, the material mechanical properties of these plates are represented in table 1

plate	*Young's Modulus (GPa)*	*Poisson's ratio*	*Mass Density (g/cm^3)*	*Flexural Rigidity (Pa m^3)*
aluminium	68	0.33	2.71	6.31
stainless steel	197	0.29	7.9	17.9

Table 1: *Material mechanical properties of the plates used experiments. It should be noted that their Poisson's ratio is rather similar, whereas the other material properties deviate much more drastically*

5.1 Experimental Setup

In order to conduct this experiment, there are several elements necessary: the plates described previously, an electric driven for our signal, a loudspeaker, an apparatus attaching the plate to the speaker such that it lies perpendicular to the speakers movement, and a fine, granular substance such as sand or salt for visualizing the nodal lines.

As seen in figure 5.1, our overall experimental set up is rather simple. To the left we have our oscilloscope, which generates our driving frequency. Here we manually change the frequency, which we can see on a screen, until we hit a natural frequency. Here you can change, most importantly; the wave form, coarseness, amplitude,and frequency range. IN this experiment we In the middle volt meter, which measures the incoming voltage. This is important for an experiment with electrical signals because you must keep track of your voltage so as to not overload the system. Our wires continue their circuit towards our loud speaker, which then translates the impulse into vibration. This vibration causes a small attachment to move the plate transverse to the propagation of the wave. The plate is screwed onto the attachment with a bolt, which holds it stable so that our surface is as level as possible. On the surface of our plate we have our granular media, in this case it is salt.

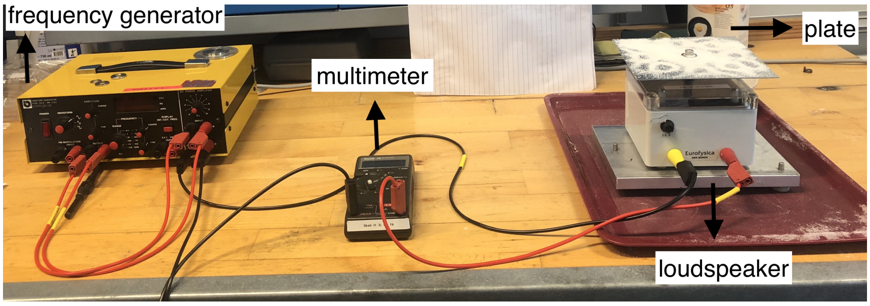

Figure 5.1: *An overview of our set up. To the left we have our oscilloscope, which generates our driving frequency. In the middle volt meter, which measures the incoming voltage. The wires continue their circuit towards our loud speaker, which then translates the impulse into vibration. This vibration causes a small attachment to move the plate transverse to the propagation of the wave.*

In this study's experiment, the mechanical vibration is driven from an electric generator and transited through a loud speaker, to give a more explicit and direct way of varying frequency. This in turn allows the identification of more precise values for different resonances corresponding to different nodal patterns. The plate lies on top of the vibrating rod of the loudspeaker such that they are perpendicular to each other and the rod is connected to the center of the plate. The plates have free boundary condition, therefore no clamping setup is required. The controlled variables of the plate are represented in the table (2). The aspect such as the exact amount of salt placed on the plate is a trivial variable, seeing as the small mass doesn't actually affect the displacement abilities of the mechanism. Additionally, the salt is merely used as a way of visualizing the nodal points, and doesn't truly affect where they are formed (2).

plate	*thickness(mm)*	*Dimensions (mm)*	*Boundary condition*	*Geometry*
Aluminum	1	150-150	free	square
Stainless steel	1	150-150	free	square

Table 2: *Characteristics of the plates used for experiments.*

5.2 Procedure for Determining Resonant Frequency

The issue of finding the resonant frequencies for plates is a matter of trial and error. Based on looking into past experiments to find the best feasible methods to determine precise resonant frequencies,and based on several trials with different methods, it was found that the optimal method is to slowly increase the frequency in small increments within the range of 10 Hz until a semblance of a pattern is observed, or when the fine substance on the plate begin to get displaced after being in a stationary state. At that stage, the frequency is fine tuned until the pattern is clear with fine nodal lines. Clear Patterns are defined by the highest contrast between nodal lines and anti-nodal regions. This was indicated by the highest amount of turbulence and disturbance along the anti-nodal points and the lowest disturbance on the nodal points. This method is verified by the research conducted in (7).

6 Results and Analysis

6.1 Collected data from experiment

From the collected experimental, 3 variables are defined, a mode , frequency, and a pattern. The patterns formed which vary by the number of elements or nodal lines and the shape and size of them, are arranged in order of modes, as the order in which they are formed on each plate as in 6.1, the resonant frequency however, is the frequency value at which a pattern is formed, which means that the same pattern can be formed on the two plates for the same frequency but not in the same order, as some patterns formed on one plate do not necessarily form on, which is what is observed in this experiment.

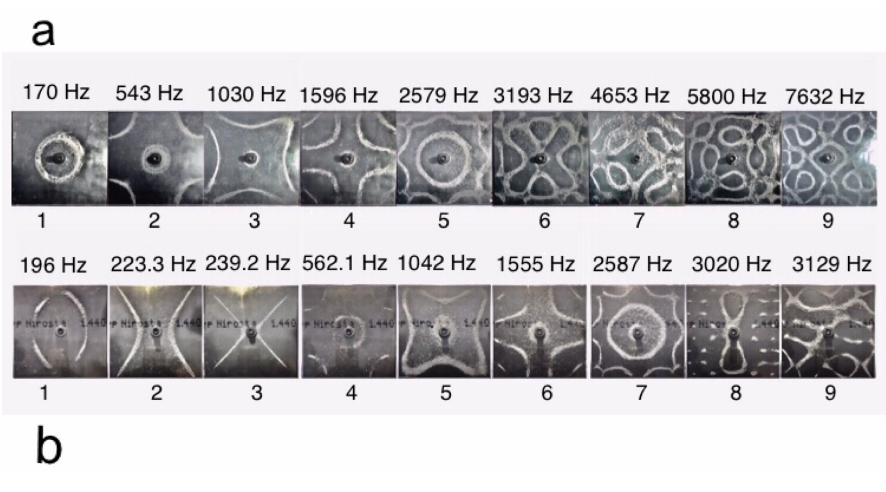

Figure 6.1: *The first 9 frequency modes of a free aluminum plate (a) and a free Stainless Steel plate (b) of the same thickness and dimensions*

Generally, it was observed that as the frequency increases the number of elements on a plate increases and the patterns become more complicated which is explained by the increase of wave crests and troughs as frequency increases which results in a higher probability of interference. There are many missing patterns when compared to other studies and the maximum number of patterns observed is about 14. However, this number is not the same for both plates, with aluminum more patterns were observed at higher frequencies which are discussed later while for stainless steel, there were more patterns observed at low frequencies within the range of 190-240 Hz, [6].

The frequency at which the first mode occurs is lower for an aluminum plate= 170 Hz and 196 Hz for stainless steel as seen in 6.1. The patterns however, look like they are forming an oval or a circle while this is clear for the aluminum plate, the patterns seem disturbed for stainless steel. Increasing the frequency after 500 Hz results in the formation of patterns that look the same and occur at similar frequency values this can be seen in the figure below 6.2.

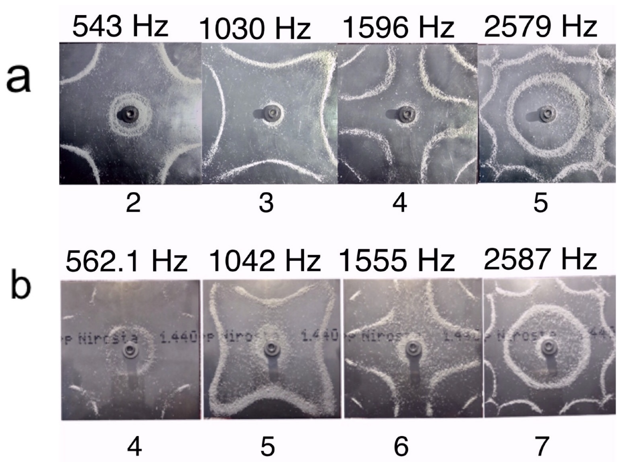

Figure 6.2: *Repeated Patterns for both plates starting form the second mode for an aluminum plate and the forth mode for a stainless steal plate*

From the four modes, it can be generalized that within this range of frequency 500-2600 Hz the same patterns occur for both plates but at lower frequencies for aluminum plate a. However, overall these modal frequencies do not vary significantly, this shows that the plates behavior is almost the same within this frequency range.

After reaching a frequency range of f= 3020-3193 Hz the patterns formed start looking significantly different as represented in 6.3. After this point, the patterns are not comparable by their order of number of modes but by their frequencies. It can be seen that in a the frequency difference for each mode is varied within at least 1000 Hz, forming patterns of different shapes until the last mode is found at f= 9198 Hz, while in b the 4 new patterns which also look different

[6]these patterns were assumed to be intermediate patterns between consecutive ones so they were omitted

from each other start forming within in a range of 3000-5524 Hz. Another observation is that after this frequency $f =$ 5524 Hz, no more modal frequencies are found for b, however, for a the maximum modal frequency reaches $f =$ 9198 Hz.

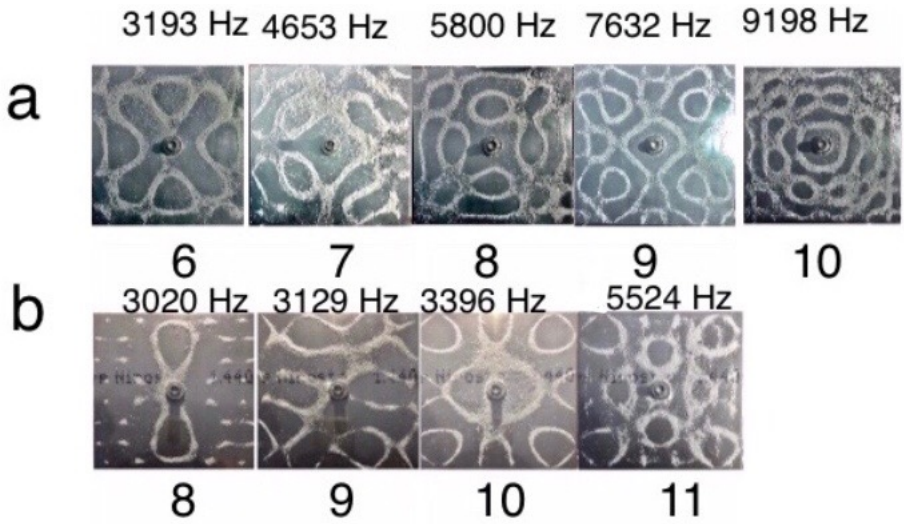

Figure 6.3: *Different Patterns formed based on order of number of modes starting from the 6th mode for aluminum plate and 8th mode for stainless steel*

6.1.1 Processed data

For data processing, a set for each plate is plotted of frequency as a function of its corresponding mode are plotted for each plate in figure 6.4 for aluminum and figure 6.5 for stainless steel, assuming that the order of the modes is true. For example, the first mode corresponds to the first pattern we see for a plate, the second mode is the next and so on . We assume there are no missing modes in between what we have recorded. Graph 6.6 is plotted to show the change in frequency between each mode for both plates.

From the 6.4 and 6.5, it can be seen that there is a trend that is true for both plates; the change between modes is slow and gradual, but around the 6th mode an accelerated rate of change of frequency between modes starts. This is illustrated by the decrease in density between points as the order of mode increases, and the increase in steepness of slope. This indicates that, for both plates, but especially for stainless steel, there are more modes in the lower range than in the higher range. Figure 6.6 shows that the most stark difference between the two plots is the range in which each plate reaches its final mode. For aluminum, there are 10 separate modes, the last of which is reached at 9198 Hz. For stainless steel, there is 13 different recorded modes [7]. However, despite the larger number of total existing modes, stainless steel can only show patterns at up to 5524 Hz.

[7] not all are pictured in the report

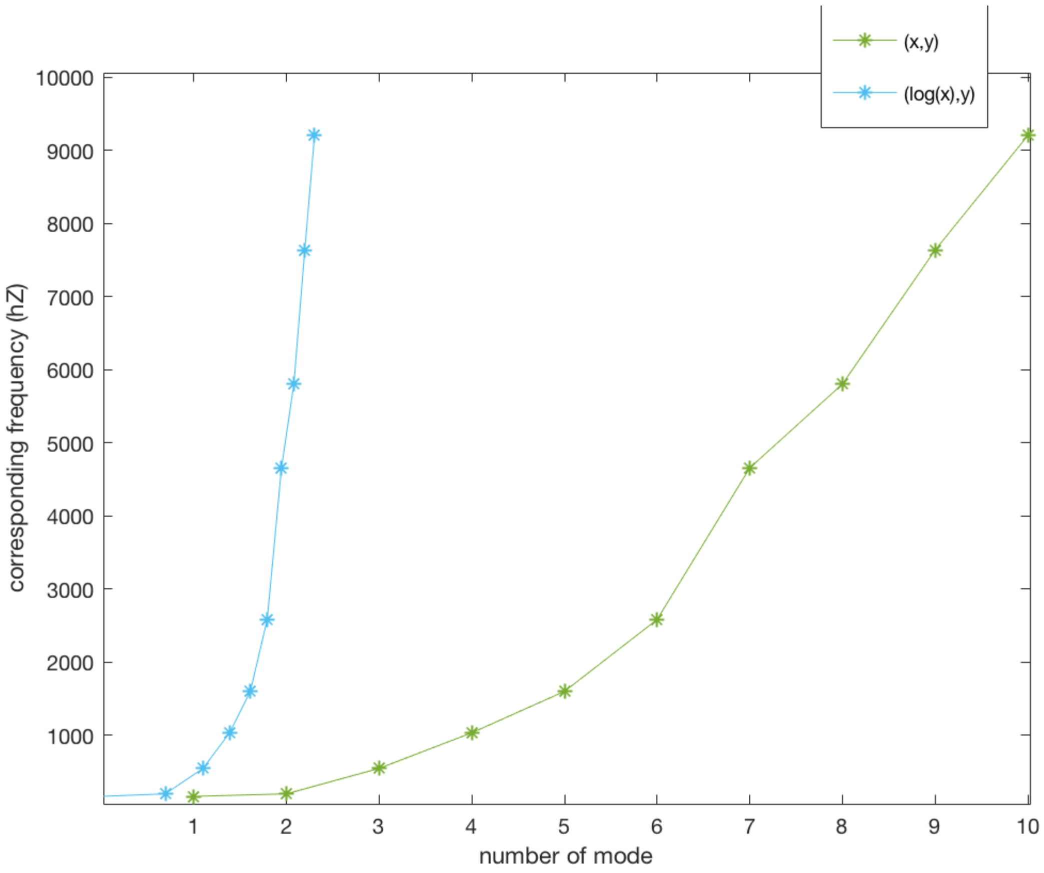

Figure 6.4: *Aluminum plate. Different Patterns formed based on order of number of modes, for a regular plot and a semi-log. This helps us visualize the rate at which the patterns change and range in which they do so.*

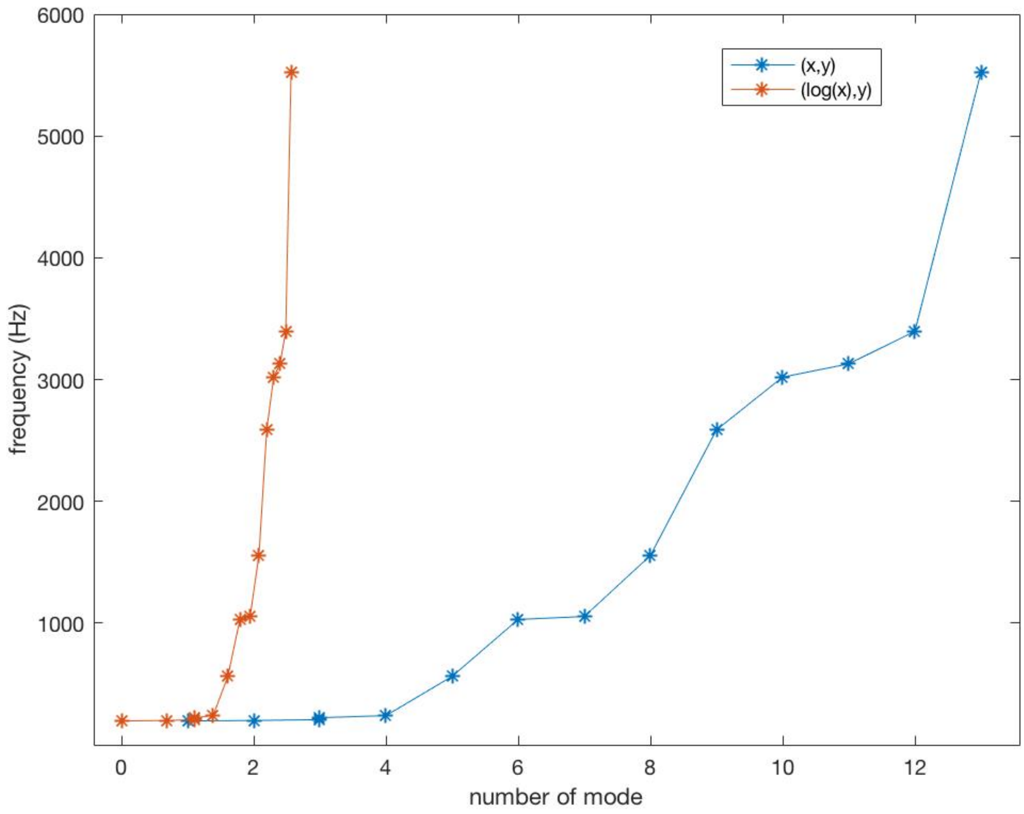

Figure 6.5: *Stainless Steel plate. Different Patterns formed based on order of number of modes, for a regular plot and a semi-log. This helps us visualize the rate at which the patterns change and range in which they do so. The purpose of the semilog plot is simply to highlight the intensity of the rate of the curve.*

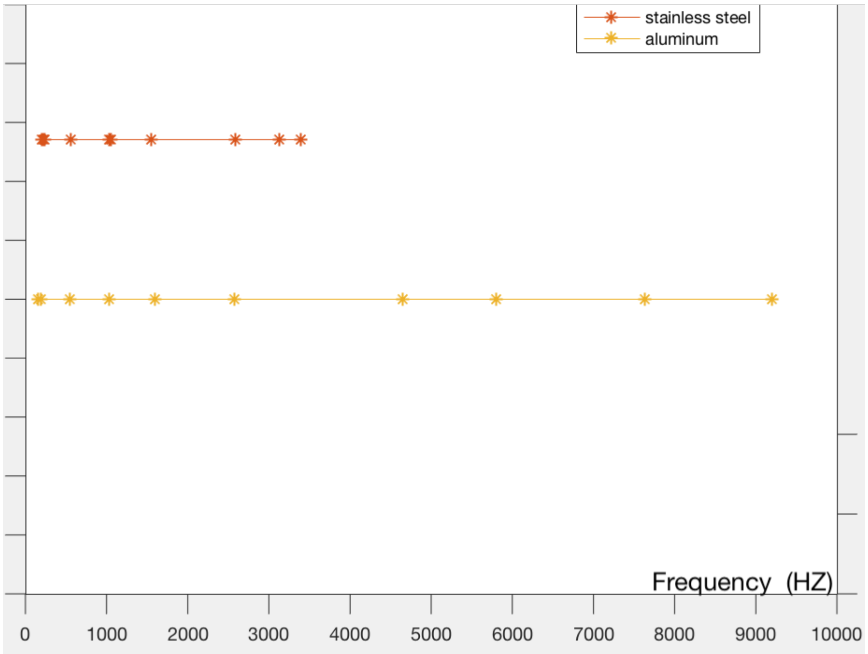

Figure 6.6: *a numberline comparing the range of frequency between plates and the rate of change between modes in said range.*

7 Discussion

This section is devoted to the interpretation of the obtained raw and processed data of the experiment conducted and presented in the previous section.
Generally, the observation of the increase of number of elements[8] for each pattern as the frequency is increased, can b related to the relationship for frequency and wavelength which is assumed to be the same for plates, that is, as the frequency increases, λ approaches zero, this means that the number of maximums and minimums of a wave increases and hence there is more probability of interference thus creating more nodal lines per unit area. So it follows that as f approaches , λ approaches zero.

From the experimental results, the fact that with controlling all variables and using two plates of different Young's modulus, mass density, and relatively different Poisson's ratio, resulted in the formation of different patterns leads to the conclusion that these properties do have an influence on the behavior of the plate, however to a certain extent and within specific ranges of frequencies. Whithin a range of $f=$ 500-2600 Hz, the same patterns were repeated for the plates with almost the same frequencies for each pattern. However, at frequencies higher that f=2600 Hz, the patterns look completely different. Besides this, the frequencies at which the

[8]elements are qualitatively defined to be the number of islands of shape per plate

patterns are formed also vary with high ranges as was shown in figure 6.3 and the graph plotted in figure 6.6. A possible explanation for this inconsistency is that these parameters' influence increases at higher frequencies while their influence is not very significant at lower frequencies. Theoretically, we cannot make any direct analogy between these properties and the range in which the patterns are formed because we don't have a mathematical model for their formation. We know that the relation between speed and frequency and wavelength can be described as:

$$v = \lambda f \tag{7.1}$$

and we know from the wave of a string, that the speed is:

$$v = \sqrt{\frac{T}{\rho}} \tag{7.2}$$

Combining these two equations we get:

$$\lambda = \frac{1}{f} \cdot \sqrt{\frac{T}{\rho}} \tag{7.3}$$

From this equation we can see that a higher density of the string will lower the wavelength of the wave[9]. And with this analog between the string and the plate our hypothesis is that the density will have an impact on at which range we will see standing waves. We can see that we will not get patterns for frequencies higher than 5524 Hz for stainless steel, but for the aluminium plate we will get patterns up to 9198 Hz[10]. And since we think there is an analog between density and stiffness, this also holds for stiffness - so if the stiffer a plate is the more difficult it will be to see patterns at higher frequencies.

Besides concluding this, it is also clear that for stainless steel, which has a higher flexural rigidity and mass density of 17.9 (pa m^3) and 7.9 (g/cm^3) respectively than that of aluminum of lower flexural rigidty and mass density of 6.3 (pa m^3) and 2.71 (g/cm^3) respectively, the patterns, after which they start forming differently, have less elements than those for aluminum (see figure 6.3). The advantage of having a similar Poisson's ratio for both plates allows for a better deduction that its either Young's modulus or mass density that are causing these differences. This should have been taken into consideration when the experiment was designed in order to deduce which of the two parameters is the most influential.
however we know from the wave equation for the plate that the flexural rigidity (D) and the mass density (ρ) has an impact on how fast the changes in the plate are occurring. This can be seen from the wave equation for the plate.

[9]with the same tension and frequency

[10]The formula $v = \lambda f$ does not hold for the plate as a hole, since the oscillation is not a harmonic one. This is indicated by the fact that there are different wavelengths occurring at the same time on the plate. Even though, we can still use the relation $v = \lambda f$ in an areal between two nodal lines.

If we then calculate the two ratios between flexural rigidity and mass densities between them, we see that:

$$\frac{D_{st}}{D_al} = \frac{17,9Pa \cdot m^3}{6,3Pa \cdot m^3} = 2,75 \tag{7.4}$$

And the fraction between the densities:

$$\frac{\rho_{st}}{\rho_{al}} = \frac{7900kg/m^3 =}{2705kg/m} 2,92 \tag{7.5}$$

So by this, we can argue that the mass density is of the same importance as the flexural rigidity. These deductions are based on a experiment limited to our visualization, therefore missing patterns does not mean that they don't exist, it only means that we cant visualize them with the specific and simple method we used.

7.1 Conclusion

The properties of a vibrating system affect its behavior and thus its natural frequencies, one interesting property which was attempted to be studied, is the stiffness of the plate defined in this research by flexul rigidty and mass density. This research aimed to answer the following research question

How does varying the stiffness properties of a free square plate affect the configuration of its nodal lines under mechanical vibration?

theoretically by considering the governing wave equation of the vibration of a thin plate and experimentally by conducting a classical method of visualizing patterns of natural frequency of an aluminum and stainless steel plate and qualitatively analyzing these patterns.

In this study, it was shown that Young's modulus and mass density influence on the number of modes and the natural frequencies of a vibration become significant above a specific range of frequency. The material with greater Young's modulus and mass density being stainless steel showed **1**) more number of patterns **2**) patterns with less elements **3**) patterns at lower natural frequencies. This leads to the conclusion the mentioned properties have an influence on the configuration of the nodal lines by not only changing their shapes and positions but also decreasing their number, and the frequency at which their are formed.

7.2 Future Perspectives

We planned to use a simulation to see if, with a vast contrast between flexural rigidities, we might get different patterns at the same frequencies. This way we could have more conclusive data on its effect on the nodal configuration.

Before we testing our theory, we wanted to see if our code could give us reliable results. In order to asses the codes ability to give us a realistic representation of a plate, we should compare the pictures of the patterns formed our experimental results to a simulated version of the 'same plate' with the same corresponding frequencies. While we still would not have a quantitative way to analyze the exact pattern, we felt that if the overall shape of where the nodal lines lay was comparable, we could use MatLab as a valid black-box for the experiment.

The code we chose was called "Vibration of Square Plates", created by The Standard NAFEMS Benchmarks. The code describes for a square plate with dimensions '10X10X1'. This of course is an issue with comparing to our simulated results, as our plate is 15X15X1. We feel that this distinction is negligible, as the surface area in a square plate has little effect on the nodal configuration, unless you are looking at a disparity of say $100cm^2$ to $1000cm^2$ (7).
In this code the author allows us to hard code our mass density, E value, v value, and a range of frequencies. A more detailed description of how this code works is presented in our appendix. This simulation would be useful for this project because we could schematically change the materials rigidity properties until we noticed a substantial change in output (in terns of nodal configuration). This was difficult to see from our experimental results, which indicated to us that we would need a much more significant contrast between the young's modulus, Poisson's ratio, and mass density in order to see a change. in this way we could make a much more quantities analysis onto how these said parameters effect the outcome, since we could essentially find a critical point at which they do.

While we did get this code to run for one plate and one set of frequencies, we ran into several problems. The first issue was that, even just through inspection, it was difficult to tell if the model was reliable for comparing to our experiments. This was because beyond the first few modes, the simulated output did not seem to match the pattern from our experimental output. The simulated results for some of the modes in an aluminum plate are included in our appendix in image 8.1 .
The more pressing issue however, was that this code required an external toolbox to solve the pde function called "PDE solver". We were able to get the code to run for a short time becuase we used a free student trial for this toolbox (which is normally 3500 DKK). However, we did not realize that the trial would end before we had collected all of our experimental data. Therefore we could not access the code and run our tests to see what might happen with more extreme values in our Flexural Rigidity.

As seen in section 3.3 for solving the wave equation for a membrane, the equation for solving the angular frequency ω becomes paramount for understanding how to solve our system and thus to find out how these patterns are formed. To remind the reader, The frequency is proportional to the length(s) a and b, of the plate, the speed of the wave c, and of course the modes m and n. The configuration of the patterns on the plate is directly influenced by the modes. For example, in a membrane, and also in our experiments, the fundamental mode (which also relates to the lowest frequency at which we see a pattern) is where m=n=1. This looks like a circle both in simulation and in our experimental results. The modes (m corresponding to the x-axis and n to the y-axis) do not have to be equal (for example, you can have a plate where m=4 and n=1), however, your displacement will be asymmetric and thus you will not see symmetric patterns. In the simulation, written by N. M. Abassi, you can alter the modes in any configuration you choose. In our experiment however, we can only see the *result* of the modal configuration, and thus have to make a reverse causal analysis of modes and frequency. This is partly why we do not use this code to compare to our experimental data, along with the fact that it models a membrane and not plate. However, as seen in section 4.2.1, We do use this code to help visualize what modes look like in a square membrane.

We discovered that the actual shape of the patterns has many more influences than what is discussed in this project. One such factor is the proportionality between the frequency, velocity of the wave, and the wavelength. In a homogeneous system, we have $f = \frac{v}{\lambda}$. However , in a dynamic system such as ours, the homogeneous equation does not apply. We think this is an r the complexity in the patterns. For example, if this proportionality was homogeneous throughput the plate, we would have the same shape of nodal points for each space along the x or y axis. This is not the case. While the patterns are symmetric, they are not uniform.

Another factor that this project does not go into detail on is how to define the shape for each mode. If there was quantitative way of defining each shape, it would be interesting to see how the complexity of the pattern (in terms of its elements) changes between frequencies. This could give us more insight, when comparing this rate of change between plates, into how material properties effect the standing wave patterns on plates.

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8 Appendix

8.1

Calculation of the flexural rigidity of aluminium and stainless steel:

$$D_{alu} = \frac{68 \cdot 10^9 Pa \cdot (10^{-3} m)^2}{12 \cdot (1 - 0,32^2)} = 6,3 Pa \cdot m^3 \tag{8.1}$$

$$D_{st} = \frac{197 \cdot 10^9 Pa \cdot (10^{-3} m)^2}{12 \cdot (1 - 0,29^2)} = 17,9 Pa \cdot m^3 \tag{8.2}$$

8.2

Code for Plates

A geometrical model is imported into the mail file. it uses a function 'solvepde' within which it takes into account the boundary conditions. This is also an external operator, using an external toolbox called "PDE solver". As seen in The wave equation solutions, we separate the variables of the characteristic wave equation is crucial for it's solution, thus to get the solution we requires advanced partial differentiation. In the editor of the main file, you may hard code different structural proprieties . Here we can write the Young's modulus, Mass Density, and Poisson's ratio of our plate. Under the operator 'pdePlot3D' we are also allowed to hard code a set of frequencies (up to 9 different ones) into the editor. These are then are used correspondingly to generate a 3D model corresponding to each frequency, whilst the structural properties are held constant. The code uses the Finite Element method (FEM) to solve the equation of motion. If a set of values exceed the array bounds, FEM modifies the given set or matrix iteratively until the numbers can be used by the functions in the code.

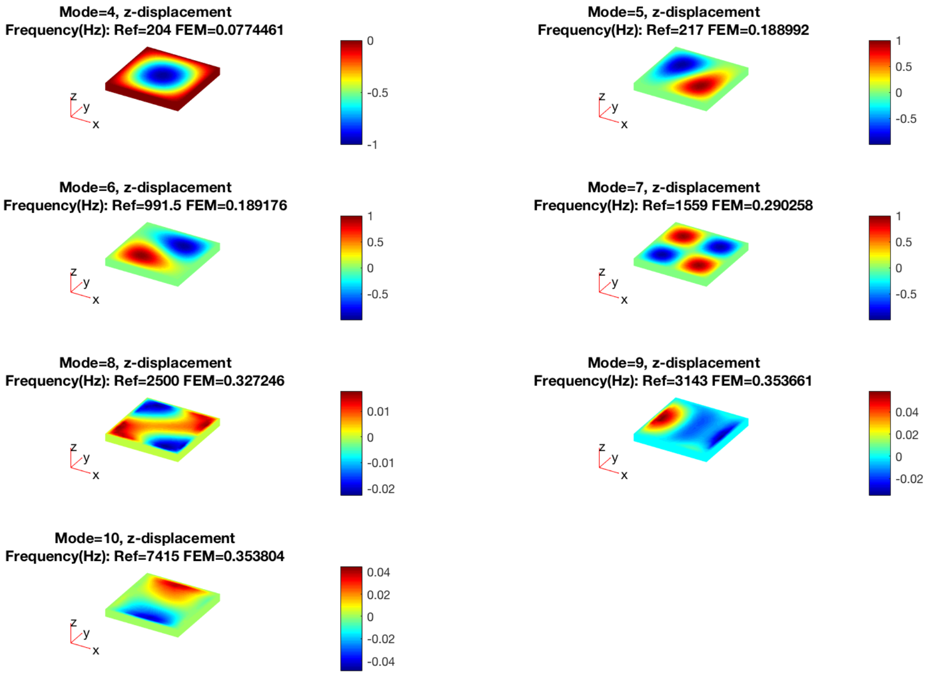

Figure 8.1: *simulated modes using Matlab code for an aluminum plate.*