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ADVANCES IN RESEARCH AND DEVELOPMENT CONCERNING MATHEMATICAL MODELLING IN MATHEMATICS EDUCATION

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Introduction and brief historical outline

Mathematical models and mathematical modelling have been on the agenda of teaching and learning of mathematics since the 1970s, albeit more so in some countries (e.g. Australia, Denmark, Germany, The Netherlands, UK) than in others, even though singular activities undertaken by individuals or small groups of educators are indeed found in other countries as well (e.g. Austria, Greece, Italy, Portugal, Sweden, USA).

The formative years, 1970-1990

In the early, formative, years, roughly 1970 to 1990 - which also saw the establishment of the International Conferences on the Teaching of Mathematical Modelling and Applications (the ICTMAs), the first of which was held in 1983 in Exeter, UK – the emphasis was on making a plea for the inclusion of mathematical applications and modelling in the teaching of mathematics, based on positive case-supported experiences.

This gave rise to a variety of foci of attention for the people involved, foci which are well represented in the series of ICTMA proceedings. First of all it became important to attend to the conceptual clarification of the notions of applications, model and modelling and to the nature and properties of these concepts. Secondly, putting forward and discussing arguments in favour of applications and modelling in mathematics education formed a significant enterprise, as did descriptions and discussions of different ways of incorporating applications and modelling in curricula and syllabi, and in special courses, in particular, at different educational levels. Here, one issue was which overall approach to teaching (and learning) should be adopted. Should applications and modelling be integrated into ‘ordinary’ mathematics teaching or should they be dealt with in separate courses or units? Moreover, presentations of cases of experimental teaching designs and curricula focusing on applications and modelling were given high priority on the agendas of national and international discourses in those early years. The same was true of producing and presenting textbooks, teaching materials and other resources, which, typically, referred to specific settings and circumstances.

During the early years, the study, in educational contexts, of *applications* of mathematics in various extra-mathematical domains and the corresponding *models* through which the applications came into being formed the core of the work done in the area. In other words, teaching and learning primarily dealt with students considering already existing applications and models rather than engaging in constructing new ones.

It follows from what has just been said that the kind of work done in the early years on applications and modelling was primarily of a developmental nature, whereas research proper

primarily concentrated on conceptual clarification and introduction of theoretical constructs and sometimes on gauging the outcomes of cases of experimental teaching or curricula.

The years of consolidation, 1990-2010

During the 1990s, emphasis gradually changed towards students' own construction of models of extra-mathematical situations, i.e. towards students' engagement in the *processes* of modelling, not only in the investigation and employment of given applications and models. This was spurred, among other things, by the repeated observation that even students with a solid knowledge and a high level of skills in pure mathematics and familiarity with existing applications and models were oftentimes unable to independently perform modelling themselves.

In other words, it became clear that undertaking mathematical modelling is difficult and demanding and that the ability to do so successfully does not automatically result from being good at pure mathematics. This gave rise to two immediate questions: if knowledge and skills in pure mathematics are not sufficient prerequisites for modelling, what more is needed, and what is sufficient for fostering and furthering this ability in students?

The good news is that *modelling* can indeed be learnt and cultivated, provided sustained efforts are being made by mathematics education researchers, curriculum authorities and teachers to design and implement rich teaching/learning environments together with the presence of necessary boundary conditions, including material and immaterial resources (e.g. time and teacher competence). Of course such efforts, if they are serious, come at a cost both in terms of increasing demands on students and teachers and in terms of teaching and learning activities that necessarily receive reduced emphasis with regard to time and demands on the part of both students and teachers. However, if you are not willing to cover the expenses entailed by the pursuit of your goals you are not really willing to pursue the goals.

Against this background, the last decade of the 20th century and the first of the 21st saw lots of developmental work that focused on designing teaching and learning environments that can serve the purpose of fostering and furthering students' ability to undertake mathematical modelling. However, this also generates the need for research on the extent to which different settings, environments and approaches to the teaching of modelling are successful in terms of students' learning to perform independent modelling, as well as on the features and factors that are co-responsible for either success or failure.

So, *research* on the teaching and learning of modelling gained momentum since the mid-1990s and is now the predominant activity in the community of mathematics educators with a special interest in mathematical modelling in mathematics education. Significant research topics include the nature of the modelling process, both in theoretical and empirical terms, characterising modelling capabilities and competencies and identifying their presence with students, assessing students' modelling achievements in response to different approaches to the teaching of modelling. In the sections to follow, we shall take a closer look at some of these topics with particular regard to the current state of affairs.

Recent advances and focal points of research on the teaching and learning of mathematical modelling

Setting the stage: basic notions and terms

In order to provide a background to the core parts of this paper it is necessary to briefly introduce some basic notions and terms.

Classically, a mathematical model of some extra-mathematical domain is constructed – and hence modelling is undertaken - in order to capture essential aspects of a context and situation belonging to that domain, with the purpose of understanding the situation, testing hypotheses, solving a problem, explaining phenomena, predicting future events, or paving the way for decision-making and so on and so forth. So, the overarching purpose is to pose and seek answers to some pertinent questions. Subjecting the situation to mathematical modelling takes place on the expectation that mathematical traits are actually or potentially present in the situation, and that this can be used to seek answers to the questions posed.

To formalise things slightly, let us denote the extra-mathematical domain by the letter D . The modeller then selects the extra-mathematical objects, and the relations amongst them, that are deemed relevant to the purpose of the modelling enterprise, chooses some mathematical domain, M , and translates the extra-mathematical objects and their interrelations, into mathematical objects in M , and mathematical relations amongst them, and - especially – translates the generating extra-mathematical questions into mathematical questions concerning M . We can metaphorically think of the translation as a mapping, f , from the extra-mathematical domain into the mathematical domain, $f: D \rightarrow M$. With this in hand we can define a *mathematical model* as a *triple* (D, M, f) , thereby making it explicit that a mathematical model is a model of something, composed of an extra-mathematical domain, a mathematical domain, and a translation mapping from the former to the latter. The process of constructing a mathematical model is what we call *mathematical modelling*. Let us take a closer look at this process.

First, the extra-mathematical domain D has to be prepared for modelling. This happens by focusing on the elements and features that are significant for the whole purpose and discarding the ones that are not. This typically involves a great deal of simplification of the complexity of the situation, and a fair amount of idealisation as well. It further involves making assumptions about the relationships amongst the elements taken into account and specifying the extra-mathematical questions as clearly as possible. These activities constitute what is oftentimes called *pre-mathematisation*. It is important to notice that in most cases the choices and decisions involved in pre-mathematisation are far from easy, obvious or automatic. It is also important to notice that even though the activities involved in pre-mathematisation are conducted entirely within the extra-mathematical domain at issue, they are also conducted with an eye on the potential involvement of different mathematical representations in relation to the purpose of the modelling enterprise.

Once pre-mathematisation has been completed, what is possibly the pivotal stage of the modelling process occurs, namely that of *mathematisation*. Mathematisation is the outcome of the stage in which the selected objects and their interrelations in the extra-mathematical domain D as well as the generating questions are mathematised, i.e. translated – by way of a mapping f – into selected mathematical objects and mathematical interrelations and questions belonging to some mathematical domain M . As mathematisation takes place at the crossroad of extra-mathematical and mathematical

domains, it can be done in a multitude of different ways and is, therefore, oftentimes difficult and demanding and involves a lot of considerations and judgments. Important considerations and judgments on the part of the modeller are to do with the potential affordances of different mathematical domains and representations within them, and, in particular, with imagining possible effective mathematical treatments that are conducive to the modelling purpose. This depends on the mathematical background and resources of the modeller and must take place prior to or integrated into the mathematisation process itself, i.e., paradoxically enough before it has been completed.

The outcome of the mathematisation stage then is a set of mathematical entities, i.e. objects, relations amongst them, and mathematical questions, belonging entirely to the domain M . In order to seek answers to the mathematical questions, posed with regard to the translated extra-mathematical questions, a *mathematical treatment* of these entities, i.e. mathematical problem solving, has to be undertaken. This means that mathematical concepts, facts, results, methods, practices, techniques, procedures and reasoning have to be put to work in order to obtain answers to the mathematical questions.

The answers to the mathematical questions thus obtained have to be translated back to the extra-mathematical domain and interpreted in terms of answers to the questions that initiated the modelling activity in the first place. What do the answers to the mathematical questions tell us with regard to answers to the extra-mathematical questions? This stage is usually called *de-mathematisation*. Sometimes de-mathematisation is entirely straightforward (e.g. consisting in amending units to numbers only), at other times more complicated and involving further analysis and reflection.

The final stage of the modelling process is to *validate* the outcomes generated by the model and to *evaluate* the model itself. Validating the outcomes amounts to relating these to the questions that the modelling activity was set out to answer and to gauging the extent to which these answers are solid and meet the purposes and goals of the enterprise. Evaluating the model amounts to analysing the range and scope of its outcomes, especially with regard to the simplifications, idealisations and assumptions made at the beginning of the process, the sensitivity of the answers to the accuracy of parameter specifications, and the generalisability of the model to cover new or modified situations and conditions.

It is usual to depict the modelling process by a diagram representing the so-called *modelling cycle*, which in an idealised form captures the stages just outlined. The literature contains several such diagrams. They all agree on the fundamentals but differ in the extent to which they pay attention to specific sub-processes. All the diagrams are analytic reconstructions of stages and aspects of the modelling process that must exist in principle. However, the modelling cycles are *not* meant to describe what individual modellers actually do when performing modelling. Figure 1 is my preferred modelling diagram, because it provides details of what happens in the somewhat blurred extra-mathematical domain – the “amoeba” - before and after mathematisation, mathematical treatment and de-mathematisation.

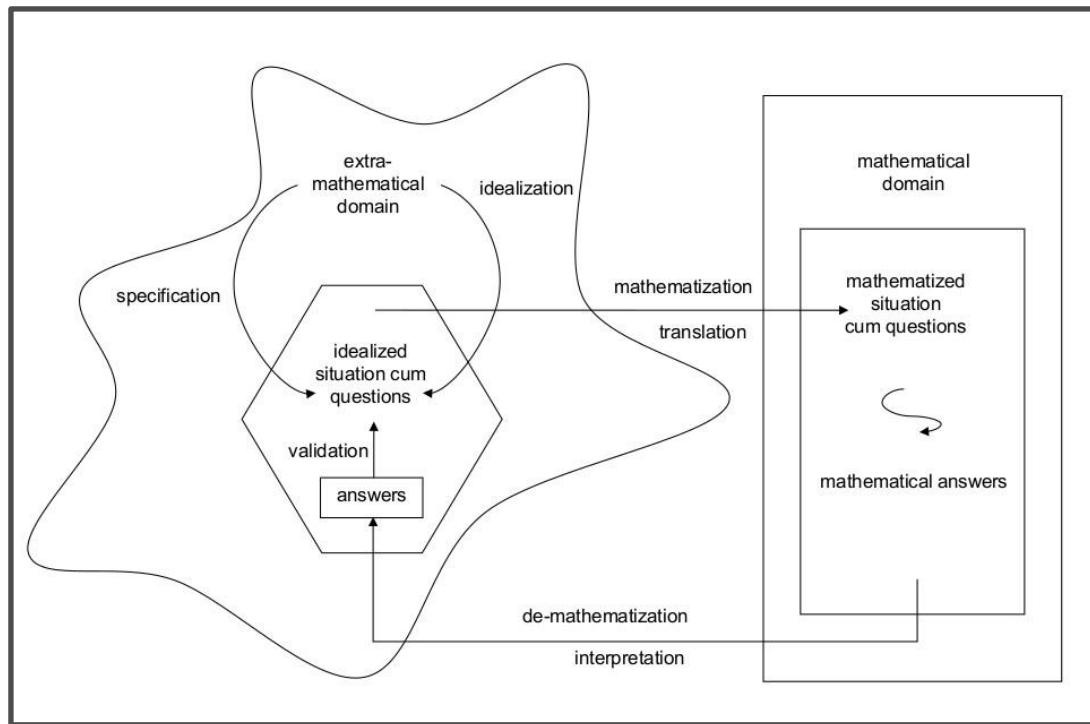


Figure 1. Mathematical modelling and its processes (based on Niss, 2010).

In what we have just outlined, the focus was on capturing aspects of a given extra-mathematical domain, context and situation by way of mathematical modelling in order to come to grips with objects, relationships, phenomenae, and processes already existing in the context and situation considered. We call such modelling *descriptive modelling*. It is about handling *existing reality*.

However, another and rather different kind of mathematical modelling, too, is omnipresent in scientific, technological and practical spheres. It is focused on structuring, designing, constructing and *creating reality*. This happens whenever scientific or practical measures such as pH, velocity and acceleration, density, growth rate, the BMI index, the Gini coefficient of socio-economic inequality, are introduced. It also happens when physical objects such as buildings, roads, furniture, tools and utensils are designed to have certain properties and to meet certain specifications. And it also happens when societal artefacts such as annuity loans, pension schemes, tariff and ticketing schemes, insurance premiums, allocation of resources, location of hospitals or other institutions, defining voting procedures and determining the outcome of elections, and so on and so forth, are being designed. In all these and myriads of other similar cases mathematics is involved in crucial roles by translating selected objects and relations from an extra-mathematical domain into objects and relations in some mathematical domain. We call such modelling *prescriptive modelling* – by some also called normative modelling. By such modelling we construct mathematical models *for* aspects of reality, whereas in descriptive modelling we construct models *of* aspects of reality. It should be stressed, though, that the difference between descriptive and prescriptive modelling lies in the different purposes of the endeavour, not in the difference between the resulting models, which may can be the same in both contexts. In other words, we do not speak of descriptive and prescriptive models.

The modelling cycle is well capable of capturing the stages of prescriptive modelling. However, the stage of validating model outcomes and evaluating a model resulting from a prescriptive modelling process is different from that stage in descriptive modelling. This is because in descriptive modelling a key element in that stage is confronting model outcomes with the corresponding known facts and properties of the extra-mathematical domain, whereas in prescriptive modelling it does not make sense to confront model outcomes with aspects of reality that only exist once they have been created by implementing these very outcomes. Instead, validation and evaluation in the context of prescriptive modelling has to focus on the sensibility, relevance, appropriateness, expediency and usefulness of the constructs obtained in relation to the purposes of the enterprise. It is worth noting that whilst the community of researchers on mathematical modelling in mathematics education has been dealing with descriptive modelling for a long time, attention to prescriptive modelling is a rather recent phenomenon (Niss, 2015).

So far in this section we have primarily considered models and modelling as such, without placing them in an educational context. This is what we shall do now.

There are basically two different, but certainly not conflicting, purposes of including mathematical modelling in mathematics education.

The first purpose is to enable and empower students to independently and successfully conduct mathematical modelling in a variety of contexts and situations within different extra-mathematical domains, because fostering and furthering this competency is considered an educational goal in itself. In other words, here, mathematical *modelling is a goal* of mathematics education. Given that the ability to undertake mathematical modelling does not follow automatically from knowledge, insight and skills regarding pure mathematics, but has to be learnt and developed, the serious inclusion of mathematical modelling in the teaching and learning of mathematics has non-trivial consequences in terms of allocation of time, resources and activities.

Next, there are lots of evidence to show that mathematical models and modelling can help provide motivation for the study of mathematics as well as sense-making, underpinning and consolidation of mathematical concepts, methods and results. So, mathematical modelling, without being a goal in itself, can support students' learning of mathematics as a discipline, which then is the second purpose of including modelling in mathematics education. Here, *modelling is a vehicle* for something else, namely the learning of mathematics. Although, as mentioned, these two purposes, modelling as a goal, and modelling as a vehicle, are not conflicting they are indeed distinct, which has important consequences for the design of teaching and learning environments and for the orchestration of activities within and outside class. For example, if modelling is a goal it is necessary to pay serious attention to dealing with the extra-mathematical domains of modelling, and the corresponding components of the modelling cycle, whereas this is typically much less important if modelling is a vehicle for the learning of mathematics at large.

Modelling competency and modelling (sub-)competencies

Ever since it became clear that mathematical modelling is difficult and demanding and hence something that has to be learnt, researchers have attempted to define and characterise the notion of *modelling competency* and to analyse its constituents - which are usually called *sub-competencies* - i.e. they have taken a top-down approach. Other researchers have taken a bottom-up approach and have focused, first, on identifying a set of crucial *modelling competencies*, which may or may not

afterwards be bundled together into one comprehensive competency, oftentimes supplemented with a meta-cognitive component focused on combining and integrating the individual modelling competencies. Such work gained increasing momentum from the late 1990s and onwards and today constitutes a core activity of the international modelling in education community. As an example of the top-down approach, consider the following definition of modelling competency (Niss & Jensen, 2002, p. 52):

This [modelling] competency consists, on the one hand, of being able to *analyse* the foundation and properties of given mathematical models and to assess their ranges and robustness. This includes being able to ‘de-mathematise’ (aspects of) given mathematical models, i.e. being able to decode and interpret model elements and outcomes with regard to the domains and situations modelled. On the other hand, the competency consists of being able to *undertake active model construction* in given contexts, i.e. to bring mathematics into play and employ it to dealing with matters extra-mathematical [My translation from Danish].

As a significant example of a bottom-up approach to modelling competencies we may take Katja Maaß’ (2006). Based on previous work of Blum and Kaiser, she identifies five modelling competencies (each with several sub-competencies): Competency to understand the real problem and to set up a model based on reality; Competency to set up a model from the real model; Competency to solve mathematical questions within the mathematical model; Competency to interpret mathematical results in a real situation: and Competency to validate the solution. However, Maaß observes that these “(sub-)competencies are not enough to run through a modelling process. Moreover, the learners should keep an overview of their proceedings and aim at a goal when modelling a problem” (p. 137).

Top-down and bottom-up approaches agree that the modelling (sub-)competencies all refer to the stages of the modelling cycle. It is also those stages and the corresponding modelling (sub-)competencies that historically have been used as a basis for assessing students’ modelling work, cf. Money & Stephens (1993) and Haines, Crouch & Davis (2001).

Against this background it is clear that endeavours to foster and further students’ modelling ability must focus on developing their modelling competency and (sub-)competencies. The question then is: how best to accomplish this? Particular attention (see e.g. Blomhøj & Jensen, 2003) has been paid to the derived question: should this happen by way of a *holistic* approach, in which students primarily work on tasks involving the full modelling cycle and all its stages, calling upon the comprehensive modelling competency, or by an *atomistic* approach in which students work on a number of tasks each of which call on a single or a few (sub-)competencies corresponding to few stages of the modelling cycle? This issue has not been definitively settled. Widespread experience and research evidence suggest (e.g. Kaiser & Brand, 2015) that both approaches are indeed significant and ought to be combined.

Modelling difficulties – implemented anticipation

If students are to develop modelling competency it is essential to help them overcome the various kinds of stumbling blocks - obstacles or blockages – we know many of them experience when engaging in modelling activities. Obviously, for this to be possible we first have to be able to

identify these stumbling blocks and then try to counteract them. For two decades researchers have worked on these issues (e.g. Ikeda & Stephens 1998; Galbraith & Stillman 2006), so we now know a good deal about them.

At first, stumbling blocks were found in the transition stages in the modelling cycle: mathematising, mathematical problem solving, de-mathematisation, and validation/evaluation (in addition to the aspect of pre-mathematisation consisting in making assumptions). Amongst these, it is no surprise that problem solving is a significant stumbling block, as is well-known from mathematical problem solving. Apart from that, multiple studies show that mathematisation is the most serious stumbling block to students' modelling endeavours. Only rarely does de-mathematisation constitute a serious stumbling block. When it does, it is typically because there is quite a distance from the answer(s) to the mathematical problem(s) to the extra-mathematical domain within which these answers have to be interpreted. Validation of modelling outcomes and evaluation of the model as such are often very demanding undertakings. However, they do not seem to constitute stumbling blocks to a corresponding extent, simply because they come last in the cycle and because students only seldom pay serious attention to this component.

More recently, many stumbling blocks have been located in several parts of the pre-mathematisation stage, even to the extent that this may entirely prevent mathematisation to be carried out (Jankvist & Niss, submitted). Tailoring and idealising the situation to be modelled, selecting the essential entities pertaining to the questions the modelling endeavour is meant to answer, and – which proves particularly demanding – to discard those that are not essential enough to be taken into account, specifying the exact questions to which answers are being sought, making simplifying, but not too simplistic, assumptions about the context and situation, finding information and data to underpin the modelling process all constitute potential stumbling blocks.

Research has further shown that there is a common underlying stumbling block to several of the ones located in the different stages of the modelling cycle. This stumbling block is the need in several places for the modeller to anticipate what can be done in subsequent stages, after the current stage has been completed, but to implement that anticipation *before* completion of the current stage. In Niss (2010) I have termed this process *implemented anticipation*. It is on the agenda when the modeller is pre-mathematising the context and situation and preparing it for mathematisation with a view to the subsequent possibilities of mathematisation, and further on to the subsequent problem solving opportunities. It is on the agenda when the modeller chooses between a multitude of possibilities and settles on a particular mathematisation with a view to the subsequent problem solving possibilities yielding mathematical answers that can be translated back into relevant answers to the original extra-mathematical questions. And it is on the agenda when the modeller looks for problem solving strategies that are likely to not only provide solutions to the mathematical problems posed but solutions that are meaningful and useful in the extra-mathematical context at issue.

Whilst initially an experience-based theoretical construct, implemented anticipation and its crucial role has later been empirically identified as key factors in actual modelling work (Stillman & Brown 2014; Niss 2017).

Conclusion

In this article we have charted the evolution of research and development concerning mathematical

modelling in mathematics education. We have shown that the ability to successfully undertake mathematical modelling is difficult to obtain and that it does not follow automatically from being good at pure mathematics. We have looked into some of the reasons why this is so, focusing on stumbling blocks occurring in the modelling process. However, mathematical modelling can be learnt provided concerted efforts are being made to design appropriate teaching and learning environments and to invest the human and other resources needed to foster and further modelling competency/cies in students.

During the last several decades we have learnt a lot about the mathematical modelling in mathematics education, but there is still a long way to go until full-fledged modelling competencies are commonplace amongst our students.

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