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Coordination

Ezzaim LAABID

As often happens, transfer of the MS-Word file from one computer (mine) to another one (that of the publisher) has reintroduced the numbering of formulas which I had eliminated. Since they appear in increasing order 1-2-3-...-94, they should be easily identified. However, in order to offer a more correct version, I append the pdf-version of my manuscript.

THE INDIAN SUMMER OF AL-ANDALUS MATHEMATICS? AN EXPANDED ADDENDUM?

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Abstract

At the 11[ième] Colloque Maghrébin in Algers in October 2013 I spoke about the *Liber mahameleth*, and I suggested this work not to be an independent compilation made by a Latin scholar but a (probably free) translation of an Arabic work presenting "*mu'āmalāt vom höheren Standpunkt aus*, 'from a higher vantage point'", written by an Arabic astronomer-mathematician, which however had no impact in later Arabic mathematics but only (through the translation, and even here modestly) in the Latin world. At the last moment of my preparation it occurred to me that this might be one of three instances of advanced arithmetic from 12th-century *al-Andalus* that only survived in Latin translation but not in Arabic, and I introduced that idea in an addendum. At the present occasion I shall present this suggestion in more depth, discussing all three instances.

A closing "addendum to the addendum" presents a discovery along the same lines which I made after the meeting and after having written what I had believed to be the final version of the paper.

INTRODUCTION

À Ahmed Djebbar, en amitié

In my contribution to the 11[ième] Colloque Maghrébin I examined the relation of the mid-twelfth-century *Liber mahameleth* to Arabic '*mu'āmalāt* mathematics and arrived at the general conclusion (as opposed to earlier workers[1]) that it was no Latin compilation made directly on the basis of genuine Arabic *mu'āmalāt* works but instead a (plausibly free) translation of that "book which in Arabic is called *Mahamalech*"[2] spoken of by Gundisalvi in the *De divisione philosophiae* [ed. Baur 1903: 93], made perhaps by Gundisalvi himself, and if not by some collaborator or contact of his. I concluded moreover that this Arabic work was not a *mu'āmalāt* book proper but a work treating of *mu'āmalāt* "von höheren Standpunkt aus" ("from a higher vantage point"), in the words of Felix Klein's lecture [1908] – that is, a work presenting (select) *mu'āmalāt* topics on a theoretically satisfactory basis, and produced by a mathematician trained in Euclid and proportion techniques (that is, almost certainly, a mathematician also versed in astronomy).

I had several reasons for this conclusion, of which only one is relevant for my present topic – namely the way algebra and proportion theory are used. I came to the conclusion that sophisticated mathematics was employed to unfold the theoretical possibilities inherent in (certain kinds of) *mu'āmalāt* mathematics, and that this was not transmitted efficiently to the Arabic world before the collapse of *al-Andalus* but only survived in a Latin translation. The fate of Jābir ibn Aflaḥ's and ibn Rušd's writings show that this would not be impossible.

Then it suddenly dawned to me that two other examples of sophisticated innovative mathematics appear to have had the same origin and the same fate. I had worked on both on earlier occasions without connecting them, even though both are known from Fibonacci's *Liber abbaci*. That "discovery" was made in the last moment: my contribution was already too long; and its focus was different. In consequence, I could only describe the discovery rather briefly. In what follows I shall therefore take up the thread, presenting the three cases one by one.

[1] See [Hoyrup 2013: 2–3].

[2] My translation, as all translations in the following when nothing else is stated.

THE *LIBER MAHAMELETH*

Several problem sequences in the *Liber mahameleth* take as their starting point and their purported subject a typical *mu'āmalāt*-problem – for instance, buying or selling, which begins by stating the rule of three and then presents the two alternatives where division precedes multiplication [ed. Vlasschaert 2010: 186].[1] That is fully traditional, and corresponds for example to what is found in al-Karajī's *Kāfī* [ed., trans. Hochheim 1878: II, 16*f*]. But then follow variations never encountered in commercial practice, nor dealt with in books presenting *mu'āmalāt* calculation. Using p and P for prices, q and Q for the appurtenant quantities, we have $\frac{q}{p}$ 1::$\frac{Q}{P}$ 2. Forgetting every consideration about dimensions and homogeneity (and thus about concrete meaning), the text first presents us with these problems in systematic order:[2]

$\frac{3}{13}$ 3::$\frac{Q}{P}$ 4, $Q+P = 60$

$\frac{3}{13}$ 5::$\frac{Q}{P}$ 6, $P-Q = 60$

$\frac{3}{8}$ 7::$\frac{Q}{P}$ 8, $Q\cdot P = 216$

$\frac{4}{9}$ 9::$\frac{Q}{P}$ 10, $\sqrt{Q}+\sqrt{P} = 7\frac{1}{2}$ 11

$\frac{4}{9}$ 12::$\frac{Q}{P}$ 13, $\sqrt{P}-\sqrt{Q} = 1\frac{1}{2}$ 14

$\frac{4}{9}$ 15::$\frac{Q}{P}$ 16, $\sqrt{Q}\cdot\sqrt{P} = 24$

[1] Since I used this edition for the Algers meeting, at a moment when [Sesiano 2014] was not yet available, and since I reviewed Vlasschaert's edition and therefore have a heavily annotated copy, this is the edition I shall refer to throughout.

[2] Closer descriptions of all of these problems and analysis of the way they are solved can be found in [Høyrup 2013: 14–20].

The main tools that are used are proportion transformations (*conversa*, *eversa*, etc.): the first problem, for instance, is transformed into $\frac{3}{3+13}$ 17::$\frac{Q}{Q+P}$ 18, after which the rule of three can be applied. Some of the transformations are more intricate, and appear to presuppose an implicit understanding of the ratios as divisions. For $\frac{4}{9}$ 19::$\frac{Q}{P}$ 20, $\sqrt{Q}+\sqrt{P} = 7\frac{1}{2}$ 21, three solutions are proposed, the last of which, when expressed in a formula, amounts to an amazing

$$\left(\sqrt{\frac{(\sqrt{P}\cdot\sqrt{Q})^2}{(P-Q)\cdot Q}+\left(\frac{\sqrt{P}\cdot\sqrt{Q}}{(P-Q)\cdot Q}\right)^2}-\frac{\sqrt{P}\cdot\sqrt{Q}}{(P-Q)\cdot Q}\right)^2 = Q.$$

A chapter follows "about the same, with [algebraic] things" (*res* – r in my symbolic translations). First comes $\frac{3}{10+r}$ 22::$\frac{1}{r}$ 23 – in words, "Three measures are given for 10 coins and a thing, but this thing is the price of one measure".

Then (in similar formulations)

$\frac{4}{20+2r}$ 24::$\frac{1\frac{1}{2}}{2r+3}$ 25

$\frac{8}{20+r}$ ::$\frac{2}{r-1}$ 26

$\frac{6}{10-r}$ 27::$\frac{2}{r}$ 28

$\frac{6}{10+r}$ 29::$\frac{2}{r+1}$ 30

$\frac{3}{20+r}$ ::$\frac{\frac{q}{2}}{\frac{2}{3}r-2}$

$\frac{6}{10-r}$ ::$\frac{2}{r}$

$\frac{4}{8-r}$ ::$\frac{2}{r+1}$ 31

$\frac{4}{20-2r}$ ::$\frac{1\frac{1}{2}}{2r-3}$ 32

Familiarity with other Arabic *mu'āmalāt* writings or Italian abbacus books might make us expect use of cross-multiplication – but then we shall be deluded. The first problem is transformed into $\frac{3}{10+r}$ ::$\frac{3}{3r}$ 33, from which it is seen that $3r = 10+r$; alternatively (a subtractive variant of *ex aequa*), $\frac{3-1}{(10+r)-r}$ ::$\frac{1}{r}$ 34, that is, $\frac{2}{10}$ ::$\frac{1}{r}$ 35, whence $\frac{1}{5}$ ::$\frac{1}{r}$ 36, etc. The following questions are treated similarly, and in the end comes the observation (p. 199) that "this kind of questions cannot be understood unless one is trained in algebra or in Euclid's book".

In yet "another chapter about an unknown in buying and selling" we then find that an unknown number of measures is sold for 93, and addition of this number to the price of one measure gives 34 – in our symbols (since no *res* occurs): $x+\frac{93}{x} = 34$. First the solution is stated as $\frac{34}{2}\pm\sqrt{(\frac{34}{2})^2-93}$ 37 (the sign depending on whether the number of measures exceeds or falls short of the price of one measure). Then a geometric argument based on a subdivided line (following the principles of *Elements* II.5) is given. Contrary to the normal predilections of the *Liber mahameleth* Euclid is not mentioned; the direct inspiration might therefore be Abū Kāmil's similar proof for the *al-jabr* case "possession plus number equals things" (Abū Kāmil's algebra is referrred to repeatedly and correctly in the *Liber mahameleth*). This is followed by one of the subtractive variants, again with alternative precriptions; the other subtractive variant is told to be solved correspondingly.

Then comes a problem $\frac{q}{p}$ 38::$\frac{Q}{P}$ 39, $pq = 6$, $PQ = 24$, $(p+q)+(P+Q) = 15$. The argument goes via a tacitly presupposed factor of proportionality s (= 2), $sp = P$, $sq = Q$ (later, a geometric argument shows how to find this factor, so it *is* really presupposed). An apparently innocuous analogue follows, $\frac{q}{p}$ 40::$\frac{Q}{P}$ 41, $pq = 10$, $PQ = 30$, $(p+q)+(P+Q) = 20$ – but this requires $s = \sqrt{3}$ and therefore entails complications and an appropriate cross-reference to the chapter about roots, and finally leads to a discussion in terms of the classification of *Elements* X, apparently expected to be familiar.

Next come a subtractive and two multiplicative variants, using similar methods (the latter two requiring a rational respectively an irrational value for s). Then two questions not defined in terms of a proportion, and where the identification of the variables as price and quantity is therefore nothing but a pretext to present them in the actual context:

$\sqrt{p}$ 42 $= 3q$, $p-q = 34$ and $\sqrt{p}$ 43 $= 2q$, $p+q = 18$

Both are solved first by a numerical quadratic completion ($\sqrt{p}$ serving as basic unknown), then by a line-based geometric proof. Then follow the two problems

$\frac{6}{4+r}$ 44::$\frac{2}{3\sqrt{4+r}}$ 45 and $\frac{6}{4-r}$ ::$\frac{2}{3\sqrt{4-r)}}$ 46

The left problem is transformed into $\frac{6}{4+r} :: \frac{6}{9\sqrt{4+r}}$ 47. The resulting equation $(4+r) = 9\sqrt{4+r}$ 48 is not made explicit but the numerical prescription corresponds to a transformation into $\sqrt{4+r}$ 49 $= 9$ and further into $4+r = 81$ – the right problem being solved correspondingly. That is, the text somehow makes use not of *al-jabr* but of equivalent patterns of thought. This interpretation is confirmed by the next question,

$$\frac{3}{x+y} :: \frac{1}{y+\frac{1}{9}y} \text{ 50}, \quad xy = 21$$

(x and y occur as "two different things"). The prescription corresponds to a transformation into $\frac{3}{x+y} :: \frac{3}{3y+\frac{1}{3}y}$ 51, whence $x+y = 3\frac{1}{3}$ 52y, $x = 2\frac{1}{3}$ 53y, $2\frac{1}{3}$ 54$y^2 = 21$, $y^2 = 9$, and finally $y = 3$, $x = 7$ (afterwards shown by a line-based argument). Alternatively, the problem is solved "according to *al-jabr*", which must hence be something different. Now the *thing* (r) takes the place of y, while the *dragma* (d) takes that of x. This time, a different but similar transformation of the proportion is used, namely $\frac{1}{\frac{1}{3}d+\frac{1}{3}r} :: \frac{1}{r+\frac{1}{9}r}$ 55, etc. This is followed by the analogue $\frac{5}{x+y} :: \frac{1}{\frac{1}{3}x+2}$ 56, $xy = 144$, solved by similar methods, which here, however, lead to a mixed second-degree problem.

Similar systematically varied problem sequences take as their starting point (or pretext) profit and interest, partnership, etc.

This is a far cry from anything that can be found in genuine *muʿāmalāt* contexts. We notice, firstly, the preponderant use of proportion techniques and of line geometry similar to what Abū Kāmil uses in his proofs of the fundamental *al-jabr* rules; secondly, the minor role played by *al-jabr* algebra; and thirdly, the use of methods that for us looks as first- and second-degree algebra but are considered distinct from *al-jabr*. It is also different from what we find in authors somehow moving in the vicinity of the *madrasah* environment – say, ibn al-Yāsamīn and ibn al-Bannāʾ.

It corresponds well, on the other hand, to what could be done by a member of the other main class of Arabic mathematicians – those whose professional upbringing had brought them through the *Elements*, the "middle books" on spherics, and the *Almagest* – if he would try his hand on the topics of *muʿāmalāt* mathematics. It is quite different, on the other hand, from what we would expect from a philosopher-theologian like Gundisalvi .

THE MANY MEANS

Chapter 15 of Fibonacci's *Liber abbaci* consists of three parts.[1] Most famous, and often discussed, is the third part, dealing with "certain problems according to the method of algebra and almuchabala, that is, by proportion and restoration". What concerns us here is the first part [ed. Boncompagni 1857: 387–397], which claims to deal with "the proportions of three and four quantities, to which the solution of many questions belonging to geometry are reduced" (p. 387). That it deals with three or four *magnitudes* in proportion is only directly wrong in so far as Fibonacci actually speaks of *numbers* afterwards; indeed, many geometric questions *are* reduced to problems about (geometric) proportions. But though not explicitly wrong the claim is misleading, since Fibonacci's text does not take up the applications to geometry, and never refers to geometric problems. Actually, when dealing with geometric problems in the next section and encountering one where a cross-reference would be adequate (p. 399) he seems to have forgotten what he has written a few pages earlier – which suggests that he is not composing independently but at least to some extent compiling from existing materials.

The section can be divided into 50 "logical paragraphs" (not always marked as paragraphs in the edition).

§§1–3 consider three numbers $P{:}Q{:}R$ in continued proportion. One of the numbers is given together with the sum of the other two. The naming of the segments that represent the numbers presupposes the Latin alphabetic order *a*, *b*, *c*, … .

The sequence §§4–38 still treats of three numbers, but now differences between the numbers are among the given magnitudes. The alphabetic order underlying naming changes to *a*, *b*, *g*, *d*, … .

[1] This section draws on what I have published in [Høyrup 2011].

§§39–50 consider four numbers in proportion, $\frac{P}{Q}$ 57 :: $\frac{R}{S}$ 58. The underlying alphabetic order is still *a*, *b*, *g*, *d*, … . At first (§39), the *e contrario* and *permutata* transformations are explained, and it is shown how any one of the numbers can be found from the three others via the product rule. Then follow problems where two of the numbers are given together with the sum of (§§40–45) respectively the difference between (§§46–49) the two others; finally, in §50, two numbers and the sum of the squares of the remaining two is given.

Most interesting is the sequence §§4–38. The Latin alphabetic sequence of §§1–3 allows the possibility that this opening was due to Fibonacci himself. The Arabic (or possibly Greek) order in the sequences §§4–38 and §§39–50 instead forces us to assume that they are copied without too much reelaboration – and if even §§1–3 should be copied (which I doubt), they must be copied from a different source (or different sources).

However, we may distinguish a fine structure. The letter *c* turns up in the *manipulations leading to the solution* in §§4–5, both of which still deal with numbers in continued proportion; moreover, these two and the observation §6 but none of the following paragraphs designate one of the segments by a single letter. Further, the continued proportion is treated again in §§27–29, without any cross-reference or apology for the repetition. Finally, §7 is preceded by the heading *modus alius proportionis inter tres numeros*. In consequence, §§4–5 may have been inserted by Fibonacci himself in continuation of the topic of §§1–3 but in emulation of the sequence which follows. The borrowed sequence should thus presumably be restricted to §§7–38 – or, in case even §§4–5 with the observation made in §6 represent a borrowing, then not from the same (ultimate) source as §§7–37.

As it turns out, all of these except §26 (on which imminently) and the observations §19 and §33 deal with the various means between two numbers discussed in ancient Greek mathematics[1] (with some deviations, on which imminently). More precisely, they show how to find the various means (Q) if the extremes P and R are given, or any of the extremes if the other extreme and a mean are given. The following scheme relates Fibonacci's problems with Pappos's and Nicomachos's presentations and order of these:[2]

	Pappos	Nicomachos	*Liber abbaci*
$\frac{R-Q}{Q-P}$ 59 :: (arithmet.)	P1	N1	
$\frac{R-Q}{Q-P}$ :: or $\frac{R-Q}{Q-P} :: \frac{Q}{P}$	P2	N2	§§27–29
$\frac{R-Q}{Q-P} :: \frac{R}{P}$	P3	N3	§§7–9
$\frac{R-Q}{Q-P} :: \frac{P}{R}$	P4	N4 (but inverted)	§§10–12 (inverted)
$\frac{R-Q}{Q-P} :: \frac{P}{Q}$	P5	N5 (but inverted)	§§34–36 (inverted)
$\frac{R-Q}{Q-P} :: \frac{Q}{R}$	P6	N6 (but inverted)	§§20–22 (inverted)
$\frac{R-P}{Q-P} :: \frac{R}{P}$	absent	N7	§§16–18
$\frac{R-P}{R-Q} :: \frac{R}{P}$	P9	N8	§§13–15
$\frac{R-P}{Q-P} :: \frac{Q}{P}$	P10	N9	§§30–32
$\frac{R-P}{R-Q} :: \frac{Q}{P}$	P7	N10	§§37–38
$\frac{R-P}{R-Q} :: \frac{R}{Q}$	P8	absent	§§23–25
$\frac{R-P}{R-Q} :: \frac{R}{Q}$ (Q unknown)	absent	absent	§26

[1] Cf. [Heath 1921: II, 85–88].
[2] Pappus, ed. [Hultsch 1876: I, 70–73, 84–87]; Nicomachos, ed. [Hoche 1886: 124–144]. Boethius, ed. [Friedlein 1867: 140–169] follows Nicomachos, his contents as well as his order.

As we see, Fibonacci agrees with Nicomachos and Boethius and not with Pappos in the cases 4–6, having $\frac{R}{P}$ :: $\frac{Q-P}{R-Q}$ 60 instead of $\frac{R-Q}{Q-P}$ :: $\frac{P}{R}$ 61, etc. However, more than the change of alphabetic order rules out that Fibonacci himself has produced a piece of theory inspired by Boethius. Firstly, he deals with the case P8 which is absent from Nicomachos's list, and his order is wholly different from both Greek authors as soon as we get beyond P4=N4, the subcontrary to the harmonic mean. Secondly, where these speak of *R–P* directly as the difference between the extremes, Fibonacci identifies it repeatedly as the sum of the first and the second difference. Thirdly, Fibonacci does not seem to have recognized the link to the ancient theory of *medietates* (which he would have known if building upon Boethius), nor to have seen that §§27–29 deal with the continued geometric proportion which was already treated in §§4–5. All of this confirms that Fibonacci uses a source whose ultimate inspiration was probably Nicomachos (who was well known by Arabic mathematicians) but which had been thoroughly reshaped, inserting missing cases, P8 as well as Fibonacci's §26,[1] omitting the uninteresting initial arithmetical mean, and transforming the list of *mere definitions* into a sequence of *problems with solutions*).

In §§39–50, single-letter naming of segments and the reappearance of the letter *c* in the manipulations suggest that this sequence may come from Fibonacci's own pen, or from a different source.

So, the sequence §§7–38 is another systematic theoretical exploration of the *Aufforderung zum Tanz* coming from a non-theoretical mathematical field. In so far it seems parallel to what we have observed in the *Liber mahameleth*. The methods used to solve the problems are also suggestive. Once again we find proportion transformations (*permutatim, conjunctim, disjunctim*, etc.); use of *Elements* II.5–6, without explicit reference to Euclid (which even Fibonacci usually likes to offer) and based on line diagrams like those of Abū Kāmil.

A GENERALIZED INHERITANCE PROBLEM

A number of Italian abbacus books contain a problem of this type:[2]

There is a gentleman who has a number of children, and it arrives that these sons of his have grown up and ask for their inheritance share because they want to be emancipated. And their father, when he sees their will, calls all of them and has a box carried in which is full of gold. And to the first he gives one mark of gold and $^1/_{10}$ of the remainder of the weight of all that which is in the box; and to the second he gives 2 marks and $^1/_{10}$ of the weight of that which is in the box; and to the third he gives 3 marks of it, and $^1/_{10}$ of the weight of that which is in the box, and in this way he divides everything stepwise, and when he comes to the last then he gives that which remains in the box, and then everyone counts what he has, and everyone finds that he has his portion precisely as that of each of the others. I ask you how many were the sons and how many the marks of each.

The fraction (henceforth φ) is almost invariably either $^1/_{10}$ or $^1/_7$. In both cases, the number N of sons equals $^1/_\varphi$–1, and the share of each is $\Delta = {}^1/_\varphi$–1. Sometimes – mostly as an alternative – the fraction is taken first, and the absolutely determined contribution second, in which case the number of sons is $^1/_\varphi$–1, and the share of each $^1/_\varphi$. On a few occasions the absolutely defined contributions start at n instead of 1, which simply means that the first n–1 shares are omitted (whence $N = {}^1/_\varphi - n$).

Outside Italy, the problem turns up in Byzantium and in the Iberian Peninsula before 1400[3] – namely in Planudes's late 13th-c. *Calculus according to the Indians, Called the Great* [ed., trans. Allard 1981: 191–194] and in the Castilian *Libro de arismética que es dicho alguarismo*, "Book about Arithmetic That is Called Algorism" (written in 1393, known from a sixteenth-century copy but building on material from no later than the early fourteenth century) [ed. Caunedo del Potro & Córdoba de la Llave 2000: 169].[4]

It is also found in Fibonacci's *Liber abbaci* (on which much more below). In extant Arabic sources,

[1] Already Heath [1921: II, 87] notices that this was omitted by the ancients; but he observes that this mean is "illusory" since it only exists if the extremes coincide; for Fibonacci and his source, who do not speak of means, the problem is fully valid, and to be treated – although this treatment then reveals its problematic character.

[2] This section draws on [Høyrup 2008: 37*f*]. The present version of the problem is taken from Paolo Gherardi's *Libro di ragioni* [ed. Arrighi 1987a: 37].

[3] For briefness I shall omit discussion of all occurrences after 1400, even though some of them might be pertinent. But see [Høyrup 2008].

[4] In [Høyrup 2008: 632] I not only overlooked this occurrence but also explicitly denied its existence, which led me to a mistaken conclusion.

however, we only find this, coming from ibn al-Yāsamīn's *Talqīḥ al-afkār fī'l 'amali bi rušūm al-ghubār* ("Fecundation of thoughts through use of *ghubār* numerals") – written in Marrakesh in c. 1190:[1]

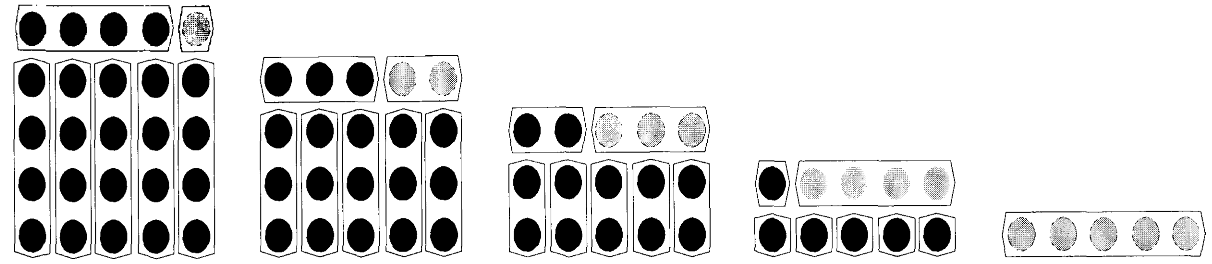
An inheritance of an unknown amount. A man has died and has left at his death to his six children an unknown amount. He has left to one of the children one dinar and the seventh of what remains, to the second child two dinars and the seventh of what remains, to the third three dinars and the seventh of what remains, to the fourth child 4 dinars and the seventh of what remains, to the fifth child 5 dinars and the seventh of what remains, and to the sixth child what remains. He has required the shares be identical. What is the sum?

The solution is to multiply the number of children by itself; you find 36, it is the unknown sum. This is a rule that recurs in all problems of the same type.

On one hand, this is earlier than any other occurrence we know of, and furthermore shows that ibn al-Yāsamīn refers to the problem he presents as a representative of *a type;* on the other, this is *not* the problem type we have discussed so far. The difference is that the latter is not a "Chinese box problem" that can be solved by reverse calculation, which that of ibn al-Yāsamīn can (betraying moreover the total number of shares): if S is what is left when the fifth share is to be taken, the fifth share is $5+\frac{1}{7}$ 62(S–5), and the sixth share is what is left after that, i.e., S–5–$\frac{1}{7}$ 63(S–5). From their equality follows that S is 12, each share thus 6, and the total therefore 6·6. Even though ibn al-Yāsamīn's version is no doubt derived from the "Italian" type, it has been reduced to a piece of normal, less astounding mathematics.[2]

The Italian version is therefore not likely to be derived from anything circulating in the Arabic world. Since we have no trace of anything similar in Italy before ibn al-Yāsamīn, we must therefore look elsewhere – and Byzantium, perhaps inheriting from late Antiquity, suggests itself. Planudes, indeed, gives the problem as an illustration of this theorem:[3]

When a unit is taken away from any square number, the left-over is measured by two numbers multiplied by each other, one smaller than the side of the square by a unit, the other larger than the same side by a unit. As for instance, if from 36 a unit is taken away, 35 is left. This is measured by 5 and 7, since the quintuple of 7 is 35. If again from 35 I take away the part of the larger number, that is the seventh, which is then 5 units, and yet 2 units, the left-over, which is then 28, is measured again by two numbers, one smaller than the said side by two units, the other larger by a unit, since the quadruple of 7 is 28. If again from the 28 I take away 3 units and its seventh, which is then 4, the left-over, which is then 21, is measured by the number which is three units less than the side and by the one which is larger by a unit,



since the triple of 7 is 21. And always in this way.

Planudes does not refer to counters or geometry, but his text fits the diagram above (reduced for

[1] My translation from Mahdi Abdeljaouad's privately communicated French translation.

[2] Another, even more reduced version is found in *al-Ma'ūna fī 'ilm al-ḥisāb al-hawā'ī* ("Assistance in the science of mental calculation"), written by ibn al-Hā'im (1352–1412, Cairo, Mecca & Jerusalem (even this one I know thanks to the kind assistance of Mahdi Abdeljaouad).

[3] [Ed. Allard 1981: 191], in very literal translation. A less literal French translation accompanies Allard's edition. The passage comes from Planudes's *Calculus according to the Indians* – but from the second part of this work, which has nothing to do with the use of Indian numerals. This part also contains material known from the probably late ancient Chapter 24 of the pseudo-Heronic conglomerate *Geometrica*.

simplicity to 5×5) to perfection. Without support by a geometric representation or by symbolic algebra (which Planudes did not have) it is difficult to see that the "theorem" holds for "any square number", and that the procedure will continue in such a way that exactly nothing remains in the end (actually, in symbolic algebra the proof of the latter point is laborious). So (and for supplementary reasons), as I argued in [2008], the problem is quite likely to be of Byzantine or late ancient Greek origin. Since this is not very important for my present topic, I shall not repeat the reasoning.

Let us now look at Fibonacci's *Liber abbaci* – more precisely at the second version from 1228 [ed. Boncompagni 1857: 279–281], since we have no evidence that this section was already in the 1202-version (nor, to be sure, any reason to believe it was not). We may designate by (α,ε|φ) the type where absolutely defined contributions $\alpha+\varepsilon i$ (i = 0, 1, ...) are taken first, and a fraction φ of the remainder afterwards; (φ|α,ε) designates the type where a fraction φ of what is at disposal is taken first and absolutely defined contributions $\alpha+\varepsilon i$ (i = 0, 1, ...) afterwards. Then Fibonacci's problems are the following (only the problems in the left columns speak about a heritage, the others are pure-number problems):

$(1,1\vert\frac{1}{7}64)$	$(1,1\vert\frac{2}{11})$	$(2,3\vert\frac{6}{31})$	$(3,2\vert\frac{5}{19})$
$(\frac{1}{7}\vert1,1)$	$(4,4\vert\frac{2}{11})$	$(\frac{6}{31}\vert2,3)$	$(\frac{5}{19}\vert3,2)$
$(3,3\vert\frac{1}{7})$	$(\frac{2}{11}\vert1,1)$		
$(\frac{1}{7}\vert3,3)$	$(\frac{2}{11}\vert4,4)$		

As we see, the first two columns contain the simple traditional problem types (with the trivial variation in column 1 that the monetary unit may be 3 or 4 *bizantii* instead of 1, whereas column 2 further presupposes the generalization that $^2/_{11} = {}^1/_{5\frac{1}{2}}$).

In the third and fourth column, on the other hand, we encounter situations where the traditional formulas ($N = {}^1/_{\varphi}-n$, etc.) do not work. In column 3, Fibonacci finds the solution to $(2,3|\frac{6}{31}65)$ by means of the *regula recta*, that is, in our terms, first-degree equation algebra with unknown *thing* (*res*). Fibonacci posits the initial total T (the number to be divided) as the *thing*, and finds by successive computation the first two shares, which he knows to be equal. The resulting equation leads to $T = 56\frac{1}{4}$ 66; the number of shares turns out to be $N = 4\frac{1}{2}$ 67; and each share $\Delta = 12\frac{1}{2}$ 68. He has thus found the *only possible* solution, but his algebraic computation does not show that the subsequent shares will also be $12\frac{1}{2}$ 69. Fibonacci does not point this out explicitly, but he makes a complete calculation step by step and so verifies that the first four shares are $12\frac{1}{2}$ 70, after which $6\frac{1}{4}$ 71 remains for the final $\frac{1}{2}$ 72-share.

In the end Fibonacci claims to "extract" the following rule from the calculation[1] ($\varphi = \frac{p}{q}$ 73):

(1ᵃ) $$T = \frac{[(\varepsilon-\alpha)\,q+(q-p)\,\alpha]-(q-p)}{p^2}\ 74\,,$$

(1ᵇ) $$N = \frac{(\varepsilon-\alpha)\,q+(q-p)\,\alpha}{\varepsilon\,p}\ 75\,,$$

(1ᶜ) $$\Delta = \frac{\varepsilon\,(q-p)}{p}\ 76\,.$$

Actually, this rule is *not* extracted. If one follows the algebraic calculation step by step, it leads to

(2ᵃ) $$T = \frac{q^2(\alpha+\varepsilon)-(q-p)q\alpha-(q-p)p\alpha-(\alpha+\varepsilon)pq}{p^2}\ 77$$

which (by means which were at Fibonacci's disposal) could be transformed into

(2ᵃ*) $$T = \frac{[q\,(\alpha+\varepsilon)-(p+q)\,\alpha]-(q-p)}{p^2}\ 78$$

but not in any obvious way into the rule which Fibonacci pretends to extract – if anything, further transformation would rather yield

(3ᵃ) $$T = \frac{[\varepsilon q-\alpha p]-(q-p)}{p^2}\ 79\,.$$

We must conclude that Fibonacci adopted a rule whose fundament he did not know, and that he pretended it to be a consequence of his own (correct but partial) solution.

This is confirmed by his treatment of the problem $(3,2|\frac{5}{19}80)$. Here, α cannot be subtracted from ε, and therefore Fibonacci replaces (1) by

(4ᵃ) $$T = \frac{[(q-p)\,\alpha-(\alpha-\varepsilon)\,q]-(q-p)}{p^2}\ 81\,,$$

(4ᵇ) $$N = \frac{(q-p)\,\alpha-(\alpha-\varepsilon)\,q}{\varepsilon p}\ 82\,,$$

(4ᶜ) $$\Delta = \frac{\varepsilon\,(q-p)}{p}\ 83\,.$$

If Fibonacci himself had reduced the algebraic solution (2ᵃ), why would he have chosen an expression which is neither fully reduced nor valid for all cases? Neither (2ᵃ) nor (2ᵃ*) nor (3ᵃ) depends on whether α<ε or α>ε.

For the case $(\frac{6}{31}84|2,3)$, Fibonacci gives the rules

(5ᵃ) $$T = \frac{[(\varepsilon-\alpha)\,q+(q-p)\,\alpha]-q}{p^2}\ 85\,,$$

(5ᵇ) $$N = \frac{(\varepsilon-\alpha)\,q+(q-p)\,\alpha}{\varepsilon p}\ 86\,,$$

[1] Obviously using the specific numbers belonging to the problem when stating the rule; but since he identifies each number by pointing to its role in the computation, the symbolic formulas map his rule unambiguously.

$$(5^c) \qquad \Delta = \frac{\varepsilon q}{p} 87\,,$$

and for $(\frac{5}{19}\,88|3.2)$

$$(6^a) \qquad T = \frac{[(q-p)\,\alpha - (\alpha-\varepsilon)\,q] - q}{p^2} 89\,,$$

$$(6^b) \qquad N = \frac{(q-p)\,\alpha - (\alpha-\varepsilon)\,q}{\varepsilon p} 90\,,$$

$$(6^c) \qquad \Delta = \frac{\varepsilon q}{p} 91\,.$$

Once again, if (1^a) had really resulted from the algebraic solution, why should he offer (5) and (6) without deriving them from algebraic operations (which could not be the same as before)?

So, not only the "simple versions" of the problem (those of columns 1 and 2) and their rules were "around"[1] but also the sophisticated versions and rules for columns 3–4. Where did they originate?

Italy can presumably be ruled out – before Fibonacci, we have no traces of anybody or any environment with the necessary mathematical skills or interests. Even though Provence is one of the regions where Fibonacci tells to have learned [ed. Boncompagni 1857: 1], that area seems to be excluded for the same reason. Since the Arabic *mu'āmalāt* culture (even generalized to the works of ibn al-Yāsamīn) did not know the problem except in a distorted and simplified version, that also seems to be excluded. The method we know from Planudes only applies to integer φ (see below), and nothing in Planudes's words suggests he knew more, nor do later Byzantine writers go beyond that.

As we have seen, Chapter 15 Part 1 of the *Liber abbaci* offers evidence that Fibonacci borrowed not only single problems or passages but also long coherent stretches of text. This is confirmed by one of the two oldest manuscripts of the *Liber abbaci* (Biblioteca Vaticana, Palat. 1343), as already noticed by Baldassare Boncompagni [1851: 32]: On fol. 47 (most recent foliation), in the transition between recto and verso, we find "hic incipit magister castellanus. Incipit capitulum no|num de baractis", so at least the initial part of the chapter on barter (perhaps the whole of it) is taken over from a Castilian book (only books, no oral instruction, have incipits).

Since Fibonacci did not know how his formulas had been derived, he must have borrowed them as a set; the only plausible origin that remains seems to be the Iberian peninsula. Would that make this expanded investigation of the unknown heritage a third case of sophisticated arithmetical theory created in twelfth-century al-Andalus and only surviving (precariously) in Latin and Romance languages?

At the general level, the style is the same: taking a piece of fairly elementary mathematics – purchase or selling according to the rule of three, the mere definition of the many kinds of means, and here a puzzling arithmetical riddle – and then looking at it "from a higher vantage point" and taking it as a pretext for developing mathematical theory systematically.

[1] Indeed, in 1370 Giovanni de' Danti [ed. Arrighi 1987b: 70] explains the solution to a problem $(1,1|^1/_{10})$ in a way that would work for any $\varphi = {}^p/_q$, that is, in column 2:

A man is dying and he has several sons, and he makes his testament and leaves his money in this way, that to the first son he leaves 1 *f* and $^1/_{10}$ of what remains, and to the second he leaves 2 *f* and $^1/_{10}$ of what remains to him when the first son has been paid, to the third son he leaves 3 *f* and $^1/_{10}$ of what remains when the first and the second have been paid, to the fourth sone he leaves 4 *f* and $^1/_{10}$ of what remains for him, and in this way step by step until everything is gone. I ask how many were the sons and how many the *f* which he left to them, that is, that each of them got as much as the others. This is the rule, because you say $^1/_{10}$, therefore detract the 1 that is above from the 10, 9 remain, divide 9 by 1 that is above in $^1/_{10}$, 9 results, and 9 were the sons. In order to know how many were the *f* he left to them, multiply 9 by itself, it makes 81, and 81 were the *f* he left to them, and it is done.

Afterwards, Giovanni describes in a similar way the solution of problem $(^1/_{10}|1,1)$. This is evidently long after Fibonacci, but the procedure suggests the same trick as the one which Fibonacci uses in column 2 rather than reduction of Fibonacci's formulas.

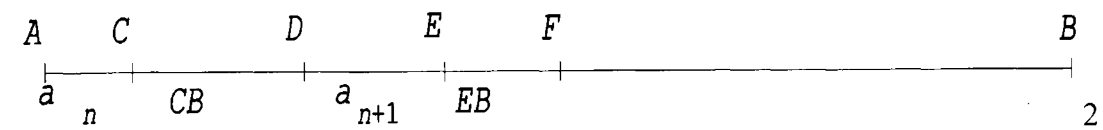
Mutatis mutandis, however, Felix Klein would do something similar some 800 years later. That is, so to speak, a thing mathematicians do. Until we dig out further similarities, all we can say is "could be". So, are the methods used in the three cases of the same kind (as we saw that they were in the first two cases)? That would increase the possibility that the similarity is historically grounded and not only an outcome of professional sociology.

Fibonacci does not help us very much. Since he does not know how his formulas were derived he obviously cannot tell. We are left with reconstruction.

Geometric diagrams of the kind suggested by Planudes could at a pinch be used to show the adequacy of the formulas *a posteriori*. In [Høyrup 2008: 627 n.16] I show this for the relatively simple case $(1,3|^2/_9)$. The example shows it to be utterly implausible that anybody would *get the idea* from such a diagram; with pebbles, which are not as easily divisible as squares, the whole matter becomes forbiddingly difficult.

Symbolic algebra could be used, but is evidently out of the question. Line diagrams, like those used by Abū Kāmil, in the *Liber mahameleth* and in Chapter 15 Part 1 of the *Liber abbaci*, are not – and they turn out to be quite fit for the task. I shall quote from [Høyrup 2008: 627*f*] the proof for the case (α,ε|φ) (the case (φ|α,ε) is easier). We look at a distribution where a number is divided in such a way that each share is the sum of some absolutely defined value and a fixed fraction φ of what remains at disposition. The aim is to show that the shares are equal if and only if the absolutely defined contributions form an arithmetical series:

for convenience I shall use letter symbols, but pointing and words could do the same:



AB represents S_n, that is, the amount that is at disposition when the *n*-th share is to be taken, *n* being arbitrary (but possible).[1] This share is *AD*, consisting of $AC = a_n$ and $CD = \varphi CB$. The following share is *DF*, consisting of $DE = a_{n+1}$ and $EF = \varphi EB$.

Since $AD = DF = \Delta$, $CB = CD+DB$, and $EB = EF+FB$, we find that

$$a_{n+1}-a_n = \varphi(CB-EB) = \varphi(CD-EF)+\varphi(DB-FB) = \varphi(a_{n+1}-a_n)+\varphi\Delta\,,$$

whence

$$(1-\varphi)\cdot(a_{n+1}-a_n) = \varphi\Delta$$

and further (in order to avoid a formal algebraic division) the proportion

$$\Delta :: (a_{n+1}-a_n) = (1-\varphi) :: \varphi\,.$$

By means, for instance, of Euclid's *Data*, prop. 2 [trans. Taisbak 2003: 254], "If a given magnitude [here Δ] have a given ratio [here (1–φ):φ] to some other magnitude [here $a_{n+1}-a_n$], the other is also given in magnitude" (or applying simply the rule of three), we find that $a_{n+1}-a_n$ has the same value irrespective of the step where we are. In consequence, the absolutely defined contributions have to constitute an arithmetical progression.

[...]

[1] The reason Fibonacci offered no proof of this kind may be that the structures of secondary logic ("for any ...", "for all ...", etc.) were not integrated in his mathematical standard language and therefore did not offer themselves readily for the construction of proofs. The present line-diagram proof, if made during or before his times, is likely not to have looked at an arbitrary step but to have started from the first and then given an argument by quasi-induction. Fibonacci, making the calculation in numbers that change from step to step, could not generalize his result in that way.

Once we are so far it is legitimate to construct the rules from the equality of the first two shares only. This can be done by somewhat laborious but simple first-degree algebra – Fibonacci shows one way to do it, but there are alternatives.

A medieval astronomer-mathematician better trained in proportion techniques than I am might possibly make more use of these than I have done. In any case it is clear, however, that the techniques used for my first two cases would also work here – while it is not easily seen which other techniques at hand at the time would do so.

SUMMING UP

So, all in all, the extrapolations of *muʿāmalāt* mathematics into the realm of higher theory and the investigation of the properties of the many means are likely to come, if not from the same hand then at least from the same environment – and Gundisalvi's reference to the "book which in Arabic is called *Mahamalech*" tells us that this environment was located in *al-Andalus*. The hypothesis that the theoretical elaboration of the unknown heritage was made in the same environment builds on indirect arguments – but as long as no credible alternative has been found, it remains the plausible assumption.

In [1993: 86], Ahmed Djebbar pointed out that there was

in Spain and before the eleventh century, a solid research tradition in arithmetic whose starting point seems to have been the translation made by Thābit ibn Qurra of Nicomachos' Introduction to Arithmetic.

The present study suggests that this research tradition survived into the twelfth century, that is, as long as al-Andalus remained scientifically productive.

AN ADDENDUM TO THE ADDENDUM

After having sent what precedes for the proceedings and expecting only to have to return to the matter for layout and proofreading I received a question from Satyanad Kichenassamy. He had analyzed Tartaglia's claims that "his derivation of the solution of some cubic equations relies on the theory of continued proportions in the form expounded by Pacioli, that provides substitutes of 'algebraic identities' that were of course not available at that time", as he summarizes a recently published of his [Kichenassamy 2015]. Having seen that one of the less obvious of these was mentioned in a paper of mine [Høyrup 2009] he asked me about the earlier history of these relations. As I was able to tell, they appear in Jacopo da Firenze's *Liber algorismi* from 1307 (the first appearance in known sources), and I had discussed them in my edition of that work [Høyrup 2007: 115–121]. They turn up as four problems dealing with the wages of the manager of a *fondaco* (Arabic *funduq*, but the word had been naturalized into Tuscan), supposed to increase in geometric proportion. Undressed, the problems are:

(1) $a:b:d$, $a+d=20$, $b=8$

(2) $a:b:d:e$, $a=15$, $e=60$

(3) $a:b:d:e$, $a+e=90$, $b+d=60$

(4) $a:b:d:e$, $a+d=20$, $b+e=30$

None of the solutions make use of *al-jabr* algebra – we only encounter prescriptions of numerical algorithms. Via the product rule, (1) reduces to the situation of *Elements* II.5, and the solutions is made accordingly. (2) finds the factor of proportionality $p=\sqrt[3]{\frac{60}{5}}$ 92, and then $b=pa$, $c=p^2a$, while (4) finds it as $\sqrt{\frac{b\cdot e}{a\cdot d}}$ 93 – after which a can be found as $\frac{a\cdot d}{1\cdot p^2}$ 94, etc. All of this is similar to what we know from the *Liber mahameleth* and *Liber abbaci*, Chapter 15 part 1. (3) introduces the rule which allowed Tartaglia to solve certain irreducible cubics:

$$ae=bd=\frac{(b+d)^3}{3(b+d)+(a+e)} \quad 1$$

once ae and bd are known, the procedure from (1) can be followed.

I do not know this formula from any earlier source, but al-Karajī's treatment of the third power of a polynomial [Woepcke 1853: 58] shows that it does not go beyond Arabic mathematical thought at its best. In any case, since nothing similar to this group of problems seems to be known from surviving Arabic sources, it may present us with yet another flower borrowed from twelfth-century al-Andalus mathematics.

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The Indian summer of al-Andalus mathematics?

An expanded addendum

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Abstract

At the 11ième Colloque Maghrébin in Algers in October 2013 I spoke about the *Liber mahameleth*, and I suggested that this work is no independent compilation made by a Latin scholar but a (probably free) translation of an Arabic work presenting "*muʿāmalāt vom höheren Standpunkt aus*, 'from a higher vantage point'", written by an Arabic astronomer-mathematician, which however had no impact in later Arabic mathematics but only (through the translation, and even here modestly) in the Latin world. At the last moment of my preparation it occurred to me that this might be one of three instances of advanced arithmetic from 12th-century *al-Andalus* that only survived in Latin translation but not in Arabic, and I introduced that idea in an addendum.

At the present occasion I shall present this suggestion in more depth, discussing all three instances.

Contents

À Ahmed Djebbar, en amitié

In my contribution to the 11ième Colloque Maghrébin I examined the relation of the mid-twelfth-century *Liber mahameleth* to Arabic *muʿāmalāt* mathematics and arrived at the general conclusion (as opposed to earlier workers[1]) that it was no Latin compilation made directly on the basis of genuine Arabic *muʿāmalāt* works but instead a (plausibly free) translation of that "book which in Arabic is called *Mahamalech*"[2] spoken of by Gundisalvi in the *De divisione philosophiae* [ed. Baur 1903: 93], made perhaps by Gundisalvi himself, and if not by some collaborator or contact of his. I concluded moreover that this Arabic work was not a *muʿāmalāt* book proper but a work treating of *muʿāmalāt* "von höheren Standpunkt aus" ("from a higher vantage point"), in the words of Felix Klein's lecture [1908] – that is, a work presenting (select) *muʿāmalāt* topics on a theoretically satisfactory basis, and produced by a mathematician trained in Euclid and proportion techniques (that is, almost certainly, a mathematician also versed in astronomy).

I had several reasons for this conclusion, of which only one is relevant for my present topic – namely the way algebra and proportion theory are used. I came to the conclusion that sophisticated mathematics was employed to unfold the theoretical possibilities inherent in (certain kinds of) *muʿāmalāt* mathematics, and that this was not transmitted efficiently to the Arabic world before the collapse of *al-Andalus* but only survived in a Latin translation. The fate of Jābir ibn Aflaḥ's and ibn Rušd's writings show that this would not be impossible.

Then it suddenly dawned to me that two other examples of sophisticated innovative mathematics appear to have had the same origin and the same fate. I had worked on both on earlier occasions without connecting them, even though both are known from Fibonacci's *Liber abbaci*. That "discovery" was made in the last moment: my contribution was already too long; and its focus was different. In consequence, I could only describe the discovery rather briefly. In what follows I shall therefore take up the thread, presenting the three cases one by one.

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[1] See [Høyrup 2013: 2–3].

[2] My translation, as all translations in the following when nothing else is stated.

The *Liber mahameleth*

Several problem sequences in the *Liber mahameleth* take as their starting point and their purported subject a typical *muʿāmalāt*-problem – for instance, buying or selling, which begins by stating the rule of three and then presents the two alternatives where division precedes multiplication [ed. Vlasschaert 2010: 186].[3] That is fully traditional, and corresponds for example to what is found in al-Karajī's *Kāfī* [ed., trans. Hochheim 1878: II, 16*f*]. But then follow variations never encountered in commercial practice, nor dealt with in books presenting *muʿāmalāt* calculation. Using p and P for prices, q and Q for the appurtenant quantities, we have $\frac{q}{p}::\frac{Q}{P}$. Forgetting every consideration about dimensions and homogeneity (and thus about concrete meaning), the text first presents us with these problems in systematic order:[4]

$\frac{3}{13}::\frac{Q}{P}$, $Q+P = 60$	$\frac{3}{8}::\frac{Q}{P}$, $Q\cdot P = 216$	$\frac{4}{9}::\frac{Q}{P}$, $\sqrt{P}-\sqrt{Q} = 1\frac{1}{2}$
$\frac{3}{13}::\frac{Q}{P}$, $P-Q = 60$	$\frac{4}{9}::\frac{Q}{P}$, $\sqrt{Q}+\sqrt{P} = 7\frac{1}{2}$	$\frac{4}{9}::\frac{Q}{P}$, $\sqrt{Q}\cdot\sqrt{P} = 24$

The main tools that are used are proportion transformations (*conversa*, *eversa*, etc.); the first problem, for instance, is transformed into $\frac{3}{3+13}::\frac{Q}{Q+P}$, after which the rule of three can be applied. Some of the transformations are more intricate, and appear to presuppose an implicit understanding of the ratios as divisions. For $\frac{4}{9}::\frac{Q}{P}$, $\sqrt{Q}+\sqrt{P} = 7\frac{1}{2}$, three solutions are proposed, the last of which, when expressed in a formula, amounts to an amazing

$$\left(\sqrt{\frac{(\sqrt{P}+\sqrt{Q})^2}{(P-Q)/Q}+\left(\frac{\sqrt{P}+\sqrt{Q}}{(P-Q)/Q}\right)^2}-\frac{\sqrt{P}+\sqrt{Q}}{(P-Q)/Q}\right)^2 = Q.$$

A chapter follows "about the same, with [algebraic] things" (*res* – r in my symbolic translations). First comes $\frac{3}{10+r}::\frac{1}{r}$ – in words, "Three measures are given for 10 coins and a thing, but this thing is the price of one measure". Then (in similar formulations)

[3] Since I used this edition for the Algers meeting, at a moment when [Sesiano 2014] was not yet available, and since I reviewed Vlasschaert's edition and therefore have a heavily annotated copy, this is the edition I shall refer to throughout.

[4] Closer descriptions of all of these problems and analysis of the way they are solved can be found in [Høyrup 2013: 14–20].

$\frac{4}{20+2r}::\frac{1½}{2r+3}$ $\frac{6}{10+r}::\frac{2}{r+1}$ $\frac{4}{8-r}::\frac{2}{r+1}$

$\frac{8}{20+r}::\frac{2}{r-1}$ $\frac{3}{20+r}::\frac{\frac{q}{2}}{\frac{2}{3}r-2}$ $\frac{4}{20-2r}::\frac{1½}{2r-3}$

$\frac{6}{10+r}::\frac{2}{r}$ $\frac{6}{10-r}::\frac{2}{r}$

Familiarity with other Arabic *muʿāmalāt* writings or Italian abbacus books might make us expect use of cross-multiplication – but then we shall be deluded. The first problem is transformed into $\frac{3}{10+r}::\frac{3}{3r}$, from which it is seen that $3r = 10+r$; alternatively (a subtractive variant of *ex aequa*), $\frac{3-1}{(10+r)-r}::\frac{1}{r}$, that is, $\frac{2}{10}::\frac{1}{r}$, whence $\frac{1}{5}::\frac{1}{r}$, etc. The following questions are treated similarly, and in the end comes the observation (p. 199) that "this kind of questions cannot be understood unless one is trained in algebra or in Euclid's book".

In yet "another chapter about an unknown in buying and selling" we then find that an unknown number of measures is sold for 93, and addition of this number to the price of one measure gives 34 – in our symbols (since no *res* occurs): $x+\frac{93}{x} = 34$. First the solution is stated as $\frac{34}{2}\pm\sqrt{(\frac{34}{2})^2-93}$ (the sign depending on whether the number of measures exceeds or falls short of the price of one measure). Then a geometric argument based on a subdivided line (following the principles of *Elements* II.5) is given. Contrary to the normal predilections of the *Liber mahameleth* Euclid is not mentioned; the direct inspiration might therefore be Abū Kāmil's similar proof for the *al-jabr* case "possession plus number equals things" (Abū Kāmil's algebra is referrred to repeatedly and correctly in the *Liber mahameleth*). This is followed by one of the subtractive variants, again with alternative precriptions; the other subtractive variant is told to be solved correspondingly.

Then comes a problem $\frac{q}{p}::\frac{Q}{P}$, $pq = 6$, $PQ = 24$, $(p+q)+(P+Q) = 15$. The argument goes via a tacitly presupposed factor of proportionality s (= 2), $sp = P$, $sq = Q$ (later, a geometric argument shows how to find this factor, so it *is* really presupposed). An apparently innocuous analogue follows, $\frac{q}{p}::\frac{Q}{P}$, $pq = 10$, $PQ = 30$, $(p+q)+(P+Q) = 20$ – but this requires $s = \sqrt{3}$ and therefore entails complications and an appropriate cross-reference to the chapter about roots, and finally leads to a discussion in terms of the classification of *Elements* X, apparently expected to be familiar.

Next come a subtractive and two multiplicative variants, using similar methods (the latter two requiring a rational respectively an irrational value for s). Then two questions not defined in terms of a proportion, and where the identification of the variables as price and quantity is therefore nothing but a pretext to present them in the actual context:

$$\sqrt{p} = 3q,\ p–q = 34 \qquad \text{and} \qquad \sqrt{p} = 2q,\ p+q = 18$$

Both are solved first by a numerical quadratic completion ($\sqrt{p}$ serving as basic unknown), then by a line-based geometric proof. Then follow the two problems

$$\frac{6}{4+r} :: \frac{2}{3\sqrt{(4+r)}} \qquad \text{and} \qquad \frac{6}{4-r} :: \frac{2}{3\sqrt{(4-r)}}$$

The left problem is transformed into $\frac{6}{4+r} :: \frac{6}{9\sqrt{(4+r)}}$. The resulting equation $(4+r) = 9\sqrt{4+r}$ is not made explicit but the numerical prescription corresponds to a transformation into $\sqrt{4+r} = 9$ and further into $4+r = 81$ – the right problem being solved correspondingly. That is, the text somehow makes use not of *al-jabr* but of equivalent patterns of thought. This interpretation is confirmed by the next question,

$$\frac{3}{x+y} :: \frac{1}{y+\frac{1}{9}y},\ xy = 21$$

(x and y occur as "two different things"). The prescription corresponds to a transformation into $\frac{3}{x+y} :: \frac{3}{3y+\frac{1}{3}y}$, whence $x+y = 3\frac{1}{3}y$, $x = 2\frac{1}{3}y$, $2\frac{1}{3}y^2 = 21$, $y^2 = 9$, and finally $y = 3$, $x = 7$ (afterwards shown by a line-based argument). Alternatively, the problem is solved "according to *al-jabr*", which must hence be something different. Now the *thing* (r) takes the place of y, while the *dragma* (d) takes that of x. This time, a different but similar transformation of the proportion is used, namely $\frac{1}{\frac{1}{3}d+\frac{1}{3}r} :: \frac{1}{r+\frac{1}{9}r}$, etc. This is followed by the analogue $\frac{5}{x+y} :: \frac{1}{\frac{1}{3}x+2}$, $xy = 144$, solved by similar methods, which this time lead to a mixed second-degree problem.

Similar systematically varied problem sequences take as their starting point (or pretext) profit and interest, partnership, etc.

This is a far cry from anything that can be found in genuine *muʿāmalāt* contexts. We notice, firstly, the preponderant use of proportion techniques and of line geometry similar to what Abū Kāmil uses in his proofs of the fundamental *al-jabr* rules; secondly, the minor role played by *al-jabr* algebra; and thirdly, the use of methods that for us looks as first- and second-degree algebra but are considered distinct from *al-jabr*. It is also different from what we find in authors somehow moving in the vicinity of the *madrasah* environment – say, ibn al-Yāsamīn and ibn al-Bannāʾ.

It corresponds well, on the other hand, to what could be done by a member of the other main class of Arabic mathematicians – those whose professional upbringing had brought them through the *Elements*, the "middle books" on spherics, and the *Almagest* – if he would try his hand on the topics of *muʿāmalāt* mathematics. It is quite different, on the other hand, from what we would expect from a philosopher-theologian like Gundisalvi.

The many means

Chapter 15 of Fibonacci's *Liber abbaci* consists of three parts.[5] Most famous, and often discussed, is the third part, dealing with "certain problems according to the method of algebra and almuchabala, that is, by proportion and restoration". What concerns us here is the first part [ed. Boncompagni 1857: 387–397], which claims to deal with "the proportions of three and four quantities, to which the solution of many questions belonging to geometry are reduced" (p. 387). That it deals with three or four *magnitudes* in proportion is only directly wrong in so far as Fibonacci actually speaks of *numbers* afterwards; indeed, many geometric questions *are* reduced to problems about (geometric) proportions. But though not explicitly wrong the claim is misleading, since Fibonacci's text does not take up the applications to geometry, and never refers to geometric problems. Actually, when dealing with geometric problems in the next section and encountering one where a cross-reference would be adequate (p. 399) he seems to have forgotten what he has written a few pages earlier – which suggests that he is not composing independently but at least to some extent compiling from existing materials.

The section can be divided into 50 "logical paragraphs" (not always marked as paragraphs in the edition).

§§1–3 consider three numbers *P:Q:R* in continued proportion. One of the numbers is given together with the sum of the other two. The naming of the segments that represent the numbers presupposes the Latin alphabetic order *a*, *b*, *c*, ...

The sequence §§4–38 still treats of three numbers, but now differences between the numbers are among the given magnitudes. The alphabetic order underlying naming changes to *a*, *b*, *g*, *d*, ...

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[5] This section draws on what I have published in [Høyrup 2011].

§§39–50 consider four numbers in proportion, $\frac{P}{Q} :: \frac{R}{S}$. The underlying alphabetic order is still *a*, *b*, *g*, *d*, ... At first (§39), the *e contrario* and *permutata* transformations are explained, and it is shown how any one of the numbers can be found from the three others via the product rule. Then follow problems where two of the numbers are given together with the sum of (§§40–45) respectively the difference between (§§46–49) the two others; finally, in §50, two numbers and the sum of the squares of the remaining two is given.

Most interesting is the sequence §§4–38. The Latin alphabetic sequence of §§1–3 allows the possibility that this opening was due to Fibonacci himself. The Arabic (or possibly Greek) order in the sequences §§4–38 and §§39–50 instead forces us to assume that they are copied without too much reelaboration – and if even §§1–3 should be copied (which I doubt), they must be copied from a different source (or different sources).

However, we may distinguish a fine structure. The letter *c* turns up in the *manipulations leading to the solution* in §§4–5, both of which still deal with numbers in continued proportion; moreover, these two and the observation §6 but none of the following paragraphs designate one of the segments by a single letter. Further, the continued proportion is treated again in §§27–29, without any cross-reference or apology for the repetition. Finally, §7 is preceded by the heading *modus alius proportionis inter tres numeros*. In consequence, §§4–5 may have been inserted by Fibonacci himself in continuation of the topic of §§1–3 but in emulation of the sequence which follows. The borrowed sequence should thus presumably be restricted to §§7–38 – or, in case even §§4–5 with the observation made in §6 represent a borrowing, then not from the same (ultimate) source as §§7–37.

As it turns out, all of these except §26 (on which imminently) and the observations §19 and §33 deal with the various means between two numbers discussed in ancient Greek mathematics[6] (with some deviations, on which imminently). More precisely, they show how to find the various means (Q) if the extremes P and R are given, or any of the extremes if the other extreme and a mean are given. The following scheme relates Fibonacci's problems with Pappos's and Nicomachos's presentations and order of these:[7]

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[6] Cf. [Heath 1921: II, 85–88].

[7] Pappus, ed. [Hultsch 1876: I, 70–73, 84–87]; Nicomachos, ed. [Hoche 1886: 124–144]. Boethius, ed. [Friedlein 1867: 140–169] follows Nicomachos, his contents as well as his order.

	Pappos	Nicomachos	*Liber abbaci*
$\frac{R-Q}{Q-P} :: \frac{R}{R}$ (arithmet.)	P1	N1	
$\frac{R-Q}{Q-P} :: \frac{R}{Q}$ or $\frac{R-Q}{Q-P} :: \frac{Q}{P}$	P2	N2	§§27–29
$\frac{R-Q}{Q-P} :: \frac{R}{P}$	P3	N3	§§7–9
$\frac{R-Q}{Q-P} :: \frac{P}{R}$	P4	N4 (but inverted)	§§10–12 (inverted)
$\frac{R-Q}{Q-P} :: \frac{P}{Q}$	P5	N5 (but inverted)	§§34–36 (inverted)
$\frac{R-Q}{Q-P} :: \frac{Q}{R}$	P6	N6 (but inverted)	§§20–22 (inverted)
$\frac{R-P}{Q-P} :: \frac{R}{P}$	absent	N7	§§16–18
$\frac{R-P}{R-Q} :: \frac{R}{P}$	P9	N8	§§13–15
$\frac{R-P}{Q-P} :: \frac{Q}{P}$	P10	N9	§§30–32
$\frac{R-P}{R-Q} :: \frac{Q}{P}$	P7	N10	§§37–38
$\frac{R-P}{R-Q} :: \frac{R}{Q}$	P8	absent	§§23–25
$\frac{R-P}{R-Q} :: \frac{R}{Q}$ (Q unknown)	absent	absent	§26

As we see, Fibonacci agrees with Nicomachos and Boethius and not with Pappos in the cases 4–6, having $\frac{R}{P} :: \frac{Q-P}{R-Q}$ instead of $\frac{R-Q}{Q-P} :: \frac{P}{R}$, etc. However, more than the change of alphabetic order rules out that Fibonacci himself has produced a piece of theory inspired by Boethius. Firstly, he deals with the case P8 which is absent from Nicomachos's list, and his order is wholly different from both Greek authors as soon as we get beyond P4=N4, the subcontrary to the harmonic mean. Secondly, where these speak of $R-P$ directly as the difference between the extremes, Fibonacci identifies it repeatedly as the sum of the first and the second difference. Thirdly, Fibonacci does not seem to have recognized the link to the ancient theory of *medietates* (which he would have known if building upon Boethius), nor to have seen that §§27–29 deal with the continued geometric proportion which was already treated in §§4–5. All of this confirms that Fibonacci uses a source whose ultimate inspiration was probably Nicomachos (who was well known by Arabic mathematicians) but which had been thoroughly reshaped, inserting missing cases, P8 as well as Fibonacci's §26,[8] omitting the uninteresting initial arithmetical mean, and transforming the list of *mere definitions* into a sequence of *problems with solutions*).

In §§39–50, single-letter naming of segments and the reappearance of the letter *c* in the manipulations suggest that this sequence may come from Fibonacci's own pen, or from a different source.

So, the sequence §§7–38 is another systematic theoretical exploration of the *Aufforderung zum Tanz* coming from a non-theoretical mathematical field. In so far it seems parallel to what we have observed in the *Liber mahameleth*. The methods used to solve the problems are also suggestive. Once again we find proportion transformations (*permutatim*, *conjunctim*, *disjunctim*, etc.); use of *Elements* II.5–6, without explicit reference to Euclid (which even Fibonacci usually likes to offer) and based on line diagrams like those of Abū Kāmil.

A generalized inheritance problem

A number of Italian abbacus books contain a problem of this type:[9]

> There is a gentleman who has a number of children, and it arrives that these sons of his have grown up and ask for their inheritance share because they want to be emancipated. And their father, when he sees their will, calls all of them and has a box carried in which is full of gold. And to the first he gives one mark of gold and $^{1}/_{10}$ of the remainder of the weight of all that which is in the box; and to the second he gives 2 marks and $^{1}/_{10}$ of the weight of that which is in the box; and to the third he gives 3 marks of it, and $^{1}/_{10}$ of the weight of that which is in the box, and in this way he divides everything stepwise, and when he comes to the last then he gives that which remains in the box, and then everyone counts what he has, and everyone

[8] Already Heath [1921: II, 87] notices that this was omitted by the ancients; but he observes that this mean is "illusory" since it only exists if the extremes coincide; for Fibonacci and his source, who do not speak of means, the problem is fully valid, and to be treated – although this treatment then reveals its problematic character.

[9] This section draws on [Høyrup 2008: 37*f*]. The present version of the problem is taken from Paolo Gherardi's *Libro di ragioni* [ed. Arrighi 1987a: 37].

finds that he has his portion precisely as that of each of the others. I ask you how many were the sons and how many the marks of each.

The fraction (henceforth ϕ) is almost invariably either $^1/_{10}$ or $^1/_7$. In both cases, the number N of sons equals $^1/_\phi$–1, and the share of each is $\Delta = {}^1/_\phi$–1. Sometimes – mostly as an alternative – the fraction is taken first, and the absolutely determined contribution second, in which case the number of sons is $^1/_\phi$–1, and the share of each $^1/_\phi$. On a few occasions the absolutely defined contributions start at n instead of 1, which simply means that the first n–1 shares are omitted (whence $N = {}^1/_\phi - n$).

Outside Italy, the problem turns up in Byzantium and in the Iberian Peninsula before 1400[10] – namely in Planudes's late 13th-c. *Calculus according to the Indians, Called the Great* [ed., trans. Allard 1981: 191–194] and in the Castilian *Libro de arismética que es dicho alguarismo*, "Book about Arithmetic That is Called Algorism" (written in 1393, known from a sixteenth-century copy but building on material from no later than the early fourteenth century) [ed. Caunedo del Potro & Córdoba de la Llave 2000: 169].[11]

It is also found in Fibonacci's *Liber abbaci* (on which much more below). In extant Arabic sources, however, we only find this, coming from ibn al-Yāsamīn's *Talqīḥ al-afkār fī'l 'amali bi rušūm al-ghubār* ("Fecundation of thoughts through use of *ghubār* numerals") – written in Marrakesh in c. 1190:[12]

An inheritance of an unknown amount. A man has died and has left at his death to his six children an unknown amount. He has left to one of the children one dinar and the seventh of what remains, to the second child two dinars and the seventh of what remains, to the third three dinars and the seventh of what remains, to the fourth child 4 dinars and the seventh of what remains, to the fifth child 5 dinars and the seventh of what remains, and to the sixth child what remains. He has required the shares be identical. What is the sum?

The solution is to multiply the number of children by itself, you find 36, it is the unknown sum. This is a rule that recurs in all problems of the same type.

On one hand, this is earlier than any other occurrence we know of, and furthermore shows that ibn al-Yāsamīn refers to the problem he presents as a representative of *a type;* on the other, this is *not* the problem type we have

[10] For briefness I shall omit discussion of all occurrences after 1400, even though some of them might be pertinent. But see [Høyrup 2008].

[11] In [Høyrup 2008: 632] I not only overlooked this occurrence but also explicitly denied its existence, which led me to a mistaken conclusion.

[12] My translation from Mahdi Abdeljaouad's privately communicated French translation.

discussed so far. The difference is that the latter is not a "Chinese box problem" that can be solved by reverse calculation, which that of ibn al-Yāsamīn can (betraying moreover the total number of shares): if S is what is left when the fifth share is to be taken, the fifth share is $5+\frac{1}{7}(S–5)$, and the sixth share is what is left after that, i.e., $S–5–\frac{1}{7}(S–5)$. From their equality follows that S is 12, each share thus 6, and the total therefore $6{\cdot}6$. Even though ibn al-Yāsamīn's version is no doubt derived from the "Italian" type, it has been reduced to a piece of normal, less astounding mathematics.[13]

The Italian version is therefore not likely to be derived from anything circulating in the Arabic world. Since we have no trace of anything similar in Italy before ibn al-Yāsamīn, we must therefore look elsewhere – and Byzantium, perhaps inheriting from late Antiquity, suggests itself. Planudes, indeed, gives the problem as an illustration of this theorem:[14]

> When a unit is taken away from any square number, the left-over is measured by two numbers multiplied by each other, one smaller than the side of the square by a unit, the other larger than the same side by a unit. As for instance, if from 36 a unit is taken away, 35 is left. This is measured by 5 and 7, since the quintuple of 7 is 35. If again from 35 I take away the part of the larger number, that is the seventh, which is then 5 units, and yet 2 units, the left-over, which is then 28, is measured again by two numbers, one smaller than the said side by two units, the other larger by a unit, since the quadruple of 7 is 28. If again from the 28 I take away 3 units and its seventh, which is then 4, the left-over, which is then 21, is measured by the number which is three units less than the side and by the one which is larger by a unit, since the triple of 7 is 21. And always in this way.

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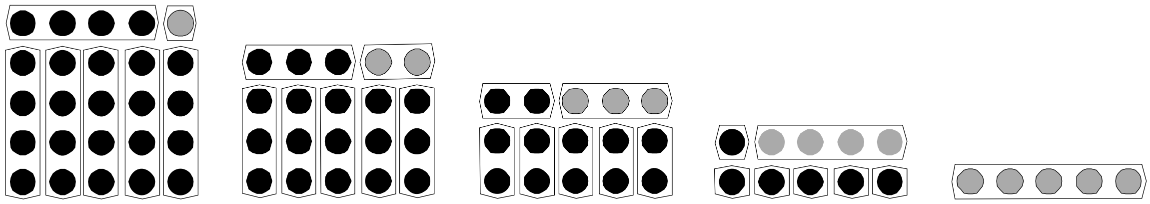
[13] Another, even more reduced version is found in *al-Maʿūna fī ʿilm al-ḥisāb al-hawāʾī* ("Assistance in the science of mental calculation"), written by ibn al-Hā'im (1352–1412, Cairo, Mecca & Jerusalem (even this one I know thanks to the kind assistance of Mahdi Abdeljaouad).

[14] [Ed. Allard 1981: 191], in very literal translation. A less literal French translation accompanies Allard's edition.

The passage comes from Planudes's *Calculus according to the Indians* – but from the second part of this work, which has nothing to do with the use of Indian numerals. This part also contains material known from the probably late ancient Chapter 24 of the pseudo-Heronic conglomerate *Geometrica*.



Planudes does not refer to counters or geometry, but his text fits the diagram above (reduceed for simplicity to 5×5) to perfection. Without support by a geometric representation or by symbolic algebra (which Planudes did not have) it is difficult to see that the "theorem" holds for "any square number", and that the procedure will continue in such a way that exactly nothing remains in the end (actually, in symbolic algebra the proof of the latter point is laborious). So (and for supplementary reasons), as I argued in [2008], the problem is quite likely to be of Byzantine or late ancient Greek origin. Since this is not very important for my present topic, I shall not repeat the reasoning.

Let us now look at Fibonacci's *Liber abbaci* – more precisely at the second version from 1228 [ed. Boncompagni 1857: 279–281], since we have no evidence that this section was already in the 1202-version (nor, to be sure, any reason to believe it was not). We may designate by $(\alpha,\varepsilon\,|\,\phi)$ the type where absolutely defined contributions $\alpha+\varepsilon i$ (i = 0, 1, ...) are taken first, and a fraction ϕ of the remainder afterwards; $(\phi\,|\,\alpha,\varepsilon)$ designates the type where a fraction ϕ of what is at disposal is taken first and absolutely defined contributions $\alpha+\varepsilon i$ (i = 0, 1, ...) afterwards. Then Fibonacci's problems are the following (only the problems in the left columns speak about a heritage, the others are pure-number problems):

$(1,1\,\|\,\frac{1}{7})$	$(1,1\,\|\,\frac{2}{11})$	$(2,3\,\|\,\frac{6}{31})$	$(3,2\,\|\,\frac{5}{19})$
$(\frac{1}{7}\,\|\,1,1)$	$(4,4\,\|\,\frac{2}{11})$	$(\frac{6}{31}\,\|\,2,3)$	$(\frac{5}{19}\,\|\,3,2)$
$(3,3\,\|\,\frac{1}{7})$	$(\frac{2}{11}\,\|\,1,1)$		
$(\frac{1}{7}\,\|\,3,3)$	$(\frac{2}{11}\,\|\,4,4)$		

As we see, the first two columns contain the simple traditional problem types (with the trivial variation in column 1 that the monetary unit may be 3 or 4 *bizantii* instead of 1, whereas column 2 further presupposes the generalization that $^{2}/_{11} = {}^{1}/_{5½}$).

In the third and fourth column, on the other hand, we encounter situations where the traditional formulas ($N = {}^1/_\phi - n$, etc.) do not work. In column 3, Fibonacci finds the solution to $(2,3 \mid \frac{6}{31})$ by means of the *regula recta*, that is, in our terms, first-degree equation algebra with unknown *thing* (*res*). Fibonacci posits the initial total T (the number to be divided) as the *thing*, and finds by successive computation the first two shares, which he knows to be equal. The resulting equation leads to $T = 56\frac{1}{4}$; the number of shares turns out to be $N = 4\frac{1}{2}$; and each share $\Delta = 12\frac{1}{2}$. He has thus found the *only possible* solution, but his algebraic computation does not show that the subsequent shares will also be $12\frac{1}{2}$. Fibonacci does not point this out explicitly, but he makes a complete calculation step by step and so verifies that the first four shares are $12\frac{1}{2}$, after which $6\frac{1}{4}$ remains for the final $\frac{1}{2}$-share.

In the end Fibonacci claims to "extract" the following rule from the calculation[15] ($\phi = \frac{p}{q}$):

$$(1^a) \qquad T = \frac{[(\varepsilon-\alpha)\,q+(q-p)\,\alpha]\cdot(q-p)}{p^2}\ ,$$

$$(1^b) \qquad N = \frac{(\varepsilon-\alpha)\,q+(q-p)\,\alpha}{\varepsilon p}\ ,$$

$$(1^c) \qquad \Delta = \frac{\varepsilon\,(q-p)}{p}\ .$$

Actually, this rule is *not* extracted. If one follows the algebraic calculation step by step, it leads to

$$(2^a) \qquad T = \frac{q^2(\alpha+\varepsilon)-(q-p)q\alpha-(q-p)p\alpha-(\alpha+\varepsilon)pq}{p^2}$$

which (by means which were at Fibonacci's disposal) could be transformed into

$$(2^{a*}) \qquad T = \frac{[q\,(\alpha+\varepsilon)-(p+q)\,\alpha]\cdot(q-p)}{p^2}$$

[15] Obviously using the specific numbers belonging to the problem when stating the rule; but since he identifies each number by pointing to its role in the computation, the symbolic formulas map his rule unambiguously.

but not in any obvious way into the rule which Fibonacci pretends to extract – if anything, further transformation would rather yield

$$(3^a) \qquad T = \frac{[\varepsilon q - \alpha p]\cdot(q-p)}{p^2} \ .$$

We must conclude that Fibonacci adopted a rule whose fundament he did not know, and that he pretended it to be a consequence of his own (correct but partial) solution.

This is confirmed by his treatment of the problem $(3{,}2\,|\,\frac{5}{19})$. Here, α cannot be subtracted from ε, and therefore Fibonacci replaces (1) by

$$(4^a) \qquad T = \frac{[(q-p)\alpha-(\alpha-\varepsilon)q]\cdot(q-p)}{p^2} \ ,$$

$$(4^b) \qquad N = \frac{(q-p)\alpha-(\alpha-\varepsilon)q}{\varepsilon p} \ ,$$

$$(4^c) \qquad \Delta = \frac{\varepsilon\,(q-p)}{p} \ .$$

If Fibonacci himself had reduced the algebraic solution (2^a), why would he have chosen an expression which is neither fully reduced nor valid for all cases? Neither (2^a) nor (2^{a*}) nor (3^a) depends on whether $\alpha<\varepsilon$ or $\alpha>\varepsilon$.

For the case $(\frac{6}{31}\,|\,2{,}3)$, Fibonacci gives the rules

$$(5^a) \qquad T = \frac{[(\varepsilon-\alpha)\,q+(q-p)\alpha]\cdot q}{p^2} \ ,$$

$$(5^b) \qquad N = \frac{(\varepsilon-\alpha)q+(q-p)\alpha}{\varepsilon p} \ ,$$

$$(5^c) \qquad \Delta = \frac{\varepsilon q}{p} \ ,$$

and for $(\frac{5}{19}\,|\,3{,}2)$

$$(6^a) \qquad T = \frac{[(q-p)\alpha-(\alpha-\varepsilon)q]\cdot q}{p^2} \ ,$$

$$(6^b) \qquad N = \frac{(q-p)\alpha-(\alpha-\varepsilon)q}{\varepsilon p} \ ,$$

$$(6^c) \qquad \Delta = \frac{\varepsilon q}{p} \ .$$

Once again, if (1^a) had really resulted from the algebraic solution, why should he offer (5) and (6) without deriving them from algebraic operations (which could not be the same as before)?

So, not only the "simple versions" of the problem (those of columns 1 and 2) and their rules were "around"[16] but also the sophisticated versions and rules for columns 3–4. Where did they originate?

Italy can presumably be ruled out – before Fibonacci, we have no traces of anybody or any environment with the necessary mathematical skills or interests. Even though Provence is one of the regions where Fibonacci tells to have learned [ed. Boncompagni 1857: 1], that area seems to be excluded for the same reason. Since the Arabic *muᶜāmalāt* culture (even generalized to the works of ibn al-Yāsamīn) did not know the problem except in a distorted and simplified version, that also seems to be excluded. The method we know from Planudes only applies to integer ϕ (see below), and nothing in Planudes's words suggests he knew more, nor do later Byzantine writers go beyond that.

As we have seen, Chapter 15 Part 1 of the *Liber abbaci* offers evidence that Fibonacci borrowed not only single problems or passages but also long coherent stretches of text. This is confirmed by one of the two oldest manuscripts of the *Liber abbaci* (Biblioteca Vaticana, Palat. 1343), as already noticed by Baldassare Boncompagni [1851: 32]: On fol. 47 (most recent foliation), in the transition between recto and verso, we find "hic incipit magister castellanus. Incipit capitulum no | num de baractis", so at least the initial part of the chapter on barter (perhaps the whole of it) is taken over from a Castilian book (only books, no

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[16] Indeed, in 1370 Giovanni de' Danti [ed. Arrighi 1987b: 70] explains the solution to a problem $(1,1\,|\,/^1/_{10})$ in a way that would work for any $\phi = {}^p/_q$, that is, in column 2:

> A man is dying and he has several sons, and he makes his testament and leaves his money in this way, that to the first son he leaves 1 *ƒ* and $^1/_{10}$ of what remains, and to the second he leaves 2 *ƒ* and $^1/_{10}$ of what remains to him when the first son has been paid, to the third son he leaves 3 *ƒ* and $^1/_{10}$ of what remains when the first and the second have been paid, to the fourth sone he leaves 4 *ƒ* and $^1/_{10}$ of what remains for him, and in this way step by step until everything is gone. I ask how many were the sons and how many the *ƒ* which he left to them, that is, that each of them got as much as the others. This is the rule, because you say $^1/_{10}$, therefore detract the 1 that is above from the 10, 9 remain, divide 9 by 1 that is above in $^1/_{10}$, 9 results, and 9 were the sons. In order to know how many were the *ƒ* he left to them, multiply 9 by itself, it makes 81, and 81 were the *ƒ* he left to them, and it is done.

Afterwards, Giovanni describes in a similar way the solution of problem $(^1/_{10}\,|\,1,1)$. This is evidently long after Fibonacci, but the procedure suggests the same trick as the one which Fibonacci uses in column 2 rather than reduction of Fibonacci's formulas.

oral instruction, have incipits).

Since Fibonacci did not know how his formulas had been derived, he must have borrowed them as a set; the only plausible origin that remains seems to be the Iberian peninsula. Would that make this expanded investigation of the unknown heritage a third case of sophisticated arithmetical theory created in twelfth-century al-Andalus and only surviving (precariously) in Latin and Romance languages?

At the general level, the style is the same: taking a piece of fairly elementary mathematics – purchase or selling according to the rule of three, the mere definition of the many kinds of means, and here a puzzling arithmetical riddle – and then looking at it "from a higher vantage point" and taking it as a pretext for developing mathematical theory systematically.

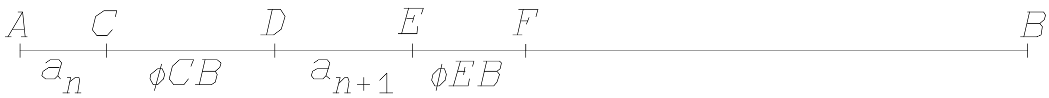
Mutatis mutandis, however, Felix Klein would do something similar some 800 years later. That is, so to speak, a thing mathematicians do. Until we dig out further similarities, all we can say is "could be". So, are the methods used in the three cases of the same kind (as we saw that they were in the first two cases)? That would increase the possibility that the similarity is historically grounded and not only an outcome of professional sociology.

Fibonacci does not help us very much. Since he does not know how his formulas were derived he obviously cannot tell. We are left with reconstruction.

Geometric diagrams of the kind suggested by Planudes could at a pinch be used to show the adequacy of the formulas *a posteriori*. In [Høyrup 2008: 627 n.16] I show this for the relatively simple case $(1{,}3\,|\,{}^{2}/_{9})$. The example shows it to be utterly implausible that anybody would *get the idea* from such a diagram; with pebbles, which are not as easily divisible as squares, the whole matter becomes forbiddingly difficult.

Symbolic algebra could be used, but is evidently out of the question. Line diagrams, like those used by Abū Kāmil, in the *Liber mahameleth* and in Chapter 15 Part 1 of the *Liber abbaci*, are not – and they turn out to be quite fit for the task. I shall quote from [Høyrup 2008: 627*f*] the proof for the case $(\alpha,\varepsilon\,|\,\phi)$ (the case $(\phi\,|\,\alpha,\varepsilon)$ is easier). We look at a distribution where a number is divided in such a way that each share is the sum of some absolutely defined value and a fixed fraction ϕ of what remains at disposition. The aim is to show that the shares are equal if and only if the absolutely defined contributions form an arithmetical series:

for convenience I shall use letter symbols, but pointing and words could do the same:



AB represents S_n, that is, the amount that is at disposition when the *n*-th share is to be taken, *n* being arbitrary (but possible).[17] This share is *AD*, consisting of $AC = a_n$ and $CD = \phi CB$. The following share is *DF*, consisting of $DE = a_{n+1}$ and $EF = \phi EB$.

Since $AD = DF = \Delta$, $CB = CD+DB$, and $EB = EF+FB$, we find that

$$a_{n+1}-a_n = \phi(CB-EB) = \phi(CD-EF)+\phi(DB-FB) = \phi(a_{n+1}-a_n)+\phi\Delta\ ,$$

whence

$$(1-\phi)\cdot(a_{n+1}-a_n) = \phi\Delta$$

and further (in order to avoid a formal algebraic division) the proportion

$$\Delta :: (a_{n+1}-a_n) = (1-\phi) :: \phi\ .$$

By means, for instance, of Euclid's *Data*, prop. 2 [trans. Taisbak 2003: 254], "If a given magnitude [here Δ] have a given ratio [here $(1-\phi):\phi$] to some other magnitude [here $a_{n+1}-a_n$], the other is also given in magnitude" (or applying simply the rule of three), we find that $a_{n+1}-a_n$ has the same value irrespective of the step where we are. In consequence, the absolutely defined contributions have to constitute an arithmetical progression.

[...]

once we are so far it is legitimate to construct the rules from the equality of the first two shares only. This can be done by somewhat laborious but simple first-degree algebra – Fibonacci shows one way to do it, but there are alternatives.

A medieval astronomer-mathematician better trained in proportion techniques than I am might possibly make more use of these than I have done. In any case it is clear, however, that the techniques used for my first two cases would also work here – while it is not easily seen which other techniques at hand at the time would do so.

[17] The reason Fibonacci offered no proof of this kind may be that the structures of secondary logic ("for any ...", "for all ...", etc.) were not integrated in his mathematical standard language and therefore did not offer themselves readily for the construction of proofs. The present line-diagram proof, if made during or before his times, is likely not to have looked at an arbitrary step but to have started from the first and then given an argument by quasi-induction. Fibonacci, making the calculation in numbers that change from step to step, could not generalize his result in that way.

Summing up

So, all in all, the extrapolations of *muʿāmalāt* mathematics into the realm of higher theory and the investigation of the properties of the many means are likely to come, if not from the same hand then at least from the same environment – and Gundisalvi's reference to the "book which in Arabic is called *Mahamalech*" tells us that this environment was located in *al-Andalus*. The hypothesis that the theoretical elaboration of the unknown heritage was made in the same environment builds on indirect arguments – but as long as no credible alternative has been found, it remains the plausible assumption.

In [1993: 86], Ahmed Djebbar pointed out that there was

> in Spain and before the eleventh century, a solid research tradition in arithmetic whose starting point seems to have been the translation made by Thābit ibn Qurra of Nicomachos' *Introduction to Arithmetic*.

The present study suggests that this research tradition survived into the twelfth century, that is, as long as al-Andalus remained scientifically productive.

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