Roskilde
University

Learning difficulties in matematics

What are their nature and origin and what can we do to counteract them?

Niss, Mogens Allan

Published in:
Cuadernos de Investigación y Formación en Educación Matemática

Publication date:
2018

Citation for published version (APA):
Niss, M. A. (2018). Learning difficulties in matematics: What are their nature and origin and what can we do to counteract them? *Cuadernos de Investigación y Formación en Educación Matemática*, *17*(Julio 2018), 127-140.

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain.
- You may freely distribute the URL identifying the publication in the public portal.

Take down policy
If you believe that this document breaches copyright please contact rucforsk@ruc.dk providing details, and we will remove access to the work immediately and investigate your claim.

Download date: 28. Feb. 2021

Learning difficulties in mathematics. What are their nature and origin, and what can we do to counteract them?[1]

Mogens Niss

Abstract

Around the world, more and more students are exposed to mathematics as a compulsory subject, also at rather advanced levels. Large numbers of students who work hard to come to grips with the subject experience severe difficulties at learning mathematics and get no real success out of their efforts. Many of them do well in other subjects, so their learning difficulties are not of a general nature. Over the last several decades, mathematics education research has made substantial progress in understanding the nature and characteristics of the learning of mathematics, and hence also of significant obstacles to mathematics learning. Less effort has been invested in utilising the insights thus gained to help students with learning difficulties in mathematics overcome or diminish their difficulties. This paper offers an account and an analysis of the nature and origins of learning difficulties in mathematics from a research perspective, and also presents a long-term research and development programme in Denmark focusing on enabling practicing mathematics teachers to become so-called "maths counsellors" who can help reduce or remove upper secondary (high school) students' learning difficulties in mathematics. This programme also offers a unique way of making research an integrated and efficient instrument for teachers' everyday practice.

Keywords

Mathematics, Mathematics Education, Learning Difficulties, Math Counsellors, Denmark.

Resumen

En todo el mundo, más y más estudiantes están expuestos a las matemáticas como asignatura obligatoria, también en niveles bastante avanzados. Un gran número de estudiantes que trabajan arduamente para enfrentarse al tema experimentan serias dificultades para aprender matemáticas y a pesar de sus esfuerzos no obtienen ningún éxito real. A muchos de ellos les va bien en otras asignaturas, por lo que sus dificultades de aprendizaje no son de naturaleza general. En las últimas décadas, la investigación en educación matemática ha avanzado sustancialmente en la comprensión de la naturaleza y las características del aprendizaje de las matemáticas y, por lo tanto, también de obstáculos significativos para el

M. Niss
IMFUFA, INM,
Roskilde University, Denmark
mn@ruc.dk

[1] Este trabajo corresponde a la conferencia plenaria dictada por el autor en el II CEMACYC, celebrado en Cali, Colombia, del 29 de octubre al 1 de noviembre de 2017.

Recibido por los editores el 16 de febrero de 2018 y aceptado el 15 de marzo de 2018.

aprendizaje de las matemáticas. Se ha invertido menos esfuerzo en la utilización de los conocimientos adquiridos para ayudar a los estudiantes con dificultades de aprendizaje en matemáticas a superar o disminuir sus dificultades. Este documento ofrece una cuenta y un análisis de la naturaleza y los orígenes de las dificultades de aprendizaje en matemáticas desde una perspectiva de investigación, y también presenta un programa de investigación y desarrollo a largo plazo en Dinamarca que se enfoca en capacitar a los profesores de matemáticas para que se conviertan en "consejeros matemáticos". ¿Quién puede ayudar a reducir o eliminar las dificultades de aprendizaje de los estudiantes de secundaria superior (secundaria) en matemáticas? Este programa también ofrece una forma única de hacer de la investigación un instrumento integrado y eficiente para la práctica diaria de los docentes.

Palabras clave

Matemáticas, Educación Matemática, Dificultades de Aprendizaje, Consejeros Matemáticos, Dinamarca.

1. Introduction and background

We shall take our point of departure in four empirical facts:

- Many students around the world, at any education level, experience severe problems in learning mathematics. This includes many students who work hard to learn mathematics but remain unsuccessful despite their efforts. In other words, the problems are not only limited to students who, for one reason or another, have given up investing time and effort in the endeavour.
- Since many such students do reasonably well in other subjects (except in those that depend heavily on mathematics), these students are not suffering from general learning problems. Their problems are specific to mathematics.
- More and more students, also at advanced levels, are forced to take compulsory mathematics, or feel the need to take mathematics because of the programme they are in, without really wishing to do so for the sake of mathematics.
- The students who experience mathematics specific learning difficulties generally are very unhappy about it; they often feel inferior and defeated and lose self-confidence, sometimes to a degree that is detrimental to them even beyond the field of mathematics.

If we – as primary, secondary or tertiary teachers, as researchers, or as educational authorities – want to help the students at issue to counteract their mathematics specific learning difficulties, we must be able to identify these and to understand their nature and origin. Fortunately some fifty years of research in mathematics education has provided us with insights and tools that allow us to do so, at least to a significant extent.

Research has taught us a lot about how mathematical *learning* takes place (or does not take place) across individuals, cultures, societies and systems. Despite considerable variation in these respects, there are important commonalities, perhaps even universalities, in the cognitive and emotive organisation of human beings that have a marked impact on their (opportunities for) mathematics learning. As Celia Hoyles has pointed out in a personal communication, this is in stark contrast to the state of affairs in mathematics *teaching*. We know much less about how teaching can be designed, organised and implemented so as to guarantee successful learning of mathematics. This is due to the fact that geographical, socio-economic, technological, political, structural and cultural conditions and factors are crucial determinants of the outcomes of mathematics teaching, and these vary greatly across institutions, towns and cities, countries, regions and continents.

Let us return to mathematical learning and look at why this is so difficult.

2. Why is it difficult to learn mathematics?

There are at least six different overarching sources of difficulty to the learning of mathematics.

The first source is the very *nature of mathematics as a discipline*. Coming to grips with mathematical concepts, with the specific nature of mathematical statements, with the modes of justification and reasoning in mathematics, with the role and forms of symbols and formalisms in mathematical processes and theoretical edifices, with the notion and characteristics of abstraction, abstractness and generalisation in the development of mathematics – all of which typically deviate from the state of affairs in other disciplines – are known to generate learning difficulties, in some cases of a severe nature.

Whilst this source of difficulty is primarily internal to mathematics, the second source, *the relationships between mathematics and the rest of the world*, goes beyond mathematics in itself. Making sense and meaning in real world terms of mathematical ideas and entities is a necessity to most learners, and the fact that this is sometimes hard to achieve, as is also true of the linking of mathematical results and theory with extra-mathematical experience, is a significant source of difficulty to the learning of mathematics. The same holds for that subtle connection between extra-mathematical domains and mathematical domains that forms the essence of mathematical modelling. For example, many learners find it hard to understand why the indisputable logical rigour of mathematical considerations within a given mathematical domain does not automatically give rise to indisputable results within extra-mathematical domains modelled by the mathematical domain under consideration, because they overlook the importance of assumptions pertaining to the extra-mathematical domain.

The third source of difficulty is *uncertainty about what it means and takes to learn mathematics*. Much of this uncertainty is related to learners' beliefs about mathematics. Beliefs are long-term and stable convictions about something that an individual holds to be true, no matter whether these convictions are in fact true, or are considered to be true by the relevant community of professionals (Phillip 2007). By being long-term and stable, beliefs are very difficult to change. Much more than manifest evidence is needed for an individual to change his or her beliefs about something. As to mathematics, beliefs concern one or more of the following (Jankvist 2015): beliefs about mathematics as a discipline, beliefs about mathematics teaching and learning, beliefs about the role of mathematics in society and culture, beliefs about the individual's relationship with mathematics - as a discipline, as an educational subject, as a field situated in culture and society. As beliefs further concern one or more of the links between these four elements, beliefs concerning mathematics can be perceived as a tetrahedron with four vertices and six edges. It is important to understand that beliefs about what it means and takes to learn mathematics are not only shaped by experiences and beliefs concerning mathematics teaching and learning, but also by beliefs concerning the other vertices and edges in the tetrahedron. This source of difficulty also deals with notions about the way(s) in which the learning of mathematics takes place (is perceived to take place), notions that are largely shaped by the learner's experiences from his or her current and previous mathematical education. Another factor at play here is whether, in the mind or the environment of the learner, mathematics is perceived as something everyone can learn if proper assistance is provided, or whether mathematics learning is seen as being reserved for just a few people in possession of an appropriate background and appropriate intellectual (perhaps even biological) prerequisites.

The *experiences that students have gained from their life spheres*, i.e. their everyday life with family and friends, life in the local community and in the surrounding society, in schooling at large, and in the world – including nature - in general, also constitute a (fourth) source of difficulty to the learning of mathematics. This is mainly because the far majority of experiences gained from those spheres are not immediately conducive to the learning of mathematics. The fact that mathematics does indeed permeate a multitude of aspects in society and culture is only rarely recognised by people, since mathematics largely remains hidden and invisible on the surface of things, except for matters related to basic arithmetic (numbers, money, weight and measure) and elementary geometry (seldom going beyond names of standard 2D-figures and 3D-solids). As Richard Noss once put it in a panel debate, the purpose of digital technology is to hide the role and significance of mathematics in friendly disguises that do not remind users of the presence of mathematics, whereas one of the tasks for mathematics educators is to uncover and reveal the role and significance of mathematics in ICT. Since everyday life for most people only offers few opportunities for non-elementary encounters with mathematics, mathematics easily becomes relegated to an isolated world of its own, which includes the mathematics classroom, governed by its own issues and rules, having no links with or bearing on "the real world".

The fifth source of difficulty to mathematical learning consists of *students' experiences from mathematics classrooms*. This may seem rather surprising, since we have just seen that the mathematics classroom is the primary environment from which students can acquire experiences with mathematics. However, this also implies that negative, limiting, distorted, or misinterpreted experiences about mathematics from the classroom can generate learning difficulties. First of all, as the classroom sets the scene for the mathematical learning experiences it also frames the opportunities to learn mathematics, in terms of teachers' background, competencies and behaviour, in terms of the material and immaterial resources and conditions available for teaching and learning - e.g. textbooks, technology, equipment, control of the level of noise, climate, or time - and in terms of the backgrounds and conduct of peers. Two classroom specific factors have a marked impact on students' classroom experiences. One is the didactical contract

(Brousseau 1997), which - established tacitly during several years in mathematics classrooms – defines the division of labour and the ensuing expectations between the teacher and the students. What can the students expect that the teacher will / will not do inside and outside the mathematics classroom: what is the teacher's role in presenting and explaining subject matter, what kinds of questions will the teacher pose to students, what is the nature and extent of the tasks students are expected to undertake, what sorts of help will the teacher provide to students, what modes of assessment will the teacher employ and when, what kind of feedback will the teacher give to the students, and so on and so forth? Conversely, what can the teacher expect the students to do inside and outside the classroom, individually, in small groups or in whole class settings? What kinds of learning activities are students supposed to engage in, and how independently are they expected to work, and for how long? What sorts of home work are students expected to accept and undertake, and under what time frames? What kinds of assessment tasks and tests will students be asked to attend to? The didactical contract greatly contributes to shaping students' mathematics classroom experiences, and hence possible learning difficulties arising from these experiences. The other factor which plays out in the individual mathematics classroom is the set of socio-mathematical norms (tacitly) established in any given mathematics classroom (Yackel & Cobb 1996). The social-mathematical norms of a given classroom combine general social norms for acceptable and encouraged behaviour in any classroom with the mathematical norms (tacitly) set for mathematical work at large. The socio-mathematical norms frame what is considered a valuable and novel contribution to the mathematical discourse in the classroom, for example a good question, a novel and effective idea for solving a problem, an interesting observation of a pattern or phenomenon, or an innovative line of reasoning. Both the didactical contract and the socio-mathematical norms are highly significant factors in students' learning of mathematics, and hence are, depending on their actual content, potential sources of mathematical learning difficulties as well.

The sixth and final source of difficulty is of a more general nature, namely the genesis, structure, organisation and functioning of *human cognition* at large. It would go far beyond the limits of this article to offer a thorough or detailed treatment of this topic. Suffice it to be mentioned that the fact that human cognition is rooted in and fundamentally governed by experiences gained from the material and social worlds of human beings gives rise to serious challenges to the coming to grips with the abstracted and abstract concepts of mathematics and with general mathematical claims covering an actual infinity of cases, all of which display relationships with the experiential world that are, at best, highly indirect. This, in and of itself, is a source of difficulty to the learning of mathematics.

3. Examples of specific learning difficulties

During the last five decades, mathematics education research – alias the didactics of mathematics – has made huge progress in identifying, analysing and understanding major specific mathematical learning difficulties pertaining to all six sources of difficulty. Here, it is only possible to zoom in on selected items.

As regards *the nature of mathematics*, an infinitude of learning difficulties are associated with coming to grips with mathematical *concepts*, even with the most elementary ones such as *number*. Many students have difficulty in accepting and understanding that 0 is indeed a number and not just a marker of absence, and in understanding the idea and functioning of place value representations of natural numbers, the relationships between fractions and rational numbers, especially in the context of the number line, the nature and role of negative numbers, the definition and meaning of decimal expansions, not to speak of irrational numbers and the distinction of these from rational numbers, and even more so of complex numbers. In what sense is it meaningful and reasonable to call all these, very different, objects *numbers* within one big hierarchy of number domains?

Similarly, justifying mathematical statements by way of *reasoning* is rather different from the way in which statements are justified within other fields or disciplines. When we want to cover infinitely many cases of a typical mathematical statement, which is usually of a general nature, this cannot be done empirically, a proof is needed. Students often don't understand or accept this. To many of them, empirical justification is not only sufficient, it is also necessary. In mathematics we not only make claims that cover infinitely many cases, we also make claims involving infinity itself, e.g. when we claim that there are infinitely many prime numbers or infinitely many solutions to a differential equation that cannot be solved in closed form. Such claims, too, cannot be justified empirically, proofs are needed. The same is true of impossibility statements, such as 'it will never ever, now or in the future, be possible to find a rational number whose square is 2', or 'a Euclidean (ruler-and-compass) trisection of an arbitrary angle will never be made'. Every one of these elements of reasoning is a source of learning difficulties for a great many students.

Perhaps the most recognised source of difficulty is the nature and role of symbols and formalisms in mathematics. Thus it is well-known that formulae have a frightening effect on lots of people. One of the key difficulties is that ordinary letters from the Latin or the Greek alphabet are stand-ins for numbers in a variety of different ways. Sometimes a letter stands for any number of a certain type (constants, parameters, variables), like in $px^2 + qx + r$ or in $(a + b)^2 = a^2 + 2ab + b^2$. At other times the letters stand only for those numbers who actually fulfil a certain relation, like a, b, and c in $a^2 + b^2 = c^2$ or m, s, x, y in $y > mx + s$. At still other times letters stand for numbers that are sought after in a given context ("unknowns"), say from equations or inequalities, like x in $3x + 5 = -x + a$ (a is an arbitrary constant, not being sought), or z in $z^2 - r < 0$. It is very difficult for students to understand that only very seldom (e.g. π, e or *sin*) do letters have substantive identities in themselves, that it does not matter which letters we use as long as we respect certain conventions ($3w + 5 = -w + a$ is the same equation as $3x + 5 = -x + a$, but $3w + 5 = -x + a$ is not). Mathematics makes use of lots of special symbols, most of which do not exist outside mathematics and science, such as: $=$, $<$, $\int$, $\sqrt{}$, $\neq$, $\in$, $\approx$, ∞, with special and often complex meanings, and mixes them with letters and numbers to construct formulae, e.g.

$$(3x^2 + \pi\sqrt{a - b/x}\,)y^a \sin(\lambda xy)$$

to take a completely arbitrary example. The equal sign, =, is particularly wicked as it stands for very different things, including *identity* ($a = a$), *definition* ($10^3 = 10 \cdot 10 \cdot 10$), an abbreviation (setting $D = q^2 - 4pr$) and *equation* ($2x = x + 8$). We further make use of historically given, but sometimes unpractical and inconsistent, notation conventions, such as ab meaning $a \cdot b$ but $1\frac{1}{2}$ meaning $1 + \frac{1}{2}$.

We use a special terminology, sometimes with words borrowed from everyday language but typically with different meanings: fraction, rational number, rectangle, angle bisector, function, direct and inverse proportionality, differentiate, decimal place, exponent, root extraction, polynomial, integral, derivative, and so on and so forth.

Coming to grips with symbols and the formalisms that regulate them is like learning a foreign language, using strange characters, words, rules, and conventions, all of which involve vocabulary, grammar, syntax, semantics. Some people take this state of affairs to imply that mathematics simply is a language. I don't agree, mathematics *is not* a language (it contains theoretical edifices and theorems, which languages do not), but it *possesses* a (crucially important) language. The fact that mathematical statements are of very different kinds causes a lot of confusion for learners. A *definition* introduces a concept by way of a specification of the exact conditions under which a given term can be employed, and hence cannot be proved or disproved. In contrast, a *theorem* is a claim that tells what necessarily must be the case in a general class of situations fulfilling certain conditions, and hence needs justification by way of a proof. An *example* is a concrete situation for which certain claims referring to that situation are true. Of particular difficulty is the role of *logical quantifiers* and *negations* in mathematical statements, like "for every … there exist a …, such that…" or "it is not the case that if … holds for all …, then … holds". Finally abstraction, abstractness, generalisation, and generality are essential features of mathematics which are notoriously difficult to capture. Abstraction and generalisations are mathematical *processes*, whereas abstractness and generality are *properties* that mathematical entities or claims may or may not possess. No wonder that the nature of mathematics in itself gives rise to a huge variety of learning difficulties! Presumably, the majority of mathematical learning difficulties are related to the nature of mathematics in some way or another, even though this is not the only source of difficulty.

The relationship between *mathematics and the rest of the world* is a source of learning difficulties in a number of different ways. One of the most important ones is the insufficient degree of reconciliation between students' experiences from the world(s) in which they live and the experiences they gather from dealing with mathematics in various contexts, but above all in and around the mathematics classrooms of educational institutions. Attributing meaning to and making sense of mathematics with regard to students' personal backgrounds and lives and seeing a bearing of mathematical theory – questions, concepts, methods, reasoning, results, organised theoretical edifices – on their experiential world is a big challenge. Where does the rule for dividing one fraction by another fraction come from? What does a decimal expansion with infinitely many non-zero decimals really mean and what does it have to do with the real world use of numbers, which is always based on finite decimal expansions? Why are geometrical objects belonging to the physical world different from corresponding geometric objects in mathematics, and why is even the most advanced and accurate high-tech measurement of angles and sides in a physical triangle not good enough for mathematical geometry? The fact that a multitude of mathematical concepts or processes do not have direct experiential counterparts causes philosophical (and motivational!) problems to many students.

We have just dealt with the problems arising from the discrepancy between the real world and the mathematical world, and the resulting experiential conflicts in students' perception of the relationship between them. However, there are also difficulties when mathematics *is* being used to dealing with extra-mathematical domains, i.e. when mathematical modelling enters the stage. The difficulties arise in the linking of an extra-mathematical domain with some mathematical domain brought about by mathematisation, in which extra-mathematical entities and relations are translated into mathematical entities and relations belonging to a mathematical domain chosen for the purpose. Whilst this linking between an extra-mathematical and a mathematical domain is the centerpiece of mathematical modelling, it is essential that the two domains are not mixed up, which has proved difficult to many learners. Another challenge is to understand that many fundamental mathematical concepts are in fact basically generated as modelling concepts. This is the case with function, additive growth, ratio (e.g. rate of change, slope, density, price-per-unit, value-for-money, gross domestic product per capita, etc.), and proportionality (e.g. multiplicity, scaling), to just mention a few.

With respect to the issue *what does it mean and take to learn mathematics?*, we have already considered the role of beliefs about mathematics and its learning. More specifically, research has investigated beliefs involving some widespread misconceptions and misunderstandings, which may be detrimental to the learning of mathematics unless they are effectively counteracted:

- Mathematics consists of nothing more than an infinity of disconnected facts, rules and procedures.
- Ordinary people can only learn mathematics by rote memorisation and by drill involving endless repetition of procedures and routines.
- Mathematics is only about solving standard problems by way of standard procedures, primarily based on calculation.
- Every single piece of mathematics, however small and strange it may be, must have a direct interpretation and practical application outside mathematics itself; otherwise it is futile and irrelevant.
- Mathematics is a dead discipline where everything is known and nothing new is happening, and hence mathematics as an educational subject is ossified and unchangeable.
- There is no such thing as creativity in mathematics, only imitation.
- I (student) know that mathematics is highly relevant in the world; unfortunately it is irrelevant to me (the *relevance paradox*).
- Only a very small minority of people, primarily boys, are able to learn mathematics with understanding. These people have strange and special brains and are socially dysfunctional, especially in relation to the opposite sex. Fortunately, I (student) am a normal human being and not one of those misfits.

Mathematicians and mathematics educators know that these beliefs are incorrect. Nevertheless they are known to be widespread and to have a massive negative impact on the learning of mathematics.

Finally, let us consider significant *experiences from the mathematics classroom*. Historically, mathematics originated and developed in response to cultural, social and intellectual needs, different in different parts of the world. It is difficult for students to obtain the kinds of experiences that gave rise to mathematics, because outside of school mathematics is largely invisible on the surface of things. However, satisfactory learning of mathematics requires links to be established between students' experiential world and that of mathematics. So, it becomes a key task of mathematics teaching to make mathematics visible.

As mentioned above, the didactical contract is tacit and therefore developed and established in students' minds through classroom experience. It greatly influences students' learning, and breaking it can have marked consequences on the part of both the students and the teacher. The same is true of socio-mathematical norms, the breach of which may lead to sanctions as well to praise. Even though the socio-mathematical norms, too, are established through tacit "negotiation" in the classroom, and hence through experience, the supreme judge on these norms is typically the teacher, who eventually is in charge of assessing the contributions to the mathematics discourse in the classroom.

Summary

We have seen that there is a multitude of deeply rooted sources and causes of mathematics specific learning difficulties, most of which have been investigated by research. As a matter of fact, in view of all the stumbling blocks we have identified, it may seem to be almost a miracle that some people are actually able to learn mathematics. Everyone, even the greatest mathematical talent, experiences difficulties and makes mistakes from time to time. Nevertheless, any "normal" person can learn (some) mathematics provided he or she invests the time and effort necessary and receives assistance from competent teachers and other professionals.

4. Counteracting learning difficulties

Introduction

It has turned out to be possible to counteract learning difficulties in mathematics. This involves two different, yet related, issues: remedying existing learning difficulties and preventing learning difficulties from arising. Here, we shall focus on the former issue, even though the latter issue is more important from a global perspective. However, insights gained from successful attempts at remedying existing learning difficulties can serve as a platform for orchestrating teaching so as to reduce the risk of the occurrence of serious learning difficulties. But - alas! – no one-size-fits-all cure is available, and no cure is miraculous. So, remedying observed learning difficulties cannot be a quick fix. It is person specific, demanding and time consuming.

The crucial step is to identify and analyse the individual's learning difficulties. Research has a lot to say about where to look. Preventing learning difficulties from arising requires deep insights into their early origin, transposed into teaching, and requires adoption of a long-term perspective. The misconceptions creating and amplifying many learning difficulties with students are often locally rational, which can both be a barrier and an advantage, an advantage because it gives access to a broader display of the manifestations of misconceptions and their interrelations.

A maths counsellor programme

An increasing number of high school students (grades 10–12) in Denmark experience severe learning problems in mathematics. A great many of them work hard to come to grips with the subject but remain unsuccessful in their endeavours.

At Roskilde University we wanted to utilise research insights to provide in-service teachers with a professional development programme that is meant to enable them to help students overcome or at least reduce their learning difficulties by becoming maths counsellors in their schools.

Such a programme was initiated in 2012 by Mogens Niss (Roskilde University) and Uffe T. Jankvist (today Aarhus University), based on earlier work by Niss and Morten Blomhøj (Roskilde University), with the following characteristics:

- The programme is research based, but very close to teachers' practice.
- In-service teachers participate part-time over three semesters.
- Each semester has a certain theme: 1. Concepts and concept formation in mathematics; 2. Reasoning, proof and proving; 3. Models and modelling.
- Schools accept to consider teachers' participation part of their work load.
- Each September a small cohort (no more than 24) of participants are admitted. Participants form a "class" for three semesters, and project teams consisting of 2–3 teachers are formed.
- In each of the three semesters two residential seminars are organised for each "class", one in the beginning and one at the end of the semester. Between these two seminars project teams work with their own students under guidance and supervision by the programme directors (MN and UTJ).
- In each semester participants are provided with a large package of selected theme specific and more general research literature in mathematics education.
- There are *three phases* of project teams' work in each semester:
  - *Identifying* students with theme specific learning difficulties, in need of assistance and willing to be assisted.
  - *Diagnosing* the nature of the learning difficulties for the students thus identified.
  - *Intervening* towards these students with regard to the difficulties diagnosed.
- For each semester we have developed a theme-specific so-called *detection test* (Jankvist & Niss 2017) aimed at assisting teachers in detecting students with learning difficulties.

- Based on the outcomes of this test and on teachers' knowledge of their students, a few students – called *focus students* – are identified who are willing to invest time and effort in improving their learning of mathematics. These students are invited to receive further assistance.
- Supported by the research literature made available to them the project team attempts to arrive at a *diagnosis* of the nature of the learning difficulties encountered with the focus students.
- Based on the diagnosis thus obtained, the project team designs and implements an *intervention programme* for the focus students. This requires creativity and inventiveness on the part of the team members, whilst the research literature provides inspiration and the directors offer guidance.
- Three kinds of intervention, which may well be used in combination, have been tried out by several project teams:
  - Interventions targeted on the *individual focus students* only.
  - Interventions targeted on (possibly several) *small groups* of students.
  - Interventions targeted on the *whole class* and implemented in a classroom setting.
- In each semester, each project group writes a joint *theme report* of their work in the semester at issue and its three phases, including findings and intervention results.
- At the final residential seminar of the semester, all project reports are presented and *peer assessed*, and project groups receive *feedback* from the programme directors.
- After the third and final semester, each team produces a *final joint project report* on their work during the three semesters.
- These reports are defended at a *final oral exam* at which participants receive individual grades. Participants who pass receive a mathematics counsellor *diploma* from the university. By January 2018, 72 counsellors have been educated.

Examples of findings and results

The findings in this section all concern students who have been focus students of the prospective mathematics counsellors at some point in the years 2012-2017.

Student A (grade 12 (sic!))

When asked to sketch the graph of the function given by $y = 2x + 5$, for real x, in a coordinate system, A gave the point $(2; 5)$ as the (complete) answer. When interviewed, A - and later several other students as well – indicated that $y = ax + b$ (x real) means that $x = a$ and $y = b$, because the a in front of x specifies the value of x, whereas b is the value left to be taken by y. A closer examination revealed that Student A had created her own, private interpretation of notation conventions, a special case of the discrepancy between concept image and concept definition studied by Tall and Vinner (1981). The intervention (15 lunch sessions of 35 minutes each) organised by the prospective mathematics counsellors aimed at making A coming to grips with notation conventions of graphical representations with specific regard to affine functions such as the one at issue. The counsellors designed a sequence of tasks with this focus. Eventually Student A passed the national exam in mathematics with respectable marks.

After having detected this "syndrome", the mathematics teachers in A's school changed their teaching of graphical representations of algebraic functions.

"0 is not a number"

When asked to count the number of integers in the interval $[-2; 5[$ many students answer "six" because they do not include 0 in the count.

A great many students claim that $a^5/a^5 = 0$, for "when all the a's are cancelled, nothing is left; hence the result is 0". It is also customary, on the same grounds, to see students stating that $(2/\sqrt{3}) \cdot \sqrt{3}/2) = 0$, even though this time we are not dealing with a variable of an unspecified value, whereas $(2/3) \cdot (3/2) = 6/6 = 1$, because the product of simple integers can be computed by way of ordinary arithmetic. Much fewer, but still some, students believe that $6/6 = 0$.

It is not unusual to encounter students who think that since 0 stands for nothing it has no effect in computations, whence $0 \cdot x = x$. The explanation why quite a few students claim that the equation $38x + 72 = 38x$ has the solution 0 is that, once we have done everything we can, we obtain $72 = 0$, and since the solution to an equation is what stands on the right hand side of the equal sign, 0 is the solution, even though the students who infer this certainly would not maintain that the number 72 is equal to 0.

This suggests, as mentioned above, that many students with learning difficulties perceive 0 as a marker of absence, not a number.

An equation is not a mathematical object

To some students the equation $x = 1$ (!) has the solution 0 (sic!), because "you have to do something" to the equation, i.e. $x - 1 = 0$, and – once again - the solution is found on the right hand side of the equal sign when you have done what you can.

When asked to solve the equation $3x - x = 2x$, many students claim that there is no solution, because reducing the equation yields $0 = 0$, and since x has disappeared there is no x to obtain a value, and hence there is no solution to the equation.

Bodin (1993) showed that surprisingly many students who were able to correctly solve the equation $7x - 3 = 13x + 15$ could not subsequently tell whether $x = 10$ is a solution or not. In the maths counsellor programme we have seen the same pattern occur with several students.

So, to lots of students an equation is nothing but a prompt to carry out certain procedures, regardless of the sense they (do not) make to the students. Neither the concept of equation nor the concept of solution exists as a mathematical entity in the minds of such students.

Student B (grade 11)

B could not come to grips with the manipulation of algebraic expressions. "There are far too many rules one has to learn by heart. I give up!". After having conducted diagnostic interviews with Student B, his two maths counsellors realised that he had no idea about the interconnected and logical coherence of the rules of transformation of algebraic expressions. So, they designed an intervention, in which they made B digest and make sense of a few basic rules, and provided tasks designed to force him to use these basic rules to derive a large set of new rules, needed to solve the problems he were given. Then all of a sudden the logic and the pattern of the rules dawned upon him, and he ended up completely changing behaviour in the mathematics classroom and being pretty successful in mathematics, receiving very respectable grades at the final national exam.

Student C (grades 11-12)

Faced with the task “A solid wooden cube, with all edge lengths 2cm, weighs 4.8g. How much does a cube made from the same wood, but with edge lengths 4cm, weigh?” Student C – and numerous other students – gave the answer 9.6 or “twice as much” (an instance of over-generalised proportionality). During the diagnosis undertaken by the maths counsellors, they found out that C had extremely weak notions of scaling, density, and ratio. Using concrete materials (including Lego blocks), they designed an intervention sequence aimed at fostering and consolidating these fundamental modelling notions with Student C. As a consequence C made considerable progress, going from receiving failing grades to passing the national final exam, albeit not with flying colours.

Summing up

The first two cohorts of maths counsellors have published books, edited by Uffe Jankvist and myself (Niss & Jankvist 2016, 2017), addressing their Danish colleagues and other mathematics educators, presenting selected cases from their work with focus students, including findings and results. At the time of writing, two more volumes are in the pipeline.

Maths counsellors educated from this programme are given key roles in their schools, in identifying students with mathematics specific learning difficulties, in counteracting these difficulties, and – as “first amongst peers” – in helping their mathematics colleagues in their school to undertake similar work.

5. Conclusion

The first part of this paper uncovered the deep and complex nature of mathematics specific learning difficulties and outlined aspects of what research has to tell us about them. The second part offered what I consider a proof of existence that it is in fact possible to

- detect students with mathematics specific learning difficulties;
- identify students who are willing and able to be assisted by maths counsellors;
- diagnose the nature, and sometimes the causes as well, of students’ learning difficulties;
- design and implement successful intervention measures that effectively counteract some of the learning difficulties identified.

This requires and involves an intimate interplay between research, development and practice, on the part of the teachers, constituting one way of overcoming the research-practice gap so often observed and lamented upon in mathematics education.

However, we have to admit that this is demanding and time-consuming for teachers, as there are no quick fixes, but it is also immensely rewarding for participating students, teachers, researchers and schools. Last, but certainly not least, the work reported has generated and continues to generate a lot of new research (cf. the list of references) deepening our understanding of mathematics specific learning difficulties and paving the way for future successful interventions to counteract them.

References

Bodin, A. (1993). What does to assess mean? The case of assessing mathematical knowledge-In M. Niss (Ed.), *Investigations into Assessment in Mathematics Education. An ICMI Study* (pp. 113-141). Dordrecht: Kluwer Academic Publishers

Brousseau, G. (1997). *Theory of Didactical Situations*. Dordrecht: Kluwer Academic Publishers

Jankvist, U.T. (2015). Changing students' images of "mathematics as a discipline". *Journal of Mathematical Behavior*, **38**, 41-56.

Jankvist, U.T. & Niss, M. (2015). A framework for designing a research-based 'maths counsellor' teacher programme. *Educational Studies in Mathematics* **90**, 259-284.

Jankvist, U.T. & Niss, M. (2016): Fostering an intimate interplay between research and practice: Danish "maths counsellors" for upper secondary school. Invited paper at TSG-50, ICME-13, Hamburg, Germany.

Jankvist, U.T. & Niss, M. (2017). The notion and role of "detection tests" in the Danish upper secondary "maths counsellor" programme. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME10, February 1-5, 2017)*, (pp. 3825-3832). Dublin: DCU, Institute of Education.

Jankvist, U.T. & Niss, M. (2018). Counteracting destructive student misconceptions of mathematics. *Education Sciences*, 8: In press.

Jankvist, U.T. & Niss, M. (submitted). Learning difficulties in mathematical modelling with upper secondary school students.

Niss, M. & Jankvist, U. T. (Eds) (2016). *Fra snublesten til byggesten. Matematikdidaktiske muligheder*. Frederiksberg: Frydenlund.

Niss, M. & Jankvist, U. T. (Eds) (2017). *Læringsvanskeligheder i matematik – hvordan kan de forstås og afhjælpes?* Frederiksberg: Frydenlund.

Phillip, R.A. (2007). Mathematics teachers beliefs and affect. In F.K. Lester, Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning*, (pp. 257-315). Charlotte, NC: Information Age Publishing.

Tall, D. & Vinner, S. (1981). Concept image and concept definition with particular reference to limits and continuity. *Educational Studies in Mathematics*, **12**, 151-169

Yackel, E. & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy ion mathematics. *Journal for Research in Mathematics Education,* **27**, 458-477