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invariants and modulo equivalence [Extended version with proofs]
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Confluence of CHR revisited:

Invariants and modulo equivalence

[Extended version with proofs]

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Confluence of CHR revisited:  
invariants and modulo equivalence*  

[Extended version with proofs]

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Abstract. Abstract simulation of one transition system by another is  
introduced as a means to simulate a potentially infinite class of similar  
transition sequences within a single transition sequence. This is useful for  
proving confluence under invariants of a given system, as it may reduce  
the number of proof cases to consider from infinity to a finite number.  
The classical confluence results for Constraint Handling Rules (CHR) can  
be explained in this way, using CHR as a simulation of itself. Using an  
abstract simulation based on a ground representation, we extend these  
results to include confluence under invariant and modulo equivalence,  
which have not been done in a satisfactory way before.

Keywords: Constraint Handling Rules, Confluence, Confluence modulo equiv-
elence, Invariants, Observable confluence

1 Introduction

Confluence of a transition system means that any two alternative transition  
sequences from a given state can be extended to reach a common state. Proving  
confluence of nondeterministic systems may be important for correctness proofs  
and it anticipates parallel implementations and application order optimizations.  
Confluence modulo equivalence generalizes this so that these “common states”  
need not be identical, but only equivalent according to an equivalence relation.  
This allows for redundant data representations (e.g., sets as lists) and procedures  
that search for an optimal solution to a problem, when any of two equally good  
solutions can be accepted (e.g., the Viterbi algorithm analyzed for confluence  
modulo equivalence in [7]).

We introduce a notion of abstract simulation of one system, the object sys-
tem, by another, the meta level system, and show how proofs of confluence (under  
invariant, modulo equivalence) for an object system may be expressed within a  
meta level system. This may reduce the number of proof cases to be considered,
often from infinity to a finite number. We apply this to the programming lan-
guage of Constraint Handling Rules, CHR [13, 14, 15], giving a clearer exposition
of existing results and extending them for invariants and modulo equivalence.

By nature, invariants and state equivalences are meta level properties that
in general cannot be expressed in its own system: the state itself is implicit and
properties such as groundness (or certain arguments restricted to be uninstanti-
ated variables) cannot be expressed in a logic-based semantics for CHR. Using
abstract simulation we can add the necessary enhanced expressibility to the
meta level, and the ground representation of logic programs, that was studied
in-depth in the late 1980s and -90s in the context of meta-programming in logic
(e.g., [5, 18, 17]), comes in readily as a well-suited and natural choice for this. The
following minimalist example motivates both invariant and state equivalence for
CHR.

Example 1 ([6, 7]). The following CHR program, consisting of a single rule, col-
clects a number of separate items into a set represented as a list of items.

\[ \text{set}(L), \text{item}(A) \iff \text{set}([A|L]). \]

This rule will apply repeatedly, replacing constraints matched by the left hand
side by the one indicated to the right. The query

\[ ?- \text{item}(a), \text{item}(b), \text{set}(\{\}). \]

may lead to two different final states, \{set([a, b])\} and \{set([b, a])\}, both
representing the same set. Thus, the program is not confluent, but it may be
confluent modulo an equivalence that disregards the order of the list-elements.
Confluence modulo equivalence still requires an invariant that excludes more
than one \text{set}/1 constraint, as otherwise, an element may go to an arbitrary of
those.

1.1 Related work

Some applications of our abstract simulations may be seen as special cases of
abstract interpretation [9]. This goes for the re-formulation of the classical con-
fluence results for CHR, but when invariants are introduced, this is not obvious;
a detailed argument is given in Section 5. It is related to symbolic execution
and constraint logic programming [21], where reasoning takes place on compact
abstract representations parameterized in suitable ways, rather than checking
multitudes of concrete instances. Bisimulation [25], which has been applied in
many contexts, indicates a tighter relationship between states and transitions
of two systems than the abstract simulation: when a state \(s_0\) is simulated by
an abstract state \(s'_0\) and there is a transition \(s_0 \rightarrow s_1\), bisimulation would re-
quire the existence of an abstract transition \(s'_0 \rightarrow' s'_1\), which may not be case as
demonstrated by Example 6.

Previous results on confluence of CHR programs, e.g., [1, 2, 3], mainly refer
to a logic-based semantics, which is well-suited for showing program properties,
but it does not comply with typical implementations [19, 27] and applies only
for a small subset of CHR programs. Other works [6,7] suggest an alternative operational semantics that lifts these limitations, including the ability to handle Prolog-style built-in predicates such as \texttt{var/1}, etc. To compare with earlier work and for simplicity, the present paper refers to the logic-based semantics.

As long as invariants and modulo equivalence are not considered, the logic-based semantics allows for elegant confluence proofs based on Newman's Lemma (Lemma 1, below). A finite set of critical pairs can be defined, whose joinability ensures confluence for terminating programs. Duck et al. [12] proposed a generalization of this approach to confluence under invariant, called observable confluence; no practically relevant methods were suggested, and (as the authors point out) even a simple invariant such as groundness explodes into infinitely many cases.

Confluence modulo equivalence was introduced and motivated for CHR by [6], also arguing that invariants are important for specifying meaningful equivalences. An in-depth theoretical analysis, including the use of a ground representation, is given by [7] in relation the alternative semantics mentioned above. However, it has not been related to abstract simulations, and the proposal for a detailed language of meta level constraints in the present paper is new. Repeating the motivations of [6,7] in the context of the logic-based semantics, [16] suggested to handle confluence modulo equivalence along the lines of [12], thus inheriting the problems of infinitely many proof case pointed out above.

An approach to show confluence of a transition system, by producing a mapping into another confluent system, is described by [10] and extended to confluence modulo equivalence by [22]; the relationship between such two systems is different from the abstract simulations introduced in the present paper. Confluence, including modulo equivalence, has been studied since the first half of the 20th century in a variety of contexts; see, e.g., [7,20] for overview.

1.2 Contributions

We introduce abstract simulation as a setting for proofs of confluence for general transitions systems and demonstrate this specifically for CHR. We recast classical results (without invariant and equivalence), showing that they are essentially based on a simulation of CHR's logic-based semantics by itself, and we can pinpoint, why it does not generalize for invariants (see Example 4, p. 9).

These results are extended for invariants and modulo equivalence, using an abstract simulation; it is based on a ground meta level representation and suitable meta level constraints to reason about it.

1.3 Overview

Sections 2 and 3 introduce basic concepts of confluence plus our notion of abstract simulation. Section 4 gives syntax and semantics of CHR along with a discussion of how much nondeterminism to include in a semantics used when considering confluence. Section 5 re-explains the classical results in terms of abstract simulation. Section 6 extends these results for invariants and modulo
equivalence. The concluding Section 7 gives a summary and explains briefly how standard mechanisms, used to prevent loops by CHR’s propagation rules, can be added.

2 Basic concepts, confluence, invariants and equivalences

A transition system $D = (S, \rightarrow)$ consists of a set of states $S$, and a transition is an element of $\rightarrow: S \times S$, written $s_0 \rightarrow s_1$ or, alternatively, $s_1 \leftarrow s_0$. A transition sequence or path is a chain of transitions $s_0 \rightarrow s_1 \rightarrow \cdots \rightarrow s_n$ where $n \geq 0$; if such a path exists, we write $s_0 \overset{*}{\rightarrow} s_n$. A state $s_0$ is final (or normal form) whenever $\not\exists s_1 \overset{*}{\rightarrow} s_1$, and $D$ is terminating whenever every path is finite. To anticipate the application for logic programming systems, a given transition system may have a special final state called failure.

An invariant $I$ for $D = (S, \rightarrow)$ is a subset $I \subseteq S$ such that

$$s_0 \in I \land s_0 \rightarrow s_1 \Rightarrow s_1 \in I.$$  

We write a fact $s \in I$ as $I(s)$ and refer to $s$ as an $I$ state. The restriction of $D$ to $I$ is the transition system $(I, \overset{I}{\rightarrow})$ where $\overset{I}{\rightarrow}$ is the restriction of $\rightarrow$ to $I$. A set of allowed initial states $S' \subseteq S$ defines an invariant of those states reachable from some $s \in S'$, i.e., $reachable(S') = \{s' \mid s \in S' \land s \overset{*}{\rightarrow} s'\}$. A (state) equivalence is an equivalence relation over $S$, typically denoted $\approx$. In the context of an invariant $I$, the relations $\approx$ and $\rightarrow$ are understood to be restricted to $I$.

The following $\alpha$ and $\beta$ corners$^1$ were introduced in [6,7], being implicit in [20]. An $\alpha$ corner is a structure $s_1 \leftarrow s_0 \rightarrow s_2$, where $s_0, s_1, s_2 \in S$ and the indicated relationships hold; $s_0$ is called a common ancestor and $s_1, s_2$ wing states. A $\beta$ corner is a structure $s_1 \approx s_0 \overset{*}{\rightarrow} s_2$, where $s_0, s_1, s_2 \in S$ and the indicated relationships hold. In the context of an invariant $I$, the different types of corners are defined only for $I$ states.

Two states $s_1, s_2$ are joinable (modulo $\approx$) whenever there exist paths $s_1 \overset{*}{\rightarrow} s'_1$ and $s_2 \overset{*}{\rightarrow} s'_2$ with $s'_1 = s'_2$ ($s'_1 \approx s'_2$). A corner $s_1 \rel s_0 \overset{*}{\rightarrow} s_2$ is joinable (modulo $\approx$) when $s_1, s_2$ are joinable (modulo $\approx$); $\rel \in \{\leftarrow, \approx\}$.

A transition system $D = (S, \rightarrow)$ is confluent (modulo $\approx$) whenever

$$s_1 \leftarrow s_0 \overset{*}{\rightarrow} s_2 \Rightarrow s_1 \text{ and } s_2 \text{ are joinable (modulo } \approx).$$

It is locally confluent (modulo equivalence $\approx$) whenever all its $\alpha$ ($\alpha$ and $\beta$) corners are joinable. The following properties are fundamental.

Lemma 1 (Newman [24]). A terminating transition system (under invariant $I$) is confluent if and only if it is locally confluent.

Lemma 2 (Huet [20]). A terminating transition system (under invariant $I$) is confluent modulo $\approx$ if and only if it is locally confluent modulo $\approx$.

$^1$ In recent literature within term rewriting, the terms peaks and cliffs have been used for $\alpha$ and $\beta$ corners, respectively.
These properties reduce proofs of confluence (mod. equiv.) for terminating systems to proofs of the simpler property of local confluence (mod. equiv.), but still, this may leave an infinite number of corners to be examined. The notion of abstract simulation introduced below may reduce this to a finite number.

3 Abstract Simulation

Consider two transition systems, \(D_O = \langle S_O, \rightarrow_O \rangle\) and \(D^M = \langle S^M, \rightarrow^M \rangle\), referred to as object and meta level systems. A replacement is a (perhaps partial) function \(\rho: S^M \to S_O\); the application of \(\rho\) to some \(s \in S^M\) is written \(s\rho\). For any structure \(f(s_1,\ldots,s_n)\) with states \(s_1,\ldots,s_n\) of \(D^M\) (a transition, a tuple, etc.), replacements apply in a compositional way. \(f(s_1,\ldots,s_n)\rho = f(s_1\rho,\ldots,s_n\rho)\). For a family of replacements \(P = \{p_i\}_{i \in \text{lex}}\), the covering (or concretization) of a structure \(f(s_1,\ldots,s_n)\) is defined as

\[
[f(s_1,\ldots,s_n)]^M_O = \{f(s_1,\ldots,s_n)\rho \mid \rho \in P\}.
\]

Notice that \(P\) is left implicit in this notation, as in the context of given object and meta level systems, there will be one and only one replacement family.

**Definition 1.** An abstract simulation of \(D_O\) by \(D^M\) with possible invariants \(I_O\), resp., \(I^M\), and equivalences \(\approx_O\), resp., \(\approx^M\), is defined by a family of replacements \(P = \{p_i\}_{i \in \text{lex}}\) which satisfies the following conditions.

\[
s_0 \rightarrow^M s_1 \Rightarrow \forall \rho \in P:\ s_0\rho \rightarrow_O s_1\rho \lor s_0\rho = s_1\rho
\]

\[
I^M(s) \Rightarrow \forall \rho \in P:\ I_O(s\rho)
\]

\[
s_0 \approx^M s_1 \Rightarrow \forall \rho \in P:\ s_0\rho \approx_O s_1\rho
\]

Notice that an abstract simulation does not necessarily cover all object level states, transitions, etc.

**Example 2.** Let \(A = \{a_1, a_2, \ldots\}\), \(B = \{b_1, b_2, \ldots\}\) and \(C = \{c_1, c_2, \ldots\}\) be sets of states, and \(O\) and \(M\) the following transition systems.

\[
O = \langle A \cup B \cup C, \{a_i \rightarrow_O b_i \mid i = 1, 2, \ldots\} \cup \{a_i \rightarrow_O c_i \mid i = 1, 2, \ldots\}\rangle
\]

\[
M = \langle \{a, b, c\}, \{a \rightarrow_M b, a \rightarrow_M c\}\rangle
\]

Assume equivalences \(b \approx^M c\) and \(b_i \approx_O c_i\), for all \(i\). Then the family of replacements \(P = \{p_i\}_{i=1,2,\ldots}\), where \(a \rightarrow^M = a_i\), \(b \rightarrow^M = b_i\) and \(c \rightarrow^M = c_i\), defines a simulation of \(O\) by \(M\). It appears that \(O\) and \(M\) are not confluent, cf. the non-confluent corners \(1 \leftarrow_O a \rightarrow_O c\) and \(b \leftarrow_M a \rightarrow_M c\), but both are confluent modulo \(\approx_O\) (\(\approx^M\)).

A meta level structure \(k\) covers an object structure \(k\) whenever \(k \in [m]^M_O\). When \([m]^M_O = \emptyset\), \(m\) is inconsistent. When \([m]^M_O \subseteq [m]^M\), \(m\) is a substate/subcorner, etc. of \(m\), depending on the inherent type of \(m\). When \(D_O\) and \(D^M\) both include failure, it is required that \([\text{failure}]^M_O = \{\text{failure}\}\). A given meta level state \(S\) is mixed whenever \([S]^M_O\) includes both failure and non-failure states. Transitions are only allowed from consistent and neither failed nor mixed states.

The following is a consequence of the definitions.
Lemma 3. An object level corner, which is covered by a joinable (mod. equiv.) meta level corner, is joinable (mod. equiv.).

When doing confluence proofs, we may search for a small set of critical meta level corners, whose joinability guaranteesjoinability of any object level corner, i.e., any other object level corner not covered by one of these is seen to be joinable in other ways. For term rewriting systems, e.g., [4], and previous work on CHR, such critical sets have been defined by explicit constructions.

We introduce a mechanism for splitting a meta level corner \( \Lambda \) into a set of corners, which together covers the same set of object corners as \( \Lambda \). This is useful when \( \Lambda \) in itself is not joinable, but each of the new corners are. In some cases, splitting is necessary for proving confluence under an invariant as shown in Section 5 and exemplified in Examples 4 and 6.

Definition 2. Let \( s \) be a meta level state (or corner). A set of states (or corners) \( \{s_i\}_{i \in I_{\text{m}}} \) is a splitting of \( s \) whenever \( \bigcup_{i \in I_{\text{m}}} [s_i]^M = [s]^M \). A corner (set of corners) is split joinable (mod. equiv.) if it (each of its corners) is joinable (mod. equiv.), inconsistent, or has a splitting into a set of split joinable (mod. equiv.) corners.

Corollary 1. An object level corner, which is covered by a split joinable (mod. equiv.) meta level corner, is joinable (mod. equiv.).

4 Constraint Handling Rules

Most actual implementations of CHR are fully deterministic, i.e., for a given query, there is at most one answer state (alternatively, the program is non-terminating). In this light, it may be discussed whether confluence is an interesting property, and if so, to what extent the applied semantics should be nondeterministic. Our thesis is the following: choice of next constraints to be tried and which rule to be used should be nondeterministic. Thus a confluent program can be understood by the programmer without considering the detailed control mechanisms in the used implementation; this also anticipates parallel implementations. We see only little interest in considering confluence for the so-called refined CHR semantics [11] in which only very little nondeterminism is retained.

Similarly to [6,7], we remove w.l.o.g. two redundancies from the logic-based semantics [1,15]: global variables and the two-component constraint store.

- Global variables are those in the original query. Traditionally they are kept as a separate state-component, such that values bound to them can be reported to the user at the end. The same effect can be obtained by a constraint \( \text{global}/2 \) that does not appear in any rule, but may be used in the original query: writing \( ?- p(X) \) as \( ?- p(X), \text{global}('X',X) \), means that the value

\[\text{In the literature, the term critical pair is used for the pair of wing states of our critical corners.}\]
of the variable named ‘X’ can be read out as the second argument of this
constraint in a final state.
– We avoid separating the constraint store into query and active parts, as the
transition sequences with or without this separation are essentially the same.

4.1 Syntax

Standard first-order notions of variables, terms, predicates atoms, etc. are as-
sumed. Two disjoint sets of constraint predicates are assumed, user constraints
and built-in constraints; the actual set of built-ins may vary depending on the
application. We use the generalized simpagation form [15] to capture all rules of
CHR. A rule is a structure of the form

\[ H_1 \setminus H_2 \iff G \mid C \]

where \( H_1 \setminus H_2 \) is the head of the rule, \( H_1 \) and \( H_2 \) being sequences, not both
empty, of user constraints; \( G \) is the guard which is a conjunction of built-in
constraints; and \( C \) is the body which is a sequence of constraints of either sort.
When \( H_2 \) is empty, the rule is a simplification, which may be written \( H_1 \iff G \mid C \);
when \( H_2 \) is empty, it is a propagation, which may be written \( H_2 \implies G \mid C \);
any other rule is a simpagation; when \( G = \text{true} \), \((G1)\) may be left out. The head
variables of a rule are those appearing in the head, any other variable is local.
The following notion is convenient when defining the CHR semantics and its
meta level simulation.

**Definition 3.** A pre-application of a rule \( r = (H_1 \setminus H_2 \iff G \mid C) \) is of the form
\( (H'_1 \setminus H'_2 \iff G' \mid C')\sigma \) where \( r' = (H'_1 \setminus H'_2 \iff G' \mid C') \) is a variant of \( r \) with fresh
variables and \( \sigma \) is a substitution to the head variables of \( r' \), where, for no variable
\( x, x\sigma \) contains a local variable of \( r' \).

The operator \( \uplus \) refers to union of multisets, so that, e.g., \( \{a, a\} \uplus \{a\} = \{a, a, a\} \);
for difference of multisets, we use standard notation for set difference, assuming
it takes into account the number of copies, e.g., \( \{a, a\} \setminus \{a\} = \{a\} \).

4.2 The logic-based operational semantics for CHR

The semantics presented here is essentially identical to the one used by [1] and
the so-called abstract operational semantics \( \omega \) of [15], taking into account the
simplifications explained above. Following [26], we define a state as an equivalence
class, abstracting away the specific variables used and the different ways
the same logical meaning can be expressed by different conjunctions of built-ins.\(^3\)
A logical theory \( B \) is assumed for the built-in predicates.

A state representation (s.repr.) is a pair \((S, B)\), where the constraint store
\( S \) is a multiset of constraint atoms and the built-in store \( B \) is a conjunction of

\(^3\) Raiser et al [26] defined “state” similarly to what we call state representation, and
they defined an operational semantics over equivalence classes of such states. We
have taken the natural step of promoting such equivalence classes to be our states.
built-ins; any s.repr. with an unsatisfiable built-in store is considered identical to failure. Two s.repr.s \( ⟨S, B⟩ \) and \( ⟨S', B'⟩ \) are variants whenever, either⁴

- they are both failure, or
- there is a renaming substitution \( ρ \) such that
  \[ B \models \forall(Bρ → ∃(Sρ = S' ∧ B')) ∧ B \models ∃(Sρ = S' ∧ Bρ) \]

A state is an equivalence class of s.repr.s under the variant relationship. For simplicity of notation, we typically indicate a state by one of its s.repr.s.

A rule application w.r.t. to a non-failure state \( ⟨S, B⟩ \) is a pre-application \( H_1 \triangleleft H_2 \triangleleft G\upharpoonright C \) for which \( B \models B \rightarrow ∃L G \), where \( L \) is the list of its local variables. There are two sorts of transitions, by rule application and by built-in.

\[
\begin{align*}
\langle H_1 \downarrow H_2 \uplus S, B⟩ &\rightarrow_{\text{logic}} \langle H_1 \uplus C \uplus S, G \land B⟩ \\
\text{when there exists a rule application } H_1 \downarrow H_2 \triangleleft G\upharpoonright C, \\
\langle \{b\} \uplus S, B⟩ &\rightarrow_{\text{logic}} \langle S, b \land B⟩ \quad \text{for a built-in } b.
\end{align*}
\]

5 Confluence under the logic-based semantics re-explained, and why invariants are difficult

Here we explain the results of [1,2], also summarized in [15], using abstract simulation. Object and meta level systems coincide and are given by a CHR program under the logic-based semantics. Two rules give rise to a critical corner if a state can be constructed in which one rule consumes constraints that the other one needs to be applied; in that case, rule applications do not commute and a specific proof of joinability must be considered. We anticipate the re-use of the construction, when invariants are introduced: in a pre-corner, the guards are not necessarily satisfied (but may be so in the context of an invariant).

**Definition 4.** Consider two rules \( r : H_1 \downarrow H_2 \triangleleft G \upharpoonright C \) and \( r' : H_1' \downarrow H_2' \triangleleft G' \upharpoonright C' \) renamed apart, and let \( A \) and \( A' \) be non-empty sets of constraints such that \( A \subseteq H_2 \), \( A' \subseteq H_1' \downarrow H_2' \) and \( B \models ∃(A = A') \). In that case, let

\[
\begin{align*}
\bar{H} & = (H_1 \downarrow H_2 \uplus H_1' \uplus H_2') \setminus A \\
s_0 & = \langle \bar{H}, (G \land G' \land A = A') \rangle \\
s & = \langle \bar{H} \downarrow H_2 \uplus C, (G \land G' \land A = A') \rangle \\
s' & = \langle \bar{H} \downarrow H_2' \uplus C', (G \land G' \land A = A') \rangle
\end{align*}
\]

When \( s \neq s' \), \( s_0 \) is a critical, common ancestor state, and \( s \leftrightarrow_{\text{logic}} s_0 \rightarrow_{\text{logic}} s' \) is a critical \( α \) pre-corner; the constraints \( A \) (or \( A' \)) is called the overlap of \( r \) and \( r' \). When, furthermore, \( B \models ∃(G \land G' \land A = A') \), it is a critical \( α \) corner.

⁴ An equation between multisets should be understood as an equation between suitable permutations of their elements.
The simulation is given by the following cover function.

\[(S, B)^\text{logic} = \{ (S \cup S^+, B \land B^+) | S^+ \text{ is a multiset of user and built-in constraints, } B^+ \text{ is a conjunction of built-ins} \}\]

\[(S, B) \mapsto \text{logic} (S', B') = \{ (S \cup S^+, B \land B^+) \mapsto \text{logic} (S' \cup S^+, B' \land B^+) | S^+ \text{ is a multiset of user and built-in constraints, } B^+ \text{ is a conjunction of built-ins, } \exists(B \land B^+) \text{ holds} \}\]

It is easy to check that this definition satisfies the conditions for being an abstract simulation given in Section 3, relying on monotonicity: \( B \models B \land B^+ \rightarrow \exists_L G \).

When a rule application with guard \( G \) can apply to a meta level state \((S, B)\), we have \( B \models B \rightarrow \exists_L G \), and the monotonicity property \( B \models B \land B^+ \rightarrow \exists_L G \) yield that the rule also applies for any consistently extended state \((S \cup S^+, B \land B^+)\).

For built-in steps, the property is trivial since the selected constraint is moved uninspected.

It can be shown that any object corner not covered by a critical corner (Definition 4) is trivially joinable:

- there are no object \( \beta \) corners;
- an object \( \alpha \) corner wherein one or both wing states are formed by a built-in step is trivially joinable as the evaluation of logical built-ins commute;
- an object \( \alpha \) corner formed by two rule applications which is not an instance of a critical corner is trivially joinable since its rule applications commute;
- any other object corner is covered by some critical corners (Definition 4).

Thus, according to Lemmas 1 and 3, the program under investigation is confluent whenever it is terminating and this set of critical corners is joinable. The set of critical corners is finite and that allows for automatic confluence proofs by checking the critical corners, one by one, e.g., [23].

**Example 3.** Consider the one-rule set-program of Example 1, ignoring invariant and state equivalence. There are two critical corners, given by the two ways, the rule can overlap with itself:

\[
\begin{align*}
\langle \{\text{set}(X1|L), \text{item}(X2)\}, \text{true} \rangle & \xrightarrow{\text{logic}} \langle \{\text{set}(X1|L), \text{set}(L2)\}, \text{true} \rangle \\
\langle \{\text{item}(X1), \text{set}(L), \text{item}(X2)\}, \text{true} \rangle & \xrightarrow{\text{logic}} \langle \{\text{set}(L1), \text{item}(X), \text{set}(L2)\}, \text{true} \rangle \\
\langle \{\text{item}(X1), \text{set}(X2|L)\}, \text{true} \rangle & \xrightarrow{\text{logic}} \langle \{\text{set}(L1), \text{set}(X2|L)\}, \text{true} \rangle \\
\langle \{\text{set}(L1), \text{set}(X2|L)\}, \text{true} \rangle & \xrightarrow{\text{logic}} \langle \{\text{set}(L1), \text{set}(X2|L)\}, \text{true} \rangle
\end{align*}
\]

None of these corners are joinable, so the program is not confluent.

The simulation defined above, relying on monotonicity, do not generalize well for confluence under invariant, referred to as “observable confluence” in [12].

**Example 4.** Consider the CHR program consisting of the following four rules.
It is not confluent as its single critical corner \( q(X) \leftarrow p(X) \rightarrow r(X) \) is not joinable (the built-in stores are \( \text{true} \) and thus omitted). However, adding the invariant “reachable from an initial state \( p(n) \) where \( n \) is an integer” makes it confluent. We indicate the set of all non-trivial object level corners as follows, with the dashed transitions proving each of them joinable.

These object corners and their proofs of joinability obviously fall in two groups of similar shapes, but there is no way to construct a finite set (of, say, one or two elements) that covers all object corners. In other words, the smallest set of meta level corners that covers this set is the set itself. This was also noticed in [12] that used a construction that essentially reduces to the abstract simulation shown above.

The abstract simulation given by \( [[ - \triangleright \text{logic} ]] \) of Definition 4 above defines an abstract interpretation, whose abstract domain is the complete lattice of CHR states ordered by the substate relationship (Section 3). Referring to Example 4, for instance the join of the infinite set of states \( \{ \langle p(t), b \rangle | t \text{ is a term}, b \text{ is a conjunction of built-ins} \} \) is \( \langle p(X), \text{true} \rangle \). When the grounding invariant is introduced, the join operator is not complete; an attempt to join, say, \( \langle p(0), \text{true} \rangle \) and \( \langle p(1), \text{true} \rangle \) would not satisfy the invariant.\(^5\)

### 6 Invariants and modulo equivalence

A program is typically developed with an intended set of queries in mind, giving rise to a state invariant, which may make an otherwise non-confluent program observably confluent (mod. equiv.). We can indicate a few general patterns of invariants and their possible effect on confluence.

- Elimination of non-joinable critical corners that do not cover any object corner satisfying the invariant. This was shown in Example 4 above, and is also demonstrated in the continuation of Example 3 (Ex. 7, below): “only one \text{set} constraint allowed”.
- Making it possible to apply a given rule, which otherwise could not apply, e.g., providing a “missing” head constraint or enforcing guard satisfaction:
  1. “if a state contains \( p(\text{something}) \), it also contains \( q(\text{the-same-something}) \)”
  2. “if a state contains \( p(\text{something}) \), this \text{something} is a constant \( > 1 \)”

\(^5\) Such an attempt might be \( \langle p(X), (X=0 \lor X=1) \rangle \); notice that \( X \) is a variable, thus breaking the invariant.
An invariant of type 1 ensures confluence mod. equiv. of a version of the Viterbi algorithm [7]; an invariant of type 2 is indicated in Example 4 and formalized in Example 6, below.

As shown in Example 4 above, invariants block for a direct re-use CHR’s logical semantics as its own meta-level and, accordingly, existing methods and confluence checkers. In some cases, it is possible to eliminate invariants by program transformations, so that rules apply exactly when the invariant and the original rule guards are satisfied; this means that the transformed program is confluent if and only if the original one is confluent under the invariant.

Example 5. Reconsidering the program of Example 4, the following is an example of such a transformed program; the constants a and b are introduced as representations of positive, resp., non-positive integers.

\[
\begin{align*}
    p(a) &\iff q(a). & p(a) &\iff r(a). & p(a) &\iff r(a). \\
    p(b) &\iff q(b). & p(b) &\iff r(b). & r(b) &\iff q(b).
\end{align*}
\]

Such program transformations become more complex when the guards describe more involved dependencies between the head variables. More importantly, invariants that exclude certain constraints in a state cannot be expressed in this way, for example “only one set constraint allowed” (Examples 3 and 7). Thus we refrain from pursuing a transformational approach. To obtain a maximum degree of generality, we introduce a meta level formalization of CHR’s operational semantics that include representations as explicit data objects of states and their components, possibly parameterized by constrained meta variables.

6.1 The choice of a ground representation

Invariants and state equivalences are inherently meta level statements, as they are about states, and may refer to notions inexpressible at the object level, e.g., that some part being ground or a variable. Earlier work on meta-interpreters for logic programs, e.g., [5,17,18], offers the desired expressibility in terms of a ground representation. Any object term, formula, etc. is named by a ground meta level term. Variables are named by special constants, say X by ‘X’, and any other symbol by a function symbol written the same way; e.g., the non-ground object level atom p(A) is named by the ground meta level term p(A'). For any such ground meta level term mtl, we indicate the object it names as \([\text{Gr} mtl]\). For example, \([p('A')]_{\text{Gr}} = p(A)\) and \([p('A') \land 'A'>2]_{\text{Gr}} = (p(A) \land A>2)\).

For a given object entity e, we define its lifting to the meta level by 1) selecting a meta level term that names e, and 2) replacing variable names in it consistently by fresh meta level variables. For example, \(p(X) \land X>2\) is lifted to \(p(x) \land x>2\), where X and x are object, resp., meta variables. By virtue of this overloaded syntax, we may read such an entity e (implicitly) as its lifting.

A collection of meta level constraints is assumed whose meanings are given by a theory \(\mathcal{M}\). We start defining meta level states without detailed assumptions about \(\mathcal{M}\), that are postponed to Definition 6 below. We assume object level built-in theory \(\mathcal{B}\), invariant \(I_{\text{logic}}\) and state equivalence \(\approx_{\text{logic}}\).
Definition 5. A constrained meta level term is a structure of the form
(mt where M), where mt is a meta level term and M a conjunction of \( \mathcal{M} \) constraints. We define

\[
[M] = \{ \sigma \mid \mathcal{M} \models M \sigma \}, \\
[mt \ where \ M]_{\text{meta}}^{\text{logic}} = \{ [[mt \sigma]]^{Gr} \mid \sigma \in [M] \}.
\]

A meta level state representation (s.repr.) is a constrained meta level term st where M for which [st where M]_{\text{meta}}^{\text{logic}} is a set of object level states. Two meta level s.repr.s SR1, SR2 are variants whenever each object level s.repr. in [SR1]_{\text{meta}}^{\text{logic}} is a variant of some object level s.repr. in [SR2]_{\text{meta}}^{\text{logic}} and vice versa. A meta level state is an equivalence class of meta level s.repr.s under the variant relationship. For structures of meta level states (transitions, corners, etc.), we apply the following convention, where \( f \) may represent any such structure.

\[
[f(mt_1 \ where \ M_1, \ldots, mt_n \ where \ M_n)]_{\text{meta}}^{\text{logic}} = [f(mt_1, \ldots, mt_n) \ where \ M_1 \wedge \cdots \wedge M_n]_{\text{meta}}^{\text{logic}}
\]

Meta level invariant \( I_{\text{meta}}^{\text{logic}} \) and equivalence \( \approx_{\text{meta}}^{\text{logic}} \) are defined as follows.

- \( I_{\text{logic}}^{\text{logic}}(s) \) whenever \( I_{\text{logic}}(s) \) for all \( s \in [S]_{\text{meta}}^{\text{logic}} \).
- \( S_1 \approx_{\text{logic}} S_2 \) whenever \( s_1 \approx_{\text{logic}} s_2 \) for all \( (s_1, s_2) \in [(S_1, S_2)]_{\text{meta}}^{\text{logic}} \).

As before, we may indicate a meta level state by a representation of it.

Definition 6. The theory \( \mathcal{M} \) includes at least the following constraints.

- \( =/2 \) with its usual meaning of syntactic identity,
- Type constraints \( \text{type}/2 \). For example \( \text{type}(\text{var}, x) \) is true in \( \mathcal{M} \) whenever \( x \) is the name of an object level variable; \( \text{var} \) is an example of a type, and we introduce more types below when we need them.
- Modal constraints \( \Box F \) and \( \Diamond F \) defined to be true in \( \mathcal{M} \) whenever \( B \models [[F]]^{Gr} \), resp., \( B \models [[\neg F]]^{Gr} \).
- We define two constraints \( \text{inv} \) and \( \text{equiv} \) such that \( \text{inv}(\Sigma) \) is true in \( \mathcal{M} \) whenever \( [\Sigma]^{Gr} \) is an I_{\text{logic}} state (representation) of the logical semantics, and \( \text{equiv}(\Sigma_1, \Sigma_2) \) whenever \( [[\Sigma_1, \Sigma_2]]^{Gr} \) is a pair of states (representations) \( (s_1, s_2) \) of the logical semantics such that \( s_1 \approx_{\text{logic}} s_2 \).
- \( \text{freshVars}(L, T) \) is true in \( \mathcal{M} \) whenever \( L \) is a list of all different variables names, none of which occur in the term \( T \); \( \text{freshVars}(L_1, L_2, T) \) abbreviates \( \text{freshVars}(L_{12}, T) \) where \( L_{12} \) is the concat. of \( L_1 \) and \( L_2 \).

Definitions 5 and 6 comprise the first steps towards a simulation of the logic-based semantics, and we continue with the last part, the transition relation.

Definition 7. Consider a (lifted version of a) pre-application \( H_1 \backslash H_2 \leftrightarrow G \mid C \) with local variables \( L \) and a consistent meta level state \( S \ where \ M \) with \( S = (H_1 \cup H_2 \cup S^+, B^+) \) and

\[
\mathcal{M} \models M \rightarrow (\text{inv}(S) \wedge \Box B^+ \wedge \Box (B^+ \rightarrow \exists L G) \wedge \text{freshVars}(L, S)).
\]
Then the following is a meta level transition by rule application.

$S \text{ where } M \rightarrow^\text{meta}_{\text{logic}} (H_1 \psi C \omega S^+, G \wedge B^+)$ where $M$

Consider a (lifted version of a) built-in $b$ of $B$ and a consistent meta level state $(S \text{ where } M)$ with $S = \langle \{b\} \omega S^+, B^+ \rangle$ and

$$M \models M \rightarrow (\text{inv}(S) \wedge \Box B^+).$$

Then the following is a meta level transition by built-in.

$\langle \{b\} \omega S^+, B^+ \rangle \text{ where } M \rightarrow^\text{meta}_{\text{logic}} \langle S^+, b \wedge B^+ \rangle \text{ where } M$

Notice that for both sorts of transitions, the implication of $\Box B^+$ excludes transitions from failed and mixed states. For built-in transitions, the resulting states may be non-failed, failed or mixed.

**Lemma 4.** For a given CHR program with $I_{\text{logic}}$ and $\approx_{\text{logic}}$, the definitions of meta level states and transitions $\rightarrow^\text{meta}_{\text{logic}}$, $I_{\text{meta}}$ and $\approx_{\text{meta}}$, together with $[-]_{\text{logic}}$ comprise an abstract simulation of the logic-based semantics.

**Proof.** A fixed CHR program is assumed with given invariant $I_{\text{logic}}$ and state equivalence $\approx_{\text{logic}}$. The lemma states the meta-level transition system defined in Section 6, given by $\rightarrow^\text{meta}_{\text{logic}}$, $I_{\text{meta}}$ and $\approx_{\text{meta}}$, is an abstract simulation of the logical semantics given by $\rightarrow_{\text{logic}}$, $I_{\text{logic}}$ and $\approx_{\text{logic}}$.

To do that, we must specify a suitable set of state replacements (mapping meta level states to object level states) and verify the six conditions of Definition 1 (p. 5). The replacements are generated from the set of all meta level substitutions as follows.

For given meta-level state representation $(s.\text{repr}) \Sigma' = (mt \text{ where } M)$ and meta level substitution $\sigma$, we define the replacement $\Sigma' \rho^\sigma = [mt \sigma]^{Gr}$ whenever $\sigma \in [M]$ and $\Sigma' \rho^\sigma$ undefined otherwise. Recall that the states of the transition systems in question are defined as equivalence classes of $s.\text{repr}s$, so for meta and object level states $\Sigma$ and $s$, we define $\Sigma' \rho^\sigma = s$ whenever there exists $s.\text{repr} s'$ of $\Sigma$ and $s'$ of $s$ for which $\Sigma' \rho^\sigma = s'$. Notice that the function $[-]_{\text{logic}}$ defined in Section 6 is in fact the covering functions given by these state replacements.

First of all, we notice that $I_{\text{logic}}$ and $\approx_{\text{logic}}$ are defined specifically such that the conditions concerning invariant and equivalence do hold, so the remaining part of the proof concentrate on the two conditions that concern transitions. Finally, we need to prove the first condition of Definition 1,

$$\Sigma_0 \rightarrow^\text{meta}_{\text{logic}} \Sigma_1 \quad \Rightarrow \quad \forall (s_0, s_1) \in [\{\Sigma_0, \Sigma_1\}]_{\text{logic}} : s_0 \rightarrow_{\text{logic}} s_1.$$ 

There are two subcases, transition by rule and by built-in. We start by rule, and referring to Definition 7, we assume thus a (lifted version of a) pre-application $Pre = (H_1 \wedge H_2 \iff G \mid C)$ with local variables $L$, $\Sigma_0 = (\langle H_1 \psi C \omega S^+, B \rangle \text{ where } M)$, $\Sigma_1 = (\langle H_1 \psi C \omega S^+, B \rangle \text{ where } M)$ such that

$$M \models M \rightarrow (\text{inv}(S) \wedge \Box B^+ \wedge \Box (B^+ \rightarrow \exists L G) \wedge \text{freshVars}(L, S)).$$
Whenever $\sigma \in [M]$, and $(s_0, s_1) = \left[ \left( \Sigma_0 \sigma, \Sigma_1 \sigma \right) \right]_{Gr}$, the detailed meta level constraints implied by $M\sigma$, i.e., $\mathsf{inv}(S\sigma)$ etc., are defined exactly such that the conditions for $s_0 \mapsto \mathsf{logic} s_1$ holds, i.e., $I_{\mathsf{logic}}(s_0)$, $s_0$ is not a failure state, the guard is satisfied, and $[\mathsf{Pre}]_G \sigma$ is an application instance whose head constraints appear in $s_0$ and whose local variables are fresh variables.

The proof of the second subcase, meta-level transition by built-in goes as follows. Assume thus a (lifted version of a) built-in $b$ of $B$ and that $\Sigma_0 = \left( \left\{ b \right\}_w S^+, B^+ \right)$ where $M$ is consistent and neither failure nor mixed, and $\Sigma_0 = \left( (S^+, b \wedge B^+) \right.$ where $M$). With $\sigma \in [M]$ and $(s_0, s_1) = \left[ \left( \Sigma_0 \sigma, \Sigma_1 \sigma \right) \right]_{Gr}$, we can argue as above that $s_0 \mapsto \mathsf{logic} s_1$ is in fact an object level transition; notice that $s_1$ may be non-failed or failed.

Transitions are not possible from a mixed or failed meta level state, but modal constraints are useful for restricting to the relevant substate, such that transitions are known to exists. This is expressed by the following propositions that are immediate consequences of the definitions.

**Proposition 1.** Let $r : H_1 \setminus H_2 \leftrightarrow G \setminus C$ be a (lifted version of a) pre-application with local variables $L$ and $\Sigma = \left( \left( S, B \right) \right.$ where $M$ a meta level state with $H_1 \setminus H_2 \subseteq S$. Whenever the meta level state $\Sigma^{\square} = \left( \left( S, B \right) \right.$ where $M \wedge \overline{M}$ is consistent, with $\overline{M} = \mathsf{inv}(S, B) \wedge \Box B \wedge \Box (B \rightarrow \exists L G) \wedge \mathsf{fresVariables}(L, \Sigma)$, there exists a meta level rule application by $r$,

$$
\Sigma^{\square} \xrightarrow{\mathsf{meta} \mathsf{logic}} (S \setminus H_2 \setminus C, B \wedge G) \text{ where } M \wedge \overline{M}.
$$

Furthermore, $\Sigma^{\square}$ is the greatest substate of $\Sigma$ to which $r$ can apply.

**Proposition 2.** Let $b$ be a (lifted version of a) built-in and $\Sigma = \left( \left( S, B \right) \right.$ where $M$ a meta level state with $b \in S$. When $\Sigma^{\square} = \left( \left( S, B \right) \right.$ where $M \wedge \overline{M}$ is consistent, with $\overline{M} = \mathsf{inv}(S, B) \wedge \Box B \wedge \Box (B \rightarrow b)$, there is a meta level trans.,

$$
\Sigma^{\square} \xrightarrow{\mathsf{meta} \mathsf{logic}} (S \setminus \left\{ b \right\}, B \wedge b) \text{ where } M \wedge \overline{M}^{\square}.
$$

Whenever $\Sigma^{\square} = \left( \left( S, B \right) \right.$ where $M \wedge \overline{M}^{\square}$) is consistent, with $\overline{M}^{\square} = \mathsf{inv}(S, B)) \wedge \Box B \wedge \Box (B \rightarrow b)$, there is a meta level transition by $b$,

$$
\Sigma^{\square} \xrightarrow{\mathsf{meta} \mathsf{logic}} (S \setminus \left\{ b \right\}, B \wedge b) \text{ where } M \wedge \overline{M}^{\square}.
$$

The state $\Sigma^{\square}$ (resp. $\Sigma^{\square}$) is the greatest substate of $\Sigma$ for which the meta level transition by $b$ leads to a non-failure and non-mixed (resp. failed) state.

With Propositions 1 and 2 in mind, we define meta level critical corners from the critical corners of Definition 4.
Definition 8. Let $\langle S_1, B_1 \rangle \leftarrow \text{logic } \langle S_0, B_0 \rangle \rightarrow \text{logic } \langle S_2, B_2 \rangle$ be a (lifted version of a) critical $\alpha$-pre-corner given by Def. 4, in which the leftmost (rightmost) rule application has local variables $L_1$ ($L_2$) and guard $G_1$ ($G_2$). Assume $S^+$ and $B^+$ are fresh meta level variables and let, for $i = 0, 1, 2,$

$$
\Sigma_i = \langle S_i \oplus S^+, B_i \land B^+ \rangle \\
M = \text{inv}(\Sigma_0) \land \Box B_0 \land \Box(B_0 \land B^+ \rightarrow \exists L_1 G_1) \land \Box(B_0 \land B^+ \rightarrow \exists L_2 G_2) \land \text{freshVars}(L_1, L_2, \Sigma)
$$

When $(\Sigma_0 \text{ where } M)$ is consistent, the following is a critical meta level $\alpha$ corner.

$$(\Sigma_1 \text{ where } M) \leftarrow \text{meta } (\Sigma_0 \text{ where } M) \rightarrow \text{meta } (\Sigma_2 \text{ where } M)$$

Example 6. (Continuing Ex.4) The invariant is formalized at the meta level as states of the form $\langle \langle \text{pred}(n) \rangle, true \rangle$ where type(int,$n$) where pred is one of $p$, $q$ and $r$. Below is shown the non-joinable critical meta level corner generated by Def. 8. It is split-joinable as demonstrated by its splitting into two corners; each shown joinable by the indicated dotted transition. Let $M$ stand for the meta-level constraint type(int,$n$), $M_1$ for $M \land \Box n \leq 0$ and $M_2$ for $M \land \Box n > 0$.

$$
\begin{array}{c}
(p(n), true) \\
q(n), true
\end{array}
\begin{array}{c}
r_1
\downarrow
\text{where } M
\end{array}
\begin{array}{c}
r_2
\downarrow
\text{where } M
\end{array}
\begin{array}{c}
\langle q(n), true \rangle \\
\text{where } M
\end{array}
\begin{array}{c}
r_1
\downarrow
\text{where } M_1
\end{array}
\begin{array}{c}
r_2
\downarrow
\text{where } M_1
\end{array}
\begin{array}{c}
\langle q(n), true \rangle \\
\text{where } M_2
\end{array}
\begin{array}{c}
r_1
\downarrow
\text{where } M_2
\end{array}
\begin{array}{c}
r_2
\downarrow
\text{where } M_2
\end{array}
$$

According to Lemma 5 shown below, the program is confluent.

When, furthermore, a state equivalence $\approx_{\text{logic}}$ is assumed, we need also show joinability of $\beta$ corners, i.e., those composed by an equivalence and a transition.

Definition 9. Let $H \land H' \iff G \land C$ be a (lifted version of a) variant of a rule with local variables $L$. Assume $S^+$, $B^+$ and $\Sigma_1$ are fresh meta-variables, and let

$$
\Sigma_0 = \langle H \land H' \land S^+, B^+ \rangle \\
M = \text{inv}(\Sigma_0) \land \Box B \land \Box(B \land B^+ \rightarrow \exists L G) \land \text{freshVars}(L, \Sigma_0) \land \text{equiv}(\Sigma_0, \Sigma_1)
$$

When $(\Sigma_0 \text{ where } M)$ is consistent, the following is a critical meta level $\beta$ corner by rule application.

$$(\Sigma_1 \text{ where } M) \approx_{\text{logic}} (\Sigma_0 \text{ where } M) \rightarrow_{\text{logic}} (\Sigma_2 \text{ where } M)$$

Let $b$ be a (lifted version of a) built-in atom whose arguments are fresh variables. Assume $S^+$, $B^+$ and $\Sigma_1$ are fresh meta-variables, and let

$$
\Sigma_0 = \langle [b] \land S^+, B^+ \rangle \\
\Sigma_2 = \langle S^+, b \land B^+ \rangle \\
M = \text{inv}(\Sigma_0) \land \Box B \land \text{freshVars}(L, \Sigma_0) \land \text{equiv}(\Sigma_0, \Sigma_1)
$$

15
When \((\Sigma_0 \text{ where } M)\) is consistent, the following is a critical meta level \(\beta\) corner by built-in.

\[(\Sigma_1 \text{ where } M) \approx^{\text{meta}} \text{ logic } (\Sigma_0 \text{ where } M) \mapsto^{\text{meta}} \text{ logic } (\Sigma_2 \text{ where } M)\]

**Lemma 5.** Let a terminating CHR program \(\Pi\) with invariant \(I_{\text{logic}}\) (and state equivalence \(\approx_{\text{logic}}\)) be given. Then \(\Pi\) is confluent (modulo \(\approx_{\text{logic}}\)) if and only if its set of critical corners (Defs 8–9) is split-joinable w.r.t. \(I_{\text{logic}}^{\text{meta}}\) (modulo \(\approx_{\text{logic}}^{\text{meta}}\)).

**Proof (Lemma 5).** A fixed, terminating CHR program is assumed with given invariant \(I_{\text{logic}}\) and state equivalence \(\approx_{\text{logic}}\), and let \(C\) be its set of meta level critical corners given by Defs 8–9. The lemma states that the program is confluent under \(I_{\text{logic}}\), modulo \(\approx_{\text{logic}}\), if and only if all corners in \(C\) are split joinable.

The “only if” part goes as follows. Assume that the program is confluent under \(I_{\text{logic}}\) modulo \(\approx_{\text{logic}}\), and let

\[A = ((S_1 \text{ where } M_1) \ Rel \ (S_0 \text{ where } M_0) \mapsto (S_2 \text{ where } M_2)) \in C,\]

for any \(\lambda = (s_1 \ Rel \ s_0 \mapsto s_2) \in [A]_{\text{meta}}^{\lambda}\), let \((s_1, s_0, s_2)\) be a lifting with fresh variables of \((s_1, s_0, s_2)\) to the meta level, and let \(E_{\lambda}\) be the meta-level equation \((s_1, s_0, s_2) = (S_1, S_0, S_2)\). Now, the joinability of the following corner \(A_{\lambda}\) follows from the joinability of \(\lambda\).

\[A_{\lambda} = ((S_1 \text{ where } M_1 \wedge E_{\lambda}) \ Rel \ (S_0 \text{ where } M_0 \wedge E_{\lambda}) \mapsto (S_2 \text{ where } M_2 \wedge E_{\lambda}))\]

The detailed argument involves lifting the transitions used for joining \(\lambda\) into a meta level transitions shows \(A_{\lambda}\) joinable. Thus \(\{ A_{\lambda} | \lambda \in [A]_{\text{meta}}^{\lambda}\}\) is a splitting into a set of corners that are all joinable, thus \(A\) is split joinable. (This is not the most “economical” splitting since it is infinite, and in most interesting cases, it should be possible to identify a finite splitting.)

The “if” part goes as follows. Assume that \(C\) is split joinable, and we need to show that the program is confluent, which we do by showing any object level corner joinable. By Lemmas 3 and 4, any object level corner covered by a corner in \(C\) is joinable, so it suffices to show that any object level corner not covered in this way is joinable. We go through the different ways that this may be the case; in the following, let \(\mapsto^r_{\text{logic}}\) (with possible index) refer to a transition by a rule application being an instance of rule \(r\), and \(\mapsto^r_{\text{logic}}\) (with possible index) refer to a transition by built-in.

\[s_1 \mapsto^r_{\text{logic}} s_0 \mapsto^r_{\text{logic}} s_2\]

This is possible in three ways. a) \(s_1 = s_2\) which makes the corner trivially joinable. b) There is no overlap, cf. Definition 4, between \(r_1\) and \(r_2\), which means that the rule applications commute. When a) and b) are not the case, it means there is an overlap, but then we can argue that it is covered by some critical meta-level corner as follows, which is a contradiction. Refer to the corner in question as \(\lambda\), and the existence of an overlap means that we can construct a critical object level corner \(\lambda'\) (Definition 4) from \(r_1\) and \(r_2\).
such that $\lambda \in [\lambda']^\text{logic}$, construct in turn a critical meta-level corner $\Lambda$ from $\lambda'$ (Definition 8) and it can be shown that $\lambda \in [\Lambda]^\text{meta}$.

$s_1 \xleftarrow{b_1} s_0 \xrightarrow{b_3} s_2$

Monotonicity (p. 9) of the logical built-ins means that the two built-in applications commute, which implies joinability; this argument extends also to cases in which $s_1$ and/or $s_2$ are failure states.

$s_1 \approx s_0 \xrightarrow{r} s_2$

Monotonicity (p. 9) of the logical built-ins means that $b$ and the evaluation of the guard commutes, which implies joinability.

$s_1 \approx s_0 \xrightarrow{r} s_2$

Refer to the corner in question as $\lambda$, and let $\Lambda$ be a critical meta-level corner constructed from $r$ (Definition 9, first part). It is straightforward to show that $\lambda \in [\Lambda]^\text{meta}$; thus there are no such object level $\beta$ corner that is not covered by some critical $\beta$ meta-level corner.

The proof is exactly as in the previous case.

**Example 7 (Cont. Ex. 3; adapted from [7]).** The invariant is formalized at the meta level as states of the form

$$\langle \{\text{set}(L)\} \sqcup S, \text{true} \rangle$$

we assume types $\text{const}$ for all constants, $\text{constList}$ for all lists of such, and $\text{constItems}$ for sets of constraints of the form $\text{item}(c)$ where $c$ is a constant.

The state equivalence is formalized at the meta level as the relationships of states of the following form, where $\text{perm}(L_1, L_2)$ means that $L_1$ and $L_2$ are lists being permutations of each other; and $M^\approx$ stands for $\text{type}(\text{constList}, L_1) \wedge \text{type}(\text{constList}, L_1) \wedge \text{perm}(L_1, L_2) \wedge \text{type}(\text{constItems}, S)$,

$$\langle \{\text{set}(L_1)\} \sqcup S, \text{true} \rangle \text{ where } M^\approx \approx^\text{meta} \approx \langle \{\text{set}(L_2)\} \sqcup S, \text{true} \rangle \text{ where } M^\approx$$

The critical object level corner with two set constraints in the states does not give rise to a critical meta level corner as the invariant is not satisfied. The other one is shown here, including (with dotted arrows) its proof of joinability modulo equivalence; $M^\preceq$ stands for $\text{type}(\text{const}, x_1) \wedge \text{type}(\text{constList}, L) \wedge \text{type}(\text{const}, x_2) \wedge \text{type}(\text{constItems}, S)$.

$$\langle \{\text{item}(x_1), \text{set}(L), \text{item}(x_2)\} \sqcup S, \text{true} \rangle \text{ where } M^\preceq$$
We consider the following critical meta level $\beta$ corner. $M^\beta$ stands for $\text{type}(\text{const}, x) \land \text{type}(\text{constList}, L_1) \land \text{type}(\text{constList}, L_2) \land \text{perm}(L_1, L_2) \land \text{type}(\text{constItems}, S)$.

$$
\langle\{\text{item}(x), \text{set}(L_1)\}\cup S, \text{true}\rangle \text{ where } M^\beta
$$

$$
\langle\{\text{item}(x), \text{set}(L_2)\}\cup S, \text{true}\rangle \text{ where } M^\beta
$$

$$
\langle\{\text{set}(\{x\mid L_1\})\}\cup S, \text{true}\rangle \text{ where } M^\beta
$$

All critical corners are joinable modulo equivalence, and since the program is obviously terminating, Lemma 5 gives that the program is confluent mod. equiv.

7 Conclusion

We generalized the critical pair approach using a meta level simulation to prove confluence under invariant and modulo equivalence for Constraint Handling Rules. We have demonstrated how this principle makes it possible to express natural invariants and equivalences, that cannot be expressed in CHR itself, in a formal way at the meta level, anticipating machine supported proofs using a meta level constraint solver, based on a ground representation. A constraint solver is currently under development, partly inspired by [5]. Depending on the complexity of the invariants and equivalences – and of the CHR programs under investigation – it may be difficult to obtain a complete solver.

For simplicity of notation, we did not include mechanisms to prevent loops caused by propagation rules; [7] has included this in a meta level representation for the Prolog based semantics, and is easily adapted for the logic based semantics exposed in the present paper.

For comparison with earlier work on confluence for CHR, we used here a logic-based CHR semantics, which has nice theoretical properties, but is incompatible with standard implementations of CHR and applies only for a limited set of programs. In [8], we have defined meta level constraints and a simulation for an alternative CHR semantics [6,7] that reflects CHR’s Prolog based implementation, including a correct handling of Prolog’s non-logical devices (e.g., var/1, nonvar/2, is/2) and runtime errors.

We could argue that the abstract simulations used for the classical CHR confluence results are special cases of abstract interpretations. When invariants are introduced – or when considering full CHR including Prolog-style non-logical devices, cf. [8] – this correspondence does not hold.

The concept of abstract simulations and their use for proving confluence (mod. equiv.) seem obvious to investigate for a large variety of rewrite based systems, e.g., constrained term rewriting, conditional term rewriting, interactive theorem provers, and rule-based specifications of abstract algorithms.

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