The use of Theory in Teachers' Modelling Projects – Experiences from an In-sevice Course

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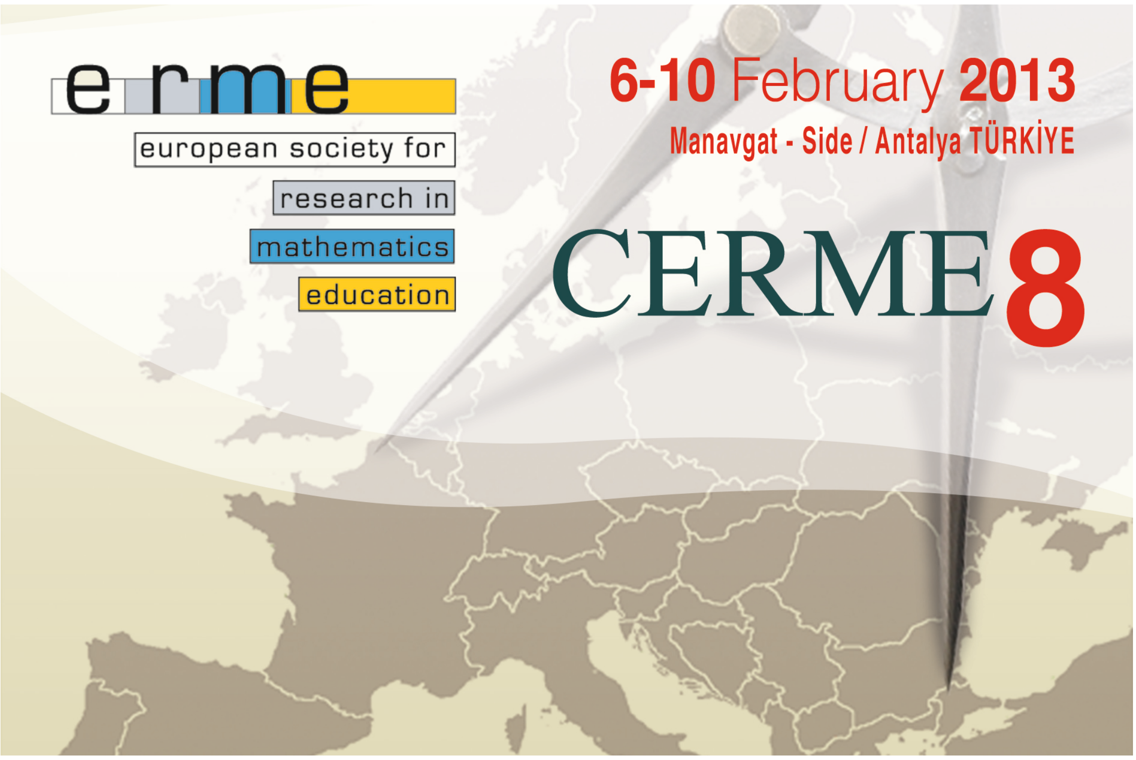

Proceedings of the EightÈ Congress of the European Society for Research in Mathematics Education

Editors

Behiye Ubuz, Çiğdem Haser, Maria Alessandra Mariotti

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Research Papers

PROCEEDINGS OF THE EIGHTH CONGRESS OF THE EUROPEAN SOCIETY FOR RESEARCH IN MATHEMATICS EDUCATION: GENERAL INTRODUCTION

Maria Alessandra Mariotti — University of Siena
Ferdinando Arzarello — University of Turin

We are glad to present the Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education (ERME), which took place 6-10 February 2013, at Manavgat-Side/Antalya in Turkey.

The story of ERME begins at its first congress in Osnabrueck, Germany, in 1998 and develops all along the CERME congresses that have taken place every two years since CERME2 in 2001. The vision shared by the first group of founders was that of establishing a community to promote communication, cooperation and collaboration in mathematics education research in Europe. At the very beginning of the ERME story, considerable time was spent talking about the nature of our conferences. Especially we wondered how were we going to achieve the communicative, cooperative and collaborative spirit we envisaged. It was clear that the conference should offer more than just a platform for presenting and listening to papers, as many other conferences do. We wanted that CERME could allow groups of researchers in a particular scientific area really to *work* together on their area of research, with sufficient time to get to know each other, to share and discuss their research and to engage in deep scholarly debate. At the same time we wanted to support the scientific development of young researchers fostering their active participation to our research community.

Therefore it was decided that CERME should abandoned the common format of parallel research report presentations and adopt a new format based on Working Group activities where participation by all who attend the congress could be promoted. Such a format has been developed stating a clear policy for the organization and the management of thematic Working Groups. At CERME participants spent most of the time in discussion and debate within the thematic Working Groups (WGs), during 6 or 7 *working* sessions of 90-120. Each CERME participant selects the membership of just one such Group, on the base of her/his personal scientific interests. For each WG, a team of leaders is nominated by the Scientific Program Committee, the leaders have in charge the complex organization of the WGs, preparing and managing what will happen at the Conference. Though participation to the conference is completely free, prospective participants are encouraged to contribute with submitting a paper or a poster. The leaders' team organizes the peer review process among the member of the Group according significantly devolved and distributed responsibility in criticising but also supporting the elaboration of the single contributions. This process aims not only at rising the quality of the papers but also at developing a sense of belonging to a community, for all participants. At the end of this first phase all the accepted paper will be posted on the website of the conference and participants are expected to read all the papers related to their own WGs, before attending the conference. This corpus of papers will constitute the first working material for the WGs activities, and a great deal of time and intellectual efforts are spent by the leaders to outline the structure of the working sessions where the different contributions will be fully discussed and related to the other contributions.

At CERME 8, different ways of organizing the working group sessions were set up by the different leaders teams. The main objective was always that of fostering the discussion exploiting the richness of the contributions. In some cases the discussion was structured according to subthemes focussing on specific clusters that emerged from the variety of the papers. Other times the leaders proposed the participants specific questions that were sent in advance to the authors of the papers who were requested to focus a short contribution on this question. The variety in the organization structure witnesses of the complexity of the task that the leaders team are asked to face but also of their passion and commitment in accomplishing their work, for which the ERME community is highly grateful to them. In the introduction to the collection of papers of each WG, the reader will find a description of the different organizations that were adopted.

The particular format of the conference gives the participants the opportunity of getting fruitful feedbacks that can enlarge and enrich their own perspectives; thus, after the conference the authors have the possibility to further revise their papers, integrating significant elements emerged from their WG's discussion: this will be the latest form in which papers will pass the final review process and when accepted will appear in the proceedings. The double review process that is used at CERME congresses - papers are firstly accepted for discussion in the WGs and than their final version has to be accepted for being published in the proceedings - not only aims at raising the quality of the papers but also at assuring a fair balance between quality and inclusion, two goals that seem to pull in different directions, and may create tension, sometimes frustration. However, the attainment of a good balance between *quality* and *inclusion* constitutes the main challenge of our community according to our main objectie: to ensure the ERME spirit‘ of communication, cooperation and collaboration.

The number of WGs increased in the years and since CERME7 we have 17 WGs, and excepting of the WG 15 and WG 17, the number of participants in each is around 25-30 on average, including about 4 WG leaders.

The themes of the WGs are as follows:

WG1: Argumentation and proof

WG2: Arithmetic and number systems

WG3: Algebraic thinking

WG4: Geometrical thinking

WG5: Stochastic thinking

WG6: Applications and modelling

WG7: Mathematical potential, creativity and talent

WG8: Affect and mathematical thinking

WG9: Mathematics and language

WG10: Cultural diversity and Mathematics Education

WG11: Comparative studies in Mathematics Education

WG12: History in Mathematics Education

WG13: Early Years Mathematics

WG14: University mathematics education

WG15: Technologies and resources in mathematics education

WG16: Different theoretical perspectives / approaches in research in mathematics education

WG17: From a study of teaching practices to issues in teacher education

The success of the ERME Conferences is witnessed by the constant increasing number of participants and presentations. In Manavgat 520 participants attended the congress, from 45 countries within and beyond Europe.

In addition to the WG activities, the congress was enriched by in a number of plenary scientific activities, and a varied social and cultural program.

The opening session included a plenary address by Paolo Boero who proposed a deep reflection on how to deal, as researchers, with the unavoidable complexity of big problems concerning the teaching and learning of mathematics in our societies. On the base of a long personal elaboration, strictly and functionally interwoven with the evolution of the experimental activity in the school carried out with the Genoa research group since the seventies, Boero offered us some answers to those big questions emerging from complex phenomena, particularly those concerning societal needs and values and related educational choices.

As at previous CERME congresses, two other plenary talks were given by former WGs leaders, Alain Kuzniak and Candia Morgan.

Kuzniak presented a vivid account of what are today the core items and the contributions of researches in the didactics of geometry, and he did it in the light of the rich discussions which have been occurring in the CERME Working Group on geometry from its beginning in 1999. Candia Morgan delineated a superb survey of the complex field of study of language in mathematics education. As she said, she offered *her* map, her personal and critical account, on previous studies in this field, and especially a theoretical elaboration as it emerged from the active discussion taking place at the CERME Working Group on Language and Mathematics over the years.

Three papers corresponding to these three plenary addresses are included in these proceedings.

Though these proceeding do not contain any document related to it, let me mention another fundamental event that took place one day before the opening of the Congress: the YERME (Young European Researchers in Mathematics Education) day. This is now a constant appointment where young researchers – doctoral students or post-doctoral researchers - meet expert scholars in thematic discussion groups. This event, together with the YERME Summer School (YESS), is based on the volunteering of some members of the society. At CERME 8 the organization of the YERME day was coordinated by João Pedro da Ponte, Ferdinando Arzarello and Behiye Ubuz; the activities were led by professors Paolo Boero, Uffe Thomas Jankvist, Barbara Jaworski, Ester Levenson, Maria Alessandra Mariotti, João Pedro da Ponte, Susanne Prediger, Mario Sanchéz, Susanne Schnell, Behiye Ubuz. (http://cerme8.metu.edu.tr/yerme.html)

As said, our Conference has a very particular format, it promotes the active involvement of all the participants and its success highly depends on their contributions; however, success also depends on the commitment of those who made this involvement possible, to them we want to express our gratitude in behalf of the ERME community: to the members of the Scientific Program Committee, for the inspiration and support that they offered in the scientific planning of the conference, to the Leaders of the WGs, for the competence, the energy and the engagement that they invested in their responsibility, and last but not least to the President, Behiye Ubuz, and the members of the Local Organizing Committee, for the incredible work done in preparing and supervising the organization of the conference, they allowed all the participant enjoy the conference days of intensive intellectual work in a efficient, comfortable and delightful place. Their attentive support did not finished with the end of the conference but continued in the patient and competent work of editing these proceedings.

We are certain that the reader will appreciate the richness of the contributions collected in this text that we hope will offer the opportunity to share with us something of the exciting experience of our congress, and encourage interested researchers to meet us at the next CERMEs.

Maria Alessandra Mariotti (Chair of the program Committee)

Ferdinando Arzarello (ERME President)

Information on-line

The CERME website was at : http://cerme8.metu.edu.tr/

These proceedings can be accessed online from: http://www.mathematik.uni-dortmund.de/~erme/doc/cerme7/CERME7.pdf

EDITORIAL INTRODUCTION FOR THE EIGHT CONGRESS OF THE EUROPEAN SOCIETY FOR RESEARCH IN MATHEMATICS EDUCATION

http://www.cerme8.metu.edu.tr/

Behiye Ubuz and Çiğdem Haser
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The Eight Congress of the European Society for Research in Mathematics Education (CERME8) was held at the Starlight Convention Center, Thalasso & Spa Hotel in Manavgat-Side, Antalya, Turkey from 6th to 10th February, 2013, chaired by Prof. Dr. Behiye Ubuz (Local Organizer Chair) and Prof. Dr. Maria Alessandra Mariotti (International Program Committee Chair). It aimed to promote the development of mathematics education through intellectual communication and cooperation by attending thematic working groups, plenary talks, poster sessions, and so forth. At CERME8 there were 3 invited plenary talks given by Paolo Boero, Alain Kuzniak, and Candia Morgan together with 17 thematic working groups (WGs). The main work of the congress took place in these Thematic Working Groups, facilitated by some 73 Working Group leaders. The congress was preceded by a meeting of Young European Researchers in Mathematics Education (YERME) on 5th and 6th February 2013.

A total of 375 Research Papers and 90 poster proposal were submitted for the congress. Following peer review, 310 Research Papers and 57 poster proposals were accepted for publication in the proceeding.

A view from Starlight Convention Center, Thalasso & Spa Hotel

Around 520 participants attended the congress, from 45 countries within and beyond Europe. Many participants from European countries were there: UK (31), Portugal (29), Germany (98), Italy (22), Greece (10), Finland (6), Spain (30), Netherlands (8), Sweden (54), Cyprus (2), Denmark (16), Norway (24), Austria (1), Czech Republic (5), France (41), Ireland (5), Romania (1), Russia (4), Belgium (4), Iceland (4), Estonia (2), Latvia (1), Poland (2), and

Switzerland (4). Moreover, there were 34 researchers from Turkey, 12 from Israel, 20 from the US and 16 from Canada, and 1 from Australia, 1 from the Far East (Japan), 27 from Latin America (Brazil, Chile, Colombia, Mexico) and several others from non-affluent countries, such as Iran (2), South Africa (1), Saudi Arabia (3), Algeria (1), Kuwait (1), Tunisian (2), Lebanon (1), and Zaire (1).

In the first section of this proceeding, the plenary talks by Paolo Boero, Alain Kuzniak, and Candia Morgan are presented. They kindly accepted the invitation of the scientific program committee and provided a written account of the ideas they presented in their plenary talks. The second section documents the research papers and poster communications accepted for publication in the proceeding. The papers and the posters are presented under each WG following the introduction written by each WG's leaders. Introductions summarize the scope of the WGs' works and the value of the studies presented.

A view from Side-Antalya (Taken from http://www.resim11.com/Antalya.html)

CERME8 must surely be regarded as a great opportunity for teachers, mathematics educators, teacher educators, and policy makers around the world and in Turkey who are interested in mathematics education and its development. It also provided a valuable development opportunity for the young researchers through YERME.

The proceeding for CERME8 is produced electronically both in CD format and in the website http://www.mathematik.uni-dortmund.de/~erme/doc/cerme8/CERME8.pdf . You can access the individual research papers and poster contributions via the hyperlinks provided on the contents page. We hope that every participant enjoyed the conference and their stay in Turkey!

Behiye Ubuz
(Chair of Local Organising Committee)

Çiğdem Haser
(Congress Secretariat)

Maria Alessandra Mariotti
(Chair of the Program Committee)

PLENARY LECTURES

MATHEMATICS EDUCATION TODAY:
SCIENTIFIC ADVANCEMENTS AND SOCIETAL NEEDS

Paolo Boero

Università di Genova

How to deal, as researchers in mathematics education, with big, complex problems related to societal needs? Through a chronological presentation of some steps of my personal and group research trajectory, I will try to show the potential inherent in establishing a dynamic relationship between: the design, experimentation and analysis of broad, long term classroom activities, planned according to wide scope theoretical constructs; and research activities performed in that context according to well established research methodologies. The research generative power of such an approach will be demonstrated, together with the necessity of further elaboration concerning local integration of different theoretical perspectives. As an example, a problem related to the PISA definition of mathematical literacy will be dealt with.

INTRODUCTION

Thanks to the organizers of CERME-8, this is an occasion for me to reflect on how to deal, as researchers, with the unavoidable complexity of big problems concerning the teaching and learning of mathematics in our societies. I will try to get some answers by considering my personal forty years experience and that of the Genoa research group in mathematics education.

Focusing on a few variables and low levels of complexity allowed mathematics education researchers to perform rigorous, reproducible experiments and, thus, get partial insights into many phenomena. In several cases, methodology was borrowed from other disciplines - particularly experimental psychology - through adaptations to classroom teaching and learning situations: see many Research Reports in the first two decades of PME conferences; see also the volume on the first thirty years of PME research, Gutierrez & Boero (2006). In most cases results were used to try to improve teacher education and teaching practices within the current perspective of schooling in western societies, without putting into question that perspective.

The difficulty met when we want to move towards more complex phenomena, particularly those concerning societal needs and values and related educational choices, depends on the usual scientific methods. Such methods seem to be too limited to deal with non-reproducible phenomena involving a long time span, a lot of inter-related variables, a lot of agents and cultural influences (be they on the scene of the classroom, or outside). As a result of the limit of usual research methods, researchers often tend to refuse the constraints of rigorous research methodology when they want to deal with those complex phenomena. Hence, we have a shift from ordinary scientific papers in the field of mathematics education to political or ideological or philosophical essays.

As an example of the difficulty to deal with complex, big problems, let us consider the following PISA definition of Mathematical Literacy (retrieved from page 4: http://www.oecd.org/pisa/pisaproducts/46961598.pdf):

> *Mathematical literacy* is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts, and tools to describe, explain, and predict phenomena. It assists individuals to recognize the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens.

In many countries, given the importance attributed to mathematical literacy, the feeling of inadequacy of the educational system to face the challenges of the so-called globalization puts into question the traditional content to be taught (usually, a set of notions and techniques) and the traditional methods of teaching mathematics. In those countries (and in others too), PISA comparative studies and PISA - related elaborations, like the above definition, are going to be very influential on long-term educational choices, teaching methods, and also cultural values and the answer to societal needs. The above definition is not neutral: it implies a specific personal relationship to mathematics, specific ways of dealing with "phenomena", a specific role of mathematics in making "well-founded judgments and decisions", and also a specific status of mathematics as an universal cultural entity. As such, the definition may result in specific curricula, teaching strategies, teacher preparation.

A big, complex problem inherent in the PISA definition of mathematical literacy is the identification of its specific epistemological and cultural features, of the consequences that it may have (if coherently adopted) on teachers, students, culture, and of possible alternatives to it (if it is the case of).

In this paper I will try to motivate and present the idea that the interplay between wide-scope theoretical framing and practices (rooted in philosophical, epistemological, anthropological, psychological elaborations), resulting in activity and reflection on activity in school, and related "local" scientific studies, may contribute: to a scientific approach to big, complex problems (like the one related to PISA definition of mathematical literacy) in mathematics education; and also to the generation of research hypotheses, further constructs, and new actions (planning and experimentation of innovative teaching sequences).

The above idea is the conclusion that I am able to derive now from a long personal and group elaboration, strictly and functionally interwoven with the evolution of our experimental, extensive activity in the school since the seventies. Therefore, I think that the best way to present the above idea is to follow a chronological thread, in order to put into evidence the origin and the "why" of it, and "how" it may contribute to develop research and to tackle big educational problems.

Forty years ago we tried to build an alternative, in primary and lower secondary school, to the "New Mathematics" movement, a big challenge at that time! At the end of the eighties, after more than fifteen years of work in hundreds of classrooms, as well as our involvement in the international community of mathematics educators (ICME, PME and CIEAEM conferences; French Summer Schools of Didactics of Mathematics), and our readings in anthropology, history and epistemology of mathematics, the need for theoretical clarification and framing emerged. This need resulted in the wide-scope construct of Field of Experience (FoE). We will present the FoE construct as a tool to frame the teaching of mathematics as a component of a broader enculturation process, according to the need of that time (and present time too!) to overcome the fragmentation of the cultural offer of school and provide opportunities to tackle some important educational problems in mathematics education, like that of the approach to proof.

Afterwards, we will show how the extensive experience of classroom work and related analyses in the perspective of the FoE construct suggested to revisit, fifteen years later, the Vygotskian *everyday concepts/scientific concepts* dialectics. In particular we will see how some "local" studies put into evidence the potential variety of ways of dealing with concepts in a *scientific* way (inside and outside mathematics), related to their roots in cultural practices and personal experiences, and to different cultural horizons. Our answer to the needs emerging from such work was the adaptation of Habermas' elaboration on rationality to educational purposes; we will present it, with some examples of its use in research.

In the perspective of the adapted construct of rationality, the teacher must act, in particular, as an interpreter, mediator and promoter of rationalities. Specific implications for teacher education derive from it; we will see how the competence of Cultural Analysis of the Content to be taught is needed for such a teacher.

As an example of the use of our present theoretical toolkit, we will show how it allows to tackle the problem related to the PISA definition of Mathematical literacy; the elaboration on that problem will result in some elements for an alternative, possible definition of Mathematical literacy.

The paper will end with some general “method” conclusions and research directions, particularly those related to the networking and, possibly, integration with other theoretical constructs.

THE FIELD OF EXPERIENCE DIDACTICS

For me (and for the research group that I lead at the Genoa University) one of the main research aims was since the beginning, and still is, to widen the borders of scientific investigation in mathematics education in order to encompass big problems raising from the reality of ordinary classrooms: they are situated in a socially and culturally more and more fragmented and diverse society, in spite of the pressures towards globalization exercised through media and political and economic decisions.

In the seventies, the initial challenge for our group was to react in a constructive way against the strong political and cultural pressures in OCSE countries aimed at reforming school teaching of mathematics according to the New Mathematics perspective. During the sixties New Mathematics had been advocated as a universal means to promote an effective teaching of mathematics, overcome social differences and cultural discriminations in mathematics education, and convey the real flavour of mathematics. But in those countries (like France) where New Mathematics had been adopted in national programs, by the half of the seventies it was already clear that all those aims were very far from being achieved, and the quality and extension of learning results were rather scarce. Available epistemological and didactical analyses, like those by Freudenthal, helped us to take distance from the theoretical background of New Mathematics and better understand its consequences. Our alternative was initially inspired by the fascinating work of Emma Castelnuovo ("Mathematics in the reality"- see Castelnuovo & Barra, 1980) and by what would have become, following Freudenthal's seminal work, the Realistic Mathematics Education movement. Since 1976, one project for the integrated teaching of mathematics and sciences in grades 5-8 and one project for teaching mathematics and other main disciplines in elementary school (grades 1-5) were elaborated and implemented in school, with the help of specialists in experimental sciences, linguistics, economy, history and psychology. In the period 1976-1985, more than one hundred teachers were involved in the design and experimentation of the teaching units.

During the eighties, problems arising from the design and experimentation of the teaching units (see Boero, 2011) brought us to refer to Vergnaud' s elaboration on concepts (with the key role of reference situations as depositary of their "sense": Vergnaud, 1990), to Vygotsky's everyday concepts/scientific concepts dialectics, and to Bishop's perspective of "mathematical enculturation": a thoughtful analysis of how, in the history of mankind, basic mathematical knowledge is rooted in everyday practices (Bishop, 1988). The resulting action-oriented construct of "Field of experience" (FoE) was presented in my plenary lecture at PME-XIII in 1989 (see Boero, 1989) and further elaborated on the educational and epistemological sides (see Boero, 1992; Boero & al, 1995; Dapueto & Parenti, 1999).

The FoE construct was proposed as a means to frame the teaching and learning of mathematics according to the aim of providing students with the opportunity of both accessing mathematical knowledge, and developing the knowledge of natural and social reality. We may note that, at the moment of its elaboration, the FoE perspective did not escape a pretension of universal objectivity, since it aimed at promoting students’ access to "the" knowledge.

A FoE is made up of three evolutionary components: the student's inner context (specific experiences, mental representations, schemes, ways of reasoning, and so on); the teacher's inner context; and the external context (artefacts, material and social constraints, social practices, etc.). The FoE didactics consists of suitable actions performed by the teacher in order to promote the evolution of the students' inner context according to the teacher's aims and expectations. At the core of FoE didactics are argumentative activities that make reference to specific features and opportunities offered by the external context.

FoEs are cultural domains characterized by specific practices, constraints, ways of behaving and knowing. In our Project for grades 1-5, non-mathematical FoEs deal with subjects like Money and purchases, Time and calendar, Seasonal changes, Sun shadows, Machines. As concerns mathematical FoE, arithmetic and geometry gradually become FoEs: they develop in classroom, thanks to the mediating role of the teacher, around those mathematical tools of knowledge that are needed to deal with non-mathematical problem situations.

The FoE didactics is based on the fundamental didactic cycle (Douek, 1999; Boero & Douek, 2008), which consists of three phases: an initial task requiring written individual productions (possibly supported by the teacher, when needed); the subsequent classroom comparison and discussion, orchestrated by the teacher (Bartolini Bussi, 1996), of some individual texts selected by the teacher, which allows the teacher to play a direct or indirect (i. e. based on students' productions) mediating role; the collective production, under the guide of the teacher, of a provisional synthesis, which may open a new cycle of activity. Such phases allow an equilibrium between the students' constructive involvement in the activity, and the mediating role of the teacher. While practicing FoE didactics, the fundamental didactic cycle is sometimes integrated with other activities according to the specificity of the subject to be dealt with: a preliminary classroom discussion may prepare the initial individual task; small group discussions on individual productions may precede the classroom discussion mediated by the teacher.

In the perspective of the FoE didactics, mathematics is a "culture" (Hatano & Wertsch, 1991), consisting of activities, artefacts, transmissible practices: an alternative to mathematics as a tool of knowledge, or as an ontologically established domain of knowledge, or as a cultural construction reflecting minds' structures.

Non-mathematical FoEs are cultural domains with their own criteria of validity and problem solving strategies; at the beginning of the ninetieths we engaged in studies aimed at ascertaining their potential for mathematics education as sources of mathematical concepts and ways of reasoning (rooted in their typical practices). The first one was a longitudinal study in primary school, reported in Boero (1990), concerning, in particular, the emergence of conditional reasoning in written texts in non-mathematical and mathematical FoE contexts. The implications of those studies for solving important mathematics education problems as well as their research developments are exemplified in the next sub-section.

A research contribution in the context of the FoEs didactics: The Cognitive Unity of Theorems

As a first example of the research generative power of the FoE construct, I report here what happened in our group during a study aimed at identifying the mechanisms of production and argumentative validation of hypotheses in the FoE didactics. The analysis of the relationships between those processes in space problem situations in the FoE of Sun shadows suggested the idea of a Cognitive Unity of theorems: the continuity that for some theorems may be established by students between arguments used for generation and plausibility of the conjecture, and arguments used for proving. Such continuity may work as a facilitator of proving, thus leading students to a smooth access to mathematical proof (Boero, Garuti, Lemut & Mariotti, 1996) - a very demanding aim for teaching in secondary school. Our perspective was in contrast with current ideas in those years about the unavoidable cognitive and epistemological gap between argumentation and mathematical proof (cf Duval, 1991).

At the beginning, the idea of Cognitive Unity arose when analyzing VIII-grade students' written productions during their work related to the following task:

> In the past years we observed that the shadows of two vertical sticks on the horizontal ground are always parallel. What can be said of the parallelism of shadows in the case of a vertical stick and an oblique stick? Can shadows be parallel? At times? When? Always? Never? Formulate your conjecture as a general statement.

More than one half of the students were able to produce a "conjecture" and, according to the didactical contract, some reasons for its plausibility. After the comparison and discussion of the conjectures, two shared statements, corresponding to two ways of reasoning in students' productions, were formulated under the guide of the teacher:

> If sun rays belong to the vertical plane of the oblique stick, shadows are parallel. Shadows are parallel only if sun rays belong to the vertical plane of the oblique stick.

> If the oblique stick is on a vertical plane containing sun rays, shadows are parallel. Shadows are parallel only if the oblique stick is on a vertical plane containing sun rays.

Then the individual task: "What do you think about the possibility of testing our hypotheses by experiment?" and the 2-hours discussion of the students' answers brought students to realize that "an experimental testing is very difficult, because one should check what happens in all the infinite positions of the sun and in all the infinite positions of the sticks". These activities prepared students to engage in the individual production of a general justification of the statements.

By comparing the production of the "conjectures" with the production of the "proofs", we found several couples of texts like those reported below, in which the arguments developed to find the "conjecture" and justify its plausibility were re-invested in the production of the "proof".

Formulation of the conjecture with the shifting of the stick (Beatrice):

> I tried to put one stick straight and the other in many positions (right, left, back, front) and with a ruler I tried to create the parallel rays. I sketched the shadows on a sheet of paper and I saw that: if the stick moves right or left shadows are not parallel; <u>if the stick is moved forward and back shadows are parallel.</u> Shifting the stick along the vertical plane, forward and back, the two sticks are always on the same direction, that is to say

> they meet the rays in the same way, therefore shadows are parallel. Whereas shifting the stick right and left the two sticks are not on the same direction anymore and therefore do not meet the Sun rays in the same way and shadows in this case are not parallel. Shadows are parallel if the oblique stick is moved forward and back in the direction of sunrays.

Proof:

> Shadows are parallel because, as we already said, Sun rays belong to the vertical plane of the oblique stick. But all this does not explain to us why this is true. First of all, though the sticks stand one in an oblique and the other in a vertical position, they are aligned in the same way and if the oblique stick is moved along its vertical plane and is left in the point in which it becomes vertical itself we see that they are parallel and, as a consequence, their shadows must naturally be also parallel, and also parallel with the shadow of the oblique stick, which has the same direction of that produced by the imaginary, vertical stick.

In this case the justification produced during the conjecturing phase *("*meet the Sun rays in the same way*")* is reworked in the subsequent proof, where Beatrice imagines the oblique stick moving along the vertical plane containing Sun rays.

Formulation of the conjecture with the movement of the Sun (Sara):

> They could be parallel if I imagine being the Sun that sees and I must place myself in the position so as to see two parallel sticks. In this way the sun sends its parallel rays to enlighten the sticks. But if the Sun changes its position it will not see the parallel sticks and, therefore, their shadows will not be parallel either. Shadows can be parallel if the oblique stick is on the same vertical plane as the Sun rays.

Proof:

> If the Sun sees the straight stick and the oblique stick parallel it is as if there were another vertical stick at the base of the oblique stick. If this stick is in front of the oblique stick its shadow covers the shadow of the oblique stick. These shadows are on the same line, therefore, the oblique and vertical sticks shadows are parallel.

In this case the initial idea "I imagine being the Sun" seems to suggest the main argument of the proof (the shadow of the imaginary, vertical stick covers the shadow of the oblique stick).

The construct of Cognitive Unity of theorems (further elaborated through other teaching experiments - see Garuti, Boero & Lemut, 1998) contributed to clarify the nature of the conjecturing and proving processes, to put into evidence the necessary tension between the direction of the proving process and the features of the product to be achieved, and to open the way to further studies. In particular Pedemonte (2007, 2008) introduced the distinction between the continuity that in the case of the Cognitive Unity of theorems exists between the content of the arguments used in the conjecturing and in the proof construction phases, and the structural break that may happen (particularly in the case of synthetic geometry theorems) between the abductive or inductive nature of argumentation in the conjecturing and early proving phases, and the deductive nature of the proof to be achieved.

The teaching experiment reported in Boero & al. (1996) not only suggested the possibility of the Cognitive Unity of theorems as a facilitator of the proving process, but offered also an example of how in non-mathematical FoEs students can develop ways of reasoning, which are consistent with those that are fundamental in mathematics. Indeed we can see how in the case of the "two sticks problem" the FoE of Sun shadows offers a physical counterpart for a space geometry problem situation of conjecturing and proving: if we replace sticks, parallel Sun rays, Sun shadows with segments, parallel straight lines, parallel projections on a plane surface we get a conjecture and a semantically-based proof in space geometry. In this case we can say that the logical-linguistic organization of mathematical proof develops in a non-mathematical context suitable for an immediate transfer to a mathematical setting. FoE didactics offers several opportunities for such kind of transfer, significant for solving important *educational problems arising within mathematics education*, like that just considered, of a smooth access to mathematical proof. Some evidence will be provided also through the first examples of the next section, which at the same time aims at introducing the discourse concerning further theoretical developments related to *more general educational problems*.

FROM STUDIES IN THE EDUCATIONAL CONTEXT OF FoE DIDACTICS TO THE NEED OF FURTHER THEORETICAL DEVELOPMENTS

The following episodes and excerpts are derived from past or ongoing studies performed in the educational context of the FoE didactics since the end of the ninetieths. This section is aimed at showing the necessity of re-interpreting the Vygotskian everyday concepts - scientific concepts dialectics, in order to deal (in the *specific* field of mathematics education) with *general*, important and complex educational problems such as:

> How to cope with the necessity that students acquire cultural tools deriving from the present dominating scientific culture, at the same time avoiding students' alienation from their cultural roots?
>
> How to exploit the potential richness of students' personal contributions and social cultural background, in order to develop a culture related to personal dispositions and societal needs?

Examples in the FoEs of Money and purchases, and Calendar

The first two examples refer to grade 1 and grade 3 students engaged in reflective activities on the writing of numbers in the decimal - position system. In the first case the teacher helps the students to deal with some semiotic aspects of the FoE of arithmetic. In the second case she wants to develop students' knowledge and awareness of the decimal-position system.

The first example was collected at the beginning of December, in grade 1: after two months of work in the FoEs of "Money and purchases" and "Calendar", the teacher exploited one child's mistake (Anna had written *tredici* - thirteen - as 31) to ask students: "Explain Anna why her writing is a mistake”. Four types of individual productions (oral sentences dictated to the teacher) were collected:

It is a mistake because…

...three and one are exchanged;

...in a month, the day thirteen comes much before the day thirty one;

...with thirteen cents I can buy less than with thirty one cents;

...thirteen cannot be written as we write thirty one, otherwise we could not understand which number it is.

The second example was collected in another class, at the beginning of grade 3.

Mario wrote *centodieci* (one hundred and ten) as 1010. The teacher asked the students to produce an individual text: "Explain to Mario why his writing is a mistake". Five types of individual productions were collected.

It is a mistake because there are four digits instead of three, you should delete the second digit;

1010 means that the number is one thousand and ten, it cannot mean another number at the same time;

1010 means a big quantity of money, you can buy a bike, while with 110 you can buy only a roller;

Mario, you have made a mistake because you have thought: one hundred and one, 101, one hundred and two, 102,..., one hundred and 9, 109, then one hundred and ten, 1010; you had to move 1 to the left, in the place of tenths;

1010 means one thousand, zero hundreds, one tenth, zero units, differently from centodieci, while 110 is composed by one hundred, one tenth and zero units, exactly like centodieci.

According to the didactical contract (see below), both tasks engaged students in explanations. In both cases, the explanations given by the students are of different kinds: in particular some of them refer to the shape of signs (morphologic aspect: like in the first texts of both cases), others to pragmatic reasons (the third texts in both cases), others to the meaning of digits according to the decimal-position system of written representation of numbers (like in the last text of the second case). Different epistemic criteria and related ways of thinking emerge as indicators of different potential directions for the development of conceptualisation towards consciousness and explicitness.

Where these productions come from? The didactical contract plays a crucial role: in the FoE didactics, since the very beginning of Grade 1 children are used to engage in supporting positions (or explaining the why of mistakes) of their schoolmates, and to move from the individual effort to the discovery of different positions and search for consensus, or the identification of unbridgeable differences. The fact that in Italy usually the same teacher teaches the same group of students over a period of five years (in the case of primary school, grades 1-5) or of three years (in the case of lower secondary school, grades 6-8) amplifies the effects of the didactical contract on students' intellectual maturation.

What to do with this kind of productions? In the perspective of a straightforward approach to structural knowledge of the decimal-position system of writing numbers, the teacher might drive the attention on those contributions that are oriented towards it. But pragmatic arguments (and even morphological arguments!) are important to enter a reflective, conscious, intentional attitude towards knowledge; thus, after comparison and discussion of the different arguments (i.e. an exposure to the others' reasoning) the conclusion should be that different reasons (conveniently reported) can be advocated to explain the mistake. While valuing different personal contributions, this might represent a first step in the direction of becoming aware that different ways of reasoning can offer different (possibly, complementary) keys to ascertain the truth (or fallacy) of a statement.

Example in the field of experience of Sun shadows

In a study reported in Boero (2002), two sets of grade 3 students' written productions concerning Sun shadows are compared and analysed.

After observations and games with Sun shadows in the courtyard in some subsequent days, students write a report about what they have discovered. The percentage of those who juxtapose "low Sun" - "long shadows" (with sentences such as "In the early morning, the Sun is low and its shadows are long") is higher than 70%. Then the teacher gradually introduces, on the basis of students' drawings and observations, the elementary geometric model of Sun shadows and its use in some problem solving activities: e.g. to establish whether the lengths of the shadows of two equal vertical sticks in two near courtyards are equal or not at the same moment.

After those activities the wording of the phenomenon dramatically changes: more than 70% produce a causal or conditional description, with sentences such as "In the early morning the shadows are long because the Sun is low", but also: "If the Sun is low, the shadows are long".

This example shows how suitable activities led by the teacher in a given FoE, resulting, in this case, in the appropriation by students of a new sign, may allow students to move to a higher level of understanding of the same phenomenon; and it shows also how the choice of suitable non-mathematical fields of experience may allow students to access high level mental activities in a "natural" way.

Anyway, in a grade 7 classroom in Eritrea some completely different productions were collected:

> The shadows are long in the early morning because the strength of the Sun is not yet sufficient to win the darkness, while the shadows will become shorter and shorter till midday because the light has gradually taken strength, and then again darkness will prevail, and it will happen everyday.

This conception allowed students to solve several problems (e.g. the problem of the shadows of the sticks in two near courtyards) in a straightforward way; afterwards they used the geometrical model introduced by the mathematics teacher to accomplish their duty as "mathematics students".

Here we see how FoE didactics may put into evidence significant aspects of students' background culture~~.~~ Taking background culture into account and developing the classroom debate on it could help students becoming aware of the differences with the culture brought by the teacher and, finally, appreciating the richness of cultural diversity. Indeed we may observe that for Eritrean students the phenomenon was framed within a broader system of knowledge, based on a cyclic, dynamic equilibrium: something that traditional farmers currently use; that was already conceived in pre-socratic Greek philosophy, particularly in Heraclitus' notion of dynamic dualism (Graham, 2008); that we find in the history of Chinese culture (again as a dynamic version of dualism, see Cheng, 1997); that Western mathematics succeeded to model in the second decade of the XX Century in the case of Lotka-Volterra predator-prey differential model. At that time the idea of a cyclic, dynamic equilibrium gained full scientific legitimacy in western culture by relying on the role of mathematics. This point reminds me of my experience as a mathematics teacher in university courses for natural sciences students. Usually those students experienced many difficulties in entering the cyclic dynamic equilibrium perspective, before my introduction of the Lotka-Volterra model and the related graphical representations. The cyclic dynamic equilibrium perspective seemed completely alien for them.

Example in the FoE of arithmetic in primary school

In Boero, Douek & Garuti (2003) the following excerpt is reported; it comes from a 5 grade classroom dealing (within the FoE of arithmetic) with the problem of "how many numbers exist between 1 and 2"; students are discussing the hypothesis of "infinite numbers" proposed by a schoolmate:

Valentina — What does it mean to say that infinite numbers exist, if we cannot count them because we must die?

Stefano — I do agree, man is not everlasting but life is everlasting.

Valeria — The woman's body ends, but she creates another woman, and so life goes on to infinity.

Emanuele — Numbers create other numbers, to infinity, by multiplying. Each number is finite, but an infinite list is produced.

Valeria's metaphor, conceived outside mathematics, works as a grounding metaphor (Nunez, 2000) for Emanuele, who exploits it to produce a strong argument in favour of the infinity of numbers. This is an interesting example of contamination between different domains. In the educational context of the FoE didactics such kinds of phenomena frequently occur in our classes. Still as concerns contamination between mathematical and non mathematical domains, we may consider grounding metaphors derived from concrete physical situations (like the balance) to support the solution of quadratic inequalities in grade 8 (see Boero, Bazzini & Garuti, 2001).

Example in the FoE of arithmetic in lower secondary school

As an example of different, potential evolutions of spontaneous students' productions, which could be driven by the teacher towards different scientific outcomes, we may consider the case of the "Think a number" game in Grade 7 (see Morselli & Boero, 2011). Students try to understand why the teacher is able to discover the result of a sequence of operations performed by students on a number chosen by them; the sequence is aimed at introducing algebra as thinking tool. Under the guide of the teacher, students move from verbal productions describing the game, towards the representation of the sequence of calculations performed on a given number and then to the representation of the sequence of calculation on a letter, that is to say “any number”. In some classes two kinds of mathematical representations emerged: a procedural one, consisting of a more or less complete sequence of instructions like $N=N+3$; and a relational one, consisting of one more or less correct algebraic expression. Starting from their own productions, students may arrive, under the guide of the teacher, to two different and correct models of the game, which exemplify two different ways (procedural and relational) of representing a sequence of calculations in general. Further activities could bring to an early approach to two different mathematical domains (computational mathematics and algebra).

REVISITING THE EVERYDAY CONCEPTS / SCIENTIFIC CONCEPTS DIALECTICS

Several studies in the perspective of the Field of Experience didactics brought us, through episodes and reflections such as those reported in the previous section, to reconsider the everyday concepts /scientific concepts dialectics hypothesized by Vygotsky as the key component of the enculturation process. With reference to Vergnaud' definition of concepts (Vergnaud, 1990), Nadia Douek's in-depth work on conceptualization (Douek, 1989; 2003; 2006) suggested that:

- "scientific" is not an ontological quality of concepts; it consists of specific (intentional, conscious, systemic, explicit) relationships that the subject develops with concepts. In the classroom those relationships depend on the cultural and didactical choices of the teacher, as we put into evidence in the discussion of previous cases. We may add that scientific relationships to a concept frequently evolve in the long term towards an everyday use of it;

- everyday concepts may evolve towards different, even conflicting "scientific" horizons under the influence of different agents (the teacher, but also the cultural environment) - see in the previous case of Sun shadows: the evolution of students' ways of thinking the phenomenon after the introduction of the model by the teacher; and an alternative, non geometric consistent conception, rooted in students' culture;

• the evolution of everyday concepts in a mathematical FoE may depend on non-mathematical arguments elaborated in different FoEs as grounding metaphors (Nunez, 2000) for their systemic links and inferences - see the example on the infinity of numbers;

• moreover, "pragmatic" (but still intentional, conscious and explicit) relationships to a concept (cf Verillon, 2011; Verillon & Rabardel, 1995) pose the research problem of the relationships between two possible kinds of evolution of conceptualization: the pragmatic one and the scientific one - see the cases of the writing of numbers.

All the above points concern the relationships between the potential inherent in students' dispositions and their cultural background, on the one hand, and the educational aims to be achieved in school, on the other; thus they seem to be relevant in order to deal *in mathematics education* with the enculturation problems evoked at the beginning of the third section. But even our enlarged interpretation of Vygotsky's everyday concepts - scientific concepts dialectics seems insufficient to analyze, and take decisions in those situations where students' cultural diversity and their personal contributions pose problems like: how to identify and characterize students' resources, in the perspective of possible, different evolutions of their everyday concepts? How to frame the educational choices and plan the classroom activities aimed at promoting: the evolution of students' everyday concepts towards scientific concepts; and their awareness of the potential and limitations inherent in different cultural perspectives and tools, inside and outside mathematics?

A COMPREHENSIVE FRAMEWORK FOR DIVERSITY OF "SCIENTIFIC" CONCEPTS AND WAYS OF ACCESSING THEM

Cultural contaminations and different, possible "scientific" horizons for the evolution of students' everyday concepts towards scientific concepts required a new toolkit in order to deal with such a diversity in an educational perspective. The key idea to get it derived from the observation that diversity concerns strategies to achieve the aim of the activity, ways of communicating with other people, and criteria of validation of statements, as three inter-related components of the same process. For instance, in the case of the writing of numbers in the decimal-position system some students refer to an extra-mathematical experience and to social constraints in order to invalidate Anna's and Mario's written representations, while other students' reasoning remains within arithmetic and refers to structural properties of the writing system. In the case of the debate on the infinity of numbers it is remarkable the shift from ordinary language to natural language in the mathematical register (cf Halliday, as quoted in Boero, Douek & Ferrari, 2008, p. 265), and from the everyday experience with its well known true facts to arithmetic properties and related inferences.

These reflections brought us to consider Habermas' elaboration on rational behaviour as a possible analytical tool to identify and compare different ways of solving problems, validating statements and communicating in mathematical FoEs, in mathematical modelling and when dealing with non-mathematical subjects. Indeed, Habermas (1999; 2001, Ch. 2, pp. 100-107) proposed a definition of rational behavior in discursive practices based on three inter-related criteria:

• ***epistemic rationality***: it concerns accounting for validation of statements according to shared principles and rules of inference ("shared" in a given cultural context);

• ***teleological rationality***: it concerns accounting for the choice of tools and strategies to achieve the aims of the activity;

• ***communicational rationality***: it concerns the intentional choice of means to communicate with others in a given social context.

Initially, we adapted Habermas' idea of rational behaviour in order to encompass the complexity of mathematical activities, which develop effectively when the adopted strategies keep into account the epistemic constraints of the product to be achieved (e.g.: the case of *proving* and *proof*). Then we extended its use to plan and analyze classroom activities aimed at promoting students' rational behaviours in mathematics and in mathematical modelling. At present, we use (or foresee the use of) Habermas' construct in order to deal with different speculative and operational aims, in mathematics education and in general education.

As regards mathematics education the construct can be used:

• to characterize and compare different kinds of rational behaviours within mathematical activities (e.g. rational behaviours in analytic geometry and in Euclidean geometry are very different - cf Boero, Guala & Morselli, 2013, in press);

• to put into evidence the different levels and kinds of awareness needed to behave rationally in mathematics, which concern the three components of rational behaviour; for instance, on the epistemic side of proving, awareness concerns particularly the rules of inference and the nature of definitions, axioms and theorems in a theory, and also the role of examples;

• to promote and analyze the evolution of students' behaviour towards the intended mathematical behaviour, according to the components of rational behaviour;

• to compare the student's actual behaviour with the expected one; and to realize how, in some cases, the student behaves in a rational way, but according to a rationality that is different from the expected one.

As regards general education (thus, with a different level of zooming) the construct can be used:

• to characterize and compare rational models of behaviour of different disciplines, or even informal cultural domains (like traditional agriculture), and establish relationships between them;

• to help teachers and students to cross the borders between different domains of knowledge in the perspective of a wide-scope enculturation, thanks to the possibility of identifying common requirements of rationality and differences.

We may note that the problems quoted at the beginning of the third section can be re-formulated in terms of rationalities: how to combine the possibility of keeping the contact with the rationalities rooted in the students' cultural background, with the necessity of accessing rationalities developed within the global system of cultural production (including present mathematicians' mathematics); and how to promote the development of rationalities by exploiting the potential inherent in the diversity of individual dispositions and cultural backgrounds.

The elaboration concerning the adapted construct of rational behaviour was exploited to perform studies involving students at different age levels. The construct was initially used to plan and analyze specific classroom mathematical activities, particularly those intended to approach conjecturing and proving with/without the use of algebraic language; afterwards, it was used also for planning and promoting a reflection on different rationalities within mathematics.

As regards the first line of research, concerning rationality in conjecturing and proving, I mention the following studies:

- in Morselli & Boero (2011), the authors adapt Habermas' construct of rational behaviour to deal with the case of the use of algebraic language in conjecturing and proving, and describe teaching experiments planned, performed and analyzed according to that construct. In particular, epistemic constraints concern two different aspects of those mathematical activities: checking for correctness of algebraic manipulations according to syntactic rules of algebraic language; and checking for validity of algebraic formalization and interpretations (i.e. for the passage from a non-algebraic situation to its linguistic representation through the algebraic language; and for the interpretation of the algebraic expressions deriving from suitable transformations);

- in Boero, Douek, Morselli & Pedemonte (2010), the authors show how rational behaviour in conjecturing and proving may work as an useful theoretical construct to design, manage and analyze suitable classroom activities aimed at the students' approach to the culture of theorems (i.e. to the knowledge of some crucial features of a theorems as a statement and its proof within a theory - see Mariotti, Bartolini Bussi, Boero, Ferri & Garuti, 1997).

As regards different rationalities within mathematics and their relevance in mathematics education, Boero & al (2013, in press) considered the following problem, belonging to a selective test for candidates (having a MD in Mathematics) to become high school mathematics teachers:

> To characterize analytically the set ***P*** of (non degenerated) parabolas with symmetry axis parallel to the ordinate axis, and tangent to the straight line $y=x+1$ in the point (1,2).

> To establish for which points of the plane does it exist one and only one parabola belonging to the set ***P.***
>
> To find straight lines that are parallel to the ordinate axis and are not symmetry axes of parabolas belonging to the set ***P.***

Candidates met big difficulties in solving the problem; most of them were not able to choose and exploit well known tools from synthetic geometry, analytic geometry, algebra, which would have allowed them to get straightforward answers to the three questions or to identify their mistakes in an easy way. Even discussing "how to solve it" in a-posteriori interviews was difficult for them. In Boero & al (2013, in press) difficulties are interpreted in terms of different rationalities inherent in those mathematical domains, which contribute to a rigid, unilateral approach to the problem, and of lack of awareness about the potential and limitations (on the teleological and epistemic sides) of the tools at disposal in the different domains.

In general, in the already performed studies the development of students' awareness about what they would like to do, or might do, or should do seems to be an educational aim difficult to achieve; but awareness (so important in both Habermas' elaboration on the idea of rationality, and Vygotsky's characterization of "scientific concepts") is a crucial requirement for a mature relationship with the scientific enculturation promoted by the teacher.

The early development of awareness is a key educational aim in an ongoing teaching experiment, which concerns another use of the construct of rationality. It consists in the approach to, and promotion of, different kinds of rational behaviour to deal with the same extra-mathematical subject. In the case of the FoE of Sun shadows we are developing, in two classes (grades 3 and 4), a multi-disciplinary approach (involving mathematics, physics, natural sciences, visual arts, literature) to the study and representation of the phenomenon. Students experience different ways of validating statements, of posing and solving problems, of communicating; they reflect on the differences under the guide of their teachers. This experiment is intended to explore the possibility of making students aware of what distinguishes a discipline from the others in terms of rationality, just at the beginning, in Italy, of the systematic study of different disciplines.

Concerning the relationships between different cultural domains, we are also engaged in the analysis of analogies and differences between the rationalities inherent in different FoEs, in order to search for new, more "natural" connections for students between mathematics and other domains of knowledge. Two FoEs emerge as candidates from already performed activities in primary school. The first is the FoE of Grammar rules; students' activities consist of reflective work on texts, aimed at searching for regularities, rules and exceptions. The second is the FoE of Rules concerning students' behaviours in the school; rules are motivated, negotiated and written in the classroom under the guide of the teacher. Inevitably the consideration of these fields of experience recalls one of the possible "externalist" interpretations of the raising of mathematical rationality in Greece, around the V-th Century B.C., as rooted in the philosophical debate on truth and language (Sophists) and the formulation and interpretation of laws (see Szabo, 1978). In early grades, when argumentative activities in the fields of arithmetic and geometry are limited to easy questions, the domain of rules seems particularly promising, because it allows to develop rather complex argumentation, as it is shown in the following episode.

After an incident, grade 2 students are debating about a rule that concerns how the class should go from the second floor to the ground floor at the end of the school time. Barbara had proposed the following text:

> Children must be in couples, from the exit of the classroom to the exit of the school; on the staircase each child keeps a schoolmate with his hand.

Danilo reacts in this way:

> I do not agree with Barbara's rule for two reasons: first, if one child falls down also his schoolmate falls down; second, which schoolmate? Because 'a schoolmate' means any schoolmate, not 'the' schoolmate of the couple!

While Danilo's first criticism comes from figuring out possible dangerous consequences of the rule given by Barbara, the second one refers to a grammatical rule with a high sensitivity to the logical features of natural language. Both criticisms by Danilo offer evidence for the emergence of skills that are important for the development of mathematical argumentation: to find the scope and consequences of a given rule; and to identify the logical meaning of articles and conjunctions in a text.

The educational context of the FoE didactics allows to perform such kind of studies under appropriate conditions, due to its specific didactical contract and the fact that teaching involves different FoEs and disciplines.

According to our intentions, these studies should prepare us to deal in the future with two much more demanding and complex problems, strictly related to the big educational problems quoted at the beginning of the third section: how to deal with different rationalities brought in the classroom by students belonging to different cultures; how to develop mathematical rationalities, in pure mathematics as well as in the application of mathematics to other subjects, taking into account different cultural environments.

Evoking such past, present and (possibly) future uses of the construct of rational behaviour outlines a broad research perspective, to be necessarily shared by experts of different disciplines. In such a perspective the richness of everyday concepts carried by learners, with their different sources (ethnic, familial, personal), should be exploited to develop knowledge and awareness of the variety and potential of human cultures, and to prevent breaking the roots with students' native cultures. Today this educational aim looks like one of the necessary conditions for exercising the informed freedom of choice, advocated by Habermas in his introduction to the elaboration on rationality, and a culturally-based tolerance as well.

THE ROLE AND THE COMPETENCES OF THE TEACHER

Dealing with the everyday concepts/scientific concepts dialectics in the perspective of different rationalities needs a "competent" mathematics teacher (as an interpreter, promoter and mediator of rationalities) with additional competencies, if compared with those usually considered in teacher education literature.

Shulman's Pedagogical Content Knowledge, as well as Mathematical Knowledge for Teaching (Ball & Bass, 2003), are defined in terms of knowledge the teachers should be provided with in teacher education programs. In our perspective of multiple rationalities within mathematics and in comparison with other cultural domains, the teacher must have the competence of Cultural *(=epistemological, anthropological, historical*) Analysis of the Content to be taught (CAC: see Boero & Guala, 2008). Such a competence should enable the teacher to recognize the cultural potential inherent in students' productions and make short term and long term conscious choices concerning: in which direction, and in which way, to drive students' attention; when and how to mediate pieces of established mathematical knowledge, criteria for validating statements, strategies, ways of communicating knowledge, etc..

In our elaboration CAC is a competence to be developed by practicing it through suitable tasks in a suitable mathematics teacher education context, and not a set of pieces of knowledge. In the CAC perspective, the teacher must be able to bring the cultural dimension of mathematics into the classroom. Under the guide of the competent teacher, who has experienced such kind of reflections and practices at an adult level, students may become aware of the potential and limitations of mathematical tools to deal with "real" problems, of the different rationalities inherent in mathematical activities, and of their connections/ conflicts with other rationalities.

Concerning the issue of the different rationalities inherent in mathematical activities, we are working on the design of tasks for teacher education, suitable for promoting prospective teachers' awareness about the deep differences, in terms of rationality, between activities in different mathematical domains (see Boero & al., 2013, in press). The already quoted task concerning the set of parabolas that are tangent in $(1,2)$ to the straight line $y=x+1$ is exploited by us in teacher education; it seems suitable to drive pre-service teachers' attention towards different rationalities and the necessity of avoiding to teach mathematics as a set of closed domains.

WHAT ABOUT MATHEMATICAL LITERACY?

By employing the theoretical toolkit developed during our research trajectory, what can we say now about the PISA definition of mathematical literacy?

First of all, that definition does not take in charge what may be derived from the cultural context the teachers and students belong to. Potential and obstacles inherent in the teacher's and students' inner contexts and in the external context are not explicitly considered. Indeed, culture may be a source of mathematical experiences for students (D'Ambrosio, 1999), and a source of problem situations that can be considered for a treatment with mathematical tools in the classroom (Brenner, 1998).

Second, the differences between the rationalities in different mathematical domains are not considered, as concerns both their inner epistemic and teleological characters, and their descriptive, interpretative and predictive potential and limitations.

Third, the definition conveys the idea of mathematics as a universal, privileged toolkit "to make the well-founded judgments and decisions needed".

Consequences of such a perspective on teachers and students (future citizens) may be: cutting the links with people's cultural background (a well known premise to alienation); and not supporting, or even preventing, the evolution of people's cultural background, which may result in a loss of potential richness in a historical moment in which some important aspects of dominating western culture are put into question. Furthermore, this definition may convey an image of mathematics as an absolute, homogeneous body of knowledge and tools to solve every kind of problem; this image, in turn, may be an obstacle in the choice of the appropriate tools to deal with extra-mathematical problems.

Keeping the above critical considerations and the adapted Habermas' construct into account, I have tried to write down a provisional, alternative definition of Mathematical Literacy (trying to keep it near to the length and the scope of the PISA definition, in order to make comparisons easier). The resulting, "draft" definition is:

> ML consists in the capacity of consciously moving from the subject's perception of a problem, rooted in his/her socio-cultural background and experience, to possible mathematical treatments; and at the same time it consists in the awareness of the epistemic and teleological constraints and limitations inherent in the different mathematical rationalities and of the consequences that they impose on the solving process and the related solutions, in comparison with non-mathematical rationalities.

The proposed draft definition is different from the PISA definition as concerns the conception of mathematics, the students' and adults' relationships with mathematical knowledge, the use of mathematical knowledge to deal with questions related to societal needs, and the attempt to keep into account the issues evoked at the beginning of the third section.

In our tentative definition the cultural, mathematical and non mathematical, background is brought to the fore and related to possible mathematical horizons; moreover different mathematical treatments might be compared, integrated, or even rejected. In particular our definition leaves the possibility, in the case of non-mathematical problems, of identifying non-mathematical ways of reasoning as more effective on the teleological side and even more secure on the epistemic side, in comparison with a standard mathematical treatment with mathematical tools. Concerning this issue, it is interesting to observe how today, and differently from fifty years ago, the degree of mathematization is no more considered as a measure of the quality of an investigation in some domains - not only biology, but also economics (a subject of lively debate today!) and ecology.

CONCLUSION AND FURTHER DIRECTIONS OF RESEARCH

Keeping the research trajectory described in this paper into account, I will try to offer some elements to answer the question posed at the beginning of this paper: "How to deal, as researchers in mathematics education, with big, complex problems related to societal needs?" I will also outline some further directions of research concerning questions that are emerging in our research activities.

It is evident that our research trajectory cannot be replicated: it depended non only on specific historical circumstances (e.g., at the very beginning, the failure of the New Mathematics reform and even the post-'68 climate) but also on the personalities of the protagonists, including researchers in history, economics, linguistics, psychology who plaid an important role in the initial phase of our work. However I think that from our experience some indications can be derived, in order to create the conditions that may allow to tackle "big and complex problems" in an effective way.

First of all, I think that it is important to create a research team (including school teachers as researchers) to plan and perform long term experimental activities in order to meet societal needs of the time, related to the big problems that one wants to tackle. The elaboration of a consistent, wide scope theoretical framework, with the collaboration of experts from different disciplines, is necessary to plan and analyse the experimental activities. Indeed, mathematics is not an isolated fragment of contemporary culture, but an important component of the historical development of cultures, strictly related with practical and speculative issues. As a consequence, if we want to tackle in an effective way big and complex problems in mathematics education (like those evoked at the beginning of section 3) it is very useful to experience on the field, and analyse with suitable theoretical lenses, the complexity of the school teaching of mathematics, of its relationships with other subjects and extra-school cultures, and of the cultural role of the mathematics teacher in the classroom. As concerns the role of the teachers in the research team (cf Malara & Zan, 2008), their research commitment offers the opportunity of a "research eye" on what happens in the classrooms, on students engaged in the planned activities, on their cultures, on the difficulties that they meet to achieve the intended educational aims and on their potential for intellectual development.

In such an educational and research environment, "local" studies should be performed, according to appropriate and well established methodologies, in order to answer specific research questions. Such questions may come from the long term experimental activities or from current research on related issues, in mathematics education or in other fields. In turn, such local studies might put into evidence the need (or the opportunity) for further developments of the general theoretical framework, provide the research team with results useful to tackle the big problems, and also offer opportunities to improve the effectiveness of the experimental activities for students.

As concerns our work in such a perspective, further investigation is needed in order to progressively develop a suitable theoretical toolkit to deal with the problems that are emerging now in our research trajectory. Some directions are outlined in the subsequent part of this section. What follows is partly influenced by the important work developed in the European community of mathematics educators, and particularly in the CERME context, under the name of "networking of theories". Indeed, our adaptation of Habermas' construct of rationality to the case of mathematics education does not exclude the use of other frameworks aimed at dealing with the complexity of mathematics education phenomena from different viewpoints; on the contrary, we think that complementary constructs are needed, and that some work in that direction should be made. The articulation between the FoE perspective and the construct of rational behaviour is per se an example of possible complementarity, and even integration, between different theoretical perspectives: the FoE construct offers a way of conceiving a cultural context in terms of its educational potential, the FoE didactics offers criteria for exploiting that potential. The adapted Habermas' construct works as an analytical tool to describe and compare different rationalities within a given FoE and between different FoEs, and to plan and analyse the transition from one rationality to another.

As an example of further development in a "networking perspective", we may consider the fact that both the original Habermas' construct and our adaptation do not encompass the institutional dimension, so relevant when we consider (for instance) the mechanisms that determine the kind of epistemic rationality that teachers should promote at a given school level. Here we acknowledge the necessity of taking into account the work done within the frame of Chevallard's Anthropological Theory of Didactics (ATD) (see Wozniak, Bosch & Artaud, 2012). A contact point concerns the possibility of analyzing mathematical activities according to semiotic and epistemic criteria (cf Chevallard's model of mathematical praxeologies). However, some components of rational behaviour have a marginal weight in the ATD: in particular, the teleological rationality, as concerns both intentionality and consciousness of problem solving strategies.

Another development concerns the situated cognition perspective, in particular the construct of Legitimate peripheral participation (Lave & Wenger, 1991): some aspects of the practical implementation of the FoE didactics, particularly those related to the development of specific forms of rational behaviour, can be seen as an enculturation process that develops according to a Legitimate peripheral participation model. Furthermore, that model accounts for the specific social contexts, for instance the school setting, in which rational behaviours are passed over to young generation, an issue that is not dealt with in Habermas' elaboration. For these reasons, we feel the situated cognition perspective could enrich our elaboration. On the other hand, the situated cognition perspective lacks a specific discourse concerning the epistemological side of the enculturation process, whilst this aspect would be so important to qualify, in the school setting, the mediation exercised by the teacher and the output of the enculturation process in terms of rationalities.

Another possible development is the integration with H. Simon's construct of bounded rationality (Simon, 1991), which takes into account the complexity of factors that intervene in decision making in economics and in other fields of decisions at risk. In term of Habermas' construct, Simon's construct concerns teleological rationality. In an ongoing research on the use of Habermas' construct in mathematics education, carried out between the Genoa team and the Turin team lead by F. Arzarello, we are engaged in comparing and, possibly, integrating Habermas' and Simon's constructs in order to deal with the approach to game theory in school (the Turin team is performing teaching experiments on it at different school levels).

Particularly due to the challenges deriving from ongoing experiences of teacher education and innovative teaching in different cultural contexts, which some members of our group are involved in (in Africa, Latin America, and multicultural Italian classes as well), further needs and elements for developing our theoretical framework will probably emerge. In particular power relationships and identity issues in the relationships with different cultures will be likely to oblige us to take into account related relevant theoretical elaborations (see Engeström & Sannino, 2010).

All the above examples, even if just sketched here, suggest that a single theoretical perspective is not sufficient to encompass the complexity of mathematics education phenomena and related big problems, and that a modesty attitude seems to be the most appropriate to develop a productive dialogue and, possibly, local integrations and complementarities between different theoretical perspectives.

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TEACHING AND LEARNING GEOMETRY AND BEYOND…

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Geometry and its teaching have always been a problematic and exemplar issue regardless of the period. Torn between utilitarian and idealistic visions, the very nature of geometry has moved within very wide margins from regarding it as sacred to aiming at its disappearance. Regarding mathematics education, researches on geometry have raised the attention of many prominent researchers in the domain such as for instance Freudenthal and Brousseau. What are today the core items and the contributions of researches in the didactics of geometry, a domain in which the current development of specific software has caused quick changes? We address this question in the light of the results of recent researches and also the rich discussions which have been occurring in the CERME Working Group on geometry from its beginning in 1999. We also develop some ideas about the perspective of geometric paradigms and spaces for geometric work (SGW) and show how it allows describe and change the nature of geometric activity in various educational contexts.

First of all, I would like to thank the organizers and the members of the scientific Committee for their invitation to give this plenary on geometry teaching and learning.

During this talk, I present some possible orientations for researches within the field of geometry didactics. The main point is, to me, that we should take advantage to focus on what I call Geometric work to advance and to develop new views on geometry education. And I develop, with some details, this idea and the framework related to it during the presentation. Another point is that we would get some benefits by linking geometry to other maths areas and to technological tools. That explains partially the meaning of Beyond in the title.

WHY TEACH AND LEARN GEOMETRY TODAY?

For a long time mathematics has been synonymous with geometry and questioning the usefulness of mathematics was equivalent to questioning geometry. Today, it is somehow different but we can learn from the past in order to think on the question but keeping in mind how the current situation is specific.

In *An essay on the usefulness of mathematical learning* written in 1701, Arbuthnot, an English physician, tried to persuade the rich people of his time to learn and practice mathematics. He based his argumentation on three points which are always interesting to consider:

> 1. Develop Mind and Reasoning. "Truth is the same thing to the Understanding as Music to the Ear and Beauty to the Eye", he wrote in the flourishing style of his time. This argument is classic and will be used and summarized later with the famous "For the honour of human spirit" of Jacobi (1830) quoted by Dieudonné (1987).

> 2. For their applications in a wide variety of fields. Arbuthnot favoured Trade, Navigation, Art of War...
>
> 3. To learn how to get to the results and not only the results. This means that the path is as important as the result. Arbuthnot praised mathematics and geometry as a method of freeing the mind from superstition.

The third argument keeps its value today in a world with a lot of technologies meaningless for common people and at the same time an increasing strength of superstitions.

Nearer from to-day and into the field of maths education, the “modern maths” revolution and the subsequent counter-reform have questioned geometry through the name of Euclid. “A bas Euclide” was Dieudonné’s provoking motto against traditional geometry based on an amount of triangle properties disconnected from the evolution of contemporary science. In a same way but for different reasons, teachers and researchers involved in the counter-reform rejected Euclid because he did not give any efficient method to apply it in the real world problems. Another marginal and provoking view was Brousseau's idea of considering Euclid as the first didactician. Indeed, Euclid wrote a text organizing knowledge and used, with some adaptations, as a base for textbooks during centuries and up to the beginning of the XXth century.

To view the variety of points of view, eventually conflicting, it is interesting to quote this remark by Fletcher, a well-known math educator, in an ESM special issue on geometry published in 1971

> The cry "Euclid must go!" has gained a certain notoriety in recent years. Our reaction to this in England was merely mild surprise since as far as we were concerned Euclid had already been gone for a long time. (Fletcher, ESM. 3-3, 1971).

This remark shows how the teaching traditions and the relationships with geometry are different among countries which may be geographically very close.

Nowadays, all these questions and conflicting viewpoints coexist and the teaching and learning of geometry have been to be developed in a changing context characterized by the tension between utilitarian and idealist visions on mathematics with an advantage to the utilitarian approaches. At the same time the use and potentialities of Dynamic Geometry Software (DGS) have deeply changed the way of discovering and proving in the domain and created a new relationship with Truth and Proof within maths education.

To progress in the direction of making our knowledge grow on how and what to teach and learn in Geometry, researchers in the domain can use the great amount of texts elaborated for the group on geometry which was existing in the Conferences of ERME since the first Conference. Among the numerous papers presented in the working group on Geometry, we can distinguish some recurrent and relevant points:

1. Development of spatial abilities and geometrical thinking through consecutive educational levels.

2. Geometry education and the ``real world'': geometrisation and applications

3. Instrumentation: artefacts such as, computers and the way they are used

4. Explanation, argumentation and proof in geometry education.

To this four classic topics in the domain we can add some theoretical aspects which in a certain sense are local and specific to the domain: Van Hiele's levels; Duval's registers of semiotic representation; Houdement and Kuzniak's geometrical paradigms.

The need for a common framework related to Geometry education appeared necessary in the working group in order to facilitate exchanges among members and to allow a capitalisation of knowledge in the domain. Due to collaborations initiated during Cerme meetings with colleagues from Cyprus, Spain and Canada or other from Mexico and Chile, it has been made possible to develop a theoretical framework that I will introduce. In our mind, the framework should be dedicated to study the teaching and learning of elementary geometry on the whole educational system that means during compulsory education and also teacher training. It should be neutral in the sense that is can be used to compare the teaching of geometry in different countries and institutions without any a priori on "best" directions. For that it appeared very soon, that it could be interesting to focus on the nature and form of the effective geometric work made by students and teachers in Geometry.

MATHEMATICAL WORK CONSIDERED A CRUCIAL POINT

As it has been underlined above, the notion of geometric work is central in the approach and we start by detailing what is geometric work for us. First we need to precise, more generally, our view, oriented by educational perspectives, on mathematical work.

In the special issue of ESM already quoted, Freudenthal (1971) found it useful to answer to the question "What is mathematics" before presenting his ideas on geometry education. Addressed to teachers and researchers considered as mathematicians, he put the stress on two aspects of the work in the domain: the activity of solving problems and the activity of organizing.

> Of course you know that mathematics is an activity because you are active mathematicians. It is an activity of solving problems, of looking for problems, but it is also an activity of organizing a subject matter. ESM 3-3 – 1971

In a same vein but a step further, the well-known conception of Thurston (1995), Fields medal in 1982, give a shared view on mathematics considered a human activity.

> Mathematics includes integers numbers and geometry plane and solids

> Mathematics is what Mathematicians study
>
> Mathematicians are those human beings who make advance human understanding of mathematics.

At a first glance, the definition looks circular, but it is not. Initiated on numbers and geometry plane and solids, it creates a dynamic between mathematics knowledge and people who make mathematics. Both aspects are important in this work which relates intimately epistemological and cognitive aspects through the image of a mathematician that we can consider a cognitive subject in charge of the "human understanding of mathematics".

Once mathematics is clearly defined as a human activity, it is easy to turn to the idea of mathematical work including and orienting these activities, but it remains to characterise the specificity of such work and for that, Habermas's (1985) consideration on work defined as a rational activity oriented toward an end will be useful.

> By work or rational activity relative to an end, I hear or an instrumental activity, or a rational choice, or else a combination of both.

Boero developed during his conference some aspects of what rationality is for Habermas, and I will not insist on this point but only retain, for our framework related to education, the necessity of thinking mathematical work as a rational human activity oriented toward a better understanding of specific topics.

How can we interpret and use this in Geometry education? Again, Freudenthal in his paper warns again the taste of mathematicians and educators to restrain mathematics work to organizing.

> A great part of mathematical activity today is organizing. We like to offer the results of our mathematical activity in a well organized form where no traces betray the activity by which they were created. This objectivation is a habit of mathematicians from the oldest times. It is a good habit, and it is a bad one. We freeze up the result of our activity into a rigid system, because this is objective, because it is rational, and because it is beautiful,and this we teach.

To avoid the risk of freezing up the results of mathematic work, it will be necessary to introduce the idea that several work context exist. Two of them are classically identified: a context of discovery where new results and solutions of problems are sought; a context of justification where discoveries are proved and presented to a larger community with its proper rules and style of work. We can add a context of use where the results become familiar to the user, are applied to solve problems which are not necessary mathematical. This variety of contexts need to be kept in mind when developing activities within an educational system and it implies various forms and phases of student's work: researching, presenting, practising...

GEOMETRIC WORK AND ITS SPACE

To study specifically the geometric work within the scope of education, we have introduced the idea of a space, named Space for Geometric Work (SGW), organized to ensure the work of people solving geometrical problems. The subject may be an ideal expert (the mathematician) or a student or senior student in mathematics. Problems are no part of the space but they justify it and speed up its construction.

Initially, this idea was suggested by architects' definition of work spaces as places to be built to ensure the best practice of a specific work (Lautier, 1999).

To ground SGW, we will think of it through epistemological and cognitive dimensions which structure the whole work. As we noted before, the former is in charge of the coherence of the mathematics content and the latter refers to the cognitive subject supposed to solve geometric problems.

According to the epistemological dimension, we introduced three characteristic components of the geometrical activity in its purely mathematical dimension. These three interacting components are the following ones:

A real and local space as material support with a set of concrete and tangible objects.

A set of artefacts such as drawing instruments or software.

A theoretical reference frame based on definitions and properties.

These components are not simply juxtaposed, they must be organized with a precise goal depending on the mathematical domain in its epistemological dimension. This justifies the name of epistemological plane given to this level. From the point of view of geometry considered as a mathematical theory, the theoretical frame of reference is crucial, even if for the users it is sometimes implicit or hidden.

From Duval (1995), we have adapted the idea of three cognitive processes involved in geometrical activity and structuring the cognitive level.

A visualization process connected to the representation of space and material support;

A construction process determined by instruments (ruler, compasses, etc.) and geometrical configurations;

A discursive process conveying argumentation and proofs.

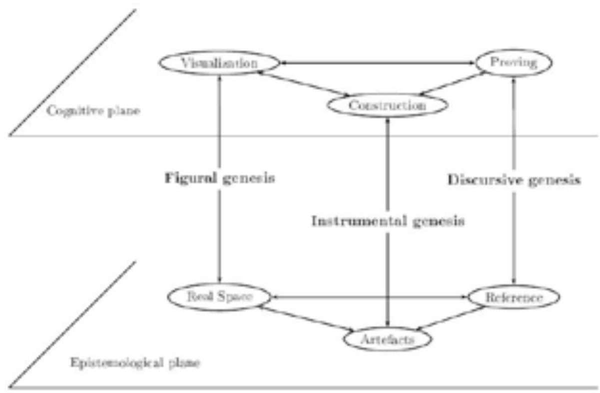
In our approach, both levels, cognitive and epistemological, need to be articulated in order to ensure a coherent and complete geometric work. This process supposes some transformations which can be pinpointed through different ways. It is possible to refer to general notion like intuition, experiment and deduction as Gonseth (1952) did in his major book on geometry. But here, in order to insist on the developmental process involved in the constitution of SGW, the notion of genesis has been used. For us, a genesis involves the development and not only the origin of a process. Strictly related to our ternary conception of each level, three genesis need to be considered:

An instrumental genesis which transforms artefacts in tools within the construction process.

A figural and semiotic genesis which provides the tangible objects their status of operating mathematical objects.

A discursive genesis of proof which gives a meaning to the properties used within mathematical reasoning.

This can be summarized and illustrated by a diagram which will play at the time a metaphoric role and be a prospective tool to think about SGW.



Space for Geometric Work and its geneses

The representation of each level by a plane does not mean that these levels are strictly plane and parallel, and the distance between the poles – the length of the processes necessary to articulate the two levels – depends on and differs from one pole to the other regarding the problem and the tools used. On the other hand, as arrows appear in the diagram, it will be necessary to see what could be the meaning of each way when we want to describe the effective work made when solving a problem.

WHAT GUIDES THE WORK? IN SEARCH OF GEOMETRIC PARADIGMS

We will start to answer the question on what guides the geometric work by giving a first example (Kuzniak and Rauscher, 2011) which among numerous others of the same kind shows that a single viewpoint on geometry would miss the complexity of the geometric work, due to different meanings that depend both on the evolution of mathematics and school institutions.

Let ABC be a triangle with a right angle in B, with AB=4 cm and BC=2 cm. The ray (Ax) is perpendicular to the line (AB). And M is a point on the ray (Ax). The purpose of this problem is to obtain particular configurations of the triangle AMC.

Question: Does a point M exist such that the triangle ACM is equilateral? Justify your answer.

This problem was given to a lot of students at different grades but especially to pre-service teachers. In this case, students have a high general and university level and no problem with reasoning and formulating an answer.

A common answer which appeared was the following:

> The correct answer is "no" and it can be shown, using compasses, that there is no third vertex on the ray (Ax) for an equilateral triangle constructed on the side (AC).

Such a response is emblematic of what we named Geometry I. The student carries out an experiment in the real, perceptible world by constructing a triangle with drawing instruments and then s/he realises that no crossing points lie on the line where they should be for the triangle to be equilateral. The argument is supported by diagrams, objects that are typical and central to Geometry I.

This response, however, does not match what is expected in French traditional education at this level. A solution without any measurement and information supported by the drawing is ruled out. And for a student, it is better to propose this kind of solutions:

> If ACM is an equilateral triangle with M on Ax, the angle MAC will measure 60° and the angle CAB 30° (sum of the three angles of a triangle) and by symmetry ∠CAC' will be 60° (C' is the symmetric of C through the line (AB)).
>
> As the triangle CAC' is isosceles in A (by symmetry), it should be equilateral. This is not true because the length of C'C is 4, which is unequal to CA and C'A (2.sqrt(5) by Pythagoras' theorem).

This solution is illustrative of Geometry II. A reasoned deductive argument is constructed on the basis of initial data and geometric theorems.

From this example, it cannot be induced that deduction does not exist in Geometry I as we can see it with the following solution:

> We can explain this by the fact that in an equilateral triangle all the angles are equals and the sum of the angles is 180°. The value of each is 60° . In this case, when we measure with a protractor, we observe that CAM is more than 60°, indeed CAM = 64◦.

This student deduced some properties belonging necessarily to the figure and then he checked directly on the drawing that the property is not true.

The notion of geometrical paradigm is useful for understanding, clarifying and organising the various and conflicting points of view observed in education. In our framework, we use the notion of paradigm according to Kuhn's definition. In his fundamental book about scientific revolution, Kuhn (1966) uses this term many times and after some approximations, he defines it by putting the stress on two aspects.

> In its most global use, the term paradigm stands for the entire constellation of beliefs, values, techniques, practices etc. shared by the members of a given community.
>
> On the other, it denotes one sort of element in that constellation, the concrete problem-solutions which, employed as models or examples, can replace explicit rules as basis for the solution of the remaining problems of normal science.

The concept of paradigm broadens the notion of theory and relates it to the existence of a community of individuals who share a common theory.

A paradigm is what the members of a scientific community share, and, a scientific community consists of men who share a paradigm (Kuhn, 1966).

Interpreted in the education world, it gives sense to the question about students' and teachers' different work in problem solving. We can argue that they are working in distinct paradigms and this epistemological difference can explain some didactic misunderstandings.

THREE ELEMENTARY GEOMETRIES

Geometrical paradigms were introduced into the field of didactics of geometry to take into account the diversity of points of view (Kuzniak and Houdement, 1999, 2003) and we summarize our findings by quoting former papers and especially (Kuzniak and Rauscher 2011, Kuzniak 2011).

To bring out geometrical paradigms, we used three viewpoints: epistemological, historical and didactical. That led us to consider the three following paradigms described below.

Geometry I: Natural Geometry

Natural Geometry has the real and sensible world as a source of validation. In this Geometry, an assertion is supported using arguments based upon experiment and deduction. Little distinction is made between model and reality and any arguments are allowed to justify an assertion and convince others of its correctness. Assertions are proven by moving back and forth between the model and the real: The most important thing is to develop convincing arguments. Proofs could lean on drawings or observations made with common measurement and drawing tools such as rulers, compasses and protractors. Folding or cutting the drawing to obtain visual proofs is also allowed. The development of this geometry was historically motivated by practical problems.

The perspective of Geometry I is of a technological nature.

Geometry II: Natural Axiomatic Geometry

Geometry II, whose archetype is the classic Euclidean Geometry, is built on a model that approaches reality. Once the axioms are set up, proofs have to be developed within the system of axioms to be valid. The system of axioms could be incomplete and partial: The axiomatic process is a work in progress with modelling as its perspective. In this geometry, objects such as figures exist only by their definition even if this definition is often based on some characteristics of real and existing objects.

Both Geometries are closely linked to real world even if it is in various ways.

Geometry III: Formal Axiomatic Geometry

To these first two approaches, it is necessary to add a third Geometry (Formal Axiomatic Geometry) which is little present in compulsory schooling but which is the implicit reference of teachers' trainers when they have studied mathematics in university, which is very influenced by this formal and logical approach. In Geometry III, the system of axioms itself, disconnected from reality, is central. The system of axioms is complete and unconcerned with any possible applications to the world. It is more concerned with logical problems and tends to complete "intuitive" axioms without any "call in" to perceptive evidence such as convexity or betweenness. Moreover, axioms are organized in families which structure geometrical properties: affine, euclidean, projective, etc.

These three approaches (and this is one original aspect of our viewpoint) are not ranked: Their perspectives are different and so the nature and the handling of problems change from one to another. More than the name, what is important here is the idea of three different approaches of geometry: Geometry I, II and III. It must also be clear that Geometry I is not a poor and small geometry for young students even if it is the first that they encounter. Abstract and complex forms of this geometry exist as it can be found in Lemoine's work on geometrography or Klein's students researches on approximation made by industrial drawing makers and using probabilistic theory to estimate the effects of errors.

Various SGW

A SGW exists only through its users, current or potential. Its constitution depends on the way users combine the two planes and their components for solving geometric problems. It also depends on the cognitive abilities of a particular user, expert or beginner. The make-up of a GWS will vary with the education system (the reference GWS), the school circumstances (the implemented or suitable GWS) and on the practitioners (students' and teachers' personal GWS). In practice, the constitution of a GWS does not rely on a single paradigm, but rather on the interplay among different paradigms and a specific study of each level is necessary. Before giving some examples we will detail these various GWS involved in Geometry education and relate them to different kinds of vigilance: epistemological, didactic and cognitive.

The reference SGW or the expected reorganization This space is normally defined and based on mathematical criteria. But it also depends on social, economical and political criteria. Studies of treatises written by mathematicians or maths educators and of the intended curriculum will allow describe this level in which an epistemological vigilance is at stake. This means that the rules of functioning of this SGW do allow knowledge to be organized in a well-defined and coherent domain.

At this point of the curriculum, the good functioning of the personal SGW is the ultimate goal of geometry teaching and learning and this point needs a cognitive vigilance. Here, the cognitive plane is concerned by a specific individual and not an epistemic or institutional subject, and to know more about its contents, conceptions, knowledge of students have to be studied through problem solving and questionnaires.

Between the two planes, and fundamental in the make-up of a coherent and global geometric work, it remains to focus on the implemented SGW concerned by the didactic vigilance which will assure that the personal student's work corresponds to what the reference SGW proposes. Indeed, when a general paradigm is accepted and the reference SGW built, it remains to teach geometry to students and for that it's necessary to organize a suitable SGW to convey the kind of geometry expected by the educational institution. The geometrical working space turns to be suitable only if it allows the user link and master the three components defining the working space. Curriculum, textbooks, observation of real class implementation and preparation will support the study.

AN EXAMPLE OF COHERENT AND QUASI ASSUMED GEOMETRY I

To show what a suitable and implemented SGW fitted to Geometry I could be, an example will be given, taken from a comparative study of the teaching of geometry in France and Chile (Guzman and Kuzniak 2006).

Following a standard model in Hispanic world, education in Chile is divided into elementary school (Básica) till Grade 8 and secondary school (Media) till Grade 12. From 1998 on, the teaching of mathematics has left aside the very abstract teaching which was in place before and turned into a more concrete and empirical way. And today, the reference SGW is underlined by Geometry I. To illustrate this and point out some differences between France and Chile, let us observe the following exercice taken from the textbook *Marenostrum* (Grade 10).

The problem is given to students starting the chapter on similarity and the solution will be given later in the same chapter:

> Alfonso is just coming from a journey in the precordillera where he saw a field with a quadrilateral shape which interested his family. We want to estimate its area. For that, during his journey, he measured, successively, the four sides of the field and he found approximately : 300 m, 900 m, 610 m, 440 m. Yet, he does not come to find the area.
>
> Working with your classmates, could you help Alfonso to determine the area of the field?

The exercise is then completed by the following hint:

> We can tell you that, when you were working, Alfonso explained its problem to his friend Rayen and she asked him to take another length of the field: a diagonal.
>
> Alfonso has come back with the datum: 630 m.
>
> Has it done right? Could we help him now, though we could not do it before?

The proof suggested in the book begins with a classical decomposition of the figure in triangles based on the indications given by the authors. But the more surprising for a French reader is to come: the authors ask to measure the missing height directly on the drawing . We recall that this way of doing is strictly forbidden at the same level of education in France.

How can we compute the area now?

Well, we determine the scale of the drawing, we measure the indicated height and we obtain the area of each triangle (by multiplying each length of a base by the half of the corresponding height).

In this case, the geometrical work is clearly within Geometry I and goes back and forth between the real world and a drawing which is a schema of the reality. Measuring on the drawing gives the missing data. The activity is logically ended by a work on the approximation closely related to a Geometry based on the possibility of measuring.

THE IMPACT OF THE SOFTWARE ON THE IMPLEMENTED FRENCH SPACES FOR GEOMETRIC WORK

In her master dissertation, Boclé (2008) described the typical situation given in French textbooks to introduce a new notion in geometry at the end of junior high school. In textbooks conceived just after 1996, the typical structure [SP1] was the following:

1. Construction of some particular figures with drawing instruments.
2. Measurement on these figures by using instruments (marked ruler or protractor).
3. Conjecture of a property.
4. Institutionalization of the property, either accepted without proof or formally proved later.

In the textbooks printed after 2005, a new tendency appears. A new notion is introduced using dynamic geometry software (DGS). The typical situation [SP2] is then the following one:

1. Construct a figure with DGS..
2. Get measures from the software.
3. Drag points to notice that the property remains true.
4. Institutionalize the property, either accepted or accepted without proof or formally proved later.

In both cases, to introduce the property, students have to construct several figures satisfying some criteria. Thanks to the measures made on the figures, it is possible to notice an invariant then to make a conjecture. In the textbooks following the 2005 syllabus, the activities of construction and measuring imply the use of DGS. At its beginning, every activity is clearly in SGW directed by Geometry I and favouring perception and instrumentation. In both approaches, with and without software, the point 4 is crucial for determining the type of geometry really used and the appropriate

SGW. If the property is only proved in a deductive way without any use of measuring, it is possible to enter Geometry II. But what happens if the property is not demonstrated? It seems that students remain in Geometry I.

These typical situations fulfil well the curricular instructions recommending the implementation of activities leading to conjecture properties. The recent emphasis on the use of DGS is taken into account in textbooks but the real contribution of the software in the transition from Geometry I to Geometry II deserves to be questioned. Indeed, the use of a DGS is justified in the textbooks by improving the measuring accuracy and the possibility of multiplying the examples. But a measure remains an approximation and therefore is not exact.

This vagueness can create a contradiction in the classroom and lead some students to become convinced by another way and then been led to prove without any measurement. By contrast, insisting on the precision of the software and its advantage with regard to ruler-and-compasses constructions risks to turn away students from the necessity of proving, which was one of the stakes expected within the reference SGW.

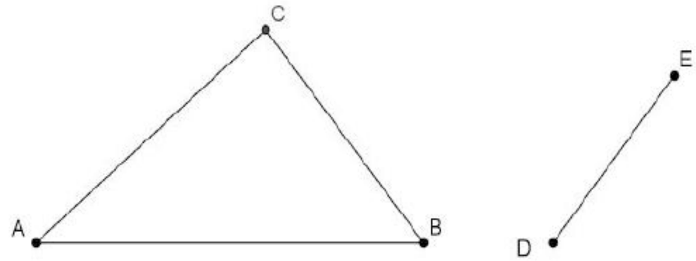
In her work, Boclé tried to see if the use of software in these typical situations favoured the transition to Geometry II or either if on the contrary it created a blocking element. She noticed that the strength of the proof by experiment overcame the classic work on demonstration with a purely deductive proof. In that case, it seems that the use of the software in standard situation stabilizes rather a SGW of Geometry I type and not a transition toward GII.

THE BREAK ACHIEVED IN GRADE 10 OR WHEN OSTENSION BECOMES DEMONSTRATION.

This contradiction is to be found again between the work expected by the institution and the work effectively set up in the teaching of similar triangles in an ordinary class at Grade 10. Similar triangles were reintroduced in French compulsory education in 2000. The notion has been removed from the syllabus since the modern maths reform and it reappeared in a quite different context in 2000 at Grade 10. Similar triangles are not considered by the programs as a new notion but as an opportunity to stabilize the geometric work at the end of compulsory education. We shall consider here only the result of a session managed by a teacher who first follows the typical way [SP1] but who changes on phase 4 (institutionalization) and then follows the process [SP2] by using the software by huimself.

The activity is the first one about similar triangles. A sheet of paper is given to the students with a drawing on and the first task is to create a triangle DEF such that $\angle BAC=\angle EDF$, $\angle ABC=\angle DEF$.

Partie I : Créer un triangle DEF tel que $\widehat{BAC}=\widehat{EDF}$, $\widehat{ABC}=\widehat{DEF}$.



Below the figure, the following questions appear on the sheet given to the students:

> What can we say about $\angle ACB$ and $\angle DFE$?
>
> Compare the sides of the triangles with your ruler. What can be noticed?
>
> Complete the sentence : We can guess that if two triangles have …. then their sides are ….

For the teacher the construction is not a problem. He anticipated two possible configurations, which seems an interesting difficulty to him. He wants to motivate in Geometry I the origin of a property which will belong completely to Geometry II when it will have been proved in the following lesson. For him, the figure is a generic example and he has not really thought about the measures given on the sheet.

The great majority of students, but not all, undertake completely the activity of construction which turns out to be long and complex. Students have difficulties with the use of their drawing instruments: the task « to make an equal angle » does not fit a well-known technique. Furthermore, the two possibilities for the final figure cause problems in the class since students are working on particular and not on general figures.

Other students understood that the construction is not important for the teacher and they quietly wait that the course goes on. They give, by abduction, purely linguistic conjectures by trying to adapt their mathematical knowledge to the situation. At the same time, students engaged in the construction task produce very different and contradictory results but actually these results and the work of these students will be left aside by the teacher who will favour the solution with DGS (Geogebra) and present it by video-projection in the class. The teacher follows the SP2 structure but without making any devolution to the students. He is the unique user of the software and he makes an institutionalization denying all the previous work of the students.

On the computer, the figure is the starting point and measures are given with five digits, even for angles. The proportion ratio calculated by the computer was 1.875 and was exactly the same for the three ratios.

The accuracy of the measures given by the computer shows to students the imperfection of their work with instruments on a very violent way. Strictly speaking, the students' work is rather useless because it is left aside by the teacher. Moreover, the accuracy of the software turns it into a tool for proof and a source of truth and, this, without the teacher knowing, as it can be seen in the following dialogue, which closes the class after the statement of the conjecture.

Teacher: Did we demonstrate the property?

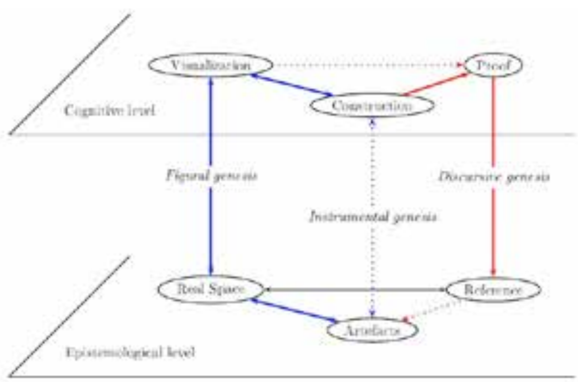
Almost all the students: Yes! We have done a demonstration.

Teacher (taken aback): Hum… No, it is too imprecise!

So after more than three years of progressive entrance in Geometry II and despite the programs which insist on the necessary awareness of the status of the statements, accepted or demonstrated, the gap between the expected work and the effective work is deep. It largely results from the appropriate SGW proposed to the students being itself very ambiguous and probably fundamentally a surreptitious Geometry I.

Within the SGW framework, it is possible to follow the break between two approaches of Geometry through various diagrams

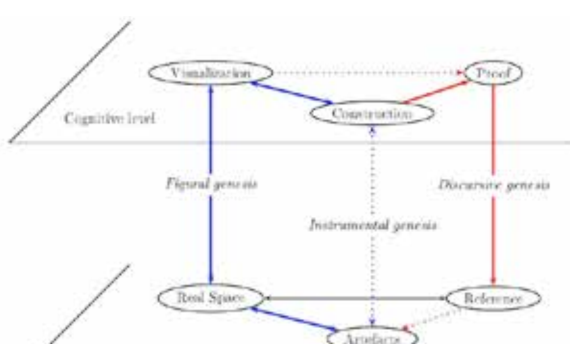
For the teacher, the construction is simple and will not cause trouble. His idea it to motivate the entrance in Geometry II by a preliminary work in Geometry I (in blue) and then to introduce a formal proof based on properties to justify the construction (in



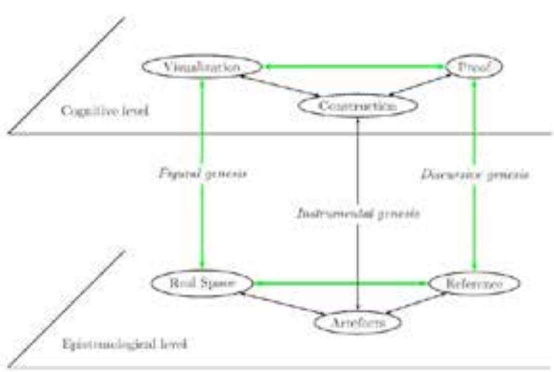
red).

Teacher's expectations

Students with low level in geometry try to construct with drawing tools. This is complex and requires a long time because the expected construction is not based on a standard technique at this grade. Moreover, due to a diversity of measurement results, a great diversity of properties are drawn by the students who consider the drawing as a particular figure without any general nature.



The construction work made by students is ignored by the teacher who gives the solution using a software, he is the only user of this tool which is really different from classical drawing tools, both in its uses and the precision of measurements. So after the "monstration" on a screen, students immediately conclude from results based on construction (in red) and the software appears as a source of Truth, but grounded on



experiment and no more on pure reasoning as in Arbuthnot's approach.

Students' geometric work without instruments

The last group of students give the conclusion without any construction work and can be summarized with the green diagram which shows an incomplete geometric work. They understand that the construction with drawing tools will be of no use for the conclusion and they complete the questions by abduction.

Even if the reference SGW always insists on a transition to GII based on GI, the implemented SGW is unstable and depends on the level of the students. For most of them, a real shift to Geometry I is favoured by the software which gives the "proof". Due to a lack of a theoretic system to ground the teaching, the reorganisation of the SGW is led only by the teacher who attempts to adapt his teaching to the supposed low level of his students. In short, a lack of epistemological vigilance conducts to a lost of cognitive vigilance.

CONCLUSION

The forms for teaching geometry and its need have always been questioned and discussed. But today, the traditional view of a geometry education useful for training logical reasoning is reconsidered by our society increasingly technological and consumerist.

With geometrical paradigms, it is possible to make explicit different stakes involved in the teaching of geometry. Each paradigm stresses a different view of "mathematical culture" considered, on the one hand, essentially practical with applications to real world or, on the other hand, more theoretical and guided by internal mathematical requirements. From this, it results that geometric work depends on various factors which can be described by the concept of SGW.

But again, the reference SGW remains difficult to determine in some countries. In France, today, the debate between the two lines is not over, even if the utilitarian line is strengthened by international institutions and evaluations like PISA which promote an empirical and utilitarian view on mathematics. So, the emergence of the suitable SGW looks very confuse and that explains partly the fickleness of personal SGWs which do not seem to reach a stable point and depend a lot on the didactical contract.

In other countries the choice of Geometry I is clearly assumed during the major part of education with a sudden change at the end of the syllabus toward a traditional SGW of type (GII/gI) with ancient forms of teaching based on Euclidean tradition.

Whatever is the chosen paradigm, a consistent geometric work has to be implemented into the classroom in order that students can solve interesting problems and are aware of the authorized tools to justify their results. So, from a Geometry I perspective, a work on approximation is essential to fix the degree of confidence to be given to the results. Similarly, in Geometry II, students and teachers need to have clear ideas on the role and use of properties. One route to be explored could be to make explicit the existence of two geometries and to seek solutions based on GI or GII for some specific problems especially in the case of modelling. Another approach is needed to overcome the conflicting viewpoints on proof and it would be interesting to relate geometric activity to other mathematical areas. Changes of areas, changes of semiotic representations have always been at the centre of mathematical work. In the case of geometry, solutions of problems are based on different sets of numbers, use of functions and all this is reinforced by new software with great semiotic potentialities. By focussing on three main geneses, semiotic – instrumental – discursive, SGW provides a framework suitable to take into account the main key points of individual mathematical work which must be linked one with another to develop a global and effective work.

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LANGUAGE AND MATHEMATICS: A FIELD WITHOUT BOUNDARIES

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It is widely acknowledged within the field of mathematics education that language plays an important (or even essential) role in the learning, teaching and doing of mathematics. However, general acceptance of the importance of language is not matched by agreement about what this role (or these roles) might be or even about what the term language itself encompasses. I propose to construct a map of the field, identifying the range of ways of conceiving of language and its relevance to mathematics education, the theoretical resources drawn upon to systematise these conceptions and the methodological approaches employed by researchers.

INTRODUCTION

In this paper I propose to offer a map of the field of study of language in mathematics education – a way of conceptualising the various sub-fields, the connections between them and their connections to other domains of study both within mathematics education and beyond. This map is inevitably idiosyncratic and partial. It is *my* map, though its components, construction and conceptualisation owe much to the work of others (not all of whom can be explicitly acknowledged), but especially all those who have shared the task of organising the CERME Working Group on Language and Mathematics over the years – a task that I think it is important to acknowledge is not merely organisational but also has demanded scientific work in planning to enable coherent discussion, bringing researchers with widely disparate foci and theoretical orientations into productive conversations.

Language has been a topic of research in mathematics education for a long time. An early review article by Austin and Howson published in Educational Studies in Mathematics in 1979 drew on research from the previous two decades to provide a "state of the art" picture of the field at that time (Austin & Howson, 1979). I hope to show some of the directions in which our thinking and knowledge have developed since then, though my intention is to attempt a schematic rather than a comprehensive review.

In more recent years there has been a massive increase in the attention paid to language in mathematics education. I see this as a consequence of three trends within the field as a whole.

1. The development of mathematics education as a mature field of study has seen serious attention to theorisation and problematisation of the components, concepts and methods of the field, including language.
2. More specifically, the development of attention to language reflects the "social turn" (Lerman, 2000) in mathematics education as a whole. An orientation to

the importance of the social environment within which mathematics education takes place has inevitably been accompanied by raised awareness of the significant roles of language and other forms of communication within that social environment. This awareness has in its turn been accompanied by changing views of and more sophisticated attention to the relationships between language, mathematics and learning.

3. As well as these trends in the theoretical orientation of the field, developments in classroom practice, professional discourse and policy have increasingly included an important role for language-rich activity in the classroom, especially talking or "discussion". While curriculum and policy developments may be informed by research and theoretical ideas about learning and language, their recontextualisation into classroom practice is often simplistic. Nevertheless, the widespread and influential notion that "discussion", "interaction", "discourse" in classrooms is a good thing, in itself strengthens the need for research that interrogates the nature and functioning of such discussion.

CHARACTERISING MATHEMATICAL LANGUAGE

First it is necessary to establish the scope of what I mean by language – and more specifically *mathematical language*. Looking at dictionary definitions we find, for example:

- the method of human communication, either spoken or written, consisting of the use of words in a structured and conventional way
- a non-verbal method of expression or communication e.g. body language
- system of communication used by a particular country or community (http://oxforddictionaries.com/definition/english/language)

Within mathematics education literature we find language used in each of these ways: dealing solely to words (referred to variously as natural language, verbal language, etc.) or including non-verbal modes of communication, especially (or indeed sometimes exclusively) mathematical symbolism, but also diagrams, graphs and other specialised mathematical modes as well as gestures and other modes of communication used in a variety of settings. There is also concern with the third sense of language both in the context of working with multilingual learners and in considering doing and learning mathematics in different national languages.

I would suggest an addition to this list in which language is used to refer to what Halliday calls register:

- the specialised method of communication used in a particular social practice

This includes mathematical language, of course, (the forms of communication used when doing mathematics) but also everyday language, scientific language, academic language, even classroom language.

While communication within the practice or practices of mathematics education and of doing mathematics is of central interest, it is also relevant to consider how this relates to other practices. On the one hand, considering how the practices of mathematicians are similar to those of other scientific or academic fields allows us to make use of knowledge about language and communication developed in those fields. On the other hand, recognising that our students are participants in a range of extra-mathematical practices may enable us to understand better their experience of communication in mathematics classrooms.

In recent years, thinking about language in mathematics education has broadened from considering primarily either words or mathematical symbolism towards a more comprehensive concern with a range of other means of communication. Again, I would suggest that this development has arisen at least in part from increasing recognition in the field as a whole of the importance of taking the social environment of learning and doing mathematics into account within research. By focusing on the social environment, the face-to-face communication that takes place in classrooms has come more into focus, moving attention away from written texts to the spoken word. This has also led to greater use of naturalistic data arising in classrooms and elsewhere together with qualitative methodologies that recognise and attempt to deal with the complexity of social situations. When observing in a classroom it is hard to miss the fact that words and mathematical symbols form only part of the communication that is going on. Whereas there are well established means of describing language, drawing on the field of linguistics, as well as attention to the syntax of mathematical symbolism (e.g., Ervinck, 1992), recognition of the multimodal nature of mathematical communication demands the development of means of describing and studying other modalities.

Of course, new technologies are changing our ways of communicating, not only introducing new semiotic resources, notably dynamic, manipulable and multiple linked representations, but also new forms of human interaction, both asynchronous (as is generally the case through email, discussion boards, blogs, podcasts etc.), and potentially synchronous (as in chat rooms, instant messaging, video conferencing). The potentialities of these new forms disrupt our established understandings of, for example, differences between spoken and written language. Developments in the general fields of communication and media studies offer some ways of theorising and analysing this wider range of resources, adding to what may be taken from linguistics, semiotics and theories of discourse. However, at the heart of any research in mathematics education we must find mathematics itself. Our conceptions of mathematics inform how we choose, use, interpret and adapt the theoretical and methodological tools offered by other fields.

We need not only to describe the language used in mathematical and mathematics education settings but also to be able to address questions such as:

- What is distinctly mathematical (or not) about the way language is being used?

- What role does the language play in the processes of doing mathematics and producing mathematical knowledge?
- How does the language function to establish what is and what is not to count as mathematics in this setting?
- How does this person's use of language position them in relation to mathematics?

and many other questions that focus on relationships between language and the activity of doing mathematics.

TRENDS IN RESEARCH

As ideas about what mathematical language encompasses have developed, I identify three related trends that have emerged (and which I will return to in the course of this presentation):

1. attention to forms and patterns of interaction in the classroom (and here I include studies of interaction mediated by new technologies).

Much of this research looks primarily at verbal language, especially the spoken word but there is also increasing interest in how gesture features in interaction and development of interactionist approaches that draw on semiotic theory to incorporate attention to a wider range of signs.

2. attention to a wider range of mathematical means of communication and the relationships between them.

There is a long tradition of study of mathematical graphs, diagrams, etc. from a cognitive perspective concerned with individual representation of mathematical concepts. This tradition has been moved on by research looking at the role of multiple representations and the cognitive demands and benefits of moving between them. As a more holistic view of language and mathematics within a social environment has come into focus, recognition of the multimodal nature of communication, including use of new technologies, has led to the development of more comprehensive and rigorous means of describing the complexities of mathematical communication and to research that seeks to describe and understand how teachers and students make use of them as mathematics is done in the classroom.

3. attention to what is achieved by using language.

A functional view of language is increasingly widespread, though sometimes alongside a representational view rather than displacing it. In other words, studying language has moved from focusing simply on what is said about mathematics to what the language achieves within a mathematical practice. This may be studied at several levels. We might characterise as micro-level a study addressing the functioning of particular words or other signs in relation to a single mathematical construct, for example, Rønning's (Rønning, 2009, 2013) work with children on fraction tasks looking at how the use of different types of semiotic resource affected the course of their problem solving.

At a meso-level, we find studies, which, while maintaining a focus on a single mathematical topic area or issue, looked at a range of data sources over several lessons to gain a wider view of how the various semiotic resources function separately and together to shape students' experiences in this area, for example, Chapman's (1995) study of lessons on function in which she showed how the teacher's and students' use of the various forms of representation of function made connections and developed over the course of a sequence of lessons. Such studies not only provide us with detailed insight into the specific topic area but also point to more general issues about the significance of the selection and coordination of representational forms.

At macro-level, there are a small number of studies looking more generally at how language use contributes to mathematical practices. Misfeld's (2007) study of how research mathematicians make use of various forms of writing during their creative work is an interesting example of this type of study, drawing on resources from the field of writing research that theorise the process of writing as problem solving or as discovery of knowledge rather than only "telling" what is already known.

The interest in how language functions thus does not focus solely on texts arising in classroom interactions but can extend to address a much wider range of texts that play a role in shaping mathematical and educational practices. In a current project in collaboration with Anna Sfard, we are addressing mathematics education at the level of the curriculum, analysing the language of high stakes examinations in the UK in order to characterise how these examinations function to construct the forms of mathematical activity expected of school students (Tang, Morgan, & Sfard, 2012).

WHAT IS THE RELATIONSHIP BETWEEN MATHEMATICS AND LANGUAGE?

Before moving on, I want to comment briefly on a major theoretical issue that informs and divides research in this field. It is claimed that language has a special role in relation to mathematics because the entities of mathematics are not accessible materially. This entails particular importance for the study of language and language use in mathematics as in some sense mathematics is done in or through using language (in the broad sense of means of communication I have discussed). However, I say "in some sense" because there are radically different theoretical conceptualisations of what the entities of mathematics actually are and how they are related to language. On the one hand, some take the position that mathematical objects have an independent existence, even though they are only experienced through language:

> We do not have any perceptual or instrumental access to mathematical objects … The only way of gaining access to them is using signs, words or symbols, expressions or

> drawings. But, at the same time, mathematical objects must not be confused with the used semiotic representations. This conflicting requirement makes the specific core of mathematical knowledge. (Duval, 2000, p.61)

On the other hand, those working with Sfard's (2008) theory of cognition and communication reject any dualist separation of mathematical object and language, arguing that mathematics is an entirely discursive activity and that mathematical objects are no more than the total of the ways of communicating about them.

These very different theorisations of language and mathematics have consequences for how researchers may think about the development of mathematical knowledge – as a process mediated by language or as the development of mathematical ways of using language – and about the place of language itself in researching mathematical thinking. Is language taken to be the means by which we get limited and partial access to learners' mathematical thinking or is the communication itself the object of study?

So far I have tried to outline some of the ways in which the field of study of language and mathematics has developed its conceptualisation of language. One recurring theme in attempting this task is the growing recognition of the complexity of the field both empirically and theoretically. I now turn to a very simple statement that still forms a starting point for much research.

"MATHEMATICAL LANGUAGE IS DIFFICULT"

The perception that language is a source of difficulty in mathematics learning has framed and continues to frame much research. This perception rests upon a dualist conceptualisation of language and mathematics as separate domains, though conceptualisations of the relationship between the domains vary from a naïve view of language as a barrier to learning that must be overcome to more sophisticated theorisation of language use embedded within particular social practices.

Early research in the field identified a number of features of mathematical language that students at all stages of education had difficulties understanding and using correctly. These included difficulties with vocabulary, with algebraic notation, with handling logical connectives but also difficulties at the level of more extended texts. Analysis of reasons for these difficulties were, however, less evident. The issue of confusions with everyday language was recognised, especially in relation to young children, for example, Durkin and Shire's (1991) analysis of ambiguities in elementary mathematics, identifying words that have different meanings in mathematical and in everyday contexts. I suggest that this relatively untheorised notion of confusion between different meanings of words may be associated with what I have called a naïve view of language as a barrier to learning. As thinking about relationships between language and learning change, ways of interpreting "confusion" between everyday and mathematical meanings also develop. We thus see more complex analyses of difficulties and attempts to theorise what happens as students encounter mathematical forms of language. While difficulty and failure to communicate effectively is still a relevant area for research, the focus now is not so much on what children cannot do or what they fail to understand as on seeking to understand what is actually happening in classroom interactions, on the nature of communication among students and teachers and on the sources and functioning of apparent miscommunication

Relationships between mathematical and everyday language continue to be a focus of research but we now see more theoretical subtlety in attempts to understand why difficulties arise. There are several notable theoretical ideas that contribute to this understanding. The first is the widespread influence of the idea of "situatedness" – the idea that people make sense and behave differently when situated in different practices. Using a word in its everyday sense may thus be seen as the result of failure to recognise the situation as mathematical rather than failure to distinguish the correct mathematical sense of the word. Moving away from dualist separation of language from mathematics, discourse theoretical developments suggest that we think of mathematics as a discursive practice: doing mathematics essentially entails speaking mathematically (or writing or using other communicational modes).

The influence of discourse theoretical approaches, provides alternative ways of thinking about miscommunication in particular as non-arbitrary combination of resources drawn from different discourses Thus, for example, in a study conducted as part of the ReMath[i] project of students' and teachers' use of gestures while working with three dimensional figures, we found we could understand some apparent student difficulties in terms of the mismatches between the ways systems of directional vocabulary, Logo formalism and systems of gestures functioned in mathematical and everyday discourses (Morgan & Alshwaikh, 2012).

The teacher-researchers, drawing on specialised mathematical discourses that include expectations of formal definitions of terms, initially assumed that there was a consistent relationship between each term in the system and the type of movement or direction referred to. However, closer analysis suggested that, while students adopted some aspects of the formal systems introduced by teacher-researchers, for some, local characteristics of their activity seemed to call up other types of (everyday) discursive resources (see Table 1).

	Specialised discourse	Everyday discourse
Linguistic/ Logo formalism	*Turn Left/Right* *Pitch Up/Down* Each instruction has a single defined function *Right/Left* and *Up/Down* are always relative to current orientation	*Turn up/right/ clockwise* *Go right/down* *Roll over/ around* Terms are multivalent and their use is context-specific, depending on the type of object that is moving and its starting orientation

		Right/left is usually relative while *up/down* is usually absolute
Gesture	Iconic, mimicking trajectory One-to-one relationship to Logo instructions	Deictic, pointing in direction of movement or iconic or hybrid Substitutable, especially to overcome physical difficulties

Table 1: Specialised and everyday language for movement in three dimensional space

For example, the student in Figure 1 was attempting to "play turtle" with her hand, a method of using gestures to support Logo programming that had been used during the introduction to the sequence of work. However, as she encountered physical difficulty in turning her right hand through more than 180°, she showed no hesitation in switching hands and changing the relationship between her gesture and the desired motion of the turtle suggesting an "everyday" type focus on direction of movement (using a discourse that allows ambiguity and substitutability) rather than the anticipated "Logo" type focus on type of turn (a discourse that demands precision and unique definition of terms).

Figure 1: Switching hands - shifting discourses

While difficulty in learning and using mathematical forms of language is still evidently an issue in classrooms and for research, efforts to understand the sources of difficulty have thus led researchers to orient towards analysing what students do communicate as well as what they do not.

Related research arising from increasing interest in the multi-semiotic nature of mathematical communication and the recognition that moving from one semiotic system to another is not a straightforward matter of translation (even if we believe that translation can be straightforward) explores how students choose from and make use of available semiotic resources to do mathematics. I have indicated here some of the theoretical approaches that address this issue. An important aspect of this is the recognition that different semiotic systems offer different possibilities for engaging with mathematical constructs. For example, a function represented by an algebraic expression lends itself:

- to analysis of its parts – does it have factors? is it a function of a function? does it contain a quadratic term? –
- to categorisation as a polynomial, a trigonometric function –
- to manipulation – can it be expressed more simply? what is it's value when x= …

while the same function represented by a graph is more likely to lend itself to consideration of global and local characteristics:

- is the function odd or even?
- is it periodic?
- is it continuous?
- does it have maximum or minimum values?
- how does y change as x increases

Again, in our ReMath project work, observing students working on an open problem solving task in a technology rich environment, we saw how their choices of communicational modes affected the ways students defined their problem and the trajectory of the problem-solving process (Morgan & Alshwaikh, 2009). This and other work shows that sensitivity to a wider range of modes of communication, including gesture and the richly multi-semiotic environments offered by new technologies, both highlights the complexity of students' encounter with mathematical discourse and offers us conceptual and analytical tools to address this complexity.

WHAT DOES LANGUAGE ENABLE US TO STUDY?

My own thinking about language and mathematics education is strongly influenced by the social semiotics of Michael Halliday (Halliday, 1978, 2003; Halliday & Hasan, 1989). An important insight offered by this theory is the recognition that the language we use not only construes the nature of our experience of the world but also our identities, relationships and attitudes. Study of language use thus offers opportunities to address many of the problems of mathematics education. I shall briefly consider three of these here:

- analysis of the development of mathematical knowledge
- tools to describe engagement in mathematical activity
- understanding processes of teaching and learning in social interactions

Analysis of the development of mathematical knowledge

Many areas of research within mathematics education have used data consisting of what students say (or other signs they produce) as evidence of their mathematical understanding. Development of thinking about language challenges some of the assumptions that lie behind such research and has also brought theoretical and methodological tools that contribute to understanding development of mathematical thinking and enable a more grounded analysis of linguistic data.

Naïve conceptions of language as a transparent means of transmission of ideas from speaker to listener have been seriously challenged by current thinking about communication. Moreover, a number of influential theoretical frameworks, including Peirceian semiotics, Wittgenstein's notion of language games, and post-structuralist theories reject any fixed relationship between word and referent. These have been taken up and developed within mathematics education to address the specific problems of mathematical learning.

Work in semiotics has offered sophisticated means of conceptualising and investigating relationships between signs and mathematical meaning making. In particular, we have seen the notion of epistemological triangle, introduced by Steinbring (2005), used as a means of describing the nature and development of mathematical knowledge in classroom situations, focusing on the role of the symbols, words, material objects and other forms of representing mathematical concepts. This notion emphasises that relationships between representations and concepts are mediated by the "reference context", including the previous knowledge and experiences of the students.

Another approach to the issue of the development of mathematical knowledge makes use of the Vygotskian notion of tool mediation (Vygotsky, 1986). From this perspective, verbal language and other semiotic systems are conceived of as psychological tools that shape the nature of human activity. This framework has been used to analyse the effects of particular tools (whether specific words or other forms of representation or more extensive semiotic systems) on the development of mathematical activity (see e.g. Bartolini Bussi & Mariotti, 2008).

Tools to describe engagement in mathematical activity

From a different tradition, current theories of language use and discourse tend to focus on what utterances achieve rather than treating them as a means of accessing inner thought or objective reality. Within mathematics education, this perspective has been developed by Anna Sfard in her communicational theory (Sfard, 2008). Here no distinction is made between speaking/writing/communicating in mathematical forms and doing mathematics/thinking mathematically. Detailed characterisation of the nature of mathematical language thus provides a means of describing the ways in which learners are engaging in mathematical activity.

Understanding processes of teaching and learning in social interactions

Developments in the study of language in mathematics education are closely related to developments in the wider field. The move to focus on practices rather than on individuals, to consider learning as a social or socially organised activity and the move from ideas of individual construction of meaning to considering meaning as something formed by individuals within social environments have opened up a space within which language oriented studies contribute to the overall project of understanding teaching, learning and doing mathematics. Many of the theoretical and methodological resources that researchers into classroom interaction draw on originate outside the field – in ethnomethodology, linguistics, pragmatics, sociology etc. – and have been developed to deal with general interactions. These ways of thinking recognise that there are patterns in any social interaction that are distinctive to particular practices and functional in shaping what gets done in the interaction. Recognising these patterns and what they achieve provides tools for analysing classroom processes and can also inform development of teaching practice. For example, the patterns of funnelling and focussing identified and discussed by Bauersfeld (1988) and Wood (1998) have proved a useful tool for working with teachers as well as a foundation for further work on identifying patterns of interaction and establishing their functions.

It is important, however, to ask what are the specifically mathematical issues that arise in studying interaction in mathematics classrooms. Why should mathematics educators be concerned? Indeed, some studies located in mathematics classrooms analyse interactions in ways that do not seem to address the teaching and learning of mathematics directly. Such studies certainly illuminate important issues, for example, how knowledge is produced in interaction or how students may be positioned differently by classroom discourse. These issues are of concern both theoretically and in practice but as a researcher in mathematics education it is not enough for me to say simply that these studies are located in mathematics classrooms. I want to know what they have to say about mathematics and about the teaching and learning of mathematics.

Studies of interaction that engage in significant ways with mathematical aspects of interaction include those using the notion of socio-mathematical norms (Cobb &

Yackel, 1996) as well as studies of specifically mathematical forms of interaction such as argumentation (Krummheuer, 1998; Planas & Morera, 2011) or group problem solving (Edwards, 2005).

In a world in which new communication technologies provide new opportunities for interaction, it seems important to develop our understanding of how technologies may affect pedagogic and mathematical communication. This is especially pertinent as funding bodies encourage the development of internet-based tools and on-line collaboration. The CERME working group has seen some interesting analyses of on-line mathematical interactions, ranging from analysis of the pedagogic and mathematical practices of participants in an on-line discussion board (Back & Pratt, 2007) to analysis of semiotic activity during paired problem solving undertaken in an internet chat-room (Schreiber, 2005). At this time, these studies of technologically mediated communication are still relatively isolated, focusing on the features of specific special contexts. This is an area that offers many opportunities for both empirical research and theoretical development as the use of communication technologies becomes more widespread in mathematics education. Again, research in mathematics education needs to be informed by the developing field of research in on-line and mobile communication while maintaining a distinct focus on mathematics.

BILINGUAL LEARNERS

The final substantive issue I shall address is that of learners studying mathematics using a language different from their mother tongue/first language. This has been of concern for many years. Indeed, current understanding of the nature of mathematical language as a whole owes much to a paper by the linguist Michael Halliday that was originally presented in 1974 as part of a UNESCO symposium addressing the issue of education in post-colonial countries (Halliday, 1974). In many of these countries the colonial language was still used as the language of instruction but there was increasing interest and political desire to make use of local languages. In many cases a mathematical register did not exist in the local languages, raising many questions for the development of mathematics education. We are still grappling with problems arising from colonialism; problems which are not only linguistic but also political.

On the one hand, questions about which language should be used for teaching and learning mathematics and about the effects on learning of using one language rather than another have been addressed by studying the affordances of a language and the issues that arise for learners. For example, Kazima (2007) has identified issues in the learning of probability concepts in Malawi due to structural differences between the local language Chichewa and English, the language of instruction in secondary schools, while Ní Riordáin (2013) offers a psycholinguistic analysis of the potential differences in cognitive processing involved in using English or Irish. Bill Barton (Barton, 2008) has provided a fascinating discussion of relationships between the characteristics of a language and the kinds of mathematical thinking that may develop through using it. His theorisation of the relationships between mathematics and languages opens up a rich field of study.

However, the practical questions about which language to use in the classroom cannot be answered fully without addressing the wider socio-political role of language. Setati's work in the context of multilingual South Africa raises an important distinction between what she calls the epistemological access to mathematical ideas that may be enabled by teaching and learning in a student's home language and the access to social, economic and political advancement enabled by developing higher levels of fluency in a world language such as English (Setati, 2005). The learners in South African classrooms and elsewhere in the world are not only learners of mathematics but are also becoming citizens of their own countries and of the world. The significant roles of language in both these domains cannot be ignored or resolved easily. In the context of this conference I am very aware of the privilege accorded to me as a native speaker of English, of the struggles that speakers of other languages go through to make themselves heard and of the complexity of the choices that have to be made by all those working in multilingual settings.

Increasingly in many European countries as well as elsewhere in the world, educators are faced with multilingual classrooms as global mobility of populations increases. This is reflected in an expanding area of research considering learning in a variety of multilingual and multicultural contexts. Within CERME this has led to certain overlap between the Working Group on "Language and Mathematics" and that on "Cultural diversity and mathematics education". As a field of study, multilingual/ multicultural teaching and learning is challenging in its complexity. The contexts considered vary enormously – culturally, linguistically and economically. Alongside issues of language, many of these contexts also involve complex issues of social deprivation, social and political exclusion and cultural differences and diversity. As might be expected in a maturing field, considerable work is being done to map out the scope and develop a coherent understanding of the theoretical diversity brought to work in this area, yet there is room for further intercommunication.

A BROADLY RELEVANT METHODOLOGICAL ISSUE

Moving on to consider how work on Language and Mathematics relates outside the narrow sub-field brings me to a methodological issue that I believe should be of concern to all those using language-based forms of data, not only those for whom language itself is a major focus. I have already referred to the privileged position of English speakers. This is not just a social and political issue but also a methodological concern. Where international conferences and journals use English as the primary language for communicating scientific studies, many researchers experience the pressure of expectations to present their work, including the data and its analysis, entirely in English. However, many of the theoretical positions regarding relationships between language, mathematics, epistemology, thought, etc. that are adopted by those whose work I have touched upon in this paper make claims about the constitutive nature of language. If the words we use and the ways in which they are combined grammatically play a constitutive role in the construction of mathematical thinking then we need to be aware of how this role may be different depending on the specific (national) language that is being used. However, there are few examples in the international English language literature that present data or analysis in other languages except in studies whose main focus is on the distinct characteristics of the (national) language of the learners.

A rare example that illustrates the importance of using the original language was given by Boero & Consogno (2007). They presented an extract of data (translated from Italian into English) to illustrate a mechanism by which individual students' contributions combine to allow joint conceptual construction and reasoning:

Maria: In the case of two as divisor, we need to move from one even number to the following one, two steps far.

Barbara: While in the case of three as divisor, we need to move from a divisible number to the next number divisible by three... three steps far

Francesco: And in the case of four, four steps far!

Lorena: The distance is growing more and more, when the divisor increases... the distance is the divisor! ... (long silence)

Roberto: So if the distance is one, the only divisor is 1.

But, in order for this extract to form an effective illustration of the mechanism, the authors' analysis needed to incorporate information about the original Italian.

> The expression "... steps far" ("...passi distante" in Italian) allows students to move from one example to another, then the idea of "distance" ("distanza" in Italian) allows to embrace all the examples in a general statement that Roberto can particularise in the case of interest for the problem situation. Note that in the Italian language students can move easily from the adjective "distante" to the noun "distance". (p. 1155)

By publishing only translated versions of interactional data, subtle yet important aspects of the functioning of language may be lost. Equally, readers of translated data are likely to form their own interpretations based on the translated words – interpretations that may have no basis in the words of the original data. I would suggest that this is an issue for research in general, not just that which focuses on language.

Clearly authors, editors, readers, conference organisers and attendees have a common interest in making forms of communication as inclusive as possible while simultaneously ensuring that the research itself is reported accurately, rigorously and meaningfully. This interest might suggest that there should be more parallel presentation of original and translated versions of data and analyses. However, we know that pressures of time, space and labour costs militate against this suggestion.

There is room for innovative solutions here – perhaps making more use of the possibilities offered by new technologies.

OUTSTANDING SUBSTANTIVE ISSUES

In spite of the widespread recognition of the difficulty that many learners have with mathematical language and the importance of language in learning mathematics, much less attention has been paid to the question of how children learn to speak or write mathematically. Detailed studies of classroom interactions sometimes demonstrate student acquisition of particular signs or ways of communicating about specific mathematical constructs but the focus here tends to be on how language use contributes to learning the mathematics rather than on acquisition of mathematical ways of speaking or writing that may be more generally applicable and acceptable in a wide range of areas of mathematics. Of course, those who adopt Sfard's rejection of the dualist separation of language from mathematics would argue that learning mathematics is identical to learning to speak in mathematical ways; even studies within this framework, however, tend to focus on particular instances of mathematical contexts rather than on the general question of the acquisition of mathematical discourse.

I suggest three areas of concern in which, while some work has been started there is a need for more substantial and coordinated research effort.

- What are the linguistic competences and knowledge required for participation in mathematical practices?
- How do students develop linguistic competence and knowledge?
- What knowledge and skills might teachers need and use in order to support the development of students' linguistic competence?

FINALLY …

As suggested in my title, in preparing this paper one challenge I have had to face is defining the scope of the field of Language and Mathematics. This is a challenge that has been implicit in the work of the CERME Working Group. The membership of the group has been very fluid as many colleagues whose main research focus may lie elsewhere have found language issues relevant to some aspect of their work and so have joined us for one year but gone to a different group in other years. This fluidity has on the one hand hampered efforts to develop greater coherence and continuity. On the other hand, it has, I believe, also enriched our work. It has provided opportunities to test out theoretical and methodological questions about language in a wide range of contexts and to disseminate them widely.

In some sense almost all studies involving language and communication in mathematics education also address other significant issues – learning, teaching, affect, identity, curriculum, assessment, etc. At the same time, it could be argued that, as most studies that locate themselves in other fields also make use of some form of textual data and communication between researchers and the participants in the research, the findings and theoretical developments related to language and communication are likely to have very broad implications. Hence my title: "A Field without Boundaries" and an open invitation to relate your own research to Language and Mathematics.

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WORKING GROUPS:

RESEARCH PAPERS
AND
POSTERS

INTRODUCTION TO THE PAPERS AND POSTERS OF WG1: ARGUMENTATION AND PROOF

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Keywords: *Argumentation; proof and proving; reasoning; logic and language; formalisation; didactics; epistemology; mathematics education from primary school to university; teacher training; activities fostering argumentation and proof skills*

INTRODUCTION

This chapter collects the contributions discussed during the working sessions by the twenty-six participants[i] from fourteen countries of the WG1 «Argumentation and proof» at CERME 8 in Antalya (Turkey). Nineteen papers[ii] and a poster[iii] from nine countries were presented and discussed. The papers were presented in seven sessions under four themes: *Epistemological and didactical issues and their relationships in proof and proving in mathematics education (one session), The role of logic and language in teaching, learning and analysing proof and proving process (two sessions), Designing activities fostering argumentation and proof skills (two sessions),Theoretical perspectives on reasoning, proof and proving (two sessions).*

In each session, we collectively discussed a main question that had been sent in advance to the presenters who were requested to focus their short presentation on this question. We present in this introduction a summary of these discussions.

EPISTEMOLOGICAL AND DIDACTICAL ISSUES

According to Lakatos, in the construction of a mathematical theory, proving and defining are intertwined processes. The presenters where asked to take in consideration the question of epistemological and cognitive aspects involved in the relations between proving, defining, truth and validity, relying on prior research and on their experience. Renaud Chorlay proposed an epistemological and didactical perspective on the making of a proof-chain, while the paper from Judith Njomgang and Viviane Durand-Guerrier presented a classical example in calculus to discuss the distinction between truth and validity, and their relationships. The main points that were raised in the discussion were the importance of considering mathematical theories as both products and processes, and to take into account the interplay between proving and defining: within a deductive chain, definitions come first, although they are usually developed last **(**Proving may lead to discover/identify unexpected properties of mathematical objects). Considering the relationship between proving and conceptualisation (formation and development of mathematical concepts), the connection with intuition, the role of argumentation and the possibility for students to have access to the key ideas of a theory have been pointed. Concerning truth and validity, there are different uses of these terms in classrooms; there are differences in considering truth for physical objects or for abstract objects; some students do not understand the link between proof and truth (they can accept a proof and reject the statement that has been proved); moreover validity is not necessarily connected to proof by students. So, it seems important to give opportunity to students to discuss about validity, truth, defining and proving in a theory. Some suggestions are provided in the third theme.

THE ROLE OF LOGIC AND LANGUAGE

Two sessions were devoted to this theme. In the first session we examined issues at the secondary level. The discussion focused on identification of aspects of logic and language likely to be an obstacle for developing proof and proving skills, and aspects that are likely to favour it. In the interest of teaching logic for fostering proof and proving competencies, Jenny Cramer discussed the possibility of *language barriers as an obstacle in the process of mathematical reasoning*. Considering that logic and language are closely related, Zoe Mesnil supported the interest, for teachers, *of logical analysis in mathematical discourse*. A crucial question is “how to do this?”. In her paper, she presents elements that are proposed for teacher training in France, where logic has been reintroduced in the high school curriculum. Pro and contra arguments were provided. Concerning the respective role of logic and language in conceptualization, there is a balance between the necessity of mathematical language, and the importance of remaining close to the natural language. But where should we draw the line? In multilingual contexts, logical analysis could have negative effects of excluding or discriminating students. On the positive side, logical argumentation may provide a bridge between natural and mathematical languages. Should we consider logical competencies and/or logic as a body of knowledge; logic as a theory modelling human reasoning and/or as a theory aiming to control validity of proof; should logical proof be considered both in terms of a final product and as a process in action? We neither reached a consensus on these questions, nor on the relevance of teaching or not teaching logic at secondary school.

In addition, in this first session, Christavgi Triantafillou presented an on-going research on *the nature of argumentation in school texts in different contexts*, pointing the role of nomo-logical, logical-mathematical, logical-empirical and empirical inferences, mediated through linguistic and non-linguistic tools.

The second session on this theme was devoted to issues for advanced mathematical thinking. According to Quine, formalisation in predicate calculus contributes to conceptual clarification. However, it appears that for many students, formalisation is an insuperable obstacle. The discussion raised the relationship between logic and formalisation. Formalisation is essential for mathematical work, in particular to control correctness. In this respect, there are both mathematical and logical formalisations. Formalisation serves the choice of aspects of a concept that fit in a theoretical way when defining it (for example perpendicularity and continuity). Such choices could have a cognitive effect on students' understandings. Educational research is concerned with the relationship between students' difficulties and formalisation. For example, Nadia Azrou analysed *students' difficulties in a proof in Algebra* in group theory. Another issue in mathematics education is the variability of formalisation in text books as presented by Faiza Chellougui *on the case of continuity at higher education.* Eva Müller-Hill discussed the *epistemic status of formalised proof* and *formalisability as a meta-discursive rule from a philosophical perspective.* In mathematics education understanding formalisation as a process is essential. An important aspect is the dialectic between syntax and semantics. There is a semantic context to syntactic work (how does a machine know what to do next when presenting a proof?). We can wonder how students could understand what a mathematical object is (i.e. group) without being able to rely on examples.

DESIGNING ACTIVITIES

In CERME 7, a crucial issue that was pointed was the necessity of sharing relevant activities fostering argumentation and proof skills across the curriculum, from kindergarten to university and in teacher education.

In CERME 8, five papers and a poster focused on this theme. Two papers and the poster proposed research situations aiming to develop proof and proving competencies. Patrick Gibel *analysed a mathematical lesson in primary school* including a situation of validation. The sequence aims to teach the rules of the 'game' in the proving process. Specifically, the sequence focuses on true and false statements but also on the way of establishing truth/falsity, in line with the relationship between truth and validity discussed in the first session. Denise Grenier of the team Math-à-Modeler presented *Research situations to learn logic and various types of mathematical reasonings and proofs*. Simon Modeste, also member of the team MathsàModeler presented a poster on the fundamental role of problem in modelling algorithmic thinking. Before the session, Denise Grenier and Simon Modeste had invited the participants to experiment a mathematical SiRC *(Research Situation for Classroom),* in line with the importance in the model they propose of experiencing the situations in order to understand what competencies they are likely to develop. The discussion was initiated by the question addressed to the presenters: in what respect the models they have elaborated were sound supports for developing new research situations oriented to the development of proof and proving skill? For researchers, models are useful for: developing situations, predicting or describing students' activities and analysing students' reasoning and their evolutions The models are also beneficial for communication with teachers and educators and for trying to promote students' engagement in proving. They also provide criteria to identify relevant mathematical problems (e.g. optimisation – number theory – discrete mathematics) and to organise situations. The dialectics between action, formulation and validation, in line with the function of reasoning, are taken into account: to make decisions; to open possibilities; to express and discuss methods. An important issue refers to the possibility of learning both new concepts and proving techniques at the same time. On the one hand, proof is constitutive of a concept; it is partly content dependent. However, there is a difference between mathematicians and students, and we could consider that if the priority is to focus on proof and proving, it could be relevant to choose problems where the involved knowledge will not raise content difficulties for students. Finally, it appears that both aspects need to be developed in the classroom. Another important point is that in such a situation, the teacher's role is crucial: favouring the engagement of students in the problem, giving the possibility to students to modify the problem, organizing exchanges among students, introducing relevant elements (new question- new games – variants, etc.), allowing the evolution of the methods and favouring the production of proofs, including counter examples. A recurrent question addressed to developers of such situation is how to evaluate the development of proof and proving skills. Some paths: along a dedicate course, evaluation should be done on such problems; one could try to evaluate if students recognize that they have produced a correct proof or not (It may occur that for teachers something is a proof but not for students, and vice versa). Many participants that were both researchers and educators reported that students aiming to become mathematics teachers express negative beliefs and attitudes toward proof. So, a crucial issue, once one considers that it is important to develop situations aiming to foster proof and proving competencies, is to identify what in-service and pre-service teachers' experiences should be in order to be able to face the educational challenge of allowing students to develop proof and proving skills. The three papers of the second session of this theme contributed to this issue. A first path is to allow them to experience the relevance and the fecundity of such situations; however, there is often a gap between knowledge and experience in teacher training and in teaching practice. This raises the question of how to make the teacher become a cultural agent in school (self-reflection is important but something more is necessary). Margo Kondratieva proposed to a group of *in-service primary and secondary teachers* to work together on the same problem and to discuss *the multiple proofs* that were provided. Ruthmae Sears presented *a case study on the enactment of proof tasks in high school geometry.* Rolf Biehler and Leander Kempen advocated the importance for teachers of recognizing generic proofs. The main question is how you know that something is generic; what is it about it that makes it generic, is complicated. Different people see different things in examples. Students face two main difficulties: seeing the generality in the specific examples and communicating this generality. A challenge for the teacher is to decide if the students recognize the generality (even if students are writing the correct algebraic formula we cannot be sure that they are aware of the generality).

THEORETICAL PERSPECTIVES

In CERME 8, six papers were discussed in this section. Theoretical perspectives have been presented in the working group on proof for a long time. In CERME 7, we discussed the use, evolution, elaboration or integration of theoretical constructs introduced at the previous CERME (e.g., cognitive unity; Toulmin's model, transparency background) and of new theoretical frames (e.g., Habermas' model for rational behaviour in proving).

Francesca Morselli's paper is in line with this perspective, crossing the theoretical framework developed by the Genova team around Paolo Boero with the cycle of Algebra in order to *approach algebraic proof at lower secondary school level* through *developing and testing an analytical toolkit*. The author enlightens the necessity for students of a goal-oriented approach for algebraic transformation, in reference to Habermas' rationality.

Following a long tradition in mathematics education consisting to question what we can learn from (brilliant) mathematicians, Gila Hanna introduced a new concept developed by Thimoty Grovers*: the width of proof;* this is also the case with Annie and John Selden's paper which introduced the concepts of *persistence and self-efficacy in proof construction*, contrasting their availability between students and mathematicians and their role in the success in proving. The discussion during the two sessions of this theme focused on the following question addressed to the six presenters: *what are the more prominent aspects of your theoretical approach that should interest the research community involved in proof and proving in an educational perspective?*

The following points emerged from the discussion: the importance of conceptual (non formal) proof; the importance of what is memorized: concepts, mathematical knowledge and methods (mathematicians draw on a repertoire of tools when they proof and to get a repertoire, one needs to collect different proofs); the importance of long term research for solving a problem. Some potentially useful actions for the problem-solving part were identified: exploring – reworking – validating; in this respect proof control occurs at a meta level, and as it has been already pointed since a long time, communication and intrinsic properties of a proof are different things.

David Reid stated that proving is done by human beings and claimed that understanding the origins of the human ability to prove helps us to identify important research questions related to proof and proving. He also considered that necessity is not a property of deduction, but a feeling people have when reasoning deductively; he hypothesized that experience and education can change this feeling. Some participants argued that deduction is inherent to our cultural construction. Markus Ruppert focused *on analogical reasoning*, while the paper from Andreas Lorange and Reinert Rinvold focused on multimodal proof in arithmetic, assuming that other modalities than the written symbolic modality can be used in proofs, allowing visualization of generality. During the discussion, many questions emerged and were not fully addressed: What is the basic nature of human thinking? When and why do people think deductively? How people use analogical reasoning? Which are the relationships between deduction and causality? Some questions concern the connection between language and deductive reasoning, while others refer to the relationship between language and level of abstraction.

CONCLUSION AND PERSPECTIVES

Regarding conclusions, we tried to shape what emerged from the discussion all along the sessions by focusing on some striking features.

First of all, epistemology, as well historical as contemporaneous one, appeared to play an important role all along the sessions, closely related with didactical issues: to understand proof and proving process*;* to consider the role of logic and language; to choose suitable problems when designing situations aiming to foster proof and proving competencies, and finally in elaboration of theoretical perspectives. Making more explicit these intertwined relationships between epistemology and didactics for research on proof and proving in mathematics education would be helpful for nurturing discussion and collaboration.

A second point that appeared in the discussion in the session devoted to *the role of logic and language in teaching, learning and analysing proof and proving process* is a kind of discrepancy between the strong interest for this question of the participants as teachers and educators, and some interrogation concerning the interest of conducting research program on this topic. This can constitute a challenge for future CERMEs to provide findings supporting the interest of developing such program through European collaboration.

The third point concerns the interest of going on sharing successful designs aiming to foster proof and proving competencies.

This opens paths for next up-coming meeting in CERME 9.

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[i] Samuele ANTONINI – Nadia AZROU – Rolf BIEHLER – Paolo BOERO – Faiza CHELLOUGUI – Renaud CHORLAY – Jenny Christine CRAMER – Viviane DURAND-GUERRIER – Patrick GIBEL – Denise GRENIER – Gila HANNA – Leander KEMPEN – Margo KONDRATIEVA – Maria Alessandra MARIOTTI – John MASON – Zoé MESNIL – Simon MODESTE – Francesca MORSELLI – Eva MULLER-HILL – David REID – Reinert A. RINVOLD – Markus RUPPERT – Ruthmae SEARS – Annie SELDEN – John SELDEN – Chrissavgi TRIANTAFFILLOU

[ii] List of the papers by themes

Epistemological and didactical issues

Renaud Chorlay (France) The making of a proof-chain: epistemological and didactical perspectives

Judith Njomgang Ngansop and Viviane Durand-Guerrier (Cameroon and France) 0, 999… = 1. An equality questionning the relationships between truth and validity

The role of logic and language

Jenny Christine Cramer (Germany) Possible language barriers in processes of mathematical reasoning

Zoe Mesnil (France) New objectives for the notions of logic teaching in high school in France: complex request for teachers

Christavgi Triantafillou (Greece) The nature of argumentation in school texts in different contexts

Nadia Azrou (Algeria) Proof in Algebra at the university level: analysis of students' difficulties

Faiza Chellougui and Rahim Kouki (Tunisia) Use of formalism in mathematical activity - case study: the concept of continuity in higher education

Eva Mueller Hill (Germany) The epistemic status of formalizable proof and formalizability as a meta-discursive rule

Designing activities

Patrick Gibel (France) Presentation and setting up of a model of analysis of reasoning processes in mathematics lessons in primary school

Denise Grenier (France) Research Situations to learn logic and various types of mathematical reasonings and proofs

Rolf Biehler and Leander Kempen (Germany) Students' use of variables and examples in their transition from generic proof to formal proof

Margo Kondratieva (Canada) Multiple proofs and in-service teachers' training

Ruthmae Sears (USA) A case study of the enactment of proof tasks in high school geometry

Theoretical perspectives

Gila Hanna (Canada) The width of a proof

Francesca Morselli (Italy) Approaching algebraic proof at lower secondary school level: developing and testing an analytical toolkit.

Annie Selden & John Selden (USA) The roles of behavioral schemas, persistence, and self-efficacy in proof construction.

David A Reid (Germany) The biological basis for deductive reasoning

Reinert Rinvold and Andreas Lorange (Norway) Multimodal proof in Arithmetic

Markus Ruppert (Germany) Ways of analogical reasoning - thought processes in an example based learning environment

[iii] Simon Modeste (France) Modelling algorithmic thinking: the fundamental notion of problem

PROOF IN ALGEBRA AT THE UNIVERSITY LEVEL: ANALYSIS OF STUDENTS DIFFICULTIES

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The reported research aims to investigate the difficulties encountered by university mathematics-major students when dealing with proof in Algebra. An exercise dealing with an equivalence relation and a subgroup, with an a-priori difficulty level compatible with the students' preparation in Algebra, assessed the students' abilities to use their definitions in some new situations and to construct simple proof steps based on mastery of concepts and their symbolic representations. According to performed analyses, students' difficulties depend on conceptual, logical and meta-mathematical factors. Some pedagogical implications are derived.

Keywords: proof, university, formalisation, logic, algebra.

I- INTRODUCTION

In the Algerian mathematical curricula, proofs become a regular activity only at the university level. After the last university reform, the time for courses and exercises sessions has been shortened for the same subjects in comparison with the past years. This change resulted in difficulties for teachers on how to teach the same material in a shorter time. In mathematical courses for engineer students, many teachers eliminated proofs from both courses and exercises to devote the 'short' time left only to definitions and statements of main theorems and results. For mathematics-major students, this was considered to be not convenient by many teachers, so they kept dealing with proofs during the exercise activities.

The teaching of logic subjects has been for years a subject of disagreement among teachers, educators and researchers (for a survey, see Durand-Guerrier &al, 2012); it comes back to discussion in Algeria these past few years, after eliminating logic and even the conscious treatment of mathematical symbols from high school programs. At present, at the university, elements of logic are taught only by some teachers. The question comes over and over again, how can we teach elements of logic so that they can help understanding formal statements from a syntactic point of view but also, and mainly and basically, from a semantic point of view. Anyway, even if this is not taught in logic lessons, it must be dealt with somewhere. The activity of proving seems to me good means for this. When students deal with proof, they must use definitions and results, construct formal steps with arguments and link all this in the final product of the proof. Weaknesses and pitfalls in doing so may be an effect of how mathematical concepts and mathematics proof are taught.

This study is intended to investigate mathematics-major students' difficulties in proving in Algebra and what are they related to in the teaching of mathematics, logic and proof. In order to perform this investigation a theoretical toolkit was derived from different sources and an exam exercise (at the end of the second Algebra course) was arranged and used, whose contents were an equivalence relation and a subgroup.

II- BACKGROUND AND THEORETICAL TOOLKIT

Mathematics education literature offers a wide spectrum of positions concerning: the relationships between the proving process, including the exploration phases, and proof as a product (as examples, see Hanna, 2000 and Pedemonte, 2007), and between argumentation and proof (with several different positions: as examples, see Duval, 1991 and Boero, Douek, Ferrari, 2008); what is relevant in the teaching and learning of proof and proving, in particular the role of logic and logical skills in proving (see Durand Guerrier, 2008 and Tanguay, 2007); classifications of students behaviours (Harel & Sowder, 1998) and interpretations of difficulties met by them in proving (as examples, see Dreyfus, 1999, Weber, 2001 and Selden&Selden, 2003). Some contributions on the above issues, in particular on those concerning formal and semantic aspects of proof, come also from mathematicians reflecting on their activities (see Thurston, 1994).

According to the aim of my study, I have considered some theoretical tools and positions that can provide different, sometimes alternative lenses to deal with data collected in my investigation. I have chosen:

- Vergnaud's theory of conceptual fields (Vergnaud, 1991), to account for Algebra as a conceptual field and to deal with the problem of meaning in Algebra. We may recall that according to Vergnaud the mastery of concepts and conceptual fields depends on the mastery of reference situations, operational invariants and linguistic representations. In particular, Vergnaud points out the deep difference between: knowing definitions of concepts, on one side; and dealing with problem situations that need to use properties ("operational invariants") and representations of concepts, and to make reference to previously experienced "reference situations" (a crucial component of transfer), on the other.

- The importance of the conceptual (in Vergnaud's sense) mastery of (meta-mathematical) knowledge about what a theorem and a proof are, as a component of that "culture of theorems", which should be passed on to students in order to promote their awareness of the "rules of the game" in proving activities (see Boero, Douek, Morselli, Pedemonte, 2010).

- Duval's epistemological and cognitive distinction between argumentation and proof as formal derivation (Duval, 1991), and the (partly) opposite positions presented in some more recent papers (see for instance Boero, Douek & Ferrari, 2008), which make also reference to Thurston's position (Thurston, 1994) on the crucial role of semantic aspects in the proving process and in the checking and communication of proofs.

- The crucial role of logic and logical skills in proving (Durand-Guerrier, 2008; Tanguay, 2007). In our educational perspective, we follow Durand-Guerrier & al. (2012, p. 370) concerning what we mean here by Logic:

> (...) logic as the discipline that deals with both the semantic and syntactic aspects of the organization of mathematical discourse with the aim of deducing results that follow necessarily from a set of premises. When we refer to logic as a subject, we mainly restrict ourselves to the mathematical uses of the words *and*, *or*, *not*, and *if-then* (the basis for "propositional logic"), especially in statements that involve variables, as well as *for-all*, and *there-exists* (the extension to "predicate logic")

The resulting theoretical toolkit is not homogeneous; this choice depends on the exploratory character of this study and on the need of testing different lenses to interpret students' behaviours and difficulties.

III-METHODOLOGY

A-priori Analysis

The experiment is about an exercise (presented below) that has been chosen by me as a teacher of twenty math-major students at the second year university level. It was an exercise of an exam of the first semester (2011). The course (Algebra II) is about algebraic structures, it is taught during two sessions per week, one course session and one exercise session; each session lasts one hour and a half. The students were supposed to have been provided with the principal prerequesites in the previous year, in the Algebra I course (set theory, applications: surjections, injections; group theory and a part of linear algebra). However, according to them, many of those concepts were not well mastered because of the little time devoted for each chapter; moreover dealing with proof was not among their regular tasks within the exercises proposed to them in Algebra I. I wanted, by choosing to assess students' preparation during an exam, to get the best possible results from them (students usually do prepare well to the exams) and I believe that working individually might reveal some errors that cannot appear when working in groups. I also wanted to know where (content, skills, ways of thinking) the difficulties with proof were situated - the answer was not clear for me during my semester teaching, especially when dealing with proofs (most exercises were about proofs). And the answer is crucial if we want to improve our teaching in order to help students overcome proof difficulties. Another issue to be dealt with was about the obstacle inherent in the mastery of formalisms and the relation with the teaching of logic. In mathematics courses, students show more and more difficulties in dealing with symbols (as concerns both syntax and semantics). Many teachers argue in favour of the elimination of teaching logic (in its technical aspects) because it is useless; many others insist on it as an initiation to mathematical language in such a way that students may interiorize some rules that can help them in understanding concepts, definitions, theorems, that intervene in proof production. The toolkit arranged for this study and presented in the previous section was expected to help tackling the above problems.

I have chosen questions with high formal-symbolic features to get the students writing formally (which is normal at the university level and even the objective of some chapters of the courses for mathematics-major students) but also to check what reasoning resulted in correct formal proofs and what kinds of difficulties were met by students. Productive reasoning, I assume, is strongly based on concept understanding and argumentation skills. My aim was to investigate the movement, made during the proving process, between the concepts, the formal transcription and the argumentation. Within such a wide exploratory study I have chosen to present in this paper only data and analyses concerning students' difficulties in proving.

Let G be a non commutative multiplicative group.
Let R be a relation defined in G by: $x\ R\ y \Leftrightarrow \exists\ g \in G$ such that $y=gx$.
1-Prove that R is an equivalence relation.
Let $x \in G$, we define G_x by: $G_x=\{g$ such that $gx=x\}$, prove that :
2- G_x is a subgroup of G.
3-The application defined in G by $f_g(x)=gxg^{-1}$ transforms a subgroup into a subgroup.

According to what students were supposed to have learned in their past algebra courses and exercises, the proofs should be as follows.

1- Let's prove that R is reflexive, symmetric and transitive.
Let $x \in G$, let's prove that xRx. Let's find a g in G that verifies $gx=x$, for $g=1$, we have $x=1x$. Hence $\forall x \in G$, xRx. That is R is a reflexive relation.
Let be $x,y \in G$ such that xRy, let's prove that yRx. We have xRy i.e $\exists\ g \in$ G such that $y=gx$, as G is a group, then $x=g^{-1}y$ i.e yRx. Hence R is a symmetric relation.
Let be $x,y,z \in G$ such that xRy et yRz, let's prove that xRz. We have xRy and yRz i.e $\exists\ g \in G$, $y=gx$ and $\exists\ g' \in$ G, $z=g'y$, by replacing y, we get $z=g'gx=g''x$ and $g'' \in G$ i.e xRz. Hence R is a transitive relation.

2-Let's prove that G_x is a subgroup of G.
Let's prove that $e \in G_x$, we have $x=ex$, i.e $e \in G_x$ which means that G_x is not empty.
Let's prove that if $g \in G_x$ then its inverse $g^{-1} \in G_x$. Let $g \in G_x$ i.e $gx=x$, multiplying by g^{-1} we get $x=g^{-1}x$ that is $g^{-1} \in G_x$.
Let's prove that G_x is close under multiplication of G. Let g_1, $g_2 \in G_x$, let's prove that $g_1\ g_2 \in G_x$.
We have $g_1\ x=x$ and $g_2\ x=x$, by replacing x in the first equality (or the second), we get $g_1g_2x=x$ which means that $g_1\ g_2 \in G_x$.. Hence G_x is a subgroup of G.

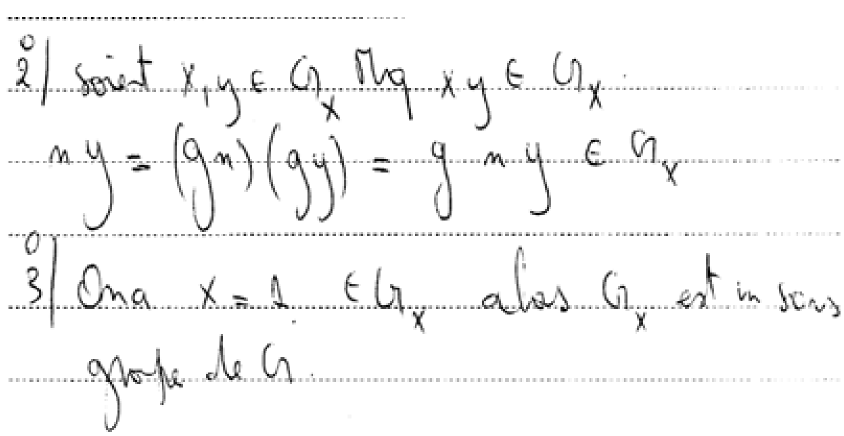
When a-priori evaluating the task, according to the exercises given to the students in both Algebra I and Algebra II courses, we arrived at the following conclusions (partly put into question by the analysis of students' performances - see Global Analysis):

- It is an easy exercise as the questions are classical and most exercises of equivalence relations and groups given to the students had the same questions.
- The mathematical content engaged in the proof is supposed to have been taught in the previous course (algebra I): groups, applications, direct image of a set by an application.
- The proofs of the statements are a direct application of definitions; there are no tricks or unusual techniques that may cause students to be stuck.
- The students are expected to show that they are able: to understand and use the symbolic language (quantifiers, implication); to make the difference between a variable and a parameter; but especially to be able to use a definition in a new situation, which (according to Vergnaud) needs to master the concept beyond its definition and may allow to build the proof steps through its properties.

IV- ANALYSIS OF STUDENTS' DIFFICULTIES

I have chosen three individual productions, which will represent the situation of the great majority of students and the most common difficulties met by them.

Production 1



2°/ soient $x, y \in G_x$ tq $xy \in G_x$

$xy = (gx)(gy) = g\,xy \in G_x$

3°/ On a $x = 1 \in G_x$ alors G_x est un sous groupe de G.

This production deals with question 2 of the exercise.

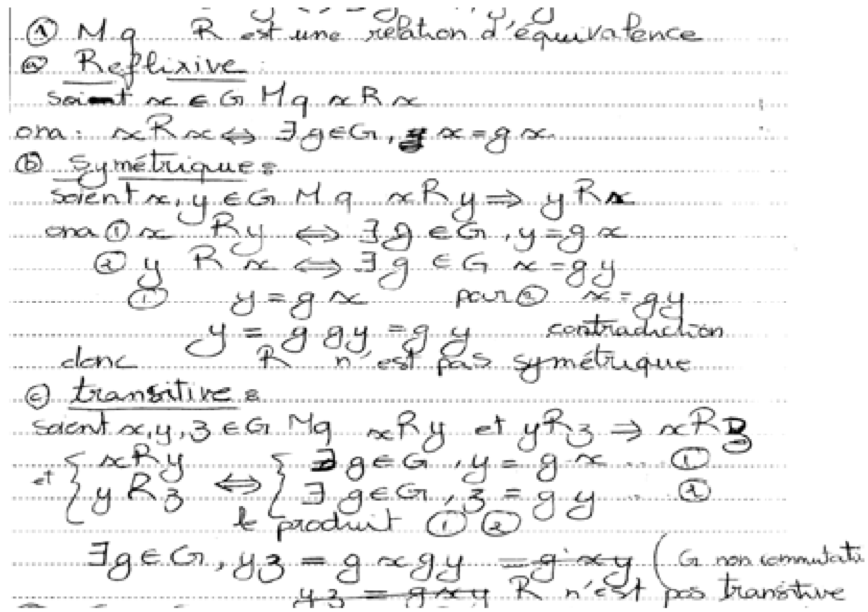
The student satisfies only two conditions among three for a subgroup, failing to prove that G_x contains the inverses of its elements. Despite of this, he finishes his proof by deducing the required result that is G_x is a subgroup; this might indicate how his conceptual (meta-) knowledge about proof is weak.

He starts by showing that G_x is closed for multiplication; he writes the definition as it is given in courses and mathematics books (with x and y) without adapting it to the exercise notations where the elements of G are denoted by g. Designing by x the elements of G is not convenient at all as x has already been used for the definition of G_x. The student shows a lack at the operatory level of transfer, probably related to an insufficient mastery of the concept of group (in the sense of Vergnaud).

In the second line, when trying to prove that the product of any two elements of G_x lies in it, the student replaces x by gx and y by gy; this shows that the elements of G_x are not clear for the student, and he is not able to identify the role of g. Here, the lack concerning the mastery of the conceptual content of the definition is clear. The last step (in line 2) is not justified by the student. I think that, being in an impasse, he wrote the last result just to finish, without being able to derive it from the previous steps. Again the lack of knowledge, of what a proof is, seems evident here.

In the third line, the lack at the operatory level of transfer is shown again by using *1* for *x*.

Production 2


1 Mq R est une relation d'équivalence
a Reflixive
Soient $x \in G$ Mq xRx
on a : $xRx \Leftrightarrow \exists g \in G, x = gx$
b Symétrique
Soient $x, y \in G$ Mq $xRy \Rightarrow yRx$
on a 1 $xRy \Leftrightarrow \exists g \in G, y = gx$
2 $yRx \Leftrightarrow \exists g \in G\ x = gy$
1 $y = gx$ par 2 $x = gy$
$y = ggy = gy$ contradiction
donc R n'est pas symétrique
c transitive
Soient $x, y, z \in G$ Mq xRy et $yRz \Rightarrow xRz$
$\left\{\begin{array}{l} xRy \\ \text{et } yRz \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \exists g \in G, y = gx \ \ldots 1 \\ \exists g \in G, z = gy \ \ldots 2 \end{array}\right.$
le produit 1 2
$\exists g \in G, yz = gxgy$ ~~$= gxy$~~ (G non commutati
~~$yz = gxy$~~ R n'est pas transitive

In this production, we deal with question 1 of the exercise.
We can see how the student proves the opposite of the required result.

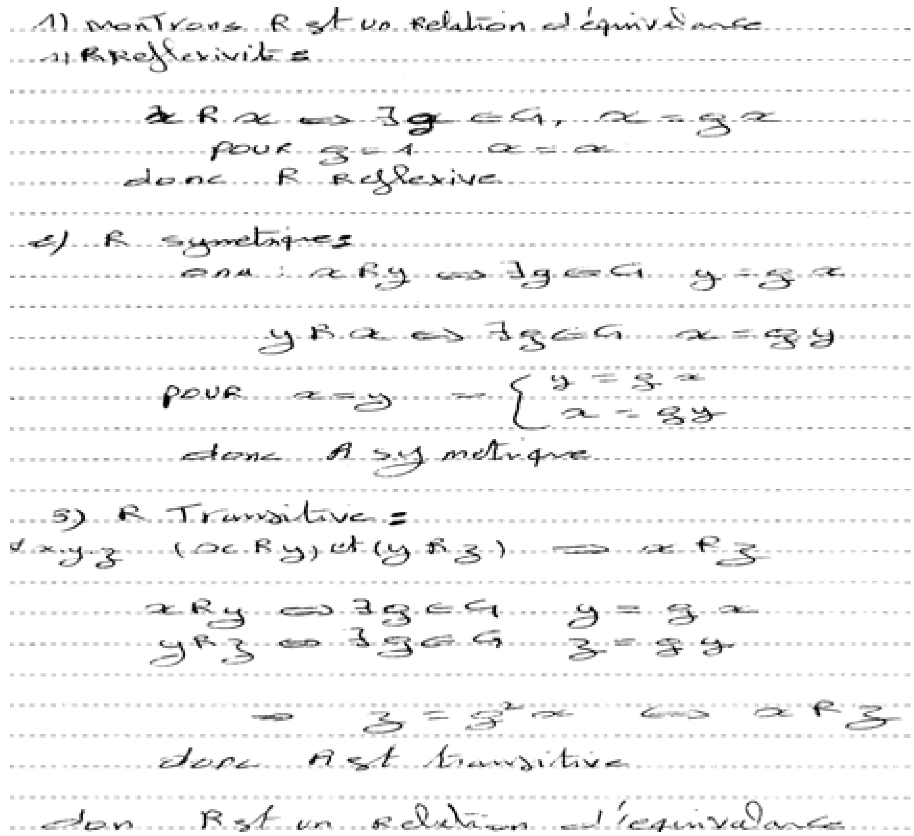
At the first step (reflexivity), the student writes correctly the definition of reflexivity and what is to be shown, but then adding nothing, shows that he is not able to adapt it to the definition given in the exercise. Lack at the operatory level of transfer, which probably depends on insufficient mastery of the concept (in Vergnaud's sense), is clear here. At the second step (symmetry), the definitions and what is to be proved are also written correctly. The student starts proving by developing both the hypothesis (*xRy*) and the result (*yRx*) of the implication according to the given definition. Then what is supposed to be proved is used as means to make the proof. Even though, the student uses the same notation for *g* for both definitions, this doesn't show whether it is the same or not. When arriving to an impasse (*y*=*gy*), the student declares that it is a contradiction and deduces the opposite result. It seems to him the 'legal' way to get away. An important question arises here. Why he does not put into question his proof instead of questioning the required result? Why does he trust his reasoning more than the text of the exercise? I think that the student is not able to see other possibilities in his proof than what he could write; if this interpretation is correct, available data show how the student's logic abilities of understanding (semantic sense) and dealing with formal statements are very limited.

At the third step, the student begins as before, by showing that his definitions are very well memorised. Then the product of the two equations is given in order to prove *xRz*.

One cannot say for sure if *g* is considered to be the same or not. The existential quantifier, at the last line, indicates that *g* is the same, but in what is crossed out (*g'*), we can tell that it is not the same. So, the student is lost again (on the logical ground, in the sense of Durand-Guerrier & al, 2012, p. 670) and gets away as before.

We can also see that this student is far from having conceptual (meta-) knowledge about what a mathematical proof is from a semantic point of view.

Production 3



1) Montrons R est un relation d'équivalence
1) Reflexivité :
$xRx \Leftrightarrow \exists g \in G, x = gx$
pour $g = 1$ $x = x$
donc R reflexive
2) R symetriques
on a : $xRy \Leftrightarrow \exists g \in G$ $y = gx$
$yRx \Leftrightarrow \exists g \in G$ $x = gy$
pour $x = y \Rightarrow \begin{cases} y = gx \\ x = gy \end{cases}$
donc R symetrique
3) R Transitive :
$\forall x, y, z$ (xRy) et $(yRz) \Rightarrow xRz$
$xRy \Leftrightarrow \exists g \in G$ $y = gx$
$yRz \Leftrightarrow \exists g \in G$ $z = gy$
$\Rightarrow z = g^2 x \Leftrightarrow xRz$
donc R est transitive
donc R est un relation d'équivalence

This production deals with the question 1 (equivalence relation).
The student tries to prove that the three conditions are satisfied. He starts good, at the reflexivity step, by replacing *g* by *1* to prove that *xRx*. But then he fails later, by considering the same *g*. This leads necessarily to an absurd situation (from *y=gx* we want to get *x=gy*); to get away from this situation, a small and simple trick seems to be the key: putting *x=y,* without realising that this is in contradiction with the symmetry definition that holds for all elements. We can interpret this as due to a lack of logical-semantic mastery of the symbolic language.

At the third and last step, the student considers *g* to be the same as he gets ($z=g^2x$) when doing the product. But this is still a trap, because as *g* is the same, g^2 wouldn't work to deduce the last result, but despite of this, the student concludes by (xRz).

This behaviour on one side recalls the ritual proof scheme of Harel & Sowder (1998), on the other puts into question again the logical-semantic mastery of the symbolic language of the exercise.

Global analysis

The tasks proposed in the exercise supposed to be easy (as they belong to past courses activities) turned out to be complicated and some of them even impossible to deal with by many students, regarding to the level of their competencies revealed by the analysis (see below). Among twenty students, no one could completely solve the first part of the exercise (equivalence relation), only two could give a correct proof of some properties of the subgroup (question 2) and no one dealt with the third question (the direct image of a subgroup). Three main problems are evident in the productions: lack of transfer at the operatory level (depending on lack of mastery of the concept at state, in Vergnaud's sense), lack of logical mastery of symbolically- stated definitions and inference steps (see Durand-Guerrier et al, 2012, p. 670), and lack of (meta-) knowledge about what a proof is in mathematics. The definitions are memorized only in a formal and very superficial way which is far from sufficient when adapting them to any other new situation and even logical mastery of their symbolic presentation is lacking. Moreover, available data confirm that students face strong difficulties in proofs involving multiple quantifiers (e. g. Chellougui, F., 2009). Still concerning proof, there is evidence that several students do not know what they are expected to do in a mathematical proof, how to use definitions that they are provided with, and what are the meaning of the hypothesis and the thesis as elements related to the proving process. Most of them do nothing or meaningless operations and then deduce the final result. And some students seem not to know what "doing mathematics" means in general!

The undergraduate students tested are in their second university year and first year in mathematics as a discipline (math-majors). For these future teachers, it's clear that dealing with a proof seems to be new and strange. The formal-symbolic language is considered as a meaningless copying of definitions. Making proofs shows well how it is difficult, even impossible, to deal with a concept by considering only the formal transcription without mastering the meaning. The semantic side of a proof seems also nonexistent; it is thought that a proof should be a sequence of formal steps done by any technique, provided that it looks at the end, somehow, like the final result (cf. "ritual proof scheme" in Harel & Sowder, 1998). The students show difficulty with proof dealing with one single concept, even if they manage the beginning and some of the steps of the proof; but they are not able to start a proof that deals with many concepts (question 3: sets, direct image of a set by an application and subgroups).

V- CONCLUSION

The analysis shows a disconnection between formal transcription and conceptual mastering of the content of definitions. Nowadays the teaching of mathematics is more focused on formal-symbolic level rather than on the semantic level of mathematics content; but I believe it is necessary to teach definitions associated with all their conceptual content (in Vergnaud's sense: reference situations, operational invariants, linguistic representations) as well as their role and their usage in proving. The productions show that the students are not used to continually switching back and forth the meaning of the definitions and the meaning of the formal symbols. This activity is crucial in proving and without it; the students cannot be able to learn how to make proofs. This might be taught partially in logic (in the sense of Durand-Guerrier & al., 2012, p. 370) and this can help in the future mathematics activity.
We have also seen that the absence of meta-knowledge about the proof in mathematics prevents constructing arguments that result in a proof. This may be a consequence of a teaching process that uses proofs as means to establish theorems and results and not as an object of an activity (as is done with multiplication, continuity, integration etc) to be taught, used and mastered by students. Even when teachers make a proof at the board, most of them write formally the main steps and fail 'to write' the arguments justifying the inference steps, in order to show all that lies behind. I think as far as this is not revealed and shown clearly, the students cannot be aware of how the proving process works, which is mainly based on argumentation sustained by a deductive chain of propositions and inference steps related to their full meaning (cf. Thurston, 1994).

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STUDENTS` USE OF VARIABLES AND EXAMPLES IN THEIR TRANSITION FROM GENERIC PROOF TO FORMAL PROOF

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First-year students are supposed to be able to handle the deductive, axiomatic system of mathematics, to learn the formal symbolic language and to master different methods of proving. In this paper, we report on our findings from a redesigned bridging course lecture for preservice teachers, in which the students were asked to construct generic proofs to complete their transition to formal proof, using their mathematical knowledge from school. The students` first assignment was collected and their handling and use of examples, generic proofs, formal proofs and variables were analysed.

Keywords Generic proof, example, variable, formal proof

INTRODUCTION

The University of Paderborn offers a lecture specifically designed to help students to deal with higher mathematics. This course, "Introduction to the culture of mathematics", serves as a bridging course and was held for the first time in 2011/2012 as a requirement for the first year secondary (non grammar schools) preservice teachers, who had passed their German "Abitur" before. The course contents comprise logic, proof method, principle of induction, functions and sequences. Since the problems of the students with higher mathematics, especially proofs, are well known, the lecture`s main focus was on argumentation, refutation and proving. Yet, the mathematic was not presented in an axiomatic deductive system. On the contrary, the mathematics of the university were connected with the mathematics learned at school. In the context of proving, generic proofs were presented as a valid argumentation method for lower school grades, as a special tool for grasping the main idea of a proof and as a point of departure for formal proofs. So the generic proof was thought of as a didactic tool for enabling students to find the general argument and to fulfill a transition to formal proof by keeping the main idea and in addition using variables. However, after the correction of the first assignment, the students` work showed problems in their understanding of generic proofs and their use of variables.

Our goal in the current study is twofold: to investigate the students` problems with generic proofs and their use of examples in the proving process of statements and to document the obstacles in their use of variables as placeholders for concrete numbers in the field of elementary number theory.

THEORETICAL FRAMEWORK AND RELATED RESEARCH

Since Balacheff (1988) identified the generic proof as one of four main types in the cognitive development of proof, research on generic proofs (or generic examples) as a didactic tool for learning to prove has increased throughout the years (e.g. Leron & Zaslavsky, 2009; Mason & Pimm, 1984; Rowland, 1998 and 2002). But still Rowland is right, when he states: *[...] I am saying that the potential of the generic example as a didactic tool is virtually unrecognized and unexploited in the teaching of number theory, and I am urging a change in this state of affairs* (2002, 157). Also, the potential for teaching and learning to prove has not been exhausted yet: Firstly, generic proofs are said to be useful in order to convince students of the truth of a statement and *they enable students to engage with the main ideas of the complete proof in an intuitive and familiar context, temporarily suspending the formidable issues of full generality, formalism and symbolism* (Leron & Zaslavsky, 2009, 2-56). So secondly, they might pave the way for the transition to formal proof and help the students to handle variables and the structure of proof (e.g. Padberg, 1997). But in the transition from generic proof to formal proof two problematic areas arise: students` difficulty to generalize from the particular, to recognize and to take into account the generic quality of a generic example (Mason & Pimm, 1984; Nardi, 2008; Selden, 2012) and to adhere successfully to the mathematical language.

In this context the use of variables plays a subordinate role as they are a formal tool in the service of generalization. The research on students` use of variables has shown their difficulties with the concept of variables (e.g. Akgün & Özdemir, 2006; Cooper, Williams & Baturo, 1999; Ely & Adams, 2012; Philipp, 1992; Trigueros & Ursini, 2003). Although variables play various different roles in mathematics which cause various difficulties (Epp, 2011; Schoenfeld & Arcavi, 1988), their concept is rarely discussed in courses at university level. As Akgün and Özdemir (2006) explored in their case study, most students consider a variable in the context of equation as one specific number, even if it is used as a general number. In addition Trigueros and Ursini (2003) argued, that first-year undergraduates cannot distinguish between a variable as a specific unknown and a variable as a general number and that they have serious difficulties when variables are related to each other. They conclude: *Students` understanding of the concept of variable lacks the flexibility that is expected at this educational level (2003, 18).*

RESEARCH QUESTIONS

Our central research questions concern students` use of examples and variables in their transition from generic proof to formal proof:

1. How do first-year students argue when they are asked to construct “generic proofs”?
2. Do they also use the general argument found in the generic case in their formal proof?

3. What characterizes students` use of variables when formulating a formal statement and a formal proof?

THE LEARNING SEQUENCE

Before the students had to submit their first assignment, two lectures and one tutorial were given. In the first lecture, the aims of the first section ("Discovery and Proof in Arithmetic") were named: Getting to know the process of discovering and proving and distinguishing between verifications of a statement with concrete examples, with generic proofs and with formal proofs including variables. A research process was initiated by the question: "Someone claims: The sum of three consecutive natural numbers is always odd. Is this correct?". The statement was tested with some examples which led to the conjecture that the sum is always three times the mean number. As verification and explanation, a generic proof was presented (see table 1) and discussed by the students until the following statement was arrived at: "In this example, we are performing operations with concrete numbers, which are also possible with all (natural) numbers. Thus, this argumentation differs from our previous examples. It is a "generic proof", which includes a general argument. So here we have got a verification for the statement and an explanation, why the sum is always three times the mean number." Thus, "generic proof" was introduced by the lecturer and not invented by the students themselves.

Then, the general argument was used in the following formal proof (see table 2).

$$
\begin{aligned}
1+2+3 &= (2-1)+2+(2+1)\\
&= 2+2+2\\
&= 3\cdot 2\\
10+11+12 &= (11-1)+11+(11+1)\\
&= 11+11+11\\
&= 3\cdot 11
\end{aligned}
$$

TABLE 1: The generic example

$$
\begin{aligned}
k\in\mathbb{N}:\ (k-1)+k+(k+1) &= k+k+k\\
&= 3\cdot k
\end{aligned}
$$

TABLE 2: The following formal proof

After this, the "research process" in the lecture continued until the following surprising conjecture was fount: „The sum of k consecutive natural numbers is divisible by k, if and only if k is an odd number". The sum of k consecutive numbers with initial number n was defined as $S_{n,k}$ and finally the statement was formally proven.

In the tutorial groups, most of the time was needed to practice the representation of odd and even numbers by variables ($2n$ and $2n$-1, n ϵ N), because it was needed in the first assignment, and the difference between an implication and a biconditional. In the following task in the tutorial group, the students had to use the representation of odd numbers as $2n$-1 to prove that a certain product is even. In the second task a proof by contradiction was needed, which one could obtain by using the formal representation of an even number.

TASK

The participants in the course were supposed to solve problems as a weekly assignment in order to get the permission to participate in the final test. The assignments of 64 students from four tutorial groups were scanned and their solutions for the following task were analyzed in an exploratory way to investigate the acceptance of the generic proof in a proof-oriented course and the students` use of variables. The task was as follows:

> *Prove the following statement with a generic proof and a formal proof. Before starting the formal proof, formulate the statement mathematically:*
> *The sum of an odd natural number and its double is always odd.*

TASK ANALYSIS AND EXPECTED SOLUTIONS

The generic proof

First, a generic proof consists of operations within concrete examples that can be generalized. Moreover, one has to find a generic argumentation, why the assumption is true in these specific examples. Afterwards one has to explain why this argumentation also fits all possible cases.

The generic proof (1):

$$1 + 2 \cdot 1 = 3 \cdot 1 = 3 \qquad 3 + 2 \cdot 3 = 3 \cdot 3 = 9 \qquad 5 + 2 \cdot 5 = 3 \cdot 5 = 15$$

Comparing the equations, one can recognize that the result must always be three times the initial number. Since three times an odd number is always odd, the result is an odd number.

The generic proof (2):

$$1 + 2 \cdot 1 = 1 + 2 = 3 \qquad 3 + 2 \cdot 3 = 3 + 6 = 9 \qquad 5 + 2 \cdot 5 = 5 + 10 = 15$$

Comparing the equations one can recognize that the second sum will always contain an odd and an even addend, because two times an odd number is always even. Since the sum of an odd and an even number is always odd, the result must be an odd number.

Formulating the statement mathematically

Formulate the statement mathematically: version (1):

$Let\ a\ \in N\ be\ an\ odd\ number. Then\ the\ sum\ a + 2a\ is\ also\ an\ odd\ number.$

Here the representation of an odd and an even number is not used, however, the use of one variable is necessary. This solution is in line with the expected level of knowledge of the students and the socio-mathematical norms created in the lecture.

Formulate the statement mathematically: version (2):
$For\ all\ n \in N\ there\ exists\ an\ m \ \in N\ with$:
$(2n-1)+2\cdot(2n-1)=2m-1.$

This is a more advanced solution, in which the use of two variables is necessary. Since n represents the first odd number, a second variable m is required. This statement also explicitly contains a universal statement and an existential quantifier.

The formal proof

Formal proof (1) - following the generic proof (1) and the statement (1):
$Let\ a\ be\ an\ odd\ number. Then\ a+2a=3a.$
$Since\ three\ times\ an\ odd\ number\ is\ always\ odd\ (*) the\ statement\ is\ proven.$

In this proof one can transfer the argumentation of the generic proof directly to the formal proof. The letter is used as a generalized number, as a generic element of a set of values. For the implication (*) we have to consider two possibilities: Either one can argue that the statement (*) is well-known in the sense of self-evident and true, which would be in line with the provided norms of the lecture, or one has to prove it, since it has not been proven before. In order to do this, one can argue: “3a = (a + a) + a”, “The sum of two odd numbers, a + a, is always even” and “The sum of an even and an odd number is always odd”.

Formal proof (2) - following the statement (2):
$For\ all\ n \ \in N: (2n-1)+2(2n-1)=6n-3=2(3n-1)-1=2m-1;$
$where\ m := 3n-1 \ \in N.$

Here the existential statement has to be shown for all $n \in N$.

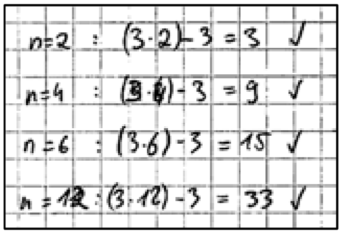
RESULTS

The students` use of variables and their solutions for the generic proof, the formal proof and the formulation of the statement were categorized. But due to the size of this paper, we will just describe the categories for the generic proof and the formal proof in detail.

1. Types of students` proofs given as generic proofs

Students’ proofs were classified into four different categories:

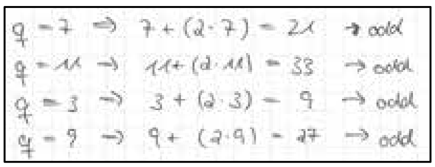
E0 The “generic proof” contains examples, which do not fit to the statement.



$n=2$: $(3\cdot 2)-3 = 3$ ✓
$n=4$: $(3\cdot 4)-3 = 9$ ✓
$n=6$: $(3\cdot 6)-3 = 15$ ✓
$n=12$: $(3\cdot 12)-3 = 33$ ✓

TABLE 3: a student proof, which belongs to the category E0

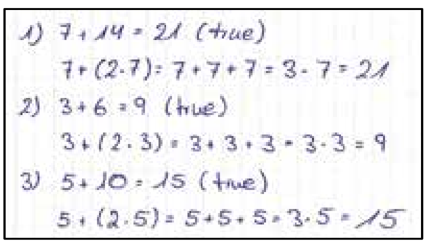
E1 The “generic proof” is just a verification by several examples without presenting the examples as generic. (These purely concrete examples are lacking explanations, further ideas or conclusions.)



$q=7 \rightarrow 7+(2\cdot 7)=21$ → odd
$q=11 \rightarrow 11+(2\cdot 11)=33$ → odd
$q=3 \rightarrow 3+(2\cdot 3)=9$ → odd
$q=9 \rightarrow 9+(2\cdot 9)=27$ → odd

TABLE 4: a student proof, which belongs to the category E1

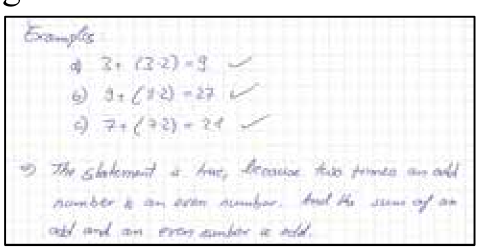
G1 The examples are presented as generic, but no further explanation is given.



1) $7+14=21$ (true)
$7+(2\cdot 7)=7+7+7=3\cdot 7=21$
2) $3+6=9$ (true)
$3+(2\cdot 3)=3+3+3=3\cdot 3=9$
3) $5+10=15$ (true)
$5+(2\cdot 5)=5+5+5=3\cdot 5=15$

TABLE 5: a student proof, which belongs to the category G1

G2 The generic proof contains operations and ideas, which are named and generalized. (Here, the students identify different findings from operations, generalize them and use their findings in their argumentation process.)



Examples
a) $3+(3\cdot 2)=9$ ✓
b) $9+(9\cdot 2)=27$ ✓
c) $7+(7\cdot 2)=21$ ✓
⇒ The statement is true, because two times an odd number is an even number. And the sum of an odd and an even number is odd.

TABLE 6: a student proof, which belongs to the category G2

The frequencies of the categories are: E0: 3 (5.6 %), E1: 36 (67.9 %), G1: 8 (15.1 %) and G2: 6 (11.3 %).

So 39 Students (73.5 %) only presented examples in their "generic proofs" without connecting them with any argumentation to verify the statement generally (E0 + E1). Obviously, they have not understood the fundamental difference between a generic proof and verification by some examples. Out of the 14 students, who presented their examples as generic (G2 + G1), only six built an argumentation upon these in order to prove the statement in the generic proof (G1). In these six solutions, where the students succeeded in constructing the generic proof, the generic proof (1) was used two times and the generic proof (2) four times (see section "task analysis and expected solutions"). Four of these students did not use algebraic operations, but they argued verbally with the correct arguments.

2. Generic proof and formal proof

When the formal proof was successfully constructed, 18 students used formal proof (1), whereas none of the students used formal proof (2) (see section "task analysis and expected solutions").

Eleven students, out of the 14 belonging to category G1 and G2, tried to construct the formal proof and eight of these were using the same argumentation in the formal proof and in their previous generic proof.

3. Formulating the statement mathematically

34 out of the 64 students tried to formulate the statement mathematically including variables. Here the version (1) was used 21 times and the version (2) 6 times. 7 students used a mixed form of these. All of the students` statements included formal mistakes.

Moreover in version (2), the hidden existential statement was not made explicit at all and only one student explicitly mentioned the universal statement in the conjecture.

4. Types of formal proofs and formal mistakes with variables

Students' solutions of the formal proof were classified into four different categories:

P1: The reasoning in the formal proof is logical and correct.

P2: The reasoning process contains gaps and/ or statements are used that are not true in general.

P3: The reasoning does not contain any argumentation.

P4: Miscellaneous

(One student tried to prove a wrong statement. His solution is placed into the fourth category "Miscellaneous".)

In most of the solutions belonging to category P2, the students did not give an explicit argument, why the term $3a$ or $3(2n-1)$ represents an odd number. We were expecting at least the argument "$3a$ (or $3(2n-1)$) is odd because both factors are odd".

The quantitative results are shown in table 7:

Type of solution	Frequency	formally correct	with formal mistakes
P1	29 (51.8 %)	7 (12.5 %)	22 (39.3 %)
P2	18 (32.1 %)	1 (1.8 %)	17 (30.4 %)
P3	8 (14.3 %)	1 (1.8 %)	7 (12.5 %)
P4	1 (1.8 %)	0	1 (1.8 %)
Sum	56 (100 %)	9 (16.1 %)	47 (83.9 %)

TABLE 7: Frequencies of answer types

29 students (51.8 %) succeeded in constructing the formal proof (P1), of which only seven students accomplished this without formal mistakes concerning variables. 18 (32.1 %) students struggled with a correct logical argumentation (P2). In the formal proofs of another 8 students, no argumentation does occur (P3). In total, there are nine formal proofs (16.1 %) constructed formally correct and 47 (83.9 %) containing formal mistakes concerning variables.

In the process of proving, many students also struggle with the distinction between conditions, conjecture and proof. Sometimes the formulation of the statement is immediately followed by algebraic manipulations. Some students use different variables in the conditions and in the following proof.

The most common mistake in using variables is not clarifying to which domain the variable belongs. However this is essential in number theory. Usually numbers have to be whole numbers and not just rational numbers. Also, many students use $2n + 1$ $n \in N$ as representation for an odd number, not considering that 1 cannot be represented hereby.

When dealing with variables the students use a mixed form of everyday language and of the symbolic language of mathematics. Many students enrich the formal mathematical language with everyday language, when they seem to struggle with the formalism of the symbolic language. Letters and word symbols are used simultaneously, often without defining a correct domain. Moreover, one can recognize an inconsistent use of the symbolic language of mathematics. In addition, mathematical symbols like “=” or “ϵ” are often used in wrong ways.

FINAL REMARKS

The results of this case study are not representative, but they shed a new light on the current discussion about the role of generic proofs in the learning process of proving. In our study, only a few students understood the idea of a generic example. This finding corresponds with the literature. It is well-known that preservice elementary teachers have difficulties in distinguishing proof and verification by examples (e.g. Martin & Harel, 1989; Recio & Godino, 2001). Furthermore, they have problems in understanding the explanatory power of generic proofs and in identifying the general idea in the particular case (Rowland, 2002). Yet, those students that recognized a common ground in the concrete examples were able to transfer it to the formal proof.

In this transition to formal proof the students struggled with the formal language of mathematics, the use of the symbols and the meaning and definition of the variables.

Formulating the statement mathematically is another important part in the process of proving. At this point it becomes clear that this process requires more than just a correct argumentation in the formal proof. To prove a statement correctly, the initial statement must be understood with all its hidden universal and/-or existential statements. This is a valid starting point of a proper proof strategy.

One can consider different reasons, why the students in this study had such problems dealing with generic proofs. First of all, more time is needed to teach the idea of a generic proof in contrast to examples and formal proofs. Nevertheless, it was surprising that one lecture plus one tutorial devoted to the topic had such limited success. Also, tutors familiar with this example-oriented proof are needed to support the (didactical) ideas. Generic proofs had not been a topic in their previous mathematical lectures they had attended. Since it is well known that first-year students struggle with formal proofs and mathematical language in general, one has to be careful in using both examples and proofs in an argumentation process and one still has to consider the barrier that the formal mathematical language presents.

For the subsequent version of the lecture – held in the winter term 2012/13 - , we redesigned the content including known misconceptions on generic proving and discussions about the generic power of the four categories named above. Further, we trained tutors in handling and explaining generic proofs and integrated new tasks for the students. In this run we obtained much better results concerning generic proofs (E0: 4%, E1: 28%, G1: 26 % und G2: 42%). These results will be discussed in detail in another paper.

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USE OF FORMALISM IN MATHEMATICAL ACTIVITY CASE STUDY: THE CONCEPT OF CONTINUITY IN HIGHER EDUCATION

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In this paper, we consider logical-mathematical formalism in mathematical statements. We examine formalism regarding the notion of continuity in higher education. The choice of this concept is based on the fact that this concept involves a large number of related variables and that its logical structure nevertheless not too complex because all quantifiers are at the top of the form (Chellougui, 2009).
First we present the didactic transposition of the continuity from knowledge learned to knowledge taught. In a second step, we consider the definition of continuity as presented in various mathematics textbooks for first year university science students.

INTRODUCTION

In Tunisian secondary mathematics education, according to official instructions, logical symbols (logical connectors: $\Rightarrow$, $\Leftrightarrow$..., and quantifiers: $\forall$, $\exists$) are not introduced, and mathematical statements (theorems, definitions) are generally expressed in natural language (Chellougui, 2003). However, from the beginning of the first year of university, scientific formalized statements are used without a specific introduction to symbolism or to the relationships between statements in natural language and formalized statements. This widespread use of formalized language at university is motivated by the supposed superiority in terms of operating statements fully or partially formalized. However, for many students, formalism seems to be an obstacle to mathematical work and therefore to conceptualization (Quine 1970). Thus, in mathematical activity in the first year of university, we identify problems of interpretation of logical-mathematical vocabulary or gap of operating order to students, specifically difficulties in manipulation of complex statements with multiple quantifications. Generally, these issues are not addressed in common textbooks. We adopt the assumption that they reflect ordinary mathematical practice of mathematics teachers (Durand-Guerrier, 2003).

To illustrate this, we chose the concept of continuity of functions which is studied in high school and again in college. For this concept, an understanding of how a definition in natural language can be expressed in form of a formal definition is required. Consider, for example, the definition of continuity of a function at any point proposed by Schwartz (1991):

(For all $a \in \mathbb{R}$) (for all $\varepsilon>0$) (there exist $\eta>0$ such that) (for all $x \in \mathbb{R}$ such that $|x-a| \leq \eta$), we have: $|f(x)-f(a)| \leq \varepsilon$. This sentence can be written formally. [...]:
$(\forall a \in \mathbb{R})\ (\forall \varepsilon>0)\ (\exists \eta>0)\ (\forall x \in \mathbb{R})\ [(|x-a| \leq \eta) \Rightarrow |f(x)-f(a)| \leq \varepsilon]$. (p.20)

Here, there are three universal quantifiers and an existential quantifier in third position. The scope of these quantifiers on implication is between parentheses. This study is concerned with the definition of the term as found in several mathematical textbooks of first year scientific university. Prior to this study, we present some elements of didactic transposition of the concept of continuity from expert knowledge to knowledge to be taught.

SOME ELEMENTS OF DIDACTIC TRANSPOSITION

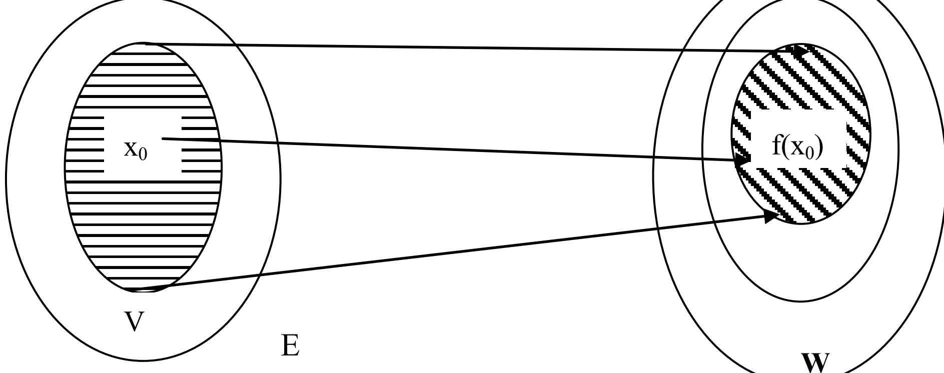
The first comprehensive outline of the Didactic Transposition Theory was developed in Chevallard (1991). This theory aims to produce a scientific analysis of didactic systems and is based on the assumption that the mathematical knowledge set up as a teaching object (‘savoir enseigné’), in an institutionalized educational system, normally has a preexistence, which is called “expert knowledge” (‘savoir savant’).

Some objects of mathematical expert knowledge are defined as direct teaching objects and constructed in the didactic system (by definition or construction), i.e. mathematical notions, such as for example addition, the circle, or second order differential equations with constant coefficients. However, there are other knowledge objects, termed para-mathematical notions, useful in mathematical activities but often not set up as teaching objects per se but pre-constructed, such as the notions of parameter, equation, or proof (Klisinska, 2009).

We try to analyze the question of the use of para-mathematical logical symbolism in mathematical activity.

We start with the general definition of continuity of a function at a point made in a Dictionary of Mathematics (Bouvier and al., 1979):

Application continuous at a point. – An application f of a topological space E into a topological space F is continuous at $x_0 \in E$ if for all neighbourhoods W of $f(x_0)$ in F, there exists neighbourhood V of x_0 in E whose image f(V) is contained in W. This is the mathematical expression of the sentence "$f(x)$ tends to $f(x_0)$ as x tends to x_0". In the case where E and F are metric spaces, f is continuous at $x_0 \in E$ if any $\varepsilon>0$, there exists $\alpha>0$ such that $d(x,x_0) \leq \alpha$ leads $d(f(x),f(x_0)) \leq \varepsilon$. (p.192)



(p.192)

This definition is given in topological spaces and neighbourhoods; it is then translated into metric spaces. It is formulated in a mixed language, an association of natural language and mathematical symbols. The authors do not use any logical symbol but illustrate this definition by charts. They were inspired by the approach of Bourbaki (1971), which itself offers the following definition:

> Definition 1. - We say that f of a topological space X into a topological space X is continuous at a point $x_0 \in X$ if any neighbourhood V' of $f(x_0)$ in X', there exists a neighbourhood V of x_0 in x such that $x \in V$ implies $f(x) \in V'$. (P.I.8)

In comparison, Durand-Guerrier and Arsac (2003) use logical symbolism and even suggest a fully formal definition to define a uniformly continuous application:

> Application uniformly continuous. – An application f of a metric space E with values in a metric space F is uniformly continuous if for all $\varepsilon>0$, there exists $\eta(\varepsilon)>0$ such that $d(x,y)\leq\eta(\varepsilon)$ implies $d(f(x),f(y))\leq\varepsilon$. This can be symbolized by:
> $(\forall\varepsilon>0)\ (\exists\eta(\varepsilon)>0)\ (\forall(x,y)\in E^2)\ (d(x,y)\leq\eta(\varepsilon) \Rightarrow d(f(x),f(y))\leq\varepsilon)$. (p.193)

In this definition the quantification is complete and dependency relationship between η and ε appears.

To give a definition of continuity in $\mathbb{R}$, it is possible to get a definition equivalent with intervals since the intervals form a basis of neighbourhood in set $\mathbb{R}$.

Haug (2000) suggests the following definition:

> **Definition 6.a** Let *E* be a set of real numbers. Let *f* be an application of *E* into IR, *a* and *b* are real numbers.
> We say that *b* is a *limit* of *f* if: any open interval J centred on *b*, there exists an open interval I centred on *a* such that $f(E\cap I)\subset J$. (p.107)

He then notes:

> Show that if we replace the above equation by the following equation we obtain an equivalent definition.
> $\forall_{\mathbb{R}_+^*}\ \varepsilon,\ \exists_{\mathbb{R}_+^*}\ \eta,\ \forall_E\ t, |a-t| \leq \eta \Rightarrow |b-f(t)| \leq \varepsilon$. (p.107)

Here, we note that there are no parentheses to express the scope of quantifiers on implication, which is a fairly common practice among authors of textbooks and mathematicians. Some students may not be aware of the difficulties related scope of quantifiers and use of parentheses. It is indeed important to understand the effects on the meaning of a statement and interpret in a mathematical theory, when changing the order of quantifiers (Dubinsky & Yiparaki, 2000).

Further, we read:

> **Definition 6.b**– Let *E* be a set of real, let *a* be an element of *E*, let *f* an application from *E* to $\mathbb{R}$.
> We say that *f* is *continuous* at *a* if *f* has a limit at *a*. (p.111)

The definition of a limit of a function is followed by a geometric representation based primarily on intervals of IR, which can enlighten the definition; there is also an explanation of the passage from the "neighbourhood" point of view to the "distance" point of view, which is not very common in other textbooks studied. Use of intervals favours the didactic transposition of the definition with neighbourhoods; it reduces the number of quantifiers and may in some cases be easier to handle (Chellougui, 2009).

STUDY OF SOME TEXTBOOKS: CONCEPT OF CONTINUITY IN THE KNOWLEDGE TO BE TAUGHT

Below we present a study of certain textbooks for students in their first year of university. We chose these textbooks because they were used by students and teachers of the Faculty of Sciences of Bizerte. The textbooks we have examined are: Chambadal and Ovaert, *Mathematics*, 1966; Arnaudies and Fraysse, 1988; Schwartz, 1991; Guégand and Gavini, 1995.

Our focus is on the different types of language used. Three phenomena emerged from this study which will be analyzed below.

Implication versus bounded quantification

In a mathematics textbook (Chambadal & Ovaert, 1966), we find the definition of a limit of a function at a point followed by that of continuity.

> **Definition 19. - Limit of a function at a point**. - Let *f* be a function defined on a part A of $\overline{\mathbb{R}}$ and x_0 an accumulation point of A. We say that *f* has a limit at x_0 if it has a limit when x tends to x_0 remaining in $P=A-\{x_0\}$. We also say more briefly that *f* has a limit when x tends to x_0. (pp.394-395)[1]

This definition, first given in natural language, is then made more explicit and formalized:

> -If x_0 is finite, so that f tends towards l when x tends to x_0, it is necessary and sufficient that:
> $\forall\varepsilon\in \mathbb{R}_+^*, \exists\ \eta\in \mathbb{R}_+^* : \forall x\in A \cap ([x_0-\eta, x_0+\eta]-\{x_0\}), |f(x) - l| \leq \varepsilon$ (1)[2]
> what writes:
> $\forall\varepsilon\in \mathbb{R}_+^*, \exists\ \eta\in \mathbb{R}_+^* : \forall x\in A, |x-x_0| \leq \eta \text{ and } x\neq x_0 \Rightarrow |f(x) - l| \leq \varepsilon.$ (2) (pp.395-396)

A beginning reader might ask where the implication that appears in (2) comes.

In another analysis textbook of first year science (Guégand & Gavini, 1995), we can learn:

> **2.1 Definition**: Let I be an interval of $\mathbb{R}$, $a \in$ I and $f : I\to \mathbb{R}$

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[1] Our translation

[2] To facilitate the study of different official statements, we have numbered (1) to (12) in the illustrations taken from different books or in our own analysis

We say that f is **continuous at *a*** if and only if f admits a limit in a equal to f(a).
Otherwise we say that f is discontinuous at *a*.
Let us clarify this definition:
f is continuous at *a*

$\Leftrightarrow \quad \forall\varepsilon>0, \exists\ \alpha>0, \forall x\in I, |x-a|<\alpha \Rightarrow |f(x)-f(a)|<\varepsilon$ (3)

$\Leftrightarrow \quad \forall\varepsilon>0, \exists\ \alpha>0, \forall x\in I, |x-a|\leq\alpha \Rightarrow |f(x)-f(a)|\leq\varepsilon$ (4)

$\Leftrightarrow \quad \forall\varepsilon>0, \exists\ \alpha>0, \forall x\in I\cap[a-\alpha,a+\alpha], |f(x)-f(a)|\leq\varepsilon.$ (5) (p.108)[3]

Here the beginning reader might wonder why the implication has disappeared in (5). This game appearance/disappearance of implication is related to mathematical practice of bounded quantification. This type of quantification is present in mathematics, but absent in the predicate calculus. For example, the mathematical writing:
$\forall x\in A\ F(x)$ is reflected in the predicate calculus by: $\forall x(x\in A \Rightarrow F(x))$.

In fact bounded quantification hides the implication the domain of quantification is limited to the elements that satisfy the antecedent of the conditional statement, which removes the implication. The practice of bounded quantification is present in textbooks, several authors provide a formulation of the definition of continuity without the conditional, often without providing explanations allowing students to be able to restore the conditional by changing the domain of quantification (e.g. Chambadal & Ovaert 1966, Guégand and Gavini 1995). Durand-Guerrier (2003) considers that being able to restore or remove correctly the conditional according with the domain of quantification in such cases contribute to the understanding of implication.

Implication versus conjunction

In another textbook of mathematics (Arnaudies & Fraysse, 1988), the authors begin defining continuity at a point with the neighbourhoods in a metric space:

Definition III.4.1 - Let A be a part of $\mathbb{R}$ and f: A $\rightarrow$ $\mathbb{R}$ a function. We say that f is continuous at $a\in A$ if and only if for every neighbourhood W of $f(a)$ there exists a neighbourhood V of a such that $f(V\cap A)\subset W$.
We say that f is discontinuous at point $a\in A$ if and only if it is not continuous at that point.
The function f is continuous if and only if it is continuous at every point of A. (p.108)[4]

After they present the classic definition using a mixed language:

[...] we obtain in particular the following definitions of continuity of f at a equivalent to the definition III.4.1:
(I) For all real $\varepsilon>0$, there exists a real $\eta>0$ such that
$(x\in A \text{ and } |x-a|\leq\eta)\Rightarrow(|f(x)-f(a)|\leq\varepsilon)$
(II) For all real $\varepsilon>0$, there exists a real $\eta>0$ such that
$(x\in A \text{ and } |x-a|<\eta)\Rightarrow(|f(x)-f(a)|<\varepsilon)$. (p.108)[5]

[3] Our translation
[4] Our translation

We can ask: why is there "and" in the antecedent of the implication?
We rather expect the following entry:
For all real $\varepsilon>0$ there exists a real $\eta>0$ such that for all $x\in A$ $(|x-a|<\eta)\Rightarrow(|f(x)-f(a)|<\varepsilon$
or:

$$\forall\ \varepsilon>0\ \ \exists\ \eta>0\ \ \forall x\in A\ \ (|x-a|<\eta)\Rightarrow(|f(x)-f(a)|<\varepsilon) \qquad (6)$$

Another formulation of the statement (II) presented in the textbook, using logical symbols for the quantification of each variable ε and η gives:

$$\forall\ \varepsilon>0\ \ \exists\ \eta>0\ \ (x\in A \text{ and } |x-a|<\eta)\Rightarrow(|f(x)-f(a)|<\varepsilon) \qquad (7)$$

Are statements (6) and (7) equivalent?

As noted above, removing the bounded quantification on the variable x to the statement (6), we obtain:

$$\forall\varepsilon>0\ \exists\eta>0\ \forall x\ [x\in A\Rightarrow(|x-a|<\eta\Rightarrow|f(x)-f(a)|<\varepsilon)].$$

Our question refers back to logical equivalence between:

$$(x\in A\ \wedge\ |x-a|<\eta)\Rightarrow(|f(x)-f(a)|<\varepsilon) \qquad (8)$$

and

$$[x\in A\Rightarrow(|x-a|<\eta\Rightarrow|f(x)-f(a)|<\varepsilon)] \qquad (9)$$

Considering only variable x, statements (8) and (9) are respectively of the form:
$(p(x)\wedge q(x))\Rightarrow r(x)$ and $[p(x)\Rightarrow(q(x)\Rightarrow r(x))]$.

It is known that in the propositional calculus, the following equivalence:
$[p\Rightarrow(q\Rightarrow r)]\Leftrightarrow[(p\wedge q)\Rightarrow r]$ is a tautology.

By extension, in predicate calculus, the two following equivalences are universally valid:

$$[p(x)\Rightarrow(q(x)\Rightarrow r(x))]\Leftrightarrow[(p(x)\wedge q(x))\Rightarrow r(x)]$$

$$\forall x\ [p(x)\Rightarrow(q(x)\Rightarrow r(x))]\Leftrightarrow[(p(x)\wedge q(x))\Rightarrow r(x)].$$

So using logical arguments, we prove the equivalence between the two statements: (6) and (7). We summarize this equivalence in the following table with a justification in logical syntax by translating writing mathematics into predicate calculus (Kouki, 2008):

<table>
<tr><th>Writing mathematical</th><th>Predicates calculus</th></tr>
<tr><td>$\forall x\in A\ (F(x)\Rightarrow G(x))$</td><td rowspan="2">$\forall x\ [x\in A\Rightarrow(F(x)\Rightarrow G(x))]\equiv$
$\forall x\ [(x\in A\wedge F(x))\Rightarrow G(x)]$</td></tr>
<tr><td>$\forall x\ [(x\in A \text{ and } F(x))\Rightarrow G(x)]$</td></tr>
</table>

[5] Out translation

Negation of formalized statements

a- It is generally recognized that formal logic is to facilitate the transition to negation (e.g. Guégand and Gavini 1995).

> *Those quantified expressions* can be easily denied. So to formulate that f does not tend to L (real) at a (real), we have (negation of the definition)
> $\exists\varepsilon>0, \forall\eta>0, \exists x\in U, |x-a|<\eta \quad |f(x)-L|\geq\varepsilon$ (10) (p.108)[6]

In the textbook *those quantified expressions* mean:

> (1') $\forall\varepsilon>0, \exists\eta>0, \forall x\in U, |x-a|<\eta \Rightarrow |f(x)-L|<\varepsilon$
> (2') $\forall\varepsilon>0, \exists\eta>0, \forall x\in U, |x-a|\leq\eta \Rightarrow |f(x)-L|\leq\varepsilon$
> (3') $\forall\varepsilon>0, \exists\alpha>0, \forall x\in U\cap[a-\alpha, a+\alpha], |f(x)-L|\leq\varepsilon$. (p.107)

In statement (10), we find a blank between the two inequalities:
($|x-a|<\eta$) and ($|f(x)-L|\geq\varepsilon$). How to fill this blank? Is it the negation of (1'), (2') or (3')?
The negation of (1') is: $\exists\varepsilon>0, \forall\eta>0, \exists x\in U, |x-a|<\eta \wedge |f(x)-L|\geq\varepsilon$
The negation of (2') is: $\exists\varepsilon>0, \forall\eta>0, \exists x\in U, |x-a|\leq\eta \wedge |f(x)-L|>\varepsilon$
The negation of (3') is: $\exists\varepsilon>0, \forall\eta>0, \exists x\in U\cap[a-\alpha, a+\alpha] \quad |f(x)-L|>\varepsilon$

Expression (10) doesn't correspond to any of the previous negations. We hypothesize that the authors aimed to negate statement (1) and did not want to use logical symbolism for the conjunction "and". If they did not put the word "and" in this blank to keep all words in formal language and to avoid using a mixed language. This is based on the fact that in mathematics we very rarely use the logical symbol "$\wedge$" which represents conjunction. This is reflected in the textbooks studied. Indeed, the logical symbol of the conjunction is identified just once among these textbooks: in the first paragraph of the part entitled: *Set Theory*, of Laurent Schwartz (1991). The author, in this section, uses the definition of continuity of a real function to illustrate the rules for handling negation: inversion of two types of quantification, negation of implication. For example:

> For express now that the function is continuous at every point, we write:
> […]
> "$(\forall a\in\mathbb{R})\ (\forall\varepsilon>0)\ (\exists\eta>0)\ (\forall x\in\mathbb{R})\ [(|x-a|\leq\eta) \Rightarrow |f(x)-f(a)|\leq\varepsilon]$ (p.20)

Further, we read:

> For example, the property for a function f of a real variable not is everywhere continuous, that is to say to be discontinuous at least one point is expressed by the single line:
> $(\exists a\in\mathbb{R})\ (\exists\varepsilon>0)\ (\forall\eta>0)\ (\exists x\in\mathbb{R})\ [(|x-a|\leq\eta) \wedge (|f(x)-f(a)|>\varepsilon)]$ (p.21)

This second statement is obtained by recursively applying transformation rules:
$\neg(\forall x\ Fx) \equiv \exists x\ \neg Fx$ (11)
$\neg(\exists x\ Fx) \equiv \forall x\ \neg Fx$ (12)

---

[6] Our translation

These rules allow change progressively the quantifiers and finally to focus on the negation of the open sentence into brackets. The application of the general rule: $\neg(p(x)\Rightarrow q(x)) \equiv p(x)\wedge\neg q(x)$ to this open sentence provides the conjunction of an atomic formula and of the negation of an another atomic formula. So, finally we need to focus only on the negation of the atomic formula: $|f(x)-f(a)|\leq\varepsilon$. Then, the logical negation symbol disappears by the equivalence between the relation "$>$" and the negation of the relation "$\leq$".

Note that these transformation process of quantifiers only apply if they are heading the formula. Indeed, a quantifier in the antecedent of an implication is not modified by the negation:

$$\neg[(\forall x\ Fx) \Rightarrow G] \equiv (\forall x\ Fx) \wedge (\neg G)$$

b-In another textbook (Arnaudies and Fraysse (1988) mentioned above), following the definition of continuity, the authors define the discontinuity noting:

> The discontinuity of f at a point $a\in$A means:
> (III) There exist $\varepsilon>0$ such that for all $\eta>0$, (x$\in$A and $|x-a|\leq \eta$) $\not\Rightarrow$ ($|f(x)-f(a)|\leq \varepsilon$)
> i.e
> (IV) There existe $\varepsilon>0$ such that for all $\eta>0$ we can fond at least one x in A such that $|x-a|\leq \eta$ and $|f(x)-f(a)|>\varepsilon$. (p.109)[7]

In statement (III), the authors use the symbol $\not\Rightarrow$ that does not conform to the syntax of logic. it is $\not\Rightarrow$. It begs leads to the question: what is negated? Especially since the universal quantification remains implicit: $[(x\in A \text{ and } |x-a|\leq\eta) \not\Rightarrow (|f(x)-f(a)|\leq\varepsilon)]$

The authors answer in statement (IV) using a given vocabulary in a language of action, where we would expect more usage of the logic symbol of the existential quantifier. This shows that what is negated is of course the implicitly universally quantified statement. This point could not be obvious for some students; indeed, some beginners could consider the following statement: $\forall x\ (x\in A \text{ and } |x-a|\leq\eta) \not\Rightarrow (|f(x)-f(a)|\leq\varepsilon)$ which is interpreted by: "None x satisfies the implication", which is not the negation of the definition.

The following negation: $\neg\ [(x\in A\wedge |x-a|\leq\eta) \Rightarrow (|f(x)-f(a)|\leq\varepsilon)]$ does not give rise to the appearance of an existential quantifier.

The negation is obviously on the universal $\neg[\forall x\ (x\in A\wedge |x-a|\leq \eta)\Rightarrow(|f(x)-f(a)|\leq\varepsilon)]$

CONCLUSION

Logical-mathematical formalizations in definitions of continuity, limits and discontinuity are different from one textbook to another. One might think that they reflect the everyday practices of mathematicians. Anyway, the authors of the textbooks

[7] Our translation

we have studied do not provide to students means to overcome the linguistic difficulties raised by the use of formalism, in particular concerning its relationships with natural language, so that it seems that there is an "illusion of transparency of mathematical language".

In some expressions, the presence of bounded quantification is indicated; practice creates a phenomenon of appearance/disappearance of involvement and quantification product entries do not conform to the syntax and logic that generates ambiguity, then the transition formalism is supposed ambiguities of ordinary language (Kouki, 2006). In addition, the use of automatic syntactic rules, not problematized, to construct recursively the negation of a sentence obscures many fundamental questions for operative use of formalism (Durand-Guerrier and al., 2012).

The introduction of logical-mathematical formalism in the learned knowledge aims to introduce a certain level of mathematical rigor in mathematical discourse in order to get rid of ambiguities, implicit assumptions and call to evidence. In the knowledge to be taught, the study showed a wide variety of formulations as well in formal language as in natural or mixed language, for which we have identified and analyzed syntactic difficulties, which are likely to affect student work.

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MATHEMATICS TEXTBOOKS

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THE MAKING OF A PROOF-CHAIN: EPISTEMOLOGICAL AND DIDACTICAL PERSPECTIVES

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We will present an epistemological narrative based on a corpus of historical texts pertaining to the theorem: the sign of the derivative of f determines the variations of f. From a didactical perspective, the main points are: (1) the texts illustrate the role of "local" counter-examples (to use Lakatosian terminology); (2) the various proof-attempts are based on at least two pretty different proof-ideas; (3) even a proper (meaning, both intuitive and formal) understanding of the concepts involved in the statement of the theorem may lead to a faulty proof scheme; (4) it helps understand how long deductive chains emerge and stabilise. On the basis of this narrative, we eventually underline connections with current research works on proof in mathematical analysis, and mention teaching and teacher-training perspectives.

Key words: epistemology, proof analysis, mathematical analysis, AMT.

RATIONALE

In the twentieth century, most tertiary-level textbooks of mathematical analysis prove the following theorem: let *f* be a differentiable real-valued function defined on an interval, if its derivative *f'* is positive, then *f* increases over this interval. Its standard proof is a rather straightforward application of the "mean value theorem"[1] ("*égalité des accroissements finis*" in French, "*Mittelwertsatz*" in German); the proof of which is a rather straightforward application of the "Rolle theorem"[2], which, in turn, depends on the fact that a continuous real-valued function defined on a closed and bounded (i.e. compact) interval has a maximum or a minimum. The latter fact, although quite intuitive, depends on not-so-trivial properties of the set of real numbers (completeness of the metric space, local compactness). Historically speaking, this *deductive chain* can be found in the textbook (Jordan, 1893, p.65-67).

With this example, we can see that the proof of a rather intuitive qualitative fact (namely: if all the tangents point upward, the curve has to move up) requires several layers of sophisticated concepts (differentiability, continuity, properties of the numerical continuum), and a few standard tricks (affine changes of variable). In this paper, we will present some of the proofs given, over the 19th century, either of this

[1] Let *f* be a differentiable real-valued function, defined over some interval $[a,b]$, there exists a value *c* between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. Geometrically speaking: on the arc of curve joining the points $(a,f(a))$ and $(b,f(b))$, there is a point where the tangent is parallel to the chord joining the two endpoints.

[2] Let *f* be a differentiable real-valued function, defined over some interval $[a,b]$, such that $f(a) = f(b)$. There is a value *c* between a and b for which the derivative vanishes.

mathematical fact, or of some key points in its proof; and some instances of critical proof-analysis.

We must stress the fact that this paper is not a work in the history of mathematics, but a work of an *epistemological* nature *based* on a historical corpus. The corpus consists of documents that we selected and translated into English (from the French or German languages). For lack of space, only part of the corpus can be presented here (for a more comprehensive presentation, see (Chorlay, 2012)). Quite a few works in maths education research have focused on similar issues (for recent examples: (Arsac & Durand-Guerrier, 2005), (Barrier, 2009)): our goals are (1) to make this new corpus available to this community of researchers; (2) to compare what this corpus helps document with current research perspectives on proof in mathematical analysis; and (3) to point to potential uses in a teaching or teacher-training context.

LAGRANGE'S PROOF (1806)

Let us quote the beginning of Lagrange's proof

A function which vanishes when the variable vanishes, will, as the variable increases positively, have finite values of the same sign as that of its derived function; or of the opposite sign if the variable increases negatively, as long as the values of the derived function keep the same sign and do not become infinite.

(...) Let us consider the function $f(x+i)$, whose general development is

$$f(x) + if'(x) + \frac{i^2}{2!}f''(x) + \cdots .$$

As we saw in the former lesson, the form of the development may be different for some specific values of x; but we saw that, as long as $f'(x)$ is not infinite, the first two terms of the expansion are exact; and that the other terms will, consequently, contain powers of i greater than the first, so that we shall have

$$f(x+i) = f(x) + i[f'(x) + V],$$

V being a function of x and i, which vanishes when $i = 0$.

So, since V vanishes when i vanishes, it is clear that, should i be made to increase from zero through imperceptible degrees, the value of V would also increase from zero by imperceptible degrees, either positively or negatively, up to a certain point, after which it may decrease; consequently, one will always be able to assign to i a value such that the corresponding value of V – regardless of the sign – is less than any given quantity, and that for lesser values of i, the values of V are also lesser.

Let D be a given quantity, which may be chosen as small as one pleases; one can always assign to i a value so small that the values of V are bounded by the limits D and –D; so, since we have

$$f(x+i) = f(x) + i[f'(x) + V],$$

It follows that the quantity $f(x+i) - f(x)$ will be bound by these two

$$i[f'(x) \pm D].$$ (Lagrange, 1884, p.86-89)

In this passage, we can see that Lagrange also had a proper numerical understanding what the value of the derivative at a given point represents, and that he did interpret limits as relationships of dependence between inequalities. For instance, he rephrased "V being a function of x and i, which vanishes when $i = 0$" as "one will always be able to assign to i a value such that the corresponding value of V – regardless of the sign – is less than any given quantity (...)".

In the part of the proof which we omitted (see (Chorlay, 2012) for a more comprehensive translation), Lagrange applied the above inequalities for x-values of type $x+i$, $x+2i$, ..., $x+(n-1)i$, determined an upper bound for the sum, then passed to the limit. In spite of the fact that the theorem Lagrange set out to prove is correct, and that the proof relied on a correct numerical understanding of the derivative construed as a limit, something does not sound right in the proof: a critical reader may spot hidden uniformity assumptions (namely: uniform derivability), and circular chains of dependent quantities (see (Barrier, 2009) for a detailed analysis of very similar cases, and (Ferraro & Panza, 2012) for recent historical work on Lagrange).

CAUCHY'S PROOF (1823)

Problem. Assuming that the function $y = f(x)$ is continuous relative to x in the neighborhood of specific value $x = x_0$, one asks whether the function increases or decreases as from this value, as the variable itself is made to increase or decrease.

Solution. Let Δx, Δy denote the infinitely small and simultaneous increments of variables x and y. The $\Delta y/\Delta x$ ratio has limit $dy/dx = y'$. It has to be inferred that, for very small numerical values of Δx and for a specific value x_0 of variable x, ratio $\Delta y/\Delta x$ is positive if the corresponding value of y' is positive and finite. (…)

This being settled, let's assume function $y = f(x)$ remains continuous between two given limits $x = x_0$ and $x = X$. If variable x is made to increase by imperceptible degrees from the first limit to the second one, function y shall increase every time its derivative, while being finite, has a positive value. (Cauchy, 1823, p.37)

Unlike Lagrange, Cauchy defined the derivative as a limit; just like Lagrange, he was able to derive proper numerical conclusions from this numerical conception of the derivative. So what makes his argument so different from Lagrange's ? Actually, they do not have the exact same understanding of what the *conclusion* to be reached is; both have implicit definitions[3] of what it is for a function to be increasing, but their definitions do not match exactly. Lagrange's definition is closer to the one we find in today's textbook: a real valued function defined over some interval I is an increasing function if, a and b being any elements of I, $a < b$ implies $f(a) < f(b)$. Lagrange's

[3] We could use the notion of "in-action definitions" (Ouvrier-Buffet, 2011). For a more detailed analysis of Cauchy's proof, see (Chorlay, 2007).

(implicit) definition reads slightly differently, since he compared the values of f at 0 and at any other *given* value.

Cauchy's implicit definition of an increasing function can be rephrased as follows: a real-valued function f defined over some interval I is an increasing function if, a being any element of I, there is a neighbourhood N_a of a such that, for any x in N_a, the order between $f(a)$ and $f(x)$ is the same as that between a and x. Lagrange's definition is global, point-wise, and refers to two (arbitrarily, independently) given points; Cauchy's definition is one in which some local property holds in the neighbourhood of every (arbitrarily) given point. It can be shown – but it takes a little work – that both definitions are actually equivalent from a mathematical viewpoint[4]. However, they differ significantly, both from an epistemological viewpoint (in which, for instance, the difference between local and global properties are put to the fore), and from a cognitive viewpoint (Chorlay, 2007).

The fact that both definitions coincide from a mathematical viewpoint does not imply that proving that the first holds involves the same kind (and amount) of work than proving that the second holds. The information we start with (sign of the derivative) being of the everywhere-local-type, a mere rephrasing of the hypotheses leads to Cauchy's definition of increasing functions, hence to the conclusion. Reaching Lagrange's conclusion involves patching up local pieces of information to reach global conclusions, an endeavour which the modern reader knows to be usually tricky.

BONNET'S PROOF (IN J.-A. SERRET'S TEXTBOOK, 1868)

Theorem I.- *Let f(x) be a function of x which remains continuous for values of x between two given limits, and which, for these values, has a well-determined derivative f'(x). If x_0 and X denote two values of x between these same limits, the following* $\frac{f(X)-f(x_0)}{X-x_0} = f'(x_1)$, *will hold, with x_1 a value between x_0 and X.*

Indeed, the ratio $\frac{f(X)-f(x_0)}{X-x_0}$ has, by hypothesis, a finite value; and, if A denotes this value, we will have (1) $[f(X) - AX] - [f(x_0) - Ax_0] = 0.$

Let $\varphi(x)$ denote the function of x defined by the formula

(2) $$\varphi(x) = [f(x) - Ax] - [f(x_0) - Ax_0],$$

then, from equality (1), $\varphi(x_0) = 0,\ \varphi(X) = 0,$

so that $\varphi(x)$ vanishes for $x = x_0$ and for $x = X$. Let us assume, for instance, that $X > x_0$, and let x increase from x_0 to X; at first, the value of $\varphi(x)$ is zero. If we assume that

[4] One must nevertheless stress the fact that if the Cauchy property holds at one given point $x = a$, it does not imply that the function is increasing in any neighbourhood of a. Consider $x + 10\,x^2 \sin 1/x$ in the neighbourhood of 0.

this function is not everywhere zero, for values of x between x_0 and X, it will have to either begin to increase, thus taking on positive values, or begin to decrease, thus taking on negative values; be it from $x = x_0$, or from some other value of x between x_0 and X. If these values are positive, since $\varphi(x)$ is continuous and vanishes for $x = X$, it is obvious that there will be a value x_1 between x_0 and X such that $\varphi(x_1)$ is greater than or equal to the neighbouring values $\varphi(x_1 - h)$, $\varphi(x_1 + h)$, h being an arbitrarily small quantity.

[Serret then proved that $\varphi'(x_1) = 0$, by a well-known argument; and stressed that this proof idea is Ossian Bonnet's]

(...) Theorem III.- *If the derivative f'(x) of function f(x) remains finite for all the values of x between the limits* x_0*, if X >* x_0*, and if x is made to increase from* x_0 *to X, the function f(x) will increase as long as the derivative f'(x) will not be negative, and it will decrease as long as f'(x) will not be positive.*

Indeed, since x lies between x_0 and X, the ratio $\frac{f(x \pm h) - f(x)}{\pm h}$ has limit $f'(x)$, which is a finite quantity; so it will of the same sign as that of the limit, for values of h between zero and some sufficiently small positive quantity ε. Consequently, for these values of h, the following will hold

$$f(x-h) < f(x) < f(x+h) \quad \text{if } f'(x) \text{ is} > 0,$$

$$\text{and } f(x-h) > f(x) > f(x+h) \quad \text{if } f'(x) \text{ is} < 0.$$

Thus, the function $f(x)$ will increase, as from any value of x for which $f'(x)$ is > 0; and decrease, as from any value of x for which $f'(x)$ is < 0. (Serret, 1900, p.17-22)

In this passage, Serret introduced Bonnet's proof of the mean value theorem, a proof idea which relied on an affine change of variable and the vanishing of the derivative at a local extremum. The existence of the extremum is not proved (at least when one compares with later rewritings of this proof), but made obvious in the *narrative* style which is so typical of the first half of the 19th century.

Strikingly, Serret did not use the mean value theorem to establish the relationship between the sign of f' and the variations of f; he relied on Cauchy's argument, hence on Cauchy's notion of functional variation.

PROOF-ANALYSIS AND REGRESSIVE ANALYSIS

Proof-analysis: the role of uniform convergence

We identified in Lagrange's proof a flaw which can be described in several ways: implicit assumption of uniform differentiability; failure to notice that some variable is dependent on some other, while trying to consider the limit of second while leaving the first fixed; exchange, without due caution, of two limiting processes. The same flaws were common to most proofs in analysis which dealt with the numerical aspect of functions (as opposed to formal aspects) (Dugac 2003) (Chorlay 2012).

At this point, one could choose to focus on texts where the new concepts of uniform convergence/continuity were first expressed with full clarity (Dugac 2003). Instead, we would like to stress the interest of texts which criticized faulty proofs, or spotted hidden lemmas. For now, let us quote just one excerpt from the correspondence between Hoüel and Darboux[5]. In this passage (dated Jan. 1875), Darboux comments on the standard argument he read in the drafts of Hoüel's textbook:

> Here is where I find fault with your reasoning, which no one deems rigorous any more. When setting
>
> $$\frac{f(x+h)-f(x)}{h} - f'(x) = \varepsilon,$$
>
> ε is a function of the two variables x and h which tends to zero when, leaving x fixed, h vanishes. But if x and h vary, as in your proof; even more, if every new subdivision $x_1 - x_0$ generates new ε quantities, I cannot see anything clearly any more, and your proof becomes only seemingly rigorous. (...) You could get out of this predicament in one of two ways, 1. By changing proofs altogether, which I advise you to do. 2. By proving that if a function always admits a derivative between x_0 and x_i, one can find a quantity h such that for all values of x between x_0 and x_i, and all values x_0 and h_1 of h less than some limit value, one has
>
> $$\frac{f(x+h)-f(x)}{h} - f'(x) < \varepsilon,$$
>
> where ε has a value which is fixed but chosen as small as one wishes; which is difficult[6]. (Gispert, 1983, p.99-100)

Regressive analysis: the role of the existence theorem for extrema

A critical mind might object to Bonnet's proof of the mean value theorem that it depends on the existence of a maximum or a minimum, an existence which is implicitly taken for granted. It seems clear that if the function is piece-wise monotonous (as seems to be assumed in the text), it will indeed admit either a local maximum or a local minimum; but a differentiable function needs not be piece-wise monotonous, as the ever useful example $f(x) = x^2 sin\frac{1}{x}$ shows.

In fact, the existence of a maximum can be grounded without piece-wise monotony, or continuous differentiability, as Weierstrass established, for instance in his 1878 lectures on the theory of functions. The following passage has nothing to do *a priori* with calculus. It comes after the construction of the set of real numbers **R** (or, more precisely, $\overline{\mathbf{R}} = [-\infty, +\infty]$) starting from rational numbers. Weierstrass derived the

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[5] This correspondence was published in (Gispert, 1983). It was discussed in (Balacheff, 1987).

[6] The fact that, if the derivative is continuous, then $\frac{f(x+h)-f(x)}{h}$ *does* tend to *f'*(*x*) *uniformly* on every closed and bounded interval was proved, for instance, in the second edition of Jordan's textbook (Jordan, 1893, p.68).

existence theorem for extrema as a consequence of a very general and abstract theorem, proved using nested intervals (we quote the theorem but skip the proof):

> Let a value y correspond to every point $(x_1, \ldots, x_n)$ of some domain; then y is also a variable quantity, hence has a lower and an upper bound; let g denote it. Then, there exists at least one point in the x-domain (that point needs not belong to the defined domain), with the following property: if we consider however small a neighbourhood of that point, and consider the values of y corresponding to that x-domain, then these values of y also have an upper bound, this upper bound being exactly g. Similarly for the lower bound.
>
> (...) One is commonly faced with the question: among the values taken on by some magnitude, is there a maximum or a minimum (maximum or minimum in the absolute sense). Let y be a continuous function of x, $y = f(x)$. Here, x must remain between two given limits a and b. In which circumstances is there a maximum and a minimum for y ? There is an upper bound for y. According to our proposition, there must be some point x_0 in the x-domain such that the upper-bound of the values of y for x between $x_0 - \delta$ and $x_0 + \delta$ is also g. Point x_0 either lies inside $a...b$, or on its border ($x_0 = a$, or $x_0 = b$).
>
> In the first case, $f(x_0)$ is a maximum. Indeed, $f(x_0)$ must be equal to g: for $f(x) - f(x_0)$ can be made as small as we wish, by choosing an adequately small $|x-x_0|$; on the other hand, since x lies between $x_0 - \delta$ and $x_0 + \delta$, $f(x)$ can be chosen arbitrarily close to g; hence $f(x_0) = g$. (If we had $f(x_0) = g+h$, we would have $f(x) - f(x_0) = f(x) - g - h$, and $f(x)$ could not come arbitrarily close to g if h was not 0).
>
> If x_0 coincided with either a or b, then we could only claim that $f(a)$ (resp. $f(b)$ is a maximum if $f(x)$ was continuously at a (resp. b) as well. (Weierstrass, 1988, p.91-92)

DIDACTICAL PERSPECTIVES

Epistemological summary

First, let us attempt to summarize the pretty intricate network of definitions, proof-ideas (or proof-germs (Downs & Mamona-Downs, 2010)), proof-techniques, and proof-analyses displayed in this sample of texts. What follows constitutes an *epistemological narrative*.

At least two *definitions* of what it means for a real-valued function to "increase" can be found in the 19th-century: a point-wise and global definition which can be found in Lagrange; a definition that relies on an everywhere-valid local property, which can be found in Cauchy. If we stick to Cauchy's definition, then the proof of the theorem about the relationship between the sign of *f'* and the variations of *f* is pretty trivial. If we want to reach the Lagrange-style conclusion, then much more work is needed, since one has to start from an everywhere-valid local property (sign of *f'*) and reach a global conclusion.

To reach that conclusion, we saw two very different *proof-ideas*, namely Lagrange's and Bonnet's. In the proof we studied, and in quite a few other parts of his work, Lagrange distanced himself from the formal manipulation of formulae (finite or infinite), and engaged in numerical proof: he relied on the correct numerical understanding of the notion of limit; on this basis, he cautiously built networks of inequalities; he finally endeavoured to ground his reasoning on the determination of upper bounds for the errors in a process of affine approximation. In the first half of the 19th century, many proofs of the most important theorems in function theory were written along this line. Distrust of this proof-scheme spread as mathematicians grew aware of the distinction between point-wise and uniform (continuity, convergence). They spread all the more slowly since the theorems were correct, the building blocks of the proofs showed a proper understanding of the notions at stake, and local counterexamples[7] were hard to find. As Darboux insightfully (but to no avail, as far as Hoüel was concerned) stressed, there were only two ways out of this predicament: either to change proof-germs, or to establish uniformity.

For the theorem on which we chose to focus, an alternative proof became available in the 1860s, which relied on a completely different proof-idea; unlike Lagrange's proof, it did not rely on what the derivative of a function at a point *is* (a limit, which provides some local affine approximation), but on a *property* of the derivative (stated in the mean value theorem). Some elements of Bonnet's proof were later seen as insufficiently grounded, in particular the existence of a minimum or a maximum; in the 1890s, mathematicians such as Jordan used Weierstrass' analysis of the set-theoretic properties of the real line to back up that weaker step in Bonnet's proof.

Didactical issues, and topics to be discussed

As to connections with current work on mathematical proof from a maths education research perspective, we would like to stress several features; we hope this will trigger discussion, possibly collaboration:

* Instead of focusing on *one* proof, the (abridged) corpus we presented and the epistemological narrative we based upon it deal with a *chain of proofs* (or *deductive* chain) which has remained stable since the beginning of the 20th century. The strong deductive nature of the final chain, with it seemingly necessary conceptual connections, makes it difficult to even imagine how it could have emerged progressively. The corpus shows how this emergence is (1) a collective phenomenon (a point emphasized in (Balacheff, 1987, p.148)), (2) gradual. As to the second point, it turns out that the various stages do not make up a linear chain of rigorization steps, in which each tentative proof would be more sea-worthy than the previous one. Indeed, the stages are of a much more varied epistemological nature, and do not make up one single line: really *different* proof ideas were at play; *several* forms of rigorization can be encountered, such as regressive analysis (finding conceptual

[7] Meaning: a counterexample to a step in a proof, not to the theorem itself.

grounds from which to derive something that was previously taken to be obvious or improvable), or conceptual distinction (between point-wise and uniform). From a teaching perspective, we feel the use of such corpuses in teacher-training could help change their *image* of maths (i.e. their view of the *nature* of mathematics). We should also discuss whether or not *meta*-level, epistemological, descriptive terms such as those we used[8] should/could be used in teacher-training.

* Here, the emphasis lies on the *reading* of proofs rather on the writing of proofs. Hence, this corpus and this narrative do not directly address central issues such as the distinction between argumentation and proof, or the question of what it takes to write a formal proof-text on the basis of a well-grounded conjecture or even a proof-idea. We focused on the analysis of proof-texts, in terms of rigour and conceptual content (i.e. concepts at play); an analysis carried out by mathematicians; an analysis which students could be asked to carry out[9].

* In the corpus we presented, we did not lay the emphasis on the use (or misuse, or lack of use) of quantifiers. It would have been possible; it would have been relevant. We focused on conceptual content rather than on deductive rigour, and claim it leads to fruitful questions. For instance, we would like to know to what extent students are able to recognize/identify fundamental theorems or concepts, when they are stated in a slightly unusual form (think of Lagrange's wording of the theorem, or his and Cauchy's use of the notion of derivative-as-a-limit); we would like them to explore whether or not, and in what respect, Lagrange's and Cauchy's views on what "increasing function" means are equivalent.

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[8] Among others: local counter-example, conceptual distinction, regressive analysis, proof-germ, in-action definition; in other texts, we relied heavily on: concept image / concept definition, change of semiotic registers etc. (Chorlay, 2007).

[9] This potential *task* was already mentioned in the conclusion of Robert and Schwartzenberger's survey paper of 1991, among tools to enhance students' proof-writing and proof-understanding skills: "Thirdly, one can suggest instruction based upon the activities of mathematicians themselves, for example through the study of historical mathematical texts. A difficulty here is the barrier of notation and language as well as the extreme difficulty of many concepts when formulated in their original contexts." (Robert & Schwartzenberger, 1991). Although the difficulties mentioned at the end of the quote are usually quite overwhelming, we would like to argue that it is not the case here.

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POSSIBLE LANGUAGE BARRIERS IN PROCESSES OF MATHEMATICAL REASONING

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The fostering of deductive reasoning within mathematical argumentation processes is a demanding task for teachers. Deductive reasoning requires not only awareness of the epistemic value of statements but also of the structural components of mathematical knowledge. The relations between different statements become visible within mathematical arguments, and the truth of propositions is established independent from their current context. The aim of this paper is to show that this requires a certain language register, characterized by context-independency and precision, to which learners in school have different levels of access.

Mathematical reasoning, academic language, migration background

INTRODUCTION

The introduction of reasoning and proof into the NCTM standards 2000 triggered a focus on argumentation and proving in the curriculum in many countries. As mathematical reasoning is closely connected to the exploration of connections between new statements and existing mathematical knowledge, it seems to be a promising way of promoting learning. More reasoning in school thus appears to be a welcome development. Research on problem solving (Lubienski 2000, 2004) and modelling (Leufer and Sertl 2010) has however shown disadvantages for children with a lower socioeconomic background and linked these findings to Bernstein's theory of different access to the language register required in education. Knipping (2012b) points out that so far, there has not been any research on the question whether similar effects can be observed for argumentation.

Limited access to the language register required in school contexts can be seen as a possible explanation of the PISA 2000 results in Germany, which showed huge deficiencies in the performances both of students with a lower socioeconomic background and of children with migration background. In this paper I will first explore the nature of deductive reasoning and its possibilities for the learning of mathematics. Secondly, I will focus on the language required in educational contexts and the possible obstacles it holds for students from a lower socioeconomic background and learners for whom the language of education is not the mother tongue. After that, I will establish a connection between the characteristics of mathematical reasoning and the features of academic language. Finally, I will present my research approach that aims at facilitating the integrated learning of mathematical reasoning and the academic language register.

ARGUMENTATION, PROOF AND DEDUCTIVE REASONING

Much has been said about the "complex, productive and unavoidable" (Boero, 1999) relationship between argumentation and proof. Following Toulmin (1958), proof can be considered as a special form of argument. Toulmin sees arguments as steps from a datum or a set of data to a conclusion, justified by a warrant for which backing may be produced if necessary. Mathematical proofs follow the same structure, ideally relying on axioms or established mathematical knowledge as data, and using rules of logical deduction in order to arrive at a conclusion. In mathematical practice however, Hanna and de Villiers (2012, p.3) state that "a proof is often a series of ideas and insights rather than a sequence of formal steps". In academic mathematics, the scientific community has been negotiating the rules for proving for a long time, and mathematicians generally have an idea about which steps in a proof they may omit (Knipping 2012b).

For school mathematics however, Knipping (2012b, p.4) points out that "there are no previously negotiated criteria for argumentation". These criteria need to be agreed upon by the community in which argumentation takes place. School mathematics cannot take an axiomatic approach to proving, or assume that the students know which types of deduction are acceptable. The role of the social community is crucial for argumentation processes. I follow Knipping's definition, which takes the social community into account and defines argumentation as "a sequence of utterances in which a claim is put forward and reasons are brought forth with the aim to rationally support this claim" (Knipping 2003, p.34, my translation). This definition encompasses proof and deductive reasoning.

As pointed out before, mathematical argumentation can be seen as a continuum, reaching from very informal arguments and inferences based solely on the authority of the speaker to strictly logical proof. These kinds of arguments differ significantly in the way in which they use deductive inferences. Not every argumentation in mathematics contains deductive reasoning. Aberdein (2012) sees mathematical argumentation as consisting of two parts. It is characterised by an underlying inferential structure, which follows strictly logical criteria, and an argumentation, which is the visible part of the argument. The argumentational part seeks to convince others that logical criteria have been obeyed and that the given inferences are valid. In order to account for this two-layered view on mathematical argumentation, Aberdein has suggested a categorization of arguments into different schemes. Arguments in which every step is a deductive transmission from the premises to a conclusion are characterized as A-scheme arguments because of the direct connection between argumentational and inferential structure. Mathematical proof falls into this category, as justifications based on logical deduction are given for each step of the argument. If the connection to the inference structure is less clear, but could theoretically be made explicit in a limited number of deductive steps, Aberdein classifies the argument as B-scheme. This type of argument is based on deductive reasoning; however, intermediate steps may be omitted or references to statements proven elsewhere may be included. In academic mathematics this is often the case in proofs presented in mathematics journals, where some steps are omitted and left for the qualified reader to complete. Deductive reasoning is a prerequisite for A- and B-scheme arguments as it ties the logic of the inferential structure to the visible argumentation. The last category for arguments proposed by Aberdein is the C-scheme. All arguments without direct or indirect reference to the inferential structure are contained in this category. Typical for this category are visual arguments and other informal argumentation techniques which make no use of deductive reasoning.

Mathematical reasoning and learning mathematics

As all of the presented argument types occur in academic mathematics, school mathematics needs to face the question, which kinds of argumentation it wants the students to engage in. C-scheme arguments can be very helpful in conjecturing processes, as they lead to assumptions about possible hypotheses. In order to systematize new mathematical knowledge however, some connection to an inferential structure needs to be established. Arguments that are solely based on informal practice or intuition cannot show what the truth of statements and their relationships depend on. They do not explain why a statement is true. However, Hanna (2000), de Villiers (1990) and others have pointed out the importance of the explanatory potential offered by a proof based on deductive reasoning. The direct link to the inferential structure also simplifies the systematization of new knowledge into the existing internal mathematical knowledge structure.

In recent years, there have been many attempts to make proving more accessible to students. As deductive reasoning is necessary for proving, much about its fostering can be learned from these approaches. Boero, Garuti, Lemut and Mariotti (1996) introduced the concept of cognitive unity between conjecturing and proving. This concept states that proving becomes easier if during the conjecturing phase, students discover arguments that they can later use in the proof. Suitable tasks must be chosen in order to enable the discovery of important arguments. Knipping (2012b) compares the restructuring of arguments found in the conjecturing phase into a deductive chain to a transition from Aberdein's C-scheme to B- or A-scheme arguments.

Another approach for dealing with proofs in school was developed by Hanna and Jahnke (2002). In this approach, the importance of hypotheses for proving is emphasized, and Freudenthal's concept of local organisation plays a major role (Hanna and Jahnke 2002, p.3). At the beginning of a new exercise, the learners are engaged in measuring and experimentation, which leads to a speculation about possible hypotheses. When several hypotheses have been collected, the students are asked to create connections between them and put them into a structure, thereby establishing a local order. Jahnke (2009) emphasizes the importance of inferences for mathematics. With Aberdein's scheme, the established structure between the hypotheses based on the newly created local order can be characterized as B-scheme or A-scheme arguments. Emphasis is not put on the validity of the hypotheses but on the certainty of the inferences. The learners work out a deductive chain that is valid as long as the hypotheses are true.

The specification of starting conditions and the insight that statements are dependent on other statements are characteristics of mathematical reasoning, and especially of deductive reasoning. Reasoning which has its origins in an inferential structure requires awareness of the mathematical background of theorems. Having established the connection of a new statement to existing mathematical structures, further exploration becomes possible. Bikner-Ahsbahs et al. (2011) have pointed out the potential of mathematical reasoning for learning mathematics. Reid (2001) has described how deductive reasoning plays a role in the acceptance of explanations. Connections and links between new knowledge and existing knowledge are forged, and bridges between previously independent islands of knowledge are established by mathematical reasoning. Schoenfeld (1994, p.68) claims that “looking to perceive structure, seeing connections, capturing patterns symbolically, conjecturing and proving, and abstracting and generalizing” are fundamental to mathematics. All of the processes mentioned in the quote are also important in mathematical reasoning. Thus, promoting the ability to reason deductively seems to be a promising path in order to enable students to learn mathematics.

Possible obstacles in the teaching of mathematical reasoning

However, more deductive reasoning in schools may also trigger some unwelcome effects. Lubienski (2000) has pointed out that not all students benefitted equally from the greater emphasis that was put on problem solving in recent years. Her findings showed that while learning was fostered for both students from lower and students from higher socio-economic backgrounds, the open and context-embedded material increased the gap between the two groups. The group of students from a lower socioeconomic background was much slower in its progress, and these students often felt insecure about the acceptability of their arguments. Many of the students uttered the wish for more support by the teacher. Furthermore, the arguments brought forth by these students were often directly linked to the context given in the task, without focussing on the intended mathematical background.

For mathematical modelling, similar problems were observed by Leufer and Sertl (2010). The application of mathematics on realistic problem situations was supposed to increase motivation and bridge the gap between school and real life, especially for students from lower social classes. However, especially these students had problems in solving the given tasks.

A possible explanation for these differences in achievement between students with higher and lower socioeconomic status given by Lubienski and by Leufer and Sertl is based on Bernstein’s sociology of education. According to Bernstein (2003), the social class of the speaker influences the language register he or she is capable of, and likely to be, using. Bernstein distinguishes between restricted and elaborated codes. The different codes are characterized by specific discourse forms and different conversation modes. Language of a restricted code takes place in situations of temporal and spatial proximity. The discourse can be classified as horizontal (Bernstein 1999), is dependent on the immediate context in which it is spoken and can, while coherent in one given context, be illogical across different contexts. Elaborated code, on the other hand, is characterized by its reference to objects that are not necessarily tangible. It makes use of vertical discourse, which is marked by context-independency, coherence, and the ability to abstract from concrete objects. Bernstein (2003, p.109) describes that, while restricted code appears in all social classes, children from are working class background are often limited to this type of language. In contrast to this, children from the middle and higher classes experience the usage of both restricted and elaborated code at home.

The considerations of Bernstein explain general deficiencies of children from a lower socioeconomic background in school, as elaborated code usually is the required language register in the educational context. In order to see which kind of language is demanded in a certain situation, Knipping (2012a) describes Bernstein's approach of necessary recognition and realisation rules. The usage of real-world contexts in mathematical tasks impedes the recognition of the expected vertical discourse. These problems of children from a lower socioeconomic background have not been analysed with a focus on argumentation yet. Mathematical reasoning requires many processes of abstraction and generalization. It can be expected that children with limited access to vertical discourse encounter problems.

ELABORATED CODE AND SECOND LANGUAGE LEARNING

Bernstein's theory of linguistic codes is concerned with children from different socioeconomic backgrounds. Social class, however, is not the only factor for language learning and achievement in school.

The results of PISA 2000 forced the German educational system into becoming aware of the fact that a successful participation in school is closely connected to the personal background of students. Both immigrant children and children with lower socioeconomic status turned out to achieve lower overall results. Heinze et al. (2011) present results from a follow-up investigation of children with migration background using the DEMAT testing material; it was shown that language proficiency in the German language has a higher influence on achievement in mathematics and cognitive performance in general than on reading skills.

If the language used in the educational system is not the speaker's mother tongue, special problems are encountered. In order to account for the specific challenges for second language learners, Cummins (2008) introduced the notions of BICS and CALP in order to distinguish between different levels of language. BICS stands for basic interpersonal communication skills, CALP means cognitive academic language proficiency. Duarte (2011, p.60) pointed to different results showing that the acquisition of BICS can be achieved within two years of being exposed to the new language, CALP abilities usually require at least five years.

In general, academic language shows the characteristics Bernstein pointed out for vertical discourse, whereas everyday speech can be compared to horizontal discourse. Duarte (2011) has given a concise overview on the characteristics of academic language compared to those of everyday speech. The features of the different language levels are listed in Table 1.

Academic language	Everyday speech
Orientation towards written language	Oriented towards spoken language
Abstract, symbolic	Concrete, factual
Context-disembedded	Context-embedded
Generalizing	Specific
Can be technical and domain-specific	Unscientific, general
Linguistically concise	Linguistically diffuse
Precise	Imprecise
Impersonal (uses personal pronouns)	Personal (usually agents are explicit)
High degree of cohesion	Partially unstructured and loose
High lexical density	Low lexical density

Table 1: Main differences between academic language and everyday speech (Duarte, 2011, p. 71, adapted and shortened)

In addition to the obstacles shown in table 1, which are true for the academic register in all languages, Duarte (2011, p.71) lists some features of German academic language. Among these are the use of sophisticated verbs instead of simple verbs with prefixes, adjectival and adverbial attributes, and nominalisations. Gogolin (2009) has introduced the term "Bildungssprache" to account for these special characteristics of German academic language. Referring to Habermas, she defines Bildungssprache as "the language register which enables to gain orientational knowledge using the means of school education" (Gogolin 2011, p.108, my translation).

Children with a migration background in Germany often come from families with a low socioeconomic status and little education (Gogolin 2009, p.267). As shown before, this leads to further disadvantages in the familiarity with elaborated code before entering school. These obstacles are important in all subjects and must be taken into account by teachers.

Language requirements for deductive reasoning

A bigger emphasis on reasoning in the mathematics classroom must take into consideration possible language barriers. Children from all backgrounds are likely to understand language based on basic interpersonal communication skills. On the other hand it is visible in the definition of Bildungssprache by Gogolin given above that academic language abilities are needed in order to gain orientational knowledge in a new context. Thus, fostering CALP should be one aim of education in all subjects, also in mathematics. In addition to this demand, mathematical reasoning has some characteristic features that make academic language not only desirable but also necessary.

Mathematical reasoning takes place *abstracted from concrete situations*. Processes of reasoning in mathematics establish a link between new knowledge and existing knowledge structures. These knowledge structures are internal and show hardly any connection to tangible objects. Furthermore, mathematical reasoning frequently makes use of *generalizing* techniques, especially in the inference rules used in deductions. Another feature of mathematical reasoning is the *precise, coherent and concise form* desired as the outcome of the reasoning process.

All of the named features are also characteristics of the academic language register. From this I conclude that academic language can hardly be avoided in the teaching and learning of reasoning. The required language register can quickly become an obstacle. I am convinced, however, that the close relationship between mathematical reasoning and academic language also offers many learning opportunities.

FIRST INSIGHTS INTO MY RESEARCH WORK

In my research, I am working as a teacher-researcher in a project for children with a migration background whose mother tongue is not German. They come to university once a week to receive support in different subjects in groups of 4-6 learners at no charge. The students come into the project from different schools. From September 2012 until July 2013 I am teaching and researching in two groups of students, one group is in their 9th year in school and the other in their 11th year.

I am collecting data from three different sources. Videotaped interviews at the start of the project, before Christmas and towards the end, combined with a reasoning task, are used as control points for the students' views on and abilities in deductive reasoning. In the first interview I found that in the mathematics classes in the schools of most of the students, hardly any reasoning takes place. For material creation and in order to have a second opinion on developments in the groups, there are weekly consultations with David Reid as an expert on reasoning, which are audiotaped. The third data source is the videotaped material from the lessons. In addition to all that, I keep a research diary in which I keep track of my experiences. The videotaped material will be evaluated at the end of the project, in order to retrace the individual development of mathematical reasoning.

Material development within the project takes place on a weekly basis, constantly taking into consideration the consultations with David Reid and the immediate impressions from the previous lessons. I am developing language sensitive material that creates opportunities for mathematical reasoning. In task creation, I am inspired by the cognitive unity approach developed by Boero et al. (1996) as well as by the approach concerning the local organisation of hypotheses by Hanna and Jahnke (2002). In both groups I detected large gaps in the knowledge from previous school years. This led to the decision of not only focussing on topics from their current grade but also including topics that the children are supposed to have dealt with in the past.

In the following, I will present some material on linear functions from my grade 9 group. In the previous lesson the children had been working with laptops, developing some hypotheses on the influences of the chosen parameters on the slope and y-intercept of linear through a game in Geogebra. In the next lesson I tried to deepen the understanding in this area in a paper task on which the students worked together. Once again, they were given two tasks with points on a coordinate grid; this time, however, the points in each grid belonged to just one function. (Fig. 1 and Fig. 2).

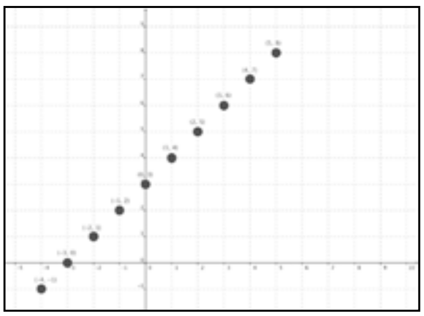

Fig. 1 First task

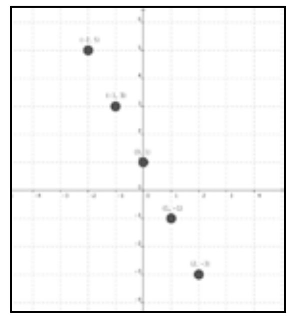

Fig. 2 Second task

Additionally, a table of values for each of the two functions was given. The students had to complete several sentences such as: "If the value for x increases by 1, …", "The intersection point of graph and y-axis is…", "If you put a 0 for x into the equation $y=m\cdot x+b$, the equation simplifies into…". These sentences make use of language from the academic register. When the students had come up with a suggestion for an equation on which they all agreed, they were allowed to check its correctness on Geogebra.

After having discussed about the influence of the chosen parameters in linear functions, a third task was given, "Find a linear function which goes through the point (0|2) and is parallel to the function from task two". The students engaged in a vivid discussion on the solution to this task. There was no agreement on a solution until one girl came up with the argument that m defines the steepness of the graph, and therefore it has to be the same in the two functions.

After six months, generalization and abstraction have turned out to be major problem areas, especially in the grade 9 group. The main difficulty seems to lie in the transition from example-based observations to the creation of general hypotheses. Further tasks will be examined in order to assess obstacles for deductive reasoning more clearly.

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PRESENTATION AND SETTING UP OF A MODEL OF ANALYSIS OF REASONING PROCESSES IN MATHEMATICS LESSONS IN PRIMARY SCHOOL

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Abstract: *The purpose of this paper is to consider reasoning processes elaborated by the pupils during a lesson "The biggest number", proposed in a class of fifth graders at primary school. We would like to describe our model of analysis, in order to highlight the main elements which structure the analysis. This research, made mainly within the framework of the theory of didactical situations in mathematics, aims at analysing on one hand pupils reasoning processes, specifying their functions, on the other hand, the conditions in which the reasoning processes are produced.*

Key words: reasoning, arithmetic, proof, situation of validation, semiotics.

INTRODUCTION

The first part of this paper aims at describing briefly the model of analysis for reasoning processes elaborated by Bloch & Gibel (2011) and establishing its relevance to analyse a mathematics lesson involving a research dimension. In the second part, we will present the lesson "The biggest number" proposed in a class of fifth graders at primary school (pupils age 10 to 11). We will make interest aspects and propose an a priori analysis of this lesson. In the third part, we will present the specificities of this lesson, explaining the different didactical situations (situation of action, situation of formulation, situation of validation) in order to identify and characterize the different shapes and functions of reasoning processes.

1. PRESENTATION OF THE MODEL OF ANALYSIS FOR REASONING PROCESSES

1.1 The theoretical tools used in the elaboration of the model

The subtle analysis of the reasoning processes produced, in a situation of validation, cannot be limited to an analysis in terms of propositional calculus based on Lorenzen's dialogical logics, as Durand-Guerrier points out (2007).

The need to take into account the semantic dimension, when analysing the reasoning processes, has contributed to sustain and justify our decision to choose the Theory of Didactical Situations (TSD), elaborated by Brousseau (1998), as foundation of our model. This theoretical framework must, however, be completed with tools of local analysis, and with an analysis of functions of the reasoning processes (Gibel, 2004) and of the signs, both formal and linguistic, which back it up. We will present the theoretical framework used to perform the semiotic analysis in the following paragraph.

1.2 The semiotic dimension of the analysis

During our previous research (Bloch & Gibel, ibid.) we underlined the fact that reasoning processes elaborated by the pupils and the teacher during a lesson, can take diverse forms: linguistic, calculative, scriptural, and graphic elements. Consequently the semiotic analysis constitutes one of the dimensions of our model, completing those previously presented: on the one hand the function of the reasoning processes and on the other hand the corresponding level of the didactical milieu.

Pierce's semiotics seems particularly appropriate for our research and will indeed enable us to study more precisely the evolution and the transformations in the signs used by different actors of the lesson.

In our application of Pierce's semiotics we will use the three designations: icon, index sign and argument-symbol. An iconic interpretation is based on intuition, sometimes based on a diagram, or using a programme of calculations; an index sign is to do with a proposition, for example, in the sequence studied, "the biggest number is obtained by multiplying the five (whole) numbers given", considered as an argument-symbol related to a mathematical proof.

1.3 The didactical repertoire and the repertoire of representations

All semiotic means used by a teacher and those he expects from his pupils, through his teaching, form the didactical repertoire of the class as defined by Gibel (2004).

The didactical repertoire of the class can be identified as being part of the mathematical knowledge that the teacher has chosen to explain, namely for validation and during institutionalization.

The repertoire of representations is a constituent part of the didactical repertoire. It is made up of signs, diagrams, symbols and shapes and also linguistic elements (oral and/or written sentences), which make it possible to name the objects encountered and to formulate properties and results.

1.4 Methodology. The model of analysis for reasoning processes

The model of structuration of the didactical milieu used in this model is that of Bloch (2006). The chart below (Table 1) sums up the levels of milieu – from M1 to M-3 – corresponding to the *experimental* situation.

The negative levels are of particular interest in the sequence studied i.e. the appearance of a proof process in the setting up of a situation involving a research dimension.

It is in terms of the articulation between the objective milieu and the reference milieu that we hope to see the expected reasoning processes appear and develop.

M1 Didactical milieu	E1: E-The universal subject	P1: P-planner	S1: Project situation
M0 Learning milieu : Institutionalization	E0: The generic student	P0: The teacher teaching	S0: Didactical situation
M-1 Reference milieu : Situation of formulation Situation of validation	E-1: E-The subject as learner	P-1: P Regulator	S-1: Learning situation
M-2 Objective milieu : Situation of action	E-2: E-The subject as an actor	P-2: P The teacher devolves and observes	S-2: Situation of reference
M-3 Material milieu	E-3: E-The epistemological subject		S-3: Objective situation

Table 1 *–Structuration of the didactical milieu*

In the previous research (Bloch & Gibel, 2011), we decided to focus on didactical analysis on three main "axes" of study used to guide our analysis of the reasoning processes.

The first axis of study is linked to the nature of the situation: in a situation involving a research dimension, the pupils produce reasoning processes which depend to a great extent on the nature of the situation (situation of action, situation of formulation or situation of validation) in relation to the level of the corresponding milieu (Table 1).

The second axis of study is the analysis of the functions of reasoning.

We will try to bring together the previous two axes of study by showing how the reasoning functions are linked to the situation specifically to the levels of milieu and how these functions also *manifest* these levels of milieu.

The third axis of study is that of observable signs and representations. These things can be observed in different forms which affect the way the situation unfolds.

The application of the model to the situation will be followed by an analysis of the milieu and semiotic analysis of the students' and teachers' productions.

In conclusion, we classify reasoning, calculations, formulas, depending on the characteristics of the situation and knowledge(s) expressed by students in the previous phase, which reflect the situation in which they are located, and the nature of signs produced. Our project consists in using our model to analyse reasoning processes in a situation of validation involving a research dimension, proposed in primary school.

2. PRESENTATION AND A PRIORI ANALYSIS OF THE DIDACTICAL SITUATION "THE BIGGEST NUMBER"

2.1 Origin of this problem

The mathematics problem below was originally proposed by G. Glaeser (1999):

> Take any five natural numbers a,b,c,d,e. What is the biggest number that can be obtained using the four elementary operations {+; -; x; ÷} applied to these numbers which can only be used once in the calculation; the same operation can, however, be used several times.

The setting up of this sequence at the C.O.R.E.M. is linked to the encounter of Brousseau and Glaeser, which was at the origin of this didactical project. The problem proposed is an open problem; G. Brousseau's idea is to get the pupils to discuss mathematical statements following rules which lead them to produce proof, and, more precisely, to search for counter-examples.

The analysis of this sequence, using our model, should enable us to provide some answers to the following questions:

What are the different forms of reasoning processes which come into play in the different phases of this sequence? What functions do they cover? What level(s) of milieu do they refer to?

2.2. Elements of a priori analysis of the sequence

2.2.1. Analysis of the nature of the expected answer to the problem proposed

To determine the nature of the answer expected answer by the teacher it is necessary to distinguish the conditions in which the answer must be given:

- If the sequence of numbers is given by the teacher, the answer expected is a number together with the programme of calculations enabling it to be obtained.
- If the five numbers are not given, that is to say if one is presented with a general case, then the proper answer will be a method of calculation. However, it must be emphasized that writing an algebraic expression will not be appropriate because it is necessary to distinguish different cases according to the sequence of numbers under consideration.

  In the second case we are led to consider the algebraic expression

$$\boxed{a \times b \times c \times d \times e}$$

where a,b,c,d and e designate any five whole natural numbers.

Yet this algebraic expression is only valid, to obtain the biggest number, from a given 5-uplet, if none of the five numbers is 0 or 1.

The field of validity of this natural algorithm is not immediately obvious, it should lead the pupils to ask themselves questions about the status of the numbers 0 and 1.

It must be pointed out that, to obtain the biggest number, it is necessary in the presence of one or several 0's to determine the biggest number of the sequence of whole numbers excluding the zero(s), and to add the zero whole number(s) afterwards, which means distinguishing special cases when formulating the method.

2.2.2. Didactical analysis of the lesson

One of the objectives of this lesson is to make it possible for pupils to move progressively from arithmetic to general statements of methods enabling them to win. The lesson also aims to teach the rules of the game of proof: it is a lesson on right and wrong but also on the way of establishing it. The main objective of the lesson is therefore to put the pupils in a situation where they are led to discuss the validity of methods for obtaining the biggest number, whatever the sequence of numbers proposed.

In this situation, it is anticipated that the rejection of a method will go together with the production of a counter-example, more precisely to the production of a sequence of five numbers and of a new method leading to the production of a new number – higher than the one obtained by the method proposed originally.

3. ANALYSIS OF THE DIDACTICAL SITUATIONS

In order to carry out an analysis of the reasoning processes produced, we use, first of all, the structure of the didactical milieu chart to distinguish the « embedded » situations corresponding to the different milieu (Brousseau et Gibel, 2005). The different didactical situations which appear in the conduct of this lesson are successively: situation of action, situation of formulation and situation of validation.

Concerning the lesson "The biggest number", our model should enable us to analyse a posteriori, the transformations occurring during the reasoning processes produced by the pupils as regards their formulation, taking into account their functions in the didactical relation.

3.1 The situation of action

In the analysis of the sequence « The biggest number », the teacher's objectives are, on the one hand to present the rules of the game, and, on the other hand, to lead the pupils to formulate the number obtained and the justification of the programme of calculation.

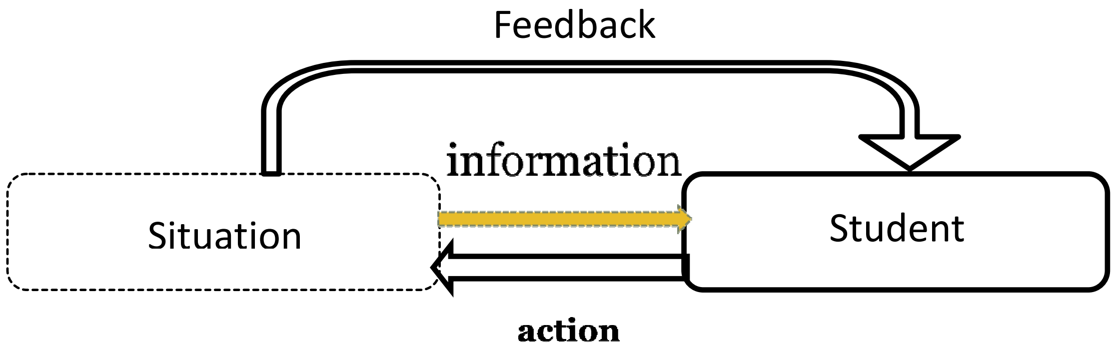

Figure 1-Situation of action

> "When the teacher devolves the objective situation to the students, they are faced to the situation of action. They take information from the objective situation, act on the situation and get feedback."(Brousseau,1998)

The situation of action (Figure 1) constitutes the process by which the students form strategies.

The dialectic of action leads pupil to produce reasoning processes corresponding to different functions: taking a decision and justifying the validity of the number obtained.

After the two games proposed to the pupils (Appendix), the teacher presents to the pupils the *situation of formulation*, which we explain in the next section.

3.2 The situation of formulation

The objective of the situation of formulation is that pupils write down a *general* method, that is to say one that can be used to obtain the biggest number whatever the sequence of five numbers proposed.

So the objective is to get the pupils to produce a method whose field of validity is the largest possible. The situation of validation aims at giving pupils the possibility to take a clear position concerning the action, and therefore to become fully aware of the decisions on which their actions are based, so that they can produce procedures whose validity can be discussed.

Each group must write a method, a *general* method, enabling them to obtain the biggest number whatever the sequence of five integers proposed.

It must be pointed out that, to obtain the biggest number, it is necessary in the presence of one or several 0's to determine the biggest number of the sequence of whole numbers excluding the zero(s), and to add the zero whole number(s) afterwards, which means distinguishing special cases when formulating the method. Moreover it is necessary to distinguish different cases, according to the numbers of 1's included in the sequence of numbers.

Denoting N, as the biggest number and distinguishing the different cases, we obtain

Case n°1 If the sequence includes one 1, 1-b-c-d-e with $1<b\leq c\leq d\leq e$ then

$$N= (b+1)cde$$

Case n°2 If the sequence includes two 1's, 1-1-c-d-e with $1<c\leq d\leq e$

If c=d=2 then N=(c+1)(d+1)e

else N= (1+1)cde

Case n°3 If the sequence includes three 1's, 1-1-1-d-e with $1<d\leq e$ then

$$N=(1+1+1)de$$

Case n°4 If the sequence includes four 1's, 1-1-1-1-e with 1<e

If e=2 then N= (1+1+1)(e+1)

else N=(1+1+1+1)e

Case n°5 If the sequence includes five 1's, 1-1-1-1-1 then N=(1+1+1)(1+1)

During the situation of formulation, the pupils use reasoning processes to formulate a general method. They produce reasoning processes which allow them to consider special cases similar to those described previously.

3.3 The situation of validation

During the situation of validation there are discussions between the groups about the validity of the methods. The didactical scheme of validation (Figure 2) motivates the student to discuss a situation and favours the formulation of their implicit validations, but their reasoning is often insufficient, incorrect, and clumsy.

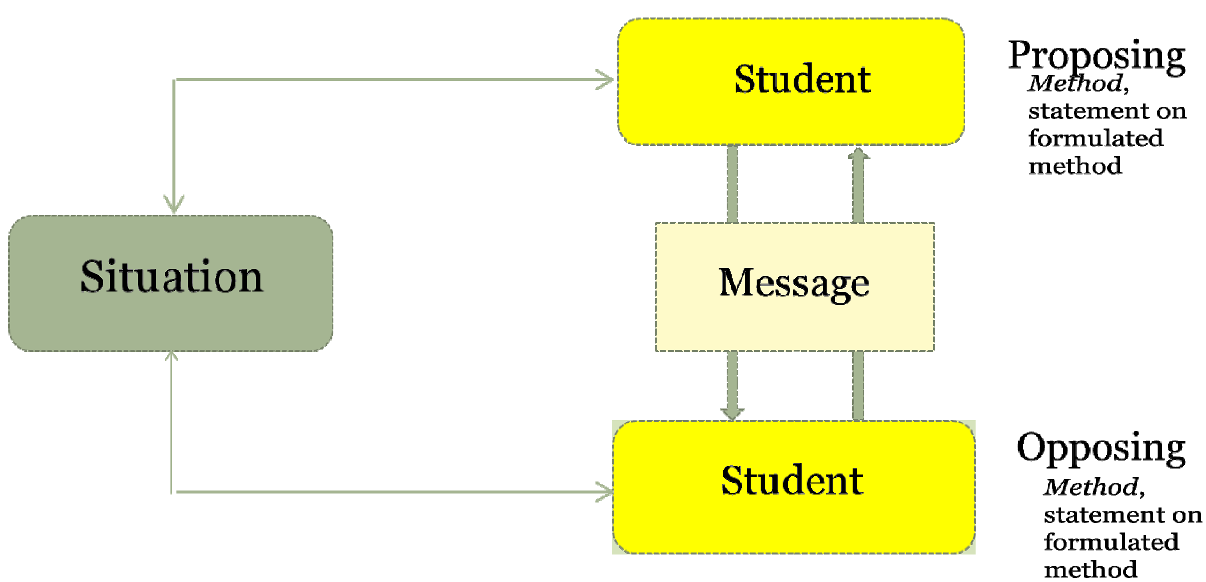

Figure 2-Situation of validation

The level (M-1), detailed in Table 1, is that of assertions. At the previous level, (M-2), we were at the level of mathematical relations, the truth was obvious, the relation was right or wrong but there was no judgment made. Whereas at level (M-1) the pupil, having the status of opponent in a situation of validation, arrives with a certain culture, and knowledge linked to the didactical repertoire he/she has at his/her disposal.

3.4 The didactical situation: the pupil and the learning situation

The didactical situation, for example a new game, based on a sequence of numbers proposed by the teacher, leads pupils to evolve or to revise their opinions. Thus the teacher can choose a sequence of numbers that leads them to develop a new method of calculations which includes specific cases that they had not previously considered.

Moreover if students cannot find a counter-example to invalidate a method, the teacher may decide to submit a new game that highlights the shortcomings of a method to get the largest number. Thus students become aware of the shortcomings of the program calculations and encouraged to write another one.

CONCLUSION

The analysis of the didactical situations allows us to highlight the different shapes of reasoning processes which appear all along the lesson. The model offers us the possibility to analyse the different functions of the reasoning processes in different didactical situations (situation of action, situation of formulation and situation of validation).

The model underlines the reasoning processes produced and links them up to previous knowledge and awareness of didactical repertoire concerning elementary operations and properties of multiplication. Concerning formulations, the model shows their evolution in relation to the different situations: one goes from giving sequences of arithmetic calculations to the formulation of general methods, of an almost algebraic nature, to end up with the production of semantic and syntactic arguments during the situation of validation. The semiotic analysis also shows the gap between the formalism introduced by the teacher and the repertoire of signs mobilised by the pupils throughout the lesson.

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Appendix Presentation of the sequence "The biggest number"

The different phases of lesson 1

Phase 1: Devolution of the game. Sequence proposed 3,8,7,5,4

Phase 2: Complementary information.

Phase 3: Individual investigation

Phase 4: Pooling. Presentation of the results and designation of the winners.

Phase 5: Comparison of methods.

Phase 6: Instructions for the second game 7,3,2,5,8

Phase 7: Individual investigation.

Phase 8: Pooling. Presentation of the results and designation of the winners.

Phase 9: Instructions concerning the proposition contest.

Phase 10: Investigation

Phase 11: Regrouping. Formulation of the propositions. Discussion concerning the propositions.

Phase 12: Phase of the game 2,5,3,2,4

Phase 13: Presentation of the results.

The different phases of lesson 2

Phase 1: Instructions concerning the proposition contest.

Phase 2: Group investigation.

Phase 3: Pooling. Explanation of the results.

Phase 4: Discussion concerning the methods.

Phase 5: Phase of the game 5,2,4,0,3

Phase 6: Presentation of the results obtained using the methods.

Phase 7: Proposition of new methods.

Phase 8: Phase of the game 8,1,3,0,0

Phase 9: Presentation of the results obtained using the methods.

Phase 10: Proposition of a new method.

Phase 11: Search for a counter-example.

Phase 12: Propositions of counter-examples. Discussion concerning the validity of the counter-examples.

Phase 13: Proposition of new methods.

Phase 14: Phase of the game (7-0-4-3-1).

Phase 15: Presentation of the results.

Phase 16: Search for counter-examples.

Phase 17: Proposition of counter-examples.

The different phases of lesson 3

Phase 1: Pooling the results following the sequence proposed by Hélène (8-1-1-1-0)

Phase 2: Discussion concerning the status of Hélène's proposition.

Phase 3: Presentation of a sequence of numbers by the teacher (1-1-1-1-1).

Phase 4: Search for the corresponding method.

Phase 5: Presentation of the methods. Explanation of the counter-example.

Phase 6: Phase of the game. (1-1-1-1-9)

Phase 7: Presentation of the results obtained using the methods.

Phase 8: Search for counter-examples.

Phase 9: Phase of individual writing down a method.

RESEARCH SITUATIONS TO LEARN LOGIC AND VARIOUS TYPES OF MATHEMATICAL REASONINGS AND PROOFS

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Abstract *– We present an analysis of the role of research processes and experimental activities in our "Research Situations", allowing students to learn fundamental knowhow in mathematics : experimentation, studying particular cases, reasoning, formulation of conjectures, examples and counterexamples, generalizing, proving, etc, as described in curricula and related documents of French junior high schools and high schools. The reasoning and proving processes are a full part of our SiRC. This will be illustrated here with examples.*

Key-words *– Research Situation, mathematics investigation, logic, reasoning, proof.*

INTRODUCTION

Routine researcher activities comprise elementary tasks such as : choosing a question, experimenting, studying special cases, choosing a solution framework, modelling, reasoning, stating conjectures, proving, defining, eventually changing the initial question, etc. *Knowledge and tools* to tackle these tasks are intrinsically part of a scientific approach, they are necessary to do mathematics and cannot be brought down to mere technics or methods.

In many countries, these basic tools are not available to a majority of scientific students. What they do and say when confronted with an « open problem » shows that their relation to mathematics is very far from the mathematician standpoint :

- « I don't know how to solve this problem, because it's new for me »
- « I don't know what to do, because I don't know the technique »
- « This problem is badly formulated, hypotheses are missing »
- « To which chapter is related this problem ? »

French mathematical curricula insist on experimentation, discovery and quality of scientific classroom activity for learning different types of reasoning and proofs, and involve some elements of mathematical logic. However, in schoolbooks and teaching practices, these objectives are not really treated, and the experimental approach and research activities are scarce. The objections raised by teachers to avoid the research situations in class are the institutional constraints and a lack of training to grasp and manage such situations. The idea that, in mathematics, one can experiment, model, study specific cases, infer conjectures from examples, is not present in the standard didactical procedures. Mathematics are seen as a service discipline, that offers a set of techniques and computation algorithms, although these can sometimes be sophisticated. Theoretical and fascinating aspects are kept inaccessible and reserved to experts.

The consequences of such positions and choices include attitudes and practices that seem to go against those which would precisely enable a significant research activity :

- Students are not allowed to change the hypotheses, nor to choose their own solution framework.
- Two strong « rules » become implicit, when students have to write or validate their proof :

« **Only** the problem data and the properties taught in class should be used in the proof »

« To complete a proof, one has to verify that **all** the hypotheses given are used »

- The mathematical activity is reduced to the technique of writing the proof, thus shrinking the research and arguing initial process.

HYPOTHESES TOWARDS GUARANTEEING A GENUINE MATHEMATICS ACTIVITY

H1. There is no real possibility of an investigation if a « toolbox » (theorems, properties, algorithms) is available and designated for the resolution of an obvious question. There is no real mathematical activity without a truth issue that can be taken and tested by students, while being non obvious to prove.

H2. It is neither necessary nor sufficient to arrange « real life » contexts (Coulange 1998) to perform a research or an experimental investigation : such contexts do not guarantee the relevance of the problem, and can even make noise that impede the investigation and the understanding of the underlying logic.

H3. It is not reasonable to propose research situations that bring into play mathematical concepts in construction. As a further advantage of avoiding such lapses, the problems will be accessible to many levels of knowledge (sometimes from primary school to university).

Our research work consists in conceiving appropriate specific situations – e.g. problems and didactical staging (in the sense of scenic design) – available in different institutional contexts, in experimenting them and analyzing their effects on the learning of mathematical reasoning, proofs, and the underlying logic.

CHARACTERIZATION OF THE MODEL “SIRC”[1]

This didactical model was already in gestation in Arsac & al. 1995 and Grenier & Payan 1998. Recall here the characterization of the SIRC model, as it has been described in Grenier & Payan 2003 & 2007. As any model, it is a reference (both epistemological and practical) for situations that we build, which can somewhat differ from the model.

[1] SiRC : “Situation de Recherche pour la classe” , ie. “Research Situation for Classroom”

• A SiRC is similar to an *actual question* in mathematical research or in a non « didactified » one. This condition, very restrictive a priori, aims to avoid the question or the answer may seem obvious or familiar. The objective is to give relevance to the research activity. This condition can be artificially recreated by the « staging » of the problem.

• The initial question is easy to understand at various levels of knowledge. Our intention is to break with the usual didactic practice that tends to attribute any problem to a specific grade level. To fulfill this requirement, the statements must in general be not as heavily mathematized. However, we try to avoid random real life « noises », which complicate the task of students in non-mathematical « concrete » problems, and sometimes prevent them from entering into actual mathematics.

• Strategies to start with are available, but they won't solve the problem completely – usual techniques or properties are not sufficient. In other words, one must ensure the devolution of the problem, by leaving space to some uncertainty that cannot be reduced just by applying known techniques or usual properties (i.e., what Brousseau described in his theory as a « good » situation). The theoretical framework of resolution is neither given nor obvious.

• One can use several strategies, such as «trials and errors », study particular cases, etc ; relevant conjectures are not obviously true, counter-examples are attainable. These points are meant to encourage the construction of conjectures by students, based on an exploration of the question investigated. These conjectures can be examined by the students, through accessible examples and counterexamples.

• Hypotheses, or the initial question, can be changed. One can change the assumptions or the original question, and grab a new problem. The initial question can lead to related questions : closing a problem through the choice of certain parameter values, or starting a new research activity.

A SiRC is characterized by some *research variables* : problem's parameters which could be didactical variables (i.e. at teacher's disposal), but that are left at the student's disposal. The *social organization* is constitutive of a SiRC : working in small groups, material to play and search, time sufficient to search and discuss. Teachers dealing with a SiRC is a specific activity : training teachers is necessary.

Methodology and experimental results

The SiRC didactical model (Research Situations for Classroom) has been studied in the "Maths-à-Modeler" team for 15 years, using classical didactical theories and methodologies : construction of situations, experiments with numerous students at many different levels of knowledge (several hundreds of students) and analyses of the results. Some of these SiRC have been integrated in certain courses at university and in secondary schools. Results have been published in articles (Grenier 2002, 2003, 2008a, 2008b, Tanguay & Grenier 2009 & 2010) and in Theses (Ouvrier-Buffet 2003, Deloustal-Jorrand 2004, Godot 2005, Cartier 2008, Giroud 2011). We give in annex 1 a list of Research Situations for Classroom which have been studied for more than ten years.

Example of different types of reasonings and proofs worked in a SiRC

In the SiRC given in annex 2, « Tiling polyminoes with dominoes », in the case where the polymino is a square and $n = 3$, at every level of knowledge, properties and *conjectures* emerge from experimentations and research (Grenier 2008b):

> A *necessary condition* to tile a square, that is *not sufficient* – this result can be proved by a very easy *counter-example ;* Then, a *necessary and sufficient condition* to tiling emerges, according to the position of the hole. The proof consists in two steps : first, a *proof of impossibility* – by a « forced solution » tiling (*reductio ad absurdum*) – and a *proof of possibility* – by exhibiting an *example* of tiling (*existence property*).

Inductive and deductive reasonings and proofs in mathematics

Leading students to these admittedly different types of reasoning - inductive and deductive reasoning – is a declared goal of French college and high school programs. Generally, inductive reasoning aims to generalize to other objects a property known for certain objects, or to build new objects. Mathematical induction differs from induction in other sciences, especially in physics, by its intrinsic validity. Here is a well known excerpt from H. Poincaré :

> The induction applied to the physical sciences, is still uncertain, because it is based on the belief in a general order of the universe, an order which is always outside ourselves. Mathematical induction, namely, proof by induction, is required instead of necessity, because it is the assertion of a property of the mind itself. (H. Poincaré, Science and Hypothesis. Flammarion).

The activity of experimenting and studying special cases plays an important role, and is almost necessary in learning inductive reasoning, because it helps to establish and justify the formulation of conjectures – rather than raising them randomly, and helps to study these conjectures by going back and forth between the experimental data and the setting up of their evidence.

Reasoning by induction has the particular characteristic of being at the junction point of the inductive and deductive procedures. In my study of students and teachers conceptions on induction (Grenier 2002 and 2003), it appears that the understanding of the concept is severely lacking in depth. Induction is reduced to one or two techniques, and is perceived as a non-constructive tool of proof, which sometimes raises doubt for its justification. Accordingly, the scope of problems that can be solved by this tool remains extremely limited. Problems of various types designed to lead to a better appraisal of induction were introduced and studied (Grenier 2012).

A RESEARCH SITUATION INVOLVING OPTIMIZATION : HUNTING BUGS

The assigned task is to protect a grid field against “bugs”, by forbidding them to land on the grid. In order to do this, traps – uniminos – are available, each covering a box.

The bugs are small polyminoes (dominoes, triminoes, etc.), and they can "land" by covering exactly some boxes of the grid. The question is to find a minimal configuration of traps that protect the field. We consider in the sequel a 5 x 5 grid field.

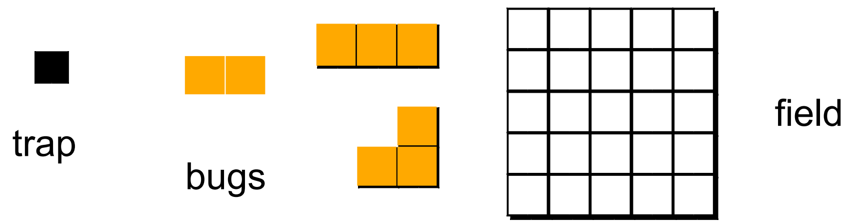


Figure 1

To solve the problem, students are equipped with appropriate material (wood, cardboard, etc.) that allows them to try and modify easily configurations without practical constraints. We are going to distinguish:

A « solution » : a set of boxes such that if one puts a trap on each of these boxes, then the field is protected. Putting a trap on each of the 25 squares is a trivial solution, but clearly a non optimal one.

An « optimal solution » : a configuration of minimum cardinality.

There may be several optimal solutions, namely sets of the same cardinality corresponding to different configurations.

The question is : for each of the three types of bugs represented above, what is the smallest number of traps that protects the area ?

Hunting Domino bugs

In that case, an optimal solution satisfies the following necessary condition : there are no two adjacent boxes without a trap. This condition is also sufficient : if there are not two adjacent boxes without a trap, then no bug can land on the field, because no domino can be put on the grid. Two « spontaneous » solutions satisfy this condition : one with 13 traps (figure 1a) and the other with 12 traps (figure 1b) – this property proves that the 13 traps solution is not optimal, but does not prove (yet) that the 12 traps configuration is optimal.

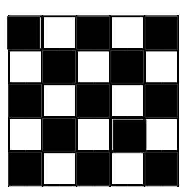


figure 2a

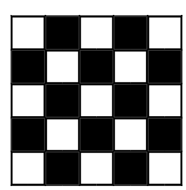


figure 2b

The following question is : Is it possible to protect the field with only 11 traps ? The answer is no. To prove this, we can study the « dual » problem, that is : what is the largest number of dominoes that can be placed on the grid area without overlap. Indeed, for a given pavement, it takes at least a trap by a domino-bug. We find easily that you can put down 12 dominoes without overlap, so at least 12 traps are necessary to protect the field. If the optimal number is denoted by N_{opt} it has been therefore established that :

12 traps are sufficient, that is, $N_{opt} \leq 12$

and 12 traps are necessary, that is, $N_{opt} \geq 12$.

So, we have proved that $N_{opt}=12$.

Numerous experiments of this problem, with students at different levels, show that a false property-in-act appears frequently, when the 13 traps solution is discovered first : « Any solution that is no longer a solution when an arbitrary trap is removed, is optimal ». figure 2b is of course a nice counter-example.

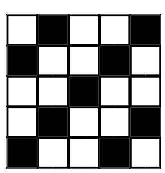
Hunting Long trimino bugs

Some of the reasonings and results that were established with dominoes can be reinvested here.

A solution satisfies the following necessary condition : there are no three adjacent boxes without a trap.

This condition is also sufficient : if there are no three adjacent boxes without a trap, then no bug can land on the field, because no long trimino can be put on the grid.

This second problem is more difficult than the first one : at every level, the experiments lead students to solutions that are far from the optimal one, such as the configuration given for the dominoes (figure 1b) – in a first step, a lot of students think that this is the optimal solution, because they cannot find any other. After performing new attempts, students frequently consider the one given by figure 3a below, with 9 traps. However, this configuration turns out not to be optimal. If no better solution is found, the teacher has to resume the situation, because otherwise students feel no clear incentive to continue the research.



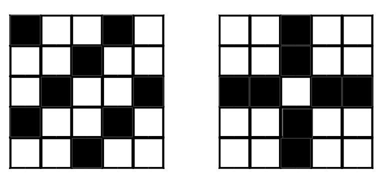


figure 3a. a 9-trap solution **figures 3b and 3c. two 8-trap solutions**

Finally, there is almost always a group that finds one of the two 8-trap solutions (figure 3b and 3c). It remains to prove that these two solutions are indeed optimal. If one reinvests the « dual » proof already encoutered in the domino problem, the majority of students finds that it is possible to put down 7 long triminos without overlap (figure 4a), so at least 7 traps are necessary to protect the field. So, we have proved that $7 \leq N_{opt} \leq 8$. Finding a pavement with 8 triminos would allow them to close the question. However, this pavement seems in practice very difficult to find by students. The pavement in figure 4b proves that 8 traps are necessary to protect the field, that is $N_{opt} \geq 8$ (hence $N_{opt}= 8$).

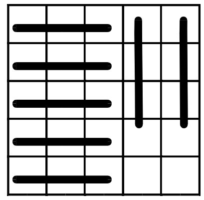


figure 4a

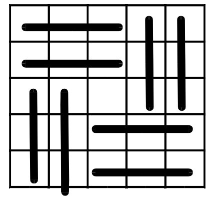


figure 4b

Hunting L-trimino bugs

Reasonings and proofs to conduct in this third problem are more complex, essentially because of the less tractable shape of the triminos : it is not easy to give a necessary condition for a solution (that protects the field). It can be a good strategy to begin by finding a pavement of the grid by a maximal number of L-triminoes (dual problem). One can easily find that there exists a pavement with 8 L-triminos (figure 5). This pavement proves that $N_{opt} \geq 8$.

A recurrent false reasoning frequently follows this discovery : a claim that $N_{opt} \leq 8$, justified by the fact that it is impossible to put down more than 8 L-triminoes on the grid, as the equality $3 \times 8 = 24$ leads to 8 being the obvious maximum.

After a substantial time for experimentation and research, in general, many students find 12-traps solution (figure 6a), then 10-traps solution (figure 6b).

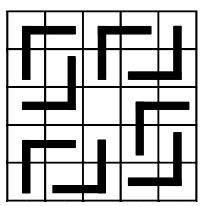


figure 5

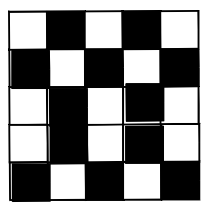


figure 6a

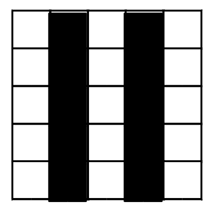


figure 6b

CONCLUSION

Learning argumentation, logical reasoning and different types of mathematical proofs requires work on appropriate specific problems, the resolution of which can be attained not just by applying techniques or formal results from the main course. We have built and experimented a number of "research situations" that should allow such acquisitions by students of various levels. We analyze especially a situation of "bug hunting" that enables students to construct conjectures and later to reformulate them in terms of implications, necessary and sufficient conditions (along with the methods of exhaustivity of cases, contraposition, reasoning by contradiction, counterexamples, etc). Our "SiRC" therefore contain all necessary ingredients to enter into genuine mathematical activities.

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ANNEX 1. SIRC EXAMPLES (MATHS À MODELER, IREM, UQAM)

Some examples of Research Situations for classroom, with the main concepts or tools that they bring into play.

Polyminos tilings /*algorithms, existence theorems* / graph theory

Hunting the bug / *optimization /* number theory, lower and upper bounds

Discrete geometrical objects / *representation, definition* / euclidian an non euclidian geometries

Moving on the discrete plane / *definition /* generating or minimal systems, linear algebra

Geometry at the mountain / *space representation /* non euclidian geometry, euclidian axioms

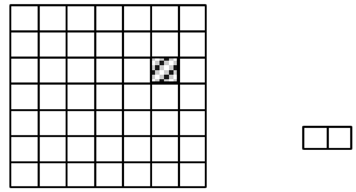
Regular 3D polyhedra / *defining, handling and proving /* 3D geometry

Regular polygons with integer vertices / *induction, ad absurdum /* combinatorial geometry

Disks in triangles or squares / optimization / combinatorial geometry, graph theory

ANNEX 2. TILING POLYMINOES

Problem P1. Rectangle with a hole in any position to tile with dominoes

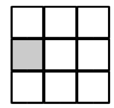


Some elements of the mathematical activity in the resolution of Problem 1

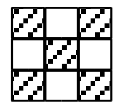
Properties and conjectures that emerge from experimentations and research

Property 1. A ***necessary condition*** to tile a square having a hole with dominoes, is that the area is even.

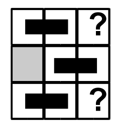
This condition is *not sufficient*. counter-example :



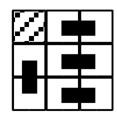
Property 2. For $n = 3$, a necessary and sufficient condition to tile with dominoes is that the hole is situated on any of the streaked position below.



proof of *impossibility* : by a « forced » tiling (ad absurdum)

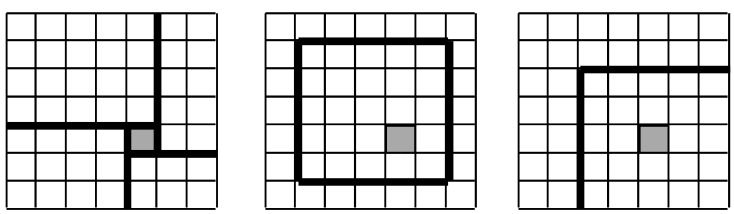


proof of *possibility* : by exhibiting a tiling (existence property)



These proofs are not transferable to any *n.*

Proof for every *n*, when tiling is possible : by structuration in rectangles with even area (without any hole), or by induction. This result is also true for rectangles

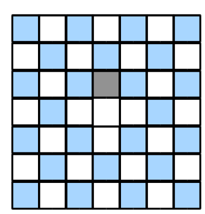


Proofs of impossibility for *n* arbitrary

Property. A necessary condition to tile is that, in a checkered coloration, the polymino is *balanced*

Proof by coloration, proof ad absurdum or by contradiction (if non balanced, then non tiling)

This condition is necessary for any type of polymino, but generally not sufficient.



THE WIDTH OF A PROOF

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The paper's aim is to discuss the concept of "width of a proof" put forward by Timothy Gowers. It explains what this concept means and attempts to show how it relates to other concepts discussed in the existing literature on proof and proving. It also explores how the concept of "width of a proof" might be used productively in the mathematics curriculum and how it might fit with the various perspectives on learning to prove.

Keywords: argumentation, explanation, memorability, proof

INTRODUCTION

The paper addresses parts of themes one and four of the call for papers, namely "The importance of *explanation, justification, argumentation* and *proof* and their relationships in mathematics and in mathematics education" and "The use, evolution, elaboration or integration of theoretical frames relevant for research argumentation and proof in an educational perspective". Its aim is to discuss the concept of "width of a proof" put forward by Timothy Gowers (2007).

Both mathematicians and mathematics teachers are well aware that "mathematical explanation" and "mathematical understanding" are elusive terms. Indeed, these concepts are recurring topics in both mathematics and mathematics education. Even though the term "understanding" is not precisely defined, it is often invoked in the context of proof and proving. Most mathematicians and educators do share the view that a proof is most valuable when it leads to understanding (Balacheff, 2010; de Villiers, 2010; Hanna, 2000; Manin, 1998; Mejia-Ramos et al, 2012; Mariotti, 2006; Knipping, 2008; Thurston, 1994). For this reason, mathematicians see proofs less as correct syntactical derivations and more as conceptual entities, consisting of a logical sequence of related mathematical ideas, in which the specific technical derivation approach is secondary.

But when the focus on proof is primarily conceptual, there arises a need for well-defined criteria, apart from the well-known syntactical ones, by which the quality of a proof can be evaluated. In the absence of such well-defined criteria, value judgements are often made on the basis of properties with an aesthetic flavour that have no precise meaning, such as "comprehensible", "ingenious", "explanatory", "elegant", "deep", "beautiful", and "insightful". Clearly, such properties go well beyond the logical correctness necessary to prove a theorem. But in both mathematical practice and mathematics education these quasi-aesthetic properties of proof, ill-defined as they may be, are extremely important, because they speak to the recognition by mathematicians and educators that proofs are “far more than certificates of truth” (Gowers, 2007, p. 37).

Aesthetically pleasing proofs enjoy a privileged position for that reason alone, of course, but Gowers notes that: “…it is remarkable how important a well-developed aesthetic sensibility can be, for purely pragmatic reasons, in mathematical research” (p. 37). The pragmatic reasons include enhanced ease of communication among mathematicians, increased interest, and the very important feature of being memorable. Memorability takes us to the concept of “the width of a proof”.

WHAT IS MEANT BY THE WIDTH OF A PROOF?

In approaching this concept, it is helpful to consider first what drove the mathematician Timothy Gowers to define the quality of mathematical proof for which he coined the term “width”, and the weight he gives to the associated notion of “memorability”. Deploring the fact that there is no well-defined language for expressing value judgments about the desirable “quasi-aesthetic” properties of proof, such as the ones mentioned above, Gowers (2007) set out by trying to clarify what is really meant by the words already in use to describe the quality of a proof. His attempt to come up with “a good theory of informal mathematical evaluation” (p. 39) led him to suggest paying more attention to memory. More specifically, he found it valuable, in formulating a good “theory of mathematical evaluation”, to consider the way mathematicians employ memory in creating and in understanding proofs.

Memorability

In the field of education the terms “memory” and “memorizing” are unfortunately associated with the undesirable notion of rote learning, which by definition eschews both explanation and understanding. But for Gowers memory has an entirely different meaning; in his discourse the term is very strongly associated with explanation and understanding, and he sees it as a crucial tool for mathematical thinking. Gowers notes, first of all, that a memorable proof is “greatly preferable to an unmemorable one” (p. 39), and adds that “memorability does seem to be intimately related to other desirable properties of proof, such as elegance or explanatory power” (p. 40). This raises the interesting question of why some proofs are easier to remember than others, and what role is played in memorability by various other non-syntactical properties such as elegance and ingenuity.

Unlike other properties so often cited, Gowers argues that the concept of “memory” is sufficiently precise to be easily investigated and eventually quantified. First, it is easy to determine whether one remembers how to prove a theorem or doesn’t. Second, it would be feasible for educators or psychologists to find out what features of proofs make them more memorable. Third, memory is clearly related to the background knowledge of the learner (or the mathematician).

Gowers may have gone too far in stating that there is a "very close connection between memorizing a proof and understanding it" (p.40), as if that were firmly established. He states:

> The fact that memory and understanding are closely linked provides some encouragement for the idea that a study of memory could lie at the heart of an explication of the looser kind described earlier. It is not easy to say precisely what it means to understand a proof (as opposed, say, to being able to follow it line-by-line and see that every step is valid), but easier to say what it means to remember one. Although understanding a proof is not the same as being able to remember it easily, it may be that if we have a good theory of what makes a proof memorable, then this will shed enough light on what it is to understand it that the difference between the two will be relatively unimportant. (Gowers, 2007, p.41.)

The concept of memorability leads Gowers to suggest looking at *width*, a term borrowed from theoretical computer science, where it refers to the amount of storage space needed to run an algorithm. (Gowers draws a parallel between "keeping in mind" and "computer storage space", in the belief that storage in computers is analogous to memory in humans.) Thus the "width of a proof" would be a measure of how many distinct pieces of information or "ideas" one has to keep in mind (to hold in your memory) in order to be able to construct or follow a proof, to understand it, and to remember it.

Width

It is important to make the distinction between the width of a proof and its length. Whereas the length of a proof refers to how many lines of argument are needed to prove a theorem, the width refers to the number of distinct ideas one has to keep in mind (or memorize) in achieving that goal. As mentioned above, for Gowers (2007) the term "memorable" is intimately connected with understanding and means "easy to memorize". Thus it becomes important to try to determine which features of a proof might contribute to making it memorable, and in his opinion:

> Some proofs need quite a lot of direct memorization, while others generate themselves …if we think about what it is that makes memorable proofs memorable, then we may find precise properties that some proofs have and others lack. … if mathematicians come to understand better what makes proofs memorable, then they may be more inclined to write out memorable proofs, to the great benefit of mathematics (p. 43).

Gowers suggests paying close attention to how mathematicians might memorize a piece of mathematics in general and a proof in particular. For this reason he goes on to expand on his notion of "memorability" by placing it in the context of mental arithmetic, discussing how one remembers sequences of numbers and how one might go about adding large numbers without writing anything down.

It is sufficient here to mention familiar arithmetic operations in which the "width" of the calculation (that is, the number of pieces of information one must keep in one's memory in order to perform a calculation) can be reduced by appropriate simplifications. For example, the mental multiplication of 47 by 52 could be greatly reduced in “width”, according to Gowers, as soon as it is perceived as the difference of two squares. One could represent the product 47 x 52 as the product of (50-3) by (50+3), minus 47, which yields (2500 – 9) - 47. The reduction in “width” occurs because this way of handling the operation means that one has to keep fewer digits in mind than when mentally performing the multiplication 47 x 52 using the customary algorithm. One has eliminated the need to remember multiple digits, at the cost of introducing a single new insightful idea, the concept of the difference of two squares. In Gowers’s terms, this new idea has transformed the original large-width operation into a lower-width one.

Examples

Gowers also illustrates his ideas by citing two examples that are more complex and go beyond the realm of numerical calculation, one being the proof of the existence of a prime factorization (a fundamental theorem of arithmetic) and the other being the proof that the square root of 2 is irrational. The latter is shown here.

Proof that the square root of 2 is irrational

There are several proofs of this theorem using different methods, such as proof by infinite descent, by contradiction, by unique factorization, and by the use of geometry. The proof by contradiction is appropriate for mathematics education, and in fact has often been discussed by mathematics educators in the context of the overall notion of proof by contradiction. The following is one version of the proof by contradiction.

> Assume that $\sqrt{2}$ is a rational number. This would mean that there are positive integers p and q with $q \neq 0$ such that $p/q = \sqrt{2}$. We may assume that the fraction p/q is in its lowest terms; it can be written $\sqrt{2} = (2-\sqrt{2})/(\sqrt{2}-1)$.
>
> Then, substituting $p/q = \sqrt{2}$; $p/q = (2-p/q)/(p/q-1) = (2q-p)/(p-q)$
>
> Because $p/q = \sqrt{2}$, it lies between 1 and 2, we have $q<p<2q$,
>
> It follows that: $0<2q-p<p$ and $0<p-q<q$
>
> We found a fraction equal to p/q but with smaller numerator and denominator. This is a contradiction, so the assumption that p/q is in lowest term ($\sqrt{2}$ is rational) must be false. ∎

Gowers’ point is that this version of the proof contains a step (step 2) that seems to “spring from nowhere”, namely the choice to write $p/q = \sqrt{2}$ as the expression “$p/q = (2-\sqrt{2})/(\sqrt{2}-1)$”. Clearly this expression is useful, but where does it come from? Clearly it would come from the memory of the trained mathematician. This is a case in which one has to have used one’s memory to store an idea for later application. But this proof is nevertheless of lower width than it would have been without this one ingenious idea, because of the greater number of calculations that would have been needed to complete it otherwise. Of course, another low width proof is the one by contradiction: Assuming $p/q = \sqrt{2}$ is rational, then $p^2 = 2q^2$ means that p is even. Assuming p and q mutually prime, q must be odd. However, the square of an even number is divisible by 4, which implies that q is even, hence a contradiction.

Of course it would help the reader of the proof to clarify why someone thought of this idea, because a step in a proof should not appear as a *deus ex machina*. Leaving that aside, however, Gowers points out that this particular step is no more than "what mathematicians normally mean by an idea" (p. 48). Ideas, according to Gowers, are in fact an intrinsic property of proofs, side by side with the property of correct logical derivation.

An example from geometry: Proof of Viviani's Theorem

> Viviani's Theorem: For a point P inside an equilateral triangle ΔABC, the sum of the perpendiculars p_a, p_b, *and* p_c from P to the sides of the triangle is equal to the altitude h.
>
> Proof: The idea is to recall that the area of a triangle is half its base times its height. This result is then simply proved as follows:
>
> $\Delta ABC = \Delta PBC + \Delta PCA + \Delta PAB$
>
> With s the side length, we have:
>
> $\frac{1}{2}\, sh = \frac{1}{2}\, sp_a + \frac{1}{2}\, sp_b + \frac{1}{2}\, sp_c$
>
> $h = p_a + p_b + p_c$ ∎

This proof of Viviani's theorem is also of low width, because it can be generated by using just one powerful idea (the area of a triangle).

Back to width

What then is the width of a proof? Gowers' first try at a definition is:

> It is tempting to define the width of a proof …as the number of steps, or step-generating thoughts, that one has to hold in one's head at any one time. … suppose that I want to convince somebody else … that a proof is valid, without writing anything down. It is sometimes possible to do this, but by no means always. What makes it possible when it is? Width is certainly important here. For example, some mathematics problems have the interesting property of being very hard, until one is given a hint that suddenly makes them very easy. The solution to such a problem, when fully written out, may be quite long, but if all one actually needs to remember, or to communicate to another person, is the hint, then one can have the sensation of grasping it all at once (p.55).

This restates the idea that the factor that determines the "width of a proof" is the number of distinct pieces of information, or ideas, needed to complete it. In Gowers' view, the fewer such pieces, the easier it should be to follow a proof or to actively devise one. A proof of low width, with few pieces of information to carry in one's head, would be superior to one of high width. Often it is a case of introducing a single new but very productive idea that makes it unnecessary to deal with a larger number of less interesting ones.

That does not mean that a proof must be short, because the length of a proof is not identical with its width, as pointed out earlier. Whereas the length of a proof usually refers to the number of deductive steps required to complete it, the width of a proof refers to the number of distinct items, or distinct ideas, that must be kept in memory in order to read, learn and understand the proof.

Reflections on Gowers' notion of width

Gowers's notion of "width" does not seem to capture in a single measure the many non-syntactical properties of proof, such as "comprehensible", "ingenious", "explanatory", "elegant", "deep", "beautiful", and "insightful". Discussion of this point is difficult because none of these terms are well defined, as Gowers points out. The main difficulty, however, seems to be that these terms are far from being synonyms or even overlapping. One could even argue that some of these properties of proof, if not opposites, are at least orthogonal to one another. Very concise proofs are often considered elegant or beautiful, for example, but they are also less likely to be comprehensible or explanatory.

Thus, even if width does prove to be a useful and potentially measurable property of proof, it would not appear to be one that could serve in place of the less measureable non-syntactical properties that mathematicians have always found meaningful and useful. To which of these properties width is most correlated remains an interesting question. An elegant proof, for example, would presumably be considered of low width (and that would be seen as a positive thing). One can imagine, however, a proof designed expressly to be highly explanatory, in which the multiplicity of ideas brought to bear would make it of high width.

The concept of width would also appear to suffer from a lack of clarity as to how different kinds of memorized information – digits, concepts, rules – are to be counted. In the above example of arithmetic multiplication, the point is made that mental use of the normal algorithm requires that many interim digits be kept in mind, implying that the procedure has a large width. The width can be reduced, as Gowers states, by using an ingenious procedure employing the difference of two squares. Now, it is true that in the ingenious procedure there are fewer digits to memorize, but on the other hand one has to have had in mind the notion of the difference of two squares. One could argue that the mental capacity to store such a relatively sophisticated mathematical notion is greater than that needed to store a few digits. Perhaps there is an opening to refine the idea of width by considering whether or not different types of information might be assigned different weights.

Another consideration might be how long a piece of information has to be kept in mind. To take the example of the calculation just discussed, the concept of the difference of two squares must be kept in mind always, so it can be called upon whenever needed, in this calculation or others, while the memory devoted to interim digits can be released as soon as the calculation is finished. The management of computer storage has a strong focus on how long a piece of storage is required. The inspiration for the term “width” came from computer storage, so it might be refined by taking into account the dimension of time.

In considering the concept of width, one is tempted by some examples to say that lower width is good, while others show the value of a richness of ideas, in which case a greater width is better. Be that as it may, mathematics educators already know that it is not enough for students, when they engage in proving, to have mastered the concepts and techniques required to construct valid sequences of logical steps. It is important for them to have a broad grasp of mathematics, so that they can draw upon a reservoir of important mathematical ideas, stored in their memory, and see how to apply them to the mathematical argument they are engaged in constructing.

Because it is a psychological concept, it is not surprising that width is difficult to define. Gowers says as much when he states that a “… precise discussion of width as a *psychological* concept is quite difficult” (p. 51, italics in the source).

An even more basic question that might be raised, however, is whether it is necessary to have quantifiable measures of non-syntactical properties. Mathematicians know an elegant proof or an ingenious proof when they see one, just as mathematics educators are quite capable of deciding which proofs are going to be more useful in conveying to their students important mathematical ideas and their connections.

In commenting on the concept of width, one must keep in mind that Gowers sees his investigation as a work in progress. While conceding that the notion of “width of a proof” is not precise and certainly still lacks a formal definition, Gowers hopes that more work will lead him to come up with a definition “more precise than subjective-sounding concepts such as ‘transparent’, or even ‘easily memorable’” (p. 57).

ASSOCIATED MATHEMATICS EDUCATION RESEARCH

A review of the current literature on proof and proving in mathematics education shows that the concept of the width of a proof is not discussed (understandably so since it is a novel one), but several research papers have dealt, as Gowers does, with the non-syntactical properties of proofs and their importance.

Mejia-Ramos et al (2012), recognizing that different proofs of the same theorem improve mathematical comprehension to different degrees, sought to devise an assessment model that could be used to measure the impact of the proof judged more promising from the point of view of comprehension.

They first reviewed the mathematics education research on the purposes of proof; and then the existing recommendations for proof that encourage comprehension, at various educational levels. Finally they interviewed nine university mathematics professors to determine what types of proof comprehension were most valued in university mathematics. Judging by their abstract, these researchers saw the importance to comprehension of the non-syntactical aspects of proof that impelled Gowers to look into the idea of the “width of a proof”.

> […]in undergraduate mathematics a proof is not only understood in terms of the meaning, logical status, and logical chaining of its statements but also in terms of the proof’s high-level ideas, its main components or modules, the methods it employs, and how it relates to specific examples. We illustrate how each of these types of understanding can be assessed in the context of a proof in number theory. (Mejia-Ramos et al, 2012, p. 3).

Hanna and Barbeau (2009) investigated the properties intrinsic to certain proofs that allow them to convey to students methods and strategies for problem solving. They also looked at the extent to which certain types of proof might yield new insights that are somewhat easier to keep in memory.

Closely related to the concept of width is that of a memorable idea in a proof, which Leron (1983) looked at from a unique perspective in his work on structured proofs. Structured proofs are those that are organized into components in such a way that students can see how each component supports the main thrust of the proof. As Leron saw it, the memorable idea was the entire proof structure, not a step within the proof.

Leron and Zaslavski (2009) discuss the strengths and weaknesses of generic proofs. One of the strengths of generic proofs is that they “enable students to engage with the main ideas of the complete proof in an intuitive and familiar context, temporarily suspending the formidable issues of full generality, formalism and symbolism.” (p. 56). Generic proofs contain identifiable (and thus potentially memorisable) main ideas. The discussion in Hemmi (2008) is also closely related to pedagogical properties of proofs and to the way in which students encounter transparency and benefit from it.

In her paper “Key ideas: what are they and how can they help us understand how people view proof?” Raman (2003) characterizes people’s views of proof by bringing together two ideas about the production and evaluation of mathematical proof, making a distinction between an essentially *public* and an essentially *private* aspect of proof, and the notion of a *key idea* related to explanatory proofs. She concurs with Gowers that for mathematicians proof is essentially about *key ideas*.

The vast research literature on proof and argumentation also contains many references to the valuable properties of proof that Gowers sought to capture with his idea of width. See the extensive surveys by Mariotti, (2006) and Durand-Guerrier et al., (2012); see also Pedemonte (2007). For example, Douek (2007) discusses the “reference corpus” that is used to back up an argument. This reference corpus might include visual arguments and experimental evidence, and might even depend on a given social context. Douek argues for non-linearity as a model of the mathematical thinking that takes place in the process of proving. This is where one sees a parallel between Douek's view of argumentation and Gowers' description of the quality of a proof, recalling that Gowers pointed out that proofs admit of "artificial" steps that look as if they "spring from nowhere" but turn out to be useful.

Jahnke (2009) argues that it is possible to bridge between argumentation and mathematical proof through explicit discussions of the role of mathematical proof in the empirical sciences, and by allowing students to bring in factual arguments. He says that "giving reasons in everyday situations means frequently to mention only the fact from which some event depends without an explicit deduction" (p. 141).

Knipping (2008) speaks of a method for revealing structures of argumentation in the classroom and invokes the concepts of local and global arguments to analyze a proof, believing that mathematical logic alone cannot capture all the processes of proving. Local arguments can certainly qualify as "proof ideas" in Gowers's sense.

In sum, Gowers' focus on memorability as a desirable property of a mathematical proof, and his introduction of the concept of width, bring a new dimension to the teaching of proof. There remains a need to investigate the relationship between "width of a proof" and memorability on the one hand, and on the other hand the many other important dimensions of proof and proving, such as mastery of mathematical concepts and techniques, grasp of proof structure, and knowledge of methods of proving. A deeper investigation of the concepts of "width of a proof" and memorability may lead to new ways of presenting and teaching proofs.

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MULTIPLE PROOFS AND IN-SERVICE TEACHERS' TRAINING

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This paper discusses a possibility to use interconnecting 'proof' problems that allow multiple solutions for teachers' professional development. Groups of teachers, which consisted of practitioners from various K-12 grade levels, were asked to produce several proofs of a given statement. I present a sampling of these proofs, which includes approaches and ways of reasoning specific for each grade level. Moments of teachers' collaboration and mutual influence are highlighted. This training method gives the participants an opportunity to make deeper mathematical connections as well as to understand better the culture of proof as a developing process along the entire mathematics curriculum across all grades.

Keywords: interconnecting task, proof development, geometry, teacher collaboration.

INTRODUCTION

Proofs and logical explanations of mathematical ideas appear in various forms at all levels of mathematics education. Many mathematical facts can be observed and hypothesized at an early stage of a child development, perhaps in the primary or elementary school. They are introduced and explained by means appropriate for that level. However, the same facts may be proved later in the secondary school using more sophisticated vocabulary and advanced methodology, and illustrated at the university level using even more rigorous and abstract ideas. A mathematical problem, in particular, one that asks to prove a given statement, is called *interconnecting* if it obeys the following conditions: (1) allows a simple formulation; (2) allows various solutions at both elementary and advanced levels; (3) may be solved by various mathematical tools from different mathematical branches, which leads to finding multiple solutions, and (4) is used in different grades and courses and can be understood in various contexts (Kondratieva, 2011a, 2011b).

The role of multiple proof tasks as a crucial pedagogical tool was highlighted in (Leikin, 2010; see also refs there). As well, Sun&Chan (2009) found these tasks to be powerful "for guiding teaching and learning" within a "spiral variation curriculum". Inspired by these works and my own practices I proposed (Kondratieva, 2011a) that a study of a succession of mathematical ideas that revolves around one *interconnecting problem* is useful for developing learners 'perception of mathematics as a consistent subject. Students familiar with the problem from their prior hands-on experience will use their intuition to support more elaborate techniques taught in the upper grades.

What is lacking about this approach, while it has a theoretical basis, is empirical evidence concerning teachers' practical reliance on it in their own classrooms. A related question is how to help teachers to adopt the interconnecting problem approach through their professional development. In this paper I discuss in-service teachers' experience with a problem that allows multiple proofs. A group of 25 mathematics teachers participated in this study in 2010 and another group of 20 teachers in 2012. Both groups took a graduate course on solving mathematical problems (I was the instructor) and this activity was one of the assignments in the course. Altogether nine subgroups of 5 students were formed, each of which included at least one primary/elementary, one junior high, and one senior high teacher. They were given one week to work on a problem stated below. Within each subgroup, the teachers were contributing solutions and explanations appropriate to the level they teach, reflecting on and checking each other's solutions. Each subgroup was required to submit at least three distinct proofs. According to their responses, this activity allowed the teachers to view the statement from various perspectives and to see how it might be used within different grade levels and mathematical topics. In the next section I discuss a theoretical background of this research project. Then I analyse mathematical ideas that emerged from the teachers' collective work. In conclusion I comment on the possibility for teachers' development offered by this study.

THEORETICAL CONSIDERATIONS

Teachers' abilities to model proofs are very important for their students' progress in understanding of mathematics. Poor or confusing instruction produces a little learning (see e.g. Hsieh, 2005 as sited in Fou-Lai Lin et al, 2012). Thus, teachers' professional development in the area of proofs and argumentation is desirable (Stylianides & Stylianides, 2009). Teacher education should involve such classroom practices as spontaneous engagement in the processes of justification and evaluation of mathematical ideas and arguments (Simon & Blume, 1996). Solving proving tasks individually or sharing and criticizing each other ideas in small groups are recommended practices helping teachers developing their understanding of proof (Zaslavsky, 2005; Stylianides & Stylianides, 2009). In the process of proof construction, both self-convincing and persuasion of others are important (Harel and Sowder, 2007; Mason et al 1982). Teachers need to have an appropriate experience of proving processes in order to successfully implement them in their classrooms, which often would require a dramatic change in existing classroom practices (Douek, 2009). Such experiences include making conjectures, moving from experimental verification to general argumentation (Kunimune et al, 2009), becoming aware of limitations of current 'justification schemas' (Harel & Sowder, 1998), and developing more sophisticated or more efficient proofs. It is also desirable that teachers were familiar with the culture of argumentation, made rational choices of mathematical tools and means of communication (Boero, 2011) and were explicit regarding transparency of proofs' presents in their classrooms discussions (Hemmi, 2008).

It had been observed that primary and secondary teachers might have distinct views on the appropriate ways and means of mathematical argumentation. This is generally consistent with cognitive stages of maturation of proof structures (Tall et al, 2012) as the learner grows from a child to adult. However, some elements of teachers' practices may cause an obstacle in their students' proper mathematical development. Teachers dealing with very young students use little symbolism and operate with 'quasi-real' mathematical objects (Wittman, 2009) and may be reluctant to accept other modes of argumentation. Elementary teachers also tend to rely of textbooks or more capable peers' information while constructing their own arguments (Simon & Blume, 1996). In contract, secondary teachers often reject verbal and visual proofs as being invalid (Biza et al, 2009) as they believe that all proofs must be formal algebraic (Dreyfus, 2000) and follow specific steps (Herbst, 2002). Consequently these teachers may focus on steps and algebraic details ignoring the overall logic of argumentation (Knuth, 2002). The idea that all proofs must be formal and rigorous leads some teachers to believe that proofs are inaccessible for grade school students, and thus excludes proofs from their pedagogical repertoire.

In this study, I was looking at a possibility to address several concerns and to adopt teachers' training recommendations found in the literature through teachers' engagement with interconnecting problems. When forming working subgroup of in-service teachers I specifically combined primary and secondary teachers in order to achieve the following outcomes: (1) expose elementary teachers to techniques and approaches employed in the secondary school; (2) expose secondary teachers to 'common logic' and intuition based explanations available at the elementary level; (3) let teachers to collaborate in solving an interconnecting proving problem, and thus let them to see and evaluate each other's concept of proof. Such collaboration could allow teachers to perceive proof as a developing and continuing process present in various forms at all stages of schooling. I was interested to collect an evidence of these processes as well as to find out what kind of assistance the teachers might need in order to benefit from solving interconnecting problems in mixed groups.

A PROBLEM AND COLLECTIVE POOL OF IDEAS

The problem for this study was chosen from (Totten, 2007) based on the criteria (1)-(4) for the interconnectivity listed in the Introduction.

<u>Problem:</u> Given a square ABCD with E the mid-point of the side CD, join B to E and drop a perpendicular from A to BE at F. Prove that the length of the segment DF is equal to the length of the side of the square.

In this section I present all ideas generated by nine groups of five teachers. Some parts of original (given in italic) students' solution are summarized to save the space.

<u>Approach 1.</u> Direct measurement and comparison using various materials including a ruler, string, Popsicle sticks, compass, or dynamic geometry software.

> *I used point D as the center of a circle and placed my compass on point C which I knew was 6 units from point D, as seen in the diagram provided. I wondered if point F would be a point on the circle's circumference. I tried it, and sure enough point F was on the circumference of the circle. Point D was the center of the circle and both points C and A were on the circumference of the circle.*

Therefore, segment DF is the same length as line AD and line DC which are 6 units in length. As well it is the same length as line AB and line BC, since all four sides of a square are equal. Below (see Figure 1, left) *is the graph with part of the circle described above drawn to show how I showed that segment DF was equal to the length of the side of the square ABCD:*

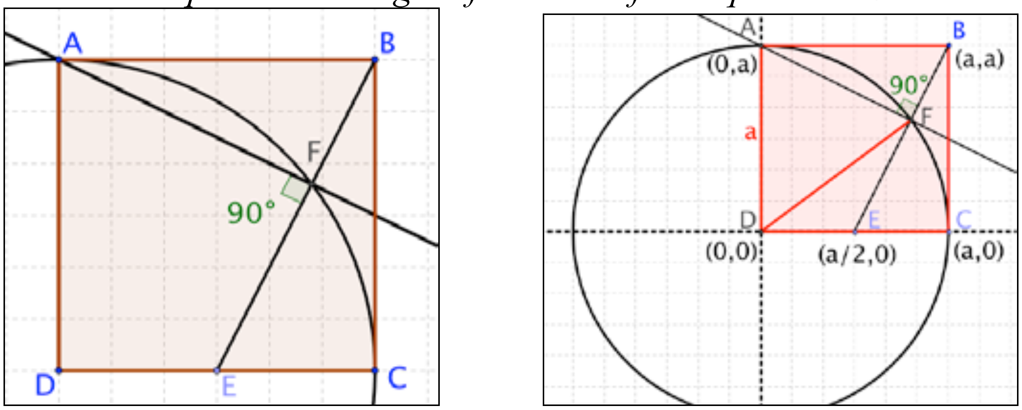

Figure 1: Approach 1 (use of compass) gives rise to Approach 2 (use of coordinates).

This approach generated an algebraic method produced by another group member.

> *I **really** liked the idea that was suggested in Method 1 of drawing a circle through the points using D as the Center, but I am not sure how to generalize that. STUCK! Then I thought that if I could find the point F, I could sub it into the equation for my circle using a side length of "a". I know how to find F by using the equation for the line that intersects to make F. I can find the slopes using "a" as my side length, but how can I find the y intercept to finish my linear equations? STUCK! If I incorporate my previous idea of using points into this it will work! AHA!*

<u>Approach 2.</u> Use of Cartesian coordinates of the points (see Figure 1, right). Let's assume that D is located at the origin. Denoting the side length of the square as a we have the points $D(0,0)$; $A(0, a)$; $B(a, a)$; $C(a,0)$. To determine the coordinates of point F we must first find the equations of the lines BE and AF. We have two points $B(a, a)$ and $E(a/2,0)$ from which we may determine slope of the line BE and ultimately the y-intercept: $y=2x-a$; Since BE is perpendicular to AF, the slope of line AF is -0.5 and the y-intercept is a. Thus AF has equation $y=-0.5x+a$. The intersection point F is found by solving the equation $2x-a= -0.5x+a$, which implies $x=0.8a$ and $y=0.6a$. The distance DF between F and the origin D is the square root from the sum of squares of the coordinates of F, which after simplifications gives a, the side length of the square. This completes the proof. In students' version of the proof it reads:

> *So point F is* $(4a/5, 3a/5)$ *. Then to see if it fell on the circle, we just need to plug it into the equation for our circle. The circle has a radius of "a" so we have our equation:* $x^2 + y^2 = a^2$*. Now we just substitute our point F and see if it works:* $(4a/5)^2 + (3a/5)^2 = (16a^2 + 9a^2)/25 = a^2$*. Thus, our point F must be on the circle. Therefore, segment DF must be the exact same length as AD and DC which are also radii of the same circle and are also the sides of the square ABCD!*

Approach 3. Recognition of similar and congruent triangles (see Figure 2, left).

I had to find yet another proof. I liked Method 2 but needed to show that AFD is an isosceles triangle by some other method... I used the properties of similar triangles and congruent triangles to show that $\overline{DF} = \overline{DA} = 6$ *units. Draw a line from point D to the midpoint of* AB. *Call this midpoint X.* DX *is parallel to* EB *because they have the same slope, so* DX *must intersect* AF *at a* 90^o *angle. Call this intersection point Y. Consider* ΔAFB *and* ΔAYX. $\angle FAB = \angle YAX$ *(common angle);* $\angle AFB = \angle AYX = 90^o$ *angles. Therefore,* $\Delta AYX \cong \Delta AFB$. *This means that* $\frac{\overline{AF}}{\overline{AY}} = \frac{\overline{BF}}{\overline{YX}} = \frac{\overline{AB}}{\overline{AX}}$. *Since:* $\overline{AX} = \overline{XB} = 3$ *units,* $\overline{AB} = \overline{AX} + \overline{XB} = 6$ *units, and the ratio* $\frac{\overline{AF}}{\overline{AY}} = \frac{\overline{AB}}{\overline{AX}} = \frac{6}{3} = 2$ *units. We know that* $\overline{AB}$ *is twice as long as* $\overline{AX}$, *so* $\overline{AF}$ *must be twice as long as* $\overline{AY}$. *Therefore,* $\overline{AY} = \overline{YF}$. *Consider* ΔDAY *and* ΔDFY. *We know:* $\overline{AY} = \overline{YF}$ *and* $\angle AYD = \angle FYD$ *are* $90°$ *angles, and* $\overline{DY} = \overline{DY}$ *(common side). Therefore,* $\Delta DAY \cong \Delta DFY$ *because of the Side-Angle-Side congruence property. So* $\overline{DF} = \overline{DA} = 6$ *units. QED*

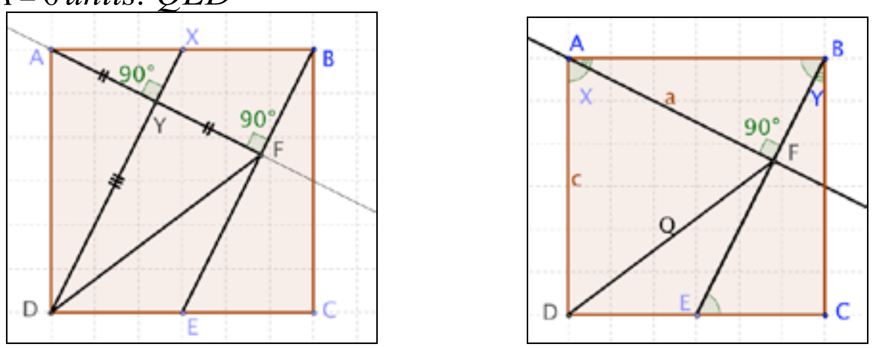

Figure 2: Approaches 3 and 4 look at triangle AFD in two different ways.

Approach 4. Use of trigonometry and the Cosine Theorem (refer to Figure 2, right).

I was looking at triangle AFD and thought that I could possibly apply the Cosine theorem to find the side of interest. Note that angles CEB, EBA and DAF are equal, call it X. Angles CBE and BAF are equal Y, and X+Y=90 degrees. Let the square has side c, $\overline{AF} = a$ *and* $\overline{DF} = Q$. *Then we obtain the following.*

From right triangle ABF we have $c = a/\sin(X)$. From right triangle BCE we find $\cot(X) = 0.5$. From triangle AFD we conclude by Cosine Theorem that $Q^2 = c^2 + a^2 - 2ac\cos(X) = c^2 + a^2 - 2a^2\cot(X) = c^2 + a^2 - a^2 = c^2$. So, $Q = c$, or $\overline{AD} = \overline{DF}$.

Reflection: *Will this work for any size square? Yes because the ratio of the sides used in ΔBEC will always be 0.5 because E is the midpoint of side DC.*

Approach 5. Based on the recognition that AFED is cyclic (Figure 3, left).

Aha! ***ADEF*** *forms a quadrilateral and opposite angles* $\angle ADE$ *and* $\angle AFE$ *are both* 90^o *so they add up to* 180^o. *(We know* $\angle AFE$ *is a right angle because it's supplementary to* $\angle AFB$**)** *This means the other two opposite angles (*$\angle DAF$ *and*

*$\angle DEF$) must also add up to 180^o since all angles in any quadrilateral add up to 360^o. Therefore, a circle can be constructed around the quadrilateral **ADEF** where each vertex, **A, D, E,** and **F** lie on the circle. Now it's likely I can prove it using arc measures.*

Now, note that angles CBE, BAF and DAE are equal, call this value y. Then $\angle DAF = 90^O - \angle BAF = 90^O - y$. From the relation for inscribed angles and arc measures we have $2\angle AFD = arc(ADE) - arc(DE) = 180 - 2y$. Thus $\angle AFD = 90^O - y$.

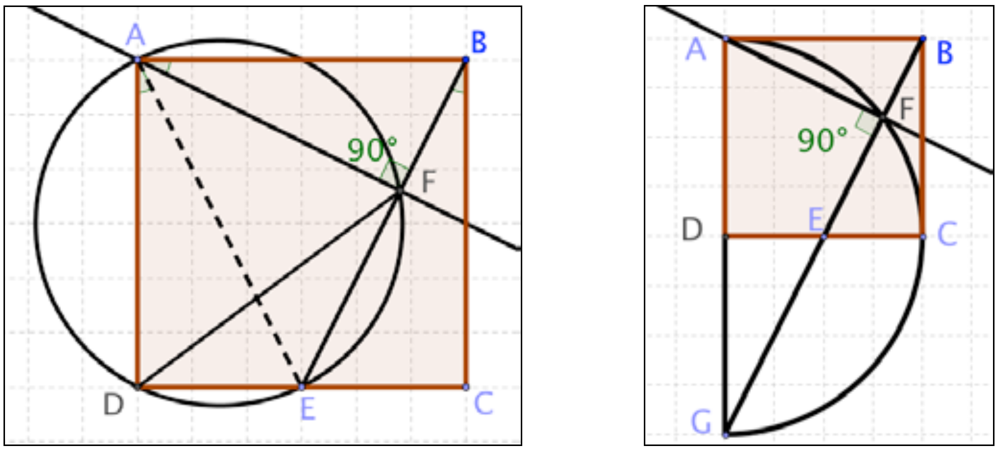

Figure 3: Approaches 5 and 6 use two different auxiliary circles.

*So line segments **AD** and **DF** must be of equal length since the isosceles triangle theorem states that sides which are opposite of equal angles in an isosceles triangle must also be equal. So **DF** is the same length as the sides of the square since **AD** is one of the sides*

ANALYSIS OF THE SOLUTIONS IN VIEW OF TEACHERS' TRAINING

Participants of this study were all enrolled in my graduate course on problem solving. Prior to solving the 'proof' problem presented in the previous section they practiced in solving several other problems individually and in groups. Their reading included the book by Mason et al (1982) regarding the stages of mathematical thinking and my paper (Kondratieva, 2011a) regarding the theory of interconnecting problems. Each group was asked to create as many as possible (but at least three) distinct solutions, individually comment on the thinking process highlighting AHA and STUCK moments (Mason, 1982), and as a group reflect on each others' approaches, identify their place in mathematics curriculum and select the most appropriate solutions for submission. Leikin (2010) suggests that 'collective solution spaces' are sources for development of 'individual solution spaces' of the group members. Teachers were asked to comment on that effect and the perceived usefulness of this training method.

Teachers' solutions and responses were examined in order to compare contributions from primary and secondary teachers as well as to observe instances of *productive* collaboration (when participants reveal and build on mathematical connections between their individual ideas). Analysis of teachers' work shows the following.

First, while there was a disagreement about sufficiency of Approach 1 that involves direct measurement, it was included in almost all groups' reports. Many primary school teachers provided detailed lesson plans on using various instruments helping students to construct and measure elements of the picture. Other group members often commented that this approach is not qualified as a proof but still is very convincing and illustrative. This result concurs with literature stating that primary and secondary teachers may disagree about adequacy of some explanations. But, remarkably, this simple approach stimulated other group members to invent more rigorous justifications. The collective effort that converts a pictorial insight onto an analytic calculation (Approach 2) reveals a *cognitive unity* (Boero et al, 1996) of an empirical argument and rigorous proof construction. Moreover, the participants agreed that due to such experiences they started to view learning to prove deductively as a gradual and multimodal process originated from reflections on empirical actions.

Second, the majority of solutions dealt with concrete numbers. As it is evident from Approach 3, the teacher uses side of length 6 throughout her solution. While teachers had read about specialization and generalization techniques (Mason, 1982) and discussed them with their peers, still the tendency to use concrete numbers without further generalization was evident in the majority of papers. However, some teachers either made a comment on how to generalize their solution (see Reflection in Approach 4), and some had a proof in a general form (see Approaches 2 and 5).

Third, many teachers used approximate calculations. For example, in the trigonometric approach they would typically write "take $\arctan(0.5) = 63.4^{O}$", and then used approximate values of $\cos(63.4^{O})$ and $\sin(63.4^{O})$ to calculate the length of segment DF despite that the equality they were proving was exact.

Fourth, many submissions were very wordy and far from being mathematically efficient in reporting their final solutions. Even though the participants were asked to submit the best possible solution, many papers contained lengthy algebraic calculations that could be easily optimized. This likely reflects teachers' belief that every little detail must be brought up. But in doing that they often unnecessarily repeated or rephrased the same idea, and explained obvious things ("$\overline{DY} = \overline{DY}$ *common side*"), as can be seen e.g. in Approach 3.

Fifth, some groups submitted several solutions that employed the same mathematical idea and differed in very little details. It seems that the group members were hesitant to make their judgement and delegated the responsibility to choose the best solution to their instructor.

And finally, while the group members' collaboration was evident on several occasions and participants as a whole had produced a great deal of approaches, still there were solutions missed by the groups, even though elements of those solutions were present in the collective pool of generated ideas. As an example, the following approach was never proposed by the teachers, but when suggested by the instructor, they agreed that they were very close to discovering it by combining ideas from their Approaches 1, 2, and 5.

Approach 6. Extend lines AD and BE and call the intersection point G (Figure 3, right). Note that DE is the midline in ABG, that is points, E and D are midpoints of sides BG and AG respectively. Since AFG is a right triangle then its vertices lie on a circle, and hypotenuse coincides with the diameter of the circle. Thus $\overline{DF}$, $\overline{DC}$ and $\overline{DA}$ are all equal to the radius of the circle.

For completeness, I give another approach that employs a bit more advanced technique and can be used to illustrate the advantage of learning some further theorems in Euclidean geometry.

Approach 7. Based on Ptolemy's Theorem for cyclic quadrilaterals.

For cyclic quadrilateral AFED (Figure 3, left), the Ptolemy's theorem reads: $\overline{AF}\cdot\overline{DE}+\overline{AD}\cdot\overline{FE}=\overline{AE}\cdot\overline{DF}$. Let $\overline{DE}=a$. Then $\overline{AD}=2a$, $\overline{AE}=\overline{BE}=\sqrt{5}a$. Set up equations $\overline{BF}+\overline{FE}=\sqrt{5}a$, and $(2a)^2-(\overline{BF})^2=5a^2-(\overline{FE})^2=(\overline{AF})^2$ from right triangles AFB and AFE. Solving the system, we get $\overline{BF}=2a/\sqrt{5}$, $\overline{FE}=3a/\sqrt{5}$, $\overline{AF}=4a/\sqrt{5}$. Substituting these values in Ptolemy's equality we find $\overline{DF}=2a$, which is the side length of the square.

CONCLUSION

This paper analyses a collection of proofs produced by groups of in-service mathematics teachers whose expertise ranged across all grade levels. Based on their responses, all participants of this study found it very informative to collaborate on one problem and produce proofs employing various methods and ideas capitalizing on the "interplay of empirical and theoretical argumentation" (Jahnke, 2008). In words of one teacher, "I never thought before of a possibility to prove the same claim in multiple ways. I was really amazed to see how many different approaches were proposed by my teacher-colleagues and how they all fit in different grades' math topics". This study reveals the potential of the use of interconnecting problems for teachers' training in mixed groups. Such setting allows teachers (1) to learn, evaluate, and criticize each other's solutions, (2) to share their ideas and to persuade their peers, (3) to collaborate on connecting intuitive and experimental methods with general argumentation, (4) to produce more efficient proofs, and (5) to choose appropriate tools and means to communicate their reasoning. Note that all these experiences are recommended in the literature for teachers' professional development. In addition, this training method allows teachers to see how different approaches are pertinent to different grades. Perceiving mathematics curriculum as a whole process of knowledge accumulation, teachers begin to acknowledge that many secondary school arguments are deeply rooted in primary/elementary level activities. At the same time, some study participants did not fully benefit from the offered exercise. This suggests the necessity of more focused supervision and advising of mathematics teachers during their training. In particular, such advising should aim at developing habit of spontaneous moving from specialization to generalization, conscious distinction between exact and approximate calculations, and reviewing one's own solutions in order to eventually present them in a more general, insightful and concise form. Resonating with Koichu (2010), this study also poses the questions: Why did all groups of teachers overlook certain approaches that clearly were within their capacity to produce? What can be done to ensure that teachers' collaboration realises the entire potential present in the individual contributions?

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NEW OBJECTIVES FOR THE NOTIONS OF LOGIC TEACHING IN HIGH SCHOOL IN FRANCE: A COMPLEX REQUEST FOR TEACHERS

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If everyone agrees that logic is needed to do mathematics, there are divergences concerning the role of mathematical logic in acquiring the necessary and sufficient knowledge in this area. In France today there are new syllabuses in which notions of logic are explicitly mentioned. How do teachers integrate them into their teaching? In this paper, through a study of the French syllabuses and textbooks, I show the strong constraints and ill-defined conditions for this teaching of logic notions. I also describe the contents of an in-service training course for teachers, which aims to give them tools for this teaching.

Key words: logic-language-reasoning-textbooks-teacher's training

INTRODUCTION

In the introduction of the proceedings of the 19th ICMI Study Conference: Proof and Proving in Mathematics Education (2009), the authors note that some research should be pursued to understand the role of logic in the teaching of proof. The experience of teaching formal logic in high school during the time of "modern mathematics" in the 1970s in different countries has shown that this approach does not directly provide students with effective tools to improve their abilities in expression and reasoning. In most of these countries, logic then disappeared from high school syllabuses, but students' difficulties in expression and reasoning persist and the debate on the role of logic in the treatment of these problems remains open.

In France, logic was briefly re-introduced in the 2001 syllabus. The 2009 syllabus, which is still in application, goes even further: it includes objectives for "mathematical notations and reasoning" which are linked to notions such as connectives AND/OR, negation, conditional propositions, equivalence, different types of reasoning. Behind these notions there are objects which are defined and studied by mathematical logic. But the aim stated by the syllabus is not to teach mathematical logic, but to convey the knowledge in logic necessary to mathematical activity. Then, two main questions emerge: How can teachers identify the logic they have to teach? What kind of (pre-service and in-service) training should they have to efficiently organize their teaching? The French context, where logic has been re-introduced in high school syllabus, is suitable for providing elements of answers to these questions.

These questions brought me to study the system of conditions and constraints in which the teacher makes his teaching choices. A more detailed study of syllabuses and textbooks from 1960 highlights the features of this system. First of all, the history of teaching logic in mathematics in France is a tormented history, and some teachers never got courses about notions of logic during their studies. Furthermore, since this teaching has been absent from syllabuses over the past years, teachers did not really have to think about it during their preservice training. Finally, a first glance at the activities proposed in the textbooks and at the content of the pages dedicated to the notions of logic mentioned in the syllabus shows diversity in the interpretation of this syllabus, but also a certain lack of knowledge of the notions at work. But prior to this study, more general questions should be raised about the links between logic and mathematics and between logic and mathematics teaching. One of the aims of this study is to contribute to the reflection on the necessary training in logic for teachers, so they could teach the underlying logic in mathematical activity.

In this paper, I will mainly present this study. It shows the difficulties encountered by French high school teachers in integrating notions of logic in their mathematics teaching. One axis of this work is to show the importance of language[1] issues in the contribution of logic to the mathematical activity. This dimension is not taken into consideration in a precise way in the documents used as resources by teachers. Logic is often seen as linked to reasoning, in general and more specifically in mathematics. But it is also linked to the setting-up and the functioning of a language which allows to describe the structure of propositions and of reasoning, and the links between both. I see mathematical logic as a reference theory dealing with objects which may be tools to analyze our mathematical discourse and to understand the ambiguities inevitably linked to the use of certain formulations which are informal or implicit in the mathematical language.

First of all, I will present some quick thoughts on logic and language, and some didactical studies that have shown difficulties for some students, probably linked to language. Later on, I will present the study of the system of conditions and constraints I have previously stated. Finally, I will briefly describe the content choices made for a training called "Initiation to logic" for teachers in activity.

LOGIC AND LANGUAGE IN THE MATHEMATICAL ACTIVITY

The study of different moments in the history of logic shows that the constitution and description of a logic is accompanied by a necessary formalization of language, in the sense of a codified formatting, whose codes vary of course depending on authors and eras. For example, in his logical work, *The Organon*, Aristotle (Greek philosopher, IV^e century BC) began to explain what he called a proposition (enunciative sentence, in which there is truth or error), and then classified these propositions according to two criteria, quality (affirmative or negative) and quantity (universal or particular). He obtained four types of propositions that are the building blocks of syllogisms, which could be treated in a formal way, because the propositions can be replaced by variables. Nevertheless, Aristotle did not want to formalize the relationship between

[1] The use of the term "language" does not refer here particularly to common language, or to the mathematical one, but to the use by someone of signs he organizes with the intention of communicating something.

the parts forming these propositions and did not use logical constants like connectives and quantifiers.

Since the beginning, some mathematicians have sought the necessary and sufficient formalization of their language that ensures the infallibility of reasoning. For some of these mathematicians, language had to be totally formalized, to reach the univocity of the meaning of expressions and the possibility of a formal treatment of these expressions, independently of their meaning. For another part of these mathematicians, too much formalization does not allow the intuitive progress of reasoning. Mathematical logic can be seen as the culmination of this research. An important step has been taken by modeling mathematical language and reasoning, allowing a further exploration of the properties of these logical systems using mathematical tools.

These epistemological considerations lead me to the assumption that mathematical logic can provide tools, probably first to the mathematician and mathematics teacher, to analyze the language they use in their mathematical activity, and to detect its ambiguities. For the teacher, an additional challenge is to provide tools for the analysis of students' reasoning, allowing to highlight another possible understanding. This can occur, for example with the propositions "if ... then ..." which are implicitly universally quantified, but quantification is not always perceived by the students. That may lead them to give a response which is not expected, and is yet the result of a correct reasoning. For example, some pupils could say that the sentence "if n is prime, then n is odd" is neither false, nor true, while almost all mathematicians say it is false (you can find a more detailed example with the "maze" task described in (Durand-Guerrier, 2004)).

In a more general way, various studies based on experiments with university students show the difficulties they have encounter in understanding and proving quantified statements (Dubinsky, Yiparaki, 2000, Arsac, Durand-Guerrier, 2003, Chellougui, 2009, Roh, 2010). For most students engaged in proving if a statement is true or false, the relationship between the quantified formulation of a statement and the framework of its proof is not clear. Thus, while recognizing the role of informal statements in memorizing mathematical results, J. Selden and A. Selden make the assumption that the ability to unpack the logic of an utterance by formally writing it is related to the ability to ensure the validity of a proof of this statement (Selden & Selden, 1995).

Unpacking the logic can be seen as writing the statements and making explicit some conventions, for example about quantification. The predicate language did not become the universal language in which the mathematicians express themselves, but a reference language, and, depending on the nature of their activities (research, drafting a communication, course ...), they use formulations whose logical structure is more or less exhibited. For example, calling a predicate P on a variable n which is a natural number, mathematicians commonly say "$P(n)$ when n is big enough." But to explain to students how to show this property, they may reformulate this as "$P(n)$ for all n beyond a certain value". And in a course, they could write "there exists N_0 such that for all $n \geq N_0$, $P(n)$". This interplay between different formulations, which is easy for a mathematician, does not always seem so simple for students! Though, I believe that a rewording work is important because it contributes to the construction of what J. Selden and A. Selden call "statement images" (Selden & Selden, 1995, p 133):

> These are meant to include all of the alternative statements, examples, nonexamples, visualizations, properties, concepts, consequences, etc., that are associated with a statement.

This invites us to think about activities for students to develop the ability to rewording, which is rarely an explicit goal of education and rarely proposed as an explicit task, and to think about the knowledge that teachers need in order to engage in such activities. S. Epp mentions the need for solid knowledge concerning language and logic (Epp, 1999, p3):

> When given by teachers with a solid command of mathematical language and logic, such feedback can be of enormous benefit to students' intellectual development.

HOW CAN A FRENCH HIGH SCHOOL MATHEMATICS TEACHER TEACH LOGIC TODAY?

Research questions, study materials and methodology

There are some goals concerning "mathematical notations and reasoning" in the first high school year[2] mathematical syllabus, launched in September 2010. In these instructions, some objects of mathematical logic are explicitly mentioned. Therefore, teachers have to build up a teaching allowing to reach these aims. While doing so, they have choices to make, for which they are submitted to a system of institutional conditions and constraints. I think it is interesting to study this system in order to understand the teachers' practices concerning logic, and in order to think about the training they need for that. I have conducted this study with the three following questions: what is the scholarly knowledge of reference for this teaching? *("savoir savant"* (Chevallard, 1985))? What are the notions to teach (the knowledge to be taught, *"savoir à enseigner")*? Where and why do we find logic in the high school mathematics teaching in France (ecological approach (Artaud, 1997))? I have looked for the answers by analyzing different documents from 1960[3] until now. I think the epistemological approach is essential to understand the current choices.

The study of syllabuses and their joined instructions contributes to answer these three questions. I have noticed the notions of logic present in these documents, as well as the terms used to speak about logic. These terms give information about the specific function attributed to logic, in relation with language and reasoning. I have completed this study by searching in the APMEP[4] periodicals for information on the reactions

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[2] 15-16 years old students.

[3] 1960 is the first year of the introduction of logic in the high school mathematics syllabuses in France.

[4] Association des Professeurs de Mathématiques de l'Enseignement Publique, the most important association

and expectations of the teachers who are on the field of teaching. I have searched for the presence or not of articles on logic, and of debates concerning its teaching. Finally, I have studied what was said about logic in the textbooks. These textbooks are seen both as a possible interpretation of syllabus, proposing a "dressed knowledge" (*"savoir apprêté"*) [Ravel, 2003] and as a resource for teachers. I have tried to determine if and how the notions of variable, proposition, connectives and quantifiers were introduced, more specifically if and how their syntaxic and semantic aspects[5] were present and linked. I have also searched what kinds of tasks were designed.

The analysis shows the complexity of the current demand: the conditions are ill-defined and the constraints are strong.

What is the scholarly knowledge of reference for the teaching of logic of mathematics?

A part of this complexity lies in the logic itself. The question that interests us here is not the teaching of mathematical logic as a branch of mathematics. The question rather has to do with the teaching of the logic of mathematics, which I defined as "the art of organizing one's speech in that discipline, seen under the double aspect of syntactic correction and semantic validity". One of the difficulties with this logic of mathematics is that there is no reference content, no consensus on the knowledge in this area that is needed to do mathematics and on the words which should be use to phrase this knowledge. Here is an example of the potential problems with the lack of common knowledge reference. In the objectives of the new syllabus, students must "be trained on examples to properly use the logical connectives "and", "or" and to distinguish their meanings from the common meanings of "and", "or" in the usual language." But neither in the syllabus, nor in the accompanying document on this subject, entitled "Resources for high school first year, Notations and mathematical reasoning" can we find a definition of the logical connectives "and", "or". However, the distinction between the use of these words in the language of mathematics and in everyday language is not limited to the inclusive or exclusive character of the connective "or". Yet it is the only thing mentioned in the textbooks or in the accompanying document.

Another essential difference lies in the fact that in mathematical language, the connectives "and", "or" link two propositions, which is not always the case in the everyday language, even if it is spoken about mathematics. For instance, if we try to show the logical structure of the proposition "Sets A and B are not empty and disjoint", there will be three "and" that will correspond to connectives between propositions (in capital letters), and one that will not match: "Set A is not empty AND set B is not empty AND sets A and B are disjoint". Not only is this distinction

of mathematics teachers in France.

[5] In its syntactic aspect, a connective is an operator on propositions and its semantic aspect is given by the truth tables.

not mentioned, but in addition to that, the study of certain exercises proposed in textbooks show confused conceptions about notions that yet seem as simple and usual in mathematics as the connectives "and", "or". This shows that even if someone has studied mathematics and therefore had used logic for this activity, it is still not enough for him to constitute a knowledge that could sustain a teaching of certain notions of logic to students. I make the hypothesis that the part of the mathematical logic which is generally called "propositional calculus and predicate calculus" is a corpus of knowledge that can be a reference to teachers in their teaching of mathematics logic. The one condition for that is for it to be linked with the study of language and reasoning practices in mathematics. No such reference is explicitly proposed nowadays, as opposed to what happened at the time of modern mathematics: in 1970, the instructions along with the new syllabus of the first high school year gave definitions and first properties of connectives and quantifiers.

What is the knowledge to be taught ?

Concepts of logic are mentioned for the first time in the mathematics syllabus for students in their first year of high school in 1960. During the middle of the twentieth century, the French mathematicians were strongly influenced by the Bourbaki group of mathematicians, whose axiomatic style spread into teaching. The reform called “modern mathematics” came into force in high school with the syllabus for first-year high school students in 1969. This syllabus was based on the idea of a unified mathematic, that could be used in experimental sciences as well as in human ones. Mastering the language of mathematics was then essential, and it was the essential function of logic. All textbooks from that time start with a first chapter on set theory and logic, which are the foundations of this language. The syntaxic and semantic aspects of the notions of logic were present. The instructions of 1970 specifically say that "the chapter *Language of sets* should become more of a practical introduction at any time in the course then of a dogmatic preamble", but they do not give examples of this practical introduction. The APMEP periodicals have published several articles between 1960 and 1975, linked to logic (theoretical presentations or stories of didactical experiences), and were spokesmen of animated debates such as the one concerning the use of quantifiers symbols. This introduction of logic in the high school mathematics syllabi then seemed like an ambitious project, linked to the acquisition of mathematical language, and was sustained by a certain wish to train teachers, not only to mathematical logic but also to modern mathematics in general. But if we look at textbooks, we can see that most of the time this logic has been reduced to simply a formal presentation of notions of logic, without linking it to mathematical activity. This presentation was not adapted to the initiation to mathematics logic for a mass of student reaching high school.

In 1981 came a radically different new syllabus. It was the time of the “counter-reform”, in which logic was explicitly excluded from mathematics teaching. Modern mathematics have been vehemently accused of being too formal, elitist and not linked to mathematics applications. Logic taught by then and some representative elements such as the tables of truth are almost symbols of this excessive formalism. This might be an explanation to the fact that none of the first high school year textbooks give tables of truth, even if some of them describe their contents, saying for instance that "the proposition **P and Q** is true only in the case where P and Q are both of them true".

This lasted until the implementation of the 2001 syllabus, which states that "training in logic is part of the requirements of high school classes". This text is included in the 2009 syllabus and is supplemented by a table setting targets for "notations and mathematical reasoning". These objectives relate to certain objects of mathematical logic, but are rather vague. It is also specified there that "the concepts and methods relevant of the mathematical logic should not be the object of specific courses but should naturally take their place in all the syllabus chapters". I will give as examples two of these objectives: "Students are trained, based on examples, to correctly use the logical connectives "and", "or" and to distinguish their meanings from the common meanings of "and", "or" in the everyday language. They are also trained to wisely use universal and existential quantifiers (the symbols ∀, ∃ are not due) and to spot implicit quantifications in certain propositions, particularly in the conditional ones." We have already seen that these objectives are not clear, concerning the logical connectives "and", "or", and that the Resources document which come with the syllabus does not take in charge all the important aspects of these notions. The second example concerns quantifications, mainly the implicit ones. We have evoked p3 certain didactical studies that have shown how these quantifications created difficulty for the students, for instance in the understanding of the propositions "if…when… ". Still, this does not appear in the Resources document. The syllabus instructions put logic both on the side of language and the reasoning one (presentation of the different reasoning types). But the links between logic and mathematics are not the same as in the time of modern mathematics. Here, it is essentially about specifying the mathematical language's particularities related to the everyday language. Furthermore, the proposition, a base element of the mathematical language, is absent from syllabuses and the Resources document.

This tormented history of logic in teaching mathematics in high school mainly has two consequences that contribute to the complexity of what is actually asked from teachers. On the one hand, all of them do not have the same training for these notions. On the other hand, there has not been a continuous thinking, particularly a didactical one, about that teaching of logic notions in high school.

Furthermore, we have seen that logic should be taught along with other notions, which represents a strong constraint as far as time is concerned. It should also be caught were it lies, which implies that teachers should be lucid and serene enough to be able to spot it.

Logic in the textbooks

I will only talk here about the current textbooks. Textbooks authors should follow the directions, even though imprecise, given in the syllabus, to provide teachers with tasks for their students. Thus, in most textbooks published in September 2010, we can find pages offering a brief overview of the concepts of logic mentioned in the syllabus. These pages are not a separate chapter. They are a sort of glossary which students and teachers can refer to during certain tasks concerning logic. An analysis of ten mathematics textbooks published in 2010 shows a diversity in the presentations. Some books have one approach that can be called "propositional", which means they constitute a kind of "grammar of mathematical propositions". Other books have an approach that can be called "natural", which means that they take common language as the starting point to speak about mathematical language, while specifying the requirements for this discipline, in particular the requirements of univocal meaning for each word. Because the syllabus demands that logic does not constitute a course on its own, very few textbooks use the terms "definitions", "properties" in their pages, even though it is exactly what they give. Exercises associated to logic are essentially "True or False" ones, and do not allow to work on the language. Finally, we can find mistakes, such as the confusion between "if…then…" and "therefore", in the pages talking about logic or in the exercises of certain textbooks. The lack of knowledge of some teachers concerning logic makes it difficult for them to spot and analyze theses mistakes.

We have seen through the study of the documents contributing to define the *knowledge to be taught* that this knowledge is not clear for the mathematics high school teacher today in France. It appears to us that it is therefore necessary to offer them trainings that will give them the tools needed for this teaching. Because logic is as essential to address language issues as it is to address the reasoning issues in the mathematical activity, I think that mathematics teachers should have tools allowing them to think about the way they speak, and about the ambiguities and implicits that lie under some usual formulations in our practice of mathematics. Mathematical logic seems a possible reference for this reflection.

AN INITIATION TO LOGIC IN THE FRAMEWORK OF A CONTINUOUS TRAINING FOR TEACHERS.

The Institut de Recherche pour l'Enseignement des Mathématiques (IREM)[6] at the Paris Diderot University proposed in 2011 a training course called "Introduction to logic" as part of the continuous training for teachers (this training course had already been organized in 2010 and renewed in 2012). This course was led in collaboration with René Cori, professor in the logic team of the Paris Diderot University. Fifty teachers (the number of places was limited) enrolled in this course, forty of them

[6] Research Institut for Mathematics Teaching

were effectively attending. The training took place during three days of 6 hours each (two consecutive days in January, then one separated day a month later). One of the training goals is to give the trainees knowledge in mathematical logic. This does not mean lecturing about mathematical logic. It is all about teaching logic for teachers, a logic in context, at the service of mathematical activity. What is proposed is an analysis and a critical look at mathematical language with which teachers are already familiar. An important place is given to the notion of variable, that we will present as being characteristic of mathematical language in relation to the common language, and the multiple ways mathematics use to implicitly quantify their statements. The logical connectives are then presented as operators on propositions, which means they allow, starting from one or two propositions, to "create" a third one. This syntactic aspect is separated from the semantic aspect broached by giving the truth tables of these connectives. The notions of tautology and propositions logically equivalent are defined and put in relation with the practices of reasoning. An important moment is dedicated to implication: establishing its truth table creates reactions. Then, it is essential to note that the negative of a conditional proposition is not a conditional proposition. We also discuss at length about the implicit universal quantification associated to the formulation "if… then…". We finally suggest few developments on the study of theories.

Another important part of the training is a more practical aspect, based on the study of the school textbooks. Basing themselves on selected parts, the trainees do a critical analysis in small groups. At the end, we share our work. This practical exercise allows to show the misunderstandings there can be about some notions, and was based on the taught theoretical components (this work was done after having spoken about variables, connectives and quantifiers). We also devote time during the third day for the trainees who wish to present activities they have done in class, so that we can discuss about it.

This training is a field for experimentation for my research. I am trying to implement data collection and analysis tools.

CONCLUSION

With this contribution, I wanted to participate to the reflection on teaching notions of logic, particularly in showing the importance of not neglecting the contribution of the logic in problematics related to language and not just to reasoning. Anyway, language and reasoning are not two isolated poles, there is an interaction between the structure of mathematical propositions and the structure of their proof. Moreover, rewording is a common activity in the proving process in mathematics.

In France today the new syllabuses for high school give goals concerning mathematical notations and reasoning, and mathematics teachers then have to explicitly teach some notions of logic. But through a study of documents defining the *knowledge to be taught*, we have shown that their choices for this teaching were made in ill-defined conditions, and were subject to strong constraints. It therefor seems essential to me that the mathematics teachers have tools allowing them to think about the way they speak, and about the ambiguities and implicits that lie under some usual formulations in our practice of mathematics. Mathematical logic seems a possible reference for this reflection.

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APPROACHING ALGEBRAIC PROOF AT LOWER SECONDARY SCHOOL LEVEL: DEVELOPING AND TESTING AN ANALYTICAL TOOLKIT

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In this contribution, I discuss two episodes from a teaching experiment performed in lower secondary school (grade 7) aimed at introducing proof and using algebraic language as a proving tool. The teaching experiment was conceived within a theoretical framework presented in a PME Research Forum (Boero et al., 2010). That theoretical framework was further developed in order to improve the a-posteriori analysis and refinement of the classroom intervention. The aim of this paper is to show how the increased theoretical framework shed new light on the students' processes. Moreover, the analysis suggested occasions for developing argumentation at the meta-level.

INTRODUCTION

In this contribution I present and discuss two episodes from a teaching experiment, performed in grade 7, aimed at introducing a "proving culture" in the classroom. The contribution is situated in the stream of research outlined in a PME Research Forum (Boero et al., 2010). From a theoretical point of view, the Research Forum proposed an integration between Toulmin's model for argumentation and Habermas' theory of rationality (see the "Background" section below). The Research Forum paper ended with a series of suggestions for further developments and implementations, which were the starting point for the teaching experiment that is the object of this contribution. In the meantime, the retrospective analysis of some teaching experiments performed in the past (see Morselli & Boero, 2011) suggested to integrate the theoretical framework presented in Boero et al. (2010) in order to better frame the modelling activity of the student when he/she moves from a problem situation (internal or external to mathematics) to its algebraic treatment. In this paper I show how the integrated framework can be used to analyse the processes carried out by the students, with a special attention to the dialectic between proof and algebra.

BACKGROUND AND THEORETICAL FRAMEWORK

According to Balacheff (1982), the teaching of proofs and theorems should have the double aim of making students understand what is a proof, and learn to produce it. De Villiers (1990) suggests that the teaching of proof should make students aware of the different functions that proof has in mathematical activity: verification/conviction, explanation, systematization, discovery, communication. Stylianides (2007) proposes the following definition of proof that can be applied in the context of a classroom community at a given time:

> "Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics: it uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification; it employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; an it is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community". (Stylianides, 2007, p. 291).

This definition brings to the fore that a smooth and meaningful approach to proof requires the students' progressive acquisition of basic content knowledge, but also the ability to manage (from a logical and linguistic point of view) the reasoning steps and their enchaining (modes of argumentation) and the ability to communicate the arguments in an understandable way. This is in line with the idea, exposed by Morselli and Boero (2009), that learning proof is approaching a specific form of rationality. The authors proposed an adaptation of Habermas' construct of rationality to the special case of proving, showing that the discursive practice of proving may be seen as made up of three interrelated components:

> "- an epistemic aspect, consisting in the conscious validation of statements according to shared premises and legitimate ways of reasoning (cf. the definition of "theorem" by Mariotti & al. (1997) as the system consisting of a statement, a proof, derived according to shared inference rules from axioms and other theorems, and a reference theory);
>
> - a teleological aspect, inherent in the problem solving character of proving, and the conscious choices to be made in order to obtain the aimed product;
>
> - a communicative aspect: the conscious adhering to rules that ensure both the possibility of communicating steps of reasoning, and the conformity of the products (proofs) to standards in a given mathematical culture". (Morselli & Boero, 2009, p. 100)

Boero et al. (2010) proposed the integration of the construct of rational behaviour, with Toulmin's model of argumentation, thus creating a comprehensive frame that allows: (1) to better analyse students' proving processes; (2) to plan and carry out innovative classroom interventions. As regards the analysis of students' processes, the integrated model allows two levels of analysis: Toulmin's model focuses on the single argumentation step, while Habermas' construct allows to study each phase of the proving process, from the exploration to the final proof construction (thus shedding light on the legitimacy of reasoning steps, on the intentions behind each step, and on the communicational constraints). As regards classroom interventions, the integration suggests the importance of developing students' awareness of the constraints inherent in the proving process. Indeed, within the integrated frame, two levels of argumentation are outlined: the meta-level, concerning the awareness of the constraints related to the three components of rational behaviour in proving, and the level concerning the proof content. Within the integrated frame, students' enculturation into the culture of theorems is a long-term process where the teacher must create occasions for meta-level argumentations aimed at promoting students' awareness of the epistemic, teleological and communicative requirements of proving.

Crucial issues are: how to create occasions for meta-level argumentation and how to manage them in the proper way. Boero (2011) analysed a mathematical discussion at university level, showing that dealing at the same time with the content level and the meta-level is quite difficult, and suggesting some a-posteriori activities so as to create *occasions* for meta-level argumentation. In the present paper, I illustrate some *occasions* emerging from another teaching experiment, aimed at the approach to proof in arithmetic. Here the approach to proof is in a dialectical relationship with the introduction of algebraic language as a proving tool (i.e. the means to perform proof).

Algebraic proof

Boero (2001) describes algebraic treatment as a cycle: the starting situation (*sem1*) is put into formula (*form1*) by formalization. The first formula (*form1*) is transformed into another one (*form2*) that may give new information to the reader (thus performing an interpretation from *form2* to *sem2*).

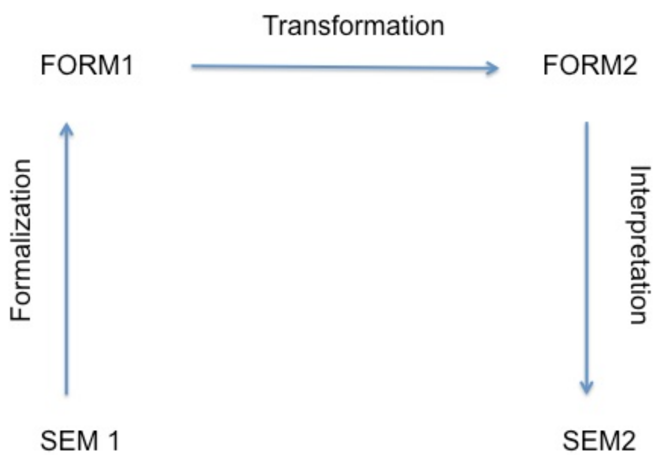

Figure 1: The cycle of algebra (Boero, 2001)

The fundamental cycle of *formalization*, *transformation* and *interpretation* is at the core of algebraic activity. In particular, algebraic proof is carried out by means of such cycles. When dealing with algebraic language as a proving tool, some crucial issues are: the choice of the formalism, that must be correct but also goal-oriented; the validity but also usefulness of the transformations; the correct and purposeful interpretation of algebraic expressions in a given context of use.

Morselli & Boero (2011), adapted the three components of a rational behavior in proving to the use of algebraic language in proving. In their elaboration, epistemic rationality consists of two distinct requirements: 1) *modelling requirements*, inherent in the correctness of algebraic formalizations and interpretation of algebraic expressions; 2) *systemic requirements*, inherent in the correctness of transformation (correct application of syntactic rules of transformation). Teleological rationality consists of the conscious choice and management of algebraic formalizations, transformations and interpretations that are useful to the aims of the activity. Communicative rationality consists of the adherence to the community norms concerning standard notations, but also criteria for easy reading and manipulation of algebraic expressions.

The contribution of this paper relies in the integration of the fundamental cycle of Algebra with the construct of rationality (in the use of algebraic language in proving) in the analysis of students' behaviours.

RESEARCH PROBLEM

This paper concerns the experimentation of a task sequence aimed at approaching proof and introducing algebra as a proving tool. The main research questions are: can the analytic tool of rational behaviour integrated with the fundamental cycle of algebra be exploited to perform more in-depth analyses and interpretations of students' behaviours?

Previous research pointed out the importance of creating *occasions* for argumentation at meta-level, so as to make students aware of the epistemic, teleological and communicative constraints of proof. More specifically, with an eye to the use of algebraic language as a proving tool, it is important to promote reflection at meta-level on the nature of the actions to perform (*formalization*, *transformation*, *interpretation*). Thus, additional research questions are: are there *occasions* for meta-level argumentation? If yes, what are the themes for such an argumentation?

METHOD

The context

The teaching experiment is situated within the research project "Language and argumentation", started in 2008, aimed at the design, experimentation, analysis and refinement of task sequences for the development of students' "proving culture". Within the project, teachers and researchers share the same theoretical references and collaborate in the design activity, as well as in the analysis of the experimentation and the progressive refinement of the tasks.

The task sequences are conceived with argumentation as a core activity. Two types of argumentation are fostered: argumentation at content level, as a part of the proving process, and argumentation at meta-level, as a means for fostering reflection on the practices of mathematical proof related to the components of rationality. To this aim, tasks encompass: formulation of conjectures; comparison between different conjectures; justification of conjectures; comparison between individual processes and between individual final products. Didactical methodologies such as group work and mathematical discussions (Bartolini Bussi, 1996) are widely used. The team also explored the importance of making students to analyse students' written individual solutions, as it is advocated within the theoretical framework of the fields of experience didactics (Boero & Douek, 2008).

The task

The task sequence "*Sum of consecutive numbers*" was conceived for grade 7 (students' age: 13-14); it encompassed exploration, conjecturing and proving in arithmetic. The approach to proof is in a dialectical relationship with the introduction of algebraic language as a proving tool. The students were at their second experience within the project. They had already experienced the task sequence "*Choose a number*". In that occasion, they had appreciated the power of algebraic language for representing generality and showing the structure of the problem (see Morselli & Boero, 2011).

The task sequence was experimented in two classes, by two teachers involved into the project. The author, a researcher in mathematics education, attended all the class sessions, acting as a participant observer. This means that she observed the class sessions, could provide further explanations, if required, during the individual and group work and could intervene in the discussion that involved all the students. She realized video recordings of the mathematical discussions and collected all the individual and group productions provided by the students.

The whole sequence lasted about 10 hours. A description of the whole task sequence, as well a comparison between the two classes, is beyond the scope of this contribution. Here we confine ourselves to the first 4 hours. The students were proposed a first task ("*What can you tell about the sum of three consecutive numbers?*"). We may note that in both classes, in line with previous experiences in arithmetic, the students interpreted the task as referring to the sum of three consecutive *natural* numbers. The fact of working with natural numbers was not discussed with the teacher. The students worked individually, shared their solutions in small groups and after compared all the group solutions within a mathematical discussion. In each class, the discussion was devoted to the comparison of the conjectures and justifications provided by the students. For the aim of the paper, I selected from each discussion the excerpt referring to the classroom discussion about *how to justify the property by means of algebra*. The description of each episode is followed by a first analysis. Afterwards, an overall discussion of the results is presented.

TWO EPISODES FROM THE TEACHING EXPERIMENT

Episode 1: Three proofs for the same property

The students from the first class worked individually and produced different conjectures. Although the norms established in the classroom require that any answer should be justified, only one student accompanied his conjecture with a justification. Elio claimed that "*the sum is a multiple of three*", performed three numerical examples (see figure 2 for the original production) and wrote down: "*Moreover, if the third number gives a unit to the first number, we have three equal numbers*".

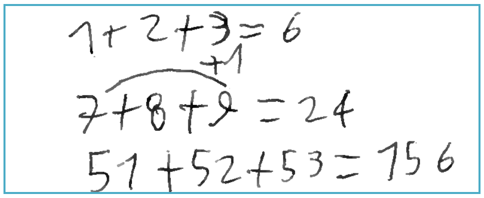

Figure 2: excerpt from Elio's individual solution

Elio's justification is firstly illustrated by means of a numerical example (7,8,9; see in figure 2 the line over the numbers, which represents the idea of 9 giving a unit to 7). This is a proof by generic example (Balacheff, 1982), since the numerical example is not aimed at "checking that the conjecture holds", rather to show "why the conjecture holds". The final sentence, although introduced by "moreover", is a justification in general terms. We may note that this proof by generic example has the function of *explanation*, not merely of *conviction*. This proof was shared by Elio to his group mates and, afterwards, presented to the whole class and discussed within a mathematical discussion. During the discussion, the observer and the teacher underlined that Elio's justification is a real *explanation* of the reasons why the conjecture holds. The observer also underlined that Elio's method shows that the property does not hold if the numbers are not consecutive, thus pointing to the function of *explanation*.

1 Observer: and in this way you understand why this is a property that not always holds. Some of you maybe tried to sum up three non-consecutive numbers. It is not sure that we still have this [*the divisibility by 3*], isn't it? This explains why we need three consecutive numbers to have it.

2 Elio: if we tried, here, instead of 503, with 504, I would get 503. I take away 1 [*from 504*] and I get 503, not 502.

3 Teacher: and you don't have anymore three equal numbers. The nice thing, using three consecutive numbers, is that if I take 1 away from the biggest number and I move it to the smallest number, I get three equal numbers. That why I always get three times the intermediate number, exactly because there is that "moving". [*they go on doing some numerical examples and applying the "taking away" strategy*]

Elio's individual solution contains also an algebraic proof:

a+a+1+a+2 could also be a+a+a+1+2

thanks to the commutative property it would be a*3+1+2

a*3+3.

During the discussion, Elio explained his choice of providing also an algebraic proof: "*But maybe they [numeric examples] did not work on great numbers and I could not do an example on all numbers*". We may observe that Elio was not completely satisfied with is proof by generic example, probably influenced by the common idea that "examples don't prove". Actually, this proof by generic example was already acceptable. We also observe that when passing from generic example to algebraic proof, Elio did not perform a translation in algebraic language of the same type of proof, rather he carried out a different proof. This fact was pointed out during the mathematical discussion. Elio, with the help of the teacher, created at the blackboard the algebraic version of his proof by generic example:

$$a+a+1+a+2=a+1 + a+1+ a+1=3(a+1)$$

In terms of cycle of algebra, the two algebraic proofs (the first one, carried out by Elio individually, and the second one, carried out during the discussion) may be modelled as it follows:

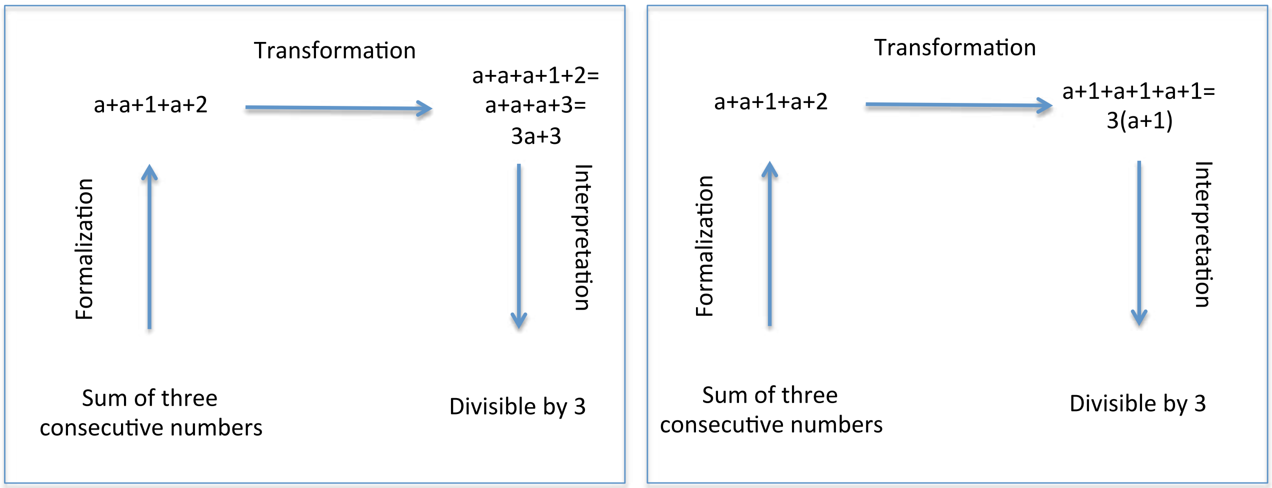

Figure 3. Elio's individual proof and the proof carried out during the discussion

Both proofs are carried out properly and each action (formalization, transformation and interpretation) involves some aspects of rationality: formalization is correct (modelling requirements of epistemic rationality), and useful, since it allows the subsequent treatment of the algebraic expression (teleological rationality). All the transformations are performed correctly (systemic requirements of epistemic rationality) and in a goal-oriented way, so as to obtain the divisibility by 3 (teleological rationality). In the second algebraic proof the strategy of "taking away" 1 is guided by the goal of getting three times the same number. Thus, transformation is even more goal-oriented (tel. rationality). Finally, in both cases interpretation is correct (systemic requirements of epistemic rationality), since the final algebraic expressions (3a+3 and 3(a+1) respectively) are read in terms of "divisibility by 3". In the first proof, 3a+3 could be more developed so as to make more evident the divisibility by 3. In the second proof the divisibility by 3 is evident and one could also note that the result is three times the intermediate number (epistemic rationality).

The analysis in terms of cycle of algebra and construct of rationality reveals some differences between the two proofs: the first one is mainly syntactical and could be carried out without having in mind the property to prove; on the contrary, the second one can be performed only under the guide of a strong anticipation (one must already have the goal of getting three times the same number); the second algebraic proof seems to be possible only in continuity with the argumentation in natural language and numerical examples (proof by generic example). Both proofs have an educational value and offer occasions for argumentation at meta-level. Indeed, the second proof is a telling example of proof as *explanation*, the first one may also convey the idea of algebraic proof as a means for *discovery*. Actually, also in second proof there is a discovery part, because also divisibility by the intermediate number turns to be evident. The second proof also highlights the importance of reflection on numbers. The analysis suggests that it would be important to promote an a posteriori comparison between them, thus fostering a meta-level argumentation on the way of carrying out algebraic proof (crucial role of *transformation*), and also on the value of algebraic proof (not only *conviction*, but also *explanation* and *discovery*).

Episode 2: struggling towards an algebraic proof

The same task was proposed in another class. One student (Edel) conjectured that "*the result is a multiple of 3*" and accompanied the conjecture by a first justification in natural language ("*because the summed numbers are three*") and by a symbolic expression (see figure 4). From the mathematical discussion, we know that Edel's intention was that of providing an algebraic proof for the property.

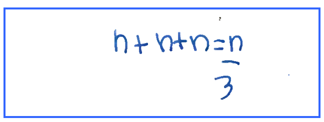

Figure 4. Excerpt from Edel's solution

In terms of cycle of algebra, Edel's attempt may be modelled in this way:

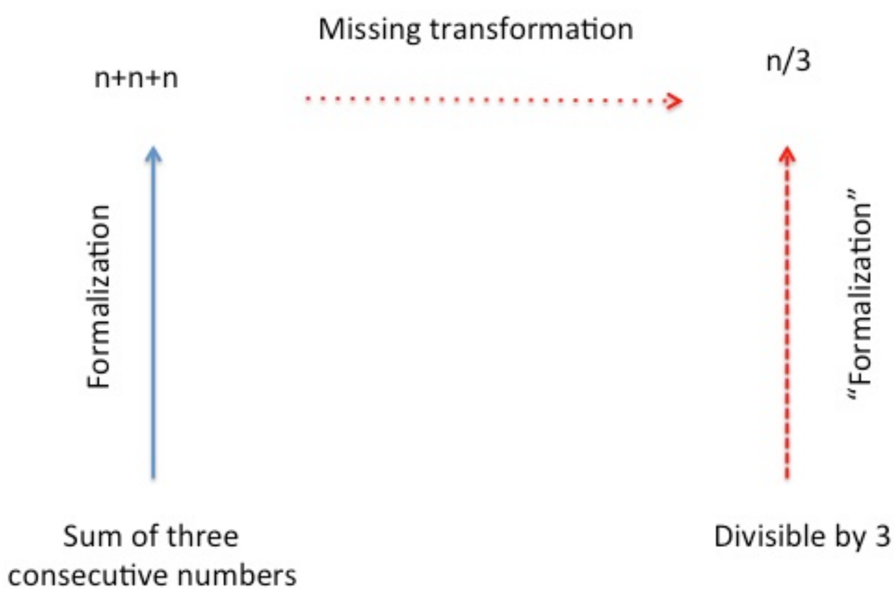

Figure 5: Edel's attempt

The cycle of algebra is not working in the proper way: formalization is not correct (since $n+n+n$ is not a correct representation of three consecutive numbers) and *form2* (n/3) is obtained by an (incorrect) formalization of the conjectured property "divisibility by 3" (*sem2*), rather than from a transformation of *form1*. Edel's difficulty in formalizing the divisibility by three may be interpreted in terms of difficulty in the first use of algebra or difficulty in dealing with multiples and divisors. Here we focus on the effect that such difficulties may have on the proving process and on the possible interventions. From the point of view of rationality, we note lacks in the modelling requirements of epistemic rationality. Anyway, we suggest that the formalization of n/3 lacks in terms of epistemic rationality, but is rational from the teleological point of view: Edel wants to translate in letters what she already discovered, that is the divisibility by 3. We may say that the missing issue is exactly the *transformational* power of algebra. What makes algebra a powerful proving tool is the possibility of passing from the starting situation to the conclusions by means of transformation. This awareness (at meta-level) is completely absent in Edel's solution. Edel's activity has a teleological rationality, but according to her own goal: translating into letters. This is linked to her "ritual" conception of algebra as a proving tool: it seems that, for her, the algebraic proof is just a symbolic translation of what is already known.

Previous analysis suggests the necessity of a reflection on how algebraic language works as a proving tool (teleological aspects). We point out that there is a rationality in the choice of using algebraic language as a proving tool, and a rationality in performing the algebraic proof. Awareness of the teleological aspects referring to the use of algebraic language as a proving tool (it is a useful proving tool because it allows to obtain the proof by means of transformation of symbolic expressions) has direct consequences on the awareness of the teleological aspects referring to algebraic activity (formalization and transformation must be goal-oriented).

CONCLUSIONS AND FURTHER DEVELOPMENTS

We described and analysed two episodes from a teaching experiment aimed at introducing algebra as a proving tool.

The new integrated framework allowed us to put the requirements of epistemic and teleological rationality in a dynamic perspective. This brought to the fore that, when proving by means of algebraic language, the student must be able to combine the adherence to syntactical rules on one side, and the goal-oriented management of the processes of formalization, transformation and interpretation, on the other.

In this way, the integrated framework allowed us to understand better the students' processes (in particular, as concerns the nature of some of their difficulties) and to detect some occasions for argumentation at meta-level, that is occasions in which students can be asked to reflect on some aspects/components of the complex process they are involved in.

Important issues, to be treated at meta-level, concern: the role and value of numerical examples and the legitimacy of proof by generic example and proof in natural language; the crucial role of transformation, and the consequent importance of transformation-oriented formalizations; the dialectic between syntactic manipulation and more creative manipulation; and the possible links with the different functions of proof (from conviction, to explanation, to discovery).

The aforementioned task and the subsequent mathematical discussions can only partially achieve the goal of improving awareness on all these points. Further developments concern the design and implementation of tasks aimed at provoking such occasions.

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THE EPISTEMIC STATUS OF FORMALIZABLE PROOF AND FORMALIZABILITY AS A META-DISCURSIVE RULE

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The first two parts of this article report on a study that was part of my dissertation project at the interface of epistemology and sociology of mathematics. The study deals with the epistemic role of formalizability, which is traditionally held to be the main epistemic feature of mathematical proofs, in actual mathematical (research) practice. As a core result, it is argued that formalizability should be understood as a feature of discursive proving actions as the true bearers of epistemic value. As I discuss in the last part of the article, this insight opens the way for a shift to an educational perspective on proof in mathematical research practice. Sfard's approach to mathematical thinking as communication, with the concept of meta-discursive rules in particular, serves well as a conceptual framework to that end.

Keywords: mathematical proof, formalizability, epistemology, sociology, meta-discursive rules

INTRODUCTION

In science education, the so-called nature of science is a widely agreed upon aspect of scientific literacy that one has to teach more or less explicitly. Important components of nature of science are epistemic features of scientific inquiry, the epistemic status of laws and theories, the tentativeness of science, the theory ladenness of observation, etc. (Akerson, Abd-El-Khalick, & Lederman, 2000). Regarding mathematical research practice and university education, proof appears to be an essential component of a “nature of mathematics”. But are the essential epistemic features of proof in mathematical research practice relevant to school mathematics? If so, how could we teach them? Though these questions can only be touched in the last part of this article, they indicate an overall framework for employing the results from research on proof in actual mathematical research practice presented here, under appropriate re-interpretation where required, in thinking about teaching proof in school mathematics.

The work (Müller-Hill, 2011) that is presented in the first two parts of this paper was concerned with the epistemic role of formalizability of mathematical proofs in actual mathematical practice, with a major focus on research mathematics and a minor on mathematics education at university[1]. Regarding these contexts, one is traditionally inclined to demand formalizability—usually without further specification—as an essential epistemic feature of proof. Hence the main research question was:

> In what sense of “formalizable” is formalizability an essential epistemic feature of proof in actual mathematical practice, and thus a necessary condition for accepting a proof?

Formalizability is a feature of informal mathematical proofs: A formalizable proof is a proof that can be transferred into a formal proof, that means it can be transferred into a formal derivation with respect to a formal axiomatic system with consistent axioms. However, this notion of formalizability is not sufficiently specified. Possible semantics of the phrase “formalizable proof” still vary in a spectrum spread between two extremes. One extreme would be the weak reading “a proof of *p* is formalizable iff the informally proven mathematical theorem is also formally derivable in a consistent formal axiomatic system”; the other extreme is the strong reading “a proof of *p* is formalizable iff it can be translated step by step into a formal proof”, which may refer to, for example, proofs that are written in some semi-formal language.

Formalizability is indeed an important feature of mathematical proofs, regarding foundational issues in the philosophy of mathematics[2]. Foundational issues, however, are not addressed in the following. What is addressed instead is the epistemic role of this feature, from the viewpoint of a socio-empirically informed philosophy of mathematics.

SOCIO-EMPIRICALLY INFORMED PHILOSOPHY OF MATHEMATICS

The epistemic role of formalizability has traditionally been investigated by analytical philosophy of mathematics. The focus of such investigations is on the project of conceptually grasping formalizability as an epistemic feature of proof. The default method of analytical philosophy is the semantic analysis of so-called ordinary language epistemic concepts like knowledge and justification. It investigates the adequateness of having a formalizable proof as a truth-condition for, e.g., knowledge attributions. The analytical philosopher thereby almost exclusively relies on his professional intuition as an expert for epistemic concepts like knowledge, belief, or justification (see recently, e.g., Glock, 2008, for an introduction to analytical philosophy).

However, concerning the question of the epistemic role of formalizable proof in actual mathematical practice, an investigation of decidable conditions under which a proof is actually acceptable because of being formalizable appears to be equally worthwhile (see also Moser, 1991).

A socio-empirically informed philosophy of mathematics aims at (ideally) establishing a reflective equilibrium between the outcomes of both kinds of investigation. An appropriate methodological framework to this end is conceptual modelling as developed in (Löwe & Müller, 2011) and (Löwe, Müller, & Müller-Hill, 2010). Conceptual modelling “of *X*” takes the form of an iterative process:

Step 1 Theory formation Guided by either a pre-theoretic understanding of *X* or the earlier steps in the iteration, one develops a structural philosophical account of *X*, including, e.g., considerations of ontology and epistemology.

Step 2 Phenomenology With a view towards Step 3, one collects data about *X* and extracts stable phenomena from them to corroborate or to question the current theory.

Step 3 Reflection In a circle between the philosophical theory, the philosophical theory formation process and the phenomenology, one assesses the adequacy of the theory and potentially revises the theory by reverting to Step 1.

In particular contrast to analytic epistemology, a socio-empirically informed epistemology of mathematical practice strengthens step 2 by including data established via accepted empirical methods from empirical sociology.

DESIGN OF AN INTERVIEW STUDY AS PART OF STEP 2

The conceptual modelling cycle displayed above was employed iteratively in my study (Müller-Hill, 2011). As one part of step 2, I developed and conducted a qualitative, so-called problem-centered, semi-standardized guideline interview study (see, e.g., Mayring, 2002) among research mathematicians. The aim was to gather detailed empirical information about practitioner's interpretations of the epistemic role of formalizable mathematical proof in actual mathematical practice.

The interview guideline (see Müller-Hill, 2011, 148 ff.) was developed with respect to certain key aspects that came out of earlier iterations of the modelling cycle, including in particular a quantitative questionnaire study reported in (Müller-Hill, 2009, 2011) on the use of epistemic attributions in actual mathematical practice.

Six mathematicians of high standing, from various fields of professional specialisation areas in mathematical research practice, from different countries and different institutions, were chosen as interviewees. The interviews were audio-recorded and transcribed. As the method of data analysis, I chose so-called phenomenological analysis. The method of phenomenological analysis, in a nutshell (cf. Mayring, 2002), is to reconstruct units of meaning in a sufficient variety of subjective interpretations and viewpoints of the matter in question as a first step. In a second step, these subjective, idiosyncratic units of meaning are synthesized, and reduced to a common, invariant essential core.

RESULTS AND INTERPRETATION

In the following, I will present some exemplary quotes from the interview transcripts, a short summary of the philosophical interpretation based on the analysis of the transcripts, and a central aspect of the subsequent conceptual theory formation (see Müller-Hill (2011) for more details on the data and its analysis).

Example quotes from the transcripts

The following examples stem from four of the six interviews.

Excerpts from interview 2:

Interviewer: And what would you call a formalizable proof?

IP 2: […] If something is obviously non formalizable, then to me it will be obviously not an acceptable proof.

Interviewer: Would you say that every proof that is accepted by the community is formalizable?[3]

IP 2: […] That's my belief, and one should say this is a hygienic belief – I mean, that's more or less a definition of what we, as mathematicians, as a community, are thinking about […] when we are talking about proofs.

Excerpts from interview 3:

Interviewer: Would you say that the proof about the classification of finite simple groups is a formalizable proof?

IP 3: […] I think in principle, it's formalizable, this classification.

Interviewer: What does 'in principle' mean?

IP 3: With the present technology, or just by hand, it may not be doable for one person in a lifetime and maybe even for the group of people who have been working on this it may not be doable, but with advanced future computer technology it may be doable.

Excerpts from interview 4:

IP 4: So one thing: maths is in a sense something personal. […] When do I think I understand something? Certainly if it's a relevant thing, during the next few weeks I would try to shoot holes in it, look at it from different directions, through different angles, ask questions why it is true, why it works this way, not that way. […]

Interviewer: What would be formal proof for you?

IP 4: […] I have to be able to understand both the global picture, to have an overview of the proof, explain to myself what the global idea is and why it works like this, and also to be able, and that's the details, to follow the proof from step to step, the logical consequences.

Excerpts from interview 5:

Interviewer: What do you mean by 'formal proof'?

IP 5: […] One of my recent graduate students is almost incapable of writing down proofs. But he really knows mathematical truths. The problem is then to extract from him why it is true, which is the proof. […] There are well respected mathematicians who are like this.

Interviewer: Can you make any sense of the term 'formalizable proof'?

IP5: It very much depends on the style of a mathematician, on a personal temper maybe, the attitudes of particular mathematicians, but it depends also on the field of mathematics. […]

Interviewer: Would you say that every proof that is accepted by the community is formalizable?

IP 5: […] Maybe I shouldn't think of myself as a mathematician if I don't believe that the proofs are formalizable. I think we act on assumption that the proofs that we produce are formalizable.

Excerpts from interview 6:

IP 6: When doing my Ph.D. thesis, I learned [...] something about the way to be careful […] If you see a proof, then you start reading it, and say, o.k., can I understand all the implications. This can be very clear, or can sometimes be a little bit blurred in that you are at the point of sort of believing it rather than actually seeing it. […]

Interviewer: Would you say that every proof that is accepted by the community is formalizable?

IP 6: […] I think they look like formalizable proofs. I don't know how they do that, but I think that all the people who have looked at these proofs think they are formalizable.

Summary of the philosophical interpretation

The philosophical interpretation developed in (Müller-Hill, 2011) on the basis of the phenomenological analysis of all interview transcripts, including those that are not displayed above, can be summarized and condensed into eight main aspects. I will only refer to the following six of these aspects here (see Müller-Hill, 2011, 205ff., for the whole list), with an emphasis on aspect (5) as a major conceptual turning point in the philosophical understanding of proofs as bearers of epistemic value.

1.In actual mathematical practice, acceptable proof includes the inevitable possibility of error. Hence the whole proving process has no final point, and epistemic attitudes of practitioners are not categorical. Nevertheless, there exist well working, situative standards for the error robustness of the core argumentation of an acceptable proof.

2.Equally besides and even beyond the aim of secure knowledge-that, mathematical practice aims at understanding in the sense of knowledge-why.

3.Explanatory proofs contribute essentially to knowledge-why. Hence, depending on the professional skill level of the epistemic subjects, acceptable proofs ought to have a meaningful argument structure. To that end, they often rely on established meta-argumentations that are not formalizable mutatis mutandis, or by mechanical means.

4.Epistemic standards for acceptable proof are gradual and context-dependent. They concern surveyability, clarity of the core argumentation, error robustness and formal correctness, the use of meta-argumentations and the possibility of perspective change.

5.Formalizability of acceptable proof should consistently be interpreted rather as an essential epistemic feature of the embedding communicative action than of the static, linguistic argument presented as a proof. (This is not to say that the mere linguistic argument presented as a proof of *p* does not bear *any* epistemic value.)

6.The concept of formal proof functions as an abstract, internalized model of proof shaping the self-image of practising mathematicians. Sophisticated mathematicians may have internalized the rules and principles of formal proving and work in agreement with these rules without explicitly and consciously employing them.

Formalizability as an epistemic feature of discursive proving actions

Moving from these results of step 2 towards step 3 of the modelling cycle, theory formation, the main question is how to specify the notion of formalizability as an epistemic feature of mathematical proof that properly fits the empirical findings. According to the interpretation of the results of the interview study, there are several essential aspects of acceptable proof in actual mathematical practice, with formalizability as one of them. However, some of these essential aspects appear to compete against certain others, such as formalizability and fallibility, formalizability and the frequent use of sophisticated meta-argumentations, or formalizability and explanatory power. Hence, to form a consistent notion of formalizability as an epistemic essential of proof within an epistemology of mathematics needs to supersede the concept of proof itself as a mere linguistic entity—an argument—by a proper alternative[4].

Such an alternative account of an epistemologically relevant notion of proof in mathematical practice that can consistently be seen as a bearer of the partly competing epistemic aspects mentioned above is provided, as I argue and develop in detail in (Müller-Hill, 2011), by the notion of discursive proving actions. I will give a brief sketch of this conception of formalizability in the following.

My account makes reference to the concept of informing dialogical communicative actions used in philosophy of language and linguistics (cf. Meggle, 1999):

An *informing dialogical communicative action* is an intentional act of communication in dialogue, with the communicative aim to make the receiver believe a certain thing.

Formalizability can thus be conceptualized as an epistemic feature of mathematical proof, in the sense that an epistemic subject *X* is justified to believe in the validity of a theorem *p* on the basis of an accepted proof, if *X* is able to carry out—given appropriate conditions—a certain kind of discursive proving action. I call these discursive proving actions “derivation indicating”[5]:

A *discursive proving action* for a mathematical theorem *p* is an informing dialogical communicative action (oral or written) where an epistemic subject *X* presents an argumentation for the validity of *p* to a certain audience under certain situative circumstances. The presentation of the argumentation includes contributing utterances of members of the audience. A discursive proving action is called *derivation indicating* iff the type of argument, the presentation of the argument, and the professional level of the audience meet the contextually given epistemic standards and *X* has sufficient

> mathematical skills to produce a appropriately formalized argument out of the given presentation. This may involve modification, correction and supplementation of the given presentation up to certain, context-sensitive levels of error robustness and stability of the core argument. (Müller-Hill, 2011, 230 f., German in the original)

FORMALIZABILITY AS A META-DISCURSIVE RULE

The general conceptual turn from mere linguistic entities to discursive proving actions in analyzing the essential epistemic features of mathematical proofs opens the way for using certain perspectives and theoretical frameworks from research in mathematics education for a re-interpretation of the results of the presented study. If this is successful, it could be a fruitful interface between a better understanding of the nature of proof in actual mathematical practice and a didactically sustainable way of teaching the nature of mathematical proof in the classroom.

Regarding research in mathematics education, Sfard's approach of mathematical thinking as communication and her concept of meta-discursive rules (Sfard, 2001, 2002, 2007, 2008) seem to grasp the core of my socio-empirically informed account of the epistemic role of formalizability particularly well. According to Sfard, meta-discursive rules, in contrast to object-level discursive rules, are rules about the discourse[6]. Formalizability, understood as a feature of discursive proving actions, can be seen as such a meta-discursive rule in actual mathematical practice. Within the scope of this article, I can only highlight two characteristics of formalizability as a meta-discursive rule in the sense of Sfard.

First, according to Sfard (2002, 30) “meta-rules are usually not anything the interlocutors would be fully aware of, or would follow consciously”. Rather, “in concert with meta-discursive rules, people undertake actions that count as appropriate in a given context and refrain from behaviours that would look out of place”. This characterization of meta-discursive rules fits well with aspect (6), formal proof as an internalized leading picture, aspect (3) regarding the role of professional skills, and aspect (4), context-sensitive epistemic standards, from the general interpretation of the interview results. The first quote from interview 2 additionally stresses that formalizability can be, precisely in the manner of Sfard's conception, a regulative for discursive decisions (Sfard, 2001, 26 f.).

Second, formalizability, when interpreted as a meta-discursive rule, can be understood as responsible both for the way and for the very possibility of successful communication (cf. Sfard, 2002, 31). The given quote from interview 2, expressing that formalizability of all accepted proofs is a “hygienic belief”, and the second quote from interview 5, can be understood in this sense. Additional, socio-historical evidence corroborates this claim:

> I interpret formalization as a symbolically generalized communicative medium, which was developed precisely when, due to profound institutional changes, the prior social-integrative mechanisms became deficient. (Heintz, 2000, 252, my translation)

Thus in research mathematics, formalization can be seen as a meta-discursive rule that guides discourse, especially in conflict situations, as a control tool: If doubt is cast on an informal argumentation regarding its principal conformity to formal rules and definitions, a higher level of formalization is chosen (which does not have to be already mechanically implicit in the original argument).

Still, formalizability does not fulfil all of Sfard's characterizing conditions of meta-discursive rules. This suggests that there are additional, competing as well as supplementing, epistemically relevant meta-discursive rules in actual mathematical practice. Some promising candidates can be read off the already presented empirical results, such as the possibility of perspective changes. Others need further empirical investigation, e.g., on use and acceptance of informal meta-argumentation strategies.

CONCLUSIONS AND THOUGHTS TO THE FUTURE

The results of the study presented here, and also the sketch of meta-discursive rules as an alternative conceptual framework for their interpretation borrowed from research in mathematics education, have several implications for understanding the concept of proof in actual mathematical practice.

Induced by, but not limited to the investigation of formalizability, a general conceptual turn from proofs as mere linguistic entities to discursive proving actions was made in analyzing the true bearers of consistently explicable epistemic features. This highlights the relevance of conducting more detailed empirical studies of discursive practices regarding oral and written communication in actual mathematical practice, both in research and in university education. A subsequent examination of concrete implications for the teaching of mathematical proof, or of possible epistemic impact of the use of formalization and proof for the teaching, understanding and clarification of mathematical concepts could, as a first step, concentrate on university level mathematics education.

Nevertheless, the concept of meta-discursive rules can also serve as a connection between the presented research on proof and school mathematics. If the essential epistemic aspects of proof, with proof being one main putative component of a "nature of mathematics", are best understood as meta-discursive rules of actual mathematical practice[7], then teaching of the nature of mathematical proof should happen within proving discourses in classrooms, and include explicit reflections on meta-discursive rules in general, and formalizability as a special meta-discursive rule in particular. The use of formalization as a control tool, the validity of picture proofs, of the use of diagrams in proofs, and of argumentation assisted by geometrical representations are example topics for developing appropriate learning environments for such reflections on formalizability in, e.g., high school level maths courses.

In turn, such explicit reflections on meta-discursive rules in general and in particular should also become an integral part of pre-service and in-service teacher education.

NOTES

1. The study was embedded in a much broader agenda of developing a transdisciplinary approach of philosophical, sociological, psychological, historical and didactic research on actual mathematical practice that has been conducted by members of the DFG scientific Network PhiMSAMP (Philosophy of Mathematics: Sociological Aspects and Mathematical Practice) 2006-2010 (cf. Löwe & Müller, 2010).

2. Note that since the works of Boole, Frege, Russell, Hilbert, and Gödel there exists a highly sophisticated discussion in (philosophy of) mathematics and logic on what kind of formal logic (e.g., regarding order, or meta-theory of the axiom system) is appropriate to formalize the foundations of mathematics (see, e.g., Hintikka, 1996, 2011). I am not concerned with this question here, but refer to today's most common foundation of mathematics within the first-order Zermelo-Fraenkel axiomatics (ZFC) of set theory. However, this assumption does neither essentially draw on the spectrum of semantics for "formalizable proof", nor on the essential features of formal proof like gaplessness and explicitness.

3. Possible bias was not taken into consideration here, as the interviewees were all professional mathematicians of high standing with arguably stable and grounded attitudes towards acceptable mathematical proofs in research.

4. Note that this is not to claim that such a conceptual turn is or should be part of the aware image of proving held by professional mathematicians; but it should be part of any consistent, empirically informed philosophical consideration.

5. Azzouni (2004) coined the technical term of a "derivation indicator", but only for single arguments as linguistic entities, and with a different meaning.

6. As an example that illustrates the distinction, regard the following: "Investigate the function $f(x)=3x^3-2x$!" (see Sfard, 2000). An object-level discursive rule is that it is not allowed to divide the equation $0=3x^3-2x$ by x to determine the nulls of f. But only meta-discursive rules determine what to do with f at all: "You are not sure whether you are supposed to list the properties of the graph (yet to be drawn!) or to admire its aesthetics; [...]; to make an investigation of the effects of real-life applications [...] or to check possibilities of transforming it, and so on." (Sfard 2000, 177)

7. Moreover, this approach offers interfaces to include a historical dimension. See, e.g., (Kjeldsen & Blomhoj, 2011) on historical case examples as a way to reflect on meta-discursive rules in school.

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0, 999… = 1
AN EQUALITY QUESTIONING THE RELATIONSHIPS BETWEEN TRUTH AND VALIDITY

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The relationship between truth in an interpretation and validity in a theory as developed in logic, is a crucial issue in mathematics. In this article, we examine how an activity based on the equality "1 = 0.999 ..." could permit students to experience this relationship. We first provide practical and epistemological reasons supporting the claim that we have here a good candidate for this purpose. We then report on an experiment with nine fresh undergraduate students in France enrolled as volunteer in a short course aiming to help them to overcome difficulties in logic, reasoning and proof met in the calculus course; we analyse some excerpts of students' discussion showing that questioning this equality could favour the emergence of discussion on truth, validity, defining and proving, proof and theoretical reference.

Key words: truth, validity, definition, decimal and real numbers, limit of series.

INTRODUCTION

The relationship between *truth* and *validity* was first pointed by Aristotle who made it explicit in the First Analytics by introducing a clear distinction between de *facto truth* and *necessary truth*[1]. Modern logicians such as Wittgenstein (1922) and Tarski (1933) developed a semantic point of view in logic and provided a theoretical framework for the distinction between truth in an interpretation and logical validity[2]. For example $[p \wedge (p \Rightarrow q) \Rightarrow q]$ is universally valid in propositional calculus (it is a tautology); that means that in any interpretation where p and q are interpreted by sentences A and B that are propositions[3], the sentence "*If A and if A, then B, then B*" is true, and this whatever the truth-value of A and B. Universally valid statements support *inference rules* that allow deduction in interpretations (Wittgenstein, 1922). The previous one supports the Modus Ponens, i.e. the well-known inference rule: ''*A*; and if *A*, then *B*; hence *B*''. While Wittgenstein restricted his approach to propositional calculus, Tarski developed it for quantified logic, by developing a semantics definition of truth that is *materially adequate* and *formally correct* (Tarski, 1933), through two crucial notions: the notion of *satisfaction* of an open formula by an element in an interpretation, and the notion of *model* of a formula. This last notion leads to the concept of *universally valid formula* as formula for which every relevant interpretation is a model.

[1] For development on this point see for example Durand-Guerrier, 2008, p. 374.
[2] See Durand-Guerrier, 2008, pp. 376-377.
[3] In logic, a proposition is a linguistic entity that is either true or false.

It seems for us rather clear that understanding this distinction between *truth in an interpretation* and *logical validity* is a clue competence for proof and proving in mathematics: indeed, the correctness of a proof relies in an essential manner on the validity of the inferences that are involved in the proof. Moreover, it is clearly a challenge for mathematics education due to the fact that, as it is well documented in the literature, students often fail to understand why a proof is needed when they are convinced of the truth of a given statement. In this respect, it is necessary to introduce doubt in order to motivate argumentation and proof (Durand-Guerrier & al. 2012). Another important aspect in Proof and Proving in mathematics education is to make students aware that to provide a mathematical proof, it is necessary to work in a theory, at least a *local theory* (axioms), with theoretical objects that need to be defined.

It is well known in mathematics education that the equality "0,999…= 1" that appears when considering that the real number set coincides with the set of terminating or non terminating decimal number looks strange for a number of students, some of them considering that it is false (Tall, 2000; Dubinsky & al. 2005). Many reasons have been advanced for explaining these difficulties (difficulties taking in consideration non terminating decimal number; difficulties with limits considered as process, not as number etc.). In this communication, we document a complementary approach on these difficulties under the light of the distinction between truth and validity, involving the interplay between proving and defining, and we provide arguments supporting the claim that working with this equality with undergraduate students may open discussion on this distinction with a benefice for both the mathematical and the meta-mathematical aspects (a better understanding of real numbers and an explicit example of the distinction between truth and validity). We first provide a brief *a priori* analysis; then we present some results of an experiment with fresh undergraduate students that we analyse through this lens.

SOME *A PRIORI* ARGUMENTS ON THE RELEVANCE OF QUESTIONING THE RELATIONSHIP BETWEEN TRUTH AND VALIDITY

For fresh undergraduate students in France, the familiar back ground for examining at first this equality is the terminating decimal numbers set, well known by students from elementary school, although it is generally not the case that they know well the specificities of this set among the others numbers sets. Anyway, they master the operative algorithms for sums, difference, product and decimal division; they have also met the case of those rational numbers whose decimal expansion has infinitely repeated sequence (repeating decimal). As a consequence, they have an empirical reference for the equality "1/3 = 0,333… " (1), that is get by executing the algorithm for decimal division. By multiplying each side of the equality by 2, one gets a new equality "2/3 = 0,666…" (2); for this equality, there is also an empirical reference by making the division; this fact supports the conjecture that the calculating rules define on terminating decimal can be applied to non terminating decimal. But of course, it does not provide a proof that it is possible to extend the rule. In this respect, we have here a first aspect of the distinction between truth and validity. On the one hand, taking the result of the division algorithm as the definition of the decimal expansion of a rational number, then equality (2) is true. Accepting this equality is in general not problematic for students, but this equality could be perceived rather as the result of the process of dividing than as equality between two numbers. On the other hand, considering the set of decimal numbers, with addition, difference and multiplication algorithms on terminating decimal, it is not possible to deduce equality (2) in that theory using addition or multiplication by 2, due to the fact that we have change the nature of number and that we have no theory, even local, in which proving that multiplication could be extended. A fortiori, it is not possible to prove that we get a new equality by multiplying each side of equality (1) by 3. As a matter of fact, there is no empirical reference relying on division allowing considering the writing 0, 9999… as referring to a concrete process. At this point, arises the question of the possibility to find a theoretical justification of equality “0, 999… = 1” (3)

A natural candidate for justification: extension of operative algorithms

The extension of the operative algorithms from terminating decimal to non-terminating decimal seems to be the more natural candidate. We have seen that the empirical reference for equality (2) supports this conjecture. However it is not granted that the operation on integers or decimals should be extended to real numbers; this necessitates elaborating a theory, in which the theoretical definitions are supposed to correspond with the empirical aspects of the concept that are formalized. Dedekind (1872) did it by developing a construction of the real number sets with cuts, providing a formalisation of the intuitive notion of continuum (Longo, 1999) in which it is possible to extend the operations on rational numbers, and hence on decimal number. However, this does not mean that it is possible to extend the addition or multiplication algorithms on finite decimal numbers. Indeed, there are practical reasons that prevents such an extension, some are well known by students:

1. The algorithm for multiplication for terminating numbers is initiated on the right digit; in a non-terminating decimal number, where should we begin?
2. The product by 3 of a terminating decimal number that is not an integer is never an integer.
3. It is not possible to extend the algorithm for comparison, due to the fact that it could occur that it does not terminates.

There are also epistemological reasons:

4. As soon as one works with infinite, strange things may appear, so it is necessary to be cautious.
5. One can wonder if a non-terminating decimal writing refers to an object or to a process.

Point 2 pleads against the extension of the multiplication algorithm due to the fact that it leads to equality (3) that violates this theorem, and comforts point 4. Moreover, point 3 prevents an algorithmic comparison, which could be useful to decide if the two writings denote the same real number. All of these aspects are likely to converge to rejecting equality (3). Point 5 open on a new question: how do we operate with process?

Process *versus* object – "tend to" *versus* "is equal to the limit"

Point 5 could be seen as an inheritance of the dividing process; although this question has already largely been discussed in the mathematics education community (e.g. Cornu 1981; Tall 2000; Dubinsky & al. 2005), it is still in debate, as it can be seen in a letter to Educational Studies in Mathematics' editors published on line on the website of the journal in 2011[4]. In this letter addressed to the editors the author discusses a sentence out of a posthumous paper from Fischbein (Fischbein 2001) arguing that in mathematics the sum of infinite multiplicity cannot be equal to a number that is finite. Referring to the sum of the relevant geometric series, she concludes that the expression **"**0.333**…** tends to 1/3**"** will be mathematically true.

What interest us in this example is the fact that although the author of the letter refers to the theoretical point of view involving the sum of series, which defines the limit as a number that of course can be finite, the author considers that 0,333… does not represent a number. We interpret this as an indication that the theoretical framework relying on sequences, series, limits and their operations could be insufficient to encompass the conception that such writing is referring to a process[5]. Notice that this point could be related to the distinction between potential and actual infinite.

Coming back to our equality "0,999… = 1", such a position leads to reject this equality and to replace it by "0,999… tends to 1":

> "In general, "limit" and "tends to" are not used in the same context. The limit designates something precise while it is possible to tend to something more vague. An example: we will say that the sequence " 0,9, 0,99, 0,999, 0,9999, … " has for limit 1" or "tends to 0,9999…" (…) For some students, an unlimited sequence has no limit … because it is unlimited. We observe that some students use the term "limit" for sequences whose limit is reached, and use the expression "tends to" when the limit is not reached" (Cornu, 1981, p. 325[6]).

The author added that even among advanced students, the initial conceptions of limit are still present.

Truth *versus* validity

Relying of the previous considerations, we consider that this equality has a potential for questioning the relationship between truth and validity in the following sense: it is possible to consider that the equality is not true due to the fact that there is not an

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[4] For the whole argumentation, see the « Letter to the editors ». http://www.springerlink.com/content/1l82j7w55h264602/

[5] This is also attested by blogs and forums that can be found on the web on this topic.

[6] Our translation

empirical reference allowing considering it as the result of a process. As we have said before, the natural candidate for a theoretical framework (extension of algorithms for non-terminating decimal) is not satisfying (introduces a new result that does not suit with a solid result for terminating decimal). On another hand, it is possible to consider that this equality is true and to be convinced by arguments that could not be accepted as proof, in particular because they rely on the extension of operative algorithms.

As a matter of fact, behind this equality lies the question of the definition of the notion of equality for real number: "two real numbers *a,* and *b* are equal" *if and only if* "for every strictly positive real number ε, the distance between *a* and *b* is strictly inferior to ε"; this definition is beyond algebraic calculation and classical comparison algorithm.

In the second part of the paper, we report on an experiment with fresh undergraduates students in France (Lyon, January 2008) where the discussion opened on some of these questions.

FRESH UNDERGRADUATE STUDENTS STRUGGLING WITH TRUTH AND VALIDITY

The experiment that we report here took place in Lyon (France) in January 2008 with nine first year university students (three girls and six boys). They had followed in fall semester a calculus course. They were volunteers for following a short course (eighteen hours in three days) aiming to help them to overcome difficulties in logic, reasoning and proof met in the calculus course. Seven of them were fresh students facing difficulties; one was in reconversion after having prepared during two years the medicine competitive exam; the last one was a very good student who wanted to deepen his logical and proof and proving competencies. We focus on a session that took place on day 2 and was devoted on proof and more precisely on the work around the equality "0,999…. = 1". The students were asked whether the equality were true.

Possible students' answers to the question

The question is to evaluate equality were one side refers clearly to an integer, while the second side is a writing referring to a non-terminating decimal "number". As said in previous paragraph, the nature of 0,999… is not obvious. Theoretically, non-terminating writings refers to real numbers, while empirically, in some cases, it refers to process; the specificity of 0,999… is that it is difficult to imagine a natural concrete process leading from integers to such writing, so that there is no available empirical argument supporting the equality. We list below the classical answers that we should expected

Answering NO, with various justifications

N1: The two numbers are different in nature

N2: That is obvious

N3: The whole parts are different

N4: It is always possible to add a 9 to the sequence

N5: 1 is the limit; it is not reached.

N1 refers to separate classes of numbers while the theoretical point of view consider that integers belong to the real number set. N2 corresponds to identification between *form* and *object*, while in a theoretical point of view different writings may refer to the same object. N3 correspond to an application of a rule valid for terminating decimal expansion, but not for non-terminating one. N4 and N5 could be interpreted as the consideration that the writing 0,999… represents a non-ending process (potential infinite) while the theoretical point of view acknowledge that this writing refers to an object (actual infinite). .

***Answering Yes, with various justification*:**

Y1: I know it because I have learned it.

Y2: "1 = 3 ×1/3 = 3 × 0,333…" This uses the implicit extension of the algorithm of multiplication for terminating decimals to non-terminating decimals. As we have said, this arises the question of the validity of this extension.

Y3: Using a classical technics for identifying the rational number associated to a periodic decimal expansion**.** Let a = 0,999…; multiplying a by 10 gives 9,99…; subtracting a to 10a gives 9a; hence 9a = 9 and then a = 1. Once again, this technic relies on the extension of the algorithms of multiplication, and also of subtraction, that should be questioned. In this respect, it is not a proof.

Y4: Showing that for every positive real number ε, $|1- 0{,}999\ldots| < \varepsilon$. This could lead to the following justification using a geometric sequence, or a geometric series.

Y5: Considering 0,999… as the limit of the sequence u_n = 0, 999…9 with n digits 9, for n≥1, and then show that the limit is equal to 1.

Y6: Considering 0, 999… as the limit of the series $\sum u_n$ where $u_n = 0{,}9\times10^{-n}$. The infinite sum $\sum_{n=0}^{+\infty} u_n$ is a theoretical mean to express the decimal expansion 0,999…; it is possible to prove that the series converge to 1.

Y1 corresponds typically to the consideration that things are true because they are said to be true. This is precisely something we should like the students to overcome during the short course. Y2 and Y3 correspond to the justification of algebraic type relying on the extension of algorithms. Y4 is a theoretical justification, whose validity is proved in the theory of real numbers, but it is difficult to use it without introducing sequences. Y5 and Y6 rely on the theoretical framework of numerical sequences and series. Their introduction along with the notion of limits, the operation on the sequences and their compatibility with limits, and the uniqueness of limit allow to provide a proof of the studied equality.

According to us, this *a priori* analysis enlightens the fact that debate on truth and validity is likely to emerge from the discussion. We present now selected excerpts of the exchanges in the different phases of the session; we will use our classification of answers and justifications as a grid for our analyses.

Analyse of students' exchanges

Students had to decide if the sentence were true or not; they worked individually for 10 minutes; then they were invited to present their production in front of the group. The session has been led by the second author of this paper; the first author attended the session as observer, took photos and audio recorded the debate. We point now some phenomena that have been observed.

"1 is equal to 1, and nothing else"

Student G1 rejected the equality, insisting that

9-G1: "1 is equal to 1 and nothing else"

We could interpret this answer as asserting a *material adequation*. Indeed, during the individual work, this student had said that: " It sees itself". Another student, F1, that had at first answered: "Yes", seemed convinced by G1 argument:

11-F1: I said yes because I was once told that it is true, but I agree with G1, it is very disturbing. I said yes because it's said to me.

Her trouble seemed to come from the fact that she knew the sentence is true, but she did not know why. Opposite with G1, she did not explicitly reject the equality.

A tentative to prove

Student G2 proposed a theoretical justification of its answer, trying to provide the " epsilon proof " relying on the equality between real numbers, which corresponds to the justification Y4.

16 G2 In fact I have said yes, but I went to euh, with a positive epsilon. In fact, the euh only, the only "$1 - 0,999$ ..." which is less than epsilon is zero.

18 G2 If we find one "1 - 0,999" which is lower than an positive epsilon; we consider that 0,99 is fixed, finished.

26 G2 We know that 0.999... . It is an infinity of 9 after zero, after the comma. But if "1 - 0.9999 ..." is less than epsilon, 0.999... is finite, which leads to an absurdity because in the beginning, it is assumed that it is ... it is assumed that it is an infinite number..

The tentative did not convince other students who address remarks to G2:

E: "I don't know why absurd is! "

E: " I don't see what is the used of epsilon!'

G2, **G3** and **T** try to clarify the project of **G2; G3** introduced the sequence $1 - u_n$ for which the sequence 10^{-n} is an upper bound. The explanation becomes clearer to others, but *epsilon* disappeared. **G2** goes back to his demonstration, reintroduced epsilon, trying to clarify its role, and explaining that he planned to show that 0, 999… is not different to 1 (proof by contradiction).

93 G3 Let us look for an epsilon such that the difference is greater than epsilon.

94 T So if we want to show that 1 is different from 0, 999… we should show that we can find an epsilon such that 1 – 0,999… is greater than this epsilon.

95 G2 This is what I wanted to do

97 G2 In fact, just to assume that it is different and show the opposite.

G2 finally did not managed to clearly expose his proof, the technical steps remaining confuse. As we said in the *a priori* analysis, the proof with epsilon is difficult if the sequences are not introduced. The other students give up following him and turn to other proofs.

"Equal to the limit versus tends to the limit"

Later in the session, a student proposed the proof *Y5*, and another one tried without success, to implement the proof *Y4*.

While she recognizes that proof *Y5* is clear, F1 is still not convinced; she engaged in a discussion with the teacher on the difference between "to be equal to the limit" or to "tend to the limit"; considering that the limit is not reached, she asserts that the proof "does not prove", and finally staked anew that she knows that it is true, but she does know why.

110 T: For you, 1 is 1. You are right, but what about the limit of u_n for you?

111 F1: It tends to 1! It tends only; it is not equal.

112 T: No, it is the sequence that tends.

113 F1: yes

114 T: But the limit?

115 F1: It is not reached

127 F1: Ben? Then, we proved nothing?

134 F1: That approaches

141 F1: I know that it is yes, but I don't understand

This position of F1 is in line with the results of Cornu (1981) and makes an echo to the position of the author of the letter to editor we presented in paragraph I.

Extension of operations on non-terminating decimal numbers

T intervenes to ask F1 providing another writing of 1/3. She gave 0,333... with infinitely many digits equal to 3; this does not disturb nor shock her, she argued that it is different:

F1 157 It is not the same thing, because if you say 1/3, it is a finite number.

Then she said

163 F1 Ah, but if you multiply everything by 3, it runs, it gives the same thing here.

At that moment, F1 seems to become convinced.

We should consider that for F1, the facts that "There is an empirical evidence for the decimal expansion of 1/3", and "Multiplying both sides by 3 provides the equality" are solid enough to ground the truth of the equality, answering her « why » pending interrogation, this although the extension of multiplication to non-terminating decimal number has to be established.

A discussion on proof Y3

Students F2 who arrived while the discussion was already engaged (round 144) asserted that the equality is true and that it can be easily proved. She proposed the proof Y3.

The teacher asked F2 if multiplying by 10 the decimal expansion is allowed. F2 then move to proof Y2, but the teacher asks again if it is possible to extend the operations; finally F2 gives up. The intervention of F2 and the teacher question introduces anew in the debate the discussion on truth *versus* validity.

178 T: [...] " What allows us to still apply the operations while we have no more the process of division? For 2/3, we had a process of division?

183 F2: We can also demonstrate by 1/3. 1/3 is 0, 333…; 2/3 is 0, 666...; 3/3 is normally 0,999….

185 F2: We have no right

At the end of the session, the student F1 remains disturbed by the equality; this leads her to accept the proofs while remaining sceptical on the truth-value of the equality.

221 F1: It always disturbs me

225 F1: It is good the proofs, it is attractive. But it is proved OK, but "one is one".

F1 seems to consider that the statement could be proved in a theory, while it would be false in a given relevant interpretation.

CONCLUSION

In this communication, we intend to show that working on the equality "0,999… = 1" is likely to open rich discussion on the relationships between truth in an interpretation and logical validity of a proof in a theory, involving the interplay between proving and defining. The exchanges reveal four main attitudes: knowing that the equality is true and knowing that there is a theoretical proof, even if exposing the proof is difficult – knowing that it is true and knowing a proof relying on the extension of operations to non-terminating decimals, without having a proof that this can be done – knowing that it is said to be truth, but without knowing why – understanding a proof, but remaining doubtful concerning the truth of the equality, opening the possibility of a discrepancy between theoretical and empirical assumptions. It is more usual for students to be sure that a statement is true, and do not understand the need for proof. In the case we have presented, it seems that students express a need to proof that implicitly engaged toward a need for theory. These results confirm our hypothesis that we have here a good candidate for discussing these topics, but it is clear that going deeper on these questions with students necessitate to design more cautiously a didactical situation, in order to allow students

> " (…) to experience not only how to validate statements according to specific reference knowledge and inference rule within a given theory, but also how the "truth" of statements depends on definitions and postulate of a reference theory." (Durand-Guerrier & al. 2012, 358)

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THE BIOLOGICAL BASIS FOR DEDUCTIVE REASONING

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This paper explores the basis of deductive reasoning from a range of psychological, biological, evolutionary and neurological perspectives. The basic structure of deductive reasoning is seen as being common to all primates and based on evolutionary pressures to perceive causes and effects. However, deductive reasoning at higher levels of abstraction is unique to humans and linked to language use. The feeling of necessity that is associated with deductive reasoning is also accounted for by the presence of somatic markers for deductive conclusions that also have an evolutionary basis.

Deductive reasoning, epistemology, necessity, abstraction, evolution

INTRODUCTION

It is clear that some human beings are capable of proving in very sophisticated ways (see, e.g., the paper by Annie and John Selden in these proceedings). There is also abundant evidence that many school and university students find proving and reading proofs difficult (see, e.g., Reid & Knipping, 2010). But proving as practiced by professional mathematicians involves the coordination of a large number of capabilities. To understand the educational task of teaching proof it is essential to understand the nature of these capabilities, how they develop with maturity and how they can be influenced by teaching. In this paper I attempt to identify capabilities essential to proving that may be common to all humans. The existence of such capabilities means that teaching related to them is a matter of developing existing capabilities rather that introducing new ones, which is a distinctly different educational task.

As my focus is on *human* capabilities my paper will draw largely on biological perspectives. Philosophical and historical approaches to epistemology have told us much about the nature of deductive reasoning, but not in relation to the *people* who actually do it. For that we need to turn to what McCulloch (1960, 1965) called "experimental epistemology", inquiry into the physiological substrate of knowledge, or what Donald Campbell (1974) called "evolutionary epistemology", "an epistemology taking cognizance of and compatible with man's status as a product of biological and social evolution." (cited in Rav, 1989, p. 51 & 2006, p. 73).

> In contrast to the various philosophical epistemologies, evolutionary epistemology attempts to investigate the mechanism of cognition from the point of view of its phylogeny. It is mainly distinguished from the traditional position in that it adopts a point of view outside the subject and examines different cognitive mechanisms comparatively. It is thus able to present objectively a series of problems [including the problems of traditional epistemologies] not soluble on the level of reason alone [but, which are

> soluble from the phylogenetic point of view]. (Riedl, 1984, p. 220; 1988, p. 287, cited in Rav, 1989, pp. 51-52 & 2006, p. 73)

Here I focus on two capabilities, being able to reason from a general rule and antecedent data to a specific consequence (reasoning represented in logic by *modus ponens*) and feeling that such consequences are necessary.

REASONING FROM A GENERAL RULE: MODUS PONENS

Modus ponens links a datum to a conclusion via a general rule. It can be considered to be the most simple instance of deductive reasoning, it is the only rule of inference needed for propositional logic, and it is difficult to imagine how anyone could prove anything without making use of *modus ponens*. The nature of *modus ponens* has been considered from a philosophical perspective by, for example, Aristotle, Bacon, Descartes, Kant, Schopenhauer, and Wittgenstein. Here, however, I am concerned with a biological question: Are all human beings capable of reasoning in this way, and if so, why?

Can humans reason deductively?

There is a large body of research that starts from the assumption that not only can humans reason deductively, deductive reasoning is the basis for all human thought (recently, Rips, 1994. Cosmides, 1989, p. 191, cites as examples Henle, 1962; Inhelder & Piaget, 1958; Johnson-Laird, 1982; and Wason & Johnson-Laird, 1972). However, such claims must account for circumstances in which humans do not reason deductively, even though the context clearly requires it.

For example, on the Wason (1966) card selection task [1] "performance is relatively poor" (Goel, 2007, p. 436). "Relatively poor" is an understatement; in some studies only 10% of subjects turn over the correct cards. Given that the task involves only deciding what data and conclusions must be checked to verify a general rule used in *modus ponens*, the difficulty many people have with this task makes it implausible that deductive reasoning is the basis for human thought. There is also a growing body of work that argues for another possible basis for human though, analogical reasoning (see, e.g., Lakoff, 1987, Hofstadter, 2001).

If deductive reasoning is not the basis for human thought, but we know that some humans can reason deductively, then we must explore two possibilities: that deductive reasoning is learned at some point, or that deductive reasoning comes with being human, having been acquired at the species level in the course of human evolution.

If deductive reasoning is learned, then is must be learned fairly early, as Richards and Sanderson (1999) report that four year old children can give a logically correct answer to the question: "All fish live in trees. Tot is a fish. Does Tot live in a tree?" Hawkins, Pea, Glick and Scribner (1984) also found that in some circumstances four year old children are capable of making deductive inferences. Stylianides and

Stylianides (2008) provide an excellent summary of the psychological research on deductive reasoning by children.

I know of no research into the process by which deductive reasoning is learned in pre-schoolers, and in fact there is evidence that humans are born already able to reason deductively, just as we can recognise faces and distinguish two from three.

Part of this evidence for is the capability of some animals to make simple deductive inferences. One experiment that has been used to test for the capability to make simple deductive inferences involves two opaque cups, one of which contains a treat. The empty cup is shaken (the animals previously experienced that the cup with food makes a noise when shaken). In order to know that the other cup contains the treat the animals must do two things. First, they must associate the noise with the treat. Then they must infer A from {A or B, not B}. Apes (Call, 2004) and grey parrots (Schloegl, Schmidt, Boeckle, Weiß & Kotrschal, 2012) are able to do this, without training, as are children (Hill, Collier-Baker & Suddendorf, 2012). No other animals have yet been found that can make this inference.

Of course, there is a significant difference between the reasoning of animals and the reasoning of humans. Humans can express their thinking in language, but the exact relationship between language and reasoning is in debate. It may be that the structure of language itself forms our thoughts into general rules (Bickerton, 1990) or that the social use of language does so (Vygotsky, 1986) or that general rules emerge from our making sense of the world and language reflects this (Devlin, 2000). In any case, the ability of people to reason using general rules is intimately tied to the ability of people to express general rules in language.

There is evidence that using language to express general rules increases markedly after a child's second birthday. Bruner and Lucariello (2006) analyse the use of words that mark "sequence" and "canonicality" or normalcy, in the monologues of Emily, a toddler whose bedtime language use was recorded from the time she was 21 months old until she was 36 months old. Of special interest here are sequence markers of causality and canonicality markers of necessity. Bruner and Lucariello contrast the number of causality and necessity markers in two periods, when Emily was 22-23 months old, and when she was 28-33 months old. They find that "just as expressions of causality emerge as the dominant, later form for binding sequences, expressions of necessity and appropriateness emerge as Emily's more advanced form of indicating canonicality" (p. 87). And in this later period, Feldman (2006) reports, she "begins to play with reasoning, to reason about fictional and invented worlds" (p. 107).

Emily's playing with reasoning, using it in imagined contexts, is very interesting. It reveals something important about the way humans use deductive reasoning, and perhaps begins to explain why a child can answer the question "Does Tot live in a tree?" while adults fail at the Wason task. Hawkins et al. (1984) asked four year olds to answer questions based on chaining two general rules using modus ponens. In "congruent" cases the correct answer conformed to the children's experiences. In "incongruent" cases it contradicted their experiences. And in "fantasy" cases it involved imaginary creatures, so prior experience did not apply. The children answered almost all congruent questions correctly, and almost all incongruent questions incorrectly (from the perspective of logic; from the perspective of their experiences their answers made sense). They were quite good at answering fantasy questions correctly, especially if those questions were asked first. This suggests that deductive reasoning and prior experiences are both used by humans, but that prior experiences carry more weight. Worlds of imagination provide contexts where deductive reasoning has priority, because experience does not apply. And interestingly, if children are prompted to use their imaginations, even when dealing with familiar objects (like fish and trees) they come to deductively correct conclusions. As Richards and Sanderson (1999) put it, "2-, 3-and 4-year-olds can solve deductive reasoning problems when they are given cues to use their imagination to create an alternative reality where different outcomes are possible" (p. B8).

The ability to abstract

To express a general rule and to invent an imaginary world both involve working with abstractions. It is worth looking more closely at that process, and especially at different levels of abstraction related to *what* is abstracted.

Devlin (2000) divides abstraction into four levels. Level 1 abstraction refers only to things that are present. My seeing many present birds as the same kind of bird is such an abstraction. This is the kind of abstraction animals engage in by perceiving in categories. Level 2 abstraction involves familiar things that are not present. The food that is not in the shaken container when it makes no noise is an abstraction of this type. Level 3 involves "real objects that the individual has somehow learned of but never actually encountered, or imaginary versions of real objects, or imaginary variants of real objects, or imaginary combinations of real objects" (p. 121) like dodo birds, fish that live in trees, unicorns and centaurs. For Devlin, level 3 abstraction amounts to having language, and so only humans can do it. Finally, level 4 abstraction involves objects that are themselves abstract.

Where do such abstract objects come from? From a process of abstracting that goes beyond forming categories. Rather than just perceiving a group of birds as "all the same" I can observe properties that are the same, for example, colour, or size, or the shape of the beak, or the pattern of the song. Such properties are also abstractions, but of a different kind. When I perceive that a cup is the same colour as a book, I create a new category of things that are that colour. The colour becomes an abstract object. "We reify our abstracting: the end of the process of abstraction—paying attention to only some of our experience—begins to be treated as a thing, an abstract thing." (Epstein, 2012, p. 252). The same thing happens with other properties, like number and shape. I form a category of all the (roughly) triangular objects I perceive, and then a triangle becomes a level 4 abstract object.

I suspect that Devlin's level 4 has further divisions. When thinking about triangles I can refer directly to a present triangular object to aid my thinking. But the same process that took me from level 2 to level 3 (recombining objects in my imagination) can be applied to abstract objects. From my abstract triangle and tetrahedron I can use my imagination to go to a four dimensional figure that is somehow like them, but at the same time even more abstract. I can model an abstract triangle with a concrete triangular object, but I cannot do that with an abstract object that is itself based on an abstract object. The higher the level of abstraction, the harder it is to think about it.

Deductive reasoning involves using general rules, and general rules involve using abstractions. This dependence means that deductive reasoning ought to be different if it involves different levels of abstraction. And it is. When dealing with level 1 and level 2 abstractions, deductive reasoning is essentially a way of describing causal relationships between things. If an ape selects the container that was not shaken it can be said to conclude "The treat is in here" from the general rules that "If there is a treat then it makes a noise when the container is shaken," "If the treat is not in one container then it is in the other" and the datum "That container did not make a noise when shaken." This is more complicated reasoning than *modus ponens*, but limited to level 2 abstractions. Of course, the ape is probably not aware of these rules, but behaves as if it knows them.

Deductive reasoning with level 3 abstractions is more complicated, as it involves imagined things expressed in language. For example, the question "Does Tot live in a tree?" (Richards & Sanderson, 1999, p. B2) and the version of the Wason task using the rule "If a person is drinking beer, then he must be over 20 years old" (Cosmides & Tooby, 1992) involve level 3 abstractions. Children prompted to use their imaginations can answer the question about Tot correctly, and most adults succeed at the Wason task if it is clearly presented in a social context. Humans can reason deductively in these contexts because humans can use language and level 3 abstractions.

Deductive reasoning with level 4 abstractions, especially abstractions that cannot be modelled easily, on the other hand, is difficult for most humans. That is why most people cannot do the abstract Wason tasks, and why most people find mathematical proofs hard to follow. It is not that they cannot reason deductively; it is that reasoning deductively gets more difficult the more abstract it becomes.

Language makes it possible to reason with level 4 abstractions, but doing so involves not only having such abstractions, but also formulating them so that they can be objects for reasoning. As Bishop (1991) puts it:

> The language of "if" and "suppose" and the conditional tense also forces an imagined reality onto the conscious level and thereby enables it to be manipulated *as if it were* an objective reality. Thus, as well as encouraging children to develop their ability to abstract, we need also to encourage them in the ways of concretising and objectivising abstract ideas. (p. 67)

In summary, *modus ponens* encapsulates a particular way of relating abstract categories back to specific cases. Language allows the relationship to be expressed, and allows *modus ponens* to be applied to abstract categories that are not based on direct experience. Because human beings evolved in a world in which relating abstract categories back to specific cases is useful, we can reason deductively, and because we can use language, we can learn to do so in abstract contexts. This suggests that the main challenge in teaching people to reason deductively in very abstract contexts like mathematics is not teaching them deductive reasoning. Deductive reasoning comes with being human. However, the abstractions of mathematics are not the context in which humans came to reason deductively, so learning to reason deductively in such contexts requires learning to cope with abstraction better.

FEELING NECESSITY

I am considering logical necessity from a biological standpoint, and so I am addressing the question: Why do (some) humans associate the conclusions of deductive reasoning with certainty, while conclusions reached in other ways are recognised as being only probable? When I come to a conclusion by deductive reasoning, I feel not only that the conclusion is so, but also that is *must be so*. This feeling does not occur with other kinds of reasoning.

Damasio's (1996ab) concept of somatic marker offers a neurological basis that can be used to account for the feeling of necessity. A somatic marker is the juxtaposition of knowledge, emotion and bodily feeling related to a decision or a thinking process. Every decision a person makes activates not only knowledge relevant to making the decision but also emotional markers triggering bodily feelings. These "somatic markers" are not activated in patients with certain kinds of brain damage, with consequences for their everyday decision making.

If a deduction is seen as a kind of decision making, then the feeling of necessity is accounted for as a somatic marker associated with any use of deductive reasoning. Why do humans feel this way about deductive reasoning? There are two possibilities: the somatic marker may be acquired through individual experience, or it may now be innate, having been acquired at the species level in the course of human evolution.

If the somatic marker is acquired though experience this would account for the observation (reported Galotti, Komatsu, & Voelz, 1997, p. 77) that while young children (about seven years old) have higher confidence in conclusions reached through deductive reasoning that conclusions reached through inductive inferences, they are less certain of the conclusions of deductive reasoning than they should be.

However, as we have seen above, the ability to reason deductively is tied to the degree of abstractness of the context, and it may be that children are less certain simply because they recognise that they are not completely fluent in abstract reasoning. Children may already have the somatic marker for necessary conclusions, but feel less certain because of unfamiliarity with abstractions. In that case, we must account for the presence of this somatic marker in humans by making reference to evolutionary pressures.

Some somatic markers (such as fear of snakes) are clearly innate, and Damasio accounts for the origins of somatic markers evolutionarily.

> Let us assume that the brain has long had available, in evolution, a means to select good responses rather than bad ones in terms of survival. I suspect that this mechanism has been co-opted for behavioural guidance outside the realm of basic survival. … It is plausible that a system geared to produce markers … to guide basic survival, would have been pre-adapted to assist with ‘intellectual’ decision making. (1996b, pp. 1416-1417)

Rav, without using the language of somatic markers, accounts for the origin of the feeling of necessity through a similar process.

> But whence comes the feeling of safety and confidence in the soundness of the schemes which formal logic incorporates? To an evolutionary epistemologist, logic is not based on conventions; rather, we look for the biological substrata of the fundamental schemes of inference. Consider for instance *modus ponens*:

$$\begin{array}{c} A \to B \\ \underline{\qquad A} \\ \square B \end{array}$$

> If a sheep perceives only the muzzle of a wolf, it flees already for its life. Here, ‘muzzle→wolf’ is ‘wired’ into its nervous system. Hence the mere sight of a muzzle—any muzzle of a wolf, not just the muzzle of a particular wolf—results in ‘inferring’ the presence of a wolf. Needless to say such inborn behavioral patterns are vital. ... The necessary character of logic, qua codified logico-operational schemes, thus receives a coherent explanation in view of its phylogenetic origin. (Rav, 1989, p. 63, 2006, p. 83)

I hypothesise that when humans developed language sufficiently to express abstract general rules in words, the rules they first articulated also existed as behavioural patterns (like that of the sheep) tied to somatic markers. The first human to say “If you see a wolf, run!” was articulating a general rule that when applied in *modus ponens* activated the somatic marker of necessity. This marker remained associated with *modus ponens* in other contexts, simply because evolution does not select out attributes that have survival value.

In summary, deductive reasoning comes with a feeling of necessity. But necessity is not a property of deduction, it is a property of deductive reasoning being done by people who have learned to feel the necessity of deductive conclusions. We cannot assume that students automatically have the somatic maker that makes them feel necessity in abstract contexts like mathematics, but I believe that all humans do in less abstract contexts, as a result of our evolutionary history. That would account for Galotti, Komatsu, and Voelz’s (1997) finding that children shows signs of associating deduction with certainty early in their schooling. But feeling necessity in mathematics might involve creating a new somatic markers, which would account for the children's lack of complete confidence. In either case, mathematics educators must explore how such a somatic marker is activated, and how it interacts with other somatic markers.

The idea of a somatic marker for necessity provides a new viewpoint from which to reexamine Fischbein's (1982) process of elaborating new intuitions.

> A new "basis of belief", a new intuitive approach, must be elaborated which will enable the pupil not only to understand a formal proof but also to believe (fully, sympathetically, intuitively) in the a priori universality of the theorem guaranteed by the respective proof. (Fischbein, 1982, p. 17)

Fischbein suggested that preformal proving (that is, deductive reasoning in less abstract contexts) might help in the development of such a basis of belief, but if the somatic marker for necessity is already present then the task is not to develop it, but rather to understand what might interfere with it when encountering formal proof. Further research in this direction is needed, supported by a theoretical framework that sees deductive reasoning in human terms.

CONCLUSION

Warren McCulloch, in his 1960 Alfred Korzybski Memorial Lecture, asked the question "What is a number, that a man may know it, and a man, that he may know a number?" (1960, p.7, 1965, p. 1) which inspired me to ask here "What is deductive reasoning, that humans can reason deductively, and what are humans, that they can reason deductively?" In other words I have sought to account for key aspects of deductive reasoning, the use of *modus ponens* and the feeling of necessity, by bringing together psychological, biological, evolutionary and neurological perspectives. The biological bases for deductive reasoning have two important implications for teaching proof. We do not need to begin by teaching students how to reason deductively. And we do not have to teach them the feeling of necessity. However, we must begin in contexts where abstraction is not an obstacle to reasoning, and we must be attentive to other somatic markers (for example associated with feeling of mathematics anxiety) that will interfere with feeling necessity.

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NOTES

1. The task is to determine which of four two-sided cards need to be turned over to verify a general rule. The task exists in many versions, based on the original in which the cards have a letter on one side and a number on the other and the rule is "If a card has a vowel on one side, then it has an even number on the other side.". The visible sides of the cards show one vowel, one consonant, one even number and one odd number and the correct answer is to pick the vowel and the odd number, corresponding to the logical proposition P & not Q.

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MULTIMODAL PROOF IN ARITHMETIC

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This theoretical paper develops further the concept of multimodal proof from the perspective of the multimodal paradigm, phenomenology and Luis Radford's theory of knowledge objectification. The study of such proof is motivated by its possible use in mathematics education, especially in school, but possibly also with adult students. We discuss one type of generic multimodal proof in arithmetic using a proof principle called schematic generalisation. It is argued that this type of proof both can establish truth in arithmetic and give phenomenologically explanations.

Key words: multimodal proof, embodied cognition, objectification, phenomenology

INTRODUCTION

The concept of multimodal proof was introduced in Rinvold and Lorange (2011). A multimodal proof is a generalized proof which beside written symbols and sentential reasoning can also include the visual modality, speech, the tactile and motor action, (p. 633). The idea of combining sentential and visual reasoning has been developed in mathematical logic and its learning by Barwise and Etchemendy (1996) under the name of heterogeneous reasoning. Their ideas have been used and followed up by several other researchers. An example is Oberlander, Monaghan, Cox, Stenning and Tobin (1999) who characterize heterogeneous proofs as multimodal. Williams et al. (2012) refers to the multi-modal nature of proof. Their study investigates how gestures and actions are related to the ascertaining and persuading phases of proof. It is about students' justifications and does not discuss what a valid multimodal proof might be as part of a mathematical community. We will now develop the concept of multimodal proof further within the multimodal paradigm of Arzarello and Robutti (2008). As far as we know, no other researchers have developed multimodal or heterogeneous proof as a concept within this framework.

The multimodal paradigm is an emerging view of thinking and reasoning. It combines the embodied mind paradigm and sociocultural theory. Mind is part of a physical body, and the cognising man acts physically and verbally in a physical and cultural world using artefacts and signs. Thinking is not only internalized speech, maybe supplemented by inner visualization, but is linked to all the senses and motor action. Thinking is made possible and restricted by our bodily life in the physical world, but has reached an advanced level through culturally developed language and artefacts. Multimodality is a direct consequence of this view of thinking. Fallacies and idiosyncrasy have been a problem with visual and intuitive proof. However, we agree with research that opens for the possibility that some non-sentential proofs are legitimate.

> The mere existence of fallacious proofs is no more a demonstration of the illegitimacy of diagrams in reasoning than it is of the illegitimacy of sentences in reasoning. Indeed, what understanding we have of illegitimate forms of linguistic reasoning has come from careful attention to this form of reasoning, not because it was self-evident without such attention. (Barwise & Etchemendy 1996, p. 6)

We study communication and learning of proof by applying the theory of knowledge objectification, Radford (2006a, 2006b, 2008). Objectification has to do with the learning of the individual when thinking is seen to have an intimate and dialectical relationship with the material and cultural world, LaCroix (2012, p. 2464). It is a process using semiotic means in order "to draw and sustain the attention of others and one's own attention to particular aspects of mathematical objects in an effort to achieve stable forms of awareness, to make apparent one's intentions, and/or to carry out actions to attain the goal of one's activity.", (ibid, p. 2464).

Our discussion is restricted to one type of proof principle in arithmetic called schematic generalisation. We look at this through an example proof which is normally described as visual or diagrammatic proof. We argue that schematic generalisation can establish truth in arithmetic. These kinds of proofs have also been studied from the perspective of generic proof, which is a less precise concept. Referring to Tall (1979), Aliebert and Thomas (1991) write that "Such a proof works at the example level but is generic in that the examples chosen are typical of the whole class of examples and hence the proof is generalizable." Tall (1979) argues that generic proofs are explanatory in the sense of Steiner (1978), which writes that "It is not, then, the general proof which explains; it is the *generalizable* proof." (p. 144)". We argue that from the perspective of multimodal proof, proof by schematic generalisation are also explanatory in another way, which we call phenomenological explanation. A problem with many formal proofs, especially algebraic ones, is that the proofs are not explanatory. Students may be able to follow the rules which are applied in the proof, but they do not get any reason why the proved theorem is true.

WHAT IS MULTIMODAL PROOF?

It is common in the literature on proof in mathematics education to make a clear distinction between proof production or proving and the final proof object. From a sociocultural point of view it is plausible that the latter is an artefact accessible for public validation. In the rest of this paper proof is seen in this way. For a long period of time proof has been written or printed on paper. In this medium visual and sentential are the only possibilities. The existence of animation, film and video give a possibility of including other modalities, but also challenges the distinction between "the proof" and proof presentation. If the presentation of a proof is videotaped, the record is an artefact. However, we do not go further into this. In this paper both proving and proofs involve artefacts and several modalities.

Multimodal or heterogeneous proving is a kind of multimodal reasoning and thinking. One core idea of multimodal thinking is that human reasoning always applies at least two modalities of thought.

> Multimodality, however, proceeds on the assumption that representation and communication always draw on a multiplicity of modes, all of which have the potential to contribute equally to meaning. (Jewitt, 2009, p. 1)

Except some proofs generated by computers, proofs are meant to be read by humans, and as such are part of communication. Because of this, words and visual diagrams are used together with mathematical formalism in the proofs. But, a common idea is that formal proof could be represented without visualisation by mathematical formalism only. The concept of visual proof indicates similarly the belief that visual arguments can be represented just by diagrams. The phrase “proofs without words”, Nelson (1993, 2000), indicates the latter. According to Barwise & Etchemendy (1996), heterogeneous proof consists of more than one mode of reasoning, in their case primarily the visual and sentential modality. By proof those researchers mean proof inside proof systems. Such a system consists of the allowed rules of inference and the allowed objects transformed by the inference rules. Since the formalism of Hilbert was developed, the objects of proof systems have mostly been formal sentences. The contribution of Barwise and Etchemendy is important for the question of legitimacy and possible acceptance of multimodal proof. Proof systems make validation of proofs easier and also support the comparison with classical proof.

THE MULTIMODAL EXAMPLE PROOF

The example proof of our further discussion is given by a visual diagram and an explanation by words. As such it is multimodal, but it is open if it can be represented by a proof only in the visual mode. The proof is not formal, in the sense that it is not based on a proof system, and that mathematical symbols are not applied.

Looking at the diagram in figure 1 below, it is not obvious what it is going to prove.

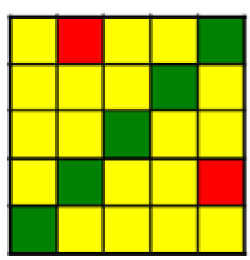


Figure 1

To be told that the statement concerns odd numbers may help some readers. The diagram shows the square of the odd number five. The diagonal in green consists of the same number of small squares as the side, and the red small squares above and below the diagonal make a pair.

Each small square in the upper triangle makes a pair with the corresponding symmetrically placed small square in the lower one. The complement of the diagonal thus is a set of disjoint pairs. Since the diagonal is a set of disjoint pairs together with a single small square disjoint from them, the large square is an odd number.

The argument shows that the square of an odd number is odd. But, we have more. The decomposition of the square into its diagonal and a pair of triangular numbers do not use that the side is odd, and hence is valid for all natural numbers. This can be used to show the opposite implication. If the square is odd, then also the diagonal is odd, for taking away a set of pairs from an odd number, results in an odd number. Since the diagonal equals the side, the side is odd when the square is odd. Formally the implications in both directions can be written

$$\forall x[\mathrm{Odd}(x)\leftrightarrow\mathrm{Odd}(x^2)]$$

RECURSIVE ω -PROOF AND SCEMATIC GENERALISATION

We argue that the proof principle used in the example proof can be formalized by the concept of recursive ω-logic, a proof principle which legitimacy hardly can be disputed. Beside this legitimacy argument, we also use ω-logic to make clear what the example proof is meant to exemplify. The origin of ω-logic is proof theory as a branch of mathematical logic, but the original use, called cut-elimination, is technical and outside the scope of this paper.

Jamnik, Bundy and Green (1997) introduced the formalization by recursive ω-logic for diagrammatic proofs like the example proof with the intention to argue that this kind of reasoning is legitimate. Each case of the theorem can be proved directly from a diagram by geometric operations. One given diagram plays a schematic role, which make it possible to generate the other diagrams and the proofs for each case. Those authors have developed a system for automated theorem proving called DIAMOND, which successfully have turned several diagrammatic proofs into recursive ω-proofs.

A recursive ω-proof of ∀n $\varphi(n)$ is a procedure which let us calculate a proof of $\varphi(n)$ for each n. In the example this means that we see the decomposition of the square as a procedure which can be done for all possible integer squares. Intuitively, this means that we can draw "the same kind of diagram" for all integer squares. In arithmetic $5^2 = 2 \cdot T_4 + 5$ is certainly not implying the general claim $n^2 = 2 \cdot T_{n-1} + n$, where T_4 and T_{n-1} are triangle numbers. What is different with the diagram is that it shows the decomposition to be more than an accidental identity between numbers.

A direct algebraic proof of φ(n) is also a recursive ω-proof of ∀n φ(n). The use of symbolic algebraic variables is based on some rules or properties which are common for all numbers in question, for instance the commutative and distributive laws in arithmetic. This is seen in the direct example proof **D1** of

$$\forall n[(n + 1)^2 = n^2 + 2n + 1]$$

The proof is given by

$$\textbf{D1}: (n + 1)^2 = (n + 1) \cdot (n + 1) = n \cdot n + n \cdot 1 + 1 \cdot n + 1 \cdot 1 = n^2 + 2n + 1$$

We normally see this as one proof of one conjecture, but an alternative is to consider it as a collection of proofs, one for each natural number. For instance the substitution of $n = 4$, gives a proof that $(4 + 1)^2 = 4^2 + 2 \cdot 4 + 1$.

Recursive ω-logic is an alternative to induction in the formalization of proofs in arithmetic and implies induction. The premises of induction is for an arithmetical formula φ the claims $\varphi(1)$ and $\forall n[\varphi(n) \rightarrow \varphi(n+1)]$. From this we get an algorithm which gives proofs of $\varphi(n)$ for each n.

$\varphi(2)$ is proved by $\varphi(1)$ and $\varphi(1) \rightarrow \varphi(2)$

$\varphi(3)$ is proved by $\varphi(1)$, $\varphi(1) \rightarrow \varphi(2)$ and $\varphi(2) \rightarrow \varphi(3)$, and so on.

This means that we have a recursive ω-proof of $\varphi(n)$, which entails $\forall n \varphi(\mathrm{n})$.

Technically, induction proofs are a good solution, but typically such proof does not give an explanation understandable by students. As an example of the latter we prove the statement related to figure 1 by an induction proof **IN** setting $\varphi(\mathrm{n})$ to be $n^2 = 2 \cdot \mathrm{T}_{n-1} + n$, where $\mathrm{T}_0 = 0$ and $\mathrm{T}_n = \mathrm{T}_{n-1} + n$ are the triangle numbers inductively defined. Using the statement proved in **D1**, we get

$$\textbf{IN}: (n + 1)^2 = n^2 + 2n + 1 = (2 \cdot \mathrm{T}_{n-1} + n) + 2n + 1 =$$
$$2 \cdot (\mathrm{T}_{n-1} + n) + (n + 1) = 2 \cdot \mathrm{T}_n + (n + 1)$$

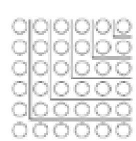
It can be seen that $\varphi(1)$ also follows. The statement related to figure 1 can be proved by a direct algebraic proof **D2** too, but this appears as a rabbit thrown from a hat:

$$\textbf{D2}: n^2 = n^2 - n + n = 2 \cdot \tfrac{1}{2} (n - 1) \cdot n + n = 2 \cdot s(n) + n$$

Now, $s(\mathrm{n}) = \tfrac{1}{2} (n - 1) \cdot n$ is always a natural number, since $(n - 1) \cdot n$ has to be an even number.

PHENOMENOLOGICAL EXPLANATION

The phenomenology of proof has to do with how proof is experienced. The example diagram gives us the experience of knowing, understanding and believing. Jamnik, Bundy & Green (1997) formulate this about the same type of diagram,



$$1 + 3 + 5 + ... + (2n – 1) = n^2$$

"Not only do we know what the diagram represents, but we also understand the proof of the theorem represented by the diagram and believe it is correct (p. 51)." But, the diagram does not only give subjective belief. It is what Kitcher (1983) calls warranted belief (p. 17). The belief has to be justified in a way that is accepted by the mathematical society. Recursive ω-logic is one way of giving such a warrant, but an alternative place to look for it is embodied cognition. Arithmetic has a phenomenological and semiotic foundation which is profounder than axiomatic formalisations. According to Longo (2005), mathematics has cognitive roots:

> We cannot separate Mathematics from the understanding of reality itself; even its autonomous, "autogenerative" parts, are grounded on key regularities of the world, the regularities "we see" and develop by language and gestures.

This physical and perceptual basis of arithmetic can be used both to argue for the legitimacy and the experienced qualities of diagrammatic proofs. The legitimacy argument is to show that the kind of reasoning used in the diagrammatic proof is also needed to verify the formal axioms of arithmetic, but it is out of the scope and space of this paper to go further into this.

The possibility of arithmetic has to do with the stability of matter, that objects has permanence and do not suddenly appear, split or disappear like clouds. It also depends on our ability to discern some things as being a collection of objects of the same type. Freudenthal (1983, p. 75) points to Euklid book VII as an origin of the set or cardinal approach to number, and cites Felix Klein for the idea of numbers as collections of things of the same type. The concept of set is based on the invariance of physical or visual collections under spatial placement. The multimodal example proof is based on perceptible sets and spatial invariance. The objects of same type are small squares which together make up a square formed lattice. The decomposition of the square in the proof is related to spatial invariance, as it can be seen as moving the triangle parts away from the diagonal part. Both the concept of natural number and the proof also depends on our faculty of visual pattern recognition. That the proof uses the perceptual roots of arithmetic can thus be a reason behind its explanatory power. It is a phenomenological explanation not only by giving the experience of explanation, but also by using the phenomena behind the conjecture to be proved.

OBJECTIFICATION IN EMBODIED COGNITION

From the embodied mind point of view mathematics originates in our perception and ordering of physical reality. Even if the connection between parts of advanced mathematics and reality is not always obvious, this view makes it natural to look for reasoning with a perceptual basis, especially for the learning of the subject. But, we know from experience and research that students, or even mathematicians, are not able to immediately grasp the intended meaning of a proof from a diagram. This can be explained by the multimodality of thinking, that more than one modality is needed, but a semiotic approach gets deeper into the learning and communication aspect. Radford's theory of knowledge objectification is a theory about how individuals can be able to notice and make sense of what they do and see.

> ..., objectification becomes related to those actions aimed at bringing or throwing something in front of somebody or at making something apparent −e.g. a certain aspect of

> a concrete object, like its colour, its size or a general mathematical property. Now, to make something apparent, students and teachers make recourse to signs and artefacts of different sorts (mathematical symbols, graphs, words, gestures, calculators and so on). These artefacts, gestures, signs and other semiotic resources used to objectify knowledge I call semiotic means of objectification... (Radford, 2006b, p. 6)

The means of objectification are actions and semiotic resources. The example proof was given by a diagram and words. The words are semiotic resources which direct the attention of the reader to the appropriate aspects of the diagram. The concepts 'triangle' and 'diagonal' help the viewer to see the large square as composed of three parts. In the diagram also colours are used to make apparent the decomposition of the square into two triangle numbers and the diagonal, and also show how the small squares above and below the diagonal make pairs.

As an alternative or supplement, a physical diagram can be made of unifix or multilink plastic cubes. Then one of the triangular parts can be laid onto the other, both showing congruence and how to make pairs of cubes. We can pair a cube with the cube lying above it. This physical approach makes the red squares superfluous. These red squares are confusing as long as communication of the decomposition is in focus, so it would be an advantage if they could be painted yellow. Showing congruence does not mean proving, but pointing to. What looks like or feels like equality, can mistakenly be taken by the student to be a proof of equality. The physical process of making pairs of corresponding upper and lower small squares in the triangles, is a proof when the side equals five, but its generalisation requires an argument. A general proof requires another way of seeing.

The explanation by words introduces a process ordered in time, in which different aspects and parts of the diagram are in the foreground. Concepts like 'odd number' and 'set of pairs' helps the viewer to see the diagram as a general pattern. It is possible for a student to grasp everything else, but to see the diagram just as the case of five times five. It is a well known misconception among students that showing one or a few cases is enough to prove a result. We know that many mathematically trained persons experience to see a general proof through diagrams like the one in the example, but we also know that this does not come easy to many students. It is necessary to see the diagram like an informal ω-proof, that is, a procedure for generating the proof in all other cases. A possibility is to let the students draw and paint the five by five square and, notice how this is done and ask them to draw and paint squares of other sizes. Alternatively, the students can build squares by coloured plastic multilink cubes. The dynamic process of painting or building makes it more likely to see an algorithm than looking at a static diagram.

Husserl made a distinction between simple and categorial intuition. The latter means to 'intuite' the conceptual, the general or the Aristotelian form through seeing something concrete. According to Cobb-Stevens (1990), "Rather than presenting some particular thing, say a red chair, categorial intuition presents the chair's *being* red, the red quality's *belonging* to the chair (p. 44)." By intuition Husserl underlines the richness or fullness of actual experience compared to thought and speech. Cobb-Stevens exemplifies this as the difference between strolling through the streets of a foreign city and vague plans of a visit (p. 43). The grasping of the general in the multimodal example proof in visual or physical version probably has the same richness compared to the induction and direct algebraic proofs **IN** and **D2**.

ALGEBRAIC AND VISUAL PROOF COMPARED

The visual or physical multimodal proof related to figure 1 seems to work considerably better than the algebraic alternatives in order to give students meaning and richness of experience when understood or objectified. The former kind of proof gives at least another kind of explanation, which for many students probably is better. Since warrant for truth is the simultaneous establishing of truth and meaning, we think that even in this aspect multimodal proof is a good alternative. However, as some hints have indicated, objectification is not straight forward to achieve. For instance it can be difficult to see the conjecture to be proved directly from diagram 1, and seeing the diagram as general is demanding. Algebraic proof has some clear advantages compared to visual proof. Algebraic proof in arithmetic has well established proof systems which have an undisputed status among mathematicians. The system of algebraic and arithmetic signs are standardised, used almost everywhere and are institutionalised by schools, universities, books etc.

Like the visual diagram 1, also the arithmetical and algebraic notations are spatial and compact. The latter are not phenomenological or iconic, but symbolic. The signs together make a system giving meaning to terms and statements. Algebraic identities like $2x + 3x = 5x$ are linked to the objectified meaning of $2 + 3 = 5$ and the addition operator. Even the algebraic system has a link through objectification to physical and perceptual phenomena behind arithmetic, but not in the direct and full way as in categorial intuition. A relevant reference for the spatial aspects of algebraic symbolism is Bergsten (1999). As long as the complexity of an algebraic statement or a diagram is restricted, both have a good potential of objectification. Ordinary language composed by words lacks spatiality and compactness and are delegated to an intermediate role in learning and objectification. Proofs given by words are not alone a good way for students to objectify proofs. But, neither algebraic formulas nor diagrams give all the necessary information directly. Visual proof has to be supplemented by symbols, standards and transformation rules which make communication, objectification and validation easier.

CONCLUSION AND FURTHER RESEARCH

The paper has contributed to clarification of the concept of multimodal proof. Multimodality means that at least two different modalities occur simultaneously in the proof. Also, a multimodal proof is an artefact, but the distinction between proving and proof is challenged. A proof performance or presentation can be turned into an artefact by the use of video recording. Still the proof will normally be the end result of refinements of more immature attempts of proving. We have restricted the attention to a type of generic visual proof which can be formalized and legitimated by recursive ω-logic.

The learning and recognition of the generality of the proofs are studied by the theory of objectification. This prescribes active learning supported by what Radford calls semiotic means of objectification. One possible action and semiotic mean is building of physical versions of the diagrams. The idea is to put the procedural aspects in the foreground, giving a link to recursive ω-logic. The explanatory power of the visual proof has been discussed from the perspectives of phenomenology, categorial intuition and objectification. The perceptual and cognitive aspect of the proof lying beyond the concept of number is suggested as a reason for the strikingly potential of explanation. A drawback of visual proof is the lack of standardisation. Proof systems and appropriate semiotic signs can also make both interpretation and validation of the proofs easier. The development of these missing aspects both theoretically and by design is central in further research.

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WAYS OF ANALOGICAL REASONING – THOUGHT PROCESSES IN AN EXAMPLE BASED LEARNING ENVIRONMENT

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This study is about analogical reasoning in problem solving situations. As one part of the study, task sequences for an example-based learning environment were developed with the aim of initiating processes of analogical reasoning. While solving these tasks the subjects were asked to verbalize their thoughts. Their problem approaches were recorded by video camera and transcribed to Think Aloud Protocols. Referring to a two-dimensional process-model of analogy the protocols were used to trace, visualize and quantify ways of analogical reasoning in order to become accessible for classification. It appears that different ways of analogical reasoning occur and there seem to be suitable attributes to describe them as different classes of ways in the two-dimensional model.

Key words: Analogy, Analogical Reasoning, Transfer, Example Based Learning, Think Aloud

INTRODUCTION

> "ALL [sic] our reasonings concerning matter of fact are founded on a species of Analogy, which leads us to expect from any cause the same events, which we have observed to result from similar causes." (Hume, 1748, Sec. IX, Par. 82)

David Hume refers to the possibility of drawing on experience to extend the (collective) knowledge as one of the main characteristics of analogical reasoning. Yet, students' transfer performances in mathematical problem solving situations are not – in general – very successful. In order to make strategies of analogical reasoning available to students, the significance of processes of analogical reasoning within the scope of mathematical activities has to be clarified.

Thus, the starting point of the below described empirical study is the following question: How do students use analogical reasoning as a possibility of drawing on mathematical experience?

TWO DIMENSIONS OF ANALOGICAL REASONING

The principle aim of analogical reasoning is to make the structure of an untapped issue (*target*) available to learners by comparison to structures within the learner's field of experience (*source*) (cf. English, 1997, p. 5). The most simple form of analogical reasoning, often used in intelligence tests measuring general intelligence *g* by analogy formation skills (e. g. Culture Fair Test CFT, CFT20 Catell & Cattell, 1963, Weiß, 2006), is seen as the establishment of a relational identity like 'A is to B, as C to D'. The concept of this identity is based upon the comparison of *object* *attributes* and especially upon the comparison of *relations* between objects involved (cf. Alexander, White & Daugherty, 1997, p. 117f; Ruppert, 2012).

Dimension 1 – Levels of analogical reasoning

Nevertheless, within the scope of learning mathematics not only the necessary skills to solve tasks, as those mentioned above, determinate the value of analogical reasoning. In fact, students are to be enabled to assess possibilities of mathematical action with regard to common structures. They finally transfer mathematical options to unfamiliar situations (‘Do you know a related problem?’, Pólya, 1949). In the latest cognition research, this process is described by the concept of *structure mapping* (cf. Gentner, 1983). In addition to finding equivalents on the levels of objects and relations, analogues to the mathematical operations that lead to a solution in the source domain have to be found (cf. Ruppert, 2010).

Thus, analogical reasoning occurs on the levels of objects and relations. In accordance with an actor oriented approach, transfer is seen as “the influence of the learners’ prior activities on their activities in novel situations.” (Lobato, 2006). Hence, analogical reasoning also occurs on the level of mathematical operations. These levels constitute the *First Dimension of analogical reasoning*.

Analogical reasoning and proving – A historic example

The relevance of analogical reasoning for proving as well as the relevance of the different levels of analogical reasoning themselves can be shown by the following excerpt of Archimedes’ “Method of Mechanical Theorems”:

> "From this theorem, to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere, I conceived the notion that the surface of any sphere is four times as great as a great circle in it; for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius." (Heath, 2007, S.20/21)

Archimedes himself states that, by transferring a given 2D situation to the 3D space, analogical reasoning on object and relation level lead him to a hypothesis about the surface of the sphere. In the further text it becomes clear that, having the proof for the circumstance of the sphere in mind, analogical reasoning on the level of mathematical operations (proving steps) lead him to a proof of his hypothesis.

Usually, once a proof is written down, the chronology of its genesis is lost. In our historic example, especially the chronology of the underlying process of analogical reasoning cannot be reconstructed in detail. Therefore it seems important to have the process character of analogical reasoning in mind if we want to know what is going on while the learner is drawing on his mathematical experience by analogy.

Dimension 2 – Components of analogical reasoning

In this regard, Sternberg (e. g. 1977) was able to identify several components of analogical reasoning by a variety of experiments. According to Sternberg, the study at hand takes the following four components as a basis (cf. Ruppert, 2012):

(1) Structuring (Sternberg: Encoding, Inferring)

(2) Mapping

(3) Applying

(4) Verifying (Sternberg: Justification, Response)

These components constitute the *Second Dimension of analogical reasoning*.

So the statements mentioned above suggest studying processes of analogical reasoning in the field of the two dimensions: ‘Level of Analogical Reasoning’ and ‘Component of Analogical Reasoning’.

As a consequence the working hypothesis of this research is: Processes of analogical reasoning can be illustrated as ‘ways’ in a two-dimension model (cf. Ruppert, 2012).

Thus, the analysis of analogical reasoning processes leads to diagrams of this type:

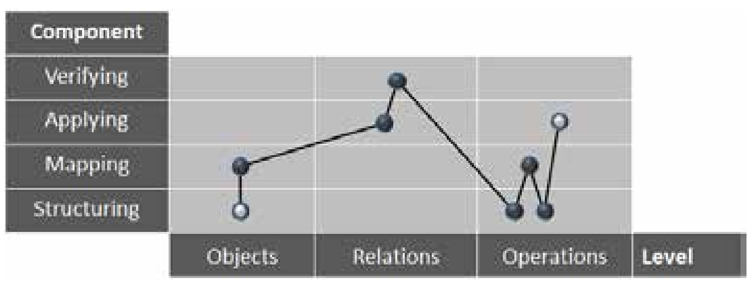

Fig. 1: Two Dimensions of Analogical Reasoning

Fig. 1 is read in the following way: The reasoning process starts with structuring statements on the level of objects and ends with the application of mathematical operations transferred from the base to the target domain. The outline itself passes different phases on different levels. The extraction of diagrams of different analogical reasoning processes defines the starting point of this present inquiry.

RESEARCH QUESTIONS

On this basis, the following questions should be particularly clarified by the given research (cf. Ruppert, 2012):

- How do specific processes of analogical reasoning appear as ‘ways’ in the outlined two-dimensional model and how do they look?
- Is it possible to classify these ‘ways’ regarding both successful and failed processes of analogical reasoning?

- Which particular importance is attached to the transitions from the structural (object and relation) level to the operation level?

RESEARCH-DESIGN: A FOUR-PHASE METHOD

To establish an empiric study on thought processes in general and especially on analogical reasoning several questions have to be answered in advance. The answers to these questions determine the research design.

1. How can processes of analogical reasoning be initiated?
2. How can processes of analogical reasoning be made visible and observable?
3. Which data can be used for the description of analogical reasoning?
4. Which criteria are suitable and necessary to systematize data on the basis of the two-dimension model?

Results of studies on *Example Based Learning* (cf. Atkinson, Derry, Renkl & Wortham, 2000) lead to a two-stage study design in which the subject is given different exercises out of one sequence of tasks.

During a 'learning phase' the *source* of analogical reasoning is established. The subject is offered sample tasks with solutions and some instructions during this phase. Afterwards, the subjects complete further tasks of this sequence single-handed during a 'testing phase'. Now the intended transfer performance in form of analogical reasoning shall occur.

In a first step, various sequences of tasks from different domains have been developed in accordance with research results on *Example Based Learning* and tested in a preliminary study.

Yet, in this inquiry on thought processes, relying only on the originated results like for instance students' documents would fall considerably short in answering the research questions formulated above. That is why the main study resorted, in compliance with results of Schoenfeld (1985) and Haastrup (1987), to the process-related, introspective method of *Pair Thinking Aloud,* a variant of the *Think Aloud* method (Ericsson & Simon, 1980, 1999), as one possible method for integrating verbal data into empirical studies. The creation of a natural situation and the necessity to keep the verbalization process of thoughts as complete as possible constituted, above all, the main reason for this kind of data collection. Haastrup (1991) writes concerning the advantages of Pair Thinking Aloud:

> „(…) by using pairs, one stimulates informants to verbalize all their conscious thought processes because they need to explain and justify their hypotheses (…) to their fellow informant. Furthermore, thinking aloud in pairs seems quite natural (…); It comes close to a real life situation." (p. 85).

So, in each case two students were asked to verbalize their thoughts loudly during the completion of tasks. Their statements were recorded by video.

In order to ensure a better reconstruction of the underlying thought processes, the data can be secured by a range of additionally measures (Haastrup, 1987; Borromeo Ferri, 2004). In the present study the following two dispositions were made:

On the one hand, the phase of task completion was divided in two phases. During a phase of *partner work* two students worked together on two tasks related to the domain they received instructions to. During a phase of *expert work* two students who did instructions and partner work on different domains worked together. Now it was their job to complete tasks on each of both domains. It was expected that the student who was familiar with the current content would assume a leadership position. In fact, it appeared that in nearly any case, even during the phase of partner work, one of the students takes over a leading role. It is assumed, that the process of observed during the study is close to the thought process of the leading student.

On the other hand, after the completion of each task one of the students was asked to track back the line of thought once again in a *Teach Back Phase* (cf. Wallach/Wolf, 2001, p. 25; Vora/Helander, 1995, p. 375).

In conclusion, the study consisted of four phases (cf. Figure 2):

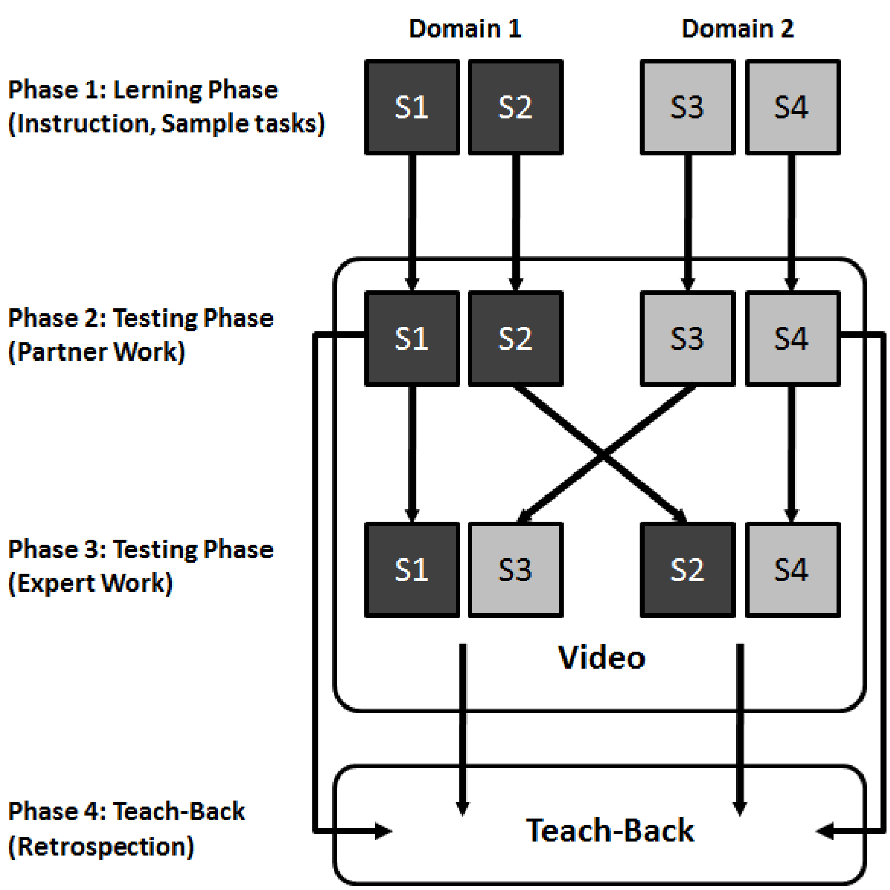

Fig. 2: Four-Phase-Design of the Study

DATA ANALYSIS

With regard to analysis the following data are available:

- transcripts of students‘ verbalizations and gestures (of Phases 2 and 3)
- video material of the *Teach Back Protocols*
- students‘ documents (solutions of tasks in written form)

On the basis of a coding guideline which was developed within the framework of another preliminary study, all verbal statements during the phases 2 and 3 of the inquiry were assigned to the different areas of the two-dimensional model (Fig.1). Therefore, different attributes of the students’ statements were outlined and anchoring examples were identified to characterize each area of the two dimension model. Teach Back Protocols, graphic material and students’ documents were used to bridge gaps in the verbalization processes as well as to substantiate the allocation to areas of the two-dimension model (for example by gestures like pointing at a distinctive part of an example task).

An example

The following dialogue is recorded during the *Testing phase*. Two students work on the problem of finding the number of different loop trails passing two villages A and B depending on the number of connecting trails between A and B. In the *Learning Phase* worked out examples of *complete graph problems* were presented.

1 S1: A and B are the vertices.
2 S2: Yes. And the trails are the edges.
3 [S2 draws a figure with two vertices and two edges]
4 S2: Loop trail means that you walk in a circle and that you don’t walk back on the same trail.
5 [S1 marks two trails between A and B to highlight a loop trail]
6 S1: Rather like this.
7 S2: Exactly.
8 S1: That means a loop trail needs two connecting trails.
9 S2: Yes.
10 [S1 writes]
11 S2: Now we have to calculate the number of days depending on n, isn’t it?
12 S2: ‘Cause we don’t know how many trails, it could be infinitely many, so to speak.
13 S1: Mhm.
14 S2: And since you want to walk one loop trail every day …
15 S1: … you need two different connecting edges a day.
16 S2: Exactly.
17 [S1 writes, S2 takes the worked examples and points a finger on example 2]

18	S2:	That means you have to, hm, …, here in the second (points on the worked example) … you mustn't divide by two.
19	S2:	Because you have to count pairs, hm?

Both students begin their argumentation (l. 1-2) by applying an analogy on object level. Commonalities in the relations between the objects are used to visualize the relational situation (l. 3, mapping on relation level) and to argue about the mapped relations in the target domain (l. 4-16, applying on relation level). The argumentation by analogical reasoning in this time segment is confirmed by the use of the terms "connecting edges" instead of "trails". For the transition to the operation level the worked examples are explicitly used (l. 17, mapping on operation level; l. 18, applying on operation level). Finally, the assumption is evaluated (l. 19, verifying on operation level).

The diagram of this short dialogue in the two dimensional model is shown in Fig. 3:

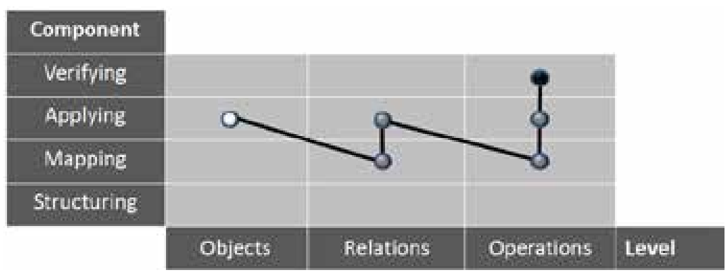

Fig. 3: Example of a "way of analogical reasoning" in the 2D-Model

Different kinds of visualization

For the coding of the data the software Videograph was used. The user interface of Videograph is partitioned into different windows which can be handled simultaneously: one window shows the video recording, a second window allows the transcription of the recorded dialogue and a third window shows a timeline on which time segments can be determined and assigned to predefined categories with respect to the coding scheme. One result of the work with Videograph was the graphic presentation of phases and levels of analogical reasoning on the timeline. In Fig. 4 the levels of analogical reasoning define the main categories and the phases appear as subcategories.

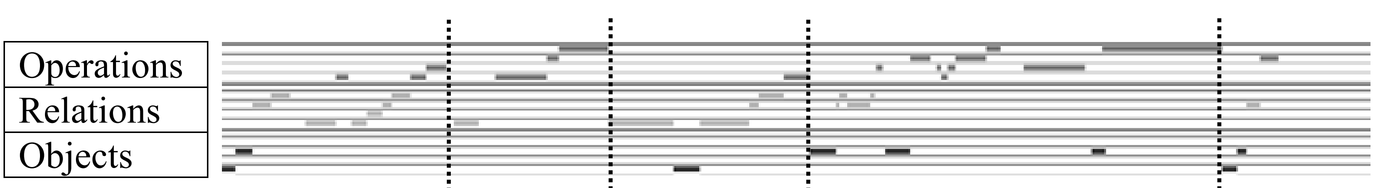

Fig. 4: Ways of analogical Reasoning in form of a graphic representation created by the software Videograph

These diagrams could be already used for a first interpretation of analogical reasoning processes. For example, it could be noticed that there are several discontinuities in the graphic presentation. Drawing the attention back on the verbalizations it could be shown, that these discontinuities often coincided with an abandonment of one line of thought (dotted lines in Figure 4). With respect to this observation the processes of analogical reasoning could be split into sections by the use of the verbal data.

Moreover, the ‘ways’ in the two-dimension model represented (like in Fig. 1) a basis for further interpretation.

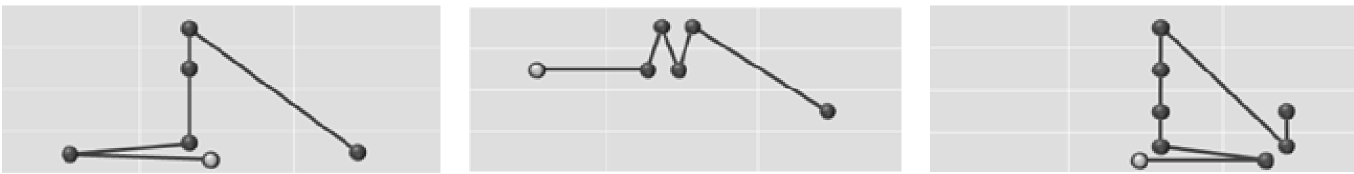

Fig. 5: Ways of analogical Reasoning in the two-dimension Model

Based on these diagrams, one can try to find a classification of ‘similar ways’ during the investigation.

In order to have quantitatively substantiated statements, the ways of the two-dimension model are translated into a ‘Stopover Matrix’, too (conf. Figure 6). By means of a cluster analysis the data should help to establish a classification of ways.

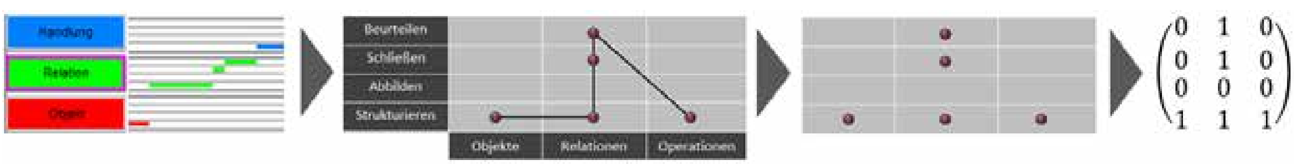

Fig. 6: The Videograph diagram (left), the corresponding ‘way’ of analogical reasoning in the two dimension model (2nd left) and the ‘Stopover Matrix’ (right)

FIRST RESULTS AND CONCLUSION

As mentioned above, the visualization of the software Videograph leads to the following interpretation (cf. Ruppert, 2012):

- Sometimes thought processes were abandoned. Thus, the processes can be divided in several sections (see above).
- In the Videograph diagram the starting point of a new section after an abandonment of the thought process is always lower than the ending point of the previous section.
- Mainly, the sections (s. a.) show an upward trend when taking them separately.

Furthermore, the interpretation of the graphical representations on the basis of the two-dimension model shows the following:

- The recorded 'ways' run, generally speaking, from 'left to right' and from 'bottom to top'.
- A new section in thinking basically starts at the object level and/or at the stage of mapping.

Now it becomes clear that the Stopover Matrices do not lose all of their way information because the matrices also have to be read "from left to right and from bottom to top". Moreover these matrices can be seen as vectors in a vector space. An appropriate distance measure makes it possible to define the grade of difference between the matrices an thus between the ways of analogical reasoning. It occurs that grouping these ways in an interpretative way almost leads to the same results as grouping the corresponding matrices by some kind of cluster analysis. Regarding first results of grouping, it appears that ways which share sections at one level or within one component of analogical reasoning often fall together in one group. Thus, at least two types of analogical reasoning can be identified: One type prefers structuring on all levels before connecting the source and the basis domains. The other type prefers analogical reasoning on the object level before reasoning on the levels of relations or operations.

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A CASE STUDY OF THE ENACTMENT OF PROOF TASKS IN HIGH SCHOOL GEOMETRY

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This paper describes how a midwestern geometry teacher enacted proof tasks with respect to levels of cognitive demands. Data were collected via teacher interviews, observation protocol, audio, and video recording of the enacted lessons and teacher artifacts. The results suggest that the guidance offered during whole class discussion often reduced the level of cognitive demand of potentially richer tasks. Furthermore, whenever the teacher talked less and allowed students to work independently or in groups, the enacted proof tasks generally maintained higher-levels of cognitive demand.

Key words: proof, geometry, high school, teaching

INTRODUCTION

Proof plays various roles in mathematics, such as to verify, explain, discover, communicate, systemize or create an intellectual challenge (De Villiers, 1999). Within the classroom setting, to carefully unpack these roles of proof, teachers must seek to provide opportunities for students to engage in proving. According to Harel and Sowder (2007), "The essentiality of opportunity to learn must be recognized not only at the intended curriculum level but also in the teachers enacted curriculum"(p.827-828). Therefore, if we seek to increase students' opportunity to engage in proving, it is important to examine proof in both the intended and enacted curriculum. However, little is known about how proof is taught, in relation to curriculum materials which conveys the intended curriculum (Mariotti, 2006). Therefore, this paper sought to provide insight into how a geometry teacher enacts proof tasks by answering the following question: How does a geometry teacher enact proof tasks, with consideration to the levels of cognitive demands?

PERSPECTIVES

The manner in which teachers use curriculum materials can impact how proof is presented and what aspects of mathematical proofs are emphasized. Therefore, it is important to consider not only the curriculum materials used, but also how they are used during instruction. McCrone, Martin, Dindyal, and Wallace (2002) acknowledged that the four teachers in their study followed the textbook rather closely to structure the enacted lesson on proof, as well as for allocating homework assignments pertaining to proof, and used technology or hands on investigation activities sparingly. Since the mathematical content emphasized in textbooks can pose a challenge to teaching authentic proofs (Cirillo, 2009), it may not always be ideal for teachers to follow the curriculum rather closely. For example, Schoenfeld (1988) conducted a year long study of teaching and learning in a 10th grade geometry course. He found that, although the teacher exhibited "good teaching", the teacher's actions might have had a negative impact on students' perceptions of proofs. He suggested that the teacher's strict adherence to the curriculum might have caused students to differentiate between constructive and deductive geometry, consider the form of the mathematical argument to be paramount, and view doing proofs as a quick activity.

Bieda (2010) conducted one of the few studies that have examined curriculum materials during the enactment of proof-related tasks during instruction. Her results highlighted that when an opportunity to prove arose, students did not provide adequate justification approximately half of the time; and that 42% of the time teachers did not provide a response, 34% of the time teachers sanctioned students conjectures, and 24% of the time teachers requested the input of the class. She acknowledged that teachers were likely to provide positive feedback for non-proof arguments as if it were general arguments. Bieda concluded that "teachers in the classrooms observed did not provide sufficient feedback to sustain discussions about students' conjectures and/or justifications…[and] when a teacher provided feedback to students' justifications, it was not sufficient to establish standards for proof in a mathematics classroom" (Bieda, 2010, p. 377).

METHOD

This case study employed qualitative methods to investigate how a geometry teacher enacted proof tasks, with consideration to levels of cognitive demands (*memorization*, *procedures without connections*, *procedures with connections*, and *doing mathematics*). It is drawn from a doctoral dissertation, which examined how 3 geometry teachers use their geometry textbooks to teach proof. During the 2011 Fall Semester, I examined how a teacher used *McDougal Littell Geometry* (Larson, Boswell, Kanold, & Stiff, 2007) to facilitate students learning to prove. Before observing lessons pertinent to reasoning and proving (Chapter 2), parallel and perpendicular lines (Chapter 3), and congruent triangles (Chapter 4), I conducted a textbook analysis of task features and levels of cognitive demand of proof tasks, for the identified chapters. Of the 977 tasks analyzed, only 13.1% were proof tasks (tasks which explicitly required students to write a complete proof, or complete a skeletal proof such that the finish product illustrated a complete proof).

The Mathematical Task Framework (Henningsen & Stein, 1997; Smith & Stein, 1998), which defines levels of cognitive demand, was used to code proof tasks as written, planned and enacted. Proof tasks coded as *memorization* reflected skeletal proofs, in which students had to fill in the blank to complete a proof argument. *Procedures without connections* proof tasks included tasks that require matching statements, or are clones of examples provided in the chapter. Proof tasks that reflected *procedures with connections* included writing proof plans, or tasks that can utilize procedures to facilitate some degree of thinking. Such tasks can help students make connections between diagrams, postulates, and symbolic representations. Finally, proof tasks coded as *doing mathematics* required students to write a complete proof that was not similar to previous tasks and examples or is not algorithmic, and may change the context or utilize a different representation. Such tasks requires great depth of critical thinking, and facilitate students engaging in evaluating the merit or lack thereof for using a particular postulate to develop a proof argument.

To triangulate data, I utilized multiple data sources: interviews, physical artifacts, audio, and video recording of the enacted lessons and an observation protocol. The observation protocol used during the enacted lesson documented the classroom climate, instructional tools used, how the tasks were facilitated, levels of cognitive demand of the tasks, and proof schemes observed. Multiple researchers assisted with coding the written tasks. We had an inter-rater reliability agreement of 89%. Furthermore, an additional researcher accompanied me to more than 25% of the observed lessons. Our coding of the observed lessons on the observation protocol was generally consistent.

Participant

Purposeful sampling was used to identify the teacher studied. Mr. Walker (pseudonym), a fifth year teacher and head of the mathematics department, taught at a rural school, and used *McDougal Littell Geometry* (Larson et al., 2007) for at least three years. Mr. Walker obtained his undergraduate degree in Statistics, and subsequently obtained a Master's degree in Mathematics Education. Additionally, he has taught high school geometry every year since he began teaching. Being one of the two mathematics teachers at the school, he taught introductory algebra, college credited algebra and statistics courses, and calculus. Mr. Walker believed that proof was needed in teaching mathematics because it fostered students gaining an appreciation for mathematics, and the work of mathematicians who contributed to the theorems and postulates that is visible within textbooks. He also believed that proof assists students to “become a little more logical in their areas of thought; not just math”. Mr. Walker asserted that teachers’ experience can influence how proof is taught, and acknowledge his preference for using the two-column proof representation. He stated, “…the two-column proofs are the easiest to see logical steps, so that is what I spend the most time teaching. Also, I may be more rigid in the steps that the students must show me... I don’t like to find missing steps in logic according to our geometric postulates or theorem” (September 29, 2011- Follow up interview- sent via email).

In Mr. Walker’s class students were first required to prove a theorem, before they could use it as supportive reasoning in a future proof. To facilitate students writing proofs, Mr. Walker required students to work in groups to construct proof arguments for proof on cards, or organize shuffle proof arguments to create logical arguments to exchange with other groups.

Mr. Walker acknowledged that the students whom he taught generally had a negative disposition towards proof tasks. He suggested that students’ peers tell them that proofs are difficult, and therefore students are biased against proofs before entering the class. He also believed that students’ negative disposition towards proofs were due in part to a lack of motivation to state their ideas with appropriate reasoning.

Furthermore, Mr. Walker was aware that proof can play multiple roles within a mathematics classroom, and suggested that the procedural nature of proof in geometry reduces the potential value of the proof.

He acknowledged that the textbook provided limited opportunity for students to engage in proving; hence, he often sought to provide supplementary proof tasks. Therefore, he was chosen as a unique case since he intentionally sought to increase the opportunity for students to engage in proving in his geometry course despite students’ disposition towards proofs, and limited amount of proof in textbooks. Moreover, he was selected because he had a greater flexibility to progress through the textbook at his own rate, unlike the other two teachers studied for the dissertation who planned instructional activities and assessments with their geometry team. I observed Mr. Walker teach 75 minutes geometry lessons, 13 times, during the 2011 fall semester, in which he sought to expose students to proving. His high school geometry class consists of 9^{th} and 10^{th} grade students. The allocation of class time was devoted to reviewing solution to homework assignments, working on proof tasks in groups, and concluding lessons by providing solutions to assigned proof tasks.

RESULTS

Mr. Walker desired for his students to learn to reason effectively, and emphasized that the order matters in how a proof argument is presented. Supportive reasoning was emphasized for each step of the proof. He gave students a list of 28 reasons and regularly quizzed students about the content on the list. The list included definitions, properties of basic operations, properties of equality (such as reflexive and symmetric), theorems about congruent, and segment and angles postulates. The list of reasons had all of the necessary information to complete proof tasks that were commonly assessed. Hence, the list could be used to complete lower-level task rather quickly, and was used as a reference point for higher-level tasks. Thus, it could be argued that the list was an implicit form of teacher intervention, even when the teacher remained silent while students worked on proof tasks independently. Most of the proof tasks posed required six or fewer steps and used the two-column proof representation.

The textbook was used to assign homework, and structure the lesson. If he deviated from the textbook, the tasks he used aligned with the lesson objective of the textbook, and were meant to facilitate students learning how to prove. Mr. Walker frequently supplemented the textbook with additional proof tasks. The supplementary tasks posed increased opportunities for students to engage in higher cognitive thinking. Based on conversations with Mr. Walker, his deviation from the textbook was due to his desire to pose more higher-level cognitive demand tasks. He acknowledged that the textbook had limitations, and he tried to overcome them. Mr. Walker remarked, “I guess, there’s just not enough like, if I look in this section in the book there’s one, there’s two proof of how we want them to be thinking about like” (November 3, 2011- Follow up interview at the end of the lesson). He also noted that sometimes the order in which content is presented in the book might not be logical, so his goal was to ensure the content progressed logically.

Although Mr. Walker’s whole class instruction often reduced higher-level cognitive demand tasks to *memorization* or *procedures without connections*, when students worked in groups higher cognitive thinking was evident. An example of Mr. Walker reducing the level of cognitive demand of a proof task was visible on November 10, 2011, in which he required students to prove two triangles were congruent, which shared a common side. He said, “All right, I’ll get you started” and proceeded to complete the proof in its entirety. In doing the proof he asked students to select one of the 6 theorems of congruency from the board to support the premise that the triangles were congruent. When a student selected an incorrect reason, the teacher continued by stating the correct reasoning and concluded the proof. Hence, the opportunity for students to engage in doing a higher-level cognitive proof task was not provided due to the excessive guidance provided by the teacher. Therefore, although the proof task had the potential to be considered a *procedures with connections*, if completed by students, when enacted by the teacher, the level of cognitive demand of the task was reduced. Mr. Walker did not pose any tasks that reflected *doing mathematics.* Although he had good intentions (which was to facilitate learning how to prove), the guidance offered during whole class discussion often reduced the level of cognitive demand of potentially richer tasks.

Cognitive Demand of Tasks during Mr. Walker's Enacted Lessons

Many of the tasks enacted in Mr. Walker's lessons were higher cognitive demand tasks. Table 1 indicates the level of cognitive demand of tasks for the original, planned, and engagement with the task during the enacted lessons as documented on the observation protocol. The original task depicted task as written, the planned task is the teacher's stated intention of how he intended to use the task during the lesson, and the engagement with the task is how the teacher actually used the task during the enacted lesson. In three lessons there existed multiple levels of cognitive demands for the various tasks posed. The shift from the original tasks to engagement with tasks suggests that when enacted the level of cognitive demand was reduced. It further suggests that half of the tasks Mr. Walker posed reflected *procedures with connections*.

Mathematical Tasks in Relations to the Levels of Cognitive Demands	Lower-Level Demands (*Memorization*)	Lower-Level Demands (*Procedures Without Connections*)	Higher-Level Demands (*Procedures with Connections*)	Higher-Level Demands (*Doing Mathematics*)
Original Tasks	2	6	8	0
Planned Tasks	2	6	8	0
Engagement with the Tasks during the Enacted Lesson	2	8	6	0

Table 1. Levels of cognitive demands observed during 13 of Mr. Walker's geometry lessons.

Mr. Walker's *memorization* tasks often required students to restate postulate, theorems, and rules. He believed that, in order for students to prove, they must know a list of reasons. For example, he reminded students that the definition of angle bisector could be used to prove that, if an angle is bisected, the two angles formed are congruent. Mr. Walker said,

> Good, definition of angle bisector. So this is on your list of 28 items. Basically what the definition of an angle bisector just says; it's a ray, or a line, or a segment that divides and angle into two congruent triangles (October 18, 2011- Enacted lesson).

He readily referenced the list as a tool to identify appropriate reasoning to support claims made.

Writing statements about congruent triangles often were *procedures without connections*. He required students to place marking on the diagrams, identify corresponding sides and angles, solve equations, and draw diagrams. For example, Mr. Walker stated,

> We've got a lot of problems with segments and whenever we do a proof with segments, and we're going to have to set an equation, there are usually two things that are going to help us set up an equation. With segments, it's either that constant to midpoint or the segmented addition postulate. With angles, it's the exact same thing except instead of, you usually have a midpoint of an angle but we've got angle bisectors so we could use an angle bisector to set up an equation or the angle addition postulate. So you're going to have to look at the given information and kind of decide which of these can I use to set up an equation. Let's keep that in mind. (September 8, 2011 - Enacted lesson)

Figure 1 shows an example of a proof he used to illustrate the procedure of using the segment addition postulate on September 8, 2011. This was categorized as a task of *procedures without connections*.

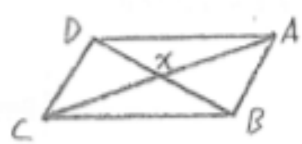
Given: AC = BD
Prove: AB = CD
Proof

Statements	Reasons
1. AC = BD	1. Given
2. AB + BC = AC	2. Segment addition postulate
3. BC +CD= BD	3. Segment addition postulate
4. AB + BC = BC +CD	4. Substitution
5. AB = CD	5. Subtraction property

Figure 1. Mr. Walker's proof tasks used to illustrate segment addition postulate.

Among the tasks that Mr. Walker posed involving *procedures with connections*, include: asking students to write complete proof, organizing shuffled proof statements and reasons to make logical proof arguments, and assigning projects in which students had to construct a town that preserved the placement of buildings in relations to parallel and perpendicular lines, or write a story that logically links 10 conditional statements.

Admittedly, enacted tasks that reflected *procedures with connections*, the teacher was a silent participant in the group discussion. Based on my classroom observations, although students evidently engaged in higher-level thinking in their respective groups, during whole class discussion, the teacher merely provided the solution to the proof without requiring students to share how they constructed the proof. Figure 2, is an example of a proof task Mr. Walker wrote (November 15, 2011- Teacher artifact) to complement Section 4.6- Use congruent triangles).


Example 4) Given: X is the midpoint of $\overline{BD}$
X is the midpoint of $\overline{AC}$
Prove: $\Delta DXC \cong \Delta BXA$

Statements	Reasons
1	1
2	2
3	3
4	4
5	5

Figure 2. Proof task Mr. Walker wrote that reflected *procedures with connections.*

IMPLICATIONS

The results suggest that Mr. Walker provided excessive guidance when he discussed solutions to proof tasks. Excessive guidance is not ideal, since it can potentially limit the opportunity for students to write proofs independently and engage in discourse about their proofs. The teacher led discussion was merely to provide solutions to proof tasks rather than have students share their reasoning and possibly critique the reasoning of others. Although he provided the opportunity for students to engage with proof tasks during the enacted lessons, his whole-class discussion provided little opportunity for students to reflect on the merit of their arguments or other means to make the same conclusions. The weak questioning strategies employed during whole class discussion, which required generally recollection of facts, did not require students to reflect on or critique the reasoning employed in constructing the proofs. Such practices could potentially devalue the importance of proof, or hinder students from conceptualizing the validity of their mathematical arguments or developing mathematical habits of minds (Cuoco, Paul Goldenberg, & Mark, 2004). Although this study is not generalizable, since it focused on only one teacher from the Midwest region of the United States, it sheds light on how a teacher enactment of proof tasks can potentially reduced the level of cognitive demands of a task.

Requiring teachers to pose *procedures with connections* and *doing mathematics* proof tasks does not guarantee that students will engage with the tasks at the same level; considering that a teacher's actions during the enacted lesson can diminish the level of cognitive demands. Therefore, future researchers ought to examine roles teachers can play to ensure teachers' enactment of proof tasks maintains higher-levels of cognitive demands. Pre-service teacher programs, and in-service professional development need to encourage teachers to pose and enact proof tasks that require critical thinking. Hence, video recordings of effective and ineffective teaching of proof in geometry are needed, such that teachers can visualize practices that should be emulated, and avoided. The videos can provide opportunities to reflect on effective questioning strategies that can be employed and instructional strategies that can increase students' engagement with proof tasks.

Additionally, textbook developers ought to increase the number of proof tasks that requires higher-level thinking, in an effort to provide students more opportunity to prove. The guidance offered to teachers in teacher's edition of textbooks ought to promote the importance of having students prove, and should suggest strategies of how to unpack proof tasks to facilitate opportunities for students to engage in *doing mathematics*.

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THE ROLES OF BEHAVIORAL SCHEMAS, PERSISTENCE, AND SELF-EFFICACY IN PROOF CONSTRUCTION

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We first discuss a theoretical perspective on proof, proof construction, and proving actions, as well as the need for persistence. Persistence is important for successful proving because it allows one to "explore", including making arguments in directions of unknown value, until one ultimately makes progress. Persistence can be supported by a self-efficacy belief, which is "a person's belief in his or her ability to succeed in a particular situation" (Bandura, 1995). We next examine actions needed for a successful proof construction of a theorem given to mid-level U.S. undergraduates in a transition-to-proof course. We contrast those actions with the actual actions of a mathematician proving the same theorem. Finally, we give some teaching implications.

Key words: proof construction, persistence, self-efficacy, undergraduates, mathematicians.

INTRODUCTION

We first discuss a theoretical perspective on proof, proof construction, and proving actions. Then, drawing on observations from a multi-year teaching experiment, we point out three proving actions that appear to be especially difficult to teach. We suggest that the teaching difficulty arises from a need for students to have a kind of persistence, which in turn may depend on students' sense of self-efficacy. After discussing self-efficacy, we illustrate the usefulness of it and of persistence by examining a hypothetical proof construction of a specific theorem that students in the teaching experiment are asked to construct. We also describe how a mathematician approached proving that theorem. Then, after a brief discussion, we end with some teaching implications.

A THEORETICAL PERSPECTIVE

We view a proof as the result of a certain kind of deductive reasoning and as a text written in a certain genre (Selden & Selden, to appear). We also treat a proof as having two parts: the formal-rhetorical part and the problem-centered part. The *formal-rhetorical* part of a proof is the part that one can write based only on logic, definitions, and sometimes theorems, without recourse to conceptual understanding, intuition, or genuine problem solving. We call this a "proof framework" and suggest to students that this framework (Selden & Selden, 1995) be written first, leaving blank spaces where needed. We call the remainder of the proof the *problem-centered* part, and it does require conceptual understanding and genuine problem solving (Selden, McKee, & Selden, 2010).

We view constructing a proof as a sequence of "actions" (Selden, McKee, & Selden, 2010). Some of these are physical, such as writing a line of the proof or drawing a diagram, and some are mental, such as focusing on the conclusion or "unpacking" its meaning. Some actions may influence the prover's own cognition, such as taking a break for "incubation." Many such actions, taken during the construction of a proof are not visible in the final written proof. This may be why it is sometimes difficult for a student to mimic a given proof when constructing another proof.

Understanding proof construction would probably be greatly aided by understanding what drives the individual actions involved. However, although one might reasonably suppose such actions generally arise from working memory that is of little help in explaining the origins of individual actions. Thus, we turn to dual-process theory, the idea that there are two systems of cognition, called S1 and S2, with S1 being more intuitive and S2 being more analytic (Leron & Hazzan, 2005). At first glance, one might view reasoning as S2 cognition. That is, the kind of cognition that is slow, evolutionarily recent, effortful, and conscious. However, we suggest that as one learns to construct proofs, the burden on working memory is reduced because much of what drives particular actions shifts to S1 cognition. That is, it shifts to cognition that is fast, evolutionarily ancient, effortless, and nonconscious. (For additional details, see Selden and Selden, 2011). This shift from System 2 to System 1 takes place because, in constructing a number of proofs, repeatedly occurring proving situations often occur immediately before corresponding repeatedly occurring actions. The situations may then become linked, in an automated way, to a tendency to carry out the corresponding actions. We see such automated situation-action pairs as persistent mental structures and call the smallest of them *behavioral schemas* (Selden, McKee, & Selden, 2010). We suggest they contribute to S1 cognition and are stored in procedural memory which, in contrast to other forms of memory, is very persistent -- indeed, it does not degrade with age.

In the above, by a *situation* we mean a reasoner's inner, or interpreted, situation as opposed to an outer situation that may be visible to an observer. However, we have found, in teaching, that we can often gauge approximately what the inner situation is from the outer, observable, situation, and the ensuing action.

Behavioral schemas

We next give an example of a behavioral schema. In situations involving a function f from X to Y and a subset A of Y, many university mathematics students will know that $f^{-1}(A)$ is defined as $\{x \mid f(x) \in A\}$. In the context of proving, some students given $b \in f^{-1}(A)$ [the situation] will automatically, without bringing the definition to mind, claim that $f(b) \in A$ [the action]. However, other students, who lack the schema, will often *not* do this.We suggest the following six properties of behavioral schemas:

1. Within a broad context, behavioral schemas are always available – they do not have to be searched for or recalled.
2. Behavioral schemas operate outside of consciousness. One is not aware of doing anything immediately prior to the resulting action.
3. One becomes aware of the resulting action of a behavioral schema as it occurs or immediately afterwards.
4. Behavioral schemas cannot be "chained together" outside of consciousness so that one only becomes aware of the final action. E.g. If the solution to a linear equation would take several steps, one cannot give the answer without being conscious of the results of some of the intermediate steps.
5. An action due to a behavioral schema depends in large part on conscious input.
6. Behavioral schemas are learned through practice. To acquire a schema, a person should carry out the appropriate action (correctly) a number of times. Changing a detrimental schema requires similar, perhaps longer, practice. (Selden, McKee, & Selden, 2010)

Three useful proving actions

In several iterations of teaching a U.S. second-year university transition-to-proof course in a modified Moore Method way (Mahavier, 1999), we have observed the following three useful proving actions that can be called upon. (1.) *Exploring.* In constructing part of a proof, one may understand both what is to be proved and what is available to use without having any idea of how to proceed. In such situations, one might reasonably try to prove something new of unknown value. However, we suspect many students are reluctant to do this, perhaps lacking confidence in their own ability to use whatever new they might prove. (2.) *Reworking an argument in the case of a suspected error or wrong direction.* In constructing a proof, one may come to suspect one has made an error or is arguing in an unhelpful direction. An appropriate response would be to rework part of the argument. However, we suspect many students are reluctant to do this, perhaps because they lack confidence in their own abilities to produce something new and better than before. (3.) *Validating a completed proof.* Upon completing a proof, one should read it carefully for correctness from the top down, checking whether each line follows from what has been said above. We suspect that few students do this, perhaps because they do not think that they are able to find errors in their own, just completed, proofs.

Some student errors may depend on a wrong belief about mathematics or logic or on a misinterpretation of a definition. Such errors can be pointed out and an explanation can be provided by a teacher. However, the above three actions in proof construction are not about correcting an error, but about habitually acting appropriately in particular situations. They seem to depend on students' views of their own abilities, that is, on a sense of self-efficacy and on persistence. We suspect that encouraging this kind of appropriate behaviour may require some kind of teaching, or facilitation, beyond merely explaining errors.

SELF-EFFICACY

Self-efficacy is "a person's belief in his or her ability to succeed in a particular situation" (Bandura, 1995). Of developing a sense of self-efficacy, Bandura (1994) stated that "The most effective way of developing a strong sense of self-efficacy is through mastery experiences," that performing a task successfully strengthens one's sense of self-efficacy. Also, according to Bandura, "Seeing people similar to oneself succeed by sustained effort raises observers' beliefs that they too possess the capabilities to master comparable activities to succeed."

According to Bandura (1994), individuals with a strong sense of self-efficacy: (1) view challenging problems as tasks to be mastered; (2) develop deeper interest in the activities in which they participate; (3) form a stronger sense of commitment to their interests and activities; and (4) recover quickly from setbacks and disappointments. In contrast, people with a weak sense of self-efficacy: (1) avoid challenging tasks; (2) believe that difficult tasks and situations are beyond their capabilities; (3) focus on personal failings and negative outcomes; and (4) quickly lose confidence in personal abilities.

Bandura's ideas "ring true" with our past experiences as mathematicians teaching courses by the Moore Method (Mahavier, 1999). Typical Moore Method (advanced undergraduate or graduate) courses are taught from a brief set of notes consisting of definitions, a few requests for examples, statements of major results, and those lesser results needed to prove the major ones. Exercises of the sort found in most textbooks are largely omitted. In class meetings, the professor invites individual students to present their original proofs and then very briefly comments on errors. Students are typically forbidden to read anything on the topic or to discuss it with anyone other than the professor. Once students are able to successfully prove the first few theorems, they often progress very rapidly in their proving ability, even without apparent explicit teaching, and even when subsequent proofs are more complex. Why should this be? We conjectured then, and also conjecture now, that students obtained a sense of self-efficacy from having proved the first few theorems successfully, and this helped them persist in explorations, re-examinations, and validations when these were needed in proving subsequent theorems.

This idea is supported by de Villiers (2012). In discussing geometry conjectures, he pointed out places where a novice might

> lose hope of getting anywhere as it's not obvious from the start this will lead somewhere useful. However, students should be encouraged to persist … and not so easily give up … One might say that a distinctive characteristic of good mathematical problem solvers [and

provers] are that they are 'stubborn', and willing to spend a long time attacking a problem from different vantage points, and not easily surrendering. (p. 8)

Thus, it would seem that a sense of self-efficacy, that is, a belief in one's ability to succeed on a particular kind of task, enables one to persist despite frustrations or wrong paths and that this is an important part of doing mathematics, and in particular, of constructing original proofs. Indeed, we suspect that it is an important part of much creative cognition in general.

A HYPOTHETICAL PROOF CONSTRUCTION

In this section we describe hypothetical proving actions for a theorem chosen from a set of notes used in our one-semester, 3-hour per week, mid-level undergraduate transition-to-proof course. In this course, the students present in class their proofs of theorems from the notes and receive substantial criticisms and advice. There is no textbook and there are no lectures. The notes include theorems about sets, functions, real analysis, and abstract algebra, as well as definitions and requests for examples. The required proofs progress from straightforward to fairly difficult. The hypothetical proof construction we describe is for one of the more difficult theorems and occurs near the end of the algebra section: *Theorem: If S is a commutative semigroup with no proper ideals, then S is a group.* We have found that constructing a proof of this theorem is challenging for mid-level undergraduate mathematics students, and indeed, for some beginning mathematics graduate students.

The relevant background information for this proof construction is quite small. A semigroup is a nonempty set S with an associative binary operation that we will write multiplicatively as xy for elements x and y of S. Associativity means that for all elements x, y, and z of S, $(xy)z = x(yz)$. S is commutative means that for all elements x and y of S, $xy = yx$. If A and B are subsets of S, we mean by AB the set of elements ab where a and b are elements of A and B respectively. In this setting, a nonempty subset I of S is an ideal of S provided SI is a subset of I. Such an ideal is called *proper* in S provided it is not all of S. In this commutative setting, S is a group if it has two additional properties. First, there must be an "identity" element e of S so that for any element s of S, $es = s$. Second, given any element s of S there must be an "inverse" element s' of S so that $s's = e$.

In constructing a proof of the above theorem, it is easy to see that if I is an ideal of S, one can conclude I is not proper, that is, $I = S$. What is not so easy is trying to construct an ideal that "looks" different from S, and what that might have to do with producing an identity element e of S and inverses, in order to prove that S is a group. Since there is nothing else to work on, one must persist in trying to find an ideal of S without any idea of whether, and how, that would be helpful. One might ask if s is any element of S, is $S\{s\}$ (also written Ss) an ideal of S, and hence equal to S? Once the idea has been articulated, it is not so hard to prove that Ss is an ideal. But how might $Ss = S$ help in proving that S is a group? Nothing in the students' notes says anything about solving equations in semigroups. However, if t is also an element of S, the above set equation means that there must be an element x of S so that $xs = t$. That is, the equation $xs = t$ can always be solved for x. It turns out that one can use the solvability of this equation in several ways to collect information which, for many students, is of unknown utility. Nevertheless this information, once collected, can be organized to show the existence of an identity element and inverses in S. To do this requires both persistence and a willingness to obtain whatever results, in the form of equations, that one can without knowing whether those results will ultimately be helpful.

We could have added two lemmas to our course notes that would have made the proof of the above theorem much easier for our students. However, the purpose of the course is to learn to construct a variety of hard, as well as easy, proofs, and having relevant experiences is important in developing the students' ability to do so. We view learning to persist in "exploring" mathematical situations by obtaining "whatever one can get," even without knowing its ultimate usefulness, as an important part of developing students' proving abilities.

While the proof of the above theorem calls for persistence and exploration, proving in general can call on a whole "tool box" of knowledge and abilities, such as the use of proof by contradiction or mathematical induction, or looking for inspiration by proving easier theorems, perhaps by adding a hypothesis such as finite-dimensional or finite. However, discussion of such topics is beyond the scope of this paper.

A MATHEMATICIAN'S PROOF CONSTRUCTION

Milos Savic (2012), investigated nine mathematics professors' proving using tablet PCs with screen capture software, as well as Livescribe pens and special paper, so that they could take the devices home and construct proofs in a naturalistic setting (without the time constraints and influences of an interview setting). All nine mathematicians' writing and speaking was recorded with time and date stamps. Several of the mathematicians acknowledged getting "stuck" on the above *Theorem 20: If S is a commutative semigroup with no proper ideals, then S is a group*, in a short set of notes containing only the material on semigroups. However, none gave up, as most students might, but persisted. One mathematician proved it the next day and another proved it after taking a break for lunch. Later, in a focus group interview, the professors indicated several ways they have of getting "unstuck" in their own research. These included getting up and walking around or doing something else for a while, as well as strengthening the hypotheses in order to prove an easier conjecture. It seems clear these mathematicians took the construction of the proofs in the semigroups notes as a positive challenge and had a sense of self-efficacy. Apparently this provided the motivation to persist, a crucial component of their success.

Several of the nine mathematicians volunteered that the material was both accessible and unfamiliar. However, they were unaware of the origin of the notes, that is, that they came from the course described above. Perhaps for this reason, several mathematicians attempted to construct counterexamples to some of the theorems. In attempting to prove Theorem 20, all nine mathematicians at some point considered "principal" ideals, a concept not in the notes, when considering the ideal Ss, where s is an element of S. This probably comes from remembering facts about ideals in rings; however, our students could not have had such memories as the course notes did not cover rings, and this course is a prerequisite for abstract algebra which would cover rings. Note that $Ss = S$ is one of two key ideas in proving Theorem 20--ideas without which it is difficult to make progress.

Dr. G's Construction of a Proof of Theorem 20

Below we describe most of the work that one of the nine mathematicians, Dr. G, did when attempting to prove Theorem 20, which he eventually did successfully. Our description is taken from transcripts of Dr. G's speaking and writing (that included metacognitive comments) while he worked alone using a Livescribe pen and special paper that recorded his writing and speaking with time and date stamps (Savic, 2012).

As seen below, Dr. G took a meandering path as he explored how to prove Theorem 20. His various "twists and turns" are indicated in bold typeface. Dr. G started at 7:02 a.m. by considering the statement of Theorem 20, but decided to **think about it** and have breakfast. At 8:07 a.m., he returned from a walk and **realized that gS** (where g is an element of S) **is an ideal**, **so *gS* = *S***. He then **thought about inverses** and **struckthrough his entire previous argument**. At 8:09 a.m., he **noted that he needed an identity element** which is not given. At 9:44 a.m., he became suspicious that **Theorem 20 might not be true**, but **noted that he had few examples** which might show that.

At 9:48 a.m., Dr. G started **"tossing around" the idea** that a [commutative] **semigroup with no proper ideals must have an identity**, in which case, he could show it is a group. However, he didn't see why S should have an identity. He **began to think that translating by a fixed element** [an idea not in the notes] would move every element, which would mean there was no identity. Consequently, he **then began to look for a counterexample**. By 9:50 a.m. he **neither saw how to prove Theorem 20 nor how to find a counterexample**.

He then looked ahead to Question 22, the final task in the notes, which has three parts that ask whether certain semigroups are isomorphic. He **saw how to answer that and then looked at** *Theorem 21: A minimal idea of a commutative semigroup is a group.* He thought that he could probably prove that, so **he went back to Theorem 20**. By 9:51 a.m. Dr. G **recalled that he had earlier rejected Theorems 3, 9, and 12** of the notes and also **did not believe that there are unique minimal ideals**. By 9:53 a.m., he **recalled that he had not been told any of the theorems were false and looked at the non-negative integers** under multiplication. He saw that {0} is a minimal ideal and noted that the non-negative integers under multiplication do not form a group. He **thought that this was a counterexample** to Theorem 21, but had interpreted Theorem 21 incorrectly – something he later discovered and fixed.

At 9:54 a.m., he **started actually answering Question 22**. By 9:58 a.m., he had answered its three parts correctly. At 9:59 a.m. Dr. G. **took a break to think about Theorem 20** and at 10:08 a.m. he again attempted a proof of it. This time **he saw that for $a \varepsilon S$, there is $e \varepsilon S$ so that $ae = a$** and saw that ***e* is "acting like …a right identity on *a*. Now why does it have to act that way on [an arbitrary] *b*?"** By 10:12 a.m. **he found e' so that $be' = b$**, but that **didn't help** since he couldn't show that $e = e'$. Then at 10:13 a.m. he **saw that there is an f so that $b = af$**, and then by 10:14 a.m., he **saw that $be = afe = aef = af = b$**. At 10:15 a.m., he **saw that *e* is the identity** element. By 10:18 a.m., he had **used a similar technique to show S has inverses** and is thus a group.

Perhaps the most important thing about the above description of Dr. G's work is what is not there. There is no evidence that Dr. G thought there was anything wrong with having gone in all of those unhelpful directions or with having thought that some theorems were false, that he later discovered were true. What seemed to matter to him was the generation of ideas. If those ideas resulted in errors, one fixed them and learned from them. He exhibited persistence and a willingness to try argument directions that he clearly didn't know ahead of time would be helpful, and he altered directions when the need arose. It seems clear that Dr. G had the needed persistence, which was probably supported by a sense of self-efficacy with respect to his own mathematical research.

DISCUSSION

We are not the first to have considered the effect of affect and self-efficacy on mathematicians', and others', proving or problem-solving success. In their study of mathematicians' problem solving, Carlson and Bloom (2005) concluded that the mathematicians' effectiveness "appeared to stem from their ability to draw on a large reservoir of well-connected knowledge, heuristics, and facts, as well as their *ability to manage their emotional responses* [italics ours]." Also, in a study of non-routine problem solving, McLeod, Metzger, and Craviotto (1989) found that both experts (research mathematicians) and novices (undergraduates enrolled in tertiary-level mathematics courses), when given different experience appropriate problems, reported having similar intense emotional reactions such as frustration, aggravation, and disappointment, but the experts were better able to control them. This suggests that the mathematicians in the two studies had a mathematical self-efficacy belief that allowed them to persist.

TEACHING IMPLICATIONS

The emphasis on actions in the theoretical perspective, described at the beginning of this paper, is linked to the view that proof construction is about doing mathematics. Thus, it can perhaps best be learned by having students independently construct a sequence of proofs of increasing difficulty. In this setting, teaching is largely a matter of facilitating students' proof constructions, encouraging helpful actions, and discouraging detrimental ones.

Further, it seems to us that in order to do things that require persistence and exploration, a student is likely to need to believe that he or she can personally benefit from his or her persistence or exploration. That calls for a self-efficacy belief, which in turn calls for, perhaps numerous, successes in what is to be done. Thus, it would be good to have students constructing their own proofs as early as possible. However, this is often delayed by an initial rather formal treatment of logic. While correct logic is essential for proof construction, we think its early and abstract treatment can be replaced by explanations of the relevant logic when needed in the context of the students' own work (in a "just-in-time" manner). This is because logic, beyond what most people know, actually occurs fairly rarely in student-constructed proofs (Savic, 2011).

While teaching logic can be integrated into discussing students' proofs and need not delay the start of their constructing proofs, there is an aspect of proving that is not usually explicitly taught early and that could be very helpful in facilitating students' successes. There is the relationship between the structure of a proof, the logical structure of the theorem being proved, and the theorems and definitions used in constructing the proof. This relationship appears as part of the final written proof, but it can be isolated and considered first. One can write it first leaving blank spaces for the remaining work. We have come to call this a *proof framework* (Selden & Selden, 1995).

Teaching students to build proof frameworks allows them to experience early successes. However, sometimes constructing a proof framework helps only a little in obtaining the final proof; this is the case for Theorem 20 discussed above. For other theorems, such as the theorem that states that the sum of two continuous real functions is continuous, constructing a proof framework can be very helpful (Selden & Selden, 2013).

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THE NATURE OF ARGUMENTATION IN SCHOOL TEXTS IN DIFFERENT CONTEXTS

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In this study we explore the substantial argumentation developed in Greek mathematical and science textbooks in specific topics related to the notion of periodicity. After analyzing a number of texts from both subjects the nature of argumentation was realized in a form of a systemic network. This network presents the complexity of the argumentative activity (the process developed and the modes of reasoning identified) in the different subjects and the tools mediate this process. Finally, by comparing and contrasting the argumentation in two texts that share a closely related thematic content we get some evidence of how the contextual activity via reasoning is shaped.

Key words: argumentation, textbook analysis, mathematics, science, modes of reasoning, nomo-logical, logical-mathematical, logical-empirical and empirical inferences

INTRODUCTION

The study of mathematical and pedagogical practices is important as these influence students' conceptions. Two factors are considered critical for the formation of students' pedagogical practices: the textbooks used and the teachers' cognitive and didactical knowledge. In most countries (Greece included) textbooks are used by teachers as the main source for their classroom activities. In textbook writing, meaning is not constructed on verbal language alone, but on the basis of graphical information and the produced argumentation and reasoning, as well. Love and Pimm (1996) note, that although the implied relation between the reader and the text is inherently passive, "*the most active invitation to any reader seems to be to work through the text to see why the particular 'this' is so*" (p. 371). Chi and her colleagues' (Chi, deLeeuw, Chiu & LaVancher, 1994) research in the science context highlighted the importance of the argumentation developed in textbooks in the meaning-making process. Specifically, they argue that students, in order to understand the text material, generate self-explanations, since even quality expositions require the reader to fill in substantial details.

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In spite of the crucial role that textbooks play in schooling and educative practices, few research studies have focused on textbook analysis and particularly on the argumentation adopted by the authors in these texts. Stacey and Vincent (2009) by analyzing the nature of reasoning presented to students in Australian mathematical textbooks in specific topics identified the following categories: (a) deductive reasoning (by using a model, or a specific or a general case); (b) empirical reasoning (concordance of a rule with a model and experimental demonstration), (c) external conviction (appeal to authority) and (d) qualitative analogy or metaphorical reasoning. Cabassut (2005) by comparing the reasoning presented in proofs in French and German school mathematics textbooks argues that deductive arguments often occur in conjunction with empirical arguments, presumably to obtain an additive effect.

The work presented in this paper is part of a research project that aims to identify epistemological and didactical aspects among different educational practices concerning the concept of "periodicity". Periodicity is an essential scientific concept because it plays a central role in the school curriculum and is expressed in different educational fields where it acquires practical importance. By adopting the position that argumentation and concepts are interwoven inside a text (Boero, Douek, Ferrari, 2008), we analyze the nature of reasoning developed in topics related to periodicity in math and science textbooks. The research on argumentation developed in school texts in different subjects for a common topic is rather limited. Analyzing the argumentation produced is didactically important since these texts are addressed to the same student who is 'responsible' for making the appropriate conceptual connections.

Our specific research questions are:

- What is the nature of argumentation that is employed in the textbooks to support the meaning of periodicity?
- How is argumentation differentiated in the mathematical and scientific context?

THEORETICAL FRAMEWORK

We adopt Vergnaud's (2009) theory of conceptual fields that addresses the process of conceptualization of reality. It is a pragmatic theory as it presupposes that knowledge acquisition is shaped by situations, problems and actions for the subject. It is, therefore, through the situations that a concept acquires meaning to a student. Vergnaud considers that a concept is a triplet of a set: **C=(S, I, L)** where **S** stands for the set of situations which give sense to a concept (*the referent*); **I** stands for the set of operational invariants associated to the concept (*the meaning*); **L** stands for the set of linguistic and non-linguistic representations which allow for the symbolic representation of a concept, its attributes, the situation to which it applies and the procedures it nourishes. In this paper, the thematic units, where the concept of periodicity appears in school texts, are considered as situations (S); the schemes of argumentation developed by the author in these units, as operational invariants (I); and the tools employed by the author in the argumentation process as linguistic and non-linguistic representations (L).

Since our interest is on argumentation practices used in textbooks to reason about the new knowledge presented we are interested in what Toulmin (1969) calls 'substantial argumentation' (p. 234). Substantial argumentation does not have the logical stringency of formal deductions but is used for gradual support of different statements. Toulmin establishes the importance of practical arguments and their logical canons, which may not be entirely secure as formal mathematical arguments are, but are however necessary tools of thinking in general. Argumentation here is taken to mean the use of reasoning for the construction of new knowledge presented in a text for the purpose of convincing the students of the truth of a conclusion. The complex interplay of reasoning and the new concepts to be learned is considered to affect students' ways of understanding and conceptualizing the field, in the case that they are consciously using and connecting the three dimensions (S, I, L) with the notion of periodicity.

Argumentation and reasoning in mathematics and in science contexts

The argumentative process that mathematicians develop to justify the truth of mathematical propositions, which is essentially a logical process, is usually called *mathematical proof* (Recio & Godino, 2001). Daily life reasoning is characterized as *informal* since people draw inferences from uncertain premises and with varying degrees of confidence; scientific reasoning, on the other hand, is based on experimental verification and has a validating intention which leads to generating scientific knowledge (Over & Evans, 2003). Szu and Osborne (2012) claim that arguments in the science context may be either *deductions* about the world based on a set of a priori premises; *inductive generalizations* when reasoning is typified by laws; or *inferences to the best explanation* as in Darwin's development of evolutionary theory.

Stinner (1992) classifies the knowledge provided in science textbooks in two planes: (a) The *logical plane,* where he encounters the finished products of science, such as laws, principles, models, theories, and the mathematical and algorithmic procedures establishing them; and (b) the *evidential plane,* where he encounters the experimental, intuitive, and experiential connections that support the logical plane. Kuhn (1962) in his influential work, 'The Structure of Scientific Revolutions', claims that textbooks are pedagogic vehicles for the perpetuation of science and identifies two levels of activity in science classes the *logical-mathematical* and the *evidential-experiential* levels of activity.

METHODOLOGY

A grounded theory research approach (Strauss & Corbin, 1998) is adopted in this study. Our methodological framework is based on the qualitative inductive content analysis. Moreover, the technique of systemic networks (Bliss, Monk & Ogborn,

1983) has been adopted not only as a form of representing our scheme of categories, but also as an analytic tool. In particular, we aim to produce a quantitative elaboration of the arguments which underlie the text in 11 Greek textbooks on topics related to the notion of periodicity.

The sample: The texts analyzed are taken from the subjects of Mathematics, Physics and applied technologies (Electrology, Electronics and Informatics) used in Greek lower secondary and upper secondary General and Vocational school. In each textbook we restrict our analysis to topics that are related to periodicity. Specifically in Mathematics the topics are trigonometry and periodic functions, in Physics the topics are related to Periodic phenomena (e.g. oscillations, simple harmonic and circular motion) while in applied technologies the topics are related to Alternate Currents. In order to implement our analytic plan, we divided the text into units of analysis by restricting analysis to all the parts which aim at delivering mathematical and scientific knowledge (we did not include worked examples, exercises and historical notes). In this paper the word ‘text’ is used to denote a section of textbook material and the accompanying visual representations.

Unit of analysis: Our unit of analysis is every conceptual thematic unit that has an independence from the rest of the text and produces an argumentation. It is conceived as a part of the topic that we analyze; it has a beginning and an end; and has a relative independence in its content: we can identify it and distinguish it from the other units. Each unit of analysis is characterized by its thematic content (e.g. “Define periodic function” or “Define periodic motion” or “Describe the generation of alternate current”) which is organized in a particular way. One unit of analysis several times coincides with a textbook unit as it is defined by the author. But in some cases we have to split the textbook unit in more units of analysis when a change in its thematic content and the argumentation produced is identified. After defining each unit of analysis we separately analyze the structure of the argumentation developed in terms of its process and its ingredients (parts). The process of argumentation is realized as a sequence of interdependent and logically connected statements. So, a secondary unit of analysis is chosen, expressed by a sequence of sentences. This usually corresponds to one or more paragraphs and the accompanying visual representations, and supports the generation of argumentation developed in the unit. Semantically, in each unit of analysis we can identify different explanations, justifications and/or proof of new knowledge. We call these types of reasoning ‘modes of reasoning’ as this term is used in Stacey and Vincent’s (2009) study.

Analysis of data: Subsequently, we analyze the modes of reasoning applied and the tools that mediated this reasoning. These tools are in the linguistic (i.e. verbal language) or non-linguistic form (i.e. physical models or mathematical representations). Within a feedback loop, the main codes were developed and negotiated among the researchers. Those codes were revised and eventually reduced to main categories and checked in respect of their reliability. As a result, categories and subcategories were formed and their interrelations were recognized by matching our emerging classification to our data. Finally, the nature of argumentation was realized in the form of a systemic network (Bliss, et al. 1983), presented in Figure 3.

RESULTS

In the first part of this section we exemplify our analysis in two texts. At the end we compare the nature of the argumentation developed in each case. The texts are from the subject of science and mathematics and share a closely related thematic content. In the second part, we present in the form of a systemic network the structure of the argumentation as it was developed after analyzing a number of texts from the above subjects.

Examples from texts

We present below a text from the subject of Physics. The text is from the topic 'Oscillations' and its specific thematic content is: "Define periodic motions"


ΠΕΡΙΟΔΙΚΕΣ ΚΙΝΗΣΕΙΣ

Figure 1: The text

> *"When you were younger you would have got into a swing many times or you would have even noticed the other kids playing with it. The swing has a high starting point, goes up and down and back to its starting point and keeps on moving in the exact same way. The yo-yo is a popular game, widely used in many countries in the world (maybe you have played with it several times). You hold the string from the one edge and you let the circle move. The string winds and unwinds around the spinning axle several times in exactly the same way.*
>
> *The movements of the swing or the yo-yo are examples of **periodic motions.** This means **that they are motions that are repeated at equal intervals.***
>
> *The normal circular motion is periodical as well as the annual motion of the Earth around the Sun. The muscle of the heart performs a periodic motion as presented at the electrocardiogram"*

(Physics, 3rd Grade of Lower Secondary School. (2008). Athens, Greece: OEDB, p. 89)

We can see how the development of argumentation in this thematic unit is produced. In the first paragraph two examples of periodic motions taken from everyday life (the swing and the yo-yo game) are presented. The properties that seem to characterize every periodic motion are presented in the key phrases: "moving in exactly the same way" or "repeated at equal intervals". In the second paragraph the definition of periodic motions are presented as generalization of the above properties. Finally, in the third paragraph additional examples of periodic motions are provided. The process of argumentation was realized as follows: introducing two special cases - providing scientific generalization with the aim of solidifying the scientific truth - giving more examples in order to reinforce students' understanding.

The parts of the argumentation are the modes of reasoning applied i.e. *empirical* based on every day experiences (when moving from the special to the general case), *nomological*, when defining periodic motions and finally *logical-empirical,* when starting from a general idea of logical type (the definition) and ending up by implementing it in certain empirical situations. These modes of reasoning as the units/parts of the argumentation developed aim to communicate to students the periodic motions, their characteristics and properties existed in a number of real life situations and to convince for their truth.

We present below a text from the topic of Trigonometry. Its thematic content is: "Define a periodic function and its period".

> *"Suppose that a ferry travels between two ports, A and B, and the graphic representation of its distance from port A as a function of time is presented in the following graph* [Figure 2a]. *We notice that every 1 and 1/2 hour the ferry repeats the exact same movement. This means that in whatever distance it is from port A in some time (t) it will be at the same distance at the time (t+1½) hours and it was at the same distance on the (t-1½) hours. Consequently, the function that presents the distance of the ferry from port A, in respect to the variable t takes the same values at t, t + 1½ and t-1½ .We suggest that this function is periodic with a period of 1 ½ hours.*

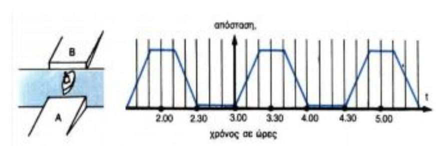

Figure 2a

> *The following graph* [Figure 2b] *is a graphic representation of the height of the swing as a function of time (t). We notice that despite the height of the swing in a certain moment (t), it will have the same height at the time (t+2)s as well as at (t-2)s. We say that the function (that models the height of the swing with respect to t) is periodic with a 2 sec period.*

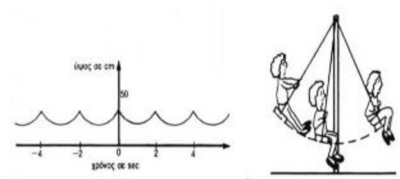

Figure 2b

> *In general: A function f with domain the set A is called* ***periodic****, when there is a real number T > 0 such that for every x ∈ A: i) x + T ∈ A, x - T ∈ A and ii) f(x + T) = f(x - T) = f(x). The real number T is called the* ***period*** *of f"* [Figure 2c].
>
> (Algebra, 2rd Grade of Upper Secondary School. (2012). Athens, Greece: OEDB, p. 73)

Examples of periodic functions are presented in the first two paragraphs while in the third paragraph the definition of periodic function comes as a generalization of the two examples. Hence, the structure of the argumentation in terms of its process is inductive since it moves from special cases of the situation to a general case (S_1, $S_2 \rightarrow G$). The modes of reasoning identified are based on empirical observations on mathematical models of periodic motions. These observations were 'evidence' identified in the graph representations where the reader must 'spot' the specific points on the graphs, identify patterns and end up in a general conclusion. We classify this reasoning as *logical-empirical* since reasoning starts from empirical situations and ends up in a general conclusion. Finally, we acknowledge this conclusion as a *nomo-logical* that emerges as a result of previous generalizations and its main aim is conceptualizing periodical behavior in the form of a mathematical object (periodic function) that satisfies certain conditions.

Comparing the structure of argumentation in the two texts

Both texts refer to periodic motions and the functions that model their behavior. The structure of the argumentation developed in mathematics and science differs in its process (S_1, S_2 $\rightarrow G \rightarrow S$) and ($S_1$, S_2 $\rightarrow G$) and its parts (the modes of reasoning employed). The modes of reasoning identified when moving from the special to the general cases were empirical in the science text and logical-empirical in the mathematical text.

The definition of periodic motions in the science text comes as a generalization of verbalized properties while the definition of periodic functions in the mathematical text comes as a generalization of mathematical and symbolic properties. Moreover, the different definitions support different perspectives of the notion. Particularly, according to Van Dormolen and Zaslavsky (2003), the science text supports a holistic perspective while the mathematics text a point-wise one. Some of these differences could easily be explained due to the difference in readers' school level (different school grades), while some others characterize the context in which each argumentation is developed.

In concluding the two reference situations differ in their goals, in the structure of the argumentation developed and in the language used by the authors when solidifying the scientific knowledge. Students have to uncover all the above differences and consciously and intentionally constructing links between them (Boero et al., 2008).

The systemic network

After analyzing a number of textual units (this research is still in progress) in our attempt to synthesize our results, we present the nature of argumentation developed in a systemic network (Fig. 3). The BAR ([) notation signifies that all the categories are mutually exclusive, whereas the BRA ({) notation signifies that any number or even all of the categories can be selected simultaneously.

The structure of the argumentation was characterized in terms of its process and its parts. The parts of the argumentation are the modes of reasoning acknowledged in this process. These two dimensions are viewed in interrelation. Four processes were identified: (a) Moving from special to general case and then exemplifying. (b) Moving from examples to general case (c) moving in the inverse way i.e. from general to special cases and (d) deductive reasoning i.e. remaining in the general case through out the entire textual unit.

The parts of the argumentation were acknowledged in terms of the kind of modes of reasoning developed by the author and the tools that mediated this reasoning. The modes of reasoning identified are the following:

- *Nomo-logical reasoning,* when a definition or a law emerges as a result of previous generalizations.
- *Logical-Mathematical reasoning,* when the inference is based on mathematical relations and techniques.
- *Logical – Empirical reasoning,* when the inferring process either starts with general statements and ends to specific situations (as in the science text), or the other way around (as in the mathematical text).
- *Empirical reasoning* when inference is based on experiences either from everyday life, or from experimental activity.

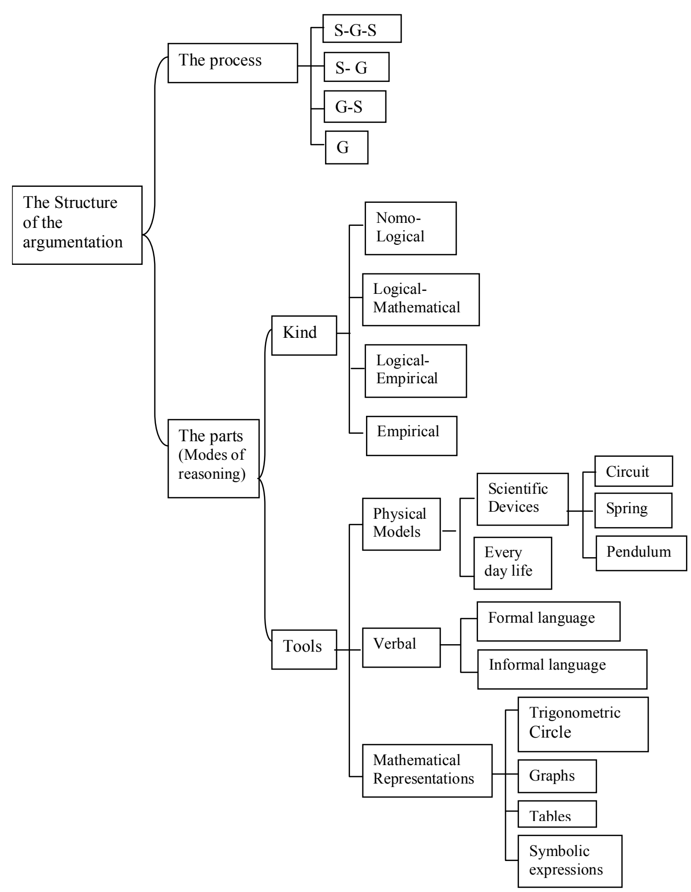

Figure 3: The systemic network

Different tools mediate the reasoning process such as physical tools (either scientific devices or every day life); mathematical representations (i.e. the trigonometric circle, graphs, tables and symbolic expressions) and verbal (i.e. informal or formal language). Finally, all the above categories and subcategories were associated and interrelated in the generation of the argumentation process.

CONCLUDING REMARKS

In this study we explore the nature of substantial argumentation developed in Greek mathematical and science textbooks in specific topics related to the notion of periodicity. In order to implement our plan we developed a methodology of analyzing the argumentation developed in texts from different subjects. Our methodology defines two units of analysis: the argumentation of each conceptual thematic unit and the mode of reasoning. In this way we can compare the argumentation, the generative activity and the tools mediated this argumentation in different texts in different grades and/or subjects.

After analyzing a number of textual units from the subjects of science and mathematics, aspects of the nature and the structure of argumentation were identified and presented in a form of a systemic network. This network is a generalized description of the complexity of the argumentative activity and its ingredients met in the different contexts. The basic modes of reasoning, parts of the argumentation process, are expressed in the form of nomo-logical, logical-mathematical, logical-empirical and empirical inferences. These inferences are mediated through a number of linguistic and non-linguistic tools. All these tools incorporate aspects of the conceptual field; while at the same time nourish the argumentation process.

Finally, by comparing and contrasting the argumentation in two texts that share a closely related thematic content, we get some evidence of how the contextual activity via reasoning is shaped in different subjects and in different grade levels. Particularly, through our analysis, we spotted differences in the argumentation produced and the linguistic and non-linguistic tools that mediate it that could illuminate aspects of the conceptual field in different ways. Such kind of evidence points out the need for a conscious connection of the above alternative procedures in order to support students' conceptualizing activity. This also implies the crucial role of mathematics and science teachers in integrating the different approaches to periodicity and in making links apparent in their classrooms.

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MODELLING ALGORITHMIC THINKING : THE FUNDAMENTAL NOTION OF PROBLEM

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The notion of a problem plays a very important role in algorithmic thinking. We propose a definition of 'problem' which is suitable to model this thinking. Then, we detail how it enables us to express the tool-object duality and describe conceptions.

Keywords : algorithm, problem, tool-object, didactical transpostion, conceptions

In many countries, algorithms and algorithmics are taking a growing importance in the curricula of mathematics[1]. This phenomenon raises many epistemological and didactical questions (Modeste & Ouvrier-Buffet, 2011). Algorithms are mathematical concepts shared with computer science and their present-day role in mathematics and their place in the mathematical activity in the classroom have to be questioned. The concept of algorithm is also strongly linked with the mathematical proof.

So, it seems important to have a better understanding of advanced algorithmic thinking and how it interplays with advanced mathematical thinking (Tall, 1991, Harel & Sowder, 2005). We propose a model of advanced algorithmic thinking from an epistemological point of view, based in particular on Knuth (1996) and Chabert (1999). This model enables to study how algorithmics is transposed in different institutions (curricula of mathematics, curricula of computer science, textbooks...).

One fundamental element of this model is the notion of problem. It links many other aspects of algorithm: proof, effectivity, complexity. It is also very important in the theoretical models of algorithm (to study decidability questions, for example).

A SPECIFIC DEFINITION OF PROBLEM?

We propose to adapt a definition which comes from the theory of algorithmic complexity. Giroud (2011) also used such a definition to describe the concept-problem in problem solving activities. Since this definition, a **problem** (e.g. finding the gcd) is:

- ***I*** a set of **instances** (e.g. N^2, all the pairs of two integers)
- ***Q*** a **question** about these instances (e.g. what is the gcd of the 2 integers?)

This definition is perfectly suitable for algorithmics: an algorithm is a systematic method which must give an answer to a question, *for all instances of the problem*, and after a finite number of steps (e.g. Euclid's algorithm solves the problem of gcd for any couple of integers). A problem is *instantiated* when one choose a particular instance ***i*** and try to answer the question ***Q(i)*** for this particular case (e.g. what is the gcd of 3654 and 76?). To deal with the concept algorithm, it is important not to solve only instantiated questions but to study a problem with all its instances.

WHAT IS THIS DEFINITION USEFUL FOR?

First, with this definition, we can rephrase the tool-object duality (Douady, 1986): Algorithm is a tool when it is used to solve a problem. Algorithm is an object when algorithms are in the instances or when questions involve algorithm (e.g. when we study the complexity of an algorithm or try to prove its termination).

It also enables us to use the cK¢ model (Balacheff, 1995) to describe conceptions in the academic knowledge (µ-conceptions) for algorithm in order to model algorithmic thinking and also to analyse people's or institutions' conceptions.

We used this model to study the didactical transposition of the concept algorithm in the French high school: the analysis of curricula, textbooks and online resources revealed a partial transposition of the concept, focused on tool aspects and programming activities and leaving apart the notion of problem.

Finally, this model is also useful to characterise problems with potential for the learning of algorithmic thinking, in order to design and study teaching situations.

NOTES

1. For example, the National Council of the Teachers of Mathematics dedicated his 1998 Yearbook to this issue (NCTM, 1998), and since 2003, algorithmics has been introduced in high-school curricula of mathematics in France.

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG 2 “TEACHING AND LEARNING OF NUMBER SYSTEMS AND ARITHMETIC”

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Keywords: *Numbers; fractions; decimals; proportion; negative numbers; arithmetic; strategies; mental calculation; flexibility; representations*

INTRODUCTION

Working Group 2 first assembled at CERME 7 as a forum for presenting and discussing theoretical and empirical research on the teaching and learning of number systems and arithmetic (including models for operations in the number systems, ratio and proportion, rational numbers and number theoretical aspects). For CERME 8 the scope was extended to grades 1-12. An emphasis was put on:

- research-based specifications of domain-specific goals (What should be learned? What can be learned? Which priority is given to particular aspects and why?)
- analysis of learning processes and learning outcomes in domain-specific learning environments and classroom cultures;
- new approaches to the design of meaningful and rich learning environments and assessments.

SYNOPSIS OF RESEARCH QUESTIONS

The papers in WG2 deal with issues related to natural numbers, fractions and decimals, proportion and negative numbers. An overview of the research questions tackled in the papers provides that most of the papers in WG 2 are mainly concerned with learners’ behaviour and thinking, related to particular mathematics contents:

- In which ways do learners understand the notion of the whole within a particular task and how does this notion unfold through different tasks? (Bednarz & Proulx)
- What are critical components for categorizing student’s response patterns in ratio tasks and what are students’ response patterns in the task? (Gomez et al.)
- Which procedures do students have for ordering integers and which previous knowledge is taken into account? (Schindler & Hußmann)
- What kind of representations do grade 3 students use and how do they relate to mathematics reasoning when the students work on a mathematical problem? (Velez & da Ponte)

- How do grade 6 students' mental computation strategies with positive rational numbers (represented as fractions) develop through a teaching experiment based on mental computation tasks with rational numbers involving the four operations and the discussion of strategies? (Carvalho & da Ponte)
- How do students use multilink cubes and mathematical signs equipped with a cultural meaning to express and communicate their thinking in social interaction? (Lorange & Rinvold)
- Which mental computation strategies do students use when solving addition and subtraction problems and how are these strategies influenced by the addition or subtraction problem situations? (Morais & Serrazina)
- Are students' error patterns in solving computational problems with fractions consistent? (Wittmann)

Only few papers are concerned with the design and evaluation of learning opportunities to tackle particular problems in students' mathematical development:

- Can an explicit focusing on the way in which children with mathematical difficulties realize number patterns and structures in mathematics education motivate a replacement of counting strategies and if so, how? (Häsel-Weide & Nührenbörger)
- How can children develop an understanding of how multi-digit-numbers are constructed from a semiotic point of view and how the written-multi-digit-numbers and the spoken-numbers denote a determined quantity? (Houdement & Chambris)
- How can we foster all students' mental constructions of the intended structural relations between part and whole by initiating activities with fraction bars? (Prediger)
- How can students develop a better understanding of the equality $0.\overline{9}=1$? (Vivier & Rittaud)

Finally, Rathgeb-Schnierer and Green introduce a new perspective on flexibility in mental calculation in their paper and implement it into a theoretical framework for a study of different degrees of flexibility in mental calculation.

The teacher was only considered as a learner, but not in its role as a teacher.

THEMES

Strategies

This synopsis of research questions of the papers of WG2 at CERME 8 shows that developing an understanding of students' strategies in terms of development, influencing factors, flexibility, and consistency related to different number domains and different ways of calculation (mental, written), has been an important topic in WG2 at CERME 8. Lorange and Rinvold analyze levels of objectification in students' strategies while solving tasks, which involve the expansion of fractions to a common denominator. Wittmann challenges the idea of error 'patterns' by analyzing the consistency of students' strategies. The results show that "students' work on computational problems quite often appears as a barely controlled and only partly aware process". As a new perspective Rathgeb-Schnierer and Green introduced a conceptualization of flexibility, which relates strategies to recognized number patterns and relationships of a given problem.

Mental calculation

Another prominent topic in the papers of WG2 at CERME 8 has been mental calculation. In the papers, mental calculation was not only treated related to natural numbers, but also related to fractions (Carvalho & da Ponte). Even related to natural numbers, it appeared that mental calculation was conceptualized differently in different papers. Consequently, defining characteristics of mental calculation is still an issue: Is mental calculation dependent on the extent to which external representations are used? This definition seems especially useful with regard to the fact that calculation is always a mental process. However it comprises the mental use of the standard algorithm and thus seems not to grasp the main idea of mental calculation. In order to avoid this problem, it seems more appropriate to define mental calculation as calculating with numbers as opposed to digits. This definition raises the question of the meaning of mental calculation in other number domains.

Conceptual development and "relative thinking"

Conceptual development and understanding, related to different concepts, has been another prominent topic in the papers of WG2 at CERME 8. Houdement and Chambris' paper is concerned with the conceptual development of (natural) multi-digit-numbers in view of a particular curricular development in France. Vivier and Rittaud deal with the conceptual development of rational numbers by focussing on a better understanding of the equality $0.\overline{9}=1$. Schindler and Hußmann address previous knowledge of students related to comparing integers as an important aspect of conceptual development. Bednarz and Proulx draw attention to the "relativity or the whole" as an important sub-construct of the "part-whole-concept".

The notion of "relativity of the whole" is an example for another topic that has emerged in the papers of WG2 and the discussions. This topic might be called 'relative thinking' and was discussed as an important aspect of conceptual development. Relative thinking is concerned with relative aspects of concepts such as the relativity of the whole in the contexts of fractions and proportion (Bednarz & Proulx; Gomez et al.) and structural relations of numbers (Häsel-Weide & Nührenbörger; Prediger) and number operations and how they are represented. Although not explicitly related to it, this theme seems to be closely related to early algebra, which is also a current theme in the research on algebra and algebraic thinking. In order to focus on relative thinking Prediger suggested "explicitly focusing structural relations" as a design principle for the development of learning environments, which aim to foster conceptual development.

CONCLUSION

In the discussion of WG2, two general topics emerged, which we would like to report, because we believe that it is worth considering them more explicitly in the future.

The first topic is related to the question of the similarity of mathematical situations. Questions such as the consistency of students' solution patterns or adaptive strategy use always have to relate to similar mathematical situations. On the one hand, it only makes sense to investigate questions like students' adaptive strategy use or the consistency of students' solution patterns if it is possible to identify similar mathematical situations. Usually similar mathematical situations are defined from a mathematical perspective by identifying similar mathematical structures. This view was challenged in the discussions of WG 2 and it was discussed, whether it seems reasonable to define similar mathematical situations psychologically from the learner's perspective. On the other hand, it might also be regarded as a goal to achieve that learners perceive mathematical similar situations as similar.

Another topic that emerged several times in the discussions of WG2 was about the cultural context of research. It appeared, that some research reacts upon curricular developments in a country and is not comprehensible, if the cultural context is not clear (see e.g. Houdement & Chambris). This also relates to the use of particular algorithms in a country (see e.g. Rinaldi) or the use of number representations and materials (see e.g. Häsel-Weide & Nührenbörger). The group agreed upon the need to make cultural contexts explicit in order to make research internationally comprehensible and useful. If we want to learn from studies in other countries it is inevitable to be aware of the cultural conditions of the research as something that is inseparable from the motivation of the research and its results.

ACKNOWLEDGEMENT

We thank Véronique Battie (France) and Luciana Bazzini (Italy) for all their work in the preparation of the conference. Both of them were official group leaders, but unfortunately were not able to participate at the conference.

LIST OF PAPERS IN WG 2

Bednarz, Nadine; Proulx, Jérome: The (relativity of the) whole as a fundamental dimension in the conceptualization of fractions.

Carvalho, Renata; da Ponte, João Pedro: Students' mental computation strategies with rational numbers represented as fractions.

Gomez, Bernardo; Monje, Javier; Pérez-Tyteca, Patricia & Rigo, Mirela: Performance on ratio in realistic discount tasks.

Häsel-Weide, Uta; Nührenbörger, Marcus: Replacing counting strategies: children's constructs working on number sequences.

Houdement, Catherine; Chambris, Christine: Why and how to introduce numbers units in 1st-and 2nd-grades.

Lorange, Andreas; Rinvold, Reinert: Levels of objectification in students' strategies

Morais, Cristina; Serrazina, Lurdes: Mental computation strategies in subtraction problem solving.

Prediger, Susanne: Focussing structural relations in the bar board – a design research study for fostering all students' conceptual understanding of fractions.

Rathgeb-Schnierer, Elisabeth; Green, Michael: Flexibility in mental calculation in elementary students from different math classes.

Rittaud, Benoît; Vivier, Laurent: Different praxeologies for rational numbers in decimal system – the $0,\overline{9}$ case.

Schindler, Maike; Hußmann, Stephan: About students' individual concepts of negative integers – in terms of the order relation.

Velez, Isabel; da Ponte, João Pedro: Representations and reasoning strategies of grade 3 students in problem solving.

Wittmann, Gerald: The consistency of students' error patterns in solving computational problems with fractions

LIST OF POSTERS IN WG 2

Reinup, Regina: Transformations via fractions, decimals, and percents.

Rinaldi, Anne-Marie : To measure with a broken ruler to understand the common technique of the substraction.

THE (RELATIVITY OF THE) WHOLE AS A FUNDAMENTAL DIMENSION IN THE CONCEPTUALIZATION OF FRACTIONS

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This paper focuses on a fundamental element in the conceptual development of fraction, the part-whole sub-construct. This focus is grounded in a research project with elementary teachers in which 3 days were spent on fractions and where issues related to the (relativity of the) whole*, occupied a significant space of learning. Through outlining the unfolding of teachers' meanings of the whole in the session, we illustrate the fundamental role of* the (relativity of the) whole *in learning about fractions, and highlight the complex ramifications that underpin its learning.*

INTRODUCTION

In the teaching and learning of mathematics, fractions have long been seen as one of the most difficult concepts to understand (see e.g. Ball, 1990; Charalambos & Pitta-Pantazi, 2007; Tobias, 2013). Following Kieren (1976) and Behr et al. (1983), these studies confirm the complexity of this concept, which is not a single but mainly a multifaceted one. This complexity of fractions has been first developed by Kieren (1976) and extended by Behr et al. (1983), creating a model of different interpretations of fractions, mainly the part-whole, ratio, operator, quotient, and measure, integrated and linked to operations, fraction equivalence, and problem solving (cf. figure 1).

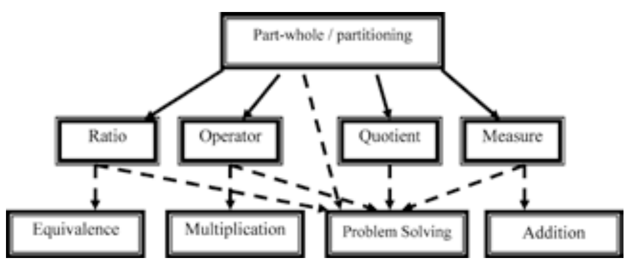


Figure 1. Behr et al. (1983) model (taken from Charalambos & Pitta-Pantazi, 2007)

While focusing on the importance of developing flexibility to handle these different interpretations, to move from one to the other in operations and problem solving (see e.g. Charalambos & Pitta-Pantazi, 2007), this model outlines that the part-whole sub-construct is of fundamental importance in the process of fraction understanding.

One aspect we focus on in this paper, and that we argue is of paramount importance in that part-whole sub-construct, is *the (relativity of the) whole*; what came to be called the "referent" by the teachers engaged in the project. Below, we show how this *(relativity of the)* whole is of particular importance and explain what we mean by it. Then, after having explained the study's objectives and methodological considerations, we detail its intricacies through analyzing its multifaceted meanings within a group of in-service elementary teachers.

PART-WHOLE SUB-CONSTRUCT AND (RELATIVITY OF THE) WHOLE

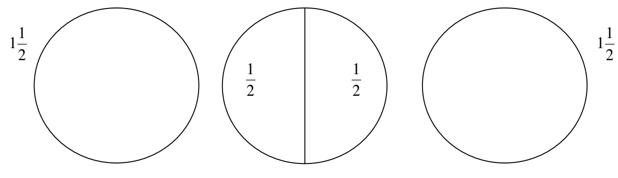
The (relativity of the) whole, albeit not explicitly outlined in Behr et al. model, has been alluded to by a number of researchers, particularly in studies involving teachers (e.g. Ball, 1990; Simon, 1993; Schifter, 1998; Prediger & Schink, 2009). For example, in Simon's study, prospective teachers were presented with a division of fraction problem for finding the number of cookies 35 cups of flour make if one cookie needs 3/8 of a cup. In their answers, numerous teachers who arrived at a remainder of 1/3 defined it as the remaining flour (1/3 of a cup of flour) instead of seeing it as 1/3 of what it takes to make a cookie (1/3 of 3/8 of a cup of flour). This raises the significant issue of referring to the *proper whole* when discussing fractions in the part-whole sub-construct. As well, more recently, Tobias (2013) developed an entire paper on the notion of the whole of a fraction, addressing it through issues of language use. Through highlighting various difficulties elementary teachers experience with fractions (e.g. Ball, 1990; Simon, 1993), she focuses on the importance, in fractions teaching and learning, to well conceptualize the whole of the fraction in order to contextualize situations, to understand the procedures to use and to interpret various solutions. She argues that a number of difficulties lived by learners with the notion of whole comes from language difficulties in defining wholes. She gives the following example (p. 2):

> In the context of subtraction, problems such as 3–2 can be stated as starting with three objects and taking away two of them. When the situation involves fractions, such as 3–½, it is incorrect to interpret this as starting with three objects and taking away half of them.

Even if we agree with Tobias' argument that language issues are of importance, we believe that there are deeper ramifications that are at play in understanding the whole of a fraction. One of these deep ramifications is related to the whole itself. We address it as *the (relativity of the) whole*.

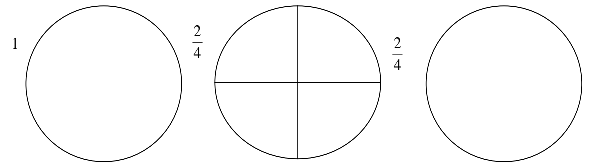
In order to clarify what we mean by *the (relativity of the) whole* and to see its importance, a Grade-4 classroom vignette is used, taken from a previous study on fractions. In this vignette, an extract of the classroom discussion related to the following problem: "Share three pizzas equally between two children" (three circled pizzas are drawn on the board for visual support) is presented. Children had to find as many ways as possible to solve it:

> *Marlene*: One child has a pizza, the other one also has an entire pizza, then one child has half, and another half. A pizza and a half each (she draws it on the board).



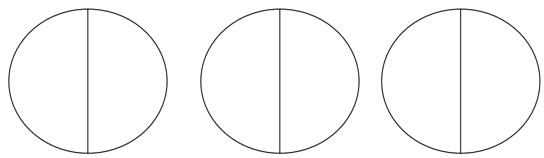
> *Teacher*: Ok, does it work? [pupils: yes!] Does anyone else have a solution?

Manon: There's one child that has one pizza, the other one has another pizza. I split the remaining one in four, one child gets two fourths and the other one two fourths. One child has a pizza and two fourths and the other one a pizza and two fourths.



Teacher: Ok, another solution?

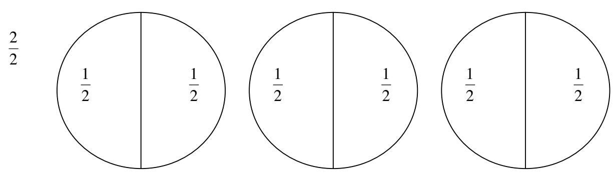
Veronique: I separated my pizzas like this: half of a pizza for a child, the other half for another child, another half for the first child, another half for the second, another half for the first, another half for the second; which means each child has three sixths.



Teacher: What do you think? Is this right?

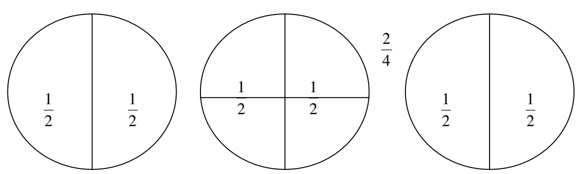
A student: No! Half plus half, that's two halves.

Teacher: Two halves (writing it on the board). Does this mean two over two?



Martin: No, one over two and one over two, that's two over four.

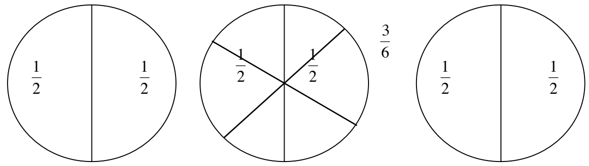
Teacher: You mean I divided a pizza in four, and I took two pieces.



Michel: No, it's not two over four [referring to Martin solution] it's one pizza and a half, because it's two halves together, one pizza plus one half, that's one pizza and a half.

Veronique: Yes, but three sixths [returning to what she said previously], that works…

Teacher: Does this mean we divided a pizza in six pieces, to find something like that?



Veronique: No, in the end it's all the same.

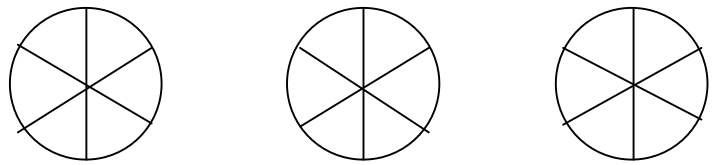
Teacher: You mean we divided by the same thing, and have the same parts of the pizza?

Veronique: We have the same result.

Teacher: Ok! Let's go back to 3/6, how do I get this? What bothers me is the three sixths.

Gabrielle: Oh, I know! There's three because there's six in total, six pieces…

Teacher: Oh, as if one pizza was divided in six pieces. Alright, it's like the pizzas were divided in six pieces, right?



Gabrielle: No, we need to divide it in two, it makes three pieces each… All the pizzas, all of them together are divided in six parts.

Another student: Does this happen by chance, because you know, there is one out of two, you take one pizza and each pizza you take one out of two, and one out of two, and one out of two, it makes three sixth.

Of interest in this vignette is not if the teacher reacted well to students' answers and questions or if students themselves understood well the concept of fraction. Of significance, within the myriad of answers given, is the possibility of the validity of these answers and on what grounds. When Manon says 1 and 2/4, the whole she refers to is one of the pizzas, her answer being 1 pizza and 2/4 of one pizza. When Veronique says 3/6, the whole she refers to is the entire three pizzas, making her answer 3/6 of the pizzas. When Martin adds ½ and ½, that gives him not 2 over 2 but 2 over 4, is he wrong or could he be adding ½ of one pizza with ½ of one pizza which gives him 2 of those halves out of 4 halves? And, again, could the student question about the coincidence of ½+½+½ giving 3/6 be along a change in the whole to represent the total? Those for us are important questions and are not answerable by a right/wrong dichotomy. This view of *the (relativity of the) whole* is closely aligned with Schifter's (1998) view, who for example queries "How can that piece of cake be 1/2 and 1/4 at the same time?", a kind of question frequently heard. In the part-whole sub-construct, fractions always need to be related to a whole. And, that whole can change, leading to the mathematical validity of what is offered. This *relativity of the whole*, mainly through relating the fraction to different possible wholes, is of fundamental importance in the mathematical reasoning about fractions. We document in this paper how this issue took form, through analyzing interpretations of a group of elementary teachers in different tasks.

RESEARCH OBJECTIVES

Though a number of studies (e.g. the ones mentioned above) have reported on issues of *whole* in teachers or students' understanding of fractions, few have detailed the ways in which this notion takes form through different tasks in learners/teachers, and how it interacts with the meaning of fraction itself. In examining the construction of *the (relativity of the) whole*, as one central aspect of the part-whole sub-construct, our report offers an analysis of different meanings of this fundamental aspect as well as of the intertwined meanings of the fraction concept itself; aspects of central importance for better understanding the phenomenon of teaching and learning fractions. In that sense, the paper is oriented by the following questions: (1) in what ways do learners, here practicing elementary teachers, understand the notion of whole within a particular task (meanings developed-in-action)? (2) In what ways does this notion unfold through different tasks?

METHODOLOGICAL CONSIDERATIONS

A group of 10 elementary teachers (Grade-4 to 6) participated in a 2-year professional development (PD) research project. This PD-research project intertwined professional development and research concerns, and can be related to a teaching experiment methodology (Steffe, 1983) through its preoccupation for documenting students' conceptualization overtime. In the teaching experiment methodology, the researcher – who is also the teacher – builds models of meanings developed-in-action by students, and confronts, during the teaching episodes, these models to the reasoning's and actions mobilized by students in new situations (leading to a continual restructuration of the models built). This methodology helps to understand a conceptual development in all its complexity and over time. In this research project, our preoccupation is similar. The intention is not to develop an in-service project for showing its potential in terms of learning outcomes. This PD-research project is an occasion to document the conceptualization about a specific arithmetic content. As teacher educators-researchers, we designed and conducted sessions, participating in the development of the mathematical understandings occurring in them, pushing their elicitation and modeling the conceptual development (as Steffe did).

The initiative is structured around day-long monthly sessions (15) during 1 ½ school years, all of them being videotaped, to keep a record of the sessions' unfolding, and a researcher journal was kept about salient events and reflections these provoke. The sessions activities revolved around "mathematics" tasks for teachers to engage with, articulated on their practice, about different mathematical topics [fractions, division, measurement (perimeter, area, volume), decimals numbers]. Teachers were invited (in small groups followed by plenary discussions) to engage with those mathematical events and to explore/discuss/make emerge the mathematical ideas inherent in these tasks. The focus here is on the first block of 3 sessions on fractions, where tasks were about the part-whole understanding of fractions through partitioning contexts.

The data analysis procedures adapted the approach proposed by Powell, Francisco, & Maher (2003), and focused on an *emergent* coding of data. Oriented by the notes from the research journal to pay attention to specific events of significance that happened in the sessions, the first stage of analysis involved (re-)becoming familiar with the sessions in full, viewing the tapes in their entirety to get a sense of their content. Specific events of importance were pointed to for orienting the continuity of the data analysis. In the second stage, the video data were described through writing brief, time-coded descriptions of each video's content and grouping it in "events". In stage three, data were reviewed anew to explore more precisely possible "significant events", which led, in stage four, to precise data transcriptions (verbal, gestures, drawings on the board, etc.) regarding these events. Each event was then analyzed in details and related to the previous and following ones to develop an evolving pattern concerning the (relativity of the) whole understandings. We outline these below.

ANALYSIS OF MEANINGS OF THE WHOLE OF FRACTION

The relevance of the whole, that came to be called the "referent" during the sessions with teachers, emerged progressively as a significant issue to consider. We present some of the tasks worked on below, and the different meanings given to the "referent" through these tasks as they emanate from the analysis. Because of space constraints, only a synthesis of the different conceptualizations, and their unfolding meanings, are outlined. The main idea is not here to document an evolution overtime from one meaning to the other during the professional development sessions. Our concern is to document how different meanings (about the whole and fractions) unfold within, and throughout, different tasks, and the way they can interact.

As teachers explored this first problem about "John and Mary" (with sample solutions), a number of meanings about the referent were engaged with.

John and Mary problem. John and Mary each have pocket money. Mary has spent ¼ of her amount and John ½ of his. Who spent more money, John or Mary?

Student solution 1:	*Student solution 2*:
John because ½ is more than ¼. Ex.: ½ of 16 = 8 but ¼ of 16 = 4	Mary ¼ John ½ = 2/4 John spent more and Mary less

(1) **Referent absent from the solving process**. In some explanations of teachers (e.g. below), the referent is *not* present in the way they make sense of the solutions, highlighting at the same time that these fractions are considered in-action as "absolutes".

> M.: We don't know how much money they [John and Mary] have, but it is not important. This child understands well the meaning of fractions, that ½ is more than ¼. And, he offers an example to confirm it.

One could be tempted to say that fractions are here treated as numbers, an important meaning in the learning process on fractions (i.e. from elementary to secondary school, from natural to rational numbers). However, the analysis of the transcripts shows that fractions were not explained as numbers but mainly taken for granted as a "things in itself", an evidence (½ is simply a bigger fraction than ¼).

(2) **Contextual referent**. Another meaning emerged, in interaction with the previous one, in which the referent is considered. Even if in this case the referent emerges as

important, this new meaning does not really affect the meaning of fraction itself, but is mainly related to the context of the problem.

> *Interaction1*.
>
> G.: My students, on the contrary, could have said that we can't answer this question because we don't know how much money they have at the beginning. Then if one has several millions dollars and the other has little money, the ¼ could be more.
>
> M.: Right. If I have 100$ and you have 10$, even if you spent half I stay richer. Our group did not see it like that.
>
> *Interaction 2*.
>
> M.: For the meaning of the fraction, this student would get a good mark. For me, he understands the meaning of a fraction, that ½ is more than ¼.
>
> J.: But, if he does not mention that there is no referent, no amount of dollars, then we can't say that his solution is good. […] If the child does not say that it depends of the beginning amount, I can't give him all the marks, since it could be different amounts.

In this case the context of the problem creates the necessity of considering *the referent* (the amount at the beginning) but this contextual referent does not affect the meaning of the fraction (½ remains bigger than ¼). It stays independent of the fraction, only impacting the resulting amount, the fraction continuing to be treated as an absolute.

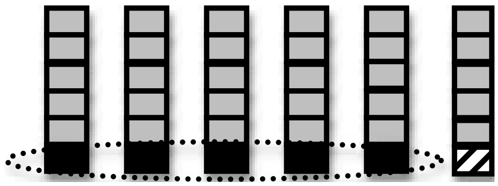
(3) **Fixed referent**. Later on in the discussions about the same task, the consideration of the referent evolved toward a new meaning influencing the meaning of fraction itself, for which not only the context provoked its consideration. This referent is, at this moment, conceived as fixed, determined in advance, so that it is possible to operate on fractions and conclude when its value is known *a priori*.

> M.: If we want to practice the meaning of fraction, then give children a referent, and stop playing with it. Hence, it would be necessary in the problem to say that each person has the same amount. Our group started with the fact that each had the same amount.
>
> Mi.: We could also, when there is missing data, work with situations where there is the same amount and other situations where there are different amounts.

Even if here there is a relation linking the fraction to a whole, there is a need to make this referent fixed so that one does not have to consider it afterwards when operating on fractions. This referent can vary from one situation to the next, but it needs to be fixed for each of those situations (like one fixes the parameters of a function when treating a specific case).

(4) **Relative referent**: This previous meaning (fixed referent) contrasts with another one (relative referent), that appeared much later in the sessions. We present here some interactions that occurred regarding the "ribbon problem" (Schifter, 1998).

Mali has 6 meters of material. She wants to make ribbons of 5/6 meters for a birthday at school. How many ribbons can she make and how much material will she be left with?



Student solution 1: 7 and 1/5

Student solution 2: We can make 7 ribbons

Student solution 3: 7 and 1/6 of a ribbon

Student solution 4: 7 and 1/7

A.: [discussing the solution 1] I mean the ribbon is finally 5/5.

Mi.: No! 5/6.

A.: No, no, no. In the end, it becomes 5/5 [M.: Oh my god!]

A.: And then the last piece remains a part of my whole, it remains 1/5.

Mi.: Yes, I understand what she means. We change the whole!

The fraction is here clearly related to a whole, and relative to this whole: the same piece can take several values (e.g. 1/5, 1/6, 1/36) depending of the whole considered (it was not the case when the referent needed to be fixed, determined). In this case, somehow the referent is seen as variable, whereas in the previous case it was conceived as parametrical (that is, changing but to be fixed for a specific problem).

This sophisticated meaning about the referent was developed, explained and refined on a long-term basis, through various tasks and interactions between participants. However, along this process, other interrelated meanings emerged. We present below, through short excerpts, three other meanings given to the referent.

(5) **Referent as pictorial representation**. For a number of tasks where students used pictures in their solutions (e.g. the ribbons problem), the referent became linked to a pictorial representation, seen as the whole. In this case of partitioning, the part needed to be completely included in the whole (in the picture of it). We found here an idea of inclusion according to which the parts are considered as elements of the whole. This is linked to a conception of fraction as a comparison between the part (a number of pieces) to the whole (the total number of pieces in which the picture is partitioned).

Interaction 1: on a problem involving a comparison of 5/6 and ¾
S.: When we have to compare fractions, we know what is the partitioning. When you understand the partitioning of the fraction, you make a drawing, dividing it e.g in four and you take 3 of them.

Interaction 2: on the “ribbons problem”, concerning the 7 and 1/5 solution
S.: It can’t be 1/5 of ribbon. A ribbon is 5/6, so it is 1/6 of ribbon, you can’t say 1/5. The remaining piece is not inside the ribbon.

It is the picture, acting as the whole, that impregnates the meanings of the referent and the fraction itself. The pieces considered must be “inside” this whole, being part of it. This referent is dependant on the drawing done, based on a view of fraction as a comparison between a part and the whole in which it is included.

(6) **Referent as common denominator**. Sometimes the referent becomes the common denominator, and is seen as the basis on which to compare fractions (as we see below in a problem involving a comparison of 5/6 and ¾).

S.: We need to compare with the same whole, oranges with oranges, apples with apples.

M.: There are quarters and sixths, so it is apples and oranges.

A.: We need to make it on twelve.

M.: You have to put them on the same denominator to compare, so it becomes our referent. We make it in twelfths and this is the referent.

Of interest here is that through making the common denominator the referent, implicitly teachers are transforming the comparison of fractions to a comparison of natural numbers [(5/6=10/12 and ¾=9/12, compare 10 (twelfths) and 9 (twelfths)].

(7) **Referent as a number linked to a fraction operator**. In the "ribbons problem", the following discussion happened concerning the remaining piece being 1/6.

N.: It means that it is 1/36 of 6 meters of material. But it can be also 1/6 of one meter.

S.: Because we can simplify the fraction, 1/6 x 36.

Teachers here use an operator to arrive at 1/6 of a meter (1/36 of 6m = 1/36 x 6 = 6/36 = 1/6). They thus agree that the same piece of material can have different values (1/36 of 6m, 1/6 of 1m), *through the operation*. Hence, it is not the consideration, for the same piece, of different wholes that leads to this acceptation, but a multiplication and simplification on numbers (fractions are here considered as operators).

DISCUSSION AND CONCLUDING REMARKS

The varied meanings given to the referent in the reconstruction over the sessions show a complex picture of its unfolding (even if more excerpts would be needed, but space constraints forbids). Different meanings emerge, interact and influence each other, taking form, being explained and refined, etc., all along the process of exploring and discussing the various tasks. These meanings, for the same or different persons, oscillate between one another; showing that these do not follow a linear path. These interacting aspects illustrate well the richness of this development. It shows that the part-whole sub-construct is a complex one, involving not only an understanding of what many researchers refer to the partitioning scheme with its different components (e.g. Charalambos & Pitta-Pantazi, 2007), but also a rich conceptualization of the whole (reasoning on the fraction in relation to the whole, seeing the relative value of the part, re-organizing the whole, etc.).

Even if many researchers point to the fundamental dimension of the sub-construct part-whole in the process of understanding fractions, the place occupied by a reflection on the meaning of the whole in this conceptual development has not been amply considered. The understanding of "the whole" acquired a broader significance through this research project. This question of referent is more than a question of language as Tobias (2013) e.g. would say. The notion of defining the whole is related, as we have seen, to the underlying meaning of the concept of fraction itself in a part-whole context. This "change" or "relativity" of the chosen whole is central in relation to comparison, equivalence, problem solving and operations (e.g. ¾ x ½, where ¾ is seen as having ½ as its whole, something often seen as "taking a part of a part of the whole"). This awareness about changing referent appears as a fundamental component to develop confirming and extending previous studies about operations on fractions (Prediger et al., 2009). In addition, even if we haven't highlighted it here, the various meanings developed about the referent are often nested with other sub-constructs. For example, issues of "referent as number on which to operate" is closely linked with the sub-construct "operator" of the fraction, as well, important issues concerning ratios were dealt with through the sessions. Those issues are to be developed in another paper.

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STUDENTS' MENTAL COMPUTATION STRATEGIES WITH FRACTIONS

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Carrying out mental computation with positive rational numbers represented as fractions may contribute to students' development of rational number sense. We analyze the development of grade 6 students' mental computation strategies with fractions through a teaching experiment based on mental computation tasks with rational numbers involving the four operations and the discussion of strategies. In the beginning of the study, the students used mainly instrumental strategies and, along the teaching experiment, they used more and more conceptual strategies.

Keywords: Mental computation, fractions, students' strategies, relational thinking, tasks.

INTRODUCTION

Mental computation with rational numbers tends to assume a marginal role in the school curriculum and in teaching practice. Given the difficulties that children have in computing mentally and in learning rational numbers, combining these two aspects may be fruitful for mathematics learning. To develop students' mental computation with rational numbers, it is important to understand students' strategies, given the kind of tasks proposed, and to realize how these strategies may evolve.

Following Reys, Reys, Nohda and Emori (1995), we regard mental computation as the process of computing an exact arithmetic result without an external support. Systematic work in mental computation may help to develop computation skills (Heirdsfield, 2011) as well as students' reasoning strategies and number and operation sense. Assuming this perspective, the mathematics curriculum for basic education (grades 1-9) in Portugal (ME, 2007) indicates that mental computation with natural numbers must be developed from grade 1 on and later be extended to other number sets. However, despite such recommendation, most students have very poor ability in mental computation with rational numbers. Our aim is to analyze the development of grade 6 students' mental computation strategies with positive rational numbers (represented as fractions) through a teaching experiment based on mental computation tasks with rational numbers involving the four operations and the discussion of strategies.

MENTAL COMPUTATION AND OPERATIONS WITH FRACTIONS

For students fractions are difficult to understand. A fraction is written using two numerals but represents only one number; although $2<4$, we have $1/2>1/4$; the same quantity may be represented by several fractions; the rules for multiplication of fractions are easy to accept (e.g., 1/2×2/5=2/10) but the rules for adding and subtracting fractions are not the most intuitive ones (1/2+1/4 is not 2/6) (Lamon, 2006) and these similarities and differences may induce students in misunderstandings about rational numbers and fractions.

For Galen, Feijs, Figueiredo, Gravemeijer, Herpen, and Keijzer (2008), fractions must receive a special attention in elementary school, as a starting point for understanding decimals and percent, giving meaning to them, and also because they play an important role in mental computation. We often think using fractions even when they are not explicitly involved. These authors call "network of relationships" the knowledge that students develop about different types of fractions and refer that this network does not develop just by practicing. They also consider that students' knowledge about fractions may be increasingly vast and formal, however it may be not specifically associated to a context but to a given fraction, that is, a student who understands that 3/4 is smaller than 4/5, is not necessarily ready to understand that 14/15 is less than 15/16.

To work with fractions with understanding, students need to be aware of the relationship between numerator and denominator. For Cramer, Wybeg and Leavitt (2009), a student with number sense is reflective about numbers, operations and results and is flexible in using comparison strategies and operations with numbers. Concerning fractions, number sense is related with the ability to use relational thinking (Empson, Levi & Carpenter, 2010), to compare and work with this rational number representation and to understand the quantities involved. Behr, Post and Wachsmuth (1986) and Cruz and Spinillo (2004) indicate that using benchmarks is important to compare and to operate with fractions. Cruz and Spinillo (2004) consider that the benchmark half plays an important role in students' initial understanding of complex logical-mathematical concepts associated with rational numbers, and the use of this benchmark may facilitate the addition of fractions.

Furthermore, students need to understand that multiplication is much more than repeated addition, and this, as Lamon (2006) suggests, brings in a new challenge. Newton and Sands (2012) consider that students must be encouraged to make conjectures about the division of fractions, leading them to use methods that are not usually taught in school. This allows them to understand efficient methods in relation to the context in which they are working. Often, students can use their knowledge about the multiplication of fractions to develop methods to divide fractions.

Empson et al. (2010), consider that learning fractions may be strongly supported if children develop relational thinking. The focus on relational thinking may help children to turn fractions into something that a drawing or a close attention to the properties of numbers and operations may lead them to reason about the quantities involved. The authors also consider that a child begins to think in a relational way about the quantities involved in a fraction, when he/she manages to relate the number of equal parts that he/she must share with the whole number of persons for which he/she must also distribute these parts. For Empson et al., (2010), relational thinking may be used to make sense of operations with fractions. When children understand fractions as a set of relationships, they are able to compose and decompose quantities for transforming expressions and simplifying them in the computation. A key point in developing a child's understanding is reached when he/she start using relational thinking in his/her strategies to make repeated additions or subtractions of fractions more efficiently by applying fundamental properties of operations and equalities to combine quantities. A child who, to compute 8 groups of 3/8, thinks that if 8 groups of 1/8 is equal to unity, then 3/8 will be three units, is making a reasoning based on the commutative and associative properties of multiplication. Thus, the development of basic knowledge to think about fractions in an efficient way integrates knowledge of properties of natural numbers, their operations and their relationships and anticipates the generalization of algebraic quantities. The authors state that the use of algorithms to develop some fluency with operations with numbers is useful, but argue that if the development of students' relational thinking is supported, the knowledge about the generalization of the properties of numbers and operations becomes more explicit and may be the basis for the learning of algebra in the subsequent school levels, reducing students' errors and misconceptions.

The student who only uses instrumental mental computation strategies, just involving known facts and memorized rules (Caney & Watson, 2003), does not show relational thinking. To compute 3/4-1/2 the student may apply the rule of subtraction of fractions with different denominators with no understanding of the quantities involved. In contrast, the student who uses knowledge about numbers, their relationships and operations provides evidence of conceptual strategies (Caney & Watson, 2003) and builds an important conceptual foundation for learning algebra (Empson et al., 2010). For example, to do this computation, a student may decompose 3/4 in 1/2+1/4 and, by subtracting the halves, see that the result is 1/4, showing some understanding about numbers. In our perspective, the use of conceptual strategies involves understanding numbers and operations but instrumental strategies may be used with no understanding. Sometimes students' strategies seem to have instrumental and conceptual features, especially when students apply a memorized rule but show some understanding of numbers and operations. However, in working with fractions that yield repeating decimals (i.e., 1/7) it may not be possible to use conceptual strategies, giving the complexity of the number involved. In this case, instrumental strategies are a powerful way of doing the computation.

The use of relational thinking is implicit in the framework of Heirdsfield (2011), who stresses that students, to compute mentally, need to understand numeration (the size and the value of numbers), recognize the effect of an operation on a number, know number facts, and make estimates to check the reasonableness of a solution. These concepts and understandings are based on number sense (McIntosh, Reys & Reys, 1992) and operation sense (Schifter, 1997; Slavit, 1999) and on a set of numerical facts that students learn in school (Wolman, 2006) and use to create their personal strategies' based on relational thinking.

Social interactions (especially collective discussions) play an important role in supporting students in sharing and building a repertoire of mental computation strategies. Thompson (2009) suggests that, to develop students' mental computation, teachers must: (i) create a classroom environment where students feel comfortable talking about their strategies; (ii) listen attentively to students' explanations about their computation methods; (iii) reinforce students' positively as they use specific strategies; (iv) enhance students' knowledge about numbers and capacity to implement effective strategies; and (v) ensure that students go through different experiences to gradually develop increasingly sophisticated strategies. In summary, the focus on the development of students' relational thinking, through the use of different representations of rational numbers, benchmarks and shared strategies in collective discussions may be an asset to learning of rational numbers.

RESEARCH METHODOLOGY

This study is qualitative (Bogdan & Biklen, 1994) with a teaching experiment design (Cobb, Confrey, diSessa, Lehere, & Schauble, 2003). The participants are a mathematics teacher and a grade 6 class with 20 children who previously worked with rational numbers in different representations (decimals, fractions, and percent) with the four operations. In the classroom setting, the first author was a participant observer. Data collection took place between February and May of 2012, through video and audio recordings of the classroom moments with mental computation tasks. Audio recordings of the preparation and reflection meetings with the teacher and researcher's notes were also taken. In data analysis the dialogues (audio and video recorded) that show students' mental computation strategies in collective discussions, were transcribed to identify how these strategies evolved during the teaching experiment. The students' mental computation strategies were categorized as instrumental or conceptual (Caney & Watson, 2003), and students' use of relational thinking (Empson et al., 2010) was analyzed.

THE TEACHING EXPERIMENT

The design of the teaching experiment was discussed with the mathematics teacher who carried it out with a grade 6 class. The anticipation of students' strategies supported the choice of tasks as well as the preparation of collective discussions. Class management, including collective discussion moments, were led by the teacher, with the first author making occasional interventions to clarify aspects related to students' presentation of strategies. The design of tasks stands on principles related to (i) context; (ii) representations of rational numbers and use of benchmarks, (iii) students' strategies; (iv) cognitive demand (Henningsen & Stein, 1997); and (v) social interactions. Concerning the context, we use tasks framed in mathematical terms and word problems because we assume that such diversity is important to develop mental computations skills. We use three representations of rational numbers (decimals, fractions and percent), in an integrated way and taking advantage of benchmarks. We favor tasks that may promote the development of mental computation strategies (based on Caney & Watson, 2003) and tasks with different levels of cognitive demand to engage students in different kinds of reasoning (Henningsen & Stein, 1997), some involving numbers which lead to simple computations and others requiring the use of numerical relationships. And, finally, we promote social interactions, especially collective discussions that we assume as a fundamental aspect in the development of mental computation strategies because they provide students with the opportunity to share ideas, reasoning and to improve their language by communicating mathematically, as they try to explain their reasoning.

The teaching experiment includes ten mental computation sets of tasks with rational numbers (with the four operations) to carry out each week for about 15-20 minutes at the beginning of a mathematics class. Seven sets of task are mathematical exercises that require students to compute the result of an operation or the value that makes a given equality true; two sets of tasks have four word problems each; and a set of tasks has a mix of exercises and problems. The rational number representation and the problems were related to the topics that students were working in class. When they were working with algebra, they computed with fractions, when working with volumes, they computed with decimals, and when working with statistics they computed with percent, fractions, and decimals. We integrate the mental computation tasks into the teacher's overall planning. This is one of the aspects that we consider important in our study compared with other studies in this field (i.e., Reys et al., 1995). The questions are presented using a timed PowerPoint, allowing 15 seconds for each exercise and 20 seconds for each problem. The students recorded their results on paper. After finishing the first five exercises, there was a collective discussion of students' strategies. Then, the students solved the second part of the task and that was followed by another collective discussion. These discussion moments allowed the students to reflect on how they think, what strategies they use, and what errors they make. This is another aspect of our study that we highlight, because this allows students to construct a collective repertoire of strategies.

In the teaching experiment, the students solved two mental computation tasks in mathematical terms using only fractions (addition/subtraction and multiplication/division), three tasks in mathematical terms with fractions, decimals and percent (with the four basics operations) and word problems where fractions appeared in combination with the other two representations. However, several times the students' used fractions in tasks that present only operations with decimals or percent, especially involving benchmarks as 0.25 or 75%. In table 1 we present some examples of tasks (exercises and problems) where we used fractions that students solved mentally throughout the teaching experiment.

The tasks were prepared taking into account several aspects important in mental computation. These tasks suggest the use of benchmarks such as ½, ¼ and ¾ (Cruz & Spinillo, 2004; Galen et al., 2008), as well as the use of halves, because recognizing and using them is a basic skill in mental computation with rational numbers represented as fractions (Callingham & Watson, 2004). We also used less common fractions such as 1/3 or 4/6 expecting that the students would apply the knowledge developed with benchmarks with these kind of fractions.

We start with addition and subtraction because these operations are the first that students study in grade 5, and then we use multiplication and division that the students learn in grade 6. The operations involve fractions with the same denominator or denominators involving multiples so that students may use equivalence.

Task 1 ¾ - ½ 3/6 + ?=1	Task 2 4/6 : 2/6 ? × 5/6=1/6	Task 3 0.75 : ¼ 8/10 - 0.2
Task 6 A tank has a capacity of 22.5 l of water. How many buckets of ½ l are needed to fill the tank completely?	Task 8 1/3 of 1/3 1/5 of ?=8	Task 9 4/6 × 6/7 2/3 × ? = 1
Task 10 Every day, 400 students eat lunch at Johns' school. Of these students, 3/4 always eat soup. How many students eat soup?		

Table 1. Example of tasks where we used the fraction representation.

In the case of word problems, the context is related to the mathematical topics that the teacher was working in the classroom (e.g., statistics, measurement, comparison of rational numbers and percent). We created problems that could lead to expressions similar to those discussed during the teaching experiment.

STUDENTS' MENTAL COMPUTATION STRATEGIES WITH FRACTIONS

In mental computation with rational numbers in fraction representation, students use mostly instrumental strategies by applying known facts and memorized rules (Caney & Watson, 2003). However, it is possible to identify the development of conceptual strategies along the teaching experiment. In these conceptual strategies, students' use equivalence, numerical relationships and properties of operations or mixed strategies combining instrumental and conceptual features.

In tasks 1 and 2, involving first addition and subtraction, and then multiplication and division, students' strategies were more instrumental. To compute 1/2+1/2, Elsa explains how she applied the rule of addition of fractions: "As the denominators are the same, I put the same denominator and I added the numerator. (...) It is equal to 1. [It is] 2/2 that is equal to 1". In this case we do not know Elsa's understanding of ½. But when Marta uses a known numerical fact (two halves form a unit) to compute the result of the same expression, we can understand that she knows that ½ means a half.

She quickly made “half plus a half” knowing that this gives 1. Marta’s strategy has instrumental and conceptual features because she applies a known fact showing some understanding of numbers. To compute 3/4x2/3, Eva explains how she used a memorized rule: “It gives 2/4. When we have the same numerator and denominator, we can cut [the equal numerator and denominator] because this fraction gives 1”. To compute 4/8:1/2 Rita used the invert-and-multiply algorithm in the division of fractions, as she explains: “I inverted the 2 with the 1 and did 4x2 that gives 8. So this yields a unit [8/8] and I wrote 1”. In dividing fractions, even when the numerators are the same, Maria computes 4/6:2/6 also using the invert-and-multiply algorithm: “I did… I changed the 2 with the 6 and then did 4x6 and 6x2. It gave me 24/12”. She shows little flexibility to find a new method to divide fractions.

In task 3, some students continue to use instrumental strategies. However, conceptual strategies begin to emerge. That may have been influenced by the collective discussions of the previous tasks and also because fractions appear in combination with decimal representations. For example, João uses a conceptual strategy to compute 3/4+0.5. He explains that: “I took 1from 3/4 [took 1/4] and, as 2/4 add with 1/2 gives 1, in the end I added what I had taken, and that gives 5/4”. He decomposed 3/4 in 2/4+1/4, and added two halves to obtain a unit (2/4+1/2=4/4) without calculating the same denominator. Finally, he added the amount taken from 3/4 and obtained 5/4.

In task 6, to solve the problem presented in table 1, Eva refers that “It gave me 45 buckets. I multiplied 22.5 by 2 (…) Because [we have] 1/2, to get the unit we have to add 0.5 twice, so, I multiply by 2”. This student thinks in a half using fraction and decimal representations, showing no confusion, and establishes a relationship between the capacity of the bucket and the capability of the tank (“with a bucket of 1l I can take 22.5 buckets. With a bucket with half of capacity, I have to take the double number of buckets”). Eva does not use the invert-and-multiply algorithm, but she uses a conceptual strategy based on numerical relationships that gives meaning to this rule (Newton & Sands, 2012).

In task 8, some students continue to use instrumental strategies (such as Ana), but also conceptual strategies (such as Maria). Ana explains how she computed 3/4 of 60: “I wrote it in a fraction way, and it gave me 180/4. I did 3x60, 180 and 4x1, 4. The division gives 45”, using the algorithm of multiplication of fractions [She considers that 60 has denominator 1]. Maria shows to have number sense by explaining that 1/3 of 1/3 it is 0.111… This student describes her reasoning: “I know that 1/3 it is 0.333… dividing by 3… is like a kind of the multiplication table by 11 (…) 11x1=11, 11x2=22… here it is 33:3 which gives 11”. She operates the decimal development based on the knowledge of the 11 multiplication table. It seems that her strategies evolved. In task 2 she used an instrumental strategy to divide fractions with the same denominators and in this task, to multiply fractions, when most of the students used instrumental strategies, Maria used a strategy based on numerical relationships.

In task 9, to compute 2.2-?=1/5, José used a conceptual strategy and to compute 2/3×?=1, Pedro used an instrumental strategy. José changed the representation: “We transform 1/5… it is 0.2" and used a property of subtraction "So, 2.2-0.2 gives 2" – to get the subtractive, he subtracted the remainder from the additive. Pedro used a known numerical fact and says that: "It is 3/2. They are inverse fractions". This student memorized that the multiplication of a fraction by its inverse gives the unit.

In task 10, to solve the problem that we presented in table 1, Ana used an instrumental strategy applying the rule of multiplication of fractions by a natural number: "400×3 gives 1200 [and] dividing by 4 it is 300". João uses equivalence showing a conceptual strategy: "300/400 is equivalent to 3/4. As 400 was the unit, it had to be 300".

In working with mental computation tasks, we underline the importance of collective discussions to promote sharing of strategies and to identify tasks that prove to be useful to promote students' conceptual strategies. Discussions allow students to make conjectures, to reflect on errors and misunderstandings, and the reasons why they made them, and to validate strategies and errors. Collective discussions also allow students to find different strategies that they did not use initially.

CONCLUSIONS

The analysis of the seven mental computation tasks that the students carried out with rational numbers represented by fractions shows that in the first two tasks their strategies were mostly instrumental (Caney & Watson, 2003). The students mostly used known numerical facts or memorized rules, for example, rules that they learned to operate with fractions. In addition and subtraction, the students computed the same denominator and added the numerators. In multiplication, they multiplied numerators and denominators and in division, they used the invert-and-multiply algorithm (as Elsa, Eva, Maria and Rita did).

From task 3 on, conceptual strategies begin to emerge (Caney & Watson, 2003) based on numerical relationships, equivalence and properties of operations, indicating increasing use of relational thinking by students (Empson et al., 2010) (as João, Eva, Maria and Ana did). We highlight the cases of Eva and Maria who, in the first tasks used instrumental strategies, and, in task 6 (Eva) and task 8 (Maria) used conceptual strategies that involve relational thinking. The strategy used by Eva is useful for making sense about the "invert-and-multiply" rule that most of the students tend to learn without understanding (Newton & Sands, 2012).

The choice of tasks and the classroom collective discussions contributed to this evolution. Using tasks in two contexts allowed students to diversify their strategies. In tasks in mathematical terms, the students tend to use instrumental strategies (as in tasks 1 and 2) and in problems they tend to use more complex strategies based on relational thinking (as in tasks 6 and 10). After task 2, students have to compute with fractions together with decimals and percent which facilitate the change of representation and equivalence, an important strategy in mental computation with rational numbers (Caney & Watson, 2003). The use of benchmarks and different representations of rational numbers in the same task along the teaching experiment helped students to make sense of rational numbers (Galen et al., 2008) since they often used fractions when operating with decimals and percent. The variation of cognitive demand of tasks, in different contexts, provided students with opportunities to develop increasingly complex personal strategies (as the case of Eva). Furthermore, social interactions (especially collective discussions) promoted sharing of strategies, contributing to the increase of the students' repertoire of strategies and for their validation of strategies, improving their capacity to conjecture and to justify how they reason in mental computation.

These results support the perspective that mental computation with rational numbers is important for the development of number and operation sense and that the work with fractions supports the understanding of decimals and percent (Galen et al., 2008). Systematic work in mental computation may promote the development of personal strategies increasingly based on numerical relationships and properties of operations, enabling students to develop relational thinking that will be useful in learning algebra. To achieve this goal, tasks must be designed so that they may facilitate the progressive development of students' relational thinking.

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PERFORMANCE ON RATIO IN REALISTIC DISCOUNT TASKS[1]

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This paper is focused on the design of a protocol for teaching and learning ratio tasks, structured around metacognitive principles and Socratic maieutic method.

The data presented in this paper illustrate the categorization of responses that a group of pre-service elementary mathematics teachers gave to a ratio discount task. This categorization outlined the students' response patterns, which is considered essential for the design of the maieutic proposal and metacognitive questioning.

***Keywords**: Ratio, Discount, Comparison, Metacognition, Maieutic.*

INTRODUCTION AND RESEARCH STRUCTURE

This study is part of a broader project – theoretical, empirical, and educational – which consists of the design, elaboration and implementation of a teaching proposal for ratio and proportion tasks, aimed at pre-service teachers.

This proposal follows the idea that "effective learners recognize the importance of reflecting on their thinking and learning from their mistakes", as pointed out by research and curricular orientation such as the NCTM (2000, p.20) in the United States. It highlights the role of metacognition in the mathematics classroom, as indicated by Lester (1985) and Schoenfeld (1985).

Teaching proposals based on metacognitive practices rest on the assumption that "metacognition demands to be taught explicitly" (Desoete, 2007 p.709). Despite this, "teachers still pay little attention to explicit metacognition teaching" (Ibid, p.709).

In this regard, Socratic maieutics (conceived by Socrates and put forward in Plato's dialogue Meno), is considered to be a suitable pedagogical method as it promotes and specifies metacognitive processes to "foster learning in students from self-recognition of their ignorance" (Rigo, 2011, p.523).

In the Socratic Method three key moments are identified: Construction, De-construction, and Re-construction (Rigo & Gómez, 2012).

In the moment of construction, the teacher poses a task that he/she knows students will answer confidently but in an incorrect or limited manner. On being asked to justify the answer, students engage in metacognitive reflection, which takes into account individual variables (confidence in the answer), task variables (knowledge of the mathematical notions involved and the degree of difficulty of such a task), and strategy variables (which strategies were available to solve the task and which was chosen and why). These reflections are necessary to prepare students for the transition towards the moment of de-construction.

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Next, the teacher leads the students to confront cognitive and metacognitive conflicts. Cognitive conflicts arise when students are confronted with the contradictions that emerge from wrong answers. Metacognitive conflicts arise when students reflect and become aware of the limitations of their answer and ideas about the subject matter. This is the so-called moment of de-construction.

Finally, at the moment of re-construction, the teacher guides the students so that they can produce a new answer that will enable them to understand what, up to that moment, has been unknown in terms of the proposed task.

Key points and research questions

For this process the chosen task must be maieutic, that is:

- Rich in interpretations and meanings
- Rich in mathematical concepts
- Can be solved in different ways
- Familiar
- The answer is likely to create a high degree of confidence, apparently simple, but can create difficulties

Furthermore, for the maieutic process to be effective and have the expected cognitive and metacognitive impact the students' response patterns are considered essential. Knowing them beforehand will allow us to design the questions introduced during maieutic stages, and to plan the cognitive conflicts which could eventually be used as learning opportunities.

Due to limited space, the data presented in this paper refers to a categorization of response patterns given by a group of pre-service elementary mathematics teachers in a numerical comparison task called "the discount comparison task". This is considered essential, as mentioned earlier, for the design of our maieutic proposal. For this purpose the questions that orient the investigation are the following:

- Which are critical components for categorizing student's response patterns in ratio tasks?
- What are student's response patterns in the task?

METHODOLOGY

To categorize the students' responses we used an interpretation scheme focused on common features to group the different responses given by pre-service elementary mathematics teachers when try to solve the tasks.

The task was presented on a worksheet (paper and pencil format) administered during a regular one-hour class session, to 314 third-year students undertaking a course in mathematics teaching (didactic) of elementary mathematics teacher training degree at the University of Valencia (Spain). These students had already completed an annual course in mathematics.

The discount comparison task

Three typical advertisements are presented to the students. The question is which discount is better (Figure 1).

Figure 1. The discount comparison task

The three advertisements are phrased differently. The first advertisement shows a percentage, “70% off second item”. The second one shows “3x2” and the third “Second item half price”.

The last two discounts are applied to a particular product: the “3x2” advertisement offers three bottles of “Rioja” wine for the price of two; the other advertisement offers the second bottle of “Extremadura” wine at half price.

In order to avoid arithmetic calculations the different offers include unit values: in 3x2, the price of each bottle is given before and after the discount: €5.58 and €3.72; and in the second bottle half price, the price of the bottle before the discount is given, €9.74 and the price of the second bottle after the discount is given, €4.87.

Critical components of the task

The difficulty in the discount comparison task lies in two critical components. One is to perceive that what is asked is not about the magnitude of the discounts; that is, the quantity to deduce from the cost, rather about the relativity of the discounts, in the sense of Freudenthal, that is, “in the sense of “in relation to”…with the criterion of comparison filled in at the dots” (1983, p. 194), in this case, in relation to the purchased items.

In the task the word relatively is not explicit, but it is implicit in the ratios 3x2, 70% off second item, second item half price.

The other critical component is that these ratios are not directly comparable, because they are formulated in different ways and have different references (price, and number of bottles).

To compare them a transformation process is required in order to express the ratios in the same way (percentage, decimal, fraction or verbal ratio “a to b”), and to “unitizing”, in order to construct a common reference unit and then to interpret the situation in terms of that unit (see Lamon, 1996, for an overview for “unitizing”).

Based on this, there are a number of strategies to answer which discount is better. For example:

1. To calculate the discount in relation to total cost of purchased items (Part-whole strategy) to determine which fraction is larger/smaller:

3×2	second item half price	70% off second
$\frac{5.58}{3\times 5.58}=\frac{5.58}{16.74}$	$\frac{9.74/2}{2\times 9.74}=\frac{4.87}{19.48}$	$\frac{\frac{70}{100}\times 5.58}{2\times 5.58}=\frac{3.9}{11.16}$ or $\frac{\frac{70}{100}\times 9.74}{2\times 9.74}=\frac{6.9}{19.48}$

2. To calculate the discount (or the cost) per item in each offer (Unit rate strategy) to determine which per-cent is larger/smaller:

3×2	second item half price	70% off second
$\frac{1}{3}=33.\hat{3}\%$	$\frac{50\%}{2}=25\%$	$\frac{70\%}{2}=35\%$

3. To calculate the discount (or the cost) of a same number of items in each offer to determine which verbal ratio (“a to b”) is larger/smaller, in a sort of “pattern recognition and replication” strategy, which was named “building up strategy” by Hart (1981), where, with the help of a table of values, children may notice a pattern which is then applied to answer the problem.

3×2	second item half price	70% off second
1 to 3 (1 save every 3)	0,5 to 2 (0.5 save every 2)	0.7 to 2 (0.7 save every 2)
2 to 6	1.5 to 6	2.1 to 6

These three strategies yield the same result: the best discount is 70 % off second item. The reason for this is that the three strategies used ratios that (although in different forms) express the same values, since a ratio express an invariant relationship between the terms of a pair, so the ratio does not change the value when the reference is changed (see the next table).

3×2	second item half price	70% off second
$\frac{5.58}{3\times5.58}=\frac{1}{3}=33.\hat{3}\%$	$\frac{9.74/2}{2\times9.74}=\frac{1/2}{2}=25\%$	$\frac{\frac{70}{100}\times5.58}{2\times5.58}=35\%=\frac{\frac{70}{100}\times9.74}{2\times9.7}$
2 to 6 is $\frac{2}{6}=\frac{1}{3}$=0.$\hat{3}$=33.$\hat{3}$%	1.5 to 6 is $\frac{1.5}{6}=\frac{0.5}{2}$= 0.25=25%	2.1 to 6 is $\frac{2.1}{6}=\frac{0.7}{2}$=0.35 = 35%.

To the majority of students this invariability is not obvious, due to the influence of the context. The influence of the number of items, 3 or 2, to which the discount applies is susceptible to discussion, because the advantage of the discounts is not applicable simultaneously if the number of purchased items is not a common multiple of 3 and 2

The scheme of response patterns

These critical components and strategies provide a scheme to categorize students' outputs, organized with categories subcategories, classes and subclasses, with the purpose to have a more detailed description of the students' response patterns.

The first criterion considered to group the responses is if students perceive the relativity of the discounts in the task or not. This determines two categories: focus on ratios and focus on specific elements of the offers.

A criterion to distinguish subcategories in this first category is if the students convert and unitize or not. This determines two subcategories: compare elements that are comparable or compare elements that are not comparable.

The criterion for distinguishing classes is if students focus on a unit value or on a same number of items; that is, if the students use the “unit rate strategy” or the “building up” strategy.

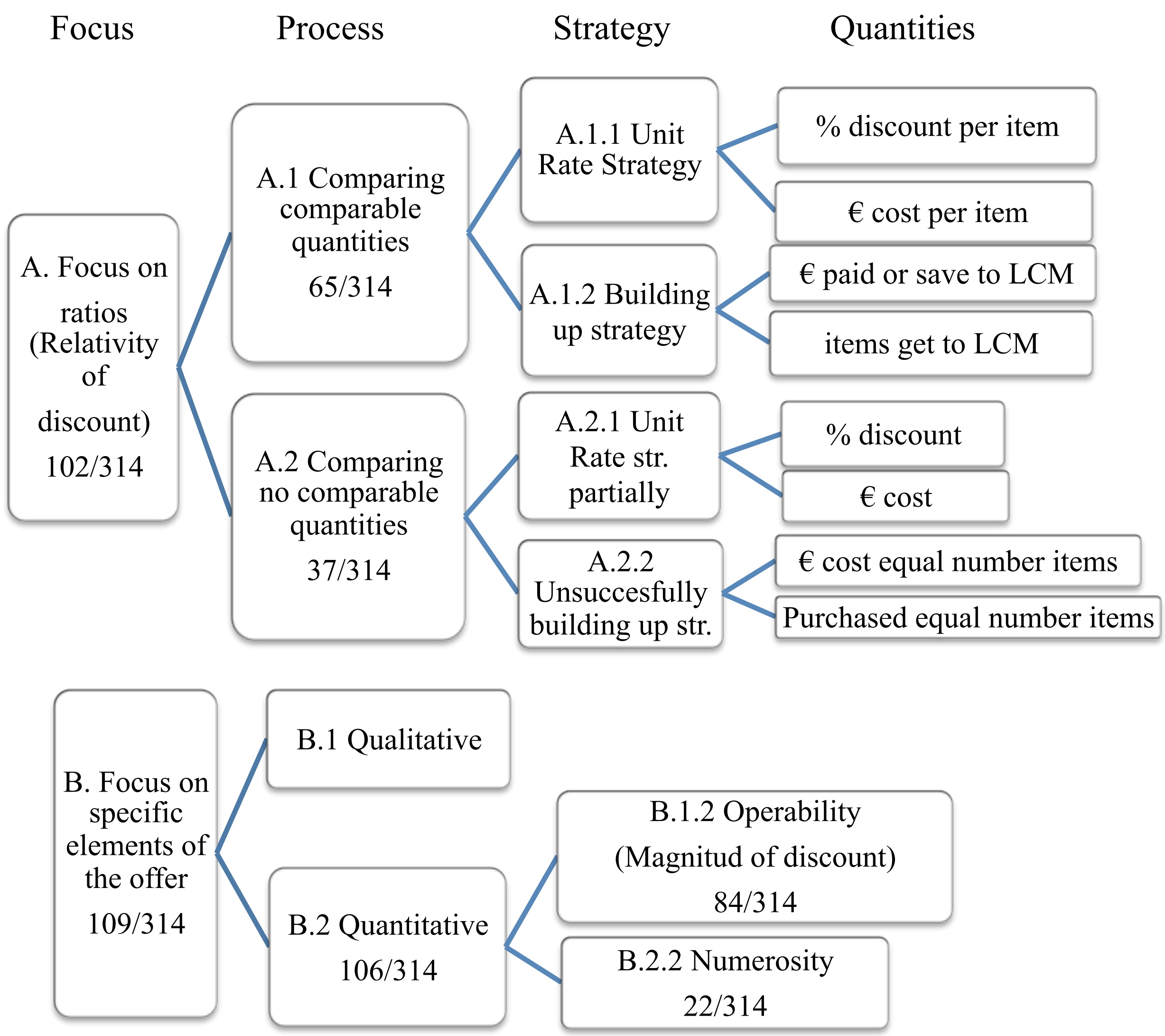
Finally, subclasses are obtained according to the quantities that are compared: discount per item or cost per item related to the unit rate strategy; cost of the LCM items or paid items purchasing the same number of items in building up strategy.

The second category comprises responses by students who perceive the discount as a feature depending on the item to be purchased.

Subcategories are established by distinguishing between qualitative and quantitative approach. The first subcategory gathers those responses that focus on aspects such as quality of the product, size of the bottle, and so on. The second subcategory includes responses that focus on particular numerical data of the discount.

In the quantitative subcategory, as a criterion to distinguish between classes, we observe whether the students focus on the operation (with the prices in the ads) or the number (how many bottles I am given).

Figure 2 graphically organizes these categories, subcategories, classes, and subclasses, as well as the frequencies obtained.



Incomplete or not identified answers were 103

Figure 2. Categories, subcategories and classes

EXAMPLES

A description of examples of each class follows.

A.1.1. Unit rate strategy

In this class there are two response patterns: in the first one the percentage of the discount is calculated and in the other one the unit price.

Example 1. In the first offer, Maria uses the rule of three to calculate the percentage paid for each item. Her calculations are: 100/5.58=x/3.72; x=100x3.72/5.58=66.66%. From this, she obtains the percentage of discount per item by subtracting 100-66≈33%. In the second offer, she divides 50%÷2 =25%. In the third offer she divides 70%÷2=35%. Finally she states that "the best deal is 70% off the second item.

Example 2. Juan says: “the item costs €5. Applying 3x2→ items €10, 10÷3 = €3.3ˆ for each item. Applying second item half price: 1st item is €5, 2nd item is €2.5, adds up to €7.5, I divide 7.5÷2 = €3.75 for each item. Applying the discount to 70% off second item, the 1st item is €5, 2nd item 70/100x5 = €3.5; 5-3.5=1.5 2nd item". Next, he writes: 5+1.5=6.5, he divides 6.5÷2=3.25, and he states: "The best deal is 70% off second item".

These students have compared comparable quantities. Both have applied the unitizing process to refer the offers to one bottle, and the converting process to transform offers in percentages. One student has used the offered price 5.58 to converted 3×2; the other, has assigned the same price €5 to all the offers.

A.1.2 Building up strategy

In this class there are also two response patterns: in the first one the number of items we get free buying the same number (LCM) of items is calculated, and in the other one, the discount or the cost to a same number (LCM) of items is calculated.

Example 3. Paco claims that "at first glance we wouldn't be able to make comparisons so we took the least common multiple of the items we purchase, which in this case is LCM (2, 3)=6. Now we can make comparisons. In the table we can see that the best deal is the third one (70% off second item)."

	Items we get	Purchased items
3x2	6	4
Second item half price	6	4.5
70% off second item	6	3.9

Example 4. José states the following: "If a bottle costs €10: 3x2 means that you get 3 and pay for 2, e.g. 3 bottles for €20, 6 bottles for €40, 9 bottles for €60. 70% off second item: 70% of €10 is €7 →second bottle €3, so 2 bottles €13, 4 bottles €26, 6 bottles €39, 8 bottles €52; 50% off second item (half price), 50% of €10 is €5→second bottle €5, so 2 bottles €15, 4 bottles €30, 6 bottles €45, 8 bottles €60. The best deal is 70 % off second item".

These students transform the ratios given in the advertisement into a verbal form in order to compare comparable quantities: "a to b", using the least common multiple of the number of bottles offered; in one case this is done without prices and in the other one the same price is assigned to all the items.

A.2.1 Unit rate strategy partially

In this class a unit rate strategy is not used in all the offers since the students end up comparing the percentage of discount or the unit price in the 3x2 offer with the percentage or cost of the discounted item (the second item) in the other two cases.

Example 5. Jesus says "the highest discount percentage is the best". First he calculates the price of three bottles of Rioja without a discount: "5.58x3 =16.74; next he calculates the price with a discount: "3.72x3=11.16". He uses the difference between these two prices (16.7-11.16=5.58) to calculate the percentage of discount by using the rule of three: 5.58x100÷16.74 =33.33%". However, he compares this with the percentages of discount in the second and third offer that only refers to the second item: "70% and 50%"

Example 6. Susana writes the following: "Price: 3€. Discounts: 70% off → 3x70÷100=2.1 which is 70% that must be subtracted from the item, 3-2.1= €0.9. R: €0.9 applying 70%. 3x2→ the product costs €3 and we would purchase 2, 3x2=6, as we get 3 products we divide: 6÷3=€2. R: €2 applying 3x2. 50%→ 3x50÷100 =1.5 which is 50%; 3-1.5= €1.5. R: €1.5 applying 50% off. As we can see the best deal is 70% off."

These students have compared no comparable quantities. Both have applied the converting process to transform 3×2 in percentage, but fail in unitizing since they do not calculate the discount per item purchased in the second and third offer and they directly deduct 50% and 70%, to compare it with the discount per unit purchased that has been calculated for the first offer.

A.2.2 Arbitrary building strategy

In this class, the students apply the discount to the same number of bottles, but they do not use a common multiple, thus making it meaningless to compare these items.

Example 7. Patricia points out for example "5 items: 70% off → you pay for 3.4 (mistake, is 3.6) items, 3x2→ you pay for 5 items, second item half price→ you pay for 4 items. The best deal is 70% off'.

Example 8. Marta writes "each item costs the same, for example €10, and in the 1st I would buy three items for €20 (€10 savings); in the 2nd three products would cost €25 (5€ savings); and in the 3rd three items would cost €27 (€3 savings); therefore the best deal is the 1st".

B.1 Qualitative

This subcategory includes those responses that are not a consequence of numerical calculations but rely on superficial aspects of the products that appear in the advertisements.

Example 9. Ana says "I would say that it depends on the product and the quality as to whether it will be cheaper or more expensive. In addition, it is important to see the amount in each bottle. With this information, relations can be established."

For this student, the concept of discount is linked to qualitative elements of the wine.

B.2.1 Quantitative, operability.

In this class the students perform arithmetic operations with the intention of calculating the magnitude of discount, without relativizing or making the required transformations to obtain comparable elements.

Example 10. Laura carries out the following operations: 3x2, 3.72x3=11.16→3 bottles. In the case of the second item half price: 4.87+9.74=14.61→2 bottles. She concludes by saying: “1st =3 bottles=€11.16 ; 2nd =2 bottles=€14.61; Here you can see the cheapest option”

This student’s solution is dependent on the prices and number of bottles that appear in each advertisement. Thus, she ends up comparing the total cost of buying three bottles of the first wine (which has a price) with the cost of buying two bottles of the other wine (which have a different price). She does not use the 70 %'s option because there no number of bottles is given it can be applied to.

B.2.2 Quantitative, numerosity.

This class groups together those responses in which the quantity of purchased items is the solution to the proposed task. The other aspects (such as those that are related with the cost) are neglected.

Example 11. Pablo writes “With 3x2 you buy two bottles and get one free, without having to pay anything. 9.74x2=€19.48 for three bottles. With 50% you pay €9.74 for one and €4.87 for the other, so you buy two and you do not get one free. With 70% you buy two bottles, one for €9.74 and the other for €2.92 so you pay for two bottles. Therefore, the best deal is 3x2 because you buy two and get one free.”

For this student the best discount depends only on the number of items you get for free. In the case of 3x2 you get three bottles and in the other two cases you only get two bottles.

FINAL REMARKS

In this paper we categorize data related to a maieutic task that involves ratio notions.

Based on the classified response patterns, we should continue with the teaching maieutic protocol, working with pre-service teachers. Now, in a same session, we can propose the task and implement the metacognitive maieutic questions because (although unknown the student’s specific answers) we know their way of thinking and their possible response patterns in advance. Consequently, the maieutic questions are not improvised, because they relate to the response patterns that have been found in this study.

In our current stage of investigation we attempt to test the maieutic protocol. With the questions for the moment of reconstruction: How did he/she solve it? What is the base he/she used? What do you think about the subject? On what does the discount depend? We expect that students are confronted with their intuitive notions of discount and ratio and become aware of the limitations of their ideas. Furthermore, we expect that students explain their reasoning, listen and make sense of other’s solutions, focus on the relativity of the discount and on the equivalence of the part-whole strategy, the unit rate strategy, and the building up strategy. Thereby, we expect that with this reflection students will not feel ashamed and will not get the feeling, that they are only competent with teachers’ help.

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REPLACING COUNTING STRATEGIES: CHILDREN'S CONSTRUCTS WORKING ON NUMBER SEQUENCES

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In mathematics didactics, consensus widely prevails on the importance of using patterns and structures for effective and flexible computation. It is equally known that children who are using counting strategies when solving problems do not perceive relations between numbers and operations. Because of this, the central objective is to replace counting strategies by realizing, recognizing and using structures. Within the context of the project ZebrA (Zusammenhänge erkennen und besprechen – Rechnen ohne Abzählen[1]) different lessons were developed to encourage children to use different interpretations of patterns and structures instead of counting. In this paper, the results of the video-based qualitative analysis of teaching/learning situations in the field of number sequences are discussed.

Key words: counting, difficulties in learning mathematics, number sequences, cooperative learning, interpretative approach

INTRODUCTION

Counting as computational strategy

Several studies have shown that young children are able to solve simple arithmetic problems using counting strategies (Krajewski & Schneider, 2009). Therefore, on the one hand, procedural knowledge of counting is needed. Fuson (1987) describes five levels in counting development beginning with string level, where the children interpret the number sequence as whole like a poem, followed by the levels unbreakable list, breakable chain, numerical chain and on highest level the bidirectional chain. Now it is possible for the children to count in stages forward and backward from a lower to a higher boundary. On the other hand, conceptual knowledge is necessary, like understanding the cardinal principle. During and in addition to the development of counting competency, an understanding of cardinals is developing and has to be developed (Desote, Ceulemann, Roeyers & Huylebroek, 2009). The approach to numbers by counting has to be combined with an approach to an understanding of quantities. As a result of both competences, children are able to see relationships between numerical quantities and between numbers in the numerical sequence (Krajewski & Schneider, 2009). The awareness of structural relations can be used for solving an addition or subtraction task, for example children calculate $8 + 6$ no longer counting six steps on from eight but rather decomposing 6 in 2 and 4 and compute like $8 + 6 = 8 + 2 + 4 = 14$. A central aim in mathematics education must be

[1] Recognizing and speaking about connections – calculating without counting out

to develop these structural relations in the numerical sequence and between quantities and use them in problem solving.

But mathematical understanding does not develop in every case in the way described above. There are children who do not build up an awareness of numbers as quantities and relations. They get caught in the interpretation of numbers as ordinals and do not use structural relations between numbers when solving problems. In an actual study (Gaidoschik, 2010), a central result is that changing the strategy from deduction to the use of memorized facts takes place significantly more often by first-graders than changing from a counting strategy to memorized facts. Gaidoschik (2010) supposes a reason in the facility of counting strategies which could lead to a persistence of counting on. In the number range up to 10 or 20, counting strategies can be used successfully instead of efficient computational strategies. But counting computation is not a strategy that works in higher number ranges. Furthermore, it often comes along with a mechanical, non-reflected procedure as well as an isolated problem solving. There is a risk that the missing insights develop into comprehensive problems in mathematics education. Children with learning disabilities in mathematics are often using persistent counting strategies as their main computational strategy (Moser Opitz, 2007).

Whereas the importance of an awareness of mathematical structures for non-counting computation is stressed and can be caused in different ways, little is known about the way in which children with mathematical difficulties realize number patterns and structures. The primary question is if and how an explicit focusing on these structures in mathematics education can motivate a replacement of counting strategies. The replacement is necessary because non-counting calculation works only when using structures. Interventions focusing on the fostering of structures have led children with difficulties in mathematics to better results in standardized tests (Dowker, 2001; Kroesbergen & van Luit, 2003). But these studies do not answer the questions as to how the children realize structures and which steps are important for them to do so. Empirical findings are missing how children who are using persistent counting as their main strategies interpret numbers, operations and how they develop a structural focus on arithmetical patterns.

Interpretation of structures as a constructive and social process

As concepts of mathematics, structures and patterns are abstract and not visible. They must be constructed individually when sighting signs (Steinbring, 2005). Numbers, operation-signs, representations can be taken as signs into consideration and also as arithmetic patterns. All signs have to be interpreted. Interpretation is an active process that each child has to perform on its own, although the community in a class is important. Studies focusing the development of new mathematical knowledge of children emphasize the relevance of interactive settings (Steinbring, 2005). The new knowledge could be built up in situations where children reflect their own perception and relate it to the perception of others. As such, the students participate in the mathematical practices in the classroom, create interpretations and negotiate meanings or resolve conflicts (Cobb, Wood & Yackel, 1991). But not every cooperative and interactive act leads to new knowledge. On the one hand, the given tasks and the cooperative discourses about them seem to have great influence, on the other hand the suggestions of the teacher seem to play a central role. Directed suggestions, interventions or instructions for cooperative organized learning situations can cause communication between children, which can lead to the construction of new mathematical knowledge (Nührenbörger & Steinbring, 2009).

PRESENT STUDY

The difficulties of children using counting as their main computational strategy can be reduced to arithmetic contents of the first two years at school (Moser Opitz, 2007). In fostering children, it is important to not only revise the contents, because an education without results becomes not more successful if it is revised in the same way (Lorenz, 2003). It must be examined if cooperative learning leads to an enhancement of individual interpretations – especially to a (more) structured focusing view of the children with persistent counting strategies.

The present study is a part of the project ZebrA. In the project, learning environments with cooperative elements for second grade in primary school or for fourth grade in special education schools have been developed, field-tested and evaluated (Häsel-Weide, Nührenbörger, Moser Opitz & Wittich, resp. 2013). Twenty units have been constructed to support children in replacing persistent counting strategies. The learning environments focus on understanding, demonstrating and imagination numbers and operations as well as the relations between them. All children of the class are taking part in the lessons. The tasks permit a fundamental awareness of mathematical structures and at the same time a deeper understanding of structures. The engaged material is sophisticated and allows learning at different stages of understanding. To initiate various interpretations, the students are working together in pairs. Each child who uses counting as its main computational strategy works with a partner who uses other strategies. The methodical design of the learning environments encourages them to exchange views on the given tasks. The tasks are given in a discursive way, so that they cause different interpretations to the end that an awareness of structures is stimulated and interpretations are enlarged. This design is innovative because fostering children usually happen in one-to-one situations or in small groups with up to 10 children outside the normal lessons. However in the ZebrA-project an integrative approach is chosen regarding different competences of children as a change for development.

The study was realised from September to December 2010. The teachers of the participating classes gave the lessons. They had taken part two times in an advanced further training, where they became confident with the concept of ZebrA. The ZebrA Project is accompanied by two empirical studies which allow focusing the replacement of persistent counting strategies from different empirical points of view. Whereas the quantitative study researches the effects of cooperative fostering

(Wittich, Nührenbörger & Moser Opitz, 2010), the study presented in this paper focused on the interpretations of children dealing with the problems and discussing with the partner. The aim of this study is to identify and describe interpretations of children with mathematical difficulties and their development during fostering.

In order to identify children using counting strategies as persistent computation strategies as well as children using non-counting strategies the data of the quantitative study was used (results of different tests, rating of the teacher). Five children and their partner – belonging to four different classes and three schools - were chosen for the qualitative study. Their work was video-graphed in ten lessons of the ZebrA-project. Corresponding transcripts were interpreted by a group of researches (Krummheuer & Naujok, 1999). The analysis has been compared in an interactive way with empirical findings of other studies and theoretical approaches, with the result, that new insights about the interpretations of children with mathematical difficulties could be constructed. This procedure allows for the development of new theoretical elements analyzing individual cases. Concerning the present study, these could match with a characterization of typical interpretation of children using persistent counting strategies.

ANALYSIS OF AN EPISODE

The procedures and interpretations of the student Kolja (fourth grader of a special education school) working on the learning environment "number sequences" are analyzed exemplarily. First, the content of the environment is explained, followed by the presentation of the documents, procedures and interpretations noticed in the discourse with his partner Medima.

Content of the learning environment: number sequences

In this lesson, the students work on number stripes, which correspond to arithmetic sequences (Fig. 1). The sequence reflects on ordinal and relational interpretations of numbers by counting development described by Fuson (1987). Some of the stripes initiate counting in steps of one, others counting in steps of two or ten. Some sequences allow counting on, some require counting back. The number sequences are given to the students on stripes and their first task is to fill them. In a second step, the students sort the stripes focusing on the relations between them (such as distance between numbers of a sequence, starting number, multiplication of number or constant difference between sequences). Afterwards, the students are asked to find other compatible number sequences and note them on free stripes. These self-productions could illustrate the insight of the relations on the one hand, and on the other hand, they enable children to work on their individual level of understanding. The use of number sequences seems to be astonishing in the first view, but number sequences allow the children to bring in their common strategies and develop them further. Children with persistent counting strategies often only are used to count on in steps of one, but filling the stripes encourage other more elaborated strategies like counting in steps. Sorting stripes and finding new fitting ones should change the attention towards the realizing of mathematical structures.

Kolja completes number sequences

Kolja, a student using persistent counting as his main computational strategy, completes the sequences as shown in figure 1[2].

10	**20**	**30**	40	50	60	70	80
1	**2**	**3**	4	5	6	7	8
6	7	8	9	10	**11**	**13**	**15**
12	**14**	**16**	18	20	22	24	26
1	2	3	4	5	**6**	**7**	**8**

Figure 1: Reconstructed number sequences filled by Kolja

The documents show that Kolja seems to be able to find sequences with the difference of one. Equally, it seems to be no problem for him to count on in steps of ten. Sequences with the difference of two, which can be found by counting on are filled correctly, too. Only the number sequence which requires counting back in steps of two seems to be difficult for him. Using the document only, it is not possible to point out if Kolja does not realize the mathematical structure in the given number sequence or if he is not able to keep the distance when counting back. But it is striking that the wrong sequence corresponds to the highest level of the counting development described by Fuson (1987). Considering Kolja's approach in the videotape, it could be seen that Kolja counts on when he finds the numbers in the sequence _, _, _, _, _, 6, 7, 8. Starting with one, he counts on and controls if the sequence fits, as it does in that case. Similarly, Kolja finds the sequence 6, 7, 8, 9, 10, **11, 13, 15**. Two times he seems to count on tapping with a pen from left to right on the free fields before he then filled the numbers starting with 10. This may suggest that Kolja has problems in counting back generally, as pointed out by Moser Opitz (2007) to be typical of children with mathematical difficulties.

Kolja and Medima sort the stripes

After completing the stripes the teacher tells Kolja and Medima that they now should sort the stripes. She asks if the stripes fit together.

1 Kolja: Yes, that (*points to the number stripe that is located at the very bottom in front of him*)

[2] The numbers in bold print were given; the other numbers were notated by Kolja. The stripes are present in chronological order.

1	2	3	4	5	**6**	**7**	**8**

fits to that (points to the number stripe that is located as the second from the top in front of him)

1	**2**	**3**	4	5	6	7	8

2	Medima	Sorry? I didn't understand.
3	Teacher	Put together in order these sequences of numbers, ok?
4	Kolja	One, one, two, two, three, three, four, four, five, five, six, six, seven, seven, eight, eight
5	Medima	Oh, yeah
6	Teacher	Why do these fit?
7	Medima	Because they are the same (*pointing alternatingly to the numbers of the upper and lower number stripe*).

Kolja's first idea to sort the number sequences is to match identical stripes. He explains this in reading out the numbers in pairs (4). Medima seems to understand immediately (5) but the teacher asks for a further explanation (6), which is given by Medima. She explains the matching with the equality of the numbers and underlines her statement in pointing to the numbers by turns.

In the further course of working, the children try to sort the other stripes in the same way, but there are no more identical sequences. Afterwards, they attempt to find matching sequences with the same starting number. Because they regard the distance between the numbers as a second factor at the same time, they do not find matching stripes. The consideration of both factors engender that they are searching for identical sequences again. To the children it is, however, not clear that considering both aspects is already the same as looking for identical stripes. They try to fit the stripes by shifting (Figure 2). But they do not find a position which allows a pair-wise order for a couple of numbers and then they give up these tries.

14	**15**	**16**	17	18	19	20	21

14	**16**	**18**	20	22	4	26	28

Figure 2: Shifted stripes

The relations considered by the children seem to be guided by the appearance of the number sequences. Exact congruence of stripes seems to be critical. They are comparing the numbers one after another with each other and are looking for identity. Matching relations like same distance between numbers or sequence initiate counting on or counting back are not discussed so far. This corresponds to other empirical findings of the ZebrA Project (Häsel-Weide, 2013). As this way to sort does not lead to results, Medima suggests a new possibility to order. She matches number sequences with the difference 10 between each position to another.

8 Medima: Found another one (*pulls two stripes over to her).*

2	4	6	8	10	**12**	**14**	**16**

12	**14**	**16**	18	20	22	24	26

9 Kolja: Yay

10 Medima: (laughs) You see? Eight (*points to the "12" on the bottom number stripe*) oh, twelve (*points once again to the "12"*), two (*points to the "2" on the top number stripe*), fourteen (*points to the "14" on the bottom number stripe*), four and so on (*moves the pair of number stripes below the one she had previously found).*

11 Kolja: (*moves two number stripes that were located in front of him together*) Look. Six, ten, seven, twenty, eight, thirty (*points each time to the number mentioned on the number stripes that are located right underneath one another*) (*points to the "9" of the top number stripe*) (.) doesn't fit

6	7	8	9	10	**11**	**13**	**15**

10	**20**	**30**	40	50	60	70	80

Medima explains to Kolja why the sequences match, in spite of the acknowledgement of Kolja before (9). In her explanation, she names the numbers pair-wise (10). She does not give a general description or an argument as she did in the dialogue with the teacher. Kolja seems to adopt the idea and tries to realize it with the residual stripes. He takes two of them and compares the numbers pair-wise, responding to their positions in the number sequence. After he has looked at three pairs, he decides that the sequences do not fit. He does not give a reason either, but reads the numbers aloud. Possibly the children check if there is a phonetic matching when reading the numbers. It is not clear if the students consider structural relations between the stripes at that time or if they confine themselves to phenomena which can be realized at the surface such as same digits at the unit position or same sound at the beginning of the numeral.

In the ongoing partner work, the children are asked to find matching sequences in free stripes. Kolja produces identical stripes, copying existing sequences. At the same time, Medima goes on with her idea to increase the numbers. Thereby she seems to mix different techniques to construct matching sequences. The single digit numbers are decupled by appending a zero, while the two digit numbers are increased by ten, changing the ten-position by one. The different chosen techniques show that Medima considers relations between pairs of numbers (3, 30 and 11, 21) more than relations between the number sequences. The different distances between the numbers do not strike her. Once all free stripes are filled out and sorted the children are approaching the teacher.

12 Teacher: I think you are so quick, you could probably think of a lot more that fit to them

…

13 Kolja What is with 100, 200, 300, 400, 500, 600, 700, 800, nine, ten

14 Teacher That is great. Where would those fit? To which that you put in order there? When you write down 100, 200, 300, 400

15 Kolja (*points to the first box of the empty stripe that is located right in front of him)*

16 Medima (*grabs the number stripe depicted below, laughs and displays it)*

10	**20**	**30**	40	50	60	70	80

17 Kolja (*fills out the empty stripe*) Yes.

100	200	300	400	500	600	700	800

…

18 Teacher Great. And now you could think of which other would fit with it. (.) How they then (*incomprehensible*).

…

19 Medima Do you already have that? Ok. (*grabs two empty number stripes that the T left on the table*) Then we have to put'em together like that (*pushes her stripe briefly underneath Kolja's stripe, then pulls it towards herself again*) But what should we then write?

20 Kolja (*pulls his stripe a little towards himself and looks at it*). Infinity, two infinity, three infinity, four infinity, five

In this episode, Kolja seems to modify Medima's idea of multiplication. Kolja's words (13) indicate that he wants to construct a sequence with big numbers and begins counting in steps starting with hundred. The question is whether Kolja realizes a relation between the existing sequences and the new ones and whether he is aware of the multiplicative relations between his sequence and an existing one. At the time when he constructs the sequence, it seems that he refers to a counting context and does not focus on relations to an existing number sequence. Medima seems to understand his idea immediately and detects a stripe that matches her idea of decupling (16). She combines Kolja's suggestion with the activity of sorting stripes. In this way, a relation between both sequences is constructed.

After the sequence is noticed, Medima asks Kolja for an idea to find another fitting stripe. He creates a new number sequence with considerably larger numbers (20). Again, it is not sure whether this sequence refers to a counting context or shows the idea "starting with a power of ten and going on in appropriate steps leads to matching sequences". Because the children have not considered the difference between the numbers in any sequence yet, the last interpretation seems perhaps too optimistic. But relations within a sequence starting with a decimal power may have been realized by Kolja.

INTERPRETATION AND CONCLUSION

The learning environments of the ZebrA Project are intended to help children using persistent counting as main computational strategy to realize mathematic structures. In the analyzed episode we were able to show that in the cooperative partner work an awareness of mathematical structures had happened.

The difficulties to count back in steps pointed out by Fuson (1987) and Moser Opitz (2007) could be seen in the solving process of Kolja. Nevertheless, Kolja has found a way to generate number sequences by testing the fitting of sequences that he generated while counting on. He may show an approach which can be observed in a similar way when children are looking for a previous number. They count on until the given number and in doing so they recognize the previous spoken one.

Sorting the stripes, but most of all finding further sequences, leads the children to dealing with the relations between the number sequences. Kolja succeeds in expanding his interpretation of equality; first he reconstructs structural relations and then uses them to find matching sequences on his own. In the realized relations, a general understanding is indicated. Both children exceed their common number space. Nevertheless, the realized relation of Kolja is limited to only one and that requires a long period of working with the material. This indicates that children with mathematical difficulties may need an extensive period of working with materials and several opportunities to become aware of mathematical structures. The ZebrA Project shows that it is possible to initiate children using persistent counting as the main computational approach to realized relations between structures, but an intensive fostering may be needed for them to become familiar with structures.

There are positive effects from Kolja's interpretation, resulting from the collaboration with Medima. She initiated an alternative interpretation and as a result the children take into consideration the mathematical relation of multiplication. The intended cooperation between a child using persistent counting as the main strategy and another works out in these instances. But Kolja is not a passive partner either: Sorting started with his idea to find equal stripes and he comes up with the number sequences with decimal power. Notwithstanding the good cooperation and communication of the students, the episode shows that the communication changes in the presence of the teacher. While the students among themselves are speaking mostly without descriptions and are using gesticulations to demonstrate their thoughts, they try to find arguments when they are asked to do so. It appears as if asking for a description or an explanation to explain the structures may be the job of the teacher.

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WHY AND HOW TO INTRODUCE NUMBERS UNITS IN 1ST-AND 2ND-GRADES

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Abstract. Learning and teaching the decimal number system is well known by researchers as more difficult than the common adult user thinks of. This paper aims at studying some semiotic problems -how are denoted the quantities and what quantity denote the whole numbers- from different points of view, epistemological, institutional, didactical. It develops a few propositions for solving them, based on the introduction of a language referring explicitly to tens (numbers units) and specific materials (hand fingers) giving meaning to this language.

INTRODUCTION

According to our function as teachers educator we were asked by the whole team of a school (grades 1 to 5) for helping with the teaching of natural numbers. This request was the pretext for pursuing the reflexion about the teaching of the written- in the Hindu-Arabic and spoken-number systems, learning from recent French didactic research, notably through institutional (Chevallard 1992) aspects.

This paper focuses on the teaching and learning of the spoken- and written-multidigit-numbers by 6-to-8-years-old students (in France, grades 1 and 2). In our paper, "26" is called a *written-number* whereas "twenty-six" is called a *spoken-number* and “2 tens 6 ones” a *numbers-units-number.* Among the knowledge young children have to learn and to understand: how the multidigit-numbers are constructed from a semiotic point of view? How the written-multidigit-numbers and the spoken-numbers denote a determined quantity? We will address the question of the teaching of spoken- and written-numbers between 1 and beyond 100 from a semiotic, didactic and institutional point of view.

Our main question is: what tasks should contain a guide in order to engage 1st-and-2nd-grades-teachers to revamp their teaching of the written- and spoken-decimal systems of numbers?

SOME ELEMENTS ABOUT THE WRITTEN AND SPOKEN NUMBERS

Written numbers: semantic and syntactic aspects

We consider the multidigit-numbers like a language with its semantic and syntactic aspects. Why this point of view? The syntax seems simple, but powerful: any digit-concatenation denote a numeral and a quantity. Each digit of the ordered list 1 to 9 denotes a quantity, one for one object, and the successor, one more than the precedent. For a quantity bigger than 9, 1 moves to the left and a digit 0 takes the right place, following by 11 12 13...: the going on written-numbers list respects again the same progression for the right digit, up to 9; then the digit on the left becomes 2 and the right one follows again the progression from 0 to 9, and so on. It is what we named the recursive aspect of the written-numbers list. A good understanding of written-numbers recursive algorithm permits to order written numbers without understanding the semantic aspects.

With regard to semantic aspect, how to know what quantity denote a multidigit-number? "Written numbers as partially iconic signs" (Ejersbo & Misfeldt, 2011, p.301), we will precise only very little iconic: number of different lengths can be easily compared; but how to understand that 52 is different and major than 25? The place value system owns genius aspects, but these aspects remain not known by young students who often consider 52 as 7.

The specific role played by the digit 0 in the written numbers is a semantic aspect: 0 indicates the lack of an isolated power of ten, e.g. 0 indicates no isolated one in 450, no isolated ten in 203, still there are 450 ones in 450 and 20 tens in 203.

A few words about spoken-numbers in French language and research

The names of the numbers units are different from those of the powers of ten, as summarized in the table below: with units, 423 is 4 "centaines" (4 hundreds) 2 "dizaines" (2 tens) 3 "unités" (3 ones), it is also 4 cent (4 hundreds) vingt (twenty) trois (three). Another difficulty: the same word "unité" is used in a context of units of measure or account, but also to describe the quantity "one".

	1	10	100	1000
English	one the ones	ten the tens	hundred the hundreds	thousand the thousands
French	un, une les unités	dix les dizaines	cent les centaines	mille les milliers

Table 1: Names for numbers and for numbers units in English and French

Mounier (2010) highlights four aspects of the spoken-numbers: "forty six" is 1) four times tens plus six: multiplicative, 2) forty plus six: additive, 3) a place in an ordinate list of words, one, two…, forty five, forty six: ordinal, 4) a place in a list of words (ten, twenty, thirty…) with counting words (one, two, three…, nine), six after forty: ordinal with marks.

NUMBERS UNITS: A DIDACTICAL NECESSITY

To tackle with the signification of spoken- and written-numbers, we stress the introduction of another system of denotation of quantities, using "numbers units" (Chambris 2008): ones, tens, hundreds, etc. - which are units of account - and not only the names of the places of the digits. Freudenthal (1983, p. 90) spoke of "decimal bundling" and "positional arrangement of the bundles", but like Fuson & al (1997), Chambris (2012) emphasizes the necessary and difficult double point of view on the bundling: 10 ones is 1 ten, 1 ten must be understandable as a multiplicity (10) and a whole (1 unit of ten), that Thanheiser (2009 p.253) names "a flexible view of unit types (e.g. ten)". This is an important stage aiming for spoken and written number systems understanding, a part of numbers "profound understanding" (Ma 1999 p. 122).

Why to introduce these numbers units? We synthesize some arguments of Chambris (2008), partially used in Van de Walle (2007). First, they enable teachers to reveal and students to approach the coherent iterative organisation concatenating digits and to estimate the "order of magnitude" of the number. Second, they were used in pedagogic arithmetic theory to describe numerations -for example Condorcet 1794- and in text books prior the new math reform. Third, they enable to express the systemic relations between the different units which are represented by places in a number: 1 ten = 10 ones = 10; 10 tens = 1 hundred = 100 ones = 100... Naturally: the use of numbers units needs the introduction of activities that help students construct the sense of these units. Fourth, they highlight many useful relations for computation: 24+58 is first 7 tens and 12 ones, and thus 82 (this explains the carries in the column operation); for the integer division of 245 by 7: the decimal aspect takes into account that 245 is also 24 tens and 5 ones, 245 doesn't contain only 4 tens, but 24 tens, and thus the quotient is 3 tens and 5 ones. Fifth, in the later teaching they will prepare and strengthen the SI units (length, weight.....). There is a sixth reason to introduce the numbers units: forty six is four tens and six ones, it may help to switch from an ordinal to a multiplicative aspect of the spoken-numbers.

Numbers units own the property to be said and written exactly the same way whereas written-numbers and spoken-numbers are often no congruent in French like in other languages (Ejersbo & Misfeldt, 2011), that is a big problem for teachers when speaking about quantities (Mounier 2010). **Numbers units bridge a gap between spoken- and written-numbers** as a flexible "*base-ten language"* (Van de Walle 2007 p189): standard oral name *forty two*; base-ten oral name *4 tens and 2* **;** standard written *42*; explicit base-ten written *4 tens and 2 ones*. *4 tens and 2 ones is also 3 tens and 12 ones*, etc. This conversion IN the base-ten-language is formed and proved by two partitions (Ross 1989) of the same collection. This is a seventh reason to teach numbers units and it has something to do with the third reason.

THE FRENCH INSTITUTIONAL STATE OF MULTI-DIGIT NUMBERS TEACHING

For US researchers and many teachers (e.g. the readers of Van de Walle 2007) numbers units and their consequences may be very usual. But in France, as we will partially see, it is not: the contexts of national curricula (contents and developments of theses contents), usual teaching and training practises, type of national dominant researches... can be very different. So, we now present how multidigit-numbers are taught at French primary school.

At pre-school, children are used to seeing and reading the written-numbers (usually from 1 to 30) like they are used to the written-language. The French curriculum texts don't give any indications about natural numbers teaching complexity: in the most recent ones (2008) "place value" or "base-ten-number system" is not written anywhere. For example, what concerns expectations:

- 1rst grade-students should "*know (write and name) natural numbers less than 100; produce and recognize additive decompositions of numbers less than 20 (addition table); compare, order, frame numbers less than 100; write numbers in ascending or descending order ; know numbers-less-than-10 doubles and numbers-less-than-20 halves*".
- 2nd grade-students should "*know (write and name) natural numbers less than 1000; place and locate these numbers on a number line, compare them, order them, frame them; write or tell sequences of numbers 10 by 10, 100 by 100, etc; know usual-numbers doubles and halves.*" (MEN 2008 p2)

Further, Chambris (2008), using the anthropological theory of didactics (Chevallard 1992), shows that in the French primary school, from the 1980's, numbers-units have "disappeared". They remain as "names places". As "units", they have been replaced by powers-of-ten, written in figures (1, 10, 100, 1000). For instance, 2x100+3x10 (or even 200+30) has replaced 2 hundreds 3 tens but 3 is still called the tens digit in 230. We don't know what is the institutional state in other countries but there are clues that similar things may have happened at other places: Fuson (1992) wondered why students wrote such writings and indicated that these types of writings do not help to learn the number words; Ma (1999), Thanheiser (2009) let us think that numbers-units may be taught as places, and not as units, in the USA.

Recent French didactic research highlights the weak knowledge of the in-service teachers. It may be a possible consequence of the curricular text that don't focus on these points: the students are taught the positional aspect of the written numeration, 52 is different from 25, they don't have the same position in the usual well-ordered number list 1, 2, 3..., 25..., 50, 51, 52; but the teachers don't systematically emphasize the decimal reason of the order of written numbers: 52 denotes 5 tens and 2 ones while 25 denotes 2 tens and 5 ones. Even in greater numbers, many teachers (especially in the studied school) reduce the tens in a number to the isolated tens, highlighting the tens digit. That teaching can create a didactical obstacle (Brousseau 1997) preventing the students from constructing and/or understanding computation algorithms. When the teachers intend to teach the decimal aspects of the written-numbers, they use the spoken numbers, especially the known order of the multiples of ten: ten, twenty, thirty... (Mounier 2010).

The question is now: how to provide teachers some tasks to teach 1st-and-2nd-grades students numbers-units in relation with the written-numbers and quantities so that they better understand the semantic of written multidigit-numbers? First we must choose materials, easy to find and to manage, giving sense to numbers units. Second we must convince them of the usefulness of the introduction of the numbers units.

HOW TO CHOOSE MATERIALS?

We share many of the "*base-ten ideas*" of (Van de Walle 2007 chapter 12: counting plays a key role, especially the dialectic between counting by ones and by tens.

Unlike Van de Walle (2007, p193) we think we don't let the students choose their way of grouping, we have to impose grouping by ten, even if grouping by two or five may be more spontaneous. Certainly grouping by five and counting orally (by five or by ten...) could be interesting and it is easier to control 5 bundles than 10. But there is a bigger semiotic distance between numbers of fives and decimal standard spoken- and written-numbers: 43 and forty-three is 8 fives and 3 singles and also 4 tens 3 ones which looks more like 43. Moreover, whatever the way of grouping, students need to learn what is *a ten*. Even if they spontaneously group by 5, they need to learn what a ten is -which is not a group, not even a group of ten but a unit of account-. So, they have to learn to count one by one (one, two, three), ten by ten (ten, twenty…) and tens (one ten, two tens…). This last way of counting does not lie in (Van de Walle 2007).

Guitel (1975, p.25) stressed the predominance of usual bases in written numerations: 10, 20, 60 as principal basis; 5, 100 and even 10000 as secondary one. From a historical point of view this common characteristic of the mankind offers a credible hypothesis for the choice of ten in many base-numerations over the world and the times. It is a track to analyse how numeration "is bodily- grounded that is, embodied within a shared biological and physical context" (Nuñez & al 1999 p.46). The English language gives another track: the word *digit* literally means finger or toe too.

Counting activities will be in the heart of our didactical proposition in 1st-and-2nd-grades. Our semiotic approach makes us vigilant on the first material we offer to students, it must integrate and naturalize so far as possible the grouping by tens : we choose the hands fingers.

Many school practitioners suggested and suggest their students to use their fingers in order to denote quantities less than 10 or to construct the different additive decompositions of 10. We agree with Ladel & Kortenkamp (2011, p.1795): the 10 fingers qualify the hands to work out questions about the decimal number system, e.g. "*How many children do we need to see 30 fingers all at once?*": 3 children

Seeing ten (or several tens) as the set of my hand-fingers (or a reunion of several-persons-fingers) would contribute (as mental picture and associated properties and processes) to create "a concrete internal mental image" (Thomas & al. 2002) of the ten unit with its double meaning: a one is a unit of account, but a ten (the ten hand-digits of a person) too, 2 tens is not 2, it is bigger than 2, it is twice ten. It is not easy to exceed 100 fingers, but the amount of one hundred will have a strong image, 10 students raise their fingers, as many students as fingers of the hands of a student. Using special expressions (numbers units), writing relations of one unit to another in the context of fingers and of numbers, expanding formally these relations contribute to reinforce the conceptual understanding of written-numbers.

Lamon (1996, p.88) shows that, for older students faced to in partitioning tasks (aiming the fractions), decomposition of a given all into small units appears quite immediately but unitization into composite pieces appears later. The hand fingers permit to work simultaneously two aspects (decomposition and unitization): showing a quantity of fingers like 32 with few children makes visible 32 singles (the fingers) and its decomposition in 3 tens (the fingers of 3 children) and 2 fingers.

Using first the fingers of the students to describe or produce a quantity owns the three properties (Stacey & al, 2001 p.201) of interesting materials for teaching young children: appeal to the students, accessibility, but especially epistemic fidelity (representing order of magnitude, bundling into tens for numbers under 99, constructing of an incorporated semiotic flexibility of the unit). Later, the students will use other “groupable” material (Van de Walle 2007) like wooden sticks so that they have opportunities to abstract the unit and transfer the ten fingers model in another context.

A DESIGN PROJECT FOR GRADES 1 & 2

The school asking for help is situated in what is named in France "a sensitive zone". The students of these zones suffer simultaneously bad socioeconomic conditions and weaker evaluation results than other schools despite more State helps. We present in this paper the design of a type of activities that respond to certain conditions: short (15 min), repetitive (each day), dynamic, very little expensive in materials and engagement of the students. These activities represent only a part of an entire design about numbers, the part that would consist to embody the concept of unit.

They aim for constructing relation between written-numbers, numbers units and quantities. The usual way the numbers are said in France is not specifically studied in this design. The students are supposed to know (spoken and written) numbers from one to ten at the beginning and also how to realise such sets with their fingers.

Examples of questions that pursue ancient questions with more little numbers:

Type 0: introducing the ten as the first unit that is not a one
The students are asked 1) how many students there are in this group (5, 7, 11, 15, the entire class), 2) to show quantity using their fingers and 3) to write or to say the number if they can. Thanks to the different numbers the students will find a way to show more than ten fingers: several students to produce the quantity... The teacher could say and write: *a student has 10 fingers, a ten of fingers. To show with fingers more than 10, more than one person is needed. Two persons with all fingers up correspond in mathematics with what is named 2 tens,* e.g. *there are 2 tens of fingers and 3 fingers again, there are 23 fingers, there are twenty five fingers.*
This collective activity is aiming for the construction of the unit, a ten, as a multiplicity (the ten fingers) and a whole: the person - with all fingers up - as a ten. The following propositions are pursuing this construction, giving the students more and more individual responsibility in their responses and so, more and more occasion to understand. Two types of collective activities are described: production activities (P), communication (C) ones. In the first ones, it is more the teacher that validates the responses; the second ones enlarge the interaction between students so that they

validate the responses and argue about the reasons. Type E refers to individual training.

Type P1: production of a set from a written- or spoken- already known number
The students form teams of 4, 5 or 6; the teacher writes a digit-number on board and asks students for showing this quantity of fingers - in the list of numbers students know - for example 8, 20, 26, 32.... Each time some configurations are observed, a realisation with the less students as possible will be encouraged.
8: 1 student with 8 fingers up (and 2 fingers down); 8 units
26: 2 students with all fingers up, 1 student with 6 fingers up (and 4 fingers down); 2 tens and 6 units... Progressively the teacher can let the team propose a solution before trying it.

Type P2: producing a way to write a quantity from a set of fingers
The teacher shows fingers and the students are asked for the quantity. The set of fingers is not showed simultaneously, but successively: producing 23 fingers consists on showing ten fingers twice (twice: fists first closed, then fast opened with all fingers up) and again 3 fingers up.
First the teacher shows quickly quantities in the list of known numbers, then quantities of bigger numbers: students can no longer count one by one, or ten by ten (ten, twenty, thirty..), they must quickly count the times when the teacher shows all her fingers raised, therefore they must count the number of tens... and the isolated ones: the constraints of the situation can change their way of thinking (Brousseau 1997)

Type P3: production of a set from a numbers-units-number
Same activities as P1, but a student can produce its set in front of the classmates, showing the tens successively as learned from the teacher in P2. The way to write or speak the number can be either found by students or given by the teacher or be pended. The teacher may conclude the activity with mathematical words and write a lesson:

8	8	*1 student with 8 fingers up*	*8 doigts*	8 unités
26	vingt-6	*2 students with all fingers up,* *1 student with 6 fingers up*	*2 dizaines de doigts et 6 doigts*	2 dizaines et 6 unités

Table 2: What must being remembered

Type C1: communication between students about quantities denotations.
The teacher gives a paper with a number in numbers units to a team, lets students of the team show their fingers (corresponding the quantity) to all the other students who write the response on their slate. The response is then compared to the original that enables to validate the communication, comparing same writings (in number-units) or both writings (in number-units and written).

The same can be played by team, two teams A and B are associated, in the first time, team A receives the number and shows the collection of fingers to team B, then the role are reversed, the teacher looks at the students

After collective discussion the number is written with numbers units, drawn fingers and digits.

3 dizaines et 4 unités	*7 dizaines*
3 tens and 4 ones	7 tens
	
34	70
Trente-quatre	Soixante-dix

The name of the number is said and written, with an oral advertising for "oral irregular" names like *soixante-dix* (notice that number names are less irregular in the Belgian French language: *septante -“seventy”- instead of soixante-dix -“sixty-ten”)* :

The chosen numbers are written by the teacher in canonical form (5 tens and 6 ones), non-canonical form (6 ones and 5 tens OR 2 tens and 16 ones) and will engage specificities: no one (6 tens OR 3 tens and 30 ones) or, more simple, no tens (5 ones).

Type E1: Individual exercises with prepared drawn hands labels and persons:
Labels of a person (conventionally supposed with all the fingers up) or labels of hand are available.
1) A numbers-units-number is given, constructing the fingers collection (with person labels or hand labels) and writing the multidigit number are asked the students. Notice that succeeding using "hand labels" reflects a better understanding of the ten unit. 2) A set of drawn fingers is given, writing the quantity with numbers units and with digits are the questions the students must face to. 3) A written-number is given, constructing the fingers collection (with digits or persons labels) and writing it with numbers units are asked the students.
It would be progressively important that the teacher proposes too "fingers not totally regrouped in hands".

Type C2: activities of communication but with more than ten ones, for example 5 tens and 12 ones. The teacher emphasizes the conclusion: 5 tens and 12 ones is also 6 tens and 2 ones. It is usually written 62, then need all the fingers of 6 persons and again 2 fingers up.

Type E2: Individual exercises

Type C3: activities of communication with more than ten tens, for example 13 tens and 7 ones. The number of persons with all fingers up is 13, and again 7 fingers. *Ten persons with all fingers up form ten tens of fingers, in mathematics a hundred of fingers*. *13 tens and 7 ones* is also *1 hundred 3 tens and 7 ones*, usually written 137.

Type E3: Individual exercises

CONCLUSION

The previous activities have a main goal: to teach the concept of unit -with the instance of ten- betting on the incorporation of the ten as privileged grouping. Using numbers units enable to improve the semantic understanding of 2-digit or even 3-

digit numbers; it also enables to speak numbers without using their difficult names. Concerning number names, the core is to memorize the relation between the tens names and how many tens in the tens names: e.g. four tens (and five) are forty (five).

The activities are expected to give progressively students more and more responsibility in the control of their responses, integrating the role of tens in written- and spoken-numbers.

We are conscious of the lack of experimental facts that would moderate our development. We would just share our a priori thinking. On the one hand, it is based on some research on learning (base-ten concepts, numbers units as mathematical and pedagogical mean...) and teaching (little trace of numbers-units in French curriculum: if they are used in usual practices it is as places...). On the other hand, it takes into account the constraints we had to face to: convincing the teachers of the utility of a new notion (numbers units), proposing short and regular meaning activities, so that they are not afraid to try them and we can examine in true the effects of their integration.

Nevertheless, some questions rose. How written- and spoken-numbers overlap in the teachers practice? How to manage with the problem of the very irregular number words in French -3 stages from 1 to 100: from ten to 10 to 19 (*dix, onze, douze...*), from 60 to 79 (*soixante* to *soixante-dix-neuf*), from 80 to 99 (*quatre-vingts to quatre-vingt-dix-neuf*)?

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LEVELS OF OBJECTIFICATION IN STUDENTS' STRATEGIES

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The aim of this paper is to identify students' strategies while solving tasks which involve the expansion of fractions to a common denominator. In this case study we follow two groups of 11 year old students and their use of the artefact multilink cubes in the solution process. The analysis of the students' strategies is based upon a semiotic-cultural framework. Five different types of strategies are reported: trial and error, factual, contextual, embodied-symbolic and symbolic. The naming of these strategy types is inspired by Luis Radford.

Key words: Artefact, common denominator, strategy and levels of objectification.

INTRODUCTION

A lot of research has been carried out involving embodied cognition and the multimodal paradigm (Arzarello & Robutti, 2008; Gallese & Lakoff, 2005). Such studies also encompass gestures and the use of various artefacts. Within the semiotic-cultural framework, learning has been formulated in terms of objectification (LaCroix, 2012; Radford, 2008), but to our knowledge this theory has not yet been applied to physical artefacts in the learning of fractions. In this paper we follow two groups of 6th grade students who use the physical artefact multilink cubes to solve tasks which involve expanding fractions to a common denominator. We focus on how students use multilink cubes and mathematical signs equipped with a cultural meaning to express and communicate their thinking in social interaction. Radford (2010b) has described mathematical thinking in the following way: "[…] thinking is considered a sensuous and sign-mediated reflective activity embodied in the corporeality of actions, gestures, and artifacts (p. XXXVI)." A main point here is that mathematical thinking entails the use of resources located outside of the brain, and that such resources play an important role in mathematical activity. Radford's theory of objectification (2006) will be a theoretical foundation for our study:

> The term objectification has its ancestor in the word *object*, whose origin derives from the Latin verb *obiectare*, meaning "to throw something in the way, to throw before". The suffix – *tification* comes from the verb *facere* meaning "to do" or "to make", so that in its etymology, objectification becomes related to those actions aimed at bringing or throwing something in front of somebody or at making something apparent − e.g. a certain aspect of a concrete object, like its colour, its size or a general mathematical property (p. 6).

An important point in this theory is that learning is closely connected to actions aimed at noticing different aspects of the mathematical object at hand. We use Radford's definition of a mathematical object (2008): "[…] mathematical objects *are fixed patterns of reflexive human activity incrusted in the everchanging world of* *social practice mediated by artifacts* (p. 222).” This definition emphasises that mathematical objects are patterns of activity closely linked with the use of artefacts. The mathematical object we study is “the expansion of two fractions to a common denominator”. This procedure is a “*fixed pattern of reflexive human activity”*, so it fits well with Radford’s definition of a mathematical object. The theory of objectification is used as an analysing tool in order to identify deeper levels of objectification in the students’ strategies of expanding two fractions to a common denominator. The research questions that guided our work were: “Which strategies do the students employ as they expand two fractions to a common denominator, and what aspects of the expansion process are at the centre of the students’ attention in these strategies?”

We regard Radford’s theory of objectification as the most appropriate for our study compared to other alternatives such as Treffer’s approach of progressive schematisation (1987) because Radford’s theory focuses more on the use of various artefacts, gestures, motor actions and the semiotic resourses the students use in the learning process. His theory also encompasses descriptions of three levels of objectification which fit our data well.

RADFORD’S LEVELS OF GENERALITY

In relation to students’ generalisation of number patterns, Radford has described the factual, contextual and symbolic layer of generality (2006, 2010a, 2010b). Radford also refers to these levels as levels of objectification. Koukkoufis and Williams (2006) have applied these levels to students’ generalisation of the compensation strategy in connection with the “dice games instruction method” (Linchevski & Williams, 1999) which is related to integer addition and subtraction through objects on a model, i.e. red and yellow cubes on a double abacus. In our paper Radfords’ levels are generalised and applied to the students’ strategies of expanding fractions. We will now give a short description of these levels of generality, and we start with the factual level (Radford, 2006). The students he referred to were to generalise a number pattern which was expressed by a visual representation, see figure 1 to the left.

Figure 1: One of the patterns in Radford’s studies.

An example of a way to determine the number of circles in a figure on the factual level of generality was: “One plus one plus three, two plus two plus three, three plus three plus three (ibid., p. 11).” Here the circles were grouped by pointing gestures of the student as described in figure 1 to the right:

No mathematical symbols were used in this generalisation. The crucial element of the generalisation is to express the variable quantity, i. e. the figure number, in some way. On the factual level of generality this quantity is not articulated in a direct way:

"[…] in factual generalizations, *indeterminacy* […] does not reach the level of enunciation: it is *expressed in concrete actions* [...] (ibid., p. 9)."

Like the factual strategy, the contextual strategy also originates from a visual approach to the process of generalisation. An example of a contextual generalisation of the number pattern shown in figure 1 is: "You have to add one more circle than *the number of the figure* in the top row, and add two more circles on the bottom row (Radford, 2010b, p. XLI)" The pivotal element of the generalisation is to express the variable quantity, i. e. the figure number, in some way. On the contextual level of generality, this quantity reaches the level of enunciation. In the example above this was done through the formulation *"the number of the figure".* According to Radford (ibid.), the explicit mentioning of the figure number is a hallmark of the contextual level of generality:

> The indeterminate object variable is now explicitly mentioned through the term "number of the figure." However, […] the new form of algebraic thinking is still contextual and "perspectival" in that it is based on a particular way of regarding something (p. XLI).

On the symbolic level of generality, a formula for the number of circles in a figure is obtained. In connection with the number pattern which is shown in figure 1, an example of a symbolic generalisation is $2n+3$. About this level of generality Radford says (2010a): "The understanding and proper use of algebraic symbolism entails the attainment of a disembodied cultural way of using signs and signifying through them (p. 56)."

METHOD

A lot of research has been carried out in the design of learning sequences in connection with rational numbers, see for example Brousseau's work on didactical situations and didactical engineering (1997), and such studies have inspired the design of the learning activities employed in this case study. However, the design of such learning activities is not a theme in this paper. The case study was carried out in a 6th grade classroom in the autumn of 2011 in Norway. In cooperation with the teacher we selected two groups of three students who were medium to high achievers. We did not choose low achievers because the students would encounter the expansion of two fractions to a common denominator one year before what is normal in Norwegian schools. Every group had 13 sessions of 45 minutes with one of the researches, and all of these sessions were videotaped. The groups were given a problem to solve by the researcher, and afterwards they were asked to explain how they reasoned as they were working with the task. The students used multilink cubes to build rectangular "chocolate bars" to depict different fractions. The brown cubes illustrated brown chocolate, and they corresponded to the numerator. The white cubes illustrated white chocolate, and the total number of cubes, regardless of colour, corresponded to the denominator. The students started to build bars corresponding to fractions like $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{2}{5}$. Figure 2 and 3 show some examples of such bars.

SOME DEFINITIONS

In order to be able to communicate and analyse the students' strategies, we will define the concepts *strip, length, height* and *congruent bars*. These definitions presuppose that the bars are oriented in the same way as in figure 2. A *strip* is a bar with *height* 1. The left part of figure 2 shows a strip that corresponds to $\frac{2}{5}$. We say that the *length* of this strip is 5. If a fraction were to be expanded, the students usually increased the *height* of the bar. The bar to the right in figure 2 is made up of two strips, and this bar corresponds to the fraction $\frac{2}{5}$ expanded by 2. The height equals the expansion factor which is 2. When we use the concept *physical* length or height, we do not mean the number of cubes, but the physical measure of the corresponding distance. Two bars are said to be congruent if the corresponding rectangles are congruent, regardless of the colour of the cubes.

Figure 2: To the left is a strip that corresponds to $\frac{2}{5}$. To the right is a bar that corresponds to $\frac{2}{5}$ expanded by 2.

THE TRIAL AND ERROR STRATEGY TYPE

The students were shown a rectangular piece of cardboard, and they were asked to solve the following task: "If this was a real chocolate bar, and you could choose between $\frac{2}{5}$ or $\frac{1}{3}$ of the whole bar, what would you choose?" They were not able to solve the problem. Then they were given a task which instructed them to build some bars where $\frac{2}{5}$ of the cubes were brown and put them in a heap, and to build some bars where $\frac{1}{3}$ of the cubes were brown and put them in another heap (see figure 3 for an example of such bars). Finally they were instructed by the task to find two congruent bars, one from each heap, and count the brown cubes in the two congruent bars. In this way they found out that $\frac{2}{5}$ is greater than $\frac{1}{3}$. This strategy corresponds to an elementary level of objectification because it is a trial and error strategy, and the students focus on building two congruent bars.

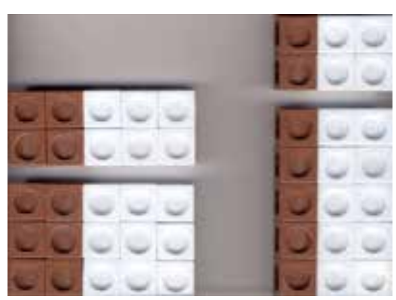

Figure 3: Bars used to order $\frac{2}{5}$ and $\frac{1}{3}$.

THE FACTUAL STRATEGY TYPE

In this section we will describe the factual strategy of expanding two fractions to a common denominator which was frequently used by the students. This strategy was invented by the students without any influence from the researcher, and it was more effective than the labour-intensive trial and error strategy. We will now give an example of this type of strategy. The students were working on the following task:

> Make a chocolate bar where $3/5$ of the chocolate is brown, and make another where $2/3$ is brown. The bars are to have the same size. Which of the bars has more brown chocolate? Which of the fractions $3/5$ and $2/3$ are the biggest?

Mary has built two congruent bars, and she was asked to pretend that she was a teacher and explain to the others what she had done.

Mary: First you build a strip with three fifths *(showing a strip which corresponds to $3/5$, picture 1)*. Then you build on as much as you think it shall be. If I for example build four of these *(picture 2)*. Then you build two thirds *(showing a strip which corresponds to $2/3$, picture 3)* and see whether it fits or not *(picture 4)*. So now you have found out how it should fit *(removes one of the four strips so that the bar corresponding to $3/5$ consist of three strips, picture 5)*. Then you enlarge it *(expanding the strip that corresponds to $2/3$ so that it gets the same size as the bar corresponding to $3/5$, picture 6).*

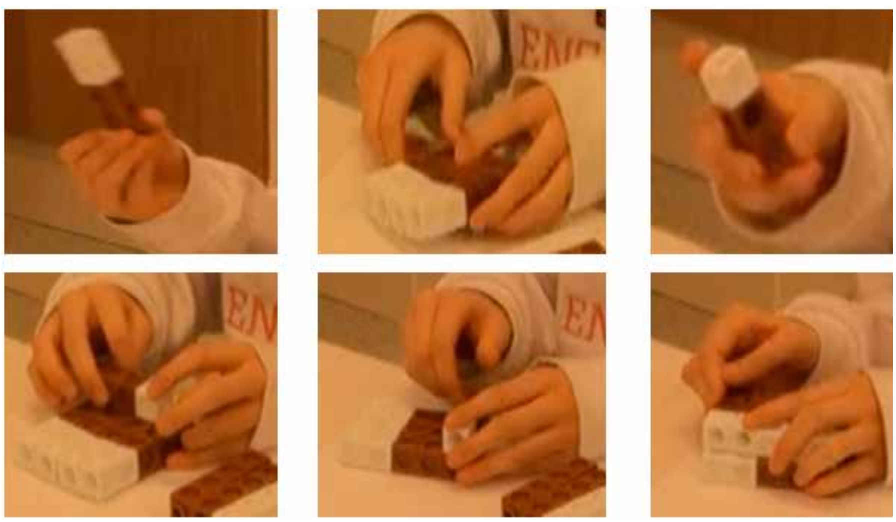

Figure 4: Picture 1 to 3 is on the first row, and picture 4 to 6 on the second.

An important element of the procedure is to find the physical heights of the two congruent bars. In the factual strategy, these heights are found through the physical lengths of the two strips which correspond to the fractions that are to be ordered. The physical heights of the bars are variable quantities in the expansion procedure which are not explicitly articulated, but expressed through actions. Thus the factual strategy reported in this section resembles Radford's factual level of generality. In the factual strategy, the students focused on the physical lengths of the strips which corresponded to the fractions that were to be expanded. This means that a deeper layer of the mathematical structure in the expansion procedure has become apparent to them, and thus they seemed to have reached a deeper layer of objectification.

THE CONTEXTUAL STRATEGY TYPE

In this section we will delineate the contextual strategy of expanding fractions to a common denominator which was often used by the students. This strategy was discovered by the students without any influence from the researcher, and it was an improvement of the factual strategy. The following task was given to the students in the example of the contextual strategy we will now analyse:

> Make a chocolate bar where $\frac{2}{3}$ of the chocolate is brown, and make another where $\frac{5}{7}$ is brown. The bars are to have the same size. Which of the bars has more brown chocolate? Which of the fractions $\frac{2}{3}$ and $\frac{5}{7}$ is the biggest?

Cathie has built two congruent bars, and she has written down an explanation as to how she built the bars which she was asked to read aloud.

Cathie: First you make a strip with five sevenths *(showing a strip that corresponds to $\frac{5}{7}$, picture 1)*. Then you make another one that shall be two thirds *(showing a strip that corresponds to $\frac{2}{3}$, picture 2).* Then you see that on two thirds, that the bottom number is three. So you build three strips with five sevenths *(gliding pointing gesture with the pencil, picture 3)*. Then you see that the bottom number in five sevenths is seven. Then you take seven lengthwise *(gliding pointing gesture with the pencil, picture 4)*.

Figure 5: Picture 1 to 4 is ordered from the left to the right.

In the factual strategy the heights of the two congruent bars were found through the physical lengths of the two strips which corresponded to the fractions that were to be expanded. In the contextual strategy these heights were found through the denominators of the fractions which the students referred to as “the bottom numbers”. These denominators are two variables in the expansion procedure which are now explicitly articulated through the words “the bottom numbers”. Thus the contextual strategy reported in this section resembles Radford’s contextual level of generality. The students focused on the denominators of the fractions which were to be expanded. This means that a deeper layer of the mathematical structure in the expansion procedure has become apparent to them, and thus they seemed to have reached a deeper layer of objectification.

THE EMBODIED-SYMBOLIC STRATEGY TYPE

At this stage of the objectification process the students had never encountered the expansion of fractions with mathematical symbols. Therefore the researcher showed them how this could be done and the connection between symbolic expansion and the expansion of bars. After this the students started to add fractions with different denominators, and the embodied-symbolic strategy arose. We will now give an example of this type of strategy. The students were working on the following task:

> One Saturday Peter makes a pizza for himself and his friends. When they have eaten, there is $\frac{1}{3}$ pizza left which he puts in the freezer. The next Saturday he also makes a pizza for his friends. Then there is $\frac{2}{5}$ pizza left which he puts in the freezer. How much pizza has Peter frozen after the two Saturdays? *(The task was accompanied by a picture of two rectangular pizzas of equal size.)*

Cathie has solved the task by building two congruent bars, and she has also carried out the calculation by writing down mathematical symbols, see figure 6. Then she was asked to explain what she had done.

$$\frac{1}{3}+\frac{2}{5}=\frac{1\cdot 5}{3\cdot 5}+\frac{2\cdot 3}{5\cdot 3}=\frac{5}{15}+\frac{6}{15}=\frac{11}{15}$$

Figure 6: Cathie's calculation of $\frac{1}{3}+\frac{2}{5}$.

Cathie: First I wrote one third plus two fifths. Then the equal sign. Then I wrote one third again with a long fraction line. Then I counted how many I had to expand it to, which was five *(gliding pointing gesture along the bar that corresponds to $\frac{1}{3}$, picture 1)*. Then I wrote times five over and under the fraction line. Then plus two fifths. Then I counted how many strips I had expanded it to *(gliding pointing gesture along the bar that corresponds to $\frac{2}{5}$, picture 2)*, which was three. Then I multiplied that *(pointing at $\frac{1\cdot 5}{3\cdot 5}$ on her sheet)* which became five fifteenths. Plus that *(pointing at $\frac{2\cdot 3}{5\cdot 3}$ on her sheet)* which was six fifteenths. Which equals eleven fifteenths.

Figure 7: Picture 1 and 2 is ordered from the left to the right.

In the factual and contextual strategy the students' attention was directed at how to build two congruent bars, but now the building procedure is no longer in focus, and the students gave no explanation as to how they built the congruent bars. Instead their attention was directed at the connection between the bars and the mathematical symbols. The expansion factors in the symbolical representation of the procedure were found through the heights of the two congruent bars. A new aspect of the expansion procedure had become apparent to them, and consequently they seemed to have reached a deeper level of objectification.

THE SYMBOLIC STRATEGY TYPE

After some time the students discovered that it was not necessary to build bars in order to add fractions with different denominators. The following task was given to the students in the example of the symbolic strategy we will now analyse:

> Each of the two brothers Bill and Benny won a chocolate bar at the charity bazaar. Bill gave away $\frac{2}{3}$ of his chocolate to his mother, and Benny gave away $\frac{1}{5}$ of his chocolate to his mother. How much chocolate did their mother get? *(The task was accompanied by a picture of two rectangular chocolates of equal size.)*

Peter had on his own initiative solved the task without using the multilink-cubes, and he had written down the mathematical symbols shown in figure 8. He had also written down an explanation of what he had done and was asked to read it aloud.

$$\frac{2}{3} + \frac{1}{5} = \frac{2 \cdot 5}{3 \cdot 5} + \frac{1 \cdot 3}{5 \cdot 3} = \frac{10}{15} + \frac{3}{15} = \frac{13}{15}$$

Figure 8: Peter's calculation of $\frac{2}{3} + \frac{1}{5}$.

Peter: First I wrote two thirds plus one fifth which equals two thirds with a long fraction line. Then I put times five on top and bottom because the denominator of the other fraction was five. Then I wrote plus one fifth with a long fraction line, times three because the denominator of the other fraction was three.

The students' attention was removed from the interplay between the bars and the mathematical symbols, and they focused only on the symbolic representation of the calculation. There was no longer any reference to the building process. Consequently they seemed to have reached a deeper layer of objectification. The symbolic strategy reported in this section resembles Radford's symbolic level of generality (Radford, 2010a) because it "entails the attainment of a disembodied cultural way of using signs (p. 56)".

CONCLUSION AND FURTHER RESEARCH

Our starting point was the following research questions: "Which strategies do the students employ as they expand two fractions to a common denominator, and what aspects of the expansion process is at the centre of the students' attention in these strategies?" At different stages of the objectification process, different aspects of the expansion procedure have been at the centre of the students' attention. In the trial and error strategy, the students' attention was directed at building two congruent bars, but this procedure was rather labour-intensive because they did not know how to find the heights of the two congruent bars. In the factual strategy the students focused on the physical lengths of the strips that corresponded to the fractions that were to be expanded. These physical lengths were used to find the physical heights of the two congruent bars that were central in the process of expanding the two fractions. In the contextual strategy, the students' attention was directed at the denominators of the two fractions. These denominators were used to find the heights of the two congruent bars. In the embodied-symbolic strategy, the students focused on the interplay between the two congruent bars and the mathematical symbols used to represent the procedure. The expansion factors in the symbolic representation of the procedure were found through the heights of the two corresponding congruent bars. In the symbolic strategy, the bars were no longer used, and the students' attention was directed at the mathematical symbols representing the calculation. In this case the expansion factors were found through the denominators of the fractions. These strategies correspond to different levels of objectification, and on these levels the students relate to the structure of the mathematical object – "the expansion of two fractions to a common denominator" – in more sophisticated ways.

In addition to generalising and applying Radford's levels of generality to the students' strategies of expanding fractions to a common denominator, we have also described the trial-and-error and the embodied-symbolic strategy. Because the two latter strategies were induced by the researcher, we have no evidence which implies that these strategies can be used in connection with other artefacts and other mathematical subjects. However, our hypothesis is that the factual, contextual and symbolic level of generality described by Radford also can be used as an analysing tool in connection with other artefacts and other mathematical subjects. The fact that Koukkoufis and Williams (2006) describe these types of generalisations in connection with integer addition and subtraction support this hypothesis. Still, there is a need for more research to elucidate to what degree these levels of generality might be applied to other fields.

The transition to the embodied-symbolic strategy was induced by the researcher because the researcher had shown the students the connection between the expansion of bars and the expansion of fractions with mathematical symbols, but the other transitions to higher levels of objectification were not initiated by the researcher. Because the building of the congruent bars entailed a lot of work, and the expansion process was repeated many times by the students, our hypothesis is that these transitions arose in order to carry out the expansion process in more efficient ways. Still, the transitions to higher levels of objectification remain somewhat opaque. We need to know more about how to design learning activities which may facilitate such transitions. Interesting questions for further research are: "What is the role of the teacher in connection with transition to higher levels of objectification, and which actions are suitable to throw a new aspect of the mathematical object into the centre of the attention of the learner?"

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MENTAL COMPUTATION STRATEGIES IN SUBTRACTION PROBLEM SOLVING

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This paper refers to part of a qualitative study made by the first author that had the main goal to understand which mental computation strategies are used by first grade pupils in addition and subtraction problem solving, namely to understand how the different addition and subtraction situations influence the mental computation strategy used in its resolution.

In this paper, we will present and discuss the strategies used at different subtraction problem situations by a pupil, Cátia, that constituted one of the three case studies held in the large study. The strategies used by Cátia seem to be related with the subtraction situations of the problems.

Keywords: number sense, mental computation, subtraction, strategies, problem solving.

INTRODUCTION

Mental computation is closely connected to one major goal in mathematics education of the elementary years: the development of number sense (e.g. NCTM, 2007).

Sowder (1992) associates number sense to an intuition and defines it as a well organized, conceptual network that allows the relationship between numbers, operations and its properties, and a flexible and creative way of problem solving. Similarly, Dehaene (1997) refers to number sense as an intuition about numerical relations, describing it as "a short-hand for our ability to quickly understand, approximate, and manipulate numerical quantities" (Dehaene, 2001, p.17).

McIntosh, Reys and Reys (1992) describe number sense as:

> "a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful strategies for handling numbers and operations" (p. 3).

Thus, in addition to other aspects, a good number sense implies a thorough and flexible understanding of numbers and their relationships, essential for the development of efficient and useful strategies, like mental computation strategies.

THEORETICAL FRAMEWORK

The importance of mental computation in the development of number sense is highlighted by several authors, but what is mental computation?

Buys (2008) briefly describes mental computation as “moving quickly and flexibly through the world of numbers” (p. 122) and characterizes it as a computation involving: a) numbers and not digits; b) computation properties and number relationships; c) a good understanding of numbers and a thorough knowledge of elementary facts up to twenty and up to one hundred; and d) the use of intermediate notes, but mainly calculating mentally. Verschaffel, Greer and De Corte (2007) add that “it’s not the presence or absence of paper and pencil, but rather the nature of the mathematical entities and actions that is crucial in our differentiation between mental arithmetic and (written) algorithms” (p.566). Should be noted that this is the understanding of mental computation we will consider throughout the paper.

The subtraction strategies used by children depend on and evolve from strategies used in this operation with numbers smaller than twenty (Fuson, Wearne, Hiebert, Murray, Human, Olivier, Carpenter & Fennema, 1997). Thus, in this field of numbers, Thompson (2009) refers the following subtraction strategies: i) count out; ii) count back from; iii) count back to; iv) count up; and v) use known subtraction facts and use derived facts. Included in the strategies with known numeric facts, the author emphasizes the importance of strategies of jumping via ten.

For subtraction of numbers higher than twenty, in Dutch literature different types of strategies[1] are identified, organized into two categories: N10 and 1010 (e.g. Beishuizen, 2009).

In the N10 category (number+10 or number-10), at the first number is subtracted a multiple of 10. In this category, distinguishes itself a more complex strategy, N10C (compensation), in which at the first number is subtracted a multiple of 10 approximated to the second number to facilitate the computation. The result is then compensated. Another type of strategy, belonging to the N10 category, is identified as A10 (adding on). In this strategy at the first number is subtracted a part of the second number, so that the result is a multiple of 10. Then, the remaining part of the second number is subtracted.

In 1010 category, numbers are split into tens and units that are subtracted separately and the final result is obtained through rearrangement of the number. A 1010’s variant is 10S (sequencial), in which the numbers are initially split into tens and units that are subtracted sequentially. Beishuizen (2009) refers that 1010 strategy may cause conflict in computations like 74-38, because pupils cannot be able to solve 4-8 and wrongly compute 8-4. The author adds that the difficulty of this type of strategy

[1] Conscious that perhaps we are using the term “strategy” to what Beishuizen refers as computation procedures, in this paper, this term refers to the strategies as N10, 1010 and their variants.

it's not the decomposition procedure but the correct rearrangement of numbers. According to the same author, N10 strategy is less vulnerable to these mistakes, so it is more efficient. However, its use requires a good ability when subtracting multiples of 10 from any number.

According to empirical research data presented by several authors (e.g. Beishuizen, 2001; Carpenter, Franke, Jacobs, Fennema & Empson, 1998; Thompson & Smith, 1999), pupils seem to prefer strategies from N10 category when solving subtraction computations. Furthermore, the success when using N10 strategies to solve subtraction computations is higher than with 1010 strategies. This last aspect, as Beishuizen (2001) states, seems to confirm the frailty of 1010 strategies, particularly regarding the lost of number sense while using the strategy.

In the elementary years, contexts provide the basis for computation (Treffers, 2008) and the support of the pupils' thinking (Ministério da Educação, 2007). For this reason, different subtraction situations were chosen to provide the basis for the use and development of mental computation strategies. There are different subtraction situations and, in this paper, we consider the situations presented by Ponte and Serrazina (2000): i) *take away*: part of a quantity is removed; ii) *compare*: two quantities are compared in order to find the difference between the two; and iii) *complete*: a value is found in order to add to a quantity so that a specific number is obtained.

METHODOLOGY

This paper refers to part of an empirical research made by the first author, as part of a master's dissertation, that had the main goal to understand how first grade pupils develop mental computation strategies, in an addition and subtraction problem solving context. To do that tried to answer three questions: a) Which mental computation strategies do pupils use when solving addition and subtraction problems?; b) How do these strategies evolve?; and c) Do the addition or subtraction problem situations influence the mental computation strategy used in its resolution?

In this paper, we will focus on mental computation strategies used in different subtraction problems.

The study was a qualitative one and three case studies were carried out. Data were collected by the first author in her first grade class, in a private school in Lisbon. Two problem chains[2] were solved in pairs[3] by pupils, between January and June 2010. All

[2] The word "chain" is use to identify the set of problems that were developed by the researcher and solved by the pupils. These sets of problems are identified as a problem chain because they were design to cover all the different subtraction problem situations. Also, the numbers involved in each problem were thoroughly selected so that they were progressively higher, increasing the difficulty of the computations.

[3] The pupils' pairs varied according to how the work was usually developed in the classroom.

problem solving lessons had the following moments: i) presentation of the problem, in which it was read by a pupil and possible doubts were clarified; ii) solving the problem in pairs, where pupils were asked to write down their strategies, through words, computations, using the empty number line, etc.; iii) presentation and discussion of the most significant solving strategies for the whole class; and iv) overview and identification of the most efficient strategies. A third and final problem chain was solved individually and outside the classroom by the three pupils who constituted the case studies, at the beginning of the second grade, in October 2010.

The study data were collected using video and audio recording (data from all lessons was fully transcribed), participant observation, pupils' records and field notes.

In the three problem chains there were 13 subtraction problems, covering the different subtraction problem situations. Table 1 shows the evolution of the magnitude of the numbers selected for each problem, as well as the presence of subtractions with and without regrouping, and the operations with values involving a different digit number. It is important to note that in the Portuguese curriculum there is no limit for the magnitude of numbers that should be worked in the first grade, and that, along the study, numbers were selected depending on the pupils' progress.

Problem chain	Subtraction situations	Computation	
1	*Compare* a)	20-6	subtraction
	Take away	15-7	subtraction
	Complete a)	28-16	subtraction without regrouping
	Complete	25-18	subtraction with regrouping
2	*Complete*	55-32	subtraction without regrouping
	Take away	49-26	subtraction without regrouping
	Compare	42-14	subtraction with regrouping
		75-48	subtraction with regrouping
	Take away a)	82-36	subtraction with regrouping
	Complete	124-47	subtraction with regrouping
3	*Compare* a)	157-43	subtraction without regrouping
	Take away	257-125	subtraction without regrouping
	Complete a)	250-135	subtraction with regrouping

a) Problems that will be described and discussed in this paper.

Table 1: Computations involved in the subtraction problems (presented by temporal order)

Content analysis was done and the categorization of computation strategies with numbers up to twenty referred by Thompson (2009) were followed, as well as the mental computation strategies with numbers higher than twenty identified by Beishuizen (2001, 2009) and Beishuizen and Anghileri (1998).

For this paper we selected five problems (indicated in Table 1) to describe and discuss the computation strategies used by Cátia, one of the three studied pupils. Cátia is a very confident pupil that has a successful academic performance not only at Mathematics but also in other curriculum areas. The problems selection was made in order to include the different semantic situations and also regrouping/not regrouping and magnitude of the numbers.

The transcriptions that will be presented were from the audio and video recordings of the lessons. During the lessons, the teacher sometimes approached Cátia and her colleague, questioning the strategy used, like she did to the other pairs of pupils in the class.

RESULTS

Problem one, 1st chain *(compare)* – The sister of Leonor and Rita is 20 years old. How many years older is she? (Leonor and Rita are twins from the class and are 6 years old.)

This was the first problem with a *compare situation* solved in the study. Before making any notes, Cátia looked around, thoughtfully, and then said "Is 14". Then, she used the number line to write down her strategy. She used a jumping via ten strategy, adding 6+4=10, and then she used basic number facts to reach 20. Finally, Cátia added the partial results (Figure 1), as she explained:

Cátia: I did like this… I know that 6 plus 4 is 10. Then I did a jump of 3 that was 13.

Teacher: Why did you make a jump of 3? Why not a jump of 4 or 2…?

Cátia: Because… I decided to do one of 3 because I thought it was a good computation to do. Then I did 4 plus 3 that is 7. Then I did another jump of 3, that it was as if this 1 [from 13] didn't exist. Then I did 7 plus 3 that was 10, and then it was just plus 4 and 10 plus 4 is 14.

Figure 1. Cátia's solution of problem one

Problem two, 1st chain *(complete)* - Marta is reading a book. She has already read 16 pages and the book has 28. How many pages are left?

Cátia used an additive A10 strategy, approaching 16 to a multiple of 10, 20. Then, she added 8 (20+8), which is a basic number fact for her.

Cátia explained her strategy to her colleague:

Cátia: Pretend that 16 was 6, and 20 was 10. I know that 6 plus 4 is 10. Then I did a jump to 28 and saw it was a jump of 8. And 8 plus 4 is 12.

Cátia didn't feel the need to write down all the numbers in the number line, she just marked the numbers of her computations (Figure 2). As in the previous problem, Cátia solves the problem adding to the given parcel a number to reach the other parcel (a+?=b) and, once again, she uses the number line to show her strategy.

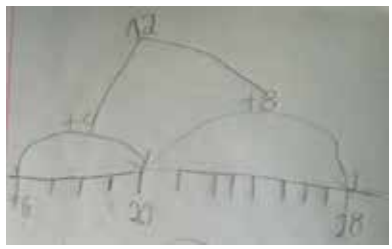

Figure 2. Cátia's solution of problem two

Problem three, 2nd chain *(take away)* - Leonor and Simão are playing a boardgame. Leonor is in house number 82. In that house she reads "How unlucky! You have to go back 36 houses." In which house is she now?

Cátia and her colleague identified easily the subtraction involved in the problem. Once again, Cátia uses a subtractive1010 strategy and subtracted without difficulty 80 minus 50. In the subtraction 2 minus 6, she splits the 6 into 2 plus 4:

Cátia: I've splitted the 6 into 2 plus 4, and 2 minus 2 equals zero… I've already spent this 2. Then, zero minus 4 is negative 4.

She seems to overcome easily the difficulty in 2-6.

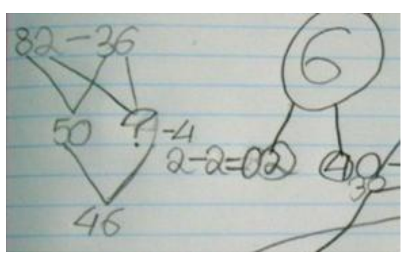

Figure 3. Cátia's solution of problem three

Cátia wrote her notes when she was talking to her colleague about what she was doing, so it is not clear if these notes were used just to show her initial strategy or if she was solving the computation as she was writing.

Problem four, 3rd chain *(compare)* - Miguel and Cláudia are playing the "Stop or Go" game. At the end, Cláudia got 157 points and Miguel got less 43 points than her. How many points did Miguel get?

Cátia starts to solve the problem using a subtractive 1010 strategy. She started to subtract the tens, then the units and finally the hundreds. However, Cátia did 100 minus 14. Unsure of the result, she added 86+43 using an additive 1010 strategy, obtaining 129.

Then, she recalculated 157 minus 43, now adding the partial differences, 100 and 14 (second computation from the right, in Figure 4), explaining:

Cátia: I tried…but I didn't know if I had to add or to subtract [first computation from left, in Figure 4]. I've subtracted but it wasn't right. Now I'm doing it again and now it's right.

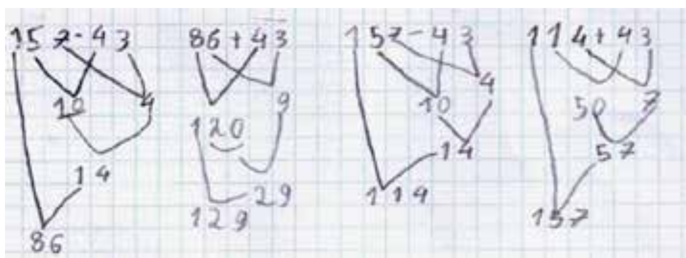

Figure 4. Cátia's attempts when solving problem four

Although Cátia reveals difficulty when rearranging the number after using a subtractive 1010 strategy, she also shows a good comprehension of the relation between addition and subtraction, using that knowledge in order to overcome her difficulty.

The role that Cátia's notes had in this problem seems unclear. As in the previous problem, it is not possible to understand if the notes were used to explicit her mental computation strategy or if she was using them to compute as she was writing.

Problem five, 3rd chain *(complete)* - Leonor went to a bookstore. That bookstore was doing a competition: the customer number 250 who enters the store wins a book collection of his choice! Leonor was the customer number 135. How many customers must enter the store until the prize is awarded?

Cátia uses an additive A10 strategy, adding parcels to 135, in order to approach 250 (Figure 5). In these partial additions, Cátia approximates the partial results to reference numbers, using basic fact numbers.

In this problem, her intermediate notes were important to write down the partial additions so she could add them to obtain the final result (115).

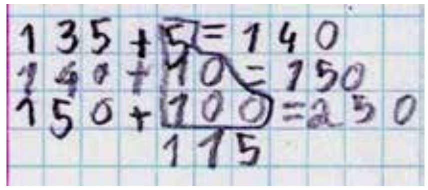

Figure 5. Cátia's solution of problem five

DISCUSSION

As seen in the results described above, the *compare* and *complete* problems were usually translated by Cátia as an expression like a+?=b, and solved mainly through additive A10 strategies. Generally, these strategies were also used by Cátia in the other *compare* and *complete* problems of the larger study. These results are consistent with the findings of several studies (Carpenter et al.,1998; De Corte & Verschaffel, 1987; Heirdsfield & Cooper, 1996) that identify this type of strategy as the most frequently used by children when solving this kind of subtraction problems. This probably is related to the semantic context involved in the problem, for example, in the first problem, the question was "How many years older is she?", seems to lead to a question as "How many more years does she have?" which is usually solved through an addition operation.

The semantic context could help explain why Cátia used a subtractive 1010 strategy in a compare problem, the fourth problem analyzed. In this problem, it was said that "(…) Miguel got less 43 points than her.", that could lead to a subtraction operation, also influencing the strategy used.

Also, in this problem it is possible to identify the weakness of 1010 strategy in subtractions with values represented with a different digit number, which led to an incorrect rearrangement of the final result. As Beishuizen (2001) stresses, this fact is due to the lost of number sense during the computation. However, Cátia overcame this difficulty through her critical analysis towards the result, using the relationship between addition and subtraction to verify the result.

In the *take away* problem, Cátia used the subtractive 1010 strategy with comprehension and without difficulty even in those situations that could cause conflict (e.g. 82-36). In the other *take away* problems, from the second and third chain, in the large study, Cátia had used this strategy with the same easiness, what seems to indicate the comprehension that she has about subtraction, particularly about the lack of commutativity of this operation. It also seems to show her understanding and mastery of negative numbers, that Thompson (2000) relates to the students with more proficiency at computation.

The notes used by Cátia had an important role in this research, as they revealed the mental computation strategy used, however, in some problems, even though all the data have been carefully analyzed, it is not clear if the use of the notes influenced the strategy actually used by the pupil.

Nevertheless, the use of notes seems to be helpful as a way of representing an inner process, as it is mental computation, and also as a way to use important supports, like the empty number line, that could help in the developing of mental computation.

Note that the mental computation strategies used by Cátia, and by the other two pupils in the large study, when solving the subtraction problems are associated in the literature to older pupils (e.g. Beishuizen, 2001; Buys, 2001; Cooper, Heirdsfield & Irons, 1995; Thompson & Smith, 1999). This aspect seems to indicate that mental computation strategies used by first grade pupils should be valued and the teacher must support their development to more complex and efficient strategies, also supporting their comprehension of numbers and operations, essential to the development of a sound number sense.

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FOCUSSING STRUCTURAL RELATIONS IN THE BAR BOARD – A DESIGN RESEARCH STUDY FOR FOSTERING ALL STUDENTS' CONCEPTUAL UNDERSTANDING OF FRACTIONS

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For fostering conceptual understanding of fractions, connecting different representations is an often used design principle. This paper shows that this design principle is necessary but not sufficient and should be complemented by focussing structural relations. In a design research study in grade 6, different strategies for this principle were developed and empirically investigated with respect to the generated learning processes. The activities were organized around the so-called fraction bar board, a visual model that allows a comprehensive structural view on fractions, order and equivalence.

Keywords: Fractions, structural relations in pictures, design research study

Students' difficulties with conceptual understanding of fractions have often been shown empirically (e.g., Hasemann, 1981; Behr et al, 1992; Aksu, 1997). As a consequence, fostering students' conceptual understanding of fractions has become a central aim for many curriculum design projects all over the world (e.g., Bokhove, et al., 1996; Cramer et al., 2009; Streefland, 1991; Prediger et al., 2013). Before developing algorithmic skills, students should develop meanings for fractions and for the basic fraction concepts like order and equivalence of fractions. These curriculum projects vary in the chosen contexts and in their priorities for different fraction models: Most emphasis is usually given to the part-whole model, whereas varying priority is attributed to other important models like measure, ratio, operator, or quotient in situations of equal sharing (see Behr et al., 1992, for an overview on models). Despite of different choices of prioritized models and of contexts, nearly all curriculum projects share the design principle of *relating multiple representations,* namely graphical, symbolic, verbal and enactive representations (Lesh, 1979). Most of them also refer to the principles of *including students' everyday experiences* by means of suitable contexts and *initiating mathematical discussions* on mathematically *rich open problems* (Freudenthal, 1983).

This paper intends to show that connecting different representations is a necessary but yet not sufficient condition for developing conceptual understanding, because especially weaker students tend not to construct the mathematically intended structural relations automatically. That is why the design principle *explicitly focussing structural relations* should complement the set of design principles. The empirical findings that support the importance of this principle and design strategies for implementing it for the topic 'order and equivalence of fractions' are drawn from several design experiments that were iteratively conducted within the long term design research project KOSIMA (Hußmann, Leuders, Barzel, & Prediger, 2011).

1. THEORETICAL AND EMPIRICAL STARTING POINTS

1.1 Fraction bars as important visual model

As developing conceptual understanding of mathematical objects necessitates to *relate different representations* (Lesh 1979), the concrete choice of contexts and concrete graphical representations is crucial for the mental models that students can develop. In line with many other curriculum projects (see above), our design approach starts with identifying and interpreting fractions in a variety of situations of equal sharing and part-whole situations, being graphically represented in rectangles, circles, bars and other pictures. In the second unit on order and equivalence of fractions (Prediger et al., 2013), the learning arrangement focuses on *fraction bars as the central visual model* that has to be connected to symbolic and verbal representations (bars are also used by others, e.g. van de Walle & Thompson, 1984; Cramer et al., 2009; Bokhove et al., 1996).

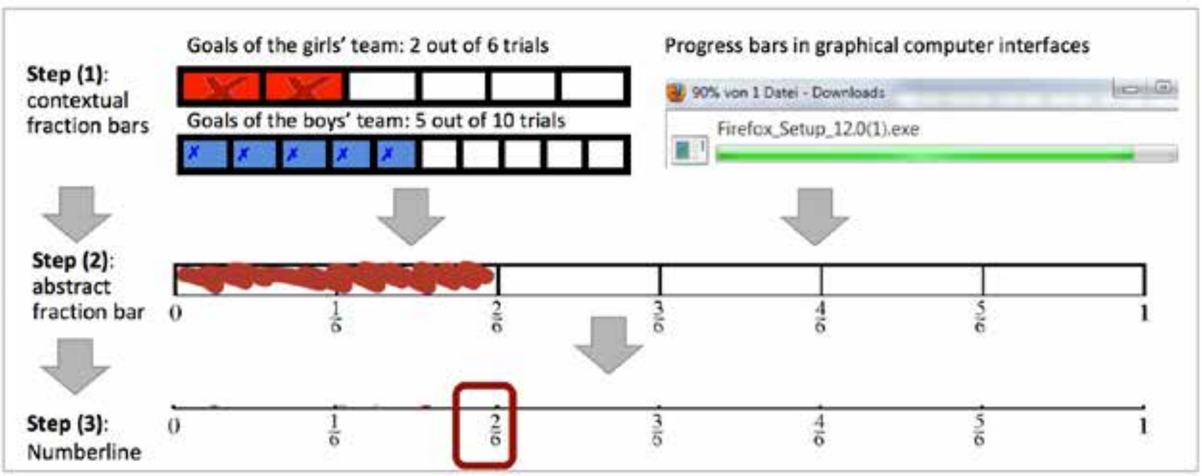

Figure 1. Three steps from contextual fraction bars to abstract fraction bars to the number line

In our curriculum, two everyday context situations support the introduction of fraction bars (Figure 1): (1a) Comparing goals for an unequal number of trials, a *situation* which provokes students to invent the mathematical concept of relative frequencies for measuring fairly, (1b) progress bars of graphical computer interfaces, showing e.g. the progress of a download. (2) From these context situations, we derive an *abstract* fraction bar as the central representation for order and equivalence. (3) In the last step (not treated in this paper), we abstract the fraction bar to the number line and use it as a bridging tool between part–whole and measure model (cf. Keijzer & Terwel, 2003, p. 288). These bridging functions are the main reason for choosing bars as main visual model.

1.2 Limits of graphical representations – a snapshot from an initial case study

Interpreting fraction bars is not evident for all students from the beginning, as could be shown in a case study on low-achieving students (Prediger, 2013). The two boys Cavit and Ismet from grade 7 followed a typical German curriculum with a focus on relating representations, but without consequent focus on meaning. In the conducted interview, both boys easily assigned the symbolic representations 3/4 and 3/5 to given fraction bars. In the transcript, Cavit explains how to find 3/5 in the bar:

9	Interviewer	... what shows you the fraction in this picture?
10	Cavit	Ehm...Ehm the ... first, you have to count the small pieces. These are five. And then, the three coloured ones, three fifth.

In spite of this correct answer, Ismet and Cavit compare the symbolic fractions 3/5 and 3/4 by commonly deciding that 3/5 was bigger (in the next sequence). The interviewer asked them to use the graphical representation to validate their judgement:

17	Interviewer	... how can you see in the picture that this fraction (*hints to 3/5*), when you say, it is greater than this *(hints to 3/4)* How can you see that in the picture?
...		
24	Ismet	[...] because here *(hints to the 3/5 bar)* it is five, and here are four *(hints to the 3/4 bar)* then you see that this *(the 3/5 bar)* is large and this *(the 3/4 bar)* is small [...]

Although line 10 showed the boys' capability to switch between the graphical and the symbolic representations without mistakes, they order the fractions idiosyncratically. Ismet's explanation in line 24 shows that relating representations does not guarantee to see the *mathematically intended structural relations* in a graphical representation: Instead of comparing the length of the coloured part, Ismet counts the pieces and argues that 3/5 is larger because its whole is divided into more pieces than the one for 3/4. This limit of graphical representations has also been found for other arithmetical topics (Steinbring, 2005): Teaching students to draw correctly is not sufficient to guarantee that they mentally construct the intended structural relations which mathematicians see into a specific graphical or symbolic representation. Cavit and Ismet could identify the right fraction bar and described the drawing procedure. However, their linguistic expression of the *relation* between the part 3 and the whole 5 was restricted to the word "and" (line 10). Ismet's idiosyncratic interpretation of the order and this vague expression are indicators for the structural relation between part and whole not being completely mentally constructed. Hence, further support is needed (cf. Prediger 2013).

The short case study illustrates the necessity to understand 'fraction' as a relational concept (Steinbring, 2005), in the sense that it grasps the structural relation between the part and the whole. Although many students mentally construct this relation simply from dealing with graphical representations, the same can be challenging for low-achieving students (similarly Moseley, 2005). From these empirical and epistemological starting points, the following design challenge and empirical research question was concluded for a design research study aiming at overcoming the limits of graphical representations:

How can we foster all students' mental constructions of the intended structural relations between part and whole by initiating activities with fraction bars?

2. METHODOLOGY OF DIDACTICAL DESIGN RESEARCH

Mathematics education research is sometimes dichotomised by *two different aims*: 1. *designing* concrete (teaching-)learning arrangements for mathematics classrooms, and 2. *understanding* and explaining teaching-learning processes. More and more researchers aim to overcome this unfruitful dichotomy and to combine empirical research and the design of learning arrangements in order to advance *both*: practical designs and theory development (e.g. van den Akker et al., 2006). In our research group, we follow the programme of Didactical Design Research as formulated by Gravemeijer and Cobb (2006) which combines the concrete design of learning arrangements with fundamental research on the initiated learning processes. By *iterative cycles of (re-)design, design experiment and analysis of learning processes,* it focuses on both: 1. creating prototypes of learning arrangements and their underlying theoretical guidelines (design principles and strategies), and 2. elaborating on an empirically grounded *subject-specific local instruction theory* that specifies the epistemological structure of the particular learning content, students' learning pathways, typical obstacles in these pathways, and conjectured conditions and effects of specific elements of the design (Prediger & Schnell, 2013; Gravemeijer & Cobb, 2006, p. 21). For investigating processes initiated within the designed learning arrangements, design experiments have proven to be a fruitful method of *data collection* (cf. Komorek & Duit, 2004; Gravemeijer & Cobb, 2006). We usually start by laboratory settings with 2-4 students as this allows in-depth insights into individual, context-specific learning pathways, obstacles or individual prerequisites. Once the arrangements have proven suitable to initiate the intended learning processes, the experiments are widened to classroom settings with regular teachers and normal resources for investigating their robustness under varying pedagogical conditions.

For *data gathering* in the topic 'order and equivalence of fractions', we successively conducted 31 design experiment series in laboratory settings (in sum n = 69 students), each with 2-6 sessions. Additionally, long-term classroom experiments were conducted with six classes (n = 123 students) and their regular teachers, encompassing about 36 sessions for the whole fraction curriculum (on basic concepts, order and equivalence for 5 sessions, then addition, multiplication etc.) All design experiments in laboratory settings were videotaped. The data corpus includes the videos, transcripts of selected video-sequences, teaching materials and students' products. The *data analysis* of the complex process data requires interpretative qualitative methods that are specified according to the research interest in each phase of the process (Prediger & Schnell, 2013). Due to space limitations, the complex, iterative design research process cannot be reported here. Instead, some selected snapshots are presented that are chosen to illustrate three design strategies for implementing the design principle 'focussing structural relations' which do not only refer to the part-whole-relationship but also to relations between fractions that appear for order and equivalence. Each of the three strategies is illustrated by one selected activity and typical moments in the learning process. These snapshots intend to contribute to a local instruction theory for fostering all students' mental constructions of structural relations for fractions, order and equivalence.

3. DESIGN STRATEGIES FOR FOCUSSING STRUCTURAL RELATIONS

3.1 Constructing relevant structural relations by contexts and systematic variation

The case of Ismet and Cavit suggests that comparing fractions might be an important activity for constructing relevant structures in the part-whole model. The *context* of computer interface progress bars (see Figure 1) was introduced into the material after analysing these initial cases. In later design experiment cycles, this connection between progress bars and the symbolic and verbal representations proved to be useful for students constructing the intended structural relations, because students activate everyday experiences and argue for example, “no, 3/5 cannot be bigger, it has less downloaded”.

In addition to these contextual supports, we drew back on the design strategy of *systematic variation* emphasized by Duval for relating representations structurally: “It is only by investigating representation variations in the source register and representation variations in a target register, that students can at the same time realize what is mathematically relevant in a representation, achieve its conversion in another register and dissociate the represented object from the content of these representations.” (Duval 2006, p. 125).

One example for a systematic variation activity is printed in Fig. 2. It was designed to support especially low-achieving students to construct the *quasi-cardinal relation* between the systematically varied fifths: 1/5, 2/5, 3/5, 4/5 and 5/5 (which is one aspect of measuring). Although the structure between the fifth is immediately clear for some students, our design experiments have shown that the task can allow an interesting discovery for lower achievers and help to make clear the difference between a divided whole and its parts (Prediger & Wessel, 2013).

More and more fifths

a) Now Kenan produces fifths with fraction bars. Complete his table.
Fraction that Kenan wants to draw: Picture:

$\frac{1}{5}$

$\frac{2}{5}$

$\frac{3}{5}$

$\frac{4}{5}$

$\frac{5}{5}$

b) Examine the table precisely and consider the following: What happens with the coloured part of the fraction stripe? Why does the coloured part change?

c) Your research: How and why does the fraction change? Write down your findings so that another student can understand what is happening with it and why it changes. …

Figure 2. Elementary task for weaker students – example for the design strategy systematic variation

For example, Hadar (12 years old) writes, "When the numerator gets bigger, one gets more fraction.". Asim (12 years old) explains the differences with reference to a contextual situation: "Because the numerator gets always bigger, that is why Kenan gets always one [piece] more [of the chocolate bar]. And the denominators stay the same." (citations from a case study in Prediger & Wessel, 2013).

3.2 Embedding structural relations in a comprehensive visual model: Bar board

For comparing fractions with respect to order and equivalence, students need to connect different fraction bars, e.g., not only fifths, but also fourths, thirds and eighths. Whereas higher achieving students tend to *mentally* construct the bars and their relations among each other quite quickly, weaker students profit from a comprehensive visual model that helps them to see many bars at the same time. Inspired by the prototype of the "fraction lift" which connects some bars (Bokhove et al., 1996), we have developed the fraction bar board as a comprehensive visual model for comparing fractions with respect to order and equivalence. Figure 3 sketches how to find fractions that are equivalent to 1/3 by vertically positioning the ruler. The lamination of the bar board guarantees its long time usability.

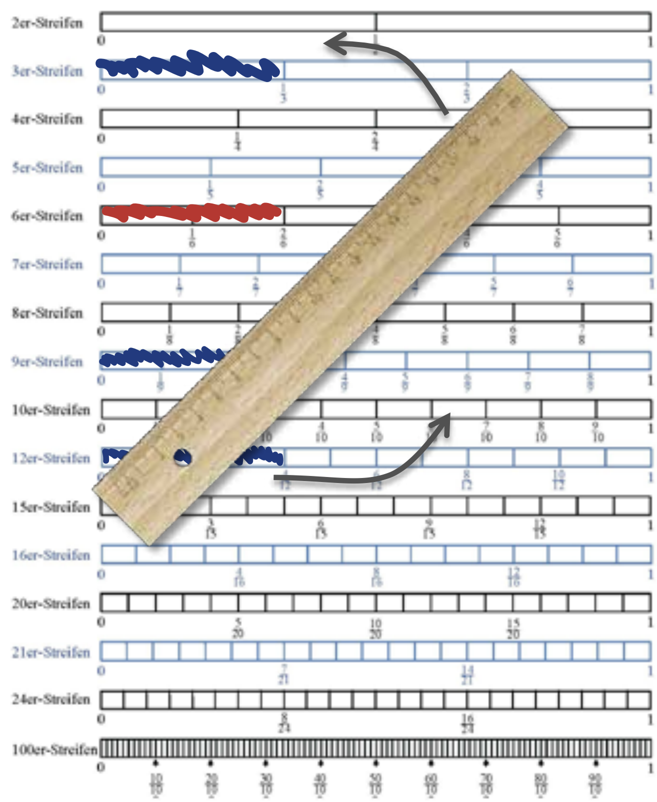

Figure 3. Extract of the fraction bar board

The classroom design experiments have shown that most students quickly learn to use the bar board and understand the ordinal relations of fractions in the interplay of symbolic, verbal and graphical representation (Prediger, 2011). The bar board allows to embed a singular ordinal relation into a comprehensive visual model. By this, students achieve an overall orientation as the example of Lisa illustrates: Having worked with the bar board for two hours, Lisa is asked to compare 1/10 with 2/3. She immediately says without watching the bar board: "Imagining them, it is evident, 1/10 is so much on the left." (cited from a classroom video).

In the next step of the curriculum, the bar board serves as starting point for the process of progressive schematization (Treffers, 1987) from visually searching equivalent fractions to extending fractions by calculating: Paul explains "If you go from 2/6 to the 12-bar, each sixth transforms into two twelfths, thus the denominator must be multiplied by two. The coloured pieces also transform from sixths into twelfths by one into two, so the numerator must also be multiplied by two." (cited from a classroom video).

3.3 Internalizing structural relations by mental practices

Unlike Lisa and Paul, who quickly internalized the relational structures inherent in the bar board, other students need more help to explicitly focus the structural relations between the bars. This is illustrated by the case of Anna and Jasmin, both 11 years old (in Prediger, 2011). Both girls worked with the bar board, but treated it only empirically, which became apparent when they searched for fractions being equivalent to 3/4:

230	Anna	9/12 here, isn't it? *(marks a sign on the 12-bar and draws a vertical line)* Well, yes, that is... is 6/8, but..... it works, it seems to work! *(controls her bar board with the ruler again, during 12 seconds)*
232		No, it is only wrong by one millimeter.

With purely empirical methods of measuring in the bar board, Anna could not convince herself whether 9/12 is equivalent to 6/8 and 3/4 or not. Whereas other children in this situation started to argue with structural relations (like Paul above), both girls only referred to the bar board as an empirical instrument (Steinbring, 2005). Even later, when they found a rule in the number patterns (multiply numerator and denominator by 2 or by 5) they could not explain the found regularity:

295	A	Yes, denominator and numerator must always be the same. That means here, 3/5 is the double of 6/10. *(8 seconds break)*
...		*(Interviewer asks for an explanation)*
303	A	That's just how it is. Like: why is a banana called a banana *(both girls laugh)*

Thus, Anna's and Jasmin's process of progressive schematization was only partly successful due to a lack of conceptual understanding *why* their rule "take the double for denominator and numerator" applied for all fraction bars. As a consequence of these empirical findings, the redesign searched for a strategy that makes Paul's insight into structural relations more explicit for all children. For this, we developed mental practice activities (Weber, 2011) for internalizing the structure of the bar board (see Figure 4).

The bar board in your head

You can also find equivalent fractions, if you only imagine the bar board in your head. Try it!

a) Imagine how to mark 2/3 on the 3 bar.
Go from the 3 bar to the 6 bar. Where is 2/3 here?
How many pieces does the 6 bar have?
How many of them are coloured?

b) Pose yourself different similar tasks. ...

c) How many 25ths are 3/5?
Explain, how you find the result even if you cannot imagine the bars.

Figure 4. Mental practice for internalizing relational structures of equivalence

4. EVALUATION

Whereas laboratory design experiments and in-depth analyses offer good opportunities to understand the *situational effects* of the design (see preceding section), a quantitative evaluation of learning effects can better contribute to evaluate *the long-term efficacy* of the curriculum. That is why we conducted a first, rough assessment of effects in long-term classroom design experiments (2008-2010) with a standardized fraction test (see Table 1 for some items). We compared the performances of students in five classes (n=108) that have worked with our fraction curriculum in grades 5 to 7, with those of five neighbour classes (n=104) from the same schools that have used the usual textbook curriculum. This *pragmatic quasi-experimental sampling by neighbour classes* suggests the approximate comparability of treatment and control group with respect to general performances and socio-economic background (as these criteria were applied for composing classes in grade 5).

For comparing the long-term efficacy of two different fraction curricula, we measured students' fraction performance ten months after finalizing the curriculum with a *standardized fraction test* being adapted from Bruin-Muurling (2010) (more details in Prediger & Wessel, 2013). The test construction of 41 items tried to limit specific training effects for the intervention group by a wide coverage of different contents: e.g., identify and draw fractions in part-whole and part-group models and on the number-line, order fractions and explain order in contextual or graphical representations, find equivalent fractions and explain, part of part-tasks, subtractions, operators.

Table 1 shows the *overall results* and those items on order and equivalence with most significant intergroup differences. The treatment group was significantly better (m = 23.49) than the control group (m = 19.52) in the whole fraction test, this difference increases enormously for the items on order and equivalence. Remarkable is also the difference in standard deviations, we interpret the higher homogeneity of the treatment group as a success in giving *also weaker students* an access to conceptual understanding.

Table 1. Comparing performances of treatment / control group for overall results and selected items

	Treatment Group	**Control Group**	**Significance**	
Sample size	n = 108	n = 104		
Mean of overall scores	m = 23.49	m = 19.52	T = 4.580	α < 0,001***
Standard deviation	SD = 5.389	SD = 7.146	F = 4.362	α < .05 **
Items with significant differences	Frequency of complete solutions			
Item 1a. Find fraction 3/4 from picture	99.1 %	86.5 %		
Item 5b. Compare 2/10 and 4/6	86.1 %	72.1 %		
Item 5d. Explain why 2/3 < 3/4	45.4 %	28.8 %		
Item 5f. Explain why 3/9 = 5/15	25.0 %	3.8 %		
Item 6a. Read on the number line 2/8	39.8 %	20.2 %		

5. CONCLUSION AND OUTLOOK

A design research project always aims at research results and design results: The central finding of the empirical *research* on students learning processes is that focussing on structural relations is an important condition for developing conceptual understanding of order and equivalence of fractions while relating different representations. Especially weaker students do not develop this focus automatically, but if they are fostered by suitable activities, they are able to.

Central results of the iterated *design* have been presented in three design strategies for implementing the design principle ‘focussing structural relations’, namely

1. constructing relevant structural relations by contexts and systematic variation,
2. embedding structural relations in a comprehensive visual model,
3. internalizing structural relations by mental practice.

The empirical snapshots (that could of course only show a minimal part of the large data set) illustrated how these design strategies can initiate learning processes. The qualitative insights into *situational effects* of the design were triangulated by a first rough evaluation of *efficacy* in a posttest-control-group design. The results show that the students who learned with our curriculum learned significantly more than the control group classes. However, this first rough evaluation has two important *methodological limits:* (1) the lack of fraction performance control in a pre-test in 2008 and (2) only partial control on the applied pedagogy in the long-term intervention in regular classrooms. Because of these methodological limits, we have started another evaluation study from 2012-2014. As the curriculum has further been developed, we hope to receive more robust results and perhaps even higher learning effects.

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FLEXIBILITY IN MENTAL CALCULATION IN ELEMENTARY STUDENTS FROM DIFFERENT MATH CLASSES

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This paper describes a study that assesses the competence of elementary students in flexible mental calculation. Based on a specific perspective on flexibility that was derived from the relevant literature and prior research, a qualitative interview was used to field test and identify degrees of flexibility in elementary students from different classrooms in South Germany and North Carolina. This paper gives theoretical background, data analysis and a quick brief summary of early results.

Key words: Flexibility in mental calculation, recognition of number patterns and relationships, elementary arithmetic

INTRODUCTION

Flexibility in mental calculation has been considered a highly desirable ability in elementary math education for at least the last decade (e.g., Anghileri, 2001; National Council of Teachers of Mathematics, 2000; Selter, 2000). The late twentieth century saw increasing interest in students' thinking and in their techniques for performing mental addition and subtraction.

With the start of the twenty-first century, researchers have begun to explore elements of flexibility in mental addition and subtraction. In this vein different results have been reported concerning students preferences for the written computing algorithms (Selter, 2000), the negative impact of learning by examples on flexibility (Beishuizen & Klein, 1998; Heirdsfield & Cooper, 2004), various influencing factors on students' strategies (Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009; Blöte, Klein, & Beishuizen, 2000; Rathgeb-Schnierer, 2006) and approaches to math education which can enhance flexibility in mental calculation (Heinze, Marschick, & Lipowsky, 2009; Rathgeb-Schnierer, 2006).

In the context of this recent research, mental calculation means solving multidigit arithmetic problems mentally without using paper and pencil procedures. For flexibility in mental calculation a deep understanding of number and operation relationships and knowledge of basic facts and fact families is required (Heirdsfield & Cooper, 2004; Threlfall, 2002).

With the ongoing study we pursue two aims: First, we want to clarify the concept of flexibility in mental calculation based on theoretical reflections and prior research. Second, we describe a study, which investigates whether students from different classrooms show different degrees of flexibility in mental calculation. Since our theoretical framework introduces a new perspective on flexibility in mental calculation, we describe this perspective and its implications for the study's design.

THEORETICAL FRAMEWORK

Process of calculation

Results of a previous study (Rathgeb-Schnierer, 2006; 2010), in which the development of strategies of mental calculation has been investigated, suggested that it is useful to examine the process of calculation in general, before focusing on flexible mental calculation. Based on this study the following model was developed to describe the process of calculation (Rathgeb-Schnierer, 2011): Solving problems is a complex interaction between different domains that implement different functions with different degrees of explication (Fig. 1).

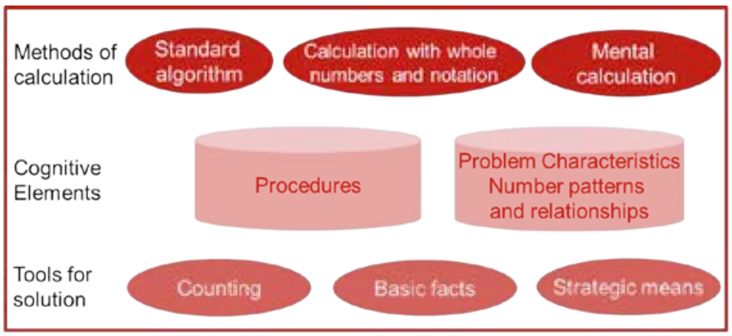

Fig. 1: Domains of calculation process (Rathgeb-Schnierer, 2011)

In our work it has been necessary to consider three distinct but interrelated domains: methods of calculation, cognitive elements, and tools for solution. Each domain is a necessary but not sufficient condition for a calculation process. Taken together, the domains allow us to examine mental flexibility independent of a problem's solution. Each domain is described more fully below.

Methods of calculation: For solving any given problem, a student can use multiple methods of calculation such as the *standard algorithm, partial sums, or mental calculation* (e.g., Selter, 2000). However, a method describes the way a solution process can be done but not how an answer is determined. For example, the standard algorithm describes a way an addition problem like 327 + 56 can be solved. First, the terms of the sum may be written correctly, one below the other to determine the sum of each column separately. This method doesn't show the actual process for adding numbers. Looking at the example above, there are many possible tools for solution that could be used to find the answer for 6 joined with 7, like counting, drawing on basic facts, or using other adaptive strategic means. In short, obtaining a problem solution *by itself* does not shed light on the mechanism(s) used to achieve that solution.

Cognitive elements: Students' solution processes are based on specific experiences that we designate with the term "cognitive elements". Such cognitive elements that sustain a solution process can be learned *procedures* (such as computing algorithms), or they can be recognized *number characteristics* (such as number patterns and relationships) (Macintyre & Forrester, 2003; Threlfall, 2009). In reality, it is difficult to reconstruct the basic cognitive elements that lie behind an exposed solution process as shown by Tanja, who solves the subtraction problem 46 – 19 (data from a prior project, Rathgeb-Schnierer, 2006):

Tanja: Um. 46 (.) minus (..) 9 – no, I do now minus 6 – and then, this equals 40 and then minus 3 equals – thirty-seven and then minus 10 equals 27.

Tanja exhibits a "Begin-With-One-Number-Method" (Fuson et al., 1997) by decomposing the subtrahend and subtracting it step by step. Whether her solution derives from a learned procedure, or from recognition and use of number patterns and relationships, or from a combination of the two cannot be determined from the information. However, for a correct assessment of Tanja's abilities in mental calculation, it is crucial to know which cognitive elements her solution process entails. A procedure-based solution can be conducted mechanically, like following steps in a recipe, whereas a solution based on number characteristics entails dynamic use of knowledge of numbers and relationships.

Tools for solution: To find the answer to a problem, cognitive elements rely on additional *tools* that are used and combined in context. Specific tools for solution may be *counting*, referring to *basic facts*, or using other adaptive "*strategic means*". Our sense of these "strategic means" is that they are not holistic strategies or cognitive menus that complete a solution path; rather, they are distinct devices that can be combined in flexible ways to modify complex problems to make them easier. Such strategic means include, for example, decomposing and composing (65+28=60+20+5+5+3), transforming a problem (46-19=47-20), deriving the solution from a known problem (if 7 joined with 7 equals 14, the answer to 7 joined with 8 equals 15, since 8 is one more than 7), and using decade analogies (if 4 joined with 5 is 9, 40 joined with 50 must be 90) (Threlfall, 2002).

Whenever students solve a problem mentally, elements of all three domains are combined, and depending on that combination, one can identify different competencies. In judging mental flexibility, there is a meaningful difference between a student who mentally replicates a mechanically learned procedure and one who dynamically applies number sense, patterns, and problem characteristics to achieve a solution. This distinction is exemplified by the reasoning Simone shows with the subtraction problem 46-19:

Simone: If I do add one (*points at the problem*), then I have got 20 here and 47 there, and then it is easier to calculate.

Interviewer: Which problem do you solve then?

Simone: 47-20. This equals 26 (.) no 27.

Interviewer: And - are you sure that you get the same answer to 47-20 and 46-19?

Simone: Yes, because I have added one to both numbers, and then I have more here (*points to 47*) and take away more there (*points to 20*).

Obviously Simone has solved the problem by mental calculation. First, she has transformed the actual problem into a new one that preserves the difference between the numbers. For Simone, the transformed problem is almost trivial. Her solution relies on a combination of subtraction basic facts and her knowledge of decade analogies. Simone has recognized the numerical proximity of 19 to 20, and she adapted this knowledge to the situation by transforming the problem. What we describe in the case of Simone is called "zeroing in" by Threlfall (2009, 47).

Flexibility in mental calculation

Multiple, inconsistent perspectives on the concept flexibility in mental calculation exist in the literature (Star & Newton, 2009). Depending on the definition used, different ways of operationalization are implied.

> We choose to navigate through this somewhat confusing terrain by […] defining flexibility as knowledge of multiple solutions as well as the ability and tendency to selectively choose the most appropriate ones for a given problem and a particular problem-solving goal (Star & Newton 2009, 558).
>
> In the present article, we will, henceforth, use the dual term 'flexibility/adaptivity' as the overall term, 'flexibility' for the use of multiple strategies, and 'adaptivity' for making appropriate strategy choices (Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009, 337/338).
>
> First […] we employed the definition of strategy flexibility as choosing among different strategies simply on the basis of the characteristics of the task, i.e. as using the compensation strategy on problems with a unit digit 8 or 9. Second, we also applied a more sophisticated definition wherein strategy flexibility is conceived as selecting the strategy that brings the child most quickly to an accurate answer to the problem (Torbeyns et al., 2009, 583).

All these definitions have in common the idea that flexibility in mental calculation is connected – explicit or implicit – with adaption and means an appropriate way of acting when faced with a problem. What exactly is meant by appropriate and how an appropriate way of acting can be identified is considered differently (Rechtsteiner-Merz in prep.). There is, on the one hand, the notion of choice in selecting an appropriate solution to a given problem (and perhaps including the choice of a solution deemed most appropriate). On the other hand, the appropriate way of acting when faced with a problem may not be determined by task characteristics but by the speed of obtaining a solution. In both views the focus is on mental outcomes, since researchers have typically regarded only one domain of the calculation process (Fig. 1), the tools for solution.

Similar to Threlfall (2002, 2009) and prior research (Rathgeb-Schnierer, 2006, 2010) we also define flexibility in mental calculation as a way of acting appropriately, but we have a different conception of what is meant by appropriate. Rather than the choice of the most suitable strategy or the quickest way of obtaining a solution, appropriate acting for us means to match the combination of strategic means (see above) to the recognized number patterns and relationships of a given problem in the context of processing a problem solution. The recognition of problem characteristics, number patterns and relationships, and their use for solving a problem again depends on a student's knowledge of numbers and operations or what Threlfall (2002, 29) terms an "interaction between noticing and knowledge":

> When faced with a fresh problem, the child or adult who follows different solution paths depending on the numbers does not do so by thinking about what the alternatives are and trying to decide which one to do. Rather, he or she thinks about the numbers in the problem, noticing their characteristics and what numbers they are close to, and considering possibilities for partitioning or rounding them. (Threlfall 2002, 41)

Our discussion leads us to posit a new perspective on flexible mental calculation because we are interested in mental processes that underlie the outcomes. That means, related to the model of process of calculation (Fig. 1), we focus on two different domains to identify the degree of flexibility in students: the tools for solution and the cognitive elements that support the solution processes. Only if the tools of solution are linked in a dynamic way to problem characteristics, number patterns, and relationships would we consider as evidence of flexibility in mental calculation. Hence our central questions about mental flexibility are: What problem characteristics, number patterns, and relationship do students recognize? And how do they use problem characteristics, number patterns, and relationships to solve problems?

OVERVIEW ON THE PROJECT

Questions and Assumptions

We investigated flexible mental calculation in elementary students from different classrooms in different countries. Referring to earlier research and our perspective on flexibility, the project is based on fundamental assumptions: First, there are different features in students' ways of acting when faced with an addition or subtraction problem that can be considered as indicators of flexibility. One feature is the recognition of problem characteristics, number patterns, and relationships in a given problem, and the other one is the dynamic use of recognized number patterns and relationships for solving a problem. Second, flexibility in calculation is not an all-or-none-phenomenon; it occurs in varying degrees (Rathgeb-Schnierer, 2010).

Research questions concerning the sorting task as research instrument: Does sorting problem cards into categories "easy" or "hard" help initiate mental flexibility when it is available? Do differences appear regarding sorting and reasoning, and can they be linked to different degrees of flexibility?

Research questions concerning patterns that appear in sorting, reasoning, and solving problems: Do students describe reasoning about problem characteristics, number patterns, and relationships? Do students link their tools for solution to recognized problem characteristics, number patterns, and relationships? Can different degrees of flexibility be identified and pooled into general types?

Research questions concerning different classrooms: Do differences and tendencies show up in sorting, reasoning, and using tools for solution for students from different math classrooms? Do differences appear regarding sorting and reasoning, and can they be linked to different degrees of flexibility?

Design

Based on the theoretical model of flexibility introduced earlier, a qualitative study has been designed and carried out. Since we were interested in mental processes underlying the process of solving a problem, we decided on probing interviews as a research instrument (see following paragraph). Second graders and fourth graders from different instructional contexts in Germany (Baden-Württemberg) and USA (Charlotte, North Carolina) were selected. In total ten classrooms were sampled, 3 second grades and 2 fourth grades in each country. We chose about eight students from each classroom. Students with learning disabilities in math or with language problems were excluded in order to achieve a minimal level of understanding of number and operation. Based on the judgment of the classroom teacher, we got a sample representing predominantly middle and high achievers and carried out interviews with a total 51 second graders and 31 fourth graders.

Interviews

We developed a qualitative, problem-oriented interview that contains twelve two-digit addition and subtraction problems. Each problem was designed to show at least one special feature, sometimes more than one. Our problems incorporated features like double and half relations, same numbers at the tens and ones place, one number close to ten, both numbers close together, numbers at the ones place equal ten, reverse problems, and problems that require regrouping. These problems were displayed on small cards: 33+33, 66-33, 56+29, 46-19, 31-29, 73+26, 88-34, 34+36, 65+35, 95-15, 47+28 and 63-25.

Interviews had three parts, the first for sorting problems and talking about the sorting procedure, the second for solving problems, and the third for comparing selected problems. In the first step, cards were mixed and laid out on the table. Students were encouraged to look carefully at the numbers in each problem and sort the problems in two categories, “easy” and “hard” (these labels were placed at each side of the table). After a card was placed either to the “hard” or the “easy” side (occasionally students decided on the middle), we asked: “Why is this problem easy or hard for you?” In the second step we asked the students to choose some problems from each side, to solve one by one and to tell us what they were thinking when they solved a problem. We always started with the easy side and skipped all the problems that had already been solved during the sorting process. In the third step we focused on selected problems (63-25 and 88-34; 47+28 and 73+26) and encouraged students to compare these in order to estimate whether one of each pair might be easier.

Students were interviewed one-on-one for 15 to 30 minutes. Video and audio recording was done for the whole interview.

Interviews were conducted by one researcher and took part in the last two months of the academic year (Germany 2010 and 2012, USA 2011). All interviews were transcribed in their original language for data analysis.

Data Analysis

Two coding systems that include both a priori and inductive meanings were used. One system was used to classify students' tools for solution; the other one was used to catalog reasoning for easy and hard problems.

The following example illustrates how a part of the coding system was developed. Data suggested that reasoning could be divided in four core categories (first level): reasoning by problem characteristics – easy (A/1), reasoning by problem characteristics – hard (A/2), reasoning by ways of solution – easy (B/1) and reasoning by ways of solution – hard (B/2).

Two different students explained why 33+33 is an easy problem:

S 1: Because there's the same numbers in each 'um space. (A/1)

S 2: Because first I add 30 and 3, and then I add 3. (B/1)

To each core category we developed codes (second level) based on possible characteristics of numbers (e.g., Fig. 2) and theoretically described solution strategies (e.g., Selter, 2000; Threlfall, 2002). The sub-codes (third level) arose from the data again. All together we got three levels of categories/codes that were invented by combining data-based and theory-based methods. With this differentiated coding system students' utterances can be exactly assigned. See student 1 in the example above: His utterance is assigned on the first level to the core category "reasoning by characteristics – easy," on the second level to the code "special numbers", and on the third level to the sub-code "double digits".

First results and outlook on further data analyses

At the moment we have transcribed 70 interviews and categorized one third of the data using both coding systems. While we cannot yet provide definite answers to our research questions, we can report on some initial patterns we have observed. We illustrate these patterns based on a sample of 21 second grade students from three German classrooms.

Core category: reasoning by characteristic for „easy“ problems (first level)	
Codes (second level)	Sub-codes (third level)
analogies between tens and ones [5]	
size of numbers	both numbers are small [1]
numbers at the ones place	• sum is less than ten [1] • remaining or regrouping is required [1] • first number smaller than second number [1] • sum of ten at the ones place [6] • number 5 at both ones place [14]
relationships of tasks	• associativity [1] • inverse problem [7]
relationships of numbers	• double or half [10] • number close to a ten [2] • range [5]
known facts	• parts of the problem are known by heart [18] • whole problem is known by heart [10]
special digits / numbers	• same numbers [5] • number with nine [2] • double digits (33) [5]

Fig. 2: Reasons for easy problems (numbers in brackets show frequency)

Overall, we have found a much larger than expected variety of patterns in students' reasoning for easy and hard problems. Already in the sample of 21 second graders various reasons for easy problems that refer to problem characteristics appeared (Fig. 2). Such variation can be considered as a hint that the sorting task encourages students to examine problems carefully; to reflect about problem characteristics, number patterns, and relationships; and to formulate solution arguments based on that recognition. In this vein, our data suggests that sorting has diagnostic value in assessing the underlying degree of flexibility in math structures.

Fig. 3 Ways of reasoning

Regarding the ways of reasoning students in our sample have displayed (Fig. 3), it appears that 2/3rd of the students tend to have preferences: 9 prefer reasoning by ways of solution; 5 prefer reasoning by characteristics; 7 students reason either way. Regarding the students individually, it appears that their pattern of reasoning can be described as either "dynamic" or "static." For instance, Bartosch (3_Ba) depicted 8 different kinds of reasons (6 characteristics and 2 solutions); he would be considered as "dynamic". In contrast, Marcel (9_Ma), who showed only two different reasons (2 solutions), would be considered as "static".

Since this is work in progress, our current perspective on the data is only on a descriptive level. For going beyond this level, we plan on constructing types for each coding system separately (Kelle & Kluge 2010). That means, we will generate two different typologies: one for the explicated ways of reasoning that gives information on students' recognized number patterns and relationships, and one for the utilized tools of solution that gives information on students' approach. In the next step we plan on linking and comparing both typologies with the purpose to see if students' ways of reasoning match the tools of solution they depicted. Based on our theoretical perspective on flexibility in mental calculation (see above), we will make use of this compliance to identify the degree of flexibility in each student. Then we will assign each student to a special degree of flexibility depending on characteristic patterns he or she displays in reasoning and solving. In a last step of analysis we plan on linking the degrees of flexibility we have found to the teachers' math instruction and dispositions to see if students' math experiences are associated with those degrees of flexibility.

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DIFFERENT PRAXEOLOGIES FOR RATIONAL NUMBERS IN DECIMAL SYSTEM – THE $0.\overline{9}$ CASE

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We report the results of an experiment, in which students were asked to make some computations involving the (ultimately periodic) decimal expansion of rational numbers and simple algorithms derived from the algorithms in use for decimal numbers. We show in which way these algorithms could be useful to increase the understanding of such a counterintuitive equality as 0.999…=1.

Keywords: rational numbers; repeating decimals; praxeology; register of representation.

INTRODUCTION

It is well known that the understanding of the double representation of decimal numbers, in base ten system, is an important issue for students, for example for its implication in analysis. The emblem is the equality $0.\overline{9}=1$ that summarizes the situation on which teachers and searchers generally focus.

Many researchers have already investigated this inequality, among them, Tall and Scharzenberger (1978),Tall (1980), Sierpinska (1985) and Dubinsky et al. (2005) are worth to be mentioned. These studies mainly focus on the infinite character of such a representation and, more specifically, on real numbers and limits. Here, we expose an alternative viewpoint on the equality $0.\overline{9}=1$ which is essentially algebraic and emphasizes rational numbers: "are repeating numbers decimal?" and "is $0.\overline{9}$ equal to 1?" are in fact the same question. The present work derives from a recent research topic in mathematics, the notion of circular words (Rittaud & Vivier, 2012).

In this paper, we consider the comparison and the sum of repeating decimals. The register of representation (Duval, 1995, 2006) allows us to take the difference between semiotic representation (repeating decimals and fractions) and mathematical objects (rational numbers) [1] into account. We exhibit two different Mathematical Organizations (Chevallard, 1999) related to (1) repeating decimals and (2) rational numbers. The main difference between them is a technology related to the identification of the double representation of decimal numbers. These two praxeologies could explain some difficulties of students when we want them to understand why $0.\overline{9}=1$. It is our opinion that an important issue of the problem is to pass from a repeating decimals praxeology to a rational numbers praxeology. In this perspective, using an algorithm for the sum (Rittaud & Vivier, submitted work [2]) of two repeating decimals, we present a sequence of two activities that we proposed to students at the beginning of university. This relates to previous teaching at secondary school and at the very beginning of university.

THE EQUALITY $0.\overline{9}=1$: REGISTERS AND TECHNOLOGY

In this section, we present the problem of the equality of $0.\overline{9}$ and 1 with registers of representation (Duval, 2006) and praxeologies (Chevallard, 1999).

Theoretical frameworks

In the Anthropological Theory of Didactics (ATD) the mathematical activity is elaborated around types of tasks, appointed by Chevallard (1999) as mathematical organization. Generally, to perform a type of tasks T, we have at least one technique τ. Type of tasks and techniques are organized in a $[T,\tau]$ appointed block of know-how or praxis. To produce and/or justify a technique τ, it is necessary to have a theoretical look at the problem posed by T. Chevallard defines a new block $[\theta,\Theta]$, called block of knowledge or logos, made up of technology and theory. Type of tasks, techniques, technology and theory form a praxeology or mathematical organization $[T,\tau,\theta,\Theta]$.

Duval (2006) starts from signs used in the mathematical work grouped into registers of semiotic representations. The essential distinction made by Duval consists in the dichotomy between treatment and conversion. Treatment is a semiotic transformation which remains within the same register of representation. Conversion is a semiotic transformation whose result is expressed in another register. Duval stresses the essential cognitive difference between treatment and conversion. Conversion is much more complex and problematic than treatments, especially for a non-congruent conversion.

Two numerical registers of representation for rational numbers: R_f and R_d

When speaking about rational numbers, one often thinks of fractions. Fractions constitute the main way of constructing rational numbers, especially in secondary teaching because of proportionality. This is the first register of representation of rational numbers, denoted here by R_f.

Duval stressed that one needs, at least, two registers in order not to mix the object with its representation. For our purpose, the interpretation of fractions *a/b* by *a* divided by *b* allows, by long division, to obtain a new register written R_d related to base ten system. The result of a long division is a repeating decimal which is, traditionally, a non finite expansion for a non decimal number and a finite expansion for a decimal number [3]. As Duval pointed out, conversions between R_f and R_d are non congruent as one can see with $1/4=0.25$ or $2/7=0.\overline{285714}$.

Non congruency also appears in treatments. Obviously, the comparison of the two rational numbers above is very different in R_f and in R_d. It is also the case in terms of the four basic operations. These calculations are well known and taught at lower secondary school in France in R_f and we do not describe them. A description of algorithms for the four basic operation in R_d is one of the aim of our submitted work. We only expose in this paper the sum (see the section devoted to the sum below).

A hidden technology

Comparing two repeating decimals, a type of tasks denoted by $T_<$, is quite simple since one just has to compare cipher by cipher from left to right, a technique denoted by $\tau_<$. It is a natural generalization of the comparison of two decimal numbers. Students have no difficulty with this technique $\tau_<$ for the type of tasks $T_<$ as we will see. But are we comparing rational numbers? It is not so obvious because $\tau_<$ deals with semiotic representations that must be interpreted in order to have mathematical objects called numbers.

Indeed, comparing $0.\overline{9}$ and 1 leads to $0.\overline{9}<1$ by $\tau_<$ technique, when computing on repeating decimals, but must lead to $0.\overline{9}=1$ for rational numbers. We identify a new technique $\tau'_<$ close to $\tau_<$ except for decimal numbers. This shows that from the type of tasks $T_<$ two praxis related to two different mathematical objects arise whose writings are exactly the same: $[T_<,\tau_<]_{rd}$ for *pure* repeating decimals and $[T_<,\tau'_<]_{\mathbf{Q}}$ for rational numbers – we do not speak here of fraction representation.

This distinction could be confusing because we see the same writings, the same ciphers, the same signs. The difference is in the interpretation of what is represented. Furthermore, the technology that justify technique $\tau_<$ and $\tau'_<$ are mainly identical: it is a technology that is grounded in the base ten system, denoted by θ_{bts}. Obviously, there is a lack for $\tau'_<$ in order to justify the equality of the two representations of decimal numbers. We denoted this *hidden* technology by $\theta_=$. One has to notice that, until now, there is no indication of $\theta_=$, it cannot appears only with the comparison and there is no indication of its nature, of its origin.

A topological technology

Actually, $\theta_=$ is an important technology of the real numbers theory, denoted by $\Theta_{\mathbf{R}}$, that is related to the topology of the set **R** of real numbers [4]. Explications within the APOS theory are quite clear as Dubinsky et al. (2005, pages 261-262) wrote:

> An individual who is limited to a process conception of .999… may see correctly that 1 is not directly produced by the process, but without having encapsulated the process, a conception of the "value" of the infinite decimal is meaningless. However, if an individual can see the process as a totality, and then perform an action of evaluation on the sequence .9, .99, .999, …, then it is possible to grasp the fact that the encapsulation of the process is the transcendent object. It is equal to 1 because, once .999… is considered as an object, it is a matter of comparing two static objects, 1 and the object that comes from the encapsulation. It is then reasonable to think of the latter as a number so one can note that the two fixed numbers differ in absolute value by an amount less than any positive number, so this difference can only be zero.

At the end of this quotation, one can note a technology, that encompasses $\theta_=$, closely related to the topology of **R**. But it is not so obvious, especially for students. And one has to notice that this technology does not stand in nonstandard analysis, nor in the monoïd of repeating decimals we will define later.

Actually, the topology of **Q** is enough. First, we can talk about limit in **Q**, but it is not interesting for teaching (except for Méray-Cantor's construction of **R**) and secondly the density of **Q** allows concluding that $0.\overline{9}=1$ because there is no number between them, but one has to assume that $0.\overline{9}$ is a rational number, which is not obvious.

In the previous quotation, one notices that it is a question of the "value" of an infinite expansion and of "numbers". Hence, before talking about topology of **R** or **Q**, we prefer to justify that we are dealing with numbers. According to (Chevallard, 1989), numbers are objects that could be, almost, compared, added, subtracted and multiplied with usual properties.

DIDACTIC UNDERSTANDING OF THE PROBLEM BY TAD THEORY

In this section, we continue the description by TAD in considering the sum type of tasks, T_+. The aim of this section is to try to justify $\theta_=$ by the sum according to: (1) the comparison is not sufficient to make $\theta_=$ arise and (2) the topology technology belongs to a higher level of knowledge. Before investigating the sum type of tasks T_+, we discuss the classical way for proving that $0.\overline{9}=1$ using calculations and an equation.

A classical way to produce and justify $\theta_=$

A classical way to produce and justify $\theta_=$ comes from calculations with repeating decimals, as if they were usual numbers. It is well known by all mathematics teachers that every repeating decimal could be converted into R_f using an equation [5]. There are others calculations that lead to the target equality, see for example (Tall & Schwarzenberger, 1978). But all these calculations rely on the operations that are supposed to be well defined – this assumption on operations, especially on subtraction, is not so obvious and is, in fact, directly linked to the problem.

The point of view is quite natural from a mathematical perspective: after generalizing objects one wants to preserve some properties. Here, one has some new *numbers* and the properties are those of the usual operations, even if we do not know if it is possible to define these operations. Obviously, this raises the problem of the consistency of the mathematics produced. It is the same perspective for the multiplication of integers within the set **Z**, and especially the sign of the product (Glaeser, 1981). But the consequences are not identical. Indeed, the result in **Z** are totally new and it is not problematic since there is no opposition with an ancient knowledge – think for example at $(-2)\times(-3)=(+6)$. But in our case it is not so simple. Of course calculations *say* that $0.\overline{9}=1$, but $\tau_<$ *say* that $0.\overline{9}<1$. Hence, there is an obvious contradiction and what a student may believe? It is natural to think that the ancient knowledge is stronger, even if a student's answer is the equality because of the didactical contract. Hence, with this type of calculations the need appears of $\theta_=$ but it also brings a contradiction and no explanation could emerge.

Four processes for the sum in R_d

Here, we are describing four processes to compute the sum of two repeating decimals. Each time, we discuss the possibility to justify $\theta_=$.

The first process consists in performing calculations by approximations and inferring the period of the sum. This process requires two technologies: *the sum of two repeating decimals is a repeating decimal* and *the sum is continuous* (according to the usual topology of **R**). For example, to compute $0.\overline{5}+0.\overline{7}$, one successively writes 0.5+0.7=1.2; 0.55+0.77=1.32; 0.555+0.777=1.332; and so on. It is not an algorithm since there are no criteria to stop the approximations (when do we get the period?). We do not retain it, first because it relies on the high-level technology we pointed out and second because this process causes some wrong writings to students (see below).

The second process is an algorithm that requires a conversion: calculating the addition after a conversion into R_f like in $0.\overline{5}+0.\overline{7}$=5/9+7/9=12/9=1+3/9=$1.\overline{3}$. The point is that the conversion of repeating decimals such as $0.\overline{9}$ requires calculations involving repeating decimals, sum and subtraction. It is close to the *doubtful* equation process (see previous section). Moreover, it is quite difficult to understand how repeating decimals could become numbers within this process since calculations are made with fractions. Hence, we also reject this process.

The third process makes an explicit use of $\theta_=$, as in $0.\overline{5}+0.\overline{7}=0.\overline{9}+0.\overline{3}=1+0.\overline{3}=1.\overline{3}$. We do not retain this algorithm since it relies strongly on $\theta_=$ itself.

The fourth process is an algorithm we proposed in (Rittaud & Vivier, submitted work). It is quite close to the algorithm of addition of two decimal numbers in base ten system: decimal points have to be in the same row, so do the periods. In the simplest case, with no carry overlapping the periodic and aperiodic parts, we get something like in the first example of figure 1. When a carry appears at the leftmost digit of the periodic part, we have to consider it twice: the first one, as usual, at the rightmost digit of the aperiodic part (this corresponds to the exceeding part), the second one, more unusual, at the rightmost digit of the periodic part (such a unusual carry is written between parenthesis). An example is given in figure 1.

$$\begin{array}{rrrrrr} & 1 & & & 1 & \\ & & 4. & 2 & \overline{4} & \overline{2} \\ + & 1 & 7. & 0 & \overline{4} & \overline{9} \\ \hline & 2 & 1. & 2 & \overline{9} & \overline{1} \end{array}
\qquad\qquad
\begin{array}{rrrr} & 1 & & (1) \\ & 0. & \overline{8} & \overline{2} \\ + & 0. & \overline{4} & \overline{1} \\ \hline & 1. & \overline{2} & \overline{\cancel{3}4} \end{array}$$

$4.2\overline{42}+17.0\overline{49}=21.2\overline{91}$ $\qquad$ $0.\overline{82}+0.\overline{41}=1.\overline{24}$

Figure 1: two examples of the algorithm

Of course, we also have to deal with sums in which the position of the decimal point or the length of the periods, are not the same. For example, to perform the sum $2.45\overline{38}+13.3\overline{192}$, we first rewrite it as $2.45\overline{383838}+13.31\overline{921921}$.

We argue that this fourth process can help making $\theta_=$ arise and, meanwhile, to make repeating decimals become numbers. We explain in the next section.

Monoïd praxeology *versus* number praxeology

The equality $0.\overline{9}=1$ is not relevant when one deals only with the comparison praxis but it is needed for calculations. Hence, we focus here on praxeologies which arise related two types of tasks: comparison, $T_<$, and sum, T_+.

We saw several techniques to solve T_+ but, except the last one given in the previous section with our algorithm, all rely upon a technology of $\Theta_\mathbf{R}$ or $\Theta_\mathbf{Q}$ that has to be accepted without any justification. Hence, we focus on our algorithm interpreted as a technique denoted by τ_+. Since students may justify the technique themselves τ_+ by θ_{bts}, the algorithm seems very interesting in order to introduce $\theta_=$ – this τ_+ justification is the main difference compared to the other techniques for the sum.

We begin to build a praxeology $[T_<,T_+,\tau_<,\tau_+,\theta_{bts}]$ related to the R_d register of the repeating decimals. It is important to notice here that one works with repeating decimals, and quite easily, whether they are interpreted as rational numbers or not.

We easily get that, for any periodic expansion a with a non-zero period, we have $a+0.\overline{9} = a+1$. Hence, in the set of periodic expansions, we do not have the usual simplification property that allows to deduce from $a+c = b+c$ the equality $a = b$. To recover it, it is necessary to identify $0.\overline{9}$ and 1. Here, this is the time for the choice of $\theta_=$: do we accept it or not? This question is directly related to the choice between two mathematical organizations linked to the comparison, $T_<$, and sum, T_+, types of tasks:

- $MO_\mathbf{Q}=[T_<,T_+,\tau'_<,\tau_+,\theta_{bts},\theta_=,\Theta_\mathbf{Q}]$ that could be extend to others operations both in R_d and R_f, that is the theory $\Theta_\mathbf{Q}$ of rational numbers.
- $MO_m=[T_<,T_+,\tau_<,\tau_+,\theta_{bts},\Theta_m]$ that could not be extend to fractions nor to other operations. Here, we only have a non regular monoïd (it is not a semi-group).

The aim of our investigation is to use this opportunity to show students where the problem is, in order to make them understand that only one of these two possible choices leads to the convenient notion of rational numbers.

AT THE BEGINNING OF UNIVERSITY

A test was given to 29 students in mathematics in the first year of university. It was presented in two steps: first an individual test (see annex 1), then, two days later, a team test (see annex 2) with three students in each group. The first step was diagnostic: understanding of the coding (questions E1 and E2), comparison (questions C1 to C4), sum (questions S1 to S6) and difference (questions D1 to D3). The team test showed the algorithm, asked for an explanation and then proposed an activity with the intent of presenting the $\theta_=$ alternative.

Students knew the two representations of rational numbers since grade 10 because of the ancient French mathematics secondary syllabus (before 2009), and heard of it again in their first semester of university. The case of $0.\overline{9}=1$ was taught as well.

This investigation follows a previous test in grade 10 and in the first year of university (Vivier, 2011): in grade 10 many of the 113 students were able to use the algorithm, but they were not able to explain it probably because the base ten system (θ_{bts}) is not understood well enough. Hence, we think that *good level* for this kind of experiment in France is at the transition between secondary and university level.

Individual and diagnostic test

The coding (E1 and E2) caused some problems to 6 students and, obviously, they did not succeed the test (they were subsequently dispatched over 6 different teams).

Comparison of the three non problematic cases was successful for all students. They wrote the numbers in extension and used $\tau_<$ for comparing them. As we expected it, $\tau_<$ is not problematic at all since it is the same technique for MO_Q and MO_m.

But, for the case of $0.\overline{9}$ and 1, only 8 students stated the equality and 21 the inequality, a ratio close to Tall(1980) where it was 14 over 36 students. Among the first ones, two students wrote that this case was seen before, one of them qualifying this case as "strange". One other student stated both the equality and the inequality, arguing for the equality that it is "because it is not a real number", and for the inequality by comparing the unit cipher of $0.\overline{9}$ and 1 (she used $\tau_<$). One can see here the gap between MO_Q and MO_m.

Unsurprisingly, 25 students computed the sums and differences by approximation. Some students used sometimes a conversion into R_f and also the equality $0.\overline{9}$=1 (see the third process above). As expected, 14 students gave *infinitesimal answers* (Margolinas, 1988) such as $0.\overline{5} + 0.\overline{7} = 1.\overline{3}2$.

Finally, 7 students concluded that 2 – 1,$\overline{9}$=0. All of them having previously declared that $0.\overline{9}$=1. 11 students gave $0.\overline{0}1$ as a result (infinitesimal answer again), 6 gave $0.\overline{1}$, 2 gave 0.0001 (with a finite number of 0) and one gave $0.\overline{10}$ as an answer. More generally, no student who affirmed $0.\overline{9}$=1 wrote an infinitesimal answer.

The equality $0.\overline{9}$=1 seems to point a more general knowledge on repeating decimals related to MO_Q. Even if repeating decimals are written in extension for treatments, the interpretation and control of results show a difference between the two mathematical organizations MO_Q and MO_m beyond the $\theta_=$ understanding.

Despite the frequent appearance of rational numbers written in base ten, in secondary teaching as well as in the university, most of the students do not agree with the equality $0.\overline{9}$=1 and about half of them write infinitesimal answers for the sum. Hence, it seems reasonable to say that previous teaching neither gave them sufficient control nor understanding of rational numbers in decimal writings, even if our experiment involved only mathematics students. MO_Q seems to provide more control and a better understanding for the sum than MO_m, even for tasks not involving $\theta_=$.

The team test

The justifications of the algorithm were quite good for 4 teams: in terms of anticipation of the carry coming from the right, existence of two carries ("local" and "inherited"), periodicity of the periods and some mentions of the stability of the periods lengths. Two other groups proposed some partial justifications, three did not write any justification and one team did not understand the algorithm.

Apart from team I in which the algorithm was not understood, teams gave the answer 1 in the third line of the table (see annex 2). Hence, the activity is adequate to set the problem that $x=0.\overline{9}+a$ even if $x-a=1$. Only the four groups in which there was at least one student who wrote $0.\overline{9}=1$ in individual test made the remark $0.\overline{9}=1$. Groups A, B and E expressed the apparent contradiction between an algebraic calculation (such as $0.\overline{9}+a-a$ is equal to 1) but concluded, as group J, that there are some approximations. Group F wrote: "if one uses the algorithm then $0.\overline{9}=1$" leading us thinking that the validity of the algorithm is quite suspicious for that group.

Hence, in spite of the opposition that emerged, our objective is not fully attained since $\theta_=$ not arose but only some contradictions related to $\theta_=$. A first explanation is that the short time at our disposal was probably not sufficient. Second, the fact that the test was given after recalling the students that $0.\overline{9}=1$ was obviously a bias that might have inhibited remarks and reflexions. It would therefore be interesting to make a comparison with secondary students (grades 11 or 12), students at the very beginning of university, and student-professors of primary education.

CONCLUSION

It appears clearly, and it is not very surprising, that both secondary and first year university teaching are not suited to understand the problem of the double representations of decimal numbers in base ten system. However, from the individual test, one may think that the understanding of the equality $0.\overline{9}=1$ is important for a general understanding of MO_Q that seems to give a powerful control on calculations.

We pointed out the following alternative: do we take $\theta_=$ or not? Besides the fact that the objective of the team session was not fully attained, overall because of a too little working time on the algorithm, we think that the use of comparison and addition could help in exhibiting and understanding this necessary choice. Indeed, from this working session, the opposition linked to the alternative arises.

Our position is close to the investigation of Weller et al (2009): they shown that working on comparison, sum and difference of rational numbers written in the decimal system – they used a software which makes calculations with fractions, hidden for the students – is important in order to understand the equality $0.\overline{9}=1$.

We intend to pursue our investigations in order to understand if and how our algorithm could foster students' understanding of the double representation of decimals. We think that another point has to be considered: conversion into fraction by long division is interesting for validation and control of the addition algorithm.

NOTES

1. In this paper we only consider numerical registers.

2. A simplified French version is available at <http://hal.archives-ouvertes.fr/hal-00593413>.

3. When dividing a by b, one has just to replace the usual condition on remainders, $0 \leq r < b$, by the new one $0 < r \leq b$ in order to obtain the alternative representation of decimal numbers, with an infinite sequence of 9. Moreover, the fact that, in R_d, two different sequences of digits could represent the same number constitute a gap with the case of decimal numbers.

4. Without identifying the double representation of decimal numbers, the obtained set is not **R** but a Cantor set.

5. This kind of exercises appears in some French textbooks at grade 10. The meaning of the equation is quite unusual since the unknown is not the number but one of its representations: for $a=0.\overline{9}$, one has $10a-a=9$ and therefore $a=1$.

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ANNEX 1: THE INDIVIDUAL TEST

CONVENTION : In the decimal writing of a rational number, we write the period by a bar above it. Hence, the number 0.127272727… with period 27 could be writen as $0.1\overline{27}$.

E1) Circle the number which is different from the other ones:

$5.0\overline{0100}$ $5.001\overline{0001}$ $5.0\overline{01000100}$ $5.0\overline{0101}$ $5.00\overline{1000}$

E2) Write in four different ways the number $14,\overline{121}$.

Circle the right answer and then justify your choice:

C1)	$8.13 < 8.\overline{13}$	$8.13 = 8.\overline{13}$	$8.13 > 8.\overline{13}$
C2)	$3.\overline{4} < 3.\overline{40}$	$3.\overline{4} = 3.\overline{40}$	$3.\overline{4} > 3.\overline{40}$
C3)	$0.\overline{9} < 1$	$0.\overline{9} = 1$	$0.\overline{9} > 1$
C4)	$45.\overline{101} < 45.1\overline{01}$	$45.\overline{101} = 45.1\overline{01}$	$45.\overline{101} > 45.1\overline{01}$

Compute the following sums:

S1) $0.\overline{24} + 0.\overline{57}$	S2) $6,7\overline{1} + 1,8\overline{5}$	S3) $0.\overline{5} + 0.\overline{7}$
S4) $0.0\overline{8} + 0.\overline{2}$	S5) $0.\overline{9} + 0.\overline{4}$	S6) $0.\overline{5} + 0.\overline{72}$

Compute the following differences:

D1) $2,1\overline{7} - 0.\overline{7}$ D2) $2 - 1,\overline{9}$ D3) $1,\overline{28} - 0.\overline{72}$

(*Two different individual tests, of equal difficulty, was given to avoid cribbing.*)

ANNEX 2: GROUP TEST

Q1) We propose to discover a new algorithm to compute the sum of two rational numbers in decimal writing. In your opinion, is this algorithm give the good result? Justify your answer. (*The two examples of figure 1 was given.*)

Q2) In trying to find all the cases, give at least five other sums involving two rational numbers in decimal writing and compute these sums with the proposed algorithm. Note that, whether the case you consider, some adaptations of the algorithm are required.

Q3) Chose, each of you, a number *a* with a period. One defines, for each number *a* chosen, the number $x = 0.\overline{9} + a$. Fill the following table:

	Student 1	Student 2	Student 3
a			
$x = 0.\overline{9} + a$			
$x - a$			

Which remark(s) this table may suggest?

ABOUT STUDENTS' INDIVIDUAL CONCEPTS OF NEGATIVE INTEGERS – IN TERMS OF THE ORDER RELATION

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In the presented study we investigated sixth graders' individual concepts of negative integers right before they were introduced to the "world" of the negatives. In order to investigate students' first ideas of negative numbers, we initially investigated their ideas concerning the order relation of integers. With a qualitative data analysis utilizing a theoretical lens concerning individual concept formation, we gained insight into the students' individual procedures and conceptions as well as into how the procedures are linked to the students' previous knowledge.

Keywords: Negative number, integer, order relation, concept formation, previous knowledge.

INTRODUCTION

In mathematics education there are many studies dealing with the development of specific content areas (i.e. algebra, fractions and functions) and other studies analyzing learners' perspectives. Many of these empirical works focus on the outcomes of students' learning, but not on the learning *processes* themselves. Furthermore the misconceptions and difficulties in individual learning processes are analyzed with respect to an ideal learning trajectory, whereas the productive potential of these individual processes is not taken into account. In our contribution the aim is to structure and understand students' learning processes in mathematics. Thereby we use a theoretical approach, which enables us to reveal key points of misunderstandings as well as main influencing factors. These constitute the basis for an adaptive structuring of the mathematical content and hence for the development of instructional designs.

THE RESEARCH OBJECTIVE

Negative integers constitute a significant topic in mathematics teaching and learning. The concept of negative integer is relevant for both, the handling of inner mathematical situations such as solving the equation x+3=1, as well as the handling of real-world situations. Although negative numbers have not been subject of research as often as e.g. fractions or other topics for a time (cf. Bruno & Martinón 1996), the number of studies and papers is increasing. With the presented study, we want to make a contribution to the current state of research by investigating one aspect, which has not been subject of research sufficiently. The aim of the presented study is to investigate the very beginning with negative numbers in mathematics class: It is our special interest to investigate the students' levels of knowledge when being introduced to negative integers in school and to analyze which parts of their previous knowledge students take into account (cf. Bruno & Martinón 1996). We focus especially on the students' understanding related to the ***order relation*** of integers, which constitutes – followed by addition, subtraction etc. – one of the first topics when learning about the negatives. The suitable continuous ordering (in terms of the formula $a<b\ (a,b \in \mathbb{N}) \Rightarrow -b<-a$) is not self-explanatory for students, who may order the numbers as well by focusing on magnitudes (in terms of the formula $a<b\ (a,b \in \mathbb{N}) \Rightarrow -a<-b$). "Ordering negative numbers is complex because there are two possible orderings that are supported by thinking about common contexts – the standard ordering and ordering by magnitude (absolute value)" (Widjaja, Stacey & Steinle 2011, 81). While there are already some findings about the ordering of negative integers (see Bruno & Cabrera 2005, Thomaidis & Tzanakis 2007, Widjaja et al. 2011), an access to the *interplay of concepts* is still missing: In order to learn more about the very beginning with negative integers, there has to be gained more insight into the parts of the students' *previous knowledge* taken into account while ordering integers, and to the reasons *why* students use the one or the other ordering.

This perspective constitutes the basis for our investigation. The expected findings are assumed to serve for a structuring and organization of the mathematical content of the introduction of negative integers, especially of ordering integers.

The research issues of the present study are the following.

(1) Which *procedures* do students have for ordering integers? Which concepts and which assumptions are interwoven in these strategies? Which difficulties do students have?

(2) Which *previous knowledge* is taken into account? Which reasons can be reconstructed for the students' approaches to order integers? Which previous-built concepts are taken into account?

In our research project we pursue these questions with regard to two main issues: *Firstly* we want to figure out, how students handle situations just shortly *before* they get to know negative integers in school in order to bring their previous knowledge to light. We intend to assess how students perform due to this previous knowledge without having had any introduction to the topic of negative integers before. *Secondly* we intend to focus on *concept formation processes*. It is our aim to investigate these processes on the one hand in a short-term perspective – to analyze how difficulties and misconceptions can be overcome and which impetus is helpful. On the other hand we focus on a long-term development over a lecture series and intend to find out how procedures and utilized concepts change. Because of the limited space, we have to restrict ourselves to the first issue, i.e. the previous knowledge, in the presented paper.

METHOD

For the purpose of our investigation, half-structured, task-orientated clinical interviews (cf. Selter & Spiegel 1997) were undertaken with eight sixth graders in German schools. The interviews were followed by a lecture series to introduce negative integers by means of a suitable learning environment, which again was followed by interviews. The qualitative analysis of video sequences and transcripts was taken as a basis for evaluating the data.

For the purpose of this study, students got different pairs of integers and had to determine the “greater” one. As negative numbers are in a way ‘fictive’ as they are not physically perceptible on their own, representations have a pivotal importance for the teaching and learning of negative integers. There are most notably four kinds of representation, which seem to be meaningful for getting to know the negatives (expanded from Bruno & Martinón 1999): These are (a) the representation on the number line or other ordinal arrangements regarding the order of integers (like ... -3 -2 -1 0 1 ...) (b) a quantity representation, which students mostly know from natural numbers (like e.g. “3 means three spots”), (b) the representation in a real-world context (e.g. temperatures, debts-and-assets) as well as (c) the symbolic representation (e.g. -6 or “minus six“). These kinds of representation and their interplay are of great importance for the learning of negative integers (cf. Bruno & Martinón 1999).

Whereas the results of some studies (e.g. Malle 1988) indicate that most of the students seem to know the symbolic representation of negative integers from the very start of the introduction to negative integers, the aim of our investigation was to find out if the students in our study also knew the symbolic representation *before* having had an introduction to negative numbers. Beyond that, we intended to find out if students are able to utilize the number line or other ordinal arrangements, and in what way they fall back on real-world contexts. Therefore, the integers were presented in a symbolic way and the use and change of representations was investigated.

Naturally, by displaying integers in a symbolic way, the knowledge students are able to activate, is limited: If integers were for example given in a contextual way, students would surely be able to handle situations more competently, using real-life knowledge more easily. But in our study, we wanted to figure out which knowledge students are able to activate when regarding numbers such as 12 or -15: if they alter the kind of representation for their reasoning, if they activate contextual knowledge etc. In the interviews students first compared a positive with a negative integer, then two negatives, followed by the comparison of a negative and a positive integer each with zero. In doing so, students received two small cards each of them displaying an integer in a symbolic representation (e.g. 12 and -15).

A GLIMPSE OF THE THEORETICAL AND ANALYTICAL FRAMEWORK

For the concern to analyze students’ individual concepts and their development we use a theoretical approach (Hußmann & Schacht 2009, Schacht 2012), which serves as a theoretical lens for analyzing concepts and from which appropriate analytical tools can be deduced. The theoretical approach is based on different influences: For the epistemological framework, Robert Brandom’s semantic Inferentialism (e.g. Brandom 1994) constitutes a basis and an essential factor from a philosophical perspective. Gérard Vergnaud’s Theory of Conceptual Fields (e.g. Vergnaud 1996) replenishes the theoretical framework from a rather psychogenetic and mathematics-didactical point of view. The background theory will be briefly outlined in the following by giving a short overview of the crucial ideas. The arising building blocks of our analytical scheme (commitments, inferences, focuses (see below)) will be illustrated and explained in detail afterwards.

In the underlying ***theoretical framework***, the individual's responsibility for his or her actions and utterances is regarded as essential for the conceptual. For the contemplation of the individual's personal responsibility, the idea of the *language game*, which can be attributed to the late Wittgenstein, plays an important role. The ability to give reasons and to ask for them in the language game is regarded as substantial for the individual's responsibility: In the language game, individuals show responsibility for their utterances and actions by being able to give reasons for them. For the analysis of the conceptual, individual commitments and individual judgments are essential: Individual *judgments* are the smallest units individuals can take responsibility for. They have propositional content, such as the judgment "Negative numbers are below zero". ***Commitments*** are those individual judgments, which are made explicit by the student as propositions (e.g. "Zero is the freezing point.").

Within our analysis, we investigate *commitments* as well as the *relations*, which exist between commitments in terms of the students' individual reasonings: the individual *inferences*. There is an ***inferential relationship*** between commitments if one commitment entitles the student to affirm another one (e.g. "Negative numbers are below zero *because* positive numbers are above zero.") and if the student accepts this inference as true. Practical reasoning discloses such inferential relationships.

Furthermore, we analyze students' individual *focuses* in terms of individual categories that are used to structure situations and to select the given information (cf. Vergnaud 1996, 225). ***Focuses*** are categories, which are used to handle and select the information of a given situation (e.g. 'the minus sign' or 'subtraction' or 'the number line' or 'changes in temperatures'). Individual commitments, inferences and focuses form students' individual *inferential webs*.

Beyond that, it is our aim to investigate if students use commitments (respectively judgments) and focuses *across situations* – to what extent they are *invariant* (cf. Vergnaud 1996). By analyzing the invariant focuses and commitments we can investigate which situations belong together in the students' eyes, which situations belong – according to the students – into the same class of situations and much more.

To sum up, the kernel of our theoretical framework signifies that understanding a concept means understanding the use of a concept with its reasons and its inferences.

In the following, the main building blocks of our ***analytical scheme*** (focus, commitment, inference) are illustrated and explained by giving a glimpse of the analysis of one student's way of proceeding during the pre-interview. The case of the student Nicole was chosen to be presented here. While working on the task to name the greater of a positive and a negative integer in the pre-interview she proceeds in a way, which is interesting as she seems to subtract the two integers and compare them by using the difference. In the following, an excerpt of the transcript of the pre-interview is presented in order to illustrate the analytical scheme. The scene took place in the very beginning of the interview.

Interviewer:	I brought along two numbers and I want you to tell me which one of them is greater (gives two small cards to the student, displaying 12 and -15)
Nicole:	(looking at the cards) (6sec) The fifteen. (6sec) No the twelve, because there at- in front of the fifteen there is- <u>minus</u> fifteen is displayed there- then it has to be the twelve.

Our approach is based on the assumption that concept formation takes place within situational conditions and that a strong interrelationship exists between situated action and conceptualization. For the understanding of students' concept formation it is indispensible to investigate the situations in which it takes place and the aspects on which students focus. By setting ***focuses*** in situations, students on the one hand deal with the situation at hand and on the other hand they simultaneously deal with the concepts they possess. While trying to handle available situations, students use individual concepts, relationships, properties etc. as categories that they developed before and that enable them to select the diverse information embodied in the given situation. In the given interview sequence Nicole seems to focus on the ***minus sign***.

Besides individual focuses the student makes assumptions concerning the situation, about relationships of the properties etc. They are interwoven with the focuses since focuses give the orientation for the assumptions to be made. In our approach, we are looking at these assumptions by examining individual ***commitments***. In the above-mentioned case of Nicole, there can be assigned the following commitments to her utterances: Firstly, she seems to commit to *"Among (-)15 and 12, (-)15 is greater." (cm01)* At this point in time, it is not yet reconstructable whether she means that 15 or minus 15 is greater. But she seems to reject this commitment by uttering "no" and committing to another, contrary commitment: she commits to *"Among (-)15 and 12, 12 is greater."(cm02)* and obviously tries to entitle this commitment by giving a reason for it. In other words, the latter commitment *(cm02)* constitutes a conclusion and she is trying to give a premise for it. In our theoretical approach these entitlings have a pivotal importance for reconstructing ***inferences*** as they concern the reasons, which students give for their commitments and focuses and finally for their approaches. In Nicole's case, she seems to entitle the commitment "*Among (-)15 and 12, 12 is greater."(cm02)* by committing to the fact that there is a minus (sign) in front of 15. The commitment *"There is a minus sign in front of 15."(cm03)* is assigned to her utterance. The inferential relationship between the commitments is assigned as an inference: *"Among (-)15 and 12, 12 is greater, <u>because</u> there is a minus in front of 15."*

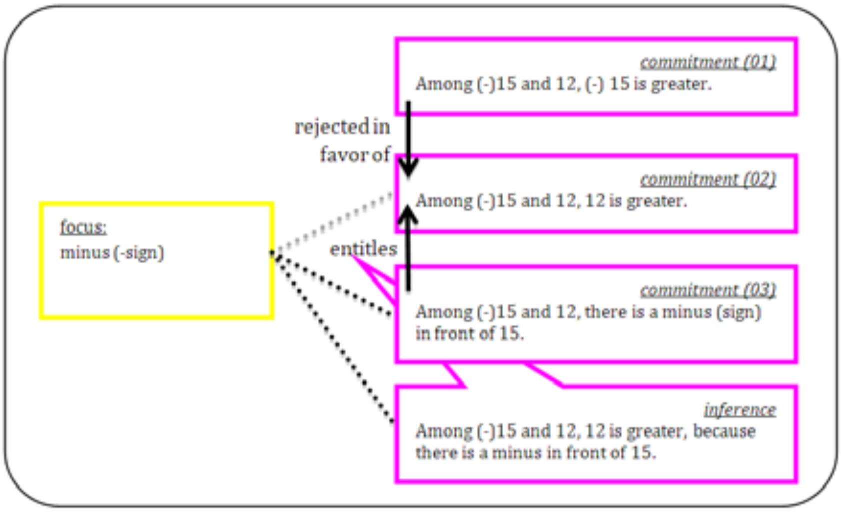

Figure 1 Nicole's focus, commitments and inference

Focuses, commitments and inferences form inferential webs. Understanding the concept of integer means to be endued with an ***inferential web*** in which the concept of integer itself and other related concepts are involved. By analyzing the students' individual inferential webs it is possible to get an insight into the above-mentioned research issues: It enables us to uncover the students' individual approaches in detail as well as their difficulties (research issue 1) and they make it possible to focus on the previous knowledge that the individual approaches rely on (research issue 2).

The outcomes of the analysis concern the above-mentioned research perspectives as they give a detailed insight into individual procedures, into difficulties and possible reasons especially for the point in time when getting to know negative integers. By examining inferential webs, it can be investigated which previous concepts are taken into account and how the inferential structures change during the learning process. This constitutes the basis for structuring the mathematical content as well as for the optimization of the developed learning environment.

Nicole's strategy of subtraction

At a later time, the analysis unfolds *why* the focus on the minus sign leads Nicole to the inference *"Among 12 and -15, 12 is greater because there is a minus in front of 15"* (see above). The commitments "*In order to determine the greater number among -15 and 12, I can subtract."* with the focus on *subtraction* as well as "*The numbers 12 and -15 form a mathematical task."* with the focus *mathematical task* that can be assigned to her utterances seem to have a pivotal importance. It appears that she perceives the situation as a mathematical task, which prompts her to subtract the two numbers. It also becomes clear *why* she seems to focus this way:

Interviewer: How did you get the idea of making a mathematical task out of it?

Nicole: Because there, well (putting the cards side by side: 12 -15), it is, let me say, cut out. Because the twelve is a number (pointing to 12) and then, the minus

> is already there (pointing to the minus sign), and then f- m- f- minus fifteen and then it is actually a mathematical task.

There are three commitments and corresponding focuses, which point out why Nicole is thinking about a mathematical task and – as a consequence – subtracts. These are (a) "12 is a number", (b) "I have got the minus of the mathematical task from the -15", (c) "I subtract 15". While regarding the numbers 12 and -15, she pays attention to what she already knows: She first focuses on the 'normal' *number* 12 as the first part of a mathematical task. Then she focuses on the *minus sign*, which is required for a mathematical task. For her, the 'minus' is linked to the focusing of *subtraction* – it indicates a subtraction in terms of a binary use of the minus sign (Bofferding 2010) as an operative sign (Vlassis 2004) – and *not* to the focusing of a signed number, where the minus-sign is a predicative and unary one (ibid.). By means of the further analysis of Nicole's inferential web, her approach concerning the comparison of the two integers as well as the influence of her previous knowledge can be increasingly detailed. In case of a negative and a positive integer the inferential web likewise shows that she uses a conspicuous strategy of first subtracting, then replacing the minuend with the result of the task und comparing the re-placed minuend (the result) with the subtrahend subsequently.

By analyzing the invariance of focuses and commitments resp. judgments we found the *individual classes of situations* that Nicole differentiates. As she does not seem to interpret the minus sign as a predictive sign, it is not surprising that the classes, which she differentiates individually, differ from classes that can be determined from a theoretical point of view. Nevertheless, it is interesting and partially surprising, which classes she is making a distinction between: Slightly different focuses, commitments and inferences, which she uses according to the absolute values of the given numbers, indicate that she differentiates, for instance, between the classes in which she compares two integers in the form a and -b ($a, b \in \mathbb{N}$, $|a|<|b|$) or in the form a and -b ($a, b \in \mathbb{N}$, $|a|>|b|$): In the latter class, she seems e.g. to focus on natural numbers as amounts and she claims that she cannot subtract in the way used before (see above). Irrespective of the differences in these individual classes, there are also many invariant commitments and focuses as the student is trying to adapt her procedures to different classes of situations. The inferential webs of other classes of situations (e.g. comparing two negative integers) show, that she uses partly the same focuses but adapted commitments according to the situations. The case of Nicole shows that she seems not yet to know the minus sign as a predictive sign. Because of that she naturally is not able to interpret the integer as negative integer, but she proceeds in a comprehensible way, which makes sense from her individual perspective.

Tom and his recourse to a real-life context

In order to give a glimpse of the broad range of the students' previous knowledge, Tom's way of comparing the numbers 12 and -15 is mentioned in the following.

Interviewer: I brought along some cards, and on the cards there are displayed two numbers. And I want you to tell me which one of them is greater (gives two small cards to the student, displaying 12 and -15)

Tom: (receiving the cards) May I -

Interviewer: Mhm (affirmative), you can take them.

Tom: (looking at the cards, holding them in his hands) The twelve is greater.

Interviewer: Mhm? (quietly)

Tom: Because (slowly) minus is below zero and the twelve is just normal, the twelve.

Interviewer: Okay. How do you imagine that?

Tom: Well at the thermometer you sometimes also see minus numbers and then, if there, er, is no minus there, then there is simply just the number. Twelve degrees or so.

When regarding the cards briefly, Tom at once seems to commit to "*Among -15 and 12, 12 is greater"*. In order to justify this commitment, he gives two reasons, which he combines. The inference "*Among -15 and 12, 12 is greater* <u>because</u> *minus numbers are below zero* <u>and</u> *12 is a normal number."* can be assigned to his utterances. Asked how he imagines that, he mentions the real-life situation of negative numbers on the ***thermometer***. Commitments like "At the thermometer there are minus numbers." and many more can be assigned to his utterances in this excerpt of the transcript and the following ones. For him, the real-life context of temperatures, especially the *thermometer*, the concept of *natural number* ('normal number') and the concept of *negative number* ('minus number') seem to constitute important focuses, which characterize his way of proceeding when trying to determine the greater of these two integers. Later in the interview it transpires that the real-life context of ***temperatures*** seems to serve as a solid basis for Tom's inferential web including negatives. Tom's experiences concerning rises and falls in temperature seem to serve as basis for his concept of integer.

THE OUTCOMES OF THE STUDY – AN OVERVIEW

The presented findings were deduced from two case studies (Nicole and Tom). The analysis of the students' inferential webs gives insight into the broad spectrum of the students' levels of knowledge before having had an introduction to negative integers. In the following, some of the main findings are outlined and reflected. They concern students' procedures and difficulties as well as the involved previous knowledge.

Concept of natural number By the analysis of Nicole's pre-interview we found that there *are* sixth graders whose inferential webs, which are activated when regarding symbolic represented integers like -12, mainly affect natural numbers. The analysis shows that Nicole does not yet seem to know about negative numbers. Consequently, she is not yet able to interpret the minus sign as a predicative sign. Instead, she activates previous knowledge in terms of arithmetic problems, which makes sense from her perspective: She knows "normal" numbers and she knows about the minus sign as indicator of subtraction which leads her to subtract the two numbers and compare them consequently. She develops a subtraction scheme with its own rules, and she tries to activate and adapt it to new situations. She seems not to know that there are negative integers beneath positive ones. Our findings seem to support the suggestion, that some of the students have to overcome (mis-)conceptions from elementary school, e.g. that there are no numbers below zero (cf. Bruno 2001, 415). The finding that students partially seem not to be able to interpret the minus sign as predictive sign, indicates that an introduction of negative integers in mathematics class should not purely be based on a symbolic representation, but probably should resume experiences in real-life contexts.

Concept of negative integer On the contrary, the case of Tom shows that there *are* students, who already have a substantial and sophisticated inferential web including negative integers at their disposal – even before they received an introduction to negative integers in school. Tom's inferential web seems to be relatively stable and to a great extent tenable from a mathematical point of view. It is interesting to have a look at the previous knowledge that Tom is activating: He utilizes contextual knowledge from the context *temperatures* (see above). This shows how powerful real-life experiences may be for conceptual development. We also found that the assigned inferential web in Tom's preliminary interview indicates an ***order relation***, which focuses on magnitudes in terms of a "divided number line" model (Mukhopadhyay 1997). The individual *focus* on the *absolute value* of integers ("the value of the number") is of vital importance for this order relation, which seems to be related to his previous knowledge in terms of natural numbers. The findings indicate that a continuous order relation for integers can not be deduced easily form real-life experiences by students on their own, but it needs educational attention and support.

In summary, it can be noted that the presented study contributes in some respects to the state of research in terms of negative numbers. We found that the students' previous knowledge is widely spread: some students were already able to interpret the symbolic representation of negative numbers before being introduced to this topic, whilst others – like Nicole – were not. Beyond that, by using our theoretical and analytical framework we gained a detailed insight of the activated knowledge, the causes of misconceptions, the interplay of activated concepts etc. Concerning the order relation, we got a detailed insight into which individual focuses the investigated students chose and how they influence students' procedures and their sustainability.

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REPRESENTATIONS AND REASONING STRATEGIES OF GRADE 3 STUDENTS IN PROBLEM SOLVING

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In this communication we analyse the representations used by grade 3 students and how they relate to mathematics reasoning on a mathematical problem. Data were collected through audio and video recording in the classroom as well as students' written productions. Students that use schematic representations are those who have more success in solving the problem, in contrast with those that use pictorial and symbolic representations. The reasoning of some students shows interesting instances of making generalizations and working in a systematic way.

INTRODUCTION

The use of mathematical representations plays a fundamental role in students' learning of this subject, which is underlined by curriculum documents in many countries (e.g., NCTM, 2000). Reasoning, another important mathematics process, is highly dependent on the use of representations (Ponte, Mata-Pereira & Henriques, 2012). However, how students use these processes is an issue under-researched, in particular at elementary school level. This communication analyses the representations used by grade 3 students and how they relate to mathematics reasoning when they work on a mathematical problem.

MATHEMATICAL REPRESENTATIONS AND REASONING

In their mathematical work, students use a variety of external representations. Webb, Boswinkel and Dekker (2008) mention three key kinds of representations: informal (produced by students themselves, closely related to the context), preformal (still connected to the context, but including some abstract and formal aspects) and formal (with mathematical notations and language). In their perspective, students begin by using informal representations and gradually move on towards formalization. Some investigations describe the difficulties of students with specific representations. For example, Deizman and English (2001) refer the difficulties that students have in using and understanding diagrams, an important visual representation. The authors consider that these difficulties are related to misunderstanding the notion of diagram (not knowing the structure of the representation and overvaluing superficial characteristics), to incapacity to elaborate adequate diagrams (choosing the diagram that represents the situation or being unable to structure in a diagram the information provided) and to incapacity of reasoning in an proper way with diagrams (making valid inferences regarding the problem).

Reasoning is the process of making proper inferences from given information (Ponte, Mata-Pereira & Henriques, 2012). Mathematics reasoning is thought of as essentially deductive, deriving statements in a logical way from given propositions, but it also may be seen as having an experimental side, making conjectures from specific cases and testing them (Pólya, 1990). Induction is the process of making a generalization by abstracting common features in a number of cases. Abduction is the process of establishing a general hypothesis that may explain a phenomenon from which only some elements are known. Rivera and Becker (2009) present in the following way the relations between abduction, induction and deduction:

> Abductive reasoning involves forming a reasonable hypothesis about the phenomenon. To form that hypothesis, we verify and test the abduced hypothesis several times to see whether it makes sense. (…). Following Peirce, abduction as a concept exists and is used only in relation to induction. Thus, a complete pattern generalization involves complementary acts of abduction and induction and, of course, justification. (p. 217)

According to Lannin, Ellis and Elliot (2011) mathematics reasoning involves processes such as conjecturing, generalizing, investigating why and developing and evaluating arguments. From these, we give especial attention to generalizing (the key process of inductive and abductive reasoning) and justifying (the key process of deductive reasoning). Teachers may understand their students' reasoning from the representations that they use, since these yield a record of their efforts to grapple with mathematics (NCTM, 2000). Mathematical reasoning takes place in solving mathematics problems. As Wickelgren (1974) indicates, solving a problem may be regarded as achieving a goal using certain operations on certain givens. In any problem, besides the explicit information provided, there is always the need to use also implicit information regarding goals, givens and operations. Solving a problem is making inferences (or reasoning) from the available explicit and implicit information in order to reach the stated goal. The global approach to solving a problem is a reasoning strategy. As this author indicates, random trial and error is the "first thing that most people do when confronted with a problem" (p. 46). A most desirable strategy to solve problems is systematic trial and error, that is a "method that automatically [produce] a mutually exclusive and exhaustive listing of all sequences of actions up to some maximum length" (pp. 46-47). As this author points out, between random and completely systematic trial and error "there can be different degrees of systematicness" (p. 47)

RESEARCH METHODOLOGY

This study follows a qualitative approach and was undertaken in a school near Lisbon with the first author as a non-participant observer (Bogdan & Biklen, 1994). The participants are the teacher Fernanda and her 19 grade 3 students (teacher and student names are pseudonyms) who have been together since grade 1. The teacher has been working in this school for the last 10 years and has tenure. She indicated that the students were used to solve problems similar to the one reported on this paper. In this communication we analyse the students' work in one task (during about 40 minutes) that we choose because we found it particularly appropriate to study students' representations and reasoning strategies.

We made content analysis (Bardin, 1977, Laville & Dionne, 1999) of the students' productions and of audio and video recorded interaction among students and between teachers and students. In the content analysis, we followed the three steps defined by Bardin (1977): Pre-analysis, material exploration and processing of results. In the first phase, we defined categories through an open model (Laville & Dionne, 1999), in the second phase, we defined the registration units by subject (Bardin, 1977) and, in the third phase, we made a qualitative approach of iterative construction of an explanation (Laville & Dionne, 1999). We chose this approach because we wanted to develop, through the data, step by step, an explanation of the phenomenon observed. In analysing students' representations we noticed two main categories (pictorial and schematic) which we further subdivided into finer categories (pictorial detailed/simplified, scheme with dashes/letters). In pictorial representations students made drawings to depict figurative images of the animals; these drawings could be detailed (showing the whole animal) or partial (showing an animal part – heads or hands). In schematic representations, students used more abstract representations like dashes or letters. Some students used drawings as decoration, but reasoned based on schemes so, we considered this as cases of schematic representations. In analysing the participants' discourse by subjects (Bardin, 1977), with a focus on students' reasoning, we came up with two main categories of strategies (working on a random way and on a systematic way) and we further subdivided the second category (working in a systematic way with a single animal first/taking into account the two animals). We come up with these categories working inductively from the data as we explain ahead in the paper. As in the previous analysis, these categories were built by observing regularities in data. In a second level of analysis, we sought to connect the information provided by both kinds of analyses – regarding representations and reasoning strategies. To understand the students' reasoning strategies we also analysed the explanations that they gave to their teacher and to their colleagues in different moments. The teacher intervention during class was not a problem as she was helping them by questioning and not telling them the solution right away. The importance of the interventions from the teacher are clear in our analysis.

STUDENTS' REPRESENTATIONS AND REASONING

Fernanda began by writing on the board the problem for the students to solve it in pairs (one student just worked by herself), providing their response in an A3 paper sheet to be latter collectively discussed by the whole class: "In a farm there are 21 ducks and rabbits. If we count the hands we know that there are 54 in total. How many ducks and how many rabbits are there?" The teacher did a brief explanation about what was intended and the students begun working. When a student pair asked for help, the teacher recalled them the statement of the problem.

Choice of representation

Some students used (informal) pictorial representations but with two levels of detail, representing the whole animal with many features or just representing parts of an animal such as hands or heads. Other students used (pre-formal) schematic representations, either with dashes (to represent hands) or with the first letter of the name of the animal.

Kind of representation	Representations used	Students
Pictorial representation	Detailed drawing of the animals	(1) Maria and Tiago, (2) Vanessa and Eloísa, (3) Joaquim and Francisco
	Simplified drawing of the animals (head or hands)	(1) Renata and Rui, (2) Guida and Júlio, (3) Núria and Daniel
Schematic representation	Scheme with dashes	(1) Dário and Kátia, (2) Ernesto and Patrick, (3) Bruna
	Scheme with letters	(1) Patrício and Sandro

Table 1. Representations used by students in their final work

For example, Maria and Tiago used pictorial representations, drawing successive animals in great detail (figure 1). At some point they begun changing an animal by another. They did not realize that this type of representation does not allow them to get the right answer on time and distract them with too many details, paying attention to irrelevant aspects and not focusing in the important issues at stake.

Figure 1. Representation of Maria and Tiago

On the other hand, Guida and Júlio and Daniel and Núria began by making detailed pictorial representations but they concluded that this was not appropriate and made a new schema, just representing animal hands. Daniel and Núria justified their change: "We were already doing drawings, but then [Núria] said: 'Let us erase all because this way we are going to take a long time!' And then we made the hands!" They showed to understand the need to find and adequate representation to solve the task.

Patrício and Sandro made a symbolic representation (vertical computation), that does not represent the givens of the problem (figure 2a). They found a number that added to 21 gives 54, but 54 is the total number of hands, not the number of animals. These two students seem to assume that symbolic representations are the most adequate and allow more efficient reasoning to solve the problem. They use sophisticated representations in an improper way and with which they could not solve the problem.

Figure 2. Representations of Patrício e Sandro: (a) First attempt; (b) Second attempt

Ernesto and Patrick presented a quite elaborated representation, with circled dashes, 2 for ducks (*pato*) and 4 for rabbits (*coelho*) (Figure 3). In this way, as we shall see, they were able to count efficiently the number of hands and the number of animals.

Figure 3. Representation of Ernesto and Patrick

Reasoning strategies and representations

All student pairs solved the problem by trial and error but used two kinds of reasoning strategies: (i) working on a random way (5 pairs); and striving to work on a systematic way, either (ii) taking into account simultaneously the two animals (3 pairs) or (iii) considering a single animal first and then striving to combine the information regarding two animals (2 pairs) (Table 2). We considered that the students used random strategies when they began to solve the problem drawing animals randomly until they meet one of the conditions of the problem. A second group of students strived to work on a more systematic way. They began assuming that they could solve the problem with only one type of animal and drew 21 animals of a particular type. Then, counting the hands, they realized that they had too much (if they drew rabbits) or too less (if they drew ducks) and started to replace some of the animals drawn by the other animal. A third group of students, assumed that the amount of ducks and rabbits was similar and began to draw them in pairs, one of each kind, until they reached the desired number of animals, and, at the same time, they were counting the hands.

Reasoning	Random strategy	Single animal first, then combining two animals	Taking into account simultaneously the two animals
Student pairs	Patrício and Sandro Joaquim and Francisco Maria and Tiago Daniel and Núria Bruna	Guida and Júlio Vanessa and Eloísa	Renata and Rui Ernesto and Patríck Dário and Kátia

Table 2. Reasoning strategies used by students in solving the problem

i) Random strategy. About a half of the student pairs solved the problem in a random way. For example, Núria and Daniel draw ducks and rabbits for a long time. As we mentioned above, at some point they realized that too much detail was not necessary and began to represent the animals in a more simplified way. They searched the solution by trial and error, but in a completely random way. They represented different combinations of animals and counted the number of hands. They were lucky because in one trial, just by random, they found, the solution of the problem. However, the other four student pairs that used a similar random strategy were not as lucky as them and never found the correct solution (Table 2).

ii) Systematic strategy considering two animals. Three student pairs that defined a systematic strategy from the beginning took into consideration the two animals. For example, Dário's last intervention in the following exchange shows in a clear way that his idea is to take into account both animals at the same time:

Dário: (…) How many rabbits there are… And ducks…

Kátia: There are 21…

Dário: 21 what?

Kátia: Ducks?

Dário: No, no, no… There are 21 ducks… And rabbits!! We must know how many ducks there are and rabbits there are, so that we have the twoooo! Get it?

Another student pair, Renata and Rui, explain to the teacher how they considered the two animals and, at the same time, counted the number of hands (Figure 4):

Renata: We drawn a duck and a rabbit and drawn another duck and another rabbit… We went doing this way…

Teacher: You draw a duck, a rabbit, a duck, a rabbit, a duck, a rabbit… Always…

Renata: Yes… And then… We stopped a little, and we counted… Counted hands and how many animals we had to do and… Counted hands to see if we had to erase or add more, or erase and let it be like this.

Figure 4. Representation used by Renata and Rui

Renata and Rui started working in an inductive way, trying out specific cases. They began by assuming that the number of ducks and rabbits should be similar, so they draw one after the other. At some point, they evaluated their work and realized that although the number of hands was high (already 50), the number of animals was much below of what was needed. So they had to make some changes and erased rabbits putting ducks on their place, so that they got 21 animals. Doing this, they ended up with the number of animals required, but they were surprised to verify that the number of hands was not enough. They were puzzled, but the teacher helped them to understand that they needed to replace back some ducks by rabbits.

Teacher: You have 21 animals, but you are short of hands… And now?

Renata: I have to put some rabbits and… (…)

Teacher: You put more rabbits? But if you draw more rabbits, there will be more than 21 animals, or not? How shall we do that? (…)

Renata: I will erase two ducks and draw one rabbit!

The statement that two ducks are equivalent (in the number of hands) to one rabbit is an important generalization. The idea that this inference (drawn from the givens of the problem in a deductive way) may be used to solve this problem is an important insight. Having done that, the students further conjecture that they would not need to do great changes and they would just need to replace 4 ducks by 2 rabbits. That was close to what was required, but still not the solution of the problem. Supported by the questions of the teacher, the students finally concluded what animals they still needed to draw to solve the problem:

Teacher: So?

Renata: We erased 4 ducks and made 2 rabbits!

Teacher: Yes… And now? How many hands we have and how many animals we have? Rui?

Rui: Now we have… 1, 2, 6 (counts hands one animal by one) 50!

Teacher: 50 hands… I want 54! We are almost there! How many animals we have?

Students (Count one by one and answer): 19.

Teacher: And how many animals we must have?

Renata: 21…

Teacher: You have 19… (…) How many animals are missing?

Renata: We need 2 animals more!

Teacher: And how many hands?

Renata: 50… 54, so we need to draw 2 ducks!!

Teacher: Why?

Renata: Because… 2 more 2 is 4 and we have 50 [hands] and we need 2 more animals and the ducks have 2 hands… I think it will work!

The drawing with the 19 animals that allows for the counting of the 54 hands provides the justification that this is a solution to the problem. It is a solution by exhibition, since the students constructed an object that satisfies the conditions of the problem. Later on, Renata provides a very explicit explanation of the relationship between the hands of ducks and rabbits, which may be regarded as the key generalization in this task:

Renata: If two ducks have four legs and one rabbit has four legs… And we had too much animals… We erased two rabbits… Oh! No! [We erased] two ducks and we draw one rabbit!

Ernesto and Patrick had an elaborated representation with circles and dashes (figure 3) and explained to the teacher their strategy to solve the problem: "We have made schemes by a trial strategy (…) We though 10 ducks and 11 rabbits. But then we saw that does not give us 54 hands and we did it again" (Patrick). They also used an inductive approach, trying out cases, and figured out the equivalence between two ducks and one rabbit.

iii) Systematic strategy beginning with one animal. Other students followed a different strategy, seeking first to fulfill one of the conditions of the problem and then doing attempts to replace some animals by the other. For example, Guida and Júlio began by drawing just ducks:

Guida: We were doing… 2 [Two hands – a duck] up to 19 [animals], but then it doesn't give us 54 hands!

They concluded that if they kept drawing only ducks they would have more animals than required. By realizing the key generalization that one rabbit is equivalent to two ducks they quickly figured out how to solve the problem replacing two ducks by one rabbit or vice-versa until they had the right number of animals and hands. Therefore, they began by representing only one kind of animal, but at some point, they evaluated the situation and they understood that they needed to combine the two kinds of animals and to use a systematic strategy of replacing an animal by another until they got the correct solution.

Vanessa and Eloísa began by drawing just rabbits, until they got 21 animals but they realized that they had much more hands than required. They randomly erased several rabbits and started drawing ducks (but not replacing one by one). They did this several times, until they got 54 hands. At this moment they realized that they had only 19 animals. With the help of the teacher, they finally found the solution:

Teacher: (counts animals with fingers one by one) 19! (…) But you need 21! And the 54 hands? (the students shake their heads meaning yes)

Teacher: What are you going to do?

Vanessa: We need to replace…

Teacher: To replace why? And by what?

Eloísa: Rabbits by ducks…

Teacher: Huummm… How many rabbits by how many ducks?

Vanessa: I am going to take… I am going to take a rabbit and then I will put… (looks at the duck) I am going to put two, because they have two hands!

After working at random for some time, these students finally noticed that a rabbit and two ducks have the same number of hands (the key generalization) and that they may change two ducks for one rabbit, keeping constant the number of hands.

In this way, the student pairs that define a systematic working strategy, either considering one or two animals, and that at some point evaluate their work and discover relationships among different elements (especially the generalization that two ducks have as many hands as one rabbit) are those who more quickly solve the task.

CONCLUSION

Solving this problem involves understanding and representing the conditions, defining a reasoning strategy, and applying and monitoring that strategy. First, the students have to understand the problem conditions and make a suitable representation. To understand the statement of the problem it is not easy for some of the students at this level. Some use pictorial (informal) representations, drawing animals in detail, a very slow, distractive and inefficient representation process. Other students use schematic representations such as dashes and hands (pre-formal), without spending time on details, what makes the solution quicker and allows them to focus their attention in important elements of the problem. Some represent ducks and rabbits in different lines or blocks, or sequences of duck-rabbit pairs, for easier counting. Others, to make sure that the number of animals is equal or easily comparable, draw them alternatively. Still some students use symbolic (formal) representations that do not represent well the problem, naturally with no success.

Based on an initial representation, the students define a solving strategy. The problem may be solved algebraically with a system of four equations and four unknowns (h=4R; k=2D; h+k=54; R+D= 21, with h and k being the number of hands of rabbits and ducks and R and D the number of rabbits and ducks) but grade 3 students cannot do it this way. They have to use trial and error, following an abductive/inductive approach, trying out different combinations of animals. The key to reach to a solution is to work in a systematic way, either taking into account simultaneously the two animals, or considering a single animal first and then combining the information regarding the two animals. However, to solve the problem the students need to recognize the four relations above. Starting with both animals means to take into account the equation R+D=21, then changing the value of one variable (the other just follows from that) and seeing what happens to the sum h+k. Starting just with one animal is similar, except that we begin with $R=0$ or $D=0$. In both cases, it is critical to notice that in replacing one rabbit by two ducks h+k remains constant (which is the key generalization in the problem) whereas R+D increases 1 unit. This may lead to formulate a hypothesis that replacing one rabbit by two ducks or vice-versa will solve the problem. All steps done by students may be justified by the givens of the problem and the allowed operations. The justification why 6 rabbits and 15 ducks is the solution is done by exhibition, counting the corresponding number of hands.

The students that use pictorial representations tend to follow a random strategy. The students that use symbolic representations also have trouble because the representations that they choose are not adequate to consider the problem conditions. The students that use schematic pre-formal representations (Webb et al., 2008) that salient important aspects of the problem (number of hands of each animal) and omit irrelevant features (e.g., the physical aspect of the animals), are those who formulate the most efficient solution strategies. The fact that students used a wide variety of representations with distinct potential for solving the problem, shows that the teacher may assume an important role, helping the students to use increasingly sophisticated representations with understanding. The difficulty of the students in keeping track of the conditions of the problem and in reasoning in a systematic way also suggests the critical role of the teacher in dialoging with students. This underlines the need to further study how teachers may support their students in learning to use mathematical representations and to develop their reasoning processes.

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THE CONSISTENCY OF STUDENTS' ERROR PATTERNS IN SOLVING COMPUTATIONAL PROBLEMS WITH FRACTIONS

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Error patterns in solving computational problems with fractions are well known. However, the question whether these patterns are consistent or not is still unanswered. This paper presents the design and results of an empirical study focussing on this question in the case of computational problems with fractions. It shows that a considerable part of error patterns and furthermore of approaches is not consistent. This implies that the concerning students do not have any strategies when dealing with computational problems. Instead, their solutions are not rationally chosen but emerge while treating the problem.

Key words: adding fractions, multiplying fractions, students' errors, empirical study

1 ERROR PATTERNS AND THE CONSISTENCY OF ERROR PATTERNS

Learning fractions in secondary mathematics education has two aspects: The students should build up conceptual knowledge (e.g. the interpretation of a fraction as part of a whole) as well as procedural knowledge (e.g. computational skills like adding or multiplying two fractions). This paper focuses on the second aspect. It investigates the consistency of students' error patterns in solving computational problems with fractions.

In that regard, a *computational problem* is a problem that can predominantly be solved by using procedural knowledge as there exist specific algorithms to solve the problem (e.g. adding two fractions by converting to a common denominator). Therefore, neither conceptual knowledge nor heuristic strategies are necessary. Beyond that, a computational problem can be solved in another manner as well: By recalling (e.g. knowing that $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$), by "seeing" the result without any calculation (e.g. $6\frac{5}{7}$ as result of $6+\frac{5}{7}$) or by using conceptual knowledge (e.g. realizing $\frac{2}{9}$ as half of $\frac{4}{9}$ and thus $\frac{2}{9}$ as result of $\frac{4}{9}\cdot\frac{1}{2}$). However, the relationship between conceptual and procedural knowledge and their interdependency still needs further research (Hallet, Nunes & Bryant, 2010).

The *research on students' errors* concerning computational problems with fractions has a long tradition. Frequently appearing errors are well-known and comprehensively documented over many years (Carpenter, Fennema & Romberg, 1993; Kerslake, 1986; Eichelmann, Narciss, Schnaubert & Melis, 2012). An *error pattern* can be identified when identically structured errors appear in the solution of two or more identically structured problems (Prediger & Wittmann, 2009).

The error patterns relevant for this study are given in table 1. Nearly all of them can be characterised as a combination of well-known steps of solution processes being inappropriate for the underlying problem. A common model to explain this fact is the "bug-repair-theory" (Brown & VanLehn, 1980).

Multiplication of two fractions with a common denominator: $\frac{a}{b} \cdot \frac{c}{b} = \frac{a \cdot c}{b}$
Addition or subtraction of two fractions: $\frac{a}{b} \pm \frac{c}{d} = \frac{a \pm c}{b \pm d}$
Addition of a fraction and an integer: $\frac{a}{b} + c = \frac{a+c}{b}$, $\frac{a}{b} + c = \frac{a}{b+c}$, $\frac{a}{b} + c = \frac{a+c}{b+c}$

Table 1: Main error patterns

An error pattern is called *consistent* referred to a student if the student deals with two or more identically structured problems within a short period of time and produces the same error pattern in each (or nearly each) of the underlying solution processes. For example, if a student deals with several computational problems of adding fractions within a test, the common error pattern 'adding the numerators and adding the denominators' may occur in all of his or her solutions – in this case it is consistent. But it may occur that only some solutions show the error pattern and the other solutions are correct – then the error pattern is not consistent. Therefore, the consistency of an error pattern has to be investigated at an individual level.

The question *whether error patterns concerning computational problems with fractions are consistent or not* still lacks research. In literature there is little evidence. Earlier studies always consider the entire test population as they aim primarily at the identification of error patterns or the frequency of their occurrence (e.g. Hart, 1981; Padberg, 1986). Solely Padberg (1986) differs between 'typical errors' (error patterns occurring frequently in the whole population) and 'systematic errors' (error patterns occurring consistently at an individual level). However, he does not provide any data answering the question if error patterns are consistent.

Recent studies concerning computational problems with fractions show a broad range of (correct or incorrect) individual solutions (Hennecke, 1999, using methods of graph theory) or demonstrate that students' procedural knowledge about fractions falls apart into several incoherent classes of problems, even the addition and the subtraction of fractions (Herden & Pallack, 2000, by means of cluster and factor analyses). Beyond that, studies on solutions of linear equations in algebra lead to the hypothesis that error patterns are often not consistent (Tietze, 1988; Stahl, 2000).

Regarding the question whether error patterns are consistent or not, a special view on students' solutions is very helpful. This study gathers students' *approaches* to computational problems with fractions. An approach is a general form of solving the problem, including many individual solutions. Taking the addition of fractions as an example, two approaches can be distinguished: first, converting both fractions to a common denominator; second, adding the numerators and the denominators separately.

Though the presented study can be seen in the long tradition of research on students' errors, the research question is formulated more generally, regarding not only errors but also correct solutions: *Are approaches to computational problems with fractions consistent?* The answer to this question can be seen as one step to understand and model the way in which students solve computational problems.

2 DESIGN OF THE EMPIRICAL STUDY

In order to examine the consistency of error patterns empirically, four sets of six problems were developed (table 2).

Multiplication of two fractions:	$\frac{4}{9}\cdot\frac{1}{2}$	$\frac{8}{5}\cdot\frac{3}{7}$	$\frac{4}{5}\cdot\frac{3}{5}$	$\frac{5}{13}\cdot\frac{3}{13}$	$\frac{5}{2}\cdot\frac{5}{2}$	$\frac{4}{15}\cdot\frac{4}{15}$
Addition of two fractions:	$\frac{1}{2}+\frac{1}{4}$	$\frac{2}{7}+\frac{4}{5}$	$\frac{2}{13}+\frac{4}{9}$	$\frac{1}{9}+\frac{1}{8}$	$\frac{2}{5}+\frac{2}{5}$	$\frac{6}{11}+\frac{6}{11}$
Subtraction of two fractions:	$\frac{3}{4}-\frac{1}{2}$	$\frac{6}{7}-\frac{2}{5}$	$\frac{7}{10}-\frac{8}{15}$	$\frac{2}{3}-\frac{2}{7}$	$\frac{13}{8}-\frac{11}{8}$	$\frac{7}{13}-\frac{7}{13}$
Addition of a fraction and an integer:	$\frac{1}{4}+1$	$\frac{2}{3}+7$	$\frac{2}{13}+6$	$4+\frac{2}{3}$	$6+\frac{5}{7}$	$7+\frac{1}{13}$

Table 2: Four sets of computational problems

The problems within a set differ from each other in the given numbers. For example, the set of problems for multiplication of two fractions contains three pairs of problems: (1) with different denominators, (2) with the same denominator but different numerators and (3) with equal fractions. Each of the pairs contains a problem with large numbers and a problem with small numbers.

Each test sheet comprises 18 tasks in three of the four sets of problems which are arranged in nine variants in a random order to avoid serial effects. Thus there are 36 different test sheets. Three more problems without any relevance were inserted to scatter a series of problems of one set caused by the random order.

The students' working time was 40 min and they were allowed to solve the problems in a free way, e.g. calculate mentally or in a written form but not with a calculator. 428 students participated in the test in July 2011. As the students only worked on three of the four sets of problems (due to the construction of the test), the number of solutions to each problem is lower. The students were 6th- and 7th-graders of Real- and Werkrealschule in Baden-Württemberg, a federal state in south-west Germany. A considerable number of low-achievers attend these schools.

Unlike traditional research on students' errors, the coding of the data focuses on approaches rather than on correct or incorrect solutions. For this reason, simple computational errors are ignored, e.g. errors concerning the multiplication tables like $7\cdot 9=61$. In order to answer the question whether an error pattern is consistent or not, these errors do not matter. This way of transforming the data reduces the data to a necessary extent and features the required precision at the same time. However, in some cases the transformation of the data is an interpretation of written calculations and hence vulnerable to artefacts. A crucial prerequisite is that different approaches can be separated clearly. In the presented study concerning fractions this succeeds very well. Table 3 shows the coding manual taking the example of multiplication.

0	Missing.
1	Correct approach (whether calculated mentally or in a written form; the solution itself may be correct or incorrect).
3	Main error pattern 'denominator maintained' (in the case of fractions with different denominators after converting to a common denominator); further errors might be encountered, e.g. errors concerning the multiplication tables.
9	Other (rare error patterns e.g. multiplication with the reciprocal or performing an operation or incomprehensible solutions).

Table 3: Coding manual taking the example of multiplication

Afterwards the data is analysed with methods of descriptive statistics: frequency tables, cross tables (supplemented by a significance test) and tables of ordered tuples of codes.

3 SELECTED RESULTS

Some results of three areas are given: multiplication of two fractions, addition of two fractions and addition of a fraction and an integer. Thereby the process of data analysis is illustrated.

3.1 Multiplication of two fractions

Table 4 shows the frequencies of the approaches concerning the six multiplication problems (N = 315).

Coding	$\frac{4}{9}\cdot\frac{1}{2}$	$\frac{8}{5}\cdot\frac{3}{7}$	$\frac{4}{5}\cdot\frac{3}{5}$	$\frac{5}{13}\cdot\frac{3}{13}$	$\frac{5}{2}\cdot\frac{5}{2}$	$\frac{4}{15}\cdot\frac{4}{15}$
0 Missing	24	31	14	21	13	21
1 Correct approach	226	228	199	167	207	176
3 'Denominator maintained'	30	25	74	102	53	91
9 Others	35	31	28	25	42	27

Table 4: Multiplication of two fractions – Frequency of approaches

A correct approach occurs more often in problems with different denominators than in problems with a common denominator while the main error pattern 'denominator maintained' pertains the opposite. Besides, some students act consistently in an extreme way: They convert fractions to a common denominator and subsequently deal with the multiplication according to the main error pattern 'denominator maintained'. So do 30 students at the first problem and 25 at the second (among them 17 at both problems).

Considering the problems with a common denominator, the main error pattern is in each of the two pairs of problems obviously more frequent when the denominator is larger. Table 5 shows a detailed analysis of this effect in the case of multiplying two equal fractions: 235 of 315 students solve both tasks by the same approach while 80 generate different approaches. Notably, 31 students performed a correct approach with the denominator 2 ('small denominator') while they maintained the denominator 15 ('large denominator'). The McNemar-Bowker-test confirms that the asymmetry of the cross table is significant ($\chi^2 = 39{,}015$; df = 5; $\alpha < 0{,}001$). This test checks against the null hypothesis that the cross table is symmetrical along the main diagonal.

		$\frac{4}{15} \cdot \frac{4}{15}$ ('large denominators')				
		0	**1**	**3**	**9**	
$\frac{5}{2} \cdot \frac{5}{2}$ ('small denominators')	**0**	11	1	0	1	13
	1	7	159	31	10	207
	3	0	2	50	1	53
	9	3	14	10	15	42
		21	176	91	27	315

Table 5: Multiplication of two equal fractions – cross table

Table 6 focuses on the approaches to the four multiplication problems with a common denominator that can be estimated/regarded as identically structured.

	So many students performed *x*-times …				
	0	**1**	**2**	**3**	**4**
Correct approach	79	28	38	35	135
'denominator maintained'	199	20	28	28	40

Table 6: Multiplication of two fractions with the same denominator – frequency of approaches

When multiplying two fractions with a common denominator, 135 of 315 students treated all four problems according to a correct approach, while e.g. 38 did so at only two out of four problems and behaved differently at the other two problems. 199 students never followed the approach 'denominator maintained' and 40 students used it each time while 20 students generated this error pattern only at one out of four problems and dealt with the remaining three problems in another way.

In addition to table 6, further interesting facts can be outlined: 11 out of 40 students who maintained the denominator at all four problems worked on the two problems with different denominators in the same way, after converting the denominators. Therefore, the error pattern is consistent regarding all six multiplication problems. Another 11 students who maintained the denominator at all four problems solved the two problems with different denominators according to the correct approach – the approaches are consistent only within the cases of common or different denominators.

3.2 Addition of two fractions

Regarding the addition of two fractions, the frequency of the main error pattern 'separate addition of numerators and denominators' varies little across all six problems, as table 7 shows (N = 347).

Category	$\frac{1}{2}+\frac{1}{4}$	$\frac{2}{7}+\frac{4}{5}$	$\frac{2}{13}+\frac{4}{9}$	$\frac{1}{9}+\frac{1}{8}$	$\frac{2}{5}+\frac{2}{5}$	$\frac{6}{11}+\frac{6}{11}$
0 Missing	19	41	116	62	16	16
1 Correct approach	227	205	116	167	224	230
3 Main error pattern	73	79	84	78	92	78
9 Others	28	22	31	40	15	23

Table 7: Addition of two fractions – frequency of the approaches

However, another effect is striking: the larger the two denominators are, the less often the right approach occurs and the less often students work on the problem (but skip it), with a range of 19 to 116 students. This can be interpreted as follows: to leave out a problem is a reaction to 'large' denominators when finding a common denominator affords a higher level of mental calculation. This reaction gives the hint that the regarding students neither have the result of the underlying multiplication problem as a retrievable fact nor can expand it briefly. It is not possible to decide if the students cannot find a common denominator or if they do not want to work on it, perhaps due to a lack of motivation or effort or a low level of frustration tolerance.

Table 8 shows the students' approaches across the six addition problems as ordered 6-tuples. For example, 110111 in the second line means that the first two and the last three problems are solved correctly whereas the third problem was not done at all. Among 347 students there are 126 different tuples: 18 of them can be found at least three times (these are shown in table 8), another 13 tuples occur only twice and 95 only once.

6-Tupel	Frequency	Percentage	Cumulative percentage
111111	83	23,9	23,9
110111	34	9,8	33,7
333333	32	9,2	42,9
110011	21	6,1	49,0
111133	7	2,0	51,0
333311	7	2,0	53,0
000000	6	1,7	54,8
110131	6	1,7	56,5
333331	5	1,4	57,9
111131	4	1,2	59,1
100011	3	0,9	59,9
110113	3	0,9	60,8
111311	3	0,9	61,7
113333	3	0,9	62,5
119911	3	0,9	63,4
199911	3	0,9	64,3
333313	3	0,9	65,1
999911	3	0,9	66,0

Table 8: Addition of two fractions – processing of six problems

The tuples 111111 and 333333 indicate consistent approaches: 83 students always have a right approach and 32 students show the main error pattern ‘separate addition of numerator and denominator’ in all six tasks. This means that a third of the approaches can be explained by the assumption of consistent approaches whereas two-thirds of the approaches are seen as non-consistent.

Amongst these two-thirds there are two very interesting tuples: Both 111133 and 333311 occur seven times. The first characterises these students who approach the four additions of fractions with two different denominators in a correct way and the two additions of fractions with common denominators according to the main error pattern. The second describes the reverse of the phenomenon. In other words, there are students who ‘fall into’ the error pattern in case of common denominators but there are also students who ‘fall into’ the error pattern in case of different denominators.

3.3 Addition of a fraction and an integer

This section briefly describes two interesting results without showing the full data.

Concerning the addition of a fraction and an integer a more detailed translation of data into code is appropriate (even though there are naturally some borderline cases): First, a mental solution is classified as a category of its own because it refers to conceptual knowledge. In fact, about a quarter to a third of the problems is solved mentally. Regarding the approaches, no differences can be seen between problems of the structure ‘fraction plus integer’ and ‘integer plus fraction’ (table 2). The hypothesis that in the case of ‘integer plus fraction’ the solution is ‘seen’ more often than in the case of ‘fraction plus integer’ (cf. Padberg, 1986) cannot be approved. On the contrary, the problem $\frac{1}{4}+1$ is solved slightly more frequently mentally than the other five. In consequence, the fact whether the denominator is ‘large’ or not has more influence on the solution process than the question whether the problem is of the form ‘integer plus fraction’ or of the form ‘fraction plus integer’.

Second, the addition of a fraction and an integer does not show only one main error pattern but several error patterns (table 1) which all occur in the test. The approaches of 28 out of 313 students show at least two different error patterns. In conclusion, these students do not possess retrievable stable algorithms but only some fragments of algorithms which they apply and combine haphazardly. Furthermore, they lack conceptual knowledge to check their results or the used algorithms.

4 DISCUSSION

The study shows that the numbers given in computational problems with fractions (especially as denominators) have an effect on (1) whether students work on the problem or skip it and (2) what kind of approaches occur. Especially the last finding is astonishing. It was clear in advance that the percentage of correct *solutions* decreases when the denominators get larger as more errors occur. But the fact that the given numbers even have an effect on the *approach* could not have been expected. Concerning error patterns, this can be interpreted as following: Problems which are identically structured in a mathematical point of view (and differing only in the given numbers) are not necessarily identically structured regarding a student’s individual approach to solve these problems. For example, in terms of multiplying fractions this is well known for fractions with a common denominator vs. fractions with different denominators. The presented study shows the same effect for the size of the given numbers. However, there is the hypothesis that not only and not primarily the size of the numbers is relevant but the question if necessary multiplication tables are known by heart or can be calculated mentally without any difficulties. This hypothesis is an aim of further research.

With regard to the consistency of approaches *four groups of students* can be distinguished for each of the four operations. *Group 1* includes the students who solve all problems of one operation according to a correct approach. About a fourth to a third of the tested population belong to this group. Within this group, two subgroups can be found: students who treat the problems in a very flexible way, adequate to the given numbers (e.g. if the numbers are small, the solution is ‘seen’ or solved mentally), and students with an extreme commitment to a certain approach regardless whether it is appropriate for the posed problem or not (‘always convert the denominators before adding two fractions’). *Group 2* contains students whose solutions show the main error pattern without exception. These students may have internalized an incorrect procedure while performing automation exercises. This group is quite small.

The first two groups can be distinguished clearly. Together they comprise those students whose approaches are absolutely consistent. Two other groups include those students whose approaches are not consistent. These two groups cannot be separated clearly but merge into each other.

Group 3 contains the students who show mainly correct approaches but occasionally proceed otherwise. Due to this small amount of different approaches this group is different to group 1. One possible explanation may be careless mistakes that are possibly fostered by the intuitive form of error patterns, by the random arrangement of problems in the test sheet and by the given numbers when they ‘fit’ into an error pattern.

Group 4 accumulates the students whose approaches are not consistent at all. Even within one set of problems there are two or more different approaches. Presumably, the influence of given numbers is dominant because no stable methods are available. Therefore, special cases (e.g. fractions with a common denominator) are treated differently than general problems. The approaches of group 4 can be interpreted as ‘calculating with digits’ and as a widely unconsidered processing of numbers given in the problem according to known elements of algorithms. The concerning students do not have any strategies when dealing with computational problems.

As a result, students’ work on computational problems quite often appears as a barely controlled and only partly aware process. This is very astonishing as all of the students attended a systematic course in fractions. The approaches are in some cases rather rationally chosen but emerge over time while treating the problem (according to Threlfall, 2009). Especially the approaches of low achievers can be described as manipulating given numbers and combining fragments of algorithms without reflecting these processes. In consequence, at least some of the students’ errors emerge, too. They are not consistent referred to a student but their appearance varies from task to task depending on the given numbers (and perhaps some other parameters). Therefore, error patterns can not be regarded as an individual phenomenon but as a phenomenon appearing in written tests which are run in larger populations.

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TRANSFORMATIONS VIA FRACTIONS, DECIMALS, AND PERCENTS

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Keywords: Secondary school mathematics, mathematical transformations

SUMMARY

The topic of percentages in the school mathematics is usually taught in the age of 12-13, i.e. on the 6th and 7th grade of the secondary school. In the phase of learning this topic the students must be able to conceive transformations from decimals to percents, from fractions to percents, from fractions to decimals, and vice versa. At this phase many students face problems. This poster presentation will describe the background of these transformations, and introduce some results of a study, carried out in fall 2008.

Many teachers and researchers (e.g. Moss, 2005; Adjiage & Pluvinage, 2007; Hallett, 2008) have mentioned that teaching and learning of fractions, ratio, and proportionality is a complex process. The difficulties are caused (1) by the ambiguousness of fractions and decimals (Moss 2005; Charalambos & Pitta-Pantazi, 2007) and the necessity to change from one semiotic system to another (Duval, 2006), and (2) by students' poor skills in transformations (Hallett, 2008; Moss 2005).

It is quite common that in case the students cannot do transformations correctly, they begin to construct answers by using the numbers that they see. Hallett (2008) found that 55% of the 13-year-olds answer to the question "Which of the following numbers: 1, 2, 19 and 21 are the closest to the sum of 12/13 + 7/8" either 19 or 21. It can be assumed that some types of transformations are simpler for students. In order to better understand students' skills and preferences in transformations, I carried out a group interview with students (aged 13 – 14) from the 7th grade. (This interview is a part of a bigger study.) The results showed that transformations from percents to decimals, and vice versa, are simpler than others. The most difficult is transformation from fractions to percents, because it needs two sequential transformations: first from fraction to decimal, and then from decimal to percent.

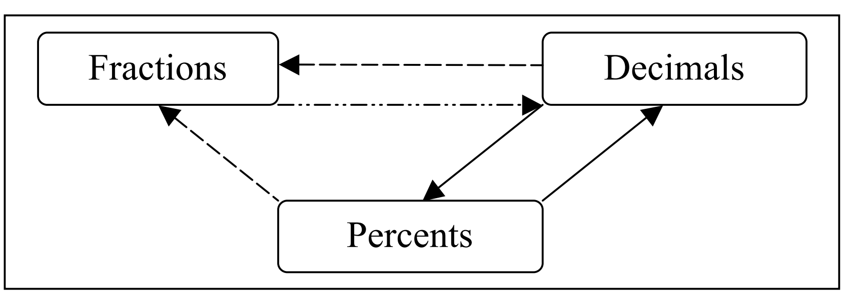

Figure 1: Strength via different transformations

THE WAY OF PRESENTING

The central part of the poster will be Figure 1, which visually presents the main conclusions from an empirical study, and illustrates the differing strengths of the skills in the transformations.

Additional visual material will be organised around this figure, e.g. tables with data from (1) students' answers (right/wrong), and (2) students' preferences in transformations, when the order of the transformations was chosen free. I have also students' written answers, which will be used as illustrating material.

The textual data in the poster will be organized in short sections: (1) Introduction, (2) Methods, (3) Results, (4) Summary, and (5) References.

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TO MEASURE WITH A BROKEN RULER TO UNDERSTAND THE COMMON TECHNIQUE OF THE SUBSTRACTION

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Keywords: Subtraction, technology, semiotic mediation

INTRODUCTION

The research that I am conducting as a PhD candidate focuses on the teaching and learning of calculus subtractive at primary school. With the poster that I am presenting, I am showing the interest to teach the notion of difference/distance and after the property of the conservation of differences for understand the usual technique subtraction in France. I suggest a sequence of learning, using as artefact a broken ruler and I have studied its potential. I am using as theoretical part of the anthropological theory of the didactic and of the semiotic mediation.

CONTENT OF THE POSTER

The poster is organised in four parts:

First box: Scientific context and motivation

I initially observed that the put-down subtractions with numbers of carry over are not mastered by pupils of the fourth year (CE2) or of the fifth year (CM1). At the first level (technical), the carried out calculations are wrong because the algorithm learned during the class is badly applied. At the second level (technological), the calculation is right, but the pupils don't know what mathematical meaning they have to give to the numbers to carry over. Should we then conclude that students do not know the differences conservation property?

The study of different manuals would lead us to believe, that one day, they have been in a situation where the property of conservation of the differences has been used but that this theoretical element is not reused in several contexts and linked up to others knowledge's.

Convinced of this lack, I think about a mathematical organisation, where the kind of proposed task will be linked up to another by a technological speech, or by the successive development of techniques. It is with this ambition that I have conceived this situation. Probably she will become an key element of the engineering that I will suggest at the end of my thesis. The fact of measuring with a ruler broken suggests that the distance between "a" and "b" is the difference between "b" and "a". (($a<b$). After it's more easy to integrate the notion of invariant difference by translation on the numerical line so the differences conservation property.

Second box: aims

To have the notion of difference between two numbers, of distance between two dots, the ruler is broken in order to not permit a direct reading of the measure. On the poster, the situation of learning will be described in its entirety. I will expose the conditions of the testing led in 2010. Finally, I will be explained the different techniques of measurement, adopted by the pupils, and exchanges between students in a group.I will show what are the techniques which have been validated or invalidated and how each pupil have could think on their own experience with the artefact and to compare some different utilisations of the same broken ruler. The last phase of the session, « the discussion of class » to resume the term of Mariotti and Maracci (2010) is also interesting for the mathematical signs. To conclude, I will present a extension of the session which will allow to introduce the property of preservation of difference by translation on the line.

Third box: Methods and results

The method is based on the confrontation between a priori analysis and a posteriori analysis. It allows establishing that the situation « to measure with a broken ruler » has a potential, under some conditions that I going to enumerate. Pupils after learning a subtraction calculation technique can take ownership, a mathematical property, which allows to justify this technique, and which also can be used in mental calculation.

Fourth box: Bibliography

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG3: ALGEBRAIC THINKING

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Keywords: *Early algebra; generalized arithmetic; equivalence; equals sign; generalization; semantic; syntactic; teaching and learning algebra; mental algebra; functions*

In CERME-8, as a long-standing group, Working Group 3 "Algebraic thinking" continued the work carried out in previous CERME conferences (Cañadas, Dooley, Hodgen & Oldenburg, 2011).

THE PAPERS AND THEMES

A total of 16 papers and 5 posters with a total of 25 group participants representing 14 countries, Canada, France, Germany, Ireland, Italy, Netherlands, Norway, Portugal, Spain, Sweden, Tunisia, Turkey, UK, USA / Romania. We considered the papers in the following themes (although, of course, there are many overlaps across the themes):

Entry to algebra

Several studies focused on the relation between algebra and arithmetic. These continue and extend discussions from previous CERME conferences (e.g., Ainley, Bagni, Hefendehl-Hebeker, & Lagrange, 2009). We note that the focus is on 'early algebra' in both primary and secondary education. Working in Portugal, where an 'early algebra' focused curriculum has been introduced, Mestre and Oliviera describe and analyse a teaching experiment in which Grade 4 children are introduced to the use of informal symbols as quasi-variables. They find that such an approach has benefits for the development of algebraic thinking, particularly in moving from equations involving specific unknowns to equations expressing generalisations about arithmetic. In contrast, Gerhard examines how a generalised number approach may hinder the development of algebraic thinking. Using her own interdependence analysis tool she examines how new patterns of action are related to old patterns of action. In particular, she argues that poorly developed patterns of action relating to multiplication (such as an over-reliance on repeated addition) may be overcome by Davydov's more geometrically-based approach. Mellone, Romano & Tortora report research with older students (Grade 10) and examine how the students tackle the problem $10^{38}+10^{37}$. They argue that, in making sense of the problem, students draw on a 'social' rationality in addition to purely mathematical reasoning and that this has implications for the relationship between arithmetic and algebra. Mata-Pereira & da Ponte's poster examines how Portuguese secondary students' reasoning about real numbers and generalized arithmetic.

Equivalence

Several papers addressed issues relating to equivalence, perhaps reflected a general resurgence of interest in the field. Drawing on Frege, Pilet argues that the French curriculum places little explicit attention to the issues of equivalence and that this exacerbates students' conceptual difficulties. She analyses a learning situation designed to do this. Zwetzschler & Prediger argue that, whilst previous research has examined the learning of equivalence in transformational situations, little attention has been devoted to the equivalence of expressions in generational activities. In a case study designed to address this gap, they find two conceptual barriers to understanding: an over-emphasis on operational rather than relational meanings, and difficulties relating to the generality of variables and figures. Vieira, Palhares & Gimenez report the findings of a pilot study that investigated Portuguese primary children's understanding of equality and the equal sign following the introduction of a new curriculum. Although the children in their study performed better than in previous studies of the equal sign, they suggest that their understanding of relational meanings and equivalence are nevertheless poor.

Structural generalisation

Unsurprisingly, generalization continues to be a theme of interest for WG3 participants. Måsøval analyses how two student teachers struggle to find an explicit general formula for a sequence. Using Brousseau, she shows how the milieu for algebraic generalisation is constrained both by the design of the task and by the teacher educator's (incorrect) presupposition of prior knowledge. Dooley examines the strategies that primary children (aged 9-10) in Ireland used when solving a quadratic problem and analyses how mediation by the teacher (and the children themselves) can enable young children to tackle relatively sophisticated algebraic problems. Postelnicu reports the results of two surveys examining secondary school and college students' difficulties with Cartesian representations of linear functions. She argues that the difficulties she finds are due to epistemological, rather than simply procedural, obstacles and these in turn relate to an overemphasis on geometric rather than analytic perspectives on slope and gradient. Rolfes, Roth & Schnotz's poster presents the early results of a teaching experiment aimed at improving secondary understanding of covariation of functions.

The relationship between semantic and syntactic understandings

Two papers focus on the relationship between syntactic and semantic understandings of algebra. Focusing on equations and inequations, Kouki & Chellougui discuss the difficulties that students encounter in mastering the syntactic rules of algebra and argue that both syntactic and semantic understanding is necessary for such mastery. Oldenburg, Hodgen & Küchemann examine whether these two aspects aspects of algebraic thinking, syntactic and semantic, can be distinguished empirically using test items. Although there are considerable difficulties in operationalizing the distinction using test items, their exploratory analysis suggests a fruitful line of analysis may be to treat the semantic aspect as consisting of two sub-dimensions, based on whether one or many meanings or interpretations appear to be required.

Teachers and teaching

Several papers and posters address issues relating to teachers and the teaching of algebra. Kilhamn examines how two teachers enact the same teaching materials and finds that outwardly similar content is addressed differently. Although these differences at first appear subtle, the two teachers approaches tackle issues such as variable in radically different ways. Çelikdemir & Erbaş analyse TIMSS 2007 data on Turkish mathematics teachers' self-reported preparedness to teach school algebra, finding that they reported that they were less well prepared to teach algebra than their counterparts around the world. Sari & Özdemir report on the effects of a teaching intervention supporting metacognitive strategy use on improving Turkish seventh grade students' conceptual and procedural knowledge on algebraic expressions and equations. They find a significant difference between experimental and control groups in terms of gain scores on conceptual and procedural knowledge in favor of experimental group. Two posters, by Wathne and Röj-Lindberg, present the methodology and early findings of the VIDEOMAT, a video study of mathematics lessons in Finland, Norway, Sweden and the USA on the introduction to algebra.

New directions

Several papers address new directions for the Algebraic Thinking Working Group at CERME. Proulx reports on an exploratory study examining mental algebra. He argues that too little attention has been devoted to understanding mental mathematics activities with objects other than numbers and discusses its potential for algebra teaching and learning. In the only paper to directly address issues involving technology, Lagrange discusses the limitations of an exclusively functional approach to algebra. He argues that both experiencing covariation and bodily activity are crucial for students' understanding of functions, showing how this can be facilitated through the Casyopée environment. Kop's poster presents early data from a study of how expert mathematicians sketch graphs of functions.

GENERAL REFLECTIONS

As we observed in our CERME7 report, algebraic thinking is a "mature" sub-domain within mathematics education research (Cañadas et al., 2011). As a result, our group discussions touched on many familiar themes, including the recontextualisation of existing research, the need for both 'young' researchers and less developed communities to 'make it their own', the relationship between different theoretical approaches and the nature of algebra / algebraic thinking / algebra as an activity. But it was also exciting to revisit epistemological debates and the implications of these for the teaching and learning of algebra.

LOOKING FORWARD TO CERME-9

Finally, in looking forward to CERME-9, the group hope that future research will address advanced as well as early algebra, the nature and design of paradigmatic tasks, the extent to which (research on) algebraic thinking is influenced by context and culture and the tensions between rigorous research and classroom practice and students' difficulties and misconceptions. We also hope to engage in continuing discussions about the nature and utility of the familiar dualities in algebra (syntactic / semantic, procedural / conceptual etc). We were surprised to have only one paper addressing digital technologies and hope that this omission will be rectified at CERME9.

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AN ANALYSIS OF TURKISH MATHEMATICS TEACHERS' SELF-REPORTED PREPAREDNESS TO TEACH ALGEBRA IN TIMSS 2007

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The purpose of the study was to investigate Turkish mathematics teachers' self-reported preparedness to teach some particular topics in school algebra and determine possible differences among groups of teachers with different teaching experience. For this purpose, the data collected from 146 Turkish mathematics teachers in TIMSS 2007 were analyzed. First of all, teachers' self-reported preparedness in each topic was analyzed using descriptive methods. Then, one-way MANOVA was run. According to the descriptive results, although majority of the Turkish teachers reported that they were very well prepared to teach the stated algebra topics, they reported that they were less well prepared to teach algebra than their counterparts around the world. On the other hand, MANOVA analysis was nonsignificant.

INTRODUCTION

Several studies have pointed out that cognitive and affective domains are related (e.g., see Ma & Kishor, 1997; Mandler, 1989; McLeod, 1992). Therefore, current reforms in mathematics education (e.g., National Council of Teachers of Mathematics [NCTM], 1989, 1990, 2000) emphasize the incorporation of affective and cognitive factors in mathematics education. However, the literature suggests that researchers place more emphasis on the cognitive domain than on the affective domain. Thompson (1992) attributes this neglect to the effect of behaviorism in mathematics education for a certain period of time.

It is obvious that trends in mathematics education are changing and developing over time and these developments guide the trends in mathematics education research. Like the neglect of affective domain in the past, teachers, compared to students, have been ignored as focus of studies (Even, 2008). There remains relatively few studies about mathematics teachers' cognitions and conceptions and the effect of these on their teaching.

There are similar gaps in research in the field of algebra, which is one of the subdomains of mathematics. Doerr (2004) noted that while there are several studies that investigated students' understanding in algebra, research on teachers' algebraic knowledge and practices is less prevalent.

Notwithstanding this, there has been a considerable effort to eliminate such shortcomings in the related literature in recent years. Even large scale international studies like TIMSS (Trends in Mathematics and Science Study) and PISA (Programme for International Student Assessment) collect data on (mathematics) teachers' perceptions. These studies provide comparative data among the participated countries and enable researchers to understand certain issue in a broad perspective. For example, TIMSS 2007 measured mathematics teachers' perceived preparedness in teaching particular topics and provided rich data for nationaland international evaluations. For this reason, the data gathered from TIMSS 2007 were used in this study. Its purpose was to investigate Turkish mathematics teachers' self-reported preparedness to teach particular topics in school algebra and algebra in general compared to their counterparts and to determine if there is any statistically significant difference among group of teachers with different teaching experience. The following research questions guided the study:

1) How do eighth grade Turkish mathematics teachers perceive their preparation to teach algebra in general and in predetermined topics, compared with the international average?

2) Does teaching experience affect teachers' perceptions about their preparation to teach algebra?

The significance of this study lies not only in yielding a better understanding of (Turkish) teachers' perceived preparedness in algebra but, also, in providing a clear picture of a possible significant relationship between preparation and experience. On the other hand, the study can provide an account of the influence of the present mathematics curriculum that developed in 2004 after Turkish students' failures in TIMSS 1999 and PIRLS 2001 (Progress in International Reading Literacy Study) and PISA 2003.

METHODS

Data Source and Sample

This study utilizes the TIMSS 2007 Turkish data collected from 146 mathematics teachers each of whom was teaching Grade 8 mathematics in a different elementary school chosen by using a two-stage cluster sampling design. Schools were determined at the first stage and one class from each school was chosen randomly at the second stage. Since the number of schools were determined using probability proportional to the size of each country, the target population was all mathematics teachers teaching eighth grade mathematics in Turkey.

Instrumentation

TIMSS 2007 used achievement tests and background questionnaires to measure students' learning in mathematics and science topics and to explain the educational context behind the scores. In the present study, data from the questionnaire that related to eighth grade mathematics teacher background were used (Erberber, Arora,

& Preuschoff, 2008). The questionnaire included 33 questions on information from mathematics teachers about their demographics, experience, attitudes, pedagogical information, instruction load, resources related to teaching mathematics, mathematics course content, and comments on the teaching of mathematics.

In particular, teachers were asked about their perceived preparedness in 18 topics in total, including 5 topics in number, 4 topics in algebra, 6 topics in geometry, and 3 topics in data and chance. For the present study, the following four topics in algebra were considered:

- numeric, algebraic, and geometric patterns or sequences (extension, missing terms, generalization of patterns),
- simplifying and evaluating algebraic expressions,
- simple linear equations and inequalities, and simultaneous (two variable) equations,
- equivalent representations of functions as ordered pairs, tables, graphs, words, or equations.

Teachers were asked to mark one of the alternatives in the four-point scale: “not applicable”, “very well prepared”, “somewhat prepared” and “not well prepared” which were coded as 1, 2, 3, and 4 respectively.

Variables and Statistical Analysis

As fit for the purpose of the study, teaching experience and teachers’ perceived preparedness to teach algebra topics stated above were taken into consideration. While teaching experience was considered as the independent variable, teachers’ preparations in specific topics in algebra were taken as dependent variables in the analysis of possible mean differences in perceived preparedness among the group of teachers and differences in teaching experience.

Since the teachers wrote their exact year of teaching experience in the questionnaire, this variable was not a categorical variable. Therefore, for the purposes of statistical analysis, data were split into four categories and coded as “1” for having less than three years of teaching experience, “2” for 3-5 years, “3” for 6-10 years, and “4” for more than 10 years. Table 1 shows the frequencies, percentages, means, standard deviations and minimum and maximum values of teaching experience in each category.

Number of years	Ω	%	*M*	*SD*	Min	Max
Less than three years	25	17.1	1.3	.5	1	2
3-5 years	35	24.0	4.3	.8	3	5
6 -10 years	31	21.2	8.0	1.2	6	10
More than 10 years	44	30.1	22.8	6.2	11	37
OMITTED	11	7.5				
Total	146	100.0				

Table 1: Teaching experience of eighth grade Turkish mathematics teachers

In order to address the first research question, percentages of teachers who perceived themselves to be very well prepared on algebra are provided and compared with the international average.

Furthermore, to answer whether the more experienced teachers perceived themselves as more prepared to teach the stated algebra topics, mean differences between the groups of teachers with different teaching experience were analyzed using a one-way multiple analysis of variance (MANOVA). Since there was one categorical independent variable and four continuous dependent variables, one-way MANOVA was chosen for the analysis. In addition, one-way MANOVA was preferred because it considers four dependent variables in combination and protects against inflated Type I errors due to multiple tests of likely correlated dependent variables (Tabachnick & Fidell, 1996). All statistical analyses were carried out using the software SPSS.

The listwise deletion method was used or data that was omitted or not administered. Accordingly, MANOVA analysis was run for 130 teachers out of 146 teachers.

RESULTS

Descriptive Results

In TIMSS 2007 (Martin, Mullis & Foy, 2008), the percentages of teachers who perceived themselves as “very well prepared” to teach four algebra topics were given for each participating country. Based on the obtained results of four algebra topics, preparedness to algebra in general was computed. In addition, the international averages for each topic and algebra in general were also calculated.

Accordingly, while the international average of teachers who perceived themselves as “very well prepared” to teach algebra is 82%, only 66% of Turkish mathematics teachers perceived themselves to be so. Comparisons of Turkish teachers and international average for each algebra topic are reported in Table 2. It is important to note that the percentages for Turkey reported in this study differ from percentages in the figures in the TIMSS 2007 International Mathematics Report (Martin et al., 2008). The standard errors in TIMSS 2007 published figures or the errors in Turkey row data can be the possible explanations for this difference.

	Patterns or Sequences	Algebraic Expressions	Linear Equations and Inequalities	Equivalent Representations of Functions
Turkey	52	81	77	40
International Avg.	70	89	88	80

Table 2: Percentage of teachers who reported very well prepared to teach algebra topics

It can be seen that although the majority of the Turkish teachers perceived themselves to be very well prepared to teach algebra topics, the percentages are considerably lower than the international averages. The highest percentages were for "Algebraic expressions" (81%) followed by "Linear equations and Inequalities" (77%) and the lowest were for 'Patterns or Sequences' (52%) followed by 'Equivalent Representations of Functions' (40%) for Turkish teachers. Similar patterns exist in the international averages. However, while the lowest percentage for Turkish teachers was for 'Equivalent Representations of Functions', the lowest percentage of the international average was for 'Patterns or Sequences'.

On the other hand, the percentages of Turkish teachers who perceived themselves as "not well prepared" or "somewhat prepared" to teach should not be ignored. Therefore, the percentages of alternative answers are given in Table 3. For the topics of 'Patterns or Sequences' and 'Equivalent Representations of Functions', the percentage of teachers who reported that they were "somewhat prepared" or "not well prepared" cannot be underestimated.

	Patterns or Sequences	Algebraic Expressions	Linear Equations and Inequalities	Equivalent Representations of Functions
Not Applicable	10	-	-	28
Very Well Prepared	52	81	77	40
Somewhat Prepared	28	11	16	24
Not Well Prepared	10	8	7	8
Total	100	100	100.0	100

Table 3: Percentages of Turkish mathematics teachers' perceived preparedness to teach the algebra topics.

MANOVA Results

Before conducting a one-way MANOVA, the following assumptions are made:

a) independent random sampling: since TIMSS used a random sampling technique, the observations are independent of each other.
b) level and measurement of the variables: while the independent variable in this study, teaching experience, was categorical, the four dependent variables were scale variables.
c) linearity of dependent variables: to examine multicollinearity, correlations between all of the four dependent variables were checked with Pearson product-moment correlations. According to Tabachnick and Fidell (1996), problems occur at higher correlations (.90 and higher). Since the highest correlation between the four dependent variables is .70, the problem was not detected.
d) Normality: all of the dependent variables are normally distributed. ($p>.05$)
e) Homogeneity of variances: the results of Box's Test (F (30, 31082) = 1.37, p = .08) were nonsignificant which means that the group variance-covariance matrices were equal.

A one-way MANOVA was conducted to explore the effect of difference in teaching experience on perceived preparedness to teach the algebra topics stated above. The results for the MANOVA were statistically nonsignificant for the teaching experience main effect, Wilks's Λ= .93 F (12, 326) = .77, p = .68. In other words, there were no statistically significant differences among group of teachers with different teaching experience on teachers' perceived preparedness in teaching algebra topics.

CONCLUSION

Based on the descriptive data, it is clear that Turkish teachers perceived themselves as less well prepared to teach all of the stated algebra topics and algebra in general compared to their counterparts around the world. This is an important observation as it may be the reason for Turkish students' low scores in algebra since research shows that teachers' confidence in their teaching affects their teaching efficacy and effective teaching affects students' achievement (Ingvarson, Beavis, Bishop, Peck, & Elsworth, 2004; Richardson, 2011; Vanek, Snyder, Hull, & Hekelman, 1996)

On the other hand, while the percentage of teachers who perceived themselves as very well prepared to teach 'Algebraic Expressions and Linear Equations' is high, the percentages of teachers who perceived themselves as very well prepared to teach 'Patterns or Sequences' and 'Equivalent Representations of Functions' are considerably lower. One possible explanation for this is the emphasis in the Turkish elementary mathematics curriculum (Talim ve Terbiye Kurulu [TTKB], 2009). Compared with patterns and sequences and functions, algebraic expressions and linear equations are given more weight in the curriculum. For this reason, teachers spend more time and energy teaching those topics. It is possible that their engagement in the topics influenced their understandings of and perceptions about teaching them (Lee, Baldassari & Leblang, 2005). Similarly, in teacher education programs, elementary mathematics teachers might have not been well prepared to teach patterns and sequences or functions in elementary school level. Moreover, pure content courses in these topics are found to be ineffective in teachers' confidence in their teaching (Beswick, 2011).

Before analyzing the data, it was expected that teachers who have more experience in teaching would perceive themselves as significantly more prepared to teach algebra topics. Although there is much research reporting the positive effect of experience on teachers' confidence in teaching (Adams, Hutchinson & Martray, 1980; Griffin, 1983), a few studies (e.g., Wessels & Nieuwoudt, 2010) have found no significant effect of experience on teachers' confidence to teach, as was the case in this study. Since there is not a consensus on this issue in the literature, it needs to be further investigated. However, this result may indicate that Turkish mathematics teachers have not been able to upgrade their understanding of algebra topics over time. In order to reach such a conclusion, further research is needed.

The findings of the study imply that regardless of their teaching experience, Turkish mathematics teachers require additional support to improve their understanding in algebra topics, especially in patterns and sequences and functions and efficacy to teach them. Therefore, professional development programs that strengthen teachers' understandings in algebra and enrich their teaching by providing multiple materials should be organized. This may influence their perceptions about teaching algebra and, in turn, may develop students' algebraic understandings.

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YOUNG PUPILS' GENERALISATION STRATEGIES FOR THE 'HANDSHAKES' PROBLEM

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In describing the strategies that young pupils employ for algebraic generalisation, most research focuses on linear rather than quadratic problem situations. The strategies identified by Lannin, Barker and Townsend (2006) for the cube sticker (linear) problem include counting, recursion, chunking and development of an explicit formula. However, erroneous solutions are often given for 'high' numbers because of the tendency by students to use inappropriate proportional reasoning in such instances. In this paper I describe the strategies that pupils aged 9 – 10 years used in whole-class conversation to solve the 'handshakes' problem, an example of a quadratic generalising situation. They used many (but not all) of the strategies that apply to linear problems and were able to verify the explicit formula that they developed with reference to the structural elements of the problem.

INTRODUCTION

The capacity of children to engage in sophisticated generalisation activities is at the core of the inclusion of the strand of algebra in primary mathematics curricula (e.g., Kaput, Blanton, & Moreno, 2008). In this regard, previous discussions at the Algebra working group of the Congress of European Research in Mathematics Education (CERME) have called for students to engage in problems the purpose of which is to express generality (e.g., Puig, Ainley, Arcavi, & Bagni, 2007). In this paper I describe the strategies that pupils aged 9 – 10 years used to solve the well-known 'handshakes' problem. It builds, in particular, on a recommendation made by Barbosa (2011, p.427) at CERME 7 that students be provided with "tasks which allow the application of a diversity of [generalisation] strategies". However central to the development of these strategies are (a) the role of the teacher and (b) task design.

GENERALISATION STRATEGIES

In early years classrooms, the emphasis in algebra is usually on the exploration of simple repeating and growing patterns. Since any variation usually occurs within the pattern itself, the focus is on single variational thinking (for example, finding relationships between *y* values rather than between *x* and *y* values). As pupils move through the primary school system, greater emphasis is placed on the formation of functional relationships and the generalisation of patterns. Functional thinking is described by Smith (2008, p.143) as follows:

> … representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of relationships across instances.

Functional thinking is intrinsic to algebraic reasoning because it allows for the generalisation of a relationship between two varying quantities. Furthermore, it lays a firm foundation for more formal algebra work in later years (Blanton & Kaput, 2011).

Although young children are capable of thinking functionally (Blanton & Kaput, 2004; Warren, 2005), there is evidence that, throughout the primary years, they focus on pattern spotting in one data set rather than on the relationship between an element of a pattern and its position. Warren (2005) suggests that that this is the case either because is cognitively easier for children or is so engrained from school experience that there is a tendency to revert to it.

Most of the research on the generalisation strategies used by pupils is based on linear problems (e.g., Lannin, Barker, & Townsend, 2006b; Rivera & Becker, 2008). An example of one such problem is the Square Matchstick Problem:

Figure 1: 'Square Matchstick' pattern

The number of matchsticks required to form the nth element of this sequence is $3n$ + 1 (for example, 13 matchsticks are required to build the fourth element).

Lannin, Barker and Townsend (2006a) developed a framework of generalisation strategies for linear problems as follows:

- Counting: Student draws a picture or constructs a model to represent the situation and then counts the desired attribute.
- Recursive: Student describes a relationship that occurs in the situation between consecutive values of the dependent variable.
- Chunking: Student builds on a recursive pattern by building a unit onto known values of the desired attribute. In the matchsticks problem, if a student knows that ten matchsticks are required for the third element, s/he might calculate the number required for the fifth element by using a strategy such as 10 + 2 (3) (because the number increases by 3 each time).
- Whole-object: Student uses a portion as a unit to construct a larger unit using multiples of the unit. For example, the fifth element requires 16 matchsticks and therefore the tenth element requires 32 – 1 (where the 1 is subtracted to take account of the first matchstick used in the first element).
- Explicit: Student constructs a rule that allows for immediate calculation of any output value.

According to Lannin et al. (2006b), recursive rules involve "recognising and using the change from term-to-term in the dependent variable" (p. 300) while explicit rules use "index-to-term reasoning that relates the independent variable to the dependent variable(s), allowing for the immediate calculation of any output variable" (ibid). For example, in the square matchsticks problem above, an example of the recursive rule is that the difference between the number of matchstick required is 3 ('going up in threes') whereas an explicit rule is '3n + 1'. The 'chunking' and 'whole object strategies' are similar and represent attempts by students to calculate values immediately. However, they are strategies that often lead to erroneous calculations. A student using a chunking strategy might add the fifth output value (16 in the matchsticks problem) to the tenth output value (31) to find the fifteenth output value (thus finding a solution of 47 instead of 46). A false 'whole-object' technique that is prevalently used by students is the application of direct proportion or linearity (for example, doubling the number of matchsticks required for ten to find that for 20 and failing to take account of the 'first' matchstick in the pattern). Stacey (1989) suggests that this tends to be evoked when 'far generalisation' (an input value that renders the step by step approach unfeasible for example, finding the number of matchsticks required in the 50th element) is required. She found this to be the case even when students have made correct use of counting or a functional rule for smaller input values. This inappropriate proportional reasoning (or the 'illusion of linearity') has been found to exist among students of different ages and in a variety of mathematical domains (De Bock, Van Dooren, Janssens, & Verschaffel, 2002).

While the strategies outlined here represent increasingly sophisticated means of generating solutions for any *n*, Lannin et al (2006a) recognise the need not only for students to formulate rules but also to engage in explanation and justification of these rules. In this regard, Rowland (1999) makes a distinction between empirical generalisation - which is achieved by considering the form of the results - and structural generalisation which is made by investigation of the underlying meanings, structures or procedures of the problem at hand. For example in the 'square matchstick' problem, a student might notice the 'going up in threes' pattern (empirical) or be able to verify the pattern in terms of the need to add three sticks in order to make a new square (structural).

There is little research on the kind of generalisation strategies that pupils might use for non-linear patterns. In research on how primary school pupils abstract mathematical entities in the context of teacher-led discussion, I taught a series of lessons in three different primary schools in Ireland (Dooley, 2010). My research design – a teaching experiment – entailed the development of a hypothetical learning trajectory in advance of each lesson (Cobb, 2000). The one that I formed in advance of a lesson on a non-linear problem (the 'handshakes' problem) was based on Lannin et al's framework. I used it in conjunction with the RBC (recognising – building with – constructing) model of abstraction (Schwartz, Dreyfus, & Hershkowitz, 2009) for analysis purposes; however, in this paper I report only on the suitability of the framework as a learning trajectory in this particular situation and not on the RBC aspect.

HYPOTHETICAL LEARNING TRAJECTORY

The lesson which is the subject of this paper is Chess (the 'handshakes' problem) a task that supports children's development of functional thinking (Blanton & Kaput, 2005). It reads as follows:

> In a chess league each participant plays a game of chess with all other participants. How many games will there be if there are 3 participants? 10 participants? 20? Is there a way to find the number of games for any number of participants?

While the 'Chess' problem is a quadratic problem situation and therefore could be expected to be more cognitively challenging than linear problems for pupils, there are many ways that the problem might be solved. I refer here to those that are most likely to emerge in a primary school setting. This is not to exclude the possibility that a primary student might notice or use others. One way, as shown in table 1, is to make a list and look for a pattern:

Number of People (*x*)	Number of Games (*y*)	Difference (*d*)
1	0	
2	1	1
3	3	2
4	6	3
5	10	4

Table 1: Table of values for 'Chess Problem'

The differences, d, between consecutive values y form the sequence of natural numbers. A related observation is that a number in the y column might be found by adding x and y values in the previous row (e.g., 4 + 6 = 10, referred to in future as the $x + y$ method). These patterns are contingent on writing x and y values in consecutive order.

The function mapping x to y is y = $x(x-1)/2$ .This might emerge from inspection of the relationship between the x and y values. The same formula emerges if one gives consideration to the symmetric nature of the activity, that is, there is one game for each 'pair' (Rowland, 2003).

Another way to solve this problem is to consider the number of games played by each person, that is, the first person plays a Chess game with seven others, the second with six more, the third with five more and so on. The solution for eight people then is 7 + 6 + 5 + 4 + 3 + 2 + 1 giving a total of 28 games. Although this method (referred to in future as 'summation') generalises for all numbers it becomes cumbersome for larger numbers, especially if the numbers are added in consecutive order.

While the lesson took place over two consecutive days, a related lesson that took place a month earlier with this class was one entitled Friendship Notes and which reads:

> As part of Friendship Week in Greenville School, each pupil writes a short note to each other pupil in his/her class. Each pupil is given one sheet of paper for each note. How many sheets of paper are needed if there are 5 pupils in a class? 10 pupils? What would be the number [of sheets] for any number of pupils?

Both Friendship Notes and Chess are characterised by non-reflexivity (that is, no element of a set relates to itself). The main difference between the activities lies in the property of symmetry. ‘Chess’ is symmetrical because if A relates to (‘competes with’) B, then it follows that B relates to A. However, in ‘Friendship Notes’, if A relates to (‘writes to’) B, the reciprocal relationship is not implied. For this reason the function mapping x (the number of people) to y (the number of notes) in the Friendships Notes is $y = x(x\text{-}1)$ while in Chess, $y = ½\, x(x\text{-}1)$ where x represents the number of people and y the number of games. Of relevance to this paper is that pupils had developed a ‘rule’ – expressed verbally – for the solution of any x in Friendship Notes.

As previously mentioned the Chess lesson took place over two consecutive days (the daily sessions will be referred to hereafter as Chess 1 and Chess 2). The format of both Chess 1 and Chess 2 was introductory whole-class discussion and small group work followed by whole-class discussion. In group work pupils worked in self-selecting pairs or triads. In Chess 1, there was a whole class lesson in which consideration was given to the number of games that would apply to 1 – 5 players. Pupils than worked in groups to consider cases of 1 – 10 players – for this part of the lesson they filled in a table similar to that shown in Table 1 above (without the ‘difference’ column). At the end of the lesson there was a plenary discussion on the number for 20 players. In Chess 2, there was a review of the previous day’s lesson. During the group-work phase they were asked to fill in a table for 11 – 20 players and during the final discussion the focus was on the generation of a rule for *any* number of players. There were 31 pupils in the class and the school was located in an area of middle socio-economic status[1]. Data collected included audiotapes of whole-class and small-group conversations, pupils written artefacts, field notes and digital photographs of activities. Video data were not collected due to ethical constraints. In this paper I report only on interactions that occurred in whole-class discussion as this was the main site for growing sophistication in solution strategies, a point to which I will return in the discussion section.

PUPILS’ SOLUTION STRATEGIES

There was a total of 712 turns in the whole-class discussion phases of both lessons. Up to turn 196, the count strategy was employed. For example, when considering the number of games for four competitors, Killian erroneously suggested five games but used a counting procedure[2]:

136	Killian:	Five.
137	TD:	Why do you think five?
138	Killian:	Cos Enda plays three and then Barry plays two, then David plays Colin , that’s ( )
139	TD:	So just say that, explain that to me again ... you’ve got …?
140	Killian:	So David plays the other three.
141	TD:	So how many games is that?

142 Killian: Three.
143 TD: Right, go on.
144 Killian: And then Barry plays the other two and then Colin plays Barry…six.

In turn 196, Anne used an incorrect recursive strategy and suggested that nine games would be played by four competitors:

196 Anne: Em, nine.
197 TD: Why are you thinking nine?
198 Anne: Because it's going up in threes.

The first time that there was any allusion to an explicit rule was in turn 206 when Fiona used the summation rule. She first counted the five pupils at the top of the room but then stated a way of working in which no mention of pupils was made:

202 Fiona: Well, if Enda would play four people and then Barry would have to play eh … three people …
203 TD: Yeah.
204 Fiona: … and then Colin would have to play two people and then David would have to play one.
205 TD: So what do you think it would be? How would you find out the answer? ... What would you do to find out the answer?
206 Fiona: Eh, four, three, two and one.

During group work, most pupils used the $x + y$ recursion method to find solutions for seven, eight, nine and ten competitors. However, when having a whole-class discussion at the end of the first session about a larger number of competitors (i.e., 20), there was evidence of inappropriate linear reasoning on the part of some pupils. For example, Desmond doubled the number of games for ten competitors to find the solution for 20:

357 Desmond: Ninety.
358 TD: Getting ninety, why do you think it's ninety for twenty people?
359 Desmond: Eh ten is forty-five.

The explicit rule that was being expressed towards the conclusion of the lesson related to 'summation'. For example, Fiona used the formula she had developed for five competitors to find the solution for 20 competitors:

402 Fiona: Well you could em you could do em add one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen up to nineteen.

Brenda, Myles and David articulated a similar strategy. Anne suggested multiplying 20 by 19 to find the solution and a short while later, David tentatively suggested a formula:

432 David: Em, well, it might not work all the time but twenty nineteens is three hundred and eighty and half that is one hundred and ninety …

However, he did not justify his conjecture structurally.

In Chess 2, the algebraic strategies most often used were ‘recursion’ and ‘explicit rule’. The counting strategy was not evident. In the lesson introduction, Liam used the $x + y$ method recursively:

447	Liam:	Just really one plus zero is one, two plus one is three, three plus three is six, four plus six is ten, five plus ten is fifteen, … eh six plus fifteen is twenty-one, eh seven plus twenty-one is twenty-eight, nine eh eight plus twenty-eight is thirty-six, nine plus thirty-six is forty-five.

During group work, pupils were given a worksheet in which they were asked to give consideration to the number of games of chess that would be played in the case of 11 to 20 competitors and thereafter to 40 competitors. The majority of them used the $x + y$ strategy to complete the worksheets for 11 to 20 competitors. However, in the follow-up discussion on the number of games for large numbers of competitors, David reiterated the rule he had found on day 1:

635	David:	Multiply it by the number less …
636	TD:	Hm, hm.
637	David:	… and then half it.

Shortly after this, Enda observed the similarity between Chess and Friendship Notes.

639	Enda:	It looks like … it’s pretty much the very same as the friendship cards, it seems kind of like that.

Barry reflected on the non-reflective nature of both activities, that is,

641	Barry:	It’s kind of the same thing as, eh, you wouldn’t have to do themselves so there’s going to be one less.

Both Myles and Colin gave consideration to the structural difference between the two activities. Colin gave the following description:

651	Colin:	Em, well cos in the friendship notes you have to give two because if there were three you would have to give one to each person …
652	TD:	Hm, hm.
653	Colin:	… and everyone has to give one to each person, so it’s the same as three by two
654	TD:	Hm, hm.
655	Colin:	Eh, and in chess you only have to play them once even if they challenge you
656	TD:	Hm, hm.

On foot of these ‘structural’ deliberations, Enda announced that

662	Enda:	Eh well, I actually definitely agree with David’s way by doing the friendship notes, the same way as the friendship notes and halving it …

It would seem that the justification offered by Barry, Myles and Colin was sufficient to convince him that the explicit formula was indeed appropriate. In follow-up reflective accounts most pupils aligned themselves with ‘David’s method’, although it is unclear if they took account of the structural verification of his formula.

DISCUSSION

The strategies that were used by this group of pupils to solve the Chess problem included counting, recursion, whole object (incorrectly) and formation of an explicit rule. They did not appear to use the ‘chunking’ strategy. The explicit rules included both summation and a more general formula. Towards the conclusion of the lesson some pupils were able to verify the explicit rule (formula) with reference to the structure of the problem. Although most pupils aligned themselves with the formula at the end of the lesson, it is not clear that all had fully understood the structural dimension. What this paper shows, however, is the variety of strategies that pupils can use to solve complex problems. It could be argued that the format of the lesson (that is the use of consecutive lower numbers initially and later ‘higher’ numbers) lent itself to a progression from count through recursion and eventually to the use of a formula. In this regard, Warren, Cooper, and Lamb (2006) have alluded to the consecutive listing of x, y values as a factor that inhibits the development of relational or functional thinking. What seems to be the case in this lesson was that, in general, pupils’ choice of strategy was based on that which seemed to be most efficient for the task at hand. However, the dominance of the $x + y$ method can be attributed to the use of a table and it would be interesting to see what kind of strategies would emerge if the lesson were designed differently.

In the lesson described in this chapter, the pupils used natural language to express the relationship between the number of players and the number of games. For David’s statement in turns 635 and 637 (“Multiply it by the number less … and then half it”) could be described in conventional algebraic terms as $\frac{1}{2}\, n\, (n\text{-}1)$. While there well might be concern about the lack of rigor in children’s use of terms, I would argue that pupils need to have the opportunity to speak thus before embarking on more conventional symbolic terms and that it is perhaps the lack of opportunity to do so that has contributed to the ‘Algebra Problem’ (Kaput, 2008). This resonates with an argument, made by Caspi and Sfard at CERME 7, that pupils’ informal discourse can serve as a powerful resource for the development of more formal algebraic ideas (Caspi & Sfard, 2011).

Lannin (2005) found that students rarely justify their generalisations in small group situations and acknowledged the role played by a teacher in pressing for such justification. It was similar in the lesson described here and, in fact, in the plenary phase at the end of each session, probes by the teacher led to students justifying their choices and building on each other’s thinking (e.g., in turn 641 Barry built on Enda’s conjecture). This was a significant factor in developing algebraic thinking. Furthermore, the use of an activity with some variation a month previously laid the ground for structural verification of the formula. It is likely that were it not for the Friendship Notes lesson, pupils’ thinking would not have progressed beyond the empirical level. To this end, the sequencing of activities by teachers is an important element in stimulating and developing pupils’ generalisation strategies.

NOTES

1. In the Republic of Ireland, indicators such as unemployment levels, housing, number of medical card holders and information on basic literacy and numeracy are used to determine socio-economic status of schools.

2. Transcript conventions (related to this paper) are: TD: the researcher/teacher (myself); … : a short pause; ( ): inaudible input.

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HOW ARITHMETIC EDUCATION INFLUENCES THE LEARNING OF SYMBOLIC ALGEBRA

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Abstract *In line with a design science project aiming at designing a theory-driven learning environment for teaching symbolic algebra by comparing geometric quantities, this paper presents first results of an investigation on how students' previous experience with arithmetic and new learned (geometrical) knowledge about symbolic algebra are intertwined.*

Keywords: *Symbolic Algebra, Didactical Contract, Algebra History*

INTRODUCTION

Various international researchers work on questions concerning problems of students' approach to school algebra (see e.g. Kieran, 2007). Hence, there is detailed knowledge about which difficulties exist and many ideas concerning how to face these difficulties. The idea of early algebra (see e.g. Carraher & Schliemann, 2007) is to introduce algebra in earlier grades. There are different reasons for following this approach. An important reason for teaching early algebra is that there is evidence for arithmetic, as it is taught in school, negatively affecting children's algebra learning ability (see McNeil, 2004). But there is little research regarding *how* exactly arithmetical education influences the learning of algebra.

THEORETICAL FRAMEWORK

Algebra as generalised arithmetic?

"When we look at school tradition in different countries [...] algebra is generalised arithmetic." (Lins & Kaput, 2004, p.50) But to tie algebra to children's experience with natural numbers provokes multifaceted difficulties. McNeil (2004) emphasised the relevance of the change-resistance account. "More specifically, the account posits that children construct knowledge on the basis of their early experiences with arithmetic operations and that this knowledge contributes to children's difficulties with more complex equations." (McNeil, 2004, p.938) She refers to three knowledge structures that are hindering a good approach to algebraic equations: solving math problems means acting out operations, the structure of maths problems is "operations=answer" and the equal sign is followed by the result. Carraher & Schliemann (2007, p. 670) point up similar findings.

A reason for these difficulties may be the focus on natural numbers, especially the number sense concerning these numbers, and the lack of concentration on a more generalised idea of number in the early years of mathematics education. But field reports from early algebra classrooms show that arithmetic can be taught effectually as part of algebra. This leads to the conjecture that not arithmetical knowledge itself, but the way of dealing with mathematics that students acquire in traditional arithmetic education is the source of these difficulties. And because this kind of handling of mathematics is acquired in arithmetic education it may be triggered whenever numbers come into play. In this paper the concept "patterns of action" is used for these ways of dealing with mathematics.

The concept "pattern of action" is based on Brousseau's didactical contract (see. Hersant & Perrin-Glorian, 2005). "Situating a problem within a certain mathematical field guarantees that certain techniques will appear natural and will be favoured whereas others will be improbable." (Hersant & Perrin-Glorian 2005, p.118). Patterns of actions are defined as this certain techniques which are bound to the *domain* in which students believe themselves to be. Patterns of actions depend on the *didactic status of knowledge*. Usually patterns of actions connected with old knowledge are rooted deeper than patterns of actions connected with new, recently taught knowledge.

There are several ideas to counter the problem caused by unwanted patterns of action that are induced by arithmetic education and hinder a good approach to algebra. One idea is to change curricula, train teachers and in doing so trying to teach arithmetic in a more algebraic way. Research literature on early algebra shows the gains from such approaches (see Carraher & Schliemann, 2007). However, changes in primary mathematics classrooms are slow and "mathematics equals calculating" is deep-seated in our society. Hence, there is also a need for research about how to teach algebra to students who have already acquired patterns of action that hinder their algebra learning.

Teaching algebra without arithmetic?

In traditional school algebra the introduction of algebra is characterised by a syntactic way of using symbolic algebra. The consequence is that students focus on the symbols and have problems to give variables meaning (see e.g. Malle, 1993). In order to give symbolic algebra meaning it is necessary to connect it with a mathematical subject students are already familiar with, like numbers and arithmetic. However, using arithmetic as base for symbolic algebra brings in the undesired patterns of action described above. As the patterns of action are bound to a domain, in this case arithmetic, a way to face this problem may be introducing symbolic algebra within a different domain and link algebra with arithmetic later, when the desired algebraic patterns of action are consolidated.

A historical point of view

To learn more about the learning of symbolic algebra an additional look at the development of symbolic algebra from a historical point of view may be helpful. The reason for looking at the history of algebra is not that the ontogenesis of students' understanding of symbolic algebra has to follow the phylogeny of symbolic algebra (see Harper, 1987). Rather a look at the history of the development of symbolic algebra can help to identify specific problems with and chances for learning to deal with symbolic algebra. Rhetoric and syncopated algebra were used over centuries before symbolic algebra emerged. Hence it is not surprising that introducing symbolic algebra is likewise challenging for students.

History shows that the development of algebra is closely connected to geometry, the so called geometric algebra. One may reason that geometric algebra was an obstacle for symbolic algebra because in order to work with symbolic algebra one has to detach oneself from the graphic power of geometry (Krämer, 1988), which is not easy. However, whilst geometric algebra supports a holistic, object-oriented view on algebra (Sfard, 1995), the introduction of symbolic algebra allowed a new view on geometry, namely analytical geometry.

Geometric algebra as an alternative entry point to algebra

This historical situation cannot be transferred one-to-one to school curriculum, because the students do not have an extensive education in geometry prior to the introduction of algebra. They only have an extensive education in arithmetic. If the (possible) roles of geometry and arithmetic for school algebra are contrasted (see Figure 1) a possible alternative route into algebra becomes apparent.

<table>
<tr><th colspan="2">Introducing symbolic algebra with...</th></tr>
<tr><th>Geometry</th><th>Arithmetic</th></tr>
<tr><th colspan="2">Advantage</th></tr>
<tr><td>A graphic basis for working with variables allows a holistic view on the task, the operations and relations.</td><td>A numerical basis for working with variables allows a pool of abstract examples for operations and relations.</td></tr>
<tr><th colspan="2">Disadvantage</th></tr>
<tr><td>“Graphic power” ties symbolic algebra to concrete geometry.</td><td>Patterns of action hinder the view on operations and relations.</td></tr>
<tr><th colspan="2">Perspective</th></tr>
<tr><td>Analytical geometry gives new insight into geometry.</td><td>“Algebraic arithmetic” gives new insight in arithmetical operations.</td></tr>
</table>

Figure 1: Relation between symbolic algebra and geometry respectively arithmetic

The idea is to use the graphic basis of geometry to teach students a holistic view of operations and relations. Because geometry is used, the already acquired patterns of action that may obstruct students view on operations and relations may not be triggered. Later, when students have achieved the desired algebraic patterns of action, algebra can be connected with arithmetic. Then the rich arithmetic pool of abstract examples for relations and operations can be used to untie algebra from concrete geometry. Additionally the new “algebraic arithmetic” may give new insight in arithmetical operations.

The crucial point of this approach towards algebra does not appear to be the teaching of algebra in a geometric way. In fact the fundamental question is what will happen when geometric algebra meets arithmetic? Will arithmetic patterns of action still dominate? How sustainable will be geometric-algebraic patterns of action? Which patterns of action will be applied if the arithmetic and the geometric-algebraic domain converge? In the following will be presented how these questions have been investigated and as well as the discussion of first results

METHODOLOGY

The aim of the design research project (see The Design-Based Research Collective, 2003) that is underlying this paper is the development of a theory-driven learning environment for introducing symbolic algebra after primary school, based on geometry. After several pre-studies (see Gerhard, 2009) a teaching experiment with 10-11 years old students of a 5th grade of a grammar school was conducted. The learning environment is based on ideas of the El'konin-Davydov-Curriculum, which was developed in Russia and refined during teaching experiments by the Measure-Up-Program at the University of Hawaii. For a detailed description of the teaching experiments see Dougherty (2008) and Davydov (1975).

The teaching experiment took place at the end of the school year. During the school year arithmetic lessons contained repeating basic arithmetic operations, place value system, calculating with decimals and prime numbers. Topics of geometry lessons were the coordinate system and basic ideas about circles and angles. The topics length, area and volume had not been taught at the time of the teaching experiment. Hence, for the teaching experiment, a key consideration was that student would need to use intuitive ideas of area and volume without using multiplication. Figure 2 shows the basic principles of the learning environment and a sample task.

Basic principles	**Sample task**
• Comparing geometric quantities of unknown size • Calculation with letters that characterise unknown sizes of geometric quantities • Visualisation with auxiliary drawings. • Analytical idea: choosing letter variables for unknown values • Finding as many relations as possible. • Write down relations as equations and inequations. • Use basic transformation rules (Malle, 1993, e.g. a+b=c ⇔ $b = c\text{-}a$ ⇔ $a = c\text{-}b$)	A piece of land has area p. The piece of land contains of two parts. One part is grassland, the other part is farmland. The grassland has area g. How big is the area of the farmland? 1) Make a drawing which should be as simple as possible. 2) Write down all equations and inequations you can think of the diagram. 3) Which equations and in-equations tell you how big the farmland is?

Figure 2: Learning Environment

The research focus of this paper is on the question how new patterns of action about symbolic algebra acquired in the teaching experiment might interact with old patterns of action acquired in previous arithmetic education. The approach to this question is explorative and therefore hypothesis generating. To allow for an investigation of the long-term effects, additional problem-centred, semi-standardised interviews were conducted 6-7 months after the teaching experiment. During the interviews students were confronted with arithmetic story problems that were modified by using letters instead of numbers. The transcripts of the interviews together with the students' written products were analysed using an analysis tool which was designed for this purpose (see Gerhard, 2011 and Figure 3).

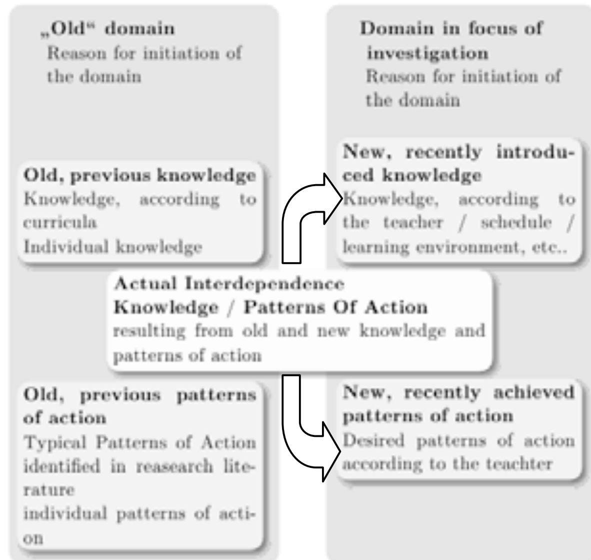
Sample interview problem	**Interdependence Analysis Tool (revised 2012)**
Running for a good cause Mrs. Gunther takes part in a run in stages. The more kilometres she runs the more donations are paid to the organizer. The run lasts from Monday to Saturday. From Monday she runs $\underline{a}$ hours per day. On Saturday she only runs half of the time. If she makes $\underline{b}$ kilometres per hour, how many kilometres does she run altogether? (Structure of the task: $5 \cdot a \cdot b + (a:2) \cdot b = L$)	

Figure 3: Interview design and revised interdependence analysis tool

There was a need for a new analysis tool because for the analysis two dimensions have to be considered. First, students previous experience that has its origin in previous arithmetic education has to be contrasted with their understanding of algebra that stems from the geometric-algebraic teaching intervention. Second, beyond the patterns of action the students' arithmetical and geometric-algebraic knowledge have to be investigated as well. In fact the patterns of action supposedly play a key role, but this does not mean that the domain-specific knowledge can be neglected.

The analysis tool is used as follows: The starting point is an interpretation concerning the reasons for how the student acts in the actual situation. These reasons are classified according to domain, knowledge and patterns of action, both old and new. The analysis starts with expected patterns of actions and knowledge in both domains and ends with the student's supposable actual algebraic patterns of action and knowledge influenced by previous arithmetic patterns of action knowledge.

THE STUDENTS

The performance of two students, Dorian and Christina, on the sample task is used to illustrate the findings. The analysis is part of case studies with twelve students, which differ in performance on arithmetic and algebra. Dorian is an outstanding student in arithmetic, while Christina is low-achieving. Both students did well in the tasks of the teaching intervention.

Dorian read the task and immediately started to write down and explain a solution of the task (see Figure 4), using letters and numbers.

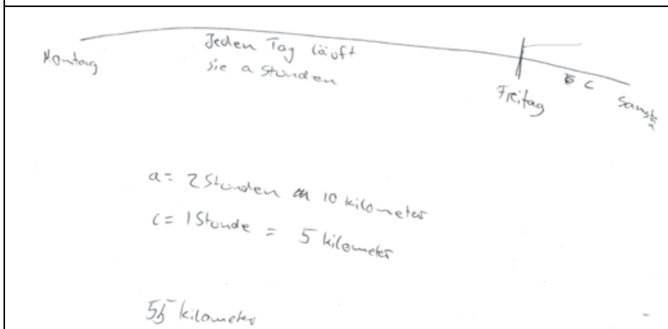
1	Hm- well, than is that here, then she runs Monday to Friday, that are 5 days every day **a** hours,

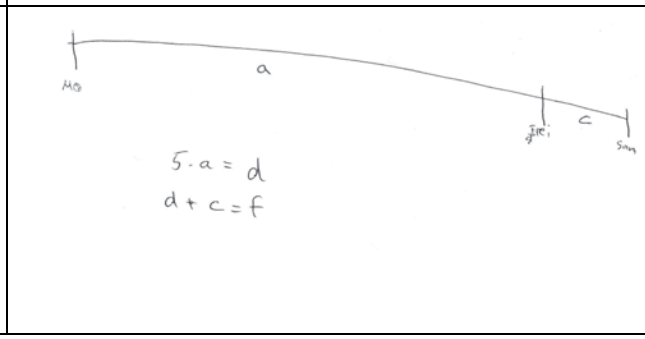
2	that means **a** plus **a** plus **a** plus **a** plus **a-** (writes ‚a+a+a+a+a‘) and then she runs only half as many and
3	that means she also runs only half as many, that means again once more plus **a** divided by 2 (writes
4	‘+a:2‘) [...] And that is then- only the time (writes ‚=T‘). If we now, if we, if now **a** hours are 2 hours,
5	then she would walk there always 2. (points to the first **a**, writes ‘2+2+2+2+2’) and here always
6	1 (writes ‘+1’) and that would be then two four six eight ten (writes ‘=11’) eleven hours. And then
7	she makes 3 Kilometres per hour, well in 2 hours then 6 plus 6 plus 6 plus 6 plus 6 plus 3 (writes
8	‘6+6+6+6+6+3’), that would be one two three four five- thirty three (writes ‘=33’). And that is so a
9	while, that is the same as this (points one after another at the first and second written number
10	equation). Then I could her, you already see, that looks nearly the same, **b** plus **b** plus **b** plus **b** plus **b**
11	plus **b** divided by 2 (writes ‘b+b+b+b+b+b:2’) and that would be kilometres (writes ‘=K’).

Figure 4: Dorian solving the task “Running for a good cause”

Christina (see Figure 5) begins with an auxiliary drawing. Then she solves the task with invented numbers, while at the same time working on the letters (line 21-25). Finally she solves the task with letters (line 31-32).

Solution with numbers	Solution with letters





21	And the, erm, has to be from Monday till Friday, erm, every day, erm, **a** hours. Well, you could maybe
22	**a- a** equals 2 (writes ‘**a**=2 Stunden’ (***a***= *2 hours*). [...] well write down. Every day she (writes ‘Jeden
23	Tag läuft sie **a** Stunden’ *(Every day she runs **a** hours)* under the line) runs **a** hours. And then, if she
24	from, erm, on Saturday runs only half of the time, erm, this has to be so 1 hour. Well here, if you, if
25	you (incomprehensible) **b**, no (writes **b** under the line and cancels it out) **c**, **c** is the one hour?
...	
31	Well, for my part 5 times **a**? [...] 5 times **a** equals- Then you shall invent letters for the result? [...]
32	5 times **a** equals **d**, for my part- and yes then **d** plus **c** equals for my part **f**.

Figure 5: Christina solving the task “Running for a good cause” with illustrating notes

RESULTS AND DISCUSSION

Dorian’s and Christina’s transcripts illustrate the following patterns of action: *Keyword-Strategy*, *Inventing-Letters-Strategy*, *Inventing-Numbers-Strategy*, *Letters-As-Quantities* and *Use of Auxiliary-Drawings*.

Keyword-Strategy

Using keywords to find out which operation to conduct is an old *pattern of action* rooted in previous mathematics education. For example the keyword “remove” leads to subtraction. Because keywords will not change when letters are used this strategy still work with letters. Dorian may have chosen the operation “:2” because of the keyword “half” (line 2-4). Keywords have a positive effect, if the calculation strategies they invoke are explicitly available. For Christina, the keyword “half” did not invoke “:2”. Instead she wrote *c* (line 25) and defined, that *c* is half of *a* (line 24). She is not doing this explicitly but via *Inventing-Number-Strategy* (see later). This may be a sign that she lacks *old arithmetic knowledge*, that she has an *insufficient operational understanding*

Students adopt the Keyword-Strategy for the use in the *new algebraic domain*, but they will only have a positive effect, if the calculation strategies they invoke are compatible with the corresponding symbolic description. In another task the students had to find out how often a distance *x* fits into a distance *y*. Transcripts of several students working on that task show that another keyword that cannot easily be expressed in symbolic language is “how often fits in”. The students learned in previous arithmetic education the *old pattern of action* that if *x* and *y* are natural numbers and ***y*** is a multiple of *x* it is possible to calculate the solution *L* with the help of repeated addition or multiplication, instead of division. But repeated addition without knowing how many addends there are is difficult to express with symbolic algebra. The corresponding symbolic description for multiplication is: $L{\cdot}x=y$. As the solution *L* of the problem is not identical with the “result” *y* of the equation, this equation is not compatible with *Process-Orientation* **(**see later), an *old pattern of action* which draws the student’s attention to results instead of relations.

Inventing-Letters-Strategy

Inventing letters directly results from a fundamental algebraic idea, the *Analytical Idea,* which was taught as *new algebraic knowledge* as part of our algebraic learning environment. It results in the *new pattern of action*, that if a value is unknown you can write a letter that can be used like a number. Thus one is able to work with the unknown value like an actual number assumed this *new algebraic knowledge* exists. Dorian is combining the analytical idea with an *old pattern of action*, *Process-Orientation***.** He is allowed to use letters for unknown values and the result of his letter calculations (line 2-4 and 11) is unknown. So he concludes that he is allowed to invent letters for results, *T* for time and *K* for kilometres. Of course an algebra teacher might welcome Dorian’s expressions $a+a+a+a+a+a{:}2$ and $b+b+b+b+b{:}2$ as result. But these expressions lack closure and students who are tied to process-orientation will hardly accept this. However, inventing a letter as result may be a good *new pattern of action* for a start.

Christina also uses this strategy as *new pattern of action* (line 25 and 31). At first she uses *b* for half of the time but realised properly that *b* is already given away (line 24-25). Instead she uses *c.* The difficulty with inventing letters for unknown numbers is that students have to relate every new letters symbolically with the variables already in use. Christina is not able to relate *c* with *a* symbolically, most likely because of *insufficient operational understanding*. However if she gains more experience with the use of variables this may help her to identify insufficient operational understanding and make explicit her *implicit old knowledge* about operations. Here it is possible that her *old arithmetical knowledge* may benefit from *new algebraic knowledge*.

Inventing-Numbers-Strategy

Dorian replaces letters by invented numbers, conducts the calculations and translates the conducted calculations back in letter expressions. Later Dorian will say that he has used the numbers as aid for thinking.

Old knowledge about *setting up and manipulating number expressions* as well as *number operations* can support *new knowledge* about *setting up and manipulating letter expressions* as well as *letter operations*. But *patterns of action* like choosing inappropriate numbers, replacing the wrong letters or inventing numbers for unknown values without keeping in mind the relations to other values, which are deeply connected with *old knowledge* may cause problems.

Christina is inventing the numbers for the hours per day and hours for half of the time considering the relation between the variables (line 22-24). But she does not explain that c is half of a. Instead she uses the numbers to make this relation explicit by explaining that a equals 2 hours, 1 is half this time and c equals 1 hours. Probably the *pattern of action* of choosing easy numbers is inappropriate to put in her mind that the half can be calculated explicitly by ":2".

Letters-as-Quantities

Dorian was taught the *new knowledge letters signify unknown numerical values of quantities.* Earlier in the interview he explained that letters stand for numerical values, but he consequently refers to the quantities as if he is talking about the value of the quantities. Therefore he uses Letter-as-Quantities as *new pattern of action.* He calls T time (line 4) and K kilometres (line 12). At first this seemed to be a metonymy because it is easier to talk about kilometres instead of the number of kilometres. Especially in geometry this double meaning is widely accepted, e.g. talking about the side s if actually meaning the length s of the side. But using multiple meanings as *old geometric pattern of action* may cause problems.

At the end of the task (line 10-11) Dorian is using b for both kilometres per hour and kilometres per day, but not for all kind of kilometres as the kilometres in total are signified as K. One reason may be that due to using the *Inventing-Number-Strategy* and consequently calculating the kilometres per day with mental arithmetic, for him the relation between hours and kilometres does not become explicit. Thus he may not realize that the status of the kilometres has changed. Christina does not clarify the relation between hours and kilometres, too, and mixes up the letters that signify the quantities. Again *new algebraic knowledge* may be a chance, here for making explicit the students' *implicit old knowledge* about relations

Using auxiliary drawings

Christina encountered the problem in beginning to solve the task and needed first to convince herself that she could use a drawing, with lines to represent the quantities.

While she was working she comprised the drawing many times (line 22-25). In the end she translates the drawing into equations (line 31-32). This suggests that using auxiliary drawings is an important n*ew pattern of action* for her. Dorian does without using a drawing but he later refers to the 'graphic power' of the equations by stating "you already see, that looks nearly the same" (line 10). He seems to be able to 'see' the explicit relations in equations without using drawings. Dorian's new algebraic knowledge benefits from this *old knowledge*. But unfortunately this *old knowledge* is not self-evident for students.

The problem with auxiliary drawings is that students can only see in the drawings what they have put into it. Christina's drawing does not contain enough information about the relation between hours and kilometres and hence she is not able to gain equations that represent the relations of the task, although the equations represent the relations of the drawing correctly. Indeed, the *old pattern of action* of *concentrating on numbers instead of relations*, reinforced by a *Process-Orientation* and an inappropriate *Keyword-Strategy* may be the reason for inadequate drawings.

PERSPECTIVE

The influence of previous arithmetical education on the learning of algebra is complex. The examples of Christina and Dorian reveal that in particular the lack of explicit knowledge about operations and relations hinder a good approach to algebra. Patterns of action and knowledge about proportions of numbers, both achieved in previous arithmetic education account for clouding this knowledge. Giving algebra a graphic basis may help to make operations and relations explicit. Christina's example shows that the drawings help her finding equations that represent the relations of the drawing. However, it also shows that teaching students to transform texts into drawings is a challenge that should not be underestimated. It remains open if a graphic basis of algebra is an effective corrective for patterns of action acquired in previous arithmetic education.

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HIDDEN DIFFERENCES IN TEACHERS' APPROACH TO ALGEBRA – A COMPARATIVE CASE STUDY OF TWO LESSONS

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Algebra is a multi-dimensional content of school mathematics. In this study a fine-grained analysis was made of two lessons on introduction of variables in two Swedish classes with teachers who followed the same curriculum. When comparing the two lessons subtle differences were found in the approach to algebra and the meaning of variable in what at first seemed to be lessons about the same content. Findings indicate that teachers shape the opportunities of learning by the approach they take.

INTRODUCTION

Introduction to algebra is known to be a problematic topics in school mathematics, and recent research claims that the problems are more related to learning conditions than to cognitive limitations (e.g. Cai & Knuth, 2011; Kaput, Carraher, & Blanton, 2008). Algebra is a large content area and can be approached from many different perspectives. There is a considerable variation between different countries concerning approaches to algebra and content for algebra teaching, and naturally these differences may lead to differences in learning outcomes (Kendal & Stacey, 2004). Differences found in curricula documents and textbooks can make comparisons of teaching and learning algebra difficult due to a lack of consensus about how they define algebra. Research on teachers' use of curriculum (Remillard, 2005) has highlighted the teacher's role in the enactment and meaning making of curriculum. The enacted curriculum is influenced by variables such as teacher's mathematical knowledge, beliefs, goals and traditions. Some such variables may be clearly outspoken, others may stay hidden and surface only in the enacted curriculum. The question raised in this comparative case study is whether two teachers who follow the same curriculum and textbook teach the same algebra, or if differences can be found on a subtle classroom level that may influence students' opportunities for learning. A fine-grained analysis of the given instruction shows that two different approaches to algebra are made although both teachers follow the same national curricula syllabus and plan the lesson influenced by the same textbook task. The specific content in the lessons is introduction to the concept of variable. When learning about variables six different ways that children interpret and use letters have been described by Küchemann (1981), of which only one is the interpretation of a letter as a variable in its proper sense: as representing a range of unspecified values. The other categories are: letter evaluated, not used, as an object, as a specific unknown and as a generalised number i.e. representing several (specific) values rather than just one.

DIFFERENT APPROACHES TO SCHOOL ALGEBRA

The discussion of how to define algebra is as multifaceted as are people's experiences of algebra and views on how to teach it. In the 1990's reports showed that school algebra was predominantly rule based and procedural (Kieran, 1992) whereas researchers suggested a broader approach including a generalisation, modelling, problem-solving and functional perspective (Bednarz, Kieran, & Lee, 1996). Although algebra is an indispensible linguistic support in the development of mathematics (Rojano, 1996), history warns us of approaching algebra as a language emphasising its syntax and "placing symbolic manipulation as an object of knowledge in advance of situations which can give rise to it" (ibid, p 61). Today algebra is described as a branch of mathematics that deals with symbolizing general numerical relationships and mathematical structures. The learning of algebra can thus be seen as both *learning to see and reason* about relationships and structures, and *learning the formal symbolic language* used to express these relationships and structures. The former is often called algebraic reasoning and is prominent in literature about early algebra (Cai & Knuth, 2011; Kaput et al., 2008). According to Kieran (2004) school algebra is constituted by three types of activities; *generational activities* such as forming expressions and equations using variables and unknowns, *transformational activities* such as simplifying, substituting or solving equations, and *global meta-level activities* where algebra is used as a tool, including problem solving, modelling, noticing structure or change and analysing relationships. In a linguistic approach generational and transformational activities dominate instruction, whereas a generalisation (Mason, 1996) or a problem-solving (Bednarz & Janvier, 1996) approach to algebra involve more global meta-level activities. The current state of school algebra varies from country to country but tend to follow either a traditional or reform-oriented program (Kieran, 2007). A traditional program has a strong symbolic orientation and approaches algebra as a language, whereas a reformist program deals more with functional situations through modelling and problem-solving activities. Using the term *curriculum* to describe the resources and guides used by teachers (Remillard, 2005), this study investigates if two lessons following the same curriculum give the same perspective on algebra or if different approaches to algebra surface through the teachers' transposition of the curricula documents.

METHOD

As part of data collected in an international comparative video study called VIDEOMAT (Kilhamn & Röj-Lindberg, 2013) two Swedish grade 6 classrooms were videotaped during a lesson on introduction to the concept of variable. Data was collected *in situ* using three cameras during a sequence of four lessons when the teachers planned to introduce algebra. Both schools were public schools, following the national curriculum (Lgr11) and using the same textbook (Carlsson, Liljegren, & Picetti, 2004). Both teachers were trained as generalist teachers for grades 1-7 with approximately 10 years of teaching experience. Explorative analyses were made of the videos and of verbatim transcripts (in Swedish and translated into English).

Similarities and differences of the two lessons were sought, particularly in relation to different approaches to algebra as described in research literature.

CURRICULA DOCUMENTS AND TEXTBOOK TASK

Before presenting the classroom analysis the curriculum that serves as a common point of departure for both teachers is described. When planning the algebra unit both teachers refer to the National Curriculum (Lgr11) and the textbook (Carlsson et al., 2004), including its teacher guide, as the foundation of their instruction. Lgr11 includes a mathematics syllabus consisting of general aims and core content in mathematics (Lgr11, pp. 59-46). Algebra is a core content, including for grades 4-6:

> "Unknown numbers and their properties and also situations where there is a need to represent an unknown number by a symbol; simple algebraic expressions and equations in situations that are relevant for pupils; methods of solving simple equations; and how patterns in number sequences and geometrical patterns can be constructed, described and expressed."

The two lessons analysed is, in both classes, the very first lesson in which the concept of variable is introduced. The textbook unit on algebra starts on pages 94-99 dealing with the meaning of the equality sign and simple equations with one unknown. On page 100, labelled '*Variables can vary',* the concept of variable is introduced using age differences with the following information given (all translations made by the author of this paper):

> "We can call Amer's age *a*. Sama is 4 years older than Amer. That makes Sama's age *a* + 4. The value of *a* changes when Amer's age changes. The value of *a* can vary, *a* is a variable." (Carlsson et al., 2004, p. 100).

This introduction is followed by a task on the same topic (fig 1), and is commented in the teacher guide in a short paragraph (italics added):

> "The word *variable* is another new word for the students. A variable is a quantity that can *vary*. On page 100 we have ages as an example of variables. All students understand that ages vary – when my sibling is one year older I am one year older. To work with variables is *very useful* in mathematics, for example all *formulas* are based on the fact that you can vary the value of the variable. Formulas are *valid for a range of values*."

Osmond is 3 years older than Mohammed.
Leyla is 5 years younger than Mohammed.

How old is Osman when Mohammed is
a) 10 years b) 15 years c) 30 years

How old is Leyla when Mohammed is
a) 10 years b) 15 years c) 30 years

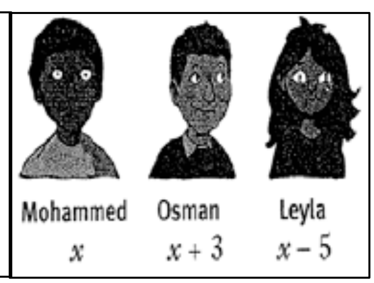

Figure 1: First task on introduction to variables (Carlsson et al., 2004, p. 100)

RESULTS

The results presented here focus similarities and differences concerning the content matter taught by the two teachers, Ms B and Ms C. Undoubtedly there are demographic differences as well as differences in classroom organization that have an impact on learning outcomes which are not in focus in this study, and no comparisons of actual learning outcomes is possible from the collected data. Instead the intent is to look closely at the art of the content matter taught and the approach taken to algebra in each lesson, highlighting similarities and differences in the first classroom activity in the lesson on *introducing variables* that surfaced in the analysis.

Ms B's lesson

Ms B starts her lesson by referring to previous tasks where a variety of symbols were used in place of a number, and asks the students to find a more simple, convenient way of writing the mathematical statement "some number added to 2". Ms B emphasises many times that the point of algebra is to write something in an efficient way. "And maths is very much about actually finding convenient ways of doing things" The term *variable* is said to be related to *varying* illustrated by the statement that "x varies [in x+2=5 and x+2=7] because it does not represent the same value".

The age relation task from the textbook (fig 1) is projected onto the whiteboard and students are asked to work with it in small groups. Later their solutions are projected onto the board and discussed in class. Ms B particularly points out that she expects them to explain how they worked out the answer: "I want you to fill in how you have reasoned using a variable" She points out that they need to know what x means "what is it that you– in the problem so to speak what is it that you have found out? It might be good to know then what, this symbolises. Because x is a symbol for something."

When Ms B discusses the students' solutions in class, much time is spent going back and forth between different representations; words ⇔ symbols. On the whiteboard Ms B first shows answers without any variable, asking if the calculations are correct, which they all agree to. One example of such an answer is:

> a) 13 years. Osman is 3 years older than Mohammed so when Mohammed is 10 years Osman is 13 years.

Then Ms B highlights answers where x is present in the explanation, ending with the student focus group (FG) who wrote:

a) 10+x+3=13
b) 15+x+3=18
c) 30+x+3=33 x=0 Osman is always 3 years older than Mohammed.

In this example the students have added Mohammed's age and Osman's age (x+3) and the difficulty does not lie in getting the correct answer but in understanding the meaning of x. Excerpt 1 shows how the group discussed the problem. One student has solved the problem of finding the sought ages straight away [1], but then the group spends another 10 minutes discussing how to write it down. They include an x

because they know it is supposed to be there [2]. They try adding the symbolic expressions of the ages of Osman and Mohammed [2-4], but the discussion of what *x* symbolises continues until the third student finally suggests that $x=0$ [5-6].

Excerpt 1: FG discussing the age task, extracts from a 10 minutes long interaction.

[1]	S1	it is just 13, 18 and 33
	S3	so we can start working out how old they are now, how old they are.
	S2	Osman is 3 years older than Mohammed
(…)		
[2]	S4	Shall I work it out in an algebra way?
	S1	no (protests and wants to go on)
	S4	okay. Well then this is what we do. That is x plus x plus 3.
(…)		
[3]	S4	(pointing to where she has written 10 + x + 3) That is Osman's age [10] and that is someone else's age [x + 3]. It is Mohammed's age
(…)		
[4]	S2	We know Osman is 3 years older than Mohammed all the time.
	S1	So it has to be 13
	S2	yes 18, and then 33
	S1	yes. I said so all along. That's how easy it is.
	S2	because I tried, we tried to work out how old they are <u>now</u>. But that is impossible. (S4 writes 10+x+3=13, 15+x+3=18, 33+x+3=33)
[5]	S3	but eh, how about the x?
	S4	but x, they, here it is an age
	S3	well but
	S4	it's one of them. It's 3
	S3	but you can't have just x there? If it's there you think it means something ( ) write something
	S4	write x plus 3 is equal to, eh, 3
(…)		
[6]	S3	(writes x=0) Do you think this is, do you think this is correct?
	S2	x is equal to zero?
	S3	yes because x is nothing it is just what his age is called.

In the following whole class discussion (excerpt 2), Ms B directs attention to the information given in the task, particularly the algebraic expressions under the picture. She wants the students to express the ages in the algebraic way; however, this is not easy since they all agree that the answer is already given in the text.

Excerpt 2: Extracts of Ms B's whole class discussion of the age task.

Ms B:	What information did you use, to arrive at what Osman was? What does Peter say?
Student:	How old the others were.
Ms B:	And how did you find that out? Marcus?
Student:	It's in the text.
(…)	(Ms B returns to the task and points at the algebraic expressions under the picture)
Ms B:	Was there any group who explained the ages this way? When you tried to find out, how old Osman was?
(…)	

Student: Eh, we, we checked what the *x*:s meant and put it together.

Ms C's lesson

Ms C starts the lesson reminding the students of equations and introducing the term variable as being "reminiscent of variation for example. Thus it is something that varies". The introduction is centred on a description of the ages of Ms C's own family members, and ways of describing those ages in relation to her. She writes ages and age differences on the whiteboard (se fig 2). Ms C uses the first letter of each name to represent that person's age. Then she relates all the members' ages to her own, "describe our ages based on a variable then. I will describe it with a, a, with letters and numbers. And I will base it on myself all the time". When describing relations Ms C introduces *formulas* (in Swedish: *formel*) as illustrated in figure 2 and says:

> "Eh, for me to describe dad's age I'll take my age and then I'll add years, because he's older than me. This– my– since J means 36 right now. Mm, eh, so I add 27 there. That means that one can calculate via this formula, if one knows that I'm 36, then, that dad is 36 plus 27, which is 63. And we can also by looking at this understand how old my dad will be when I'm 40. When I'm 40, then the same formula holds, he's always 27 years older. Eh, so then you get 40 in there. How old is my dad when I'm 40, Alex?"

M= J+27	**M**ark	**A**nn-**C**hristin	**J**enny	**E**mma	**A**nna	**L**otta
L = J-13	63 yrs	60 yrs	36 yrs	32 yrs	29 yrs	23 yrs

Figure 2: What Ms C wrote on the board during the introduction.

By asking the students to calculate the father's age at different points in time (when J is 36 and when J is 40), Ms C illustrates the idea of a constant relationship involving a variable that can take different values. Students are then asked to describe their own families in a similar fashion. When the students have represented their families they are asked to "write down a variable and start from your family. For example: mom's age, equals, my age, plus …" See figure 3 for an example of a student's work.

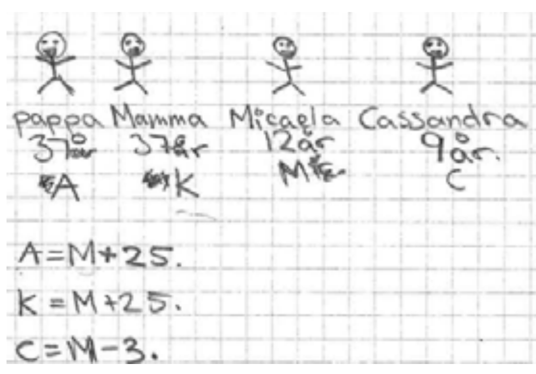

Figure 3: example of student work from Ms C's class.

Since they have all written several variables (one for each family member), what Ms C means is probably that they should decide which variable to relate to and then write down formulas that show the other members' ages in relation to the chosen one. During the following whole-class discussion of the student's results (see excerpt 3) Ms C asks: "What have you chosen as a variable", meaning which variable they have chosen as independent.

Excerpt 3: Extract from Ms C's whole class discussion on the family age relation task.

Ms C:	Then you'll tell us first about your family.
Student E:	Eh, okay, dad ( )
Ms C:	Dad, he is…
Student E:	Eh, 48.
Ms C:	48? Mm. Mom?
Student E:	Eh, 44
Ms C:	44 years old. And then you?
Student E:	Eh, I'm 12. (Ms C writes P 48, M 44, E 12).
Ms C:	You're 12. Eh, and what have you chosen as a variable?
Student E:	Eh…
Ms C:	Sh!
Student E:	I've chosen, A… Or, dad…
Ms C:	Yes.
Student E:	…equals me plus… should I say what it makes?

After some negotiation they work out 48-12=36. Ms C writes on the board: P=E+36. Here the independent variable is E (student's age) and the fathers age (P) is expressed by a formula using the variable E.

ANALYSIS OF SIMILARITIES AND DIFFERENCES

When comparing these two introductions to the concept of variable some similar features can be noticed:

- The concept of variable is introduced using age relations as suggested by the textbook, involving the use of symbolic language to represent age relations.
- A letter or other symbol for an unknown in an equation has been introduced before the concept of variable.
- Variable is described as something that varies and there is an underlying but unclear distinction between a specific unknown and a variable.

Some differences between the two lessons concerning the algebra content and the approach to algebra were found and will he be described in terms of i) the framing of the algebra task, ii) the meaning of algebra, and iii) the meaning of variable.

Framing of the age relation task

Ms B starts her lesson with group discussions based on the textbook task. In this task all relations are given both in words and symbolically and students are asked to interpret them to calculate ages in different scenarios. Only one variable is used. It is easy for the students to find the ages. The task gives the general case (the relation) and students are asked to calculate specific cases. Students *interpret and try to understand the language of algebra*. The activity is *transformational*.

Ms C asks her students to describe their own families. In that setting all specific ages are known and the task is to find a way of describing their relations, both with words and symbolically using a formula. More than one variable is used. It is a question of choosing an independent variable, of assigning symbols and of describing relations. The task gives specific cases and students are asked to express a generality. Students *use the language of algebra* as a tool to model a situation, and the activity could be characterised as *generational* and *global meta-level*.

The meaning of algebra

Ms B tells her students that algebra is a simpler, quicker, more convenient way of writing down mathematical statements. The task itself is easily solved without algebra so the introduction of algebra in the activity does not make it more simple or convenient. The meaning of algebra conveyed in this lesson is "algebra as (an efficient) language". Students first solve the task by calculating specific values and then try to express what they already know in the new language.

Ms C introduces the concept of *formula* along with the term *variable* so that relationships can be modelled. Algebra is used as a tool to model age relations and as a result formulas can be used to predict different scenarios. The meaning of algebra conveyed is "algebra as generalisation" and "algebra as a problem solving tool".

The meaning of variable:

In Ms B's lesson expressions with only one variable are used. It is implicit that the variable stands for a range of values (Osman's age as x+3), but the students are asked to calculate specific values. In the students' equations x does not vary (10+3=13; 10+x+3=13). Moreover, in Ms B's examples of a variable in the two equations x+2=5 and x+2=7, x represents a specific unknown in each case rather than a range of values. Ms B pointed out that "x is a symbol for something", but in the students talk it is unclear to them what x symbolises: "If it's there you think it means something", "x is nothing it is just what his age is called". The meaning of variable is a letter that symbolises something, unclear what, or a symbol for specific unknown numbers.

In Ms C's lesson students are asked to write formulas including two variables resulting in equations that are true for a range of unspecified values. There is some confusion as to what in the task is a variable: "write down a variable and start from your family". When Ms C says "what have you chosen for a variable" the information she seeks is actually threefold: independent variable (E), dependent variable (P) and formula describing the relation (P=E+36). For Ms C the variable is a point of departure, it is what you relate to in a formula. "describe our ages based on a variable then. I will describe it with a, a, with letters and numbers. And I will base it on myself all the time". Her use of the terms *variable* and *formula* is a bit fuzzy, but she has situated the idea of variable in the context of expressing relations. Through the activity of generating formulas with two variables, the meaning of variable is a letter that represents a range of unspecified values.

DISCUSSION

The results indicate that the two teachers approached algebra in different ways. Ms B approached algebra as if it were a language of symbols that students need to learn through activities of interpretation and translation. Eventually this new language will prove to be efficient and facilitate mathematics. The learning process involved interpreting a general statement in specific cases. Only one variable was used and students were given the opportunity to experience letters as specific unknown or generalised numbers or to solve the task sufficiently without using the letter.

Ms C approached algebra as if it were a problem-solving tool, useful to model, generalise and express relations. The learning process involved taking a specific case to a general level. Several variables were used modelling one variable as a function of another and incorporating the concept of formula. Students were given the opportunity to experience letters as representing a range of unspecified values describing a relationship between two sets of values. The description of *algebra* and *variable* given in the curriculum follow a more traditional that reformist-oriented program for school algebra, and Ms B complies with this and with the textbook task to a larger extent than Ms C, who creates a new task inspired by the curriculum but with a slightly different approach. Ms B tried to teach her students the meaning of the word variable in line with how the teachers' guide introduces it: "The word variable is another new word for the students". In contrast, Ms C lets the last part of the teachers' guide inspire her where it says: "all formulas are based on the fact that you can vary the value of the variable. Formulas are valid for a range of values". The differences in how these two teachers interpret and enact the curriculum is subtle but results in different learning opportunities for the students. An awareness of the effects of such differences may influence textbook and syllabus writers to be more specific in the choice of examples and more explicit in explanations of the intended learning.

Conjectures about differences that are made in this paper do not take into account what happened after the introductory lesson, it is not ruled out that a teacher may approach algebra differently at different points in time or consciously use different approaches separately. The intention was to illustrate differences in what *may seem* to be similar introductions to the concept of variable based on the same curriculum. It can be argued that these differences may influence opportunities of learning offered in the lesson. It might well be that the teachers themselves are unaware of these differences and that their learning goals are the same. An implication of this study is that comparative research can be useful to detect hidden differences in teaching.

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LIMIT OF THE SYNTACTICAL METHOD IN SECONDARY SCHOOL ALGEBRA

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One of the main difficulties that secondary school students have to face in Algebra is insufficient mastering of syntactic rules as often pointed out both by teachers as well as researchers. In this paper, we adopt a logical point of view on equations and inequations and we support the hypothesis that for an adequate appropriation of these two notions, it is necessary to be able to articulate syntax and semantics. We start by explaining what is meant by "a logical point of view". Then, we examine in which respect the dialectic between syntax and semantics appears in Tunisian textbooks, through an analysis relying on both the dialectic between syntax and semantics as well as mathematical praxeology. Finally, we provide an example enlightening the paramount importance, in some cases, of the semantic point of view in order to solve (in)equations.

INTRODUCTION

One could think that the main challenge in the teaching and learning of the resolution of (in)equations in high school concerns algebraic rules. However, it is not always the case that applying such rules ensures effectiveness, as we will see in the example presented.

In this paper, we adopt a logical point of view on (in)equation, and we support the hypothesis that, for an adequate appropriation of (in)equations, it is necessary to be able to articulate both syntactic and semantic aspect. We first briefly present this approach referring to predicate calculus. Then, we examine in which respect the dialectic between syntax and semantics appears in Tunisian textbooks, by analysis relying on both the dialectic between syntax and semantics aspect as well as mathematical praxeology (Chevallard 1998). Finally, we provide an example enlightening the fact that, in some cases, the lack of mobilization of the semantic point of view may prevent students of successfully solving (in)equations.

A LOGICAL POINT OF VIEW ON EQUATIONS AND INEQUATIONS

Chevallard (1989) emphasizes the essential dialectic between arithmetic and algebraic calculations that he interprets as a link between syntax and semantics. Indeed the author explains that "when, in the sixth class, the teacher moves from $2+3=5$ and $3+2=5$ to the general relationship $a+b=b+a$, he moves from computing on numbers (integer) to an algebraic calculation (integer coefficient). In other words, an algebraic calculation, that we do not define more precisely here, makes a clear syntax to which the associated computational domain provides a semantic" [1] (Chevallard, 1989, p. 51). Moreover, the author shows that the students' understanding of algebraic calculation doesn't incorporate the idea of a dialectic between manipulation of algebraic expressions and substitution of numerical values in these expressions.

Furthermore, authors such as Selden & Selden (1995), Durand-Guerrier (2003) Durand-Guerrieret al. (2000), Durand-Guerrier & Arsac (2003, 2005), Chellougui (2009), Weber & Alcock (2004) and Iannone & Nardi (2007) have pointed out the relevance of the logical point of view for the analysis of mathematical reasoning in an educational perspective, mainly in calculus. As for us, we consider that the issue of linking both perspectives, i.e, semantics and syntax, has not been extensively addressed in research, neither in research on the teaching of mathematics in general, nor in the teaching of algebra in high school.

In our research program, the main question concerns the possibility of identifying, phenomena related to the dialectical syntax / semantic in the development of concepts of equation and inequation. In order to address this issue, following Durand-Guerrier (2008), we make the assumption that the semantic conception of the truth developed by Tarski (1936, 1944) through the notion of satisfaction of an open sentence by an element of the domain of interpretation, provides a relevant framework.

In formalised languages, as predicate calculus, a formula which is neither a logical theorem nor a contradiction is neither true nor false. In order to provide a truth-value, it is necessary to choose an interpretation, namely a domain of objects, properties and/or a relation. Furthermore, it is an important fact that such a formula may be regarded as a true statement in one interpretation, and as a false statement in another one. In the case of a logical theorem, it is true whatever the interpretation, while in the case of a contradiction, it is false whatever the interpretation.

For example, let us consider the formula "$\forall x \exists y F(x,y) \Rightarrow \exists y \forall x F(x,y)$ and the following interpretation: the domain of objects is the real number set; F is interpreted by the relationship "$\leq$ " the statement that interprets the formula is "$\forall_R x \exists_R y, x \leq y \Leftrightarrow \exists_R y \forall_R x, x \leq y$"; the antecedent of the implication is true, and the consequent is false, so the statement is false. Nevertheless, if the domain of objects that is considered is an upper-bounded part of the real number set, then the corresponding statement is true.

In introducing the notion of satisfaction of a formula, Tarski explicitly refers to the notion of mathematical equation. From this perspective, an equation is an open sentence; given a domain, it may be true for some elements of the domain, but not all, or true for every element of the domain, or false for every element. Solving an (in)equation means determining the elements satisfying this open sentence. In addition, it is possible that, given an equation, no solution fits in the domain, or that there there is one or more than one solution in another domain. For example

"$x^2+1=0$" has no solution in the real number set, but two solutions in the complex number set.

On the other hand, in formalised languages, syntax is the term used in logic in a broad sense including:

1. The study of the rules of well-formedness of expressions of a given language (the grammar);

2. The set of rules of derivation in an established theory of demonstration in the formal sense of the term, opposed to the semantics which takes into account the interpretations.

For example, two equations are equivalent if and only if:

1. They are satisfied by exactly the same elements (semantic point of view);

2. It is possible to transform one equation into another one if algebraic rules preserving equivalence are applied (syntactical point of view). It is important to notice that some algebraic rules do not preserve equivalence, so that, in some cases, it is necessary to come back to the involved domain for a semantic control.

For example, let us consider the equation "$x^2-2=|3x+2|$" whose set of solutions in the real number set is $\{-3,4\}$. By applying syntactic rules and transformations without a semantic control, the set $\{-3,-1,0,4\}$ could be considered as the set of solutions although it contains elements that do not satisfy the equation.

Finally, logical semantics allow us to give precise definitions of (in)equations on the one hand, and supports our claim of the relevance of taking into account both semantic and syntactic aspects in mathematical reasoning.

CROSSING MATHEMATICAL PRAXEOLOGY AND LOGICAL SEMANTICS FOR A STUDY OF TUNISIAN SYLLABUS AND TEXTBOOKS

Following Chevallard (1989)'s perspective on didactic transposition, and in order to identify the institutional prescription concerning the knowledge to be taught on (in)equations, we have analysed the Tunisian syllabus and textbooks, through mathematical praxeology (Chevallard 1992).

According to Chevallard, all human activities, and namely, mathematical activity, can be described through praxeology. A praxeology is composed of two blocks. The first block is named the *praxis*. It refers to the practice, and has two components: *Type of the task*, what is to be done (e.g., solve a quadratic equation in complex numbers set) and *Techniques,* the ways to achieve a certain type of task (e.g., to compute discriminator; to study its sign; to apply the relevant formula to get both solutions). The second block is named the *logos*. It refers to the theory, and also has two components: the *Technology* that is, a discourse that enables justification of a technique (i.e. write down and factorize the canonical decomposition of the quadratic trinomial) and the *Theory*, which provides a justification for a technology (e.g., the axioms and properties of the complex numbers field).

However, we consider that this theoretical perspective is not sufficient to grasp the dialectics between syntax and semantics. Thus, we have enriched the categorisation by crossing praxeology (namely the praxis [2]) with the dialectics between syntax and semantics. In addition, we take into account the registers of semiotic representation (Duval, 1991) that play a crucial role in algebra and, following Robert (1998), we consider the way in which students work on notions in exercises or problems (merely applying them or being able to mobilize them when required or on their own). This led us to elaborate the bi-dimensional grid sketched by Table 1.

Technique /Task [3] **NMFK** [4]	***Semantic***	***Syntactic***	***Mixed*** [5]
Elementary	Verify / numerical-graphic	Factorize / algebraic	Solve/numerical-algebraic
Mobilized	Interpret/graphic-algebraic	Demonstrate/Analytic	Study and represent / algebraic-analytic-graphic
Available	Existence/analytic-graphic	Discuss/algebraic	Conjectur/numerical-algebraic-graphic

Table 1: Bi-dimensional grid for analysing program and textbooks

From reading this table we can say that several types of techniques, that can be semantic, syntactic or mixed, apply to a given type of task. This type of work requires a level of knowledge that can be assumed to be elementary, mobilizable or available, depending on the level of education of the student.

At a glance, our study of the Tunisian textbooks of the secondary school showed that:

- The treatment of (in)equations, in exercises and in problems of synthesis, requires mostly the use of syntactic techniques, whereas, introductive activities, that are often modelings of problem or graphic situations, take significant account of the articulation between syntax and semantics.
- The relationship between a Cartesian equation of a curve (e.g., a conic given by a not necessarily functional relation "$R(x, y) = 0$") and a graphical representation of a function (e.g., quadratics) $y = f(x)$ is not clearly stated, the conics appearing mainly, if not exclusively, as a graphic representation of functions.
- Introduction of functions mainly relies on semantic techniques: substitution values to variables, interpretation, etc. (Chevallard, 1989)

DIDACTIC INVESTIGATION

As part of a research conducted in our PhD, we submitted a questionnaire to 111 students in secondary school and 32 students in the first year of the university. [6] (cf. Kouki 2008). Through this survey we were interested in assessing which point of view (semantic and / or syntax) was preferably mobilized by the students in solving (in)equations. In addition, we tried to identify students' ability to smoothly move from one or more registers of semiotic representation to another one. [7]

The mathematical and didactic analysis of different strategies of resolution in the treatment of the various tasks were based on theoretical frameworks crossing the logical semantic perspective and the didactical anthropology of the didactics (Chevallard 1992), and took account of the system of semiotic representation and registers, for analysing students' production consisting in signs, graphics and algebraic writings (cf. Duval 1991).

For this analysis, we refer to the grid (cf. Table 1). We assigned a code to each type of technique the questionnaire: each type of technique has been denoted by t_{a-b} , indexes a, and b stand respectively for:

- a : The logical categorization of the technique in terms of syntactic, semantic or mixed respectively denoted "se", "sy" and "M".
- b : the classification of a technique in a register of semiotic representation of the type - graph, numerical, algebraic, analytical or functional, etc. respectively denoted grph, num, alg, and anl, etc.

The overall results of the questionnaire analysis lead to the conclusion that students showed a preference for syntactical techniques, even in cases where intermediate questions involving semantics treatments in numerical or graphical register were introduced. On the other hand, we have observed a fairly high percentage of pupils and students who use semantics as tools of resolution only in cases where such tools are explicitly required by the given task. Moreover, it is worth mentioning the difference between the type of procedures (syntactic/semantics) for the resolution of exercises of the questionnaire utilized by students at the same level which makes us think that it could be linked to the practice of classroom. This accords with Alcock's (2009) results at the tertiary level. Further research is required to explore this issue.

We chose to present here the results provided by the analysis of an exercise from our questionnaire. They point to the fact that, in some cases, the use of purely syntactic methods hampers solution of the given task. Indeed, the task requires adopting a semantic point of view for articulating (in)equation and curves.

In the exercise all the solutions of the "product inequation" in two real variables $(y-x)(y-x^2+3x)>0$ are sought. The difference in age-levels of the students led us to propose the exercise in two different ways. For the second group of students, i.e., those in second and third year, the exercise that was proposed is as follows:

Let f and g functions defined over IR by: $f(x)=x$ and $g(x)=x^2-3x$.

Let Γ_f and Γ_g the graphs of f and g live in an orthonormal $(O,\vec{i},\vec{j})$.

1) Representing Γ_f and Γ_g in Cartesian coordinates system $(O,\vec{i},\vec{j})$.
2) Determine the sign of $h(x,y)=(y-x)(y-x^2+3x)$ in each of the following pairs:
   - $(x,y)=(2,1)$……………………………………………………………………
   - $(x,y)=(1,3)$……………………………………………………………………
   - $(x,y)=(5,4)$……………………………………………………………………
   - $(x,y)=(-2,-1)$…………………………………………………………………
   - $(x,y)=(-1,-2)$…………………………………………………………………
   - $(x,y)=(6,7)$……………………………………………………………………
1) Place in $(O,\vec{i},\vec{j})$, points A,B,C,D and F respective coordinates of: $(2,1)$, $(1,3)$, $(5,4)$, $(-2,-1)$, $(-1,-2)$ et $(6,7)$.
2) Determine by calculation or graphically, all solutions of the inequation $(y-x)(y-x^2+3x)>0$.

The first three questions were proposed to provide indicators that could lead students to move to the register of algebraic graph, to represent the different parts of the plane determined by the two graphic representations Γ_f and Γ_g, and to make a correspondence between the sign of each values of the function h in different points A, B, C and D in each region, on the other side.

Regarding the students of “Preparatory Classes for engineering schools”, we have considered that they had acquired sufficient ability to solve the inequation based solely on graphs Γ_f and Γ_g.

Consequently, we have chosen to propose the exercise as follows:

Let f and g functions defined over IR by: $f(x)=x$ and $g(x)=x^2-3x$.

Let Γ_f and Γ_g the graphs of f and g live in an orthonormal $(O,\vec{i},\vec{j})$.

1) Representing Γ_f and Γ_g landmarks in $(O,\vec{i},\vec{j})$.
2) Determine by calculation or graphically, all solutions of the inequation $(y-x)(y-x^2+3x)>0$.

The *a posteriori* analysis of the first three issues has shown that students can mobilize different types of technical records, articulating different algebraic, numerical and analytical graphics to meet the intermediate targets.

In this paper, we focus on the resolution of the inequation $(y-x)(y-x^2+3x)>0$, which can be solved, straightforwardly, by the technique of a change of register: from the register of algebraic equations involving the straight line $y-x=0$ and the parabola $y-x^2+3x=0$ to graphs Γ_f and Γ_g which intersect at the origin of the orthonormal coordinate system $(O,\vec{i},\vec{j})$ and the point of coordinates $(4,4)$. The line divides the plane into two half planes with open upper half plane is the points

$M(x,y)$ as far as $y-x>0$and the interior of the parabola is the set of points $M(x,y)$ as $y-x^2+3x>0$. The sign of the algebraic expression $(y-x)(y-x^2+3x)$ is determined by Figure 1..

Sign

Regions					
	-	-	+	+	+
	-	+	+	-	-
	+	-	+	-	-

Figure

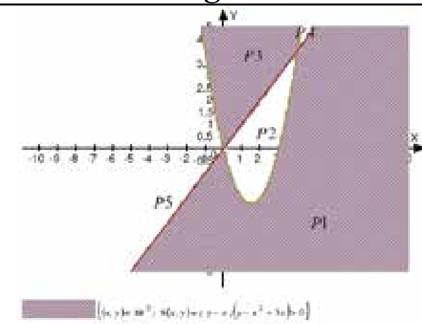

Figure 1: Sign of the "algebraic expression" in different parts of the plan

In conclusion, the set of the solutions of the inequation is the set of pairs that exactly correspond to the coordinates of the points in the region $P_1 \cup P_3$.

The *a priori* analysis of this question showed that three types of techniques could be used by students; we respectively denote these techniques T_1, T_2 and T_3.

The first technique T_1 is of type "Mixed"; it consists of interpreting graphically the sign of the two algebraic expressions "$y-x$" and "$y-x^2+3x$" and then concluding that the product is strictly positive in the region $P_1 \cup P_3$.

The second technique T_2 is of type "semantics"; it consists of linking graphic and algebraic register, affecting the sign of the factors for each point according to its position in the Cartesian plane, and then deducing which are the points in the region.

The third technique T_3 is a purely syntactic one in algebraic register that consists of attempts in order to transform the inequation $(y-x)(y-x^2+3x)>0$. This technique appears to be inoperative at the considered level.

For methodological reasons, we denote the group of students of the second year of secondary school G1 and the group of third year students mathematics section G2 and the group of students of the first year of the university G3. Concerning the last question, more than 50% (76 among 143[8]), did not give an answer, while students answering these questions have mobilized different techniques. The syntactic techniques T_3 of the algebraic register appeared in 28 answers of 67 and contained no exact answer (7 of G1, 7 of G2 and 14 of G3). The semantic techniques T_2 of the graphic register appeared in 18 copies (8 of G1, 6 of G2 and 4 of G3), among them 4 correct answers were found (2 of G2 and 2 of G3). The mixed technique T_1 was mobilized in two answers that they were correct (1 of G1 and 1 of G3). The other answers had no connection with the exercise; they were classified in "Other types of responses". We can add that for the types of tasks that require mobilized and available levels of knowledge, only students who were able to articulate semantic and syntactic aspects and to coordinate the registers of semiotic representations succeeded in this task.

The results on the three exercises highlight that pupils use the syntactic technique of resolution as soon as they are available even if intermediate questions should call for semantic treatments (graphic or numerical). On the other hand, we pointed out that a rather large percentage of pupils and students use the semantic tool of resolution only if the type of task requires it. Moreover, we observed rather remarkable differences between the procedures of resolution of the exercises of the questionnaire between students of the same level. In this respect, we think that such differences could be linked to differences in the practice of the teachers in class with respect to the dialectics between syntax and semantics.

CONCLUSION

The semantics logical approach, referring to the notion of satisfaction of an open sentence, has permitted us to identify difficulties that pupils could face when working with (in)equation and function. In particular, when students solve the expression $y = f(x)$ our investigation shows that the logical status of the variable y and the term $f(x)$ are not clear for students. A deeper investigation will be needed in order to better understand this point.

The necessity for semantics, something that is largely neglected in the Tunisian programs and textbooks is highlighted if account is taken of the relationship between equations, curves and functions. Combining the approach we have developed with the study of effective teaching practices could provide paths for examining the extent to which the syntactic or the semantics preferences that we have observed are or are not linked to teachers' preferences. Our experimental investigations have pointed out the relevance of using semantics when syntactic techniques have been shown to be ineffective. In addition, we have shown that many students fail to take account of the relationship between the equation, curves and function, in line with the fact that, as evidenced by our analysis, this relationship is not made explicit in the textbooks.

Finally, we think that the investigation of the same notions of (in)equation and function in domains others than numerical domains, which appear in advanced mathematics, would be promising for the didactic analyses in higher education. We actually investigate the interest of our logical point of view in reference to Predicate calculus for studying in an educational perspective (in)equations in various domains (matrix, vector, functions etc.) Indeed, the notion of open sentences and satisfaction by an element provides a very general frame for the concepts of (in)equation that we suspect to be fruitful.

NOTES

1. Our translation.

2. We have chosen in our work not to consider the second block in praxeology (the logos) due to the fact that in Tunisian program and textbooks, there are very few references to technology and theory.

3. By this, we mean type of technique corresponding to a definite type of task.

4. Level of operating knowledge.

5. Techniques are expected to mobilize when both syntactic and semantic techniques aremixed in the treatment of mathematical objects.

6. They were students with high achievement attending « Classes préparatoires aux grandes écoles », that prepare them to the competitive examinations for entry into Engineering Schools.

7. The different types of techniques articulating different registers are detailed in Kouki ((2008).

8. Three classes of second year secondary science section composed of 88 students, one class of third grade of secondary school of the mathematics section composed of 23 students and a preparatory class in technology specialty.

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COVARIATION, EMBODIED COGNITION, SYMBOLISM AND SOFTWARE DESIGN IN TEACHING/LEARNING ABOUT FUNCTIONS: THE CASE OF CASYOPÉE

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From the early nineties, most reformed curricula at upper secondary level chose to give functions a major position. The goal of this paper is to introduce key elements of a successful approach to the teaching of functions and to show how functions can influence research work including the design of a software environment and the evaluation of classroom situation. We will discuss how experiencing covariation and using references to body activity are crucial for students' understanding of functions. An example of a situation based on the use of the Casyopée environment is proposed as an illustration.

INTRODUCTION

The functional perspective on the teaching of algebra is seen by curricula reformers as an effective approach to consolidate post middle school students' algebraic knowledge and to prepare them to undertake courses in calculus. The use of technology, especially graphical and dynamic geometry software, is encouraged in an exploratory approach to functions. However, the topic of functions is complex and Kieran (2007, p. 710) notes that the reform gives way to "hybrid versions of programs of study that can create additional difficulties for algebra learners". She depicts these programs of study as oriented towards the solution of realistic problems and multi-representational activity, with the aid of technological tools, resulting in algebraic content that is less manipulation oriented and an approach in which there is a shift away from traditional algebraic skills. Among the objections Kieran raises, I am particularly sensitive to the strong presumption, that, in these programs, symbolic forms will be interpreted graphically, rather than dealt with, as the use of technology prioritises screen (graphical) interpretations of functions. Clearly, there is a risk that, through these programs, students will have no access to understanding symbolic forms which are at the core of algebra and will be deprived of the power they offer for solving problems and more generally "for understanding the world".

In a first part of the paper I review some issues regarding the teaching/learning of functions. Then I question the use of "standard" computer algebra in this domain, taking the example of a non-differentiable function. This is starting point for reporting on the work of the Casyopée research group that, for more than ten years, worked in France in order to promote an approach to functions encouraging students' multi-representational exploration together with work on algebraic symbolism and that designed a CAS software environment with this aim.

TEACHING/LEARNING ABOUT FUNCTIONS, KEY QUESTIONS

Within the general concern expressed above, the functional approach to algebra raises several issues documented by literature. Here I will consider three: (1) the necessity for students to understand covariation as a foundation for functions, (2) the interest in using references to body activity for students' understanding of functions (3) the understanding of the structure of the algebraic formula in a function. I will then discuss, using an example, the potentialities and limitation of a software tool.

From process-object to covariation and embodied cognition

From the early nineties most of the studies concerning students' conceptions of functions were based on the distinction between the two major stances that students adopt towards functions: the *process view* and the *object view* (Sfard, 1991). With regard to the process–object duality in students' understanding of functions, mathematics educators suggested that students' understanding of functions can be considered as moving from an initial focus on actions and processes to more object-oriented views characterized by a gradual focus on structure, incorporation of properties and reification of mathematical objects. In this vein, from the middle nineties, a number of approaches developed to describe object-oriented views of function that emphasized the covariation aspect of function (Thompson, 1994). Covariational reasoning consists in coordinating two varying quantities while attending to the ways in which they change in relation to each other. This involves a shift in understanding an expression from a single input-output view to a more dynamic way which can be described "as 'running through' a continuum of numbers, letting an expression evaluate itself (very rapidly!) at each number" (Thompson, 1994, p. 26). However, this dynamic conception of variation does not seem to be obvious for students since it is essential to take into account simultaneous variation between magnitudes at different levels emerging in an ordered succession. Furthermore, there is a need for situations that provide students with opportunities to think about the covariational nature of functions in modelling dynamic events.

The hypothesis is then that students' understanding of covariation can be built following interaction with a dynamic physical device. Researchers like Rasmussen et al. (2004) and Botzer & Yerushalmy (2008) refer to embodied cognition to characterise the sense of a mathematical notion that students can get via interaction with a physical device. A central assumption of embodied cognition is that students' reference to bodily activity in physical settings and to emotions experienced in this activity, can be a basis for deeper understanding of functional notions, compared with a pure formal approach of these notions. Covariational characteristics of functions are experienced through feelings and emotions. Moreover, elements of language used to reflect on the experience, especially gestures and words describing covariation, give students a semiotic means of expression complementing formal approaches. Rasmussen et al. (2004) give the example of a university student who knew the formal definition of acceleration, but did not fully understand this notion.

Experimenting with a rotating unbalanced wheel she identified herself with the wheel and became "friends with the acceleration".

The role of symbolism

There is evidence in the literature that the symbolism of functions is a major difficulty for students. Students' view of symbolic expressions can be of a pure input-output correspondence. In other circumstances, it can be pseudo-structural, the expressions being understood as an object in itself, not connected to functional understanding (Sfard, 1991). Slavit (1997) indicates the critical role of symbolism "confronted in very different forms (such as graphs and equations)" (p. 277) in the development of the function concept and suggests the need for students' investigation of algebraic and functional ideas in different contexts such as the geometric one. Even when students have access to basic proficiencies in algebraic symbolism, coordinating these proficiencies with an understanding of the structure of the algebraic formula in a function is critical and is particularly at stake when the function comes from a problem context. Most students fail in this coordination. Evidence of failure is given in the context of equation. For instance, van der Kooij (2010, p.122) notes that most students in a vocational high school "were able to do calculation on the pendulum equation $T = 2\pi\sqrt{\frac{1}{g}}$ while they gave no sense to an "abstract" equation $y = 2\sqrt{x}$".

Potentialities and limitation of a software tool: The case of Computer Algebra

With regard to software tools that might contribute to a functional approach to algebra, the preceding review of issues, and particularly the critical question of symbolism focus attention on Computer Algebra Systems (CAS) because these systems were created to help mathematicians to go beyond mere numerical experimentation. Used in the classroom, computer algebra might help students concentrate on the sense of transformations. The reason for this is that in paper/pencil transformational activities, algebraic manipulations and transformational skills are necessary to get a given form, possibly detracting attention from how the outcome compares with the initial form. CAS generally include graphing and tabulating capabilities, then allowing graphic and numerical exploration, as well as symbolic approaches to problem solving.

However, standard CAS are tools for mathematicians, and the representation of mathematical objects often behave differently compared to mathematical objects as defined in secondary curricula. The following example comes from a French experimentation of the TI-92 CAS calculator in the years 1990 (figure 1).

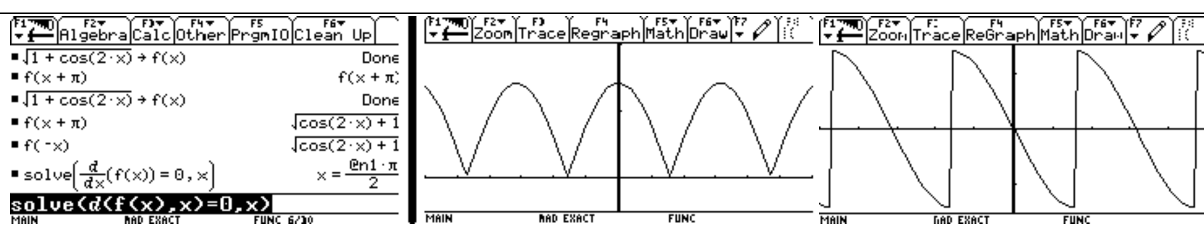


Figure 1: *Using the TI-92 to discuss a trigonometric function*

The task was to study the function **f(x)=** $\sqrt{1+\cos(2\cdot x)}$. It was set to 11^{th} graders at the end of the year. The students used a TI-92 CAS calculator throughout the year within a research project (Lagrange 1999). We expected that students, by observing that the curve has different non-zero gradients approaching these points from the right and from the left, would detect that the derivative is not defined at the points where the curve reaches the x-axis. Observation showed that the accurate interpretation of the behaviour of the function at the points where the curve reaches the **x**-axis was not possible. Students persisted in thinking of an 'ordinary minimum', wondering why repeated 'zooming in' did not show a null gradient. This idea was reinforced by the false solution of the equation **f'(x)=0** by the symbolic module, where zeros of the derivative were given for every **kπ/2**. Even the graph of the derivative (figure 1, right) was misleading because of the irrelevant line across the discontinuity. The reason for this behaviour was that in the TI-92, like other symbolic systems, functions are not defined on a domain and, in solving the equation, consideration is given to zeros of the numerators and not to possible zeros of the denominator.

Observations like this drew attention towards the importance of a careful design of software tools and of the associated classroom situations, taking into account the multiple constraints of classroom use. It was the starting point of the work of the Casyopée group, presented below.

DESGNING CASYOPÉE: OBJECTIVES AND SITUATIONS OF USE

The genesis of the Casyopée [1] group was in the years 1995-2000 with the above mentioned experimentation of the TI-92 calculator. The group was concerned with the instrumental difficulties and epistemological problems inherent in Computer Algebra Software (CAS) designed for advanced users. The group started to build a CAS tool that could be really used in the classroom. A central aim was to ensure consistency with current notations and practices at secondary level. We wanted also to avoid any command language by designing a menu and button driven interface as in Dynamic Geometry, because keywords are always difficult to handle for beginners and they create confusions with mathematical notations. These choices helped to create an innovative algebraic tool contributing to a better appreciation of CAS by teachers. The group saw the potential of this tool for students to explore and solve problems involving modelling geometrical dependencies, for instance, an area against a length. However the group was concerned that geometrical exploration and modelling had to be done separately from the work of Casyopée.

In the years 2006-2009, the group was involved in the ReMath [2] project that focused on multi-representation of mathematical objects. This was an opportunity to extend the representations in Casyopéee by adding a dynamic geometry window and representations of measures and of their covaration. This extension enabled covariations between couples of geometric magnitudes to be explored and couples that are in functional dependency to be exported into the symbolic window. The outcome of this exportation is an algebraic function modelling the dependency, likely to be treated with all the available tools. In order to help students in modelling dependencies, this exportation can be done automatically. I will refer to this functionality as “automatic modelling” below. After the ReMath project, the group worked to build a conceptual framework about functions and algebra (Lagrange & Artigue 2009). It is based on the idea that students approach the notion of functions by working on dependencies at three levels (1) activity in a physical system where dependencies are “sensually” experienced; (2) activity on magnitudes, expected to provide a fruitful domain that enhances the consideration of functions as models of physical dependencies; and (3) activity on mathematical functions, with formulas, graphs, tables and other possible algebraic representations.

Designing classroom situations: the modelling cycle

The outcome of the above reflection is that the Casyopéee group favours classroom situations where students have to build and use a function as a model of a relationship between variables in a dynamic system. Starting from a problem relative to a dynamic system, we expect that students progress through the steps of a “functional modelling cycle” (figure 2).

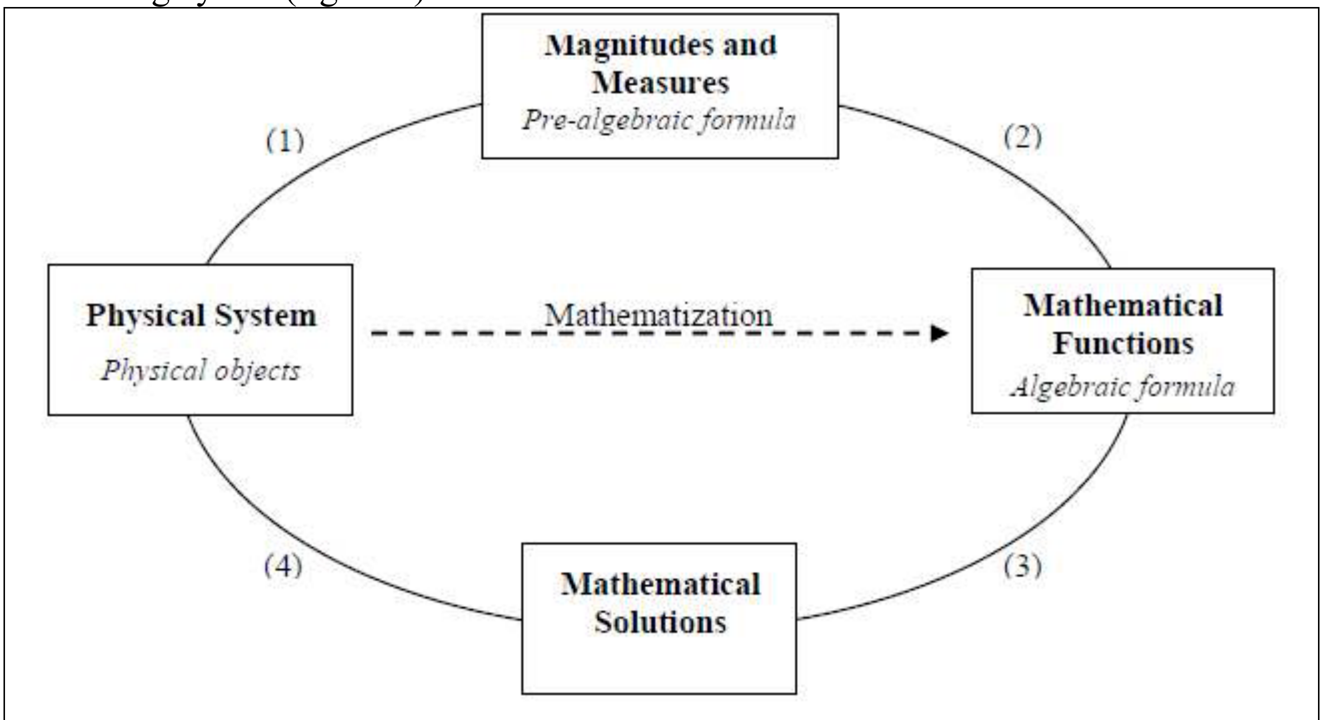


Figure 2: The functional modelling cycle (Minh, 2012)

In the physical system, students can observe, explore and perceive dependency relations between objects (step 1). Generally the problem that they have to solve (for instance, a problem of optimisation) involves quantifying the observations, and then students have to identify magnitudes whose covaration model the behaviour of the physical system. As explained above, the choice has been made in Casyopée to implement covariations between geometric magnitudes (areas, lengths, angles…)

involving geometrical objects existing in the dynamic geometry window. The consequence is that a dynamic geometry figure representing the system has to be built, followed by the determination of the magnitudes in covariation. This is far from obvious for students, and an important step in the understanding of functions. Step 2 consists of calculating a domain and a formula for a function expressing the covariation identified in step 1. The actual calculation is generally technically difficult and for the Casyopée group it is not insightful. In contrast, we favour the possibility of students exploring several covariations in order to find one which could be adequate for solving the problem. That is why the students can use the "automatic modelling" feature presented above. Step 3 consists of manipulations and transformations of the formula and algebraic proofs, helped by Casyopée's symbolic capabilities, in order to find a mathematical result that could help to solve the problem. Step 4 is the interpretation phase. Students go back to the physical system to interpret the mathematical result and get a solution. In the next section I present a classroom situation designed on this basis.

A CLASSROOM SITUATION: THE AMUSEMENT PARK RIDE

This classroom situation was designed to take up two challenges. The first one was the necessity for students to consider "irregular" functions before entering the university level. This is so because situations involving modelling dependencies generally deal with infinitely differentiable functions not questioning understanding of irregularities like discontinuities or non-differentiability. The second challenge was to test with Casyopée the above mentioned embodied cognition assumption regarding the role of bodily activity in physical settings. More precisely here the situation was designed in order that students connect properties of an irregular function with a sensual experience of movements, in order to get a deeper understanding. Another feature of the situation is that the function involved is similar to the function whose study was at stake in the task with the TI-92 calculator observed above, allowing a comparison of students' achievements in two different settings.

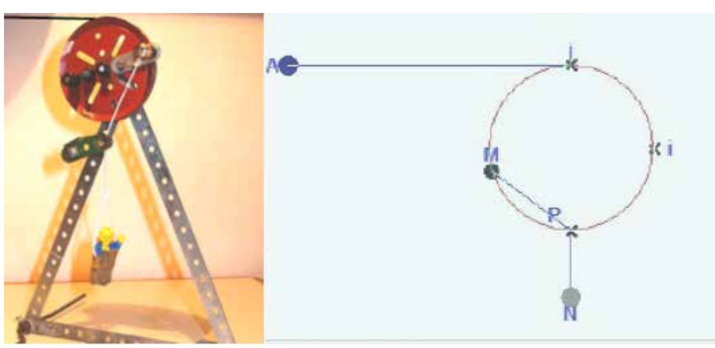


Figure 3: The Amusement Park Ride Problem

The problem was the following: a wheel rotates with uniform motion around its horizontal axis. A rope is attached at a point on the circumference and passes through a fixed guide. A car is hanging at the other end. The motion is chosen in order that a person placed in the car feels differently the transition at high and low point. It was expected that students would identify the difference, associate this with different properties of the function (non-differentiability and differentiability) after modelling the movement. The problem is given in “real life” settings, the students being able to manipulate a scaled device, and then the first step of modelling consists in building a dynamic geometry figure replicating the device (figure 3).

The following indications are given to the students: the rope is attached to the wheel in a mobile point M and the guide is on the fixed point P. The car is in N (figure 3). Step 1 implies a not trivial geometric modelling: construction of a point M in order that the circular distance IM is equal to the linear distance Aj, and of a point N in order that MP+PN is constant.

In this situation, we did not want especially to emphasize the choice of variables (step 2) because students already met this aspect of functions in previous situations. The real focus was on the properties of the function (step 3) and the interpretation (step 4). In order to ease step 2, and because working on covariation between lengths is easier for students, compared with covariation involving a variable angle, we asked students to consider that the wheel is put into rotation by pulling on a horizontal rope jA. At step 3, the focus is placed on the algebraic formula of the function. The students have to use Casyopée to get the derivative and should notice and identify precisely the points of non-differentiability. The lesson was carried out with a 12th grade class in a 90 minute session. I report first on students’ spontaneous model of the physical situation, and then on how they performed in the four steps presented above and how their understanding of the physical situation progressed after working on the algebraic function.

Students’ spontaneous understanding of the physical situation

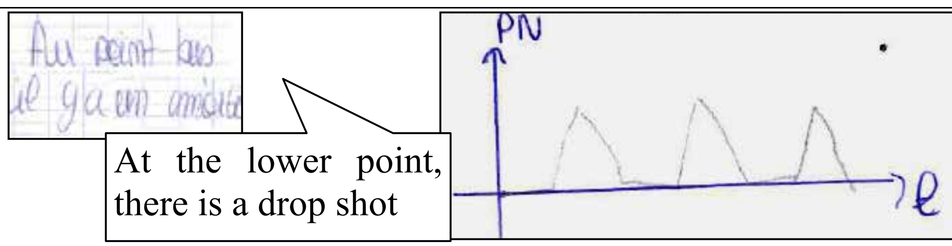


Figure 4: A student’s spontaneous understanding

Starting the session and demonstrating by animating a scaled model, the teacher asked the students to describe what was happening at the lower point and whether it was different to the high point. Figure 4 illustrates a typical answer. Students said that, at the high point, the car stopped and they had some difficulties explaining what was happening at the lower point. Some students tried to mimic the movement by getting up and down. The most commonly used expression, ‘drop shot’, is not accurate because it means that the car is arriving at a certain speed, stops and starts up again at a lower speed. Students illustrated with a graph of a piecewise linear function. Actually they thought that because the wheel rotates uniformly, the car's movement should be piecewise uniform.

Building a dynamic geometry model

This was a difficult part. Students' poor practical knowledge in trigonometry explains why they needed help in defining M in order that the circular distance IM equalled the linear distance Aj. It seems more surprising that they found it difficult to define N in order to make MP+PN=2 (the length of the rope). After the teacher indicated that PN is known when MP is known, some students used a circle centred in P with a radius of 2-MP and defined N as an intersection point with the y-axis, and others directly defined N with the coordinates (0; y_P-(2-MP)).

Choosing the dependent and independent variables

Generally the students had no difficulties in making this choice appropriately with the software. However, their explanations for this choice were not always clear. For instance a pair of students wrote in the report: "We choose distance Aj as the (independent) variable" and added "Aj is a function of the coordinates of N".

Working on the algebraic function obtained via Casyopee's automatic modelling

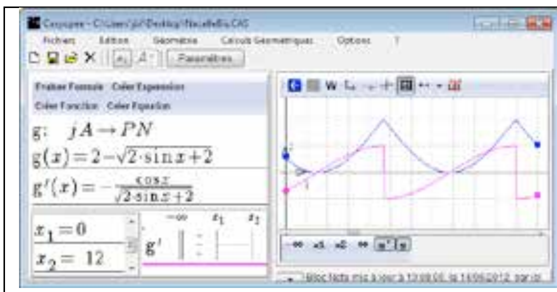


Figure 5: A graph of the derivative with wrong vertical segments

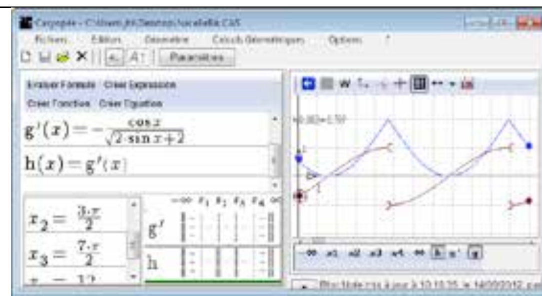
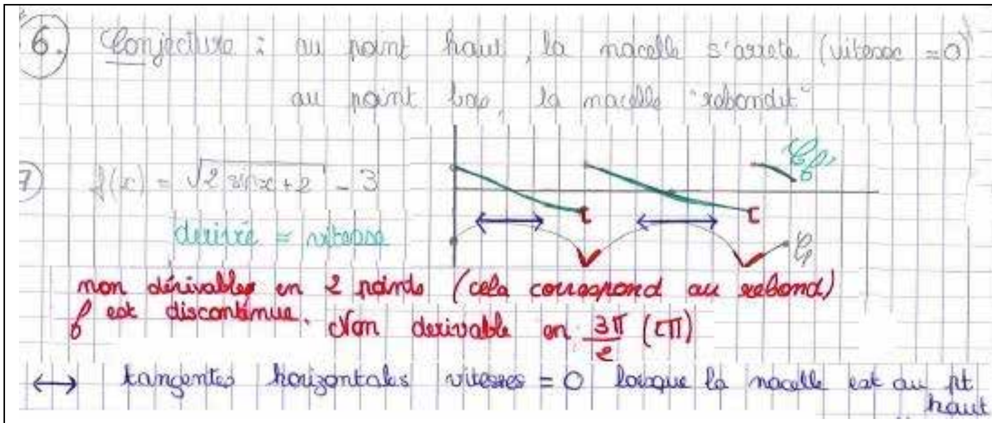


Figure 6: A correct graph of the derivative

The students obtained the derivative by using Casyopée under the form $x \rightarrow \frac{-\cos x}{\sqrt{2 \cdot \sin x + 2}}$. Casyopée issued warnings because this function is not defined everywhere. Students ignored the warnings and obtained a graph with incorrect vertical segments (figure 5). The teacher drew students' attention to these segments and asked them to make a connection with Casyopée's warning. Students recognised that there should be discontinuities of the derivative corresponding to the low points. The teacher asked them to compute the position of these discontinuities. No students did this from the formal definition of the derivative. Rather they came back to the geometrical figure or the physical device, looking for the value of Aj corresponding to the lower point of the car. After they found these values and excluded them from the definition of the derivative, they got a correct graph (figure 6).

Students' understanding of the situation after working on the algebraic function

Students' understanding was much better after working on the algebraic function. They identified the derivative and the car's speed, saying that the speed was null at the high point corresponding to a horizontal tangent on the graph of the movement. Implicitly, they recognised that at the lower point the car starts up again briskly at the same speed, speaking of “rebound” corresponding to non-differentiability points, rather than of “drop shot” implying softer stop and restart (Figure 7).



The student correctly identified the “rebound” and connected to non-dif-ferentiability.

She however confused this with discontinuity.

Figure 7: Students’ understanding after the work

CONCLUSION AND DISCUSSION

This paper started by identifying key questions about the teaching and learning of functions: experiencing covariation and using references to bodily activity is crucial for students’ concepualisation of functions, and understanding of the structure of the algebraic formula in a function is critical.

An example of a situation was presented, based on the “embodied cognition” hypothesis and on the use of Casyopée’s functionalities, especially the fact that it builds a link between a geometrical dependency and a function modelling this dependency, and the consideration of a domain for the definition of a function. The comparison with the observation of students using a TI-92 to explore the properties of a similar function is insightful: (1) in the TI-92 situation, the formula and the graph do not allow students distinguishing between the two kinds of extrema, while in the “amusement park” situation students can link the behaviour of the function with “embodied” sensations helping them to progressively grasp the different nature of the extrema (2) Casyopée’s functionalities helped students to get an accurate view of the function’s behaviour, while the TI-92, like other CAS, delivers confusing messages. These functionalities are the result of ten years of reflection and development.

Regarding Kieran’s concern mentioned in the introduction that, in recent curricula, symbolic forms will be interpreted graphically, rather than dealt with, the example shows how students can work on symbolic forms of the function at stake and are able to establish links with the magnitudes whose covariation the function models. The capacity to deal with symbolism and of the associated situations of use is a distinct design feature of Casyopée, with the aim to reconcile symbolic forms and dynamic manipulation of mathematical objects and relationships. This is at least the beginning of an answer to Kieran’s objection.

Acknowledgment: This work was done thanks to the help of the Institut Français d'Education (EducTICE).

NOTES

1. Casyopée, is an acronym for « Calcul Symbolique Offrant des Possibilités à l'Elève et l'Enseignant »

2. "Representing Mathematics with Digital Media", 6th FP, IST-4-26751-STP, 2005-2009 (http://remath.cti.gr)

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FROM RECURSIVE TO EXPLICIT FORMULA FOR THE *N*-TH MEMBER OF A SEQUENCE MAPPED FROM A SHAPE PATTERN

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This paper presents how two student teachers struggle to find an explicit formula for the general member of a sequence mapped from a shape pattern which they have successfully generalised in terms of a recursive formula. The paper shows how the milieu for algebraic generalisation in the observed episode is constrained in two senses: first, by the design of the task; second, by the teacher educator's intervention which presupposes prior knowledge that the student teachers do not have. It is indicated how manipulations on the geometrical configurations might enhance the milieu for algebraic generalisation of shape patterns and provide a link between recursive and explicit formulae.

Keywords: Recursive and explicit formulae, shape pattern, growth of sequence, decomposition, milieu, adidactical situation.

INTRODUCTION

There is a two-fold purpose of tasks on algebraic generalisation of shape patterns (Måsøval, 2011): One is to provide a physical or iconic context for algebraic generalisation where the aim is to promote students' algebraic thinking and justification. Here, algebra is approached through pattern generalisation. The other is to lead students to experience patterns as mathematical structures as an aim in itself. Here, algebra is a mediational means to represent invariant structures in the patterns. However, it is rather common that shape patterns are used only to produce a sequence of numbers which subsequently is generalised in terms of an algebraic formula without references to the elements of the pattern (Lannin, Barker & Townsend, 2006). Strategies of "guess-and-check" that involve superficial pattern spotting are frequently used with the consequence that students do not detect the generality of the formulae they find (Lannin et al., 2006).

THEORETICAL FRAMEWORK

In Brousseau's (1997) theory of didactical situations in mathematics, an *adidactical situation* is a situation in which the student takes a mathematical problem as his own and solves it on the basis of its internal logic without the teacher's guidance and without trying to interpret the teacher's intention with the problem. The *devolution* of an adidactical learning situation is the act by which the teacher encourages the student to accept the responsibility for an adidactical learning situation or for a problem, and the teacher accepts the consequences of the transfer of this responsibility (Brousseau, 1997). The student cannot engage in any adidactical situation; the teacher attempts to arrange an adidactical situation that the student can handle.

In the devolution process, which is part of the broader (didactical) situation, the teacher is faced with a system that consists of the student and a *milieu* "that lacks any didactical intentions with regard to the student" (Brousseau, 1997, p. 40). The milieu is a subset of the students' environment with only those features that are relevant with respect to the knowledge aimed at by the teacher in the didactical situation. The concept of milieu models the elements of the material or intellectual reality on which the students act and which may be an obstacle to their actions and reasoning (Laborde & Perrin-Glorian, 2005). That is, the milieu of a didactical situation is the part of the environment that can bring feedback to students' actions to accomplish a task.

An adidactical situation is part of the didactical situation that is the broader situation with the system of interaction of the students with the milieu arranged with the purpose of the students' appropriation of the target knowledge without the teacher's intervention (Brousseau, 1997). The teacher can act on the milieu by providing new information or new equipment, for example by asking a question or directing students' attention to certain factors in the classroom situation. When the teacher acts on the milieu, she changes the knowledge needed to solve the problem (Perrin-Glorian, Deblois & Robert, 2008). Whether the student can handle an adidactical situation depends upon two conditions: first, that the student has prior knowledge that enables him to engage with the situation; second, that the milieu created by the teacher provides the student with personal knowledge that enable him to develop the knowledge aimed at (by the teacher).

METHODOLOGY

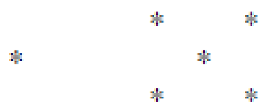
The reported research is derived from the author's PhD project (Måsøval, 2011). In the rest of the paper, "students" is used to refer to the student teachers, and "teacher" is used to refer to the teacher educator. The research question addressed in the paper is*: How do the mathematical task and the teacher's intervention constrain students' appropriation of algebraic generality in a shape pattern?* The task with which the paper deals (Figure 1) was the first of four tasks (during eight lessons) on algebraic generalisation of shape patterns. There had been a short whole-class introduction to figurate numbers (illustrated by triangular numbers) before the observed small-group lesson.

The data is a video-recorded observation of two students' collaborative engagement with Task 1 (with teacher intervention). The students are Alice and Ida, who were in their first academic year on a four-year undergraduate teacher education programme for primary and lower secondary school in Norway. The teacher is Erik, who was responsible for teaching algebra to the class of which Alice and Ida were members. The observed students worked on the same task as the rest of the class (ca. 60 students). My role during data collection was to be a non-participant observer while video-recording the interaction between the students and the teacher. The video-recorded episode has been transcribed and analysed through a process of open coding (using an adapted grounded theory approach, Strauss & Corbin, 1998) where concepts from Brousseau's (1997) theory have been used to answer the research question. For a discussion of the legitimacy of the theory of didactical situations in the analysis of the data, I refer to (Måsøval, 2011, Chapter 2.5).

An *a priori* analysis of the shape pattern in the mathematical task

Task 1 (Figure 1) has been designed by Erik for collaborative work in small groups. According to Erik, the aim of Task 1 was twofold: first, to express the regularity of the shape pattern in natural language; second, to transform the natural language expression into algebraic symbolism.

Below you see the development of the first two shapes in a pattern.



a) Draw the third and fourth shapes in this pattern. You may use the squared paper.

b) Count the number of stars in each of the shapes you have now, and put the results into a table. Explain how the number of stars increases from one shape to the next. Use this to calculate how many stars there are in the fifth shape.

c) What you have found in task b is called a recursive (or indirect) formula. Can you express it in terms of mathematical symbols?

d) Try to find a connection between the position of a shape and the number of stars in that shape. This is called an explicit formula. Can you express such a formula in terms of mathematical symbols?

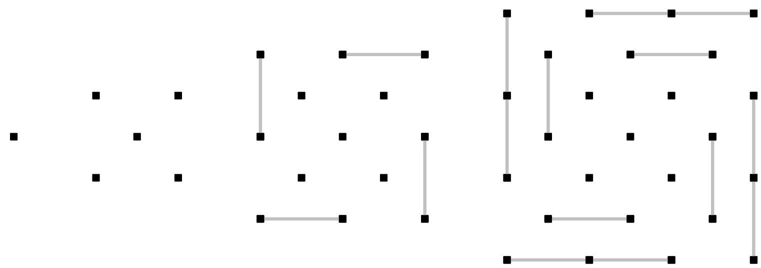
Figure 1. Task 1 given to the class for work in small groups

There are several possible ways to continue the pattern in Task 1 of which the first two geometrical configurations (elements) are given. In the *a priori* analysis I concentrate on the alternative identified by Alice and Ida (Figure 2), and relate it to the teacher's intention with the task.

Figure 2. A possible continuation of the shape pattern in Task 1

A figural approach to algebraic generality in shape patterns might involve an analysis of the invariant structure of the shape pattern by *decomposition* of its geometrical configurations according to an algorithmic rule (e.g., to isolate diagrammatically by encircling, or to paint with different colours). Below I present two possible decompositions of the actual pattern (shown in Figure 3 and Figure 4). In Figure 3 the first four elements are partitioned to illustrate that the differences of the sequence mapped from the shape pattern are multiples of four. Moreover, it is possible to see how the next element (with five dots on each side) can be made by the same rule:

Adding a line with four dots on each side of the fourth element (the same way as the other lines are placed) will complete the fifth element.



$a_1 = 1$ $a_2 = a_1 + 4 \cdot 1$ $a_3 = a_2 + 4 \cdot 2$ $a_4 = a_3 + 4 \cdot 3$

Figure 3. The first four elements illustrating that the differences are multiples of four

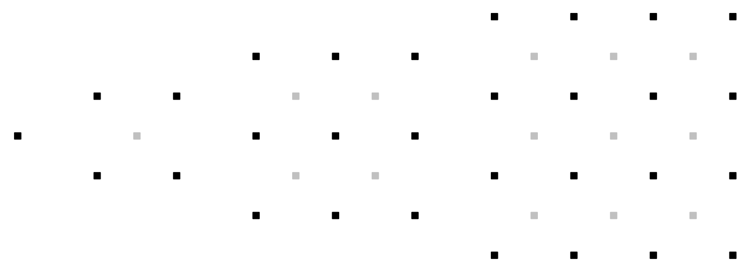
The arithmetic relations presented in Figure 3 ($a_1 = 1$, $a_2 = a_1 + 4 \cdot 1$, $a_3 = a_2 + 4 \cdot 2$, $a_4 = a_3 + 4 \cdot 3$) have references in the partitions: It is visible how each element is composed by the previous element plus four lines, each line with one dot less than the position of the current element.

Based on this, the n-th member of the sequence mapped from the shape pattern can be generalised by algebraic thinking (Mason, 1996) as the recursive formula $a_n = a_{n-1} + 4(n-1)$, with $a_1 = 1$. It can be noticed that the decomposition presented in Figure 3 can also serve as reference for the representation of the sequence shown in Table 1, where the n-th member is given in terms of an explicit formula: $b_n = 1 + \sum_{i=1}^{n-1} 4i$.

Table 1. Sequence originating from a decomposition in terms of multiples of four

Position	1	2	3	L	n
Number of dots	1	$1+4\cdot 1$	$1+4\cdot 1+4\cdot 2$	L	$1+\sum_{i=1}^{n-1} 4i$

A second decomposition is shown in Figure 4, where the components of the first four elements are drawn with different colours to illustrate that each element is a sum of consecutive squares. The arithmetic relations presented in Figure 4 ($c_1 = 1^2$, $c_2 = 2^2 + 1^2$, $c_3 = 3^2 + 2^2$, $c_4 = 4^2 + 3^2$) can be used to identify a relationship between the position of a member of the sequence and the rank of the squares to be added: The n-th member can be generalised by algebraic thinking as $c_n = n^2 + (n-1)^2$. This is an explicit formula for the n-th member of the sequence at stake. It is equivalent to b_n, but syntactically and semantically different (because it has been developed from a different route). The formula for the n-th partial sum of an arithmetic series can now be used to establish that $b_n = 1 + 4\frac{n}{2}(n-1) = 2n^2 - 2n + 1 = n^2 + (n-1)^2 = c_n$. This provides a connection between the different formulae with references to the partitions of the alternative decompositions presented.



$c_1 = 1^2$ $c_2 = 2^2 + 1^2$ $c_3 = 3^2 + 2^2$ $c_4 = 4^2 + 3^2$

Figure 4. The first four elements illustrating nested squares

Different decompositions developed by students provide opportunities to import different meanings for the algebraic symbols in formulae (corresponding to the structure of the different partitions). Moreover, it provides opportunities to engage students in meaningful manipulations of (equivalent) algebraic expressions when students show that formulae that are syntactically different can be transformed into the same expression.

ANALYSIS OF THE EPISODE

A formula for the general member of a sequence – isolated from the regularity of the shape pattern from which it is arising

The students have written the numbers mapped from the first four elements of the shape pattern down in a table where differences between successive members are written in the bottom row. Alice notices that "it is indeed the four-times table", and Ida says that "it must be sixteen and then twenty the next time." They use this to extend their table which becomes like the one shown in Table 2.

Table 2

Shape	1	2	3	4	5	6
Number of stars	1	5	13	25	41	61
	+4	+8	+12	+16	+20	

Then they reflect further on how the number of stars increases from one shape to the next (Task 1b):

28	Alice:	You increase by the four-times table in a way. Each time [Pause 1-3 s] it increases by four and four each time.
29	Ida:	Yes, for each new shape it increases by four.
30	Alice:	Then it increases by [Pause 1-3 s] the previous [increment] plus four.

Their observations of the growth, as manifested in turns 28-30, describe properties of a pattern with quadratic growth: The first differences follow a pattern of linear growth; that is, the second differences are constant (Kalman, 1997). Alice and Ida, however, do not use the terms "linear", "quadratic", "difference", or "second difference". After they have written the first eight, and then the 19th, 42nd and 99th members of the sequence as the sum of the previous member and the difference, they arrive at the following recursive formula: $(n-1)4 + s_{n-1} = s_n$. This formula is similar to a_n presented above, except that it does not display the initial condition ($a_1 = 1$).

Alice and Ida have in the above used a numerical approach to generality. This is not surprising, given the design of the task; the focus is on counting components and putting the numbers into a table (Task 1a). Further, the students have decomposed particular numbers mapped from the shape pattern in terms of arithmetic relations, where the numbers are written as a sum of their predecessor and the difference. Algebraic thinking is then used to establish the formula $(n-1)4 + s_{n-1} = s_n$. This is based on the students' observation (of the particular members) that the difference between successive members is equal to the product of the number four and the position of the least member.

In the numerical approach employed by the students, the geometrical configurations are important only as a context to produce a sequence of numbers that subsequently is generalised in terms of a formula in algebraic symbols. However, the shape pattern in the task is a *real milieu* (Brousseau, 1997) in the sense that it can be manipulated. The students draw the next elements and count their components, but they do not decompose the geometrical configurations to analyse the invariant structure of the pattern: There is no feedback in the didactical situation that makes a structural analysis of the elements necessary.

The teacher's intention with the task (to express the regularity of the shape pattern) might have been handled by making the students create references (as suggested in the *a priori* analysis) between partitions (resulting from decomposition) of the elements on the one hand, and the symbols in the formula on the other. These two objects, partition and symbol, would be the outcomes of the situations of *action* and *formulation* (Brousseau, 1997), respectively. To express the regularity of the pattern in natural language (based on a decomposition) would correspond to an informal model of the regularity of the pattern in the situation of action, whereas the algebraic formula would correspond to a formal model of the regularity in the situation of formulation.

Alice and Ida do not make references between the iconic context and the arithmetic relations they have written down. In this way, when they generalise the arithmetic relations by algebraic thinking, it is likely that they do not experience the pattern as a mathematical structure to be an aim in itself, where algebra is a mediational means to represent the invariant structure in the pattern.

An attempt to make a connection between a recursive formula and an explicit formula does not succeed

When the students subsequently (without the teacher present) worked together to find an explicit formula for the *n*-th member of the same sequence, they use an analogue approach: In search for an explicit relationship, explained in Task 1d as “a connection between the position of a shape and the number of building blocks in that shape”, they calculate the *differences* between the members of the actual sequence and their respective positions. This method I interpret as the students’ erroneous application (“overgeneralisation”) of the features of a recursive approach in an explicit approach to the general member of the sequence.

An explicit formula for the general member of a sequence mapped from a shape pattern can be defined as the numerical value of the n-th element expressed as a function of n. The method employed by the students is inappropriate because they establish an arithmetic relation (difference) between member and position, $f(n)-n$, instead of a functional relationship between member and position. Based on their calculation of the differences, $f(n)-n$, I infer that they have interpreted the word “connection” used in the task to mean “difference”. Alice and Ida’s construal is possibly influenced by their engagement with a recursive formula (Task 1c, in the same session), where they had calculated the differences between successive numbers of the sequence at stake. In search for an explicit formula (Task 1d), they produce the diagram shown in Table 3, where the commentary column (where R refers to “row”) is made by me to explain how the numbers are derived. To distinguish differences from other elements of the table, differences are written in grey.

Table 3: Diagram produced by the students in search for an explicit formula

													Commentary
	1		2		3		4		5		6	R 1	n
0		3		10		21		36		55		R 2	$f(n)-n$
	1		5		13		25		41		61	R 3	$f(n)$
		3		7		11		15		19		R 4	$(\text{R } 2)_{i+1}-(\text{R } 2)_i$
			+4		+4		+4		+4			R 5	$(\text{R } 4)_{i+1}-(\text{R } 4)_i$

The second row of Table 3 (in the commentary column symbolised by $f(n)-n$) consists of differences between members of the sequence mapped from the shape pattern (third row) and their position (first row). In constructing the numbers in the second row, the students have not focused on coordinating the referents (components and position) for the variables. If referents were added for numbers in the second row, the resulting number sentence would be: 5 [components] − 2 [position?] = 3 [components]. This is problematic since the operation of difference is not referent transforming. Further, the fourth row of Table 3 consists of first differences of $f(n)-n$, and the fifth row consists of second differences of $f(n)-n$.

When teacher Erik, who has designed the task, enters the room (on his own initiative), the students ask him if they are on the right track with respect to an explicit formula. The teacher responds by asking them what the characteristic of the recursive formula is. The students answer by describing the general nature of a recursive formula, but the teacher says that he means the particular recursive formula in Task 1. The conversation continues like this:

374	Teacher E:	It is not so easy, you know, the explicit one [laughs]. There is something, there is something about the recursive [relationship] which makes it complicated. [Pause 1-3 s] How is it if you look at the increase from one shape to the next?
375	Alice:	Ehm [Pause 1-3 s] you take the previous one and multiply. No, you take the previous increase and add four to it.
376	Teacher E:	Yes, exactly, right. You take the previous increase and add four.

When the teacher in turn 374 suggests that there is something that makes the sought explicit relationship complicated, I interpret that he attempts to explain the complexity by the *type* of growth of the sequence arising from the shape pattern. This is indicated in the same turn by the teacher's attention to the (first) differences of the sequence when he asks the students to describe "the increase from one shape to the next." He reinforces that the growth is non-constant by repeating Alice's description of the increase (turn 376). I find it plausible that his claim about the complexity of the sought explicit relationship and his attention to the fact that the first differences are non-constant, are attempts to make the students work analytically and thereby potentially deduce properties of the syntax of the explicit formula searched for. This interpretation has been approved by teacher Erik in a conversation I had with him after the observed lesson. It is consistent with a later utterance (from the transcript), where it appears that the teacher understands the numbers in the fourth row to represent the first differences of the sequence mapped from the shape pattern:

455	Teacher E:	It's just that, that [Pause 1-3 s] life would have been much easier with respect to a formula, if we had a pattern where *this* row had been a constant number [points at the fourth row in Table 3].

This interpretation by the teacher of the fourth row is possibly influenced by a comment by Alice (turn 377) about the same row of Table 3, where she claims to refer to the difference $f(6)-f(5)$ (which is equal to 20), whereas she actually refers to the difference $f(6)-6$ (which is equal to 55). Erik's interpretation of the fourth row of Table 3 as consisting of (first) differences would be in agreement with the second differences in the fifth row (which are constantly four). I interpret the teacher's intervention described in the above as an attempt to make a connection between the recursive properties of the sequence at stake (that it has *quadratic* growth) and the syntax of the desired explicit formula (that it is a polynomial of order *two*). This is however non-trivial, and there is no indication in the students' reasoning which suggests that they are able to utilise the teacher's hint: When the teacher later asks the students: "Do you have a kind of feeling which type of formulaic expressions that may emerge?" (turn 474), this is succeeded by 16 seconds of silence.

Given the school curriculum, I believe that it is most likely that Alice and Ida have no previous knowledge about different types of growth of sequences. Teacher Erik's intervention in this episode I interpret as an instance of a *metamathematical shift*

(Brousseau, 1997): It is characterised by the phenomenon that the teacher has substituted for the mathematical task (to find an explicit formula in algebraic notation) a discussion of the logic of its solution (what can be inferred about the syntax of the explicit formula from the observations about the growth of the sequence at stake). The teacher has tried to help the students improve their proficiency in establishing an explicit formula for the general member of a sequence, but the chosen method did not bring about the desired results. The unprompted utterances below indicate that the focus on a connection between recursive properties and the syntax of an explicit formula has not been helpful for the students:

611	Ida:	I'm supposed to come up with that one [explicit] [Pause 1-3 s] I'm all the time confused by the [recursive] one we figured out here.
612	Alice:	I don't see a clear distinction between recursive and explicit (Ida: no). I don't know what the different formulae are, and then I can't just shift from one to the other.

DISCUSSION

The milieu in the observed episode does not provide any feedback that requires that they analyse the pattern structurally (e.g, by decomposition) to make references between the elements of the pattern and the syntax of a formula. Quite the contrary, Task 1a focuses on counting components and putting the numbers into a table. This is in opposition to the teacher's intention, which is to express the invariant structure of the pattern. The feedback provided by the milieu with respect to an explicit formula is the concept "connection" between member and position. "Connection" is a vague (everyday) notion used instead of the mathematical concept "functional relationship" between member and position. It constitutes a weakness in the milieu because it contributes to confusion for the students in that they do not distinguish between a recursive approach (which involves *difference* between successive members) and an explicit approach (which involves member as a *function* of position).

When teacher Erik intervenes during Alice and Ida's struggle to find an explicit formula, he acts on the milieu by directing attention to a relationship between the recursive and the explicit formulae through the concept of type of growth of the sequence at stake. He thereby changes the knowledge needed to solve the problem. The analysis of the episode shows that the students cannot handle the new adidactical situation: They do not have knowledge of type of growth of sequences, neither does the milieu provide feedback that enables them to develop the knowledge necessary to utilise the teacher's intervention. However, encouraging students to connect recursive and explicit formulae is by Lannin et al. (2006) claimed to be important. Further, they recommend that tasks on generalisation of shape pattern be designed so as to promote students to remain connected to the figural representation (see also Steele, 2008). This is in line with Hewitt (1994) who warns against using contexts only to produce tables and spotting patterns in number sequences, because it does not give students insights into the structure of the original situation.

The *a priori* analysis of the actual shape pattern indicates how the process of decomposition might be used to provide feedback that could help the students remain connected to the figural representation, and hence discover and express the invariant structure of the pattern. Further, the decomposition presented in Figure 3 shows how a recursive formula and an explicit formula for the general member of the derived sequence are connected (in the way Figure 3 displays how an explicit formula is the sum of the first member and the first differences).

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DIFFERENT WAYS OF GRASPING STRUCTURE IN ARITHMETICAL TASKS, AS STEPS TOWARD ALGEBRA

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In this study we propose an analysis of some interesting solving processes of 9-grade students engaged in an arithmetical task. It originates as an item of a national test that has proven to be very critical for Italian students. By changing the way of administering the task, and also by virtue of some interviews, we got the opportunity to observe interesting students' behaviours, some of which throw, in our opinion, new light on students' sense-making processes in the borderline between arithmetic and algebra.

KEYWORDS: *arithmetic-algebra, sense-making*

INTRODUCTION

In these last years new perspectives about the teaching and learning of algebra are emerging, due, among other things, to the impact of educational technologies (see e. g. De Vries & Mottier, 2006) and to the achievements of Early Algebra (for a wide overview, see Cai & Knuth, 2011). Moreover, interesting hypotheses about the presence of a sort of algebraic discourse within the daily life media have been advanced: “*It is possible that these days algebra is simply 'in the air'* […]. *With the help of media, algebraic forms of expression may even be infiltrating colloquial discourses*” (Caspi & Sfard, 2012, p. 64).

Our research group has been working for several years on the teaching and learning of algebra with particular care to motivational aspects and sense-making processes (see e. g. Guidoni, Iannece & Tortora, 2005). We have been studying these processes mainly adopting a Vygotskian research perspective, that is arranging suitable class settings, where social interaction and immersion in culturally relevant activities are enhanced, at a secondary school level (Iannece & Romano, 2008), as well as at a primary one (Mellone, 2011), where we share the basic claims of Early Algebra.

This paper originates from a didactic activity, devised for understanding and analysing the poor performances realized in some items of the annual Italian national assessment for 10-grade students organized by INVALSI (*Istituto Nazionale per la VALutazione del Sistema educativo di Istruzione e di formazione*) in the year 2010-2011 (a complete account of the test, together with an analysis of students' difficulties can be found in http://www.invalsi.it/snv1011/documenti/Rapporto_SNV%202010-11_e_Prova_nazionale_2011.pdf).

Some poor results of the test come as no surprise, rather they confirm some of the most frequently observed difficulties met by students.

The following (D16) is one of the INVALSI test items, where the Italian students encountered major difficulties, as shown in the table below:

D16. The expression $10^{37} + 10^{38}$ is also equal to:

A. 20^{75}

B. 10^{7}

C. $11 \cdot 10^{37}$

D. $10^{37 \cdot 38}$

A (%)	B (%)	C (%)	D (%)	No answer (%)
35.0	1.9	22.0 (correct)	38.7	2.4

This result, together with the results to the whole test, acted on us as a stimulus for trying to understand more deeply their causes. Therefore, we have arranged several slightly different modalities for administrating the questions to different students. Moreover, in a second phase, a selected sample of them have been briefly interviewed. We have collected several data and are now just interpreting them, as the object of a deep study.

In this paper, of an exploratory character, we do not intend to analyze the causes of students' errors. Instead, our aim is to focus on a few episodes occurring in students' answers to item D16, mainly to underline how, when the students are (more) free to select and to express their own strategies, the spectrum of their behaviours considerably enlarge and often goes beyond any simplified classification attempt. In particular we notice that some of their arguments on the borderline between arithmetic and algebra may even suggest new reflections and ideas on the usual meaning of the algebraic notions and procedures.

THEORETICAL FRAMEWORK

Algebra is a mathematical domain in which the search for meaning is very problematic. Indeed, many meanings can be given to the algebraic symbols and to the word 'Algebra' itself in 'official' Mathematics just as much as in mathematics education. For instance, we contend that in Bourbaki's work we can find the ultimate roots of the emphasis given in the last decades by all world school curricula to the syntactical aspects of algebra and of the corresponding tendency to leave meanings in the shadow.

However, it is widely acknowledged that from a didactic point of view the link between arithmetic and algebra is the high road to support and justify the introduction and the development of algebraic skills. Therefore, the problem becomes rather to understand the different aspects of this link and to suitably manage them by means of an effective didactic mediation, in order to avoid the common and well known difficulties met by students.

In this direction, many research lines suggest that the link arithmetic-algebra cannot be reduced to a simple one-way path (see, for instance, the study Iannece, Mellone & Tortora, 2010 inspired to Davydov's, 1982, ideas). In a recent study (2011), Subramaniam and Banerjee document that as early as in the twelfth century famous Indian text 'Bhaskara', the role of algebra is explicitly viewed as a foundation rather than as a mere generalization for arithmetic, along a typical two-ways relationship: algebra "*was viewed both as a domain where the rationales for computations were grasped and as a furnace where new computational techniques were forged*" (Subramaniam & Banerjee, 2011, p. 95).

Turning to students' behaviours, the international literature about the solution of algebraic exercises and word problems offers several examples that suggest an apparent 'suspension of sense-making' (see, for example, Schoenfeld, 1991). On the other hand, according to wide evidence coming from neuroscience (see, for example, Rizzolatti & Sinigaglia, 2006), there doesn't exist in our brain an absolute *absence of meaning*, but instead the brain's way of working is always subjected to an automatic and sometimes unconscious dynamics of search for meaning. In particular, whenever the time at our disposal is quite short, what often happens is a suspension of the meanings that come from mature knowledge in favour of an appeal to previous more basic resources or to rapid analogical reasoning.

So, along a solving process, students' making-sense could be genuinely a search for solutions plausible and coherent with the text of the exercise, typically *inside* the discipline. Or, students can also look for a, so to say, *outside* sense, basing their interpretation of the task on the fact that who assigns it is an adult bearing an authority, and so trying to interpret this adult's behaviour too: "[…] *there is a gross mismatch between the goals that the teacher thinks he or she is getting for students and the goals that students actually seek to achieve. In other words, the teacher believes that the students are operating in a mathematical context when their overall goals are primarily social rather than mathematical in nature*" (Cobb, 1986, p. 8). For example, during the solving process of an algebraic exercise, it can happen that its sense be identified with the mechanical use of formulas or algorithms, because students' perception is often that the meaning of such an exercise is exactly to show that some rules have been well memorized. This is also the effect of the belief that the success in mathematics strongly depends from the ability to execute a task in a brief time, belief that often comes from the first grades of school and is supported by the many test that students meet during their lives. As claimed by Arcavi (2005), the kind of sense making followed by students in their solving processes is strongly linked to the classroom culture. Therefore, the classroom culture has a central role in behaviour or habit of the learners and this has important implications in the didactic practice. Arcavi gives some advice as to the direction that didactic practice could move, suggesting *"asking students to develop the habit not to jump to symbols right away, but to make sense of the problem, to draw a graph or a picture, to encourage them to describe what they see and to reason about it"* (Arcavi, 2005, p. 45).

METHODOLOGY AND CONTEXT

The INVALSI test for grade 10 lasts 90 minutes and contains 30 questions, in most cases multiple-choice questions, and some consisting of several items. The test is administered to all the Italian students and, as we have already noted, some of the questions were answered very poorly. So, we decided to involve a group of teachers in a research study in order to examine the reasons behind this poor performance. We are convinced that the formats of the questions (multiple-choice, above all) and the short time have strongly contributed to students' failure, triggering a sense-making oriented to a "social survival" (Cobb, 1986) more than one genuinely linked to the discipline. By social survival, we refer to a general problem facing complex human interaction within constraints formed by a variety of intentions and goals, many of which are not mathematical.

As a first step of our work, each question and the corresponding national results has been carefully studied. Then we decided to modify the way of administering some selected critical items, one of which is the D16 item, and to test them in several classes of different grades. The teachers inserted the selected questions in their monthly written tests, which usually include about 15 exercises of several kinds (open questions, closed questions, problem solving) and the students were asked to justify their answers, including those for multiple-choice questions. The time allowed for answers is generally longer than in INVALSI test. In a few cases a student has been interviewed, after the correction of the classwork, in order to better understand some passages in her/his text or to realize to what extent (s)he was aware of the legitimacy of the procedure employed. The general purpose was to reflect on students' solution processes and, possibly, to link them with the didactic strategies of the teachers. In fact, it is worth noticing that the educational approach of the teachers of our team is inspired by Vygotsky's ideas about the social and semiotic nature of learning. In particular, they often approach algebra by means of numerical problems organized around the cycle Conjecture-Argumentation-Proof: firstly, students are asked to search for regularities and to make conjectures, later they are encouraged to discuss and finally to provide a mathematical proof or a counter-example to the conjectures. Often proofs need new algebraic techniques that, in this way, aren't proposed as mere theoretical notions but rather as suitable tools to make generalizations, tools that only afterward will be stabilized as standard procedures to be used in similar problems.

In this paper we report an analysis of students' answers to the D16 item. As the item involves tasks usually faced in the 9th grade, we decided to administer it in

May 2012 to 43 students of two 9th grade classes. This was after the students were formally taught the rules of calculation with powers.

D16 is very interesting for many reasons. First of all, it is a context-free task that can be approached through different solving strategies: for example, a) it can be solved comparing the magnitudes of the numbers; b) the particular base 10 of the powers could induce students to perform arithmetical computations; c) the sum of two powers with the same base can wrongly induce students to apply powers properties, as suggested by the two distractors A and D (38% of Italian students chose D); d) the expression can be manipulated as an algebraic one, using a symbolic rule like factorization to facilitate a complex arithmetical computation, becoming in this way a good example of interplay between algebra and arithmetic. Moreover, D16 requires a transformation of an arithmetical expression: according with (Subramaniam & Banerjee, 2011), working with the operational composition of numbers is one of the key ideas marking the transition from arithmetic to algebra.

The following table contains the answers to question D16 in the two classes:

Choice	A	B	C	D	No answer
Number of students	4	0	28	11	0

Although our sample is not representative and of sufficient size to show statistical significance, we note that the percentage of correct answers is considerably higher than the national one (28/45 ≅ 62% vs. 22%). We suggest that this may be largely due to two factors: the extra time allowed to them (with respect to the official test), and to the teacher's care in giving sense to the transition arithmetic-algebra. In the next section we will analyse a few protocols, which can be seen, in a sense, as alternative to the usual arithmetical/algebraic procedures.

ANALYSIS OF SOME PROTOCOLS

Many correct answers employ standard algebraic procedures (which are also those expected by the authors of INVALSI test), as well exemplified in Lucia's protocol (Fig. 1), where the result is obtained using a factorization step, carefully expressed in algebraic terms and applied to the numerical case.

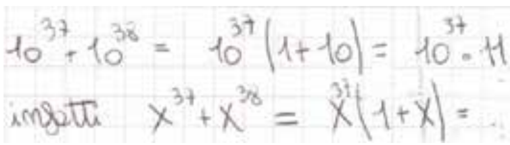

Fig. 1 - Lucia's protocol

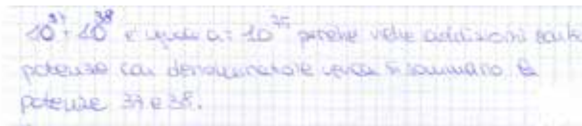

Fig. 2 - Susy's protocol. She says: "$10^{37} + 10^{38}$ is equal to $= 10^{75}$ because when we add powers with the same denominators [*sic*] the powers [*sic*] 37 and 38 are added".

There are several incorrect answers mainly due to confusions about the rules for powers. See Susy's protocol, fig. 2. Note also that, although she selects option A, she writes 10^{75} instead of 20^{75} in her justification). Also there are cases where the correct answer is achieved via incorrect procedures. See Emanuela's protocol, fig. 3, where all the passages are pure inventions but magically lead to the right conclusion $11 \cdot 10^{37}$. But what is more interesting for us is to try to understand some less normal, although still reasonable, ways followed by other students, both in the cases where they do and do not succeed.

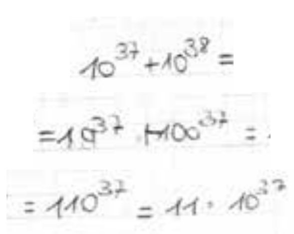

Fig. 3 - Emanuela's protocol

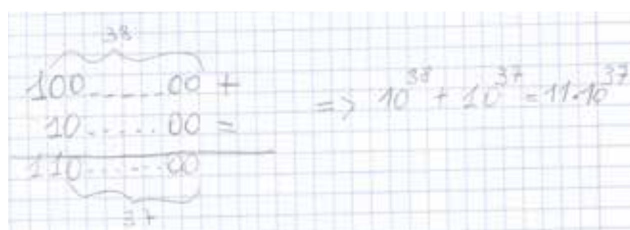

Fig. 4 - Ciro's protocol

Ciro's (Fig. 4), Rosanna's (Fig. 5) and Giuseppe's (Fig. 6) excerpts are similar, in that they catch the right answer using procedures that have some algebraic features, but do not abandon the arithmetical domain. Another common characteristic is that their results are, from a strictly logical point of view, just likely, maybe strongly likely, but not safely proved. Or, at least, it can be said that possible refinements of their reasonings, to get rigorous proofs, are not a priority for them.

Ciro's way of avoiding heavy computations is as easy as an Egg of Columbus and reveals a strong confidence with place value and related algorithms. The use of ellipsis in his decimal writing of powers of ten is a clever shortcut, which may be considered as a kind of algebraic attitude, although Ciro's focus is on structure rather than generalization. In fact, Ciro is able to recognize his result as the product $11 \cdot 10^{37}$. Moreover, we guess that the path toward forms of generalizations is wide open for Ciro. One can simply replace the numbers 37 and 38 on the braces to begin to generalise to other similar problems.

The right answer is C. In fact, if for example we figure $10^2 + 10^3$ out, we get 100 + 1000 = 1100. The other options are wrong, since they look as if the properties of powers had been applied to the expression $10^{37} + 10^{38}$, whereas they can only be applied to multiplications or divisions.

Fig. 5 - Rosanna's protocol (our translation)

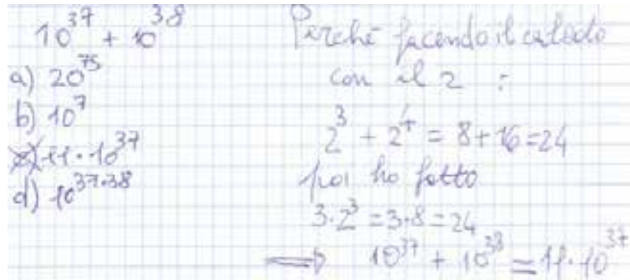

Fig. 6 - Giuseppe's protocol. He says: "Since operating with 2… then I have made…"

Rosanna's strategy is not so different, but she prefers a simplified numerical example to justify her choice. She notices that the numbers 37 and 38 are consecutive and is quite aware of the importance of this fact: again a structural aspect underlying the specific numerical case. And she is able to substitute two simpler numbers, that is 2 and 3, to see and to explain the result for the larger ones using the numbers chosen as a so called *generic* example. Moreover she skilfully detects the deceiving role of the distractors, even if we wonder whether she uses her metacognitive attitude to rely more firmly on her result or to provide a more complete answer.

Giuseppe's protocol is quite complex, and perhaps even more interesting. He renounces to the advantages of decimal representation of numbers and prefers to rely on smaller numbers and on his familiarity with operations on them. The choice made by Giuseppe brings him farther from the arithmetical reasoning, requiring a quite rich algebraic treatment of the problem. But, as a matter of fact, he succeeds in the task of controlling two simultaneous substitutions (the base 2 for the base 10 and the exponents 2 and 3 for 37 and 38), grasping the essential (structural) relationships, and succeeds in correctly applying the obtained results to the original numerical case. Here Giuseppe, like Rosanna, apparently uses a *generic* example. But his direct jump from a single case to another one without the interposed support of a general argument turns out to be a bit reckless. A deeper insight on his way of reasoning comes from the interview, whose essential passages are reported here.

Interviewer: Why did you change 10^{37} and 10^{38} into 2^3 and 2^4, instead of computing the two powers of ten?

Giuseppe: I chose smaller numbers to make easier calculations.

I.: But how just 3 and 4 as exponents?

G.: Since 37 and 38 are consecutive numbers, so I selected two consecutive numbers, but very small.

I.: Ok, and then?

G.: Well, $2^3 + 2^4$ is 24, then I noticed that, in the answer C, 10^{38} becomes 11 [sic], therefore I transformed 2^4 into 3, computed $3 \cdot 2^3 = 24$, and since the results turn out to be the same, I chose C.

I.: Oh, but you certainly remember what you have heard several times, that one example doesn't make a proof! How could you be sure of your answer, after only one example?

G.: Because I controlled the other answers.

I.: What do you mean?

G.: I computed 4^7 [Giuseppe points on the option A on his sheet], 2^7 [on B] and 2^{12} [on D] and none of them resulted in 24, then I guessed that if only the option C works for 2, then the same must hold true for 10.

As can be seen, Giuseppe's justifications of his procedure confirm what has already been deduced from his protocol, that is a solid grasping of the structural features of the numerical expression. Indeed he handles the arithmetical expressions occurring in the various answers as algebraic ones, substituting time after time the numbers of his example. But what is more interesting is the intertwining between his reasoning toward the right result and the way he uses the distractors, in particular the hypothesis that only one of the answers is correct. His mathematical (or better logical) argument is frankly convincing although quite complex, to the point that we confess some difficulty of ours in transforming it in a safe logical argument. In other words, we could say that Giuseppe's argument is not certain, but nevertheless a strongly reliable one.

Mariarosaria's protocol (Fig. 7) shows a typical example of a wrong choice: she is attracted by the option D, the most preferred by Italian students.

$10^{37} + 10^{38}$ can be written as $10^{37 \cdot 38}$ since the exponents aren't equal, therefore we cannot concentrate on equal bases, there isn't a multiplication, so we can't add the exponents, but with equal bases we multiply the exponents even if there is a sum of the bases.

Fig. 7 - Mariarosaria's protocol (our translation)

Also M. was interviewed, to better understand her strategy and in particular why she completely disregarded the right answer C.

I.: Why didn't you consider C, when you realized that something didn't work with the power properties?

M.: Well, the other options looked as more likely; C was too strange.

Perhaps there is not so much to comment on this: the distractor hit. But something can still be deepened and some 'reasonable' insights identified in Mariarosaria's words. First, the word "concentrate" belongs to the class language and therefore to classroom norms and practices, in the sense of Cobb (1986), as witnessed by the teacher; when she gives instructions about products or divisions of powers, she usually says: “when the exponents are equal, you must concentrate on bases, when bases are equal, on exponents”. Apparently, teacher’s message was received, but the emphasis on the two possible cases caused neglecting that only the multiplicative structure is involved. Anyway, it can be said that Mariarosaria views the exercise as concerning just calculation techniques of powers (she calls “strange” the option C) to the extent that she dares to invent a new rule to justify her choice.

CONCLUSION

First of all, we want to emphasize that the Vygotskian methodology used by the teacher and in particular the continuous attention devoted to students’ reasoning is the invaluable condition that allows, here as well as in general, the students to freely express their views and the teachers and researchers to observe them and to achieve new insights about students’ behaviours and their sense-making processes.

In our case, the various students’ attempts to solve the task appear to us difficult to classify using existing tools. This raises, in our opinion, a lot of didactic and research questions regarding the kind of rationality which lies behind these solving processes. Indeed, we hypothesise that some kind of ‘social’ rationality (Cobb, 1986) is active and intertwined with the more pure mathematical reasoning. As a consequence, a new question arises, namely if it is better to encourage or to discourage this kind of arguments, and moreover if we are disposed to consider them genuine mathematics.

Of course, we have no definite answer to these questions, but are convinced that they are more and more urgent, since multiple choice tests are just examples of problematic situations very frequent in today life. Technological advances mean that modern society requires a great capacity to take rapid decisions. To face these problems is, in our opinion, one of the biggest challenges of tomorrow educators.

Finally, we want to point out that the variations in the way of administrating the task were also used by teachers involved in this research as a tool to reflect on their own didactic practice. For this reason we are convinced that this kind of experience could provide suggestions for a possible wide scale work with teachers, both to support them in facing the national test, and as a good hint for in-service training.

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GENERALISING THROUGH QUASI-VARIABLE THINKING: A STUDY WITH GRADE 4 STUDENTS

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This paper presents a study from the beginning of a teaching experiment to promote grade 4 students' algebraic thinking. It aims to analyse how quasi-variable thinking contributed to the development of generalisation and to the first uses of symbolisation by the students. The data were collected from two mathematical tasks that explored computation strategies. The lessons were taught by the researcher (the first author), the data were collected using video recordings, and the collective discussion moments in the classroom were analysed. The results show how the teacher conducted the exploration of some particular numerical expressions to lead students to generalise the relationships underlying the structure of the calculation strategies. Thereby, using quasi-variable thinking, some students express the generalisation in natural language and start making a pathway to symbolisation.

Key-words: algebraic thinking, generalization, symbolization, quasi-variable thinking

INTRODUCTION

The development of algebraic thinking from the first years of schooling should be understood as a way of thinking that brings meaning, depth and coherence to the learning of other topics and has the potential of unifying the existing mathematics curriculum (NCTM, 2000). The recent Portuguese curriculum (Ministério de Educação, 2007) assumes that students should start to develop the algebraic thinking by using arithmetic as an entry point, as they work with generalizable regularities in numbers and operations and also by the study of figurative sequences.

The present paper aims to discuss how the use of quasi-variable thinking can contribute to the development of generalisation and to the beginning of symbolization process in grade 4 students. [1] Namely, we seek to understand: (i) How the exploration of computation strategies from particular numerical expressions leads the students to express generalisation?; and ii) How is generalisation starting to be expressed into symbolic mathematical language?

THEORETICAL BACKGROUND

According to Blanton and Kaput (2005), algebraic thinking can be regarded as "a process in which students generalise mathematical ideas from a set of particular instances, establish those generalisations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (p. 413). Considering the potentially algebraic character of arithmetic as one of the possible approaches for the development of algebraic thinking, the construction of the generalisation can be promoted through the exploration of numerical relationships and arithmetical operations and their properties and, also, by developing the notion of equivalence related to the equal sign (Carpenter, Franke & Levi, 2003).

Rivera (2006) suggests that numerical systems should be taught in a way that students understand the numerical relationships and proprieties of individual objects and progressively realise that those are invariant independently from the considered objects. The regularities that students find on arithmetical operations can be the basis for the exploration of generalisation about numbers and operations and also to practices as the formulation, test and proof of the produced generalisations. In this way, particular numerical expressions can be used to work general relationships. Fujii (2003) uses the expression of *quasi-variable* to describe a "number sentence or group of number sentences that indicate an underlying mathematical relationship which remains true whatever the numbers used are" (p. 59). Within this perspective numbers, in generalizable numerical expressions, act themselves as variables that allow students to focus their attention in the expressions' structure, and in identifying and discussing the algebraic generalisation before being introduced to formal algebraic symbology. This kind of quasi-variable thinking can provide an important bridge between arithmetic and algebraic thinking and, also, a gateway to the concept of variable (Fujii, 2003).

Britt and Irwin (2011) consider that algebraic thinking should provide opportunities for all students to work with several layers of awareness of generalisation. These authors suggest that a pathway for algebraic thinking develop in such a way that "students use three semiotic systems to express that generalisation: first they should work with numbers as quasi-variables, then with words and finally with the literal symbols of algebra" (p. 154). Similarly, Russell, Schiffer and Bastable (2011) advocate the introduction of algebraic notation when students already express their ideas into words and images allowing them to access the meaning of symbols. These authors contend that this new form of representation, not only provides a concise expression of students' ideas, but also offers new ways of perceiving mathematical relationships.

In this paper, we also assume the conception of generalisation as a dynamic and social situated process that can evolve through collaborative acts (Ellis, 2011). In this perspective, the classroom situations are seen as multiple process of interaction "in which the students and the teacher co-contribute to the development of meaning through their talk, shared activity, and engagement with artefacts" (p. 311). This interactionist perspective includes both teacher-student and student-student interaction and allows researchers to take into account how shared ways of interacting promote the development of generalisation.

Cobb, Boufi, McClain and Whitenack (1997) designate *reflective discourse* as a kind of a classroom discourse in which mathematical activity is objectified and becomes an explicit topic of conversation. When students engage in a collective act of reflective discourse, they have the required circumstances for mathematical learning, and their individual contributions develop the discourse that supports and sustains collective reflection. In that kind of activity, the teacher's role is very important because he "can proactively support students mathematical development" (p. 269). The relation between individual and collective learning is not a simple one, as the students' mathematical development emerges from the interactions and the cultural practices in the classroom but, it is "the individual child who has to do the reflecting and reorganising while participating" (p. 266) in the discourse. So, that reflexive discourse must support both individual and collective learning, and the teacher's role must respect these two dimensions.

METHODOLOGY

The results presented in this paper are part of a broader study, which focuses on the implementation of a year-long teaching experiment (Gravemeijer & Cobb, 2006) aiming to promote the development of algebraic thinking in one grade 4 class. The teaching experiment took place in the school year of 2010/11 and the mathematical tasks proposed to the class drew on the mathematical topics defined by the annual plan made by the school teacher. However, these tasks were innovative considering the usual teacher's practice as they accommodated the prospect of conceiving the algebraic thinking as guiding the syllabus (NCTM, 2000) through a logic of curricular integration. Focusing on some insufficiencies detected on students' number sense, we developed a sequence of tasks, aimed at the exploration of numerical relations and operations properties with the intention of promoting generalisation through natural language, and the beginning of a way towards mathematical symbolisation. The use of some informal symbolism was introduced in the tenth task, when the teacher-researcher proposed the use of the symbol "?" to express "what is the number that..." in expressions like "?x5=100".

In this paper, we focus on the moments of collective discussions in the classroom, after the students' work in pairs on two of the mathematical tasks in the teaching experiment (the 12th and the 14th). These tasks explored computation strategies with the goal of leading students to express the generalisation in natural language and to begin to express it in mathematical language. These were the first tasks where the teacher-researcher intentionally promoted the expression of generalisation into mathematical symbolic language. The lessons were videotaped, and from the analyses of the videos we choose the episodes that show how the teacher conducted the exploration of particular numerical expressions to foster the generalisation of the computation strategies involved in each of the tasks. These episodes show how some students are starting to use symbolic mathematical language in the sequence of the teacher-researcher's challenge.

RESULTS

The first task analysed in this paper - *Calculation using double* - explores the double and half relationships between the four times table and the eight times table. The teacher's intention is that students explain the relationships and generalise the strategy in natural language, and then translate it to mathematical language.

"Calculating using doubles"

In Sara's class, students were calculating products:

These students used the same strategy to calculate different products.

1. Explain their strategy.

Figure 1 - Task "Calculating using the double".

In the collective discussion of the task, it was evident that the students understood the strategy, as they were able to provide some particular cases. Then, the teacher conducts the discussion with the goal of making the students extend the strategy beyond those cases in order to induce them to express it in a generalised way.

Teacher: Okay. We have three examples, but does this strategy only fit those examples?

Students: No.

Fábio: The strategy we used is good for all computations.

Teacher: And how can we synthesise that strategy in a clear way? What strategy was that?

Diogo: We made the double computation.

Teacher: The double of what?

Diogo: Double of the result. (...)

Teacher: Which multiplication table do we want to work on?

Students: The 8 one.

Teacher: And to work with the 8 times table, we used which multiplication table?

Students: The 4's.

Rita: We can use the halves.

Teacher: And what did we find out? I can construct the 8 times table using which one?

Students: The 4's.

In this episode, these students reveal that they understand that there is a property involved in the numerical equalities present in the task. But in this moment they still refer it as a procedure they can apply to all numbers. After that, a student, Rita, was able to express the computation strategy of generalisation beyond the particular cases, but still using a confusing and repetitive language. Having this in mind, the teacher asks if the expression should be simpler and clear, but still the students answer with examples: “To know 25x8 we do from 25x4” (Carolina). Then, she asks: “To know the 8 times table, what do I do?” and the students answer: “Double the 4 times table”.

From this moment on, one student proposes the generalisation of the computation strategy in natural language and writes it on the board: “To find out the 8 times table, we do the double (x2) of the 4 times table” (João). Several students show that they identify the underlying relationship between these two multiplication tables. After that, the collective discussion was conducted to enable students to express the generalisation in mathematical language. As this was the first time that students were confronted with this issue, the teacher attempts to make them understand what it means to write the generalisation in “mathematical language”.

Teacher: Now I want that you think about the sentence João wrote on the board and try to write it in mathematical language. How can we use mathematical language?

Students: With operations.

Teacher: So, how can I write that? But pay attention because I don’t want particular cases like 6x8, 12x8 or 25x8, I want that to all numbers of the 8 times table and 4 times table.

Rita: We can do it to 7x8.

Teacher: But that is a particular case. I want it to all cases.

Rita: How is that?

Teacher: To all cases in the 8 times table. What happens in the 8 times table?

Fábio: It is always plus 8.

Teacher: Ok, it’s always plus 8. But if we use multiplication, what are we doing?

Students: We are always multiplying by 8.

Teacher: How can I write that?

Rita: We can use a question mark.

Fábio: Times 8.

The expression of generalisation in mathematical language was not immediate. First, the students tried to use particular examples. Then, the teacher conducted the class to use the structure of the computation strategy of the multiplication by eight to all cases. When Rita mentions the “question mark”, she uses the symbol that she knows from the tenth task, briefly described before. Then, Rita went to the board and wrote the expression shown hereafter:

Figure 2 – Generalising expression in mathematical language, made by Rita.

Instead of saying something about the correctness of the expression, the teacher suggests the substitution of the symbol by a specific number. In this way, she tries that students attach the meaning of that symbol and find out if the expression written by Rita was correct. Some students suggest the use of number 6 and then others numbers, and therefore, they find out that the expression was not correct. As a result most students express that they agree that the correct expression should not have the question mark at the end.

The *second task* analysed in this paper - *Afonso's strategy* - concerns the inverse computation strategy of the previous one: multiplying by five is equivalent to the half of the multiplication by ten. The particular case of 36x5 is proposed in the task as it follows:

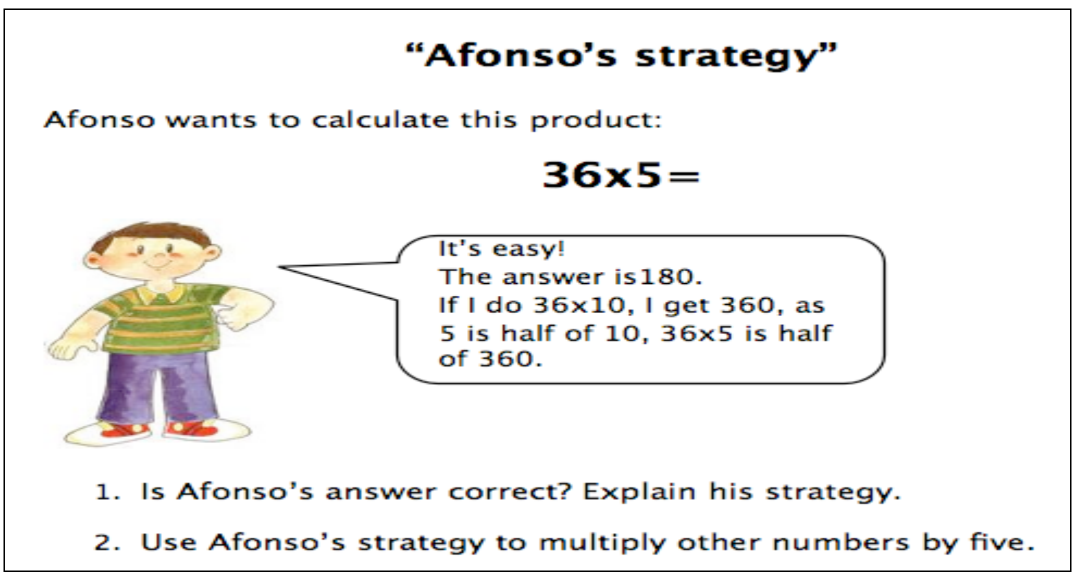

Figure 3 - Task "Afonso's strategy".

In the beginning of the collective discussion of the task, the teacher asked the students to express that strategy in natural language. Several students did it without difficulty, expressing it in the following way: "To find out the 5 times table we make half of the 10 times table". In spite of this, when the teacher requested the students to write the computation strategy in mathematical language, they still used particular cases. As in the first task, she stresses that she is not asking for particular cases but for a computation strategy that can be applied to "any number". At that point, some students suggest the use of the question mark at the beginning of the sentence, and others a few other non-numerical symbols. The difficulties they face then have to do with the representation of "half of any number". Henrique suggests a creative representation by drawing a flower divided by half of it.

Figure 4 – Henrique's attempt to express the generalisation in mathematical language.

But as this seems an odd representation to students, the teacher focused the students' attention in the exploration of the particular case presented in the statement of task, 36x5, revising the strategy previously discussed. As it was not expressed in mathematical language, but only in natural language, the teacher forwards the students to do it.

Teacher: We had 36x5... What did we do?

Rita: 35 times 10.

Teacher: And then?

Rita: We divided by 2.

Teacher: So, how can we do it? Any number times 5 is equal to...

Fábio: Is equal to half.

Teacher: What did we do first?

Gonçalo: We multiply by 10.

Teacher: We multiply by 10... But is 36x5=36x10 correct or not?

Students: No.

Gonçalo: No, then we made the half.

Teacher: (…) And how can we represent half of it?

Rita: Dividing by 2.

The teacher writes the expression 36x5=(36x10):2 on the blackboard, and then compels the students to write it for "any number, and not particularly for 36", using Rita's former representation: "?x5". In that moment, Rita says that now she knows how to do it and writes the expression in a symbolic way (fig. 5, left side). As the teacher asks for other possible representations for the question mark with the meaning of "any number", Fabio proposes the use of a flower in the expression (fig. 5, right side).

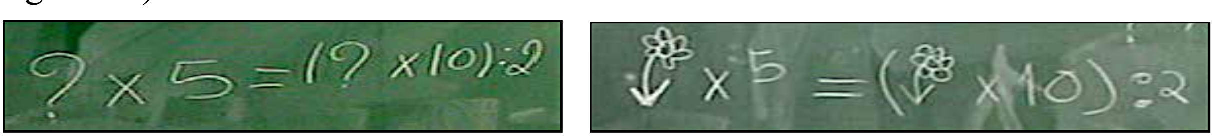

Figure 5 - Expressions of generalisation in mathematical language.

The teacher stresses that those symbols may represent any number and, at that point, Fabio mentions that: "This also works with others numbers. If you do flower times three equals six dividing by two. And also to four and eight, and to one and two", showing that he can extend the computation strategies to others multiplication tables. This is an evidence of the student's understanding of the structural property of the depicted strategy, and of the use of quasi-variable thinking in a general way.

FINAL REMARKS

This study shows that when students used particular numerical sentences regarding computation strategies they were able to identify the structure of those strategies and to express them in a more general way, as the teacher conducted the collective discussion with that intention. Actually, the computation strategies were explored in order to make explicit the underlying structure and to apply it to other numbers besides those used in the task's statement, in the meaning of quasi-variable thinking (Fujii, 2003). In this way, representing the generalisation of the computation strategy in natural language was not difficult for students because they easily identified the operation involved and translated it in words, like when one student said "to find out the 8 times table we do the double of the 4 times table".

However, when compelled by the teacher to generalise that relation in symbolic language, students had to create a new way to represent "any number" in order to represent all the numbers that satisfy the condition of the computation strategies, even in the restricted context of the multiplication tables. In a previous task, the teacher proposed the "question mark" to represent an unknown, but in the tasks analysed in this paper, she does not propose any symbol. It was one student, Rita, who suggested the "question mark". In the second task, some students created other symbols to represent their idea of "any number", using figurative representations like flowers. For instance, when they were first confronted with the need to represent "half of any number" in a symbolic way, one student shows some creativity by drawing half of a flower. This initial pathway in the use of symbols to express the idea of a general number shows that students were starting to construct their own sense and personal meaning for symbols, and were not simply reproducing rigidly the symbolism used by the teacher. In this initial phase it is important to give students the time to explore and create their own conceptions of the symbol. Although the results reported in this paper concern a very early stage of the teaching experiment, the later development of the study shows how that these initial tasks contributed to the development of the students' ability to generalise and to use meaningfully symbolic mathematical language (Mestre, 2011).

It is clear from the collective discussion in the classroom that two students had a leading role in the discourse: Fabio and Rita. As we could see, when they participate in the discourse they contribute to raise the level of the collective discussion. As Cobb, Boufi, McClain and Whitenack (1997) argue as long as these students engage in discussions, they reflect and reorganize their action and their development was explicit during the interaction with the teacher and other students. However, according to those authors, and in line with the conception of the generalisation as a collective process (Ellis, 2011), its expression in the classroom resulted from the engagement of several other students during the collective discussion. The teacher's role was crucial in this process as she promoted opportunities for students to express and to explore their ideas and conducted them to analyse the arithmetical expressions in a more algebraic way, allowing the construction of the generalisation.

The teacher's intention to have the students using symbols to express the generalisation of the depicted numerical strategies may not only contribute to assist them in expressing their ideas in a more concise way but also to foster new ways of perceiving mathematical relationships (Russell, Schiffer and Bastable, 2011). At the time of writing, we are still analysing the data from the teaching experiment and investigating this possibility, but in this paper we observe, for instance, that Fabio expands the arithmetic strategy explored from the written symbolic expression.

This study shows an alternative way to the traditional approach of arithmetic, in the Portuguese context, with the implementation of a new mathematics curriculum, which assumes the importance of algebraic thinking in the early years of schooling. By focusing the students' attention in the numerical relationships and in the study of proprieties of numbers and operations, they start to deal with algebraic ideas, developing also a deeper understanding of arithmetic. In this perspective, the use of quasi-variable thinking (Fujii, 2003), as it was evidenced in this study, can be a potential pathway for algebraic thinking since it can provide opportunities to work with several layers of awareness of generalisation (Britt & Irwin, 2011) starting from the arithmetic context which is familiar to students. The broader research from where this study emerges intends to add to the research on early algebra by showing that the exploration of quasi-variable thinking is a potential avenue to introduce algebraic symbols in young students even when their previous experiences with the mathematical generalization process were very restricted.

NOTES

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SYNTACTIC AND SEMANTIC ITEMS IN ALGEBRA TESTS – A CONCEPTUAL AND EMPIRICAL VIEW

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In this paper, we draw a distinction between syntactic and semantic aspects of algebraic thinking. We examine the hypothesis that these two aspects can be distinguished empirically using test items. We present exploratory analysis of a test of algebra based on a large sample of students aged 11-14 in England and contrast this to a previous analysis of older German students (Oldenburg, 2009). This analysis indicates that there are considerable difficulties in operationalizing the distinction using test items, but suggests a potentially fruitful line of analysis may be to treat the semantic aspect as consisting of two sub-dimensions, based on whether one or many meanings or interpretations appear to be required.

INTRODUCTION

Although research on algebra education is now at a mature stage, many aspects of student progression and learning remain poorly understood. Of particular interest, is the relationship between the syntactic understanding of the rules and procedures involved in the manipulation of symbols and the semantic understanding required to interpret and attach meaning to symbols and rules. In a previous study, Oldenburg (2009) argued that test items could be constructed to measure and distinguish these two aspects of algebraic thinking. Using this distinction, he found that students can gain some level of proficiency in one aspect while being weak in the other, but that both aspects were necessary for a sophisticated understanding to develop. Whilst accepting the utility and validity of the syntactic / semantic distinction both to describe algebraic thinking and to inform pedagogy, the other two authors of this paper were sceptical about whether this distinction could be applied to items and whether such thinking could be measured using test instruments. The present paper is a result of the authors' subsequent debates and explores the meaning, usability and limitation of this pair of constructs. To do this, we analyse the performance of different items in an algebra test administered to a nationally representative sample of students in England.

THEORY: DISTINGUISHING SYNTACTIC AND SEMANTIC TASKS

If we consider algebra as a language, then, as with any other language, algebra can be thought of as having syntax and semantics, which must be applied to any use of the algebraic language, e.g. in communication and problem solving. Since thinking is to a large extent based on language, syntax and semantics can be viewed as aspects of thinking and understanding in general. In general, any algebraic thinking involves both aspects and hence distinguishing the two is not straightforward. Nevertheless, we contend that some algebraic tasks are more amenable to syntactic approaches, whilst others are more amenable to semantic approaches. For the purposes of the present paper, we give the following working definition:

A syntactic task, or assessment item is one that can be solved by actions triggered by the syntactic structure of the expression alone without involving a mental model for interpretation, i.e. without having mental objects referred to by the symbols. For example, it is possible to solve the expansion of $(x+y)^2$ by purely syntactic thinking, because the pure lexical structure may activate the schema of the binomial theorem.

A semantic task, or assessment item is one in which the need for the interpretation of symbols (i.e. the construction of a mental model of objects denoted by symbols) is dominant in successful solutions. For example, in order to give a general expression that allows one to calculate the number of wheels a certain number of cars have, one has to activate semantic thinking to symbolize the number of cars by a letter and relate this letter in its domain of interpretation with the wheel number.

Syntactic tasks by this definition are those that can be successfully carried out by "term rewriting systems" as defined in computer science (Baader & Nipkow, 1999). Such systems cannot carry out tasks that require students to link the algebraic language with objects and concepts from outside mathematics. We contend that tasks that require at least a complete sentence in natural language are very likely to be semantic. We note that this definition is based on the anticipated processes that will be used when solving the items. Thus, in order to classify tasks as mainly syntactic or mainly semantic, one needs to anticipate the way that students tackle it.

We do not claim that our distinction between syntax and semantics is entirely unique, although other researchers' definitions differ in some key respects. Most recently, the distinction has been used to analyse different approaches to, and aspects of, the production of proof (Weber & Alcock, 2004; see also, Iannone & Nardi, 2007). Weber and Alcock distinguish between "syntactic proof production", by which they mean "one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way", and "semantic proof production", by which they mean "a proof of a statement in which the prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" (p. 210). One problem with this definition is that the precise meaning of both "stated definition" and "permissible way" is rather vague. However, Weber and Alcock's approach shares with ours a concern for the importance of the domain of reference.

Kieran (1996) draws a distinction between 'transformational', or rule-based, activity, 'generational' activity in which the objects of algebra, expressions and equations, are formed, and 'global, meta-level' activity, which includes problem-solving and modelling. Whilst she argues that transformational activities are more often based on syntactic rules, she notes that they can in certain cases be legitimated by semantic arguments (e.g., transforming $1/(1\text{-}v/c)$ to $1\text{-}v/c$ in special relativity incorporates some physical assumptions). Further, although Kieran argues that much of the "meaning-building" for algebraic objects takes place through generational activity, some generational activity can be largely rule-based (e.g., the generation of the polynomial sequence, $(x\text{-}1)^n$. Engelbrecht, Harding and Potgieter (2005) used the distinction between procedural and conceptual knowledge and found medium correlation. Note that this distinction is different from ours, as is explained in table 1. Neubrand and Neubrand (2004) draw a distinction between technical, or procedural, tasks and modeling tasks, which often require semantic, or meaning-based, constructions. However, they note that technical calculation tasks may require some meaning-based activity, particularly those involving longer algorithms. Similarly, modelling requires students to operate on symbols as well as constructing meaning for the symbols. Although neither Kieran's nor Neubrand and Neubrand's dichotomies are completely identical to the syntactic/semantic distinction, their analyses do serve to remind us that the distinction between syntactic and semantic tasks is somewhat blurred, specifically:

- Some items can be tackled both syntactically and semantically. For example, expanding $5\cdot(x+2)$ may be done either by syntactic matching to the pattern of the distributive law $a\cdot(b+c)$ or by applying the semantic way of interpreting the expression as the area of a rectangle.
- A semantic approach may be helpful to help a learner self-correct the misapplication of syntactic rules. In the case of the expansion of $(x+y)^2$ referred to above, semantic thinking (by, for example, substituting some numbers for the symbols) can help avoid the common expansion error of x^2+y^2.
- Even items that are mainly semantic in nature (e.g. "Explain the meaning of $2g+4r$ in some context") involve at least the syntactic ability to read the expression. Here, one needs the semantic understanding that $2g+4r$ is a legitimate (set of) numbers/answer not just that the expression can be read as "2 times g added to 4 times r".
- Modelling tasks (i.e. generating an expression or equation to describe a situation or relation) require at least the minimal syntactic competence to write down the expression. So, while a mental distinction may be possible, it can be blurred as soon as the communication processes are required.

Knowledge	**syntactic**	**semantic**
procedural	Distribute $2\cdot(x+y)$	Substitute values and calculate [1]
conceptual	Number of opening and closing bracket must be the same	Upon substitution for both x in $x+x$ must be replaced by the same value

Table 1: A comparison of syntactic/semantic and conceptual/procedural dimensions

One could argue that the entanglement of syntactic and semantic aspects within the domain of algebraic thinking is clear from the outset, e.g. one could read Lins and Kaput's (2004) definition of algebraic thinking that way: "First, [algebraic thinking] involves acts of deliberate generalisation and expression of generality. Second, it involves, usually a separate endeavour, reasoning based on the *forms* of syntactically-structured generalisations, including syntactically and semantically guided actions" (Lins & Kaput, 2004, p. 48). In fact, certain algebraic actions may be justified from semantic or from a syntactic perspective. Indeed we even more assume that this effect may depend on the development of algebraic thinking, i.e., we suggest that, as students' algebraic thinking becomes more sophisticated, the thinking evoked may change and tasks that previously required semantic approaches may be solved using purely syntactic approaches. Moreover, the hypothesis that items (and not a particular student's way of doing an item) can (at least to some extent) be classified as either syntactic or semantic assumes that typical students have a preferred way of tackling these problems. We note also that it is possible a students' preferred approach may be strongly influenced by the teaching approaches adopted by their teachers. Hence, whilst it is possible theoretically to distinguish between syntactic and semantic aspects of algebraic thinking, it is not clear whether these aspects can be reliably distinguished empirically through test items. If the distinction can be used successfully at all, it must be applied to small focussed test items, not to larger problems as substantial mathematical activities usually span syntactic and semantic aspects.

METHODS

In this paper, we analyse the performance of different items on an algebra test originally developed in the 1970s as part of the Concepts in Secondary Mathematics and Science (CSMS) study (Hart et al., 1981). In 2008 and 2009, these tests were administered to a nationally representative sample of 5115 students in England aged 11-14 as part of the Increasing Competence and Confidence in Algebra and Multiplicative Structures (ICCAMS) study (Hodgen, Küchemann, Brown & Coe, 2009). These data were collected as part of a larger study and not specifically designed for the study reported here.

The focus of the CSMS/ICCAMS algebra test is on generalized arithmetic (Küchemann, 1981). Drawing on, and extending, Collis's (1975) analysis of the different ways in which pronumerals can be interpreted, items were devised to bring out the following six categories (Küchemann, 1981): Letter evaluated, Letter not used, Letter as object, Letter as specific unknown, Letter as generalised number, and Letter as variable. We note that the test items were not developed specifically to address the syntactic / semantic distinction. Indeed, items in a category may involve syntactic or semantic approaches, although items in the two categories, 'letter as generalised number' and 'letter as variable', are more likely to be semantic.

The items were independently coded as syntactic, semantic or mixed by the three authors of this paper using only the definition of the categories given in the Theory section above, with mixed used for items felt to involve both aspect. Examples of classifications are given in figure 1 and 2. We assume that the classification in figure 1 can be easily agreed on, figure 2 presents items that are potentially difficult to classify. One may say that all of them have a syntactic appeal if attacked by replacing one sub-expression by a part that is known to be equal. This is what puts 5a,5b into the mixed category (as there are references to concrete numbers as well which constitute the semantic aspect). In the case of 5c, however, we assume that plugging 8 for *e*+*f* in the second equation is different from tasks the student may have met before, so that according to the above definition ("A syntactic task is one that can be solved by actions triggered by the syntactic structure of the expression alone. ") we miss the triggering effect to take place ion the student that we assume to do the item.

The results of this initial coding exercise reflect the problematic nature of applying the syntactic/semantic distinction to items. The overall inter-rater reliability (3 rater, 3 categories) measured by Cohen's kappa =0.54 which can, according to Landis and Koch (1977), be judged as moderate agreement. Inspection shows that this is mainly due to one rater who used the mixed classification a lot. Between the other two raters we find κ=0.70 which is a good inter-rater reliability. A final agreed classification was developed through discussion. In this final classification, of the 51 items in total, 18 were coded as syntactic, 25 as semantic, and only 8 as mixed.

13. $\boldsymbol{a + 3a}$ can be written more simply as $\boldsymbol{4a}$.

Write these more simply, where possible:

$2a + 5a =$	....		
$2a + 5b =$	....	$3a - (b + a) =$	....
$(a + b) + a =$	....	$a + 4 + a - 4 =$	....
$2a + 5b + a =$	....	$3a - b + a =$	....
$(a - b) + b =$	....	$(a + b) + (a - b) =$	....

Figure 1: All nine items in Question 13 were coded as syntactic

5. If $a + b = 43$	If $n - 246 = 762$	If $e + f = 8$
$a + b + 2 =$	$n - 247 =$	$e + f + g =$ ......

Figure 2: The first two items in Question 5 were coded as mixed; the third item (*e*+*f*+*g*=) was coded as semantic.

REVISITING OLDENBURG'S ORIGINAL FINDINGS

Oldenburg's (2009) original paper reports findings from a different algebra test performed with 11th graders (aged 16) in Germany. This test included many items from the ICCAMS test together with items on more advanced symbolic manipulation (e.g. simplifying square roots) and on real world applications (e.g. translating between real world contexts and algebra). The test data from 2008 indicated a rather low correlation of r=0.33 between syntactic and semantic items. In the meantime this test has been conducted with many more students and higher correlations in different groups up to r=0.54 have been found. From comparing these various groups one can deduce the rule that the correlation is higher in higher achieving schools. This result cannot be explained by a ceiling effect as even in the weakest group facility (i.e. the fraction of correct answers) on syntactic and semantic items has been 33% resp. 53% so that no ceiling effects that would reduce the correlation have to be anticipated.

One potential explanation for these low correlations is that it may be that any arbitrarily chosen groups of items of similar facilities would tend to have a similarly low correlation. In order to test this hypothesis, we performed a bootstrapping process on the data from Oldenburg (2009). [2] This produced a large set of randomly chosen scales with a higher average correlation of 0.64 (standard deviation 0.062). We conclude that the selection of syntactic items is not arbitrary.

EXAMINING THE PERFORMANCE OF SYNTACTIC AND SEMANTIC ITEMS

We now consider our analysis of the 2008/9 ICCAMS dataset. The scales or item groups for both syntactic and semantic groups were found to have good internal reliability: Cronbach's alpha was 0.87 for the syntactic scale and 0.86 for the semantic scale. However, in sharp contrast to the original study, the correlation between the two scales was r=0.75 rising to r=0.78 (when blank responses are treated as incorrect rather than missing).

First, the difference may indicate some sample dependence. For example, the relationship between syntactic and semantic approaches may be influenced by the different curricula and pedagogic approaches in England and Germany. Second, as described above, we aimed to code as many items as possible. It is clear that this approach may tend to classify 'mixed' items as either syntactic or semantic. This may be reflected in the difficulties that we encountered in the coding process. Third, unlike Oldenburg's (2009) study, the ICCAMS test items were not specifically designed to assess the syntactic / semantic distinction. In fact, the ICCAMS test was not specifically designed to examine the syntactic/semantic distinction but sought to use items that use "problems which were recognizably connected to the mathematics curriculum but which would require the child to use methods which were not obviously 'rules'." (Hart & Johnson, 1983, p.2). As a result, the test items may be biased towards syntactic items that require some semantic thinking.

To investigate the second and third of these points, we reduced the syntactic scale to a core of 9 syntactic items originally coded as syntactic by all three authors. This reduces the correlations to 0.66 (or 0.69 with missing values treated as incorrect). However, a similar process of elimination from the semantic scale does not bring the value further down, so that still the explanation is not satisfactory. We propose an explanation that is related but not identical to aspects and uses of letters as symbols.

In mathematical logic (cf. Tourlakis 2003, p. 53) semantics is defined using interpretations. An interpretation of a set of formulae of predicate calculus is given by a set S (the domain of the interpretation) and an assignment that gives an element of

S for every occurrence of an unbound variable in the formula, and functions and predicates over S for every function and predicate symbol in the formulae. After applying all these assignments the formulae reduce to statements in the domain with no unbound variables remaining. For working with specific numbers it is thus enough to consider one interpretation, but to prove that a formula is a tautology one has to show that it is true in all interpretations. This distinction is crucial for learners. Küchemann (1981) has carefully drawn the distinction between an understanding of letters as specific unknowns and a more sophisticated understanding of letters as generalized numbers or variables. We suggest that the former, although semantic, requires the learner to consider only one possible interpretation or meaning, while the latter requires one to consider multiple interpretations. This suggests that treating the semantic scale as one-dimensional may not be justified. Thus we split up the semantic scale into two:

- **SES – *Semantics with single interpretation***: These are items that require only one interpretation to be considered. Thus they can be solved by mentally replacing the x by one number. A key example is the item: Write down the smallest and the largest of these: $n + 1$, $n + 4$, $n – 3$, n, $n – 7$
- **SEM – *Semantics with multiple interpretations***: These items can only be solved if multiple interpretations of the variable are considered. A key item is: Which is larger, $2n$ or $n + 2$?

This gives the following correlations:

r	**SYN**	**SES**	**SEM**
SYN	1	0.87	0.50
SES		1	0.55

Table 1: Correlations between syntactic and two kind of semantic items

If one replaces SYN by the core of transformational items then the correlation SYN-SEM goes down to 0.45 and of SYN-SES to 0.71. Again with the standard definition of SYN, one observes an interesting dependency of the correlation SYN-SEM which is for students aged 12, 13 and 14, respectively, 0.45, 0.47, 0.54. This is consistent with the observation made above that the test from Oldenburg (2009) shows higher correlations in better performing schools. Our interpretation of this is that it is possible to gain a certain level of facility with either syntactic or semantic items without understanding the complementary aspect, but that higher achievement levels (as typically reached by older students) require a tighter integration of both aspects.

This suggests – although one should keep in mind that this claim has no broader base than the one presented –that a simple dichotomy between semantic and syntactic item may be useful as a first approximation but blurs some important distinctions. The three scales defined above seem to better model the cognitive structure of students.

In order to examine this hypothesis further, we performed a structural equation modeling of the situation with SYN, SES and SEM as latent variables. Fitting such a model enables the identification of items that do not fit well into a scale. [3] The resulting model showed a good model fit (RMSEA=0.047, CFI=0.37). See Figure 3.

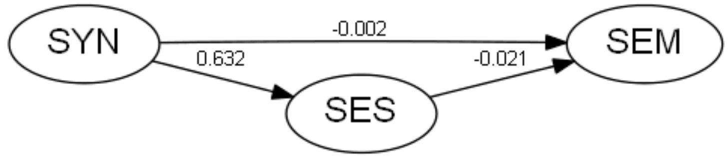

Figure 3: The structural model of three latent variables

The model equations were as follows:

SES = 0.63*SYN + error (1)

SEM = -0.002*SYN - 0.011*SES + error (2)

Equation (1) suggests that there may be a bridge from understanding the syntax of expressions to the ability to use it with specific (although unknown) numbers in one interpretation. One might conjecture that applying the interpretation (of letter as a specific but unknown number) requires at least the syntactic ability to plug in values for letters and evaluate the result. Equation (2), on the other hand, suggests that a fuller semantic understanding may develop relatively independently of syntactic and semantics with a single interpretation. [4]

CONCLUSION

As we have seen, there are considerable methodological issues in operationalizing the distinction between syntactic and semantic modes of thinking in existing test items. We found the process of coding the items to be problematic. This difficulty is reflected in the theoretical literature where the two modes are not treated as entirely distinct. Our initial approach was to attempt to code as many items as possible as either syntactic or semantic. On this basis, our analysis of the English data suggested that these data do not show the same pronounced distinction between syntactic and semantic items as was previously reported for the German data (Oldenburg, 2009). We have hypothesised possible reasons for this, such as some interdependence with different country curricula. In addition, the two samples were of different ages and it may be that the classification of items as syntactic or semantic may be age-related. Our analysis also suggests that the semantic scale may not be uni-dimensional. By classifying the semantic aspect into two sub-dimensions, based on whether one or many meanings or interpretations appear to be required, we found a relationship between syntactic and semantic items based on a single meaning, but not between either and sematic items requiring multiple interpretations. We emphasise, however, that this is exploratory analysis and further studies are needed to explore these questions deeper. Nevertheless, our analysis suggests that it may be possible to identify syntactic and semantic abilities using test items. We note, however, that this may require items designed specifically for this purpose.

NOTES

1. We are emphasising that the choice of numbers to substitute involves semantic processes.

2. The bootstrapping was conducted as follows. We drew 10,000 samples of groups of seven and 30 items, exactly the same size as the syntactic and semantic groups in Oldenburg's (2009) study. Samples were retained if the average facility was within 15% of the average facility in the syntactic (33%) and semantic (53%) item groups and otherwise discarded. The average correlation was calculated on the basis of all the retained samples.

3. We used the R package lavaan using a polychoric covariance matrix (calculated with the R package polycor) in order to compensate for the binary indicator variable used here. The binary coded items were used as indicator variables with free weights that the three latent variables load on. The variance of the latent constructs was fixed to be 1.

4. The fact that both weights are negative is not important because both numbers are very small and don't differ from 0 significantly (standard errors are 0.026 and 0.021 for the first and second coefficient, respectively).

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IMPLICIT LEARNING IN THE TEACHING OF ALGEBRA: DESIGNING A TASK TO ADDRESS THE EQUIVALENCE OF EXPRESSIONS

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In this paper, I assume that, in addition to the conceptual difficulties, some difficulties encountered by students, especially in elementary algebra, are generated by learning needs left implicit or unknown, inasmuch as they are not addressed within the learning institution. According to (Pilet, 2012), the equivalence of algebraic expressions, which plays an important part in managing, checking and anticipating algebraic transformations, is one type of learning left implicit in the French curriculum. After situating my work around the equivalence of algebraic expressions, I present a learning situation to address this notion which seems to be absent in curricula, then I analyze the implementation of this situation in a classroom.

Keywords: implicit learning, equivalence of algebraic expressions, school algebra

ORIGIN OF THE STUDY

My question about the existence of implicit learning needs which is left out in the teaching of algebra in France stems from a larger line of questioning addressed in my PhD thesis (Pilet 2012): what learning materials should be given to teachers to help them manage difficulties encountered by students in school algebra, and develop the students' personal relationships with algebra? I made the following assumption: in addition to the conceptual difficulties, *some difficulties encountered by students* in algebra may be intensified by the fact that students learn within institutions (Chevallard, 1999) where learning needs are left implicit or unknown, inasmuch as they are not addressed within the learning institution (Bosch, Fonseca, Gascon, 2004; Castela, 2008). Using an epistemological, cognitive and anthropological approach, I have shown in my thesis that the equivalence of expressions is one of these implicit learning needs in the French secondary school curriculum. However, a number of students seldom refer to the possibility of describing the same mathematical object using several algebraic expressions. Hence my question in this paper: What learning situation should be designed in order to allow students to tackle the equivalence of expressions? In this paper, I consider polynomial algebraic expressions.

After situating my work around the equivalence of algebraic expressions, I describe a learning situation[1] that I have designed to address this notion to students of a grade 9 class (14 year-olds) in France. Then, I analyze the classroom experiment of this situation. Finally, I use these analyses to reconsider my conceptual choices and question the necessary conditions under which this situation can exist in a regular classroom.

EQUIVALENCE OF ALGEBRAIC EXPRESSIONS

The notion of equivalence of expressions is present in several studies. In this section, I develop the approaches of Frege (1971) and Kieran (2007).

Frege: *sense* and *reference*

Equivalence of algebraic expressions refers to the distinction established by Frege (1971) between *Sinn* and *Bedeutung*, usually translated, respectively, as *sense* and *reference*. This distinction was re-used in didactics of algebra works for example by Drouhard (1992). A mathematical object has a unique denotation but may have different senses. For example, the expression $4(x-1)$ can be transformed as $x^2-(x-2)^2$ or as $4x-4$. These expressions have different *senses* (for example, they have different structures or they have different interpretations in geometry). They have the same *reference*, that is to say, they *refer* to same object (for example, evaluated for the same numerical values, they have the same values). Algebraic transformation is a subtle process of shifting between *sense* and *reference*. The choice of transformations of expressions is checked by *sense* (the final goal, for example, choosing the expression that can be best used for an equation) but respecting the written *reference* is an indispensible criteria. When one expression is transformed into another, we obtain two equivalent expressions. They are equal for all values, they have the same *reference*.

For many students, however, the conservation of *reference* during the transformation of an expression is unknown or absent. For instance, when a teacher addresses a common error $(x+4)^2=x^2+16$ and uses a numerical counter-example (for $x=1$, $(1+4)^2=25$ is different from $1^2+16=17$), students may not be convinced because the argument used (implicitly) refers to the conservation of *reference*. Similarly, when a teacher suggests the identity $(a+b)^2=a^2+2ab+b^2$ to replace the student's $(a+b)^2=a^2+b^2$, there can be a misunderstanding. The teacher is referring to *reference*, while for students it may seem to be a "choice" made by the teacher.

Kieran: equivalence of expressions as a theoretical control

From an international synthesis of research related to the learning of algebra, Kieran (2007) puts the equivalence of expressions at the core of the theoretical elements of transformational activity (factorization, expansion of products, rules for solving equations and inequalities, etc.).

> "One resource of algebra is a rich plurality of symbolic forms; one core notion, that of equivalence. Equivalence and transformation are linked notions, indicating sameness perceived in difference for some purposes, or indifference with respect to others? The existence of multiple expressions "for the same thing" can suggest the very possibility of transforming expressions directly to get from one to another." (Kieran 2007, p.722)

According to Kieran, equivalence of expressions has a fundamental role in theoretical control, ensuring that the transformed expression is equivalent to the second. This verification can theoretically be done in two ways: either by reference to the properties of algebra used or by linking with numerical and substituting numerical values for letters. However, students have a great deal of difficulty in identifying the properties they use when they transform algebraic expressions (Kieran, 2007). Moreover, in France, students have difficulty making the link between transformation of expressions and substitution of numerical values. Students have difficulty making connections between the arithmetical and the algebraic world.

The equivalence of expressions appears in these studies as an indispensable theoretical element of the conduct and checking of algebraic calculations. This point of view is adopted in several research works. For example, in the computing environments *Algebrista* of Cerulli and Mariotti (2009) and *APLUSIX* of Nicaud (1994), the equivalence intervenes to theoretically control algebraic transformations. Alternative points of view on equivalence are developed in other works for example (Zwetzschler and Prediger, 2013).

EQUIVALENCE OF ALGEBRAIC EXPRESSIONS: AN IMPLICIT LEARNING IN THE FRENCH CURRICULUM

In Pilet (2012), I analyzed the curriculum and several secondary school textbooks in France (for 14-15 year-olds) in order to identify the presence of equivalent expressions as a theoretical element of the transformation of expressions. The French secondary school curriculum addresses in a formal setting how to develop, factorize and simplify expressions as a goal in itself. The ideas of getting students to perceive the fact that two different expressions may represent the same object, and the idea of checking calculations, are not included in the curriculum and very rarely appear in textbooks. The equivalence of algebraic expressions is one type of learning left implicit in the French curriculum, inasmuch as its teaching is not addressed within the learning institution. Nevertheless, students are powerless when conducting and checking algebraic transformations, as they do not refer to the possibility of describing the same mathematical object using several algebraic expressions. Hence, I have designed a learning situation which aims at leading students towards understanding the fact that two expressions may denote the same mathematical object. This learning situation is presented in the next section.

DESIGN OF A LEARNING SITUATION

Context and methodology

This study is part of a larger research project, the project PépiMeP[2], which consists in developing resources (assessment, learning situation, interactive exercises) for diagnosis and differentiation of teaching about the learning of algebra towards the end of compulsory education the learning situation presented in this paper has been designed in this context. We present two components.

To start with, in this project learning situations are developed for a differentiated teaching. Teaching is differentiated in the following way: the learning objective is the same for the whole classroom—all students work on the same type of tasks—but the task is adapted to students' learning needs as identified by the diagnostic assessment *Pépite*[2] developed in PépiMeP project. In this article, the described situation focuses on the goal "Study of equivalent expressions". It is differentiated according to the specific student groups B and C established by Pépite. The differentiation in the learning situation presented in this paper is about the complexity of the algebraic expressions. Since differentiation of teaching is not of main interest in this article, I will not go further into our choices here. For more information, please see (Delozanne and al., 2008; Pilet, 2012; Grugeon and al. 2012; Pilet and al., 2013).

Secondly, I worked throughout the 2011-2012 school year in collaboration with several secondary school teachers to test the viability of our proposals in ordinary classrooms. The situation presented here was designed with these teachers. The choice of algebraic expressions, the tasks' decomposition and the predicted teaching scenario were discussed. Each situation was analyzed a priori and then tested in class by these teachers. Each experiment was analyzed a posteriori to retrace our design choices. The experiment that I present in the next section was conducted by one of these teachers.

The learning situation

The situation is shown in Figure 1. It is designed for students in a *grade nine* class (14 year-olds) in France. The learning situation focuses on the fundamental question: What are equivalent expressions? Given that the curriculum doesn't deal with the notion of equivalence, I ask students the following question: "Are these expressions equal for any value of x?".

We would like to know if the following three expressions are equal for all x:

B Group

- A: $(x-1)^2-4$
- B : $(x+1)(x-3)$
- C : $x(x-2)-x^2-2$

x	$(x-1)^2-4$	$(x+1)(x-3)$	$x(x-2)-x^2-2$
1			
-1			
0			

C Group

- A: $(x+2)^2-4$
- B : $x(x+4)$
- C : $9x-6$

x	$(x+2)^2-4$	$x(x+4)$	$9x-6$
2			
3			
0			

1) Complete the first two rows in the above table. What can you guess about the equality of these expressions?
2) Complete the third row in the above table. Does this confirm your conjecture? Otherwise makes a new guess.
3) Are the three expressions equal for any x? Justify. Does your guess hold up?

Figure. 1. A learning situation "Study equivalent expressions"

In order to lead students towards understanding the fact that two equivalent expressions have the same reference, two changes of frame ("cadre" in French) (Douady, 1986) are possible: a change from the algebraic to the numerical frame and a change from the algebraic to the functional frame. The first consists of implementing a numerical framework to evaluate the equivalent expressions for the same numerical values. The second consists of using a graphical framework to compare the representative curves of functions defined by algebraic expressions. In accordance with the French curriculum of grade 9, we chose the numerical framework.

Given that the students in groups B and C take little into account the equivalence of expressions to guide and control the algebraic transformations, the first challenge is to make sense of the fact that two expressions can be equal to any value of the letter. Therefore, the equivalence of expressions is first conjectured from numerical substitutions and proven from algebra. I now present the *a priori* analysis of the situation.

***A priori* analysis**

The learning situation consists of three questions. Three expressions are given. Expressions A and B are equivalent but not C.

The first two questions require students to make a conjecture about the equality of the expressions for any value. This step is absolutely necessary in order to bring students to understand that two different expressions can be equal to any value. In question 1, students test expressions for two numerical values chosen so that they return the same number. This leads them to formulate an initial hypothesis: the expressions are equal for any value of the letter. Then, in question 2, a third test provides a counterexample of the equivalence of expression C with A and B. A new conjecture about the equality of any value of expressions A and B can be formulated. The use of numerical tests aims to highlight the fact that the expressions either do or do not produce the same number. The layout of these tests in a table of values is a vital aspect of the task. It emphasizes the fact that two expressions can have the same *reference*. Question 3 requires students to prove this conjecture. It involves algebraic proof and the use of a numerical counterexample. In fact, students are expected to use a counterexample to prove that expression C is not equivalent to A and B. To prove the equivalence of A and B, a development is expected. In order to do this, students must detect the structure of the expressions and choose the correct rule to apply.

This task is unusual in the current curriculum. First of all, the link with numbers in order to test the expressions is seldom used. In addition, students are responsible for recognizing which proprieties to use and for deciding to transform expressions A and B so as to prove their equivalence. The proof task forces students to finalize the algebraic transformations and to justify them, which they do not do in the technical exercises.

The teacher plays an important role in helping students understand the limitations of numerical calculation and to prove the equivalence algebraically. After noticing that expressions A and B are equal for several values, some students may suggest an incorrect technique, such as “the expressions are equal for one or two values so they all are equal". The teacher is expected to rely on quantification in order to counter them: “we want a proof for any value”. The use of quantification can help students understand the necessity of proof and use algebraic calculations. The teacher may highlight that it is impossible to numerically test every possible value. Although requiring formalism is obviously not one of the task’s goals, it remains a key component of students’ being able to master the notion of equivalence of expressions.

The predicted teaching scenario involves individual work phases and collective debate phases. The fact that groups are working on different expressions is an opportunity to decontextualize the task. At the end, the institutionalization focuses on reference of algebraic expressions with the fact that two algebraic expressions can be equal for any values and the transformation of one into the other is controlled by the rules of algebra. In the next section, we discuss the implementation of this learning situation in a classroom.

EXPERIMENT

Data gathering in experiment

This learning situation was experimented in a *grade 9* class (14 year-olds) in France. The experiment’s protocol was specific to the PepiMeP research project. The teacher first had her students take the *Pépite* diagnostic assessment. Her class was divided into two groups: the C group (15 students) and the B group (6 students). She covered content related to algebra by introducing a new property of the calculation ($(a+/-b)^2=a^2+/-2ab+b^2$ and $(a+b)(a-b)=a^2-b^2$) and by developing several generalizations and proof problems. The learning situation illustrated in Figure 1 is found at the end of the chapter on algebraic expressions. It was completed during a 50-minute class period. I filmed, recorded and translated[3] individual and collective discussions between the teacher and her students.

Data analysis

The data analysis is based on the a priori analysis. Therefore, I analyze the data in several ways. First, I study the verbal exchanges about questions 1 and 2 in light of the following question: to what extent does the learning situation and its management by the teacher allow students to approach the fact that two different expressions evaluated for any number can have the same value? Then, I study the talks verbal exchanges about question 3 in light the following question: how does the teacher lead students towards understanding the limitations of numerical tests and producing an algebraic proof for the equivalence of A and B? These analyses are presented in the next section.

A posteriori analysis

Questions 1 and 2: Conjecture and reference of expressions

At the beginning, the teacher quickly informed students of the lesson’s goal: “We’ll be working on equal expressions.” She emphasized substitution with numerical values, but did not reveal the goal of this substitution. This may explain why at the beginning students focused on the simple and isolated task of substitution by a numerical value. Errors appeared such as ignoring rules for parentheses and the order of operation. Hence, the teacher allowed the use of calculators. It spared students some of the numerical calculations work. As illustrated in the following extract, students focused more on the task's ultimate goal: conjecture about the equivalence of expressions.

This is an extract from an exchange between the teacher and a pair of students in group C about question 1. Students have completed the table of values (Figure 1). They found 12 for the first line and 21 for the second for A, B and C.

Chloé: But **in fact, it should give the same results everywhere**?

Teacher: Well, sometimes it… ? [to Yann] Ok. So, what did you get, here? 12, 12, 12. And here, 21, 21, 21. So, in theory, what should we say, then?

Yann: That they… they… they are all equal.

Teacher: Yeah. The expressions are equal.

Yann: But I don't know.

Students were surprised by the fact that different expressions can have the same value when they are evaluated for the same number. The teacher leads them to conjecture the equivalence of the three expressions and to change this conjecture with question 2. Thus, the choice of appropriate numerical values and the presence of a table of values are sufficiently robust to allow students to approach the reference of expressions. However, in group exchanges in the classroom, the teacher does not return to the fact that the expressions A and B have the same numerical values when evaluated for the same number. We may wonder if this has been sufficient for students to reinvest in other cases.

Question 3: From conjecture to algebraic proof

To lead students towards understanding the limitations of numerical tests and the necessity of algebraic proof of the equivalence of A and B, the teacher used, as expected, the quantification. Here is an extract to illustrate.

Teacher: No, we don't know, do we? We just showed that it was true for two values. It doesn't necessarily mean **that it was true for all values**. So, now, you calculate for x equal to zero. […] Here we are. So, let's go, explain that! And so, your third expression, what is it? You will have to prove **that all of those expressions are equal for any value.** So, what are you going to use to be able to prove that these expressions are…?

In addition, quantification has played an important role in an attempt to show students the limits of their own technique: "it is true for two values so it must be true for all values".

Teacher: How did you manage, Ina, to prove…? Does the table allows us to assert that the expressions are equal?

Students: No.

Ina: Well, yes, because…

Théodule: No. Because we didn't use the distributive property.

Teacher: I do agree with you. Here, we find out that these are the three same values. But if you find out that both expressions are equal for three values can you assert that **they are equal for any value**?

Students: No.

Teacher: No. So, how can you manage to prove that your expressions **are equal for any value**?

Ina: Well… we calculate again.

Teacher: We should calculate again for another value?

Ina: Well, yes. Well…

Teacher : If you calculate again for another value, **it will mean that they are equal for four values. You, you want to prove that they are equal for any value**.

After several individual or collective exchanges, the teacher led her students to understand the necessity of the algebraic proof. However, students have a quite difficult time recognizing the structure of the expressions and the properties to be applied which leads to many errors in algebraic transformations.

DISCUSSION AND PERSPECTIVES

To conclude with, I return to the goal of the learning situation. Did it successfully lead students to understand the *reference* of algebraic expressions, and thus recognize equivalent expressions? Given that our analysis is only a case study, the conclusions to be drawn from it should be limited.

The choice of appropriate numerical values—the table of values—allowed students to understand the fact that expressions can produce the same number. I believe, however, that presenting a table of values with infinitely many numbers when the teacher asks students how they approached the problem could have reinforced students' understanding of the situation. The use of a spreadsheet in collective phases could play an important role in emphasizing the *reference* but differs too much from teachers' current practices. The analysis of discussions shows that the *reference* and the equivalence of expressions could be due to numerical conjecture, but the teacher did not suggest it again in question 3 or during the institutionalization. The institutionalization dealt with part of this content. The technique of numerical conjecture and algebraic proof, aiming to prove that two algebraic expressions are equal (or unequal) for any value of the letter, was presented. But the teacher did not add anything that might help students draw conclusions about the *reference* of expressions. The fact that algebraic expressions with two different written forms give the same number for any value was not presented.

This experiment shows the real potential of the designed task, as long as work has been done beforehand with teachers emphasizing their role and the task's goals. Readers should keep in mind that the designed task involves very few aspects of the current French curriculum (equivalence, links with numerical or quantification) and is quite different from teachers' current practice. The implicit learning that is essential to educational algebra needs a long period of preparation with teachers regarding two points: on one hand they must be aware of the links between implicit or ignored learning and students' difficulties in algebra and, on the other hand, they must develop their algebraic teaching practices. For this reason, I continue to work collaboratively with teachers to accompany the task with a discussion of the didactical issues at hand and their management in the classroom.

NOTES

[1] The term of "learning situation" is taken in this paper in a common sense.

[2] At present, the *Pépite* software is developed in PepiMeP project which consists of implementing computer resources in *LaboMep* platform, which has been developed by *Sésamath* to help teachers to differentiate students' learning in elementary algebra. Ile-de-France Region supports the project. *Sésamath* (http://www.sesamath.net/) is a French mathematics teachers' association, which has a central place in French online database systems. More information is given in a paper in working group 15 (Pilet et al., 2013).

[3] In this paper, I translated transcripts from French to English.

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STUDENTS' DIFFICULTIES WITH THE CARTESIAN CONNECTION

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The purpose of the study was to gain insight into students' difficulties with linear functions, particularly epistemological obstacles with the Cartesian connection, the connection between the symbolic and graphic representation. Two cross-sectional studies were conducted, one with secondary school students (Grades 8-10), and the other with college students. Participants completed several tasks and commented on their difficulties and successful algorithms. Interviews were conducted with some of the participants. Data analysis revealed that students performed poorly on tasks referring to the Cartesian connection. Moreover, findings suggest that even the few students who were successful at solving the tasks did not have the necessary mathematical principles and coherence to organize and advance their structures of knowledge.

Keywords: Epistemological obstacles, Cartesian connection, linear functions

SIGNIFICANCE AND PURPOSE

Among the key algebraic concepts are linear functions (National Mathematics Advisory Panel, 2008). During the last two decades, student difficulties with linear functions have been studied by many researchers (Knuth, 2000; Lobato & Siebert, 2002; Schoenfeld, Smith III & Arcavi, 1993; Stump, 2001; Zaslavsky, Sela, & Leron, 2002). Despite their efforts to gain insight into student difficulties and assist mathematics educators in implementing curricular and instructional changes, American students' difficulties with linear functions persist (American Diploma Project, 2010). Previous studies showed the importance of the Cartesian connection for student understanding of linear functions. The Cartesian connection states that a point (x_0, y_0) is on the graph of the line l if and only if its coordinates satisfy the equation of l, $y = mx + b$ (Schoenfeld et al., 1993), given that all the mathematical conventions associated with graphing in a Cartesian system of coordinates are respected.

The purpose of the studies reported here was to identify students' difficulties with the Cartesian connection, of interest being the "epistemological obstacles which occur because of the nature of the mathematical concepts themselves" (Cornu, 1991; Sierpinska, 1992), or those difficulties with knowledge that works satisfactory in old contexts but fails in new contexts (Brousseau, 1983). Results from two cross-sectional studies are used.

OVERVIEW OF THE FIRST STUDY

The first study (Postelnicu, 2011) was conducted with 1561 students, Grades 8-10, enrolled in mathematics courses from Pre-Algebra to Algebra II [1], and their 26 mathematics teachers. All participants completed a Mini-Diagnostic Test (MDT) on aspects of linear functions, and commented on the nature of their difficulties. The MDT tasks, illustrating connections between various representations of linear functions, were those used by Greenes, Chang, and Ben-Chaim (2007) in their study with Algebra I students from United States, Korea and Israel. Across the three countries, the most difficult tasks were those requiring the identification of slope from the graph of a line. Students' difficulties with slope formula and the change in *y* and the change in *x* represented as line segments with oriented magnitudes is seen by Schoenfeld et al. (1993) as an example of missing the Cartesian connection. American students also had difficulties determining whether a given point (x_0, y_0) is situated on a line, given the equation of the line, $y = mx + b$, in tasks that illustrate the Cartesian connection from point to line (Postelnicu, 2011).

After completing the MDT, two students from each of the 20 teachers who agreed to be interviewed were selected by the researcher and invited for interviews. In each pair of students, one student had the MDT total score above the group median, and one student had the MDT total score below the group median. Semi-structured interviews (Goldin, 2000) with think-aloud protocol were conducted. Students were asked to present their solutions to the MDT tasks and comment on their mathematical difficulties. Student interviews were coded using open coding at the level of paragraph (Strauss & Corbin, 1998) and analyzed using the Linear Conceptual Field (LCF) framework, a theoretical framework inspired by Vergnaud's theory of conceptual fields (Vergnaud, 1994). The Linear Conceptual Field (LCF) consists of:

i) Situations that can be modeled mathematically using linearity or linear functions (e.g., the car movement from Task 3, below). These situations reflect the current Intended Curriculum (Robitaille, 1980), namely what Wu (2011) calls the Textbook School Mathematics.

ii) Schemes of actions, and theorems in action needed to solve problems. Vergnaud's construct of scheme of action and the mechanism of knowledge development are borrowed from Piaget's genetic epistemology (Piaget, 1971). Some schemes of action, used to correctly solve a problem, are in fact algorithms for solving the problem. For example, one successful algorithm for Task 4 (below) has the following steps: 1) substitute the coordinates of the given points (x_1, y_1) and (x_2, y_2) into the equation of the line $y = mx + b$, and 2) solve the system of equations $\begin{cases} y_1 = mx_1 + b \\ y_2 = mx_2 + b \end{cases}$.

iii) Concepts of linearity and linear functions (e.g., the concept of slope in geometric-analytical context in Tasks 2 and 3 below), and ways of thinking (e.g., quantitative reasoning).

iv) Representations of linear functions (e.g., tabular, graphic, symbolic) and mathematical formalizations.

We present here several examples of epistemological obstacles.

Task 1 asked students whether the point with the coordinates $(2,-8)$ is on the graph of the line $y = 3x - 14$. All students described their schemes of action without referring to the Cartesian connection as a formal mathematical theorem. More than 60% of the students (N=978) missed the Cartesian connection in Task 1. Students who missed the Cartesian connection from point to line failed to identify the ordered pair $(2,-8)$ as (x_0, y_0) in $y = 3x - 14$ or to check whether the statement $-8 = 3(2) - 14$ is true. They tried (unsuccessfully) schemes of action involving the construction of graphs or symbolic manipulations of the equation of the line. Particularly difficult was the construction of the line $y = 3x - 14$. Some students asked us to provide the graph of the line (e.g., "What graph?"). Even when successful at graphing the line $y = 3x - 14$ and plotting the point $(2,-8)$, there were students who could not decide, based on their graphic representations, whether the point was on the line.

Task 2 asked to identify the slope of a line graphed in a homogeneous system of coordinates (see Figure 1).

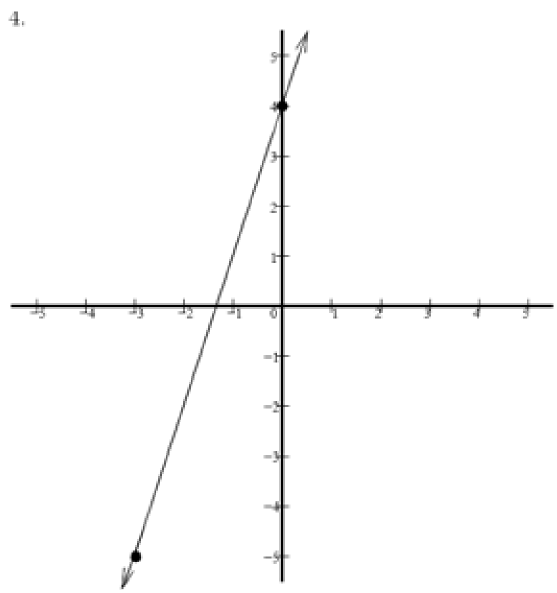

Figure 1: Task 2

To identify the slope in the geometric context from Task 2, one needs to know how to make a quantity. "Making a quantity" is the process of conceiving of a quantity as a quality of an object together with its magnitude. A magnitude is a numerical value assigned to a quantity by direct or indirect measurement. Measuring lengths on oriented axes of coordinates implies conceiving of an origin from where the measurement starts, an appropriate unit of measurement, and a sense of measurement, positive or negative (Freudenthal, 1983). In Task 2 one assigns the magnitude 9 for rise either by direct measurement (counting tick marks from $(-5,0)$ to $(4,0)$ on the Y-axis), or by evaluating the change in y, $y_2 - y_1 = 4-(-5) = 9$. "Making a ratio" is the process of conceiving of a function of an ordered pair of magnitudes, while calculating a ratio is the numerical operation of calculating the value of the function for a particular pair of magnitudes (Freudenthal, 1983). One makes a ratio for slope by assigning $(9,3)$ to the ordered pair (rise, run), and calculates the value of slope by dividing 9 by 3. The main epistemological obstacle with the concept of slope in the situation from Task 2 was making a ratio. Some students could conceive of both rise and run, but failed to assign 9/3 to the ordered pair $(9,3)$ (e.g., "I can't remember if I'm supposed to add 9 plus 3 or multiply or divide …"). Other students conceived only of the rise, ignoring the run (e.g., "I made a line up to -3 and then counted over 3 from 0").

Another epistemological obstacle occurred in Task 3 (see Figure 2), where students were asked to identify the slope of a line graphed in a non-homogeneous system of coordinates. Task 3 brings in discussion the geometric perspective of slope (slope as a property of the line), and the analytical perspective of slope (slope as a property of the linear function) (Zaslavsky et al., 2002), and the difficulty to reconcile these two perspectives.

Task 3. The graph below represents the distance that a car travelled after different number of hours.
a) What is the speed of the car in part R?
b) What is the slope of part R of the graph?

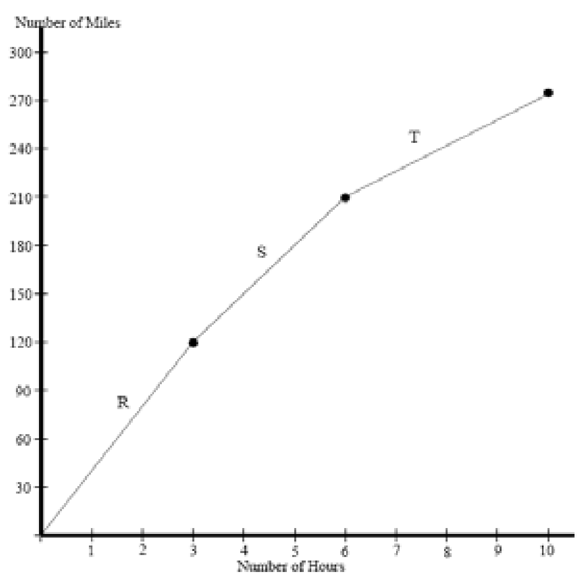

Figure 2: Task 3

Almost half of the students (N=696), correctly identified the speed of the car (*40 mph*). About half of the students who correctly identified the speed of the car failed to identify the slope of the line segment (N=336). Moreover, about 82% of those students who failed to identify the slope of the line graphed in a non-homogeneous system of coordinates correctly identified the slope of the line graphed in a homogeneous systems of coordinates in Task 2 (Postelnicu, 2011). One student expressed his difficulty this way: "In part *b* I didn't know whether to take from the values on the side or the units." During the interviews, only the students who held a geometric perspective and not an analytical perspective of slope failed to take into consideration the values of the function for rise ($y_2 = 120, y_1 = 0$) when calculating $y_2 - y_1$, and instead counted the "tick marks" on the *Y*-axis. They calculated the slope of part *R* of the graph as "rise over run." They found that the run, the line segment from the origin of the system of coordinates to (3, 0), had a length of 3 units on the *X*-axis, and the run, the line segment from the origin of the system of coordinates to (0, 120), had a length of 4 units on the *Y*-axis. They concluded that the slope of the line segment was 4/3 (rather than 120/3). When the analytic perspective of slope prevailed, and students used $\frac{y_2 - y_1}{x_2 - x_1}$ to calculate the slope, the interviewed students did not encounter difficulties.

OVERVIEW OF THE SECOND STUDY

The second study included 155 college students from a four-year programme at the university, enrolled in five undergraduate mathematics courses (Intermediate Algebra, College Algebra, Precalculus, Brief Calculus, and Discrete Mathematics [2]). A task sequence on aspects of the Cartesian connection was administered to all participants. After completing the sequence, all participants were asked to compare the tasks ("Are the tasks alike or different? Explain.") Semi-structured interviews (Goldin, 2000) were conducted with six students, randomly selected by the researcher. Three students had the total score above the group median, and three students had the total score below the group median. The interviewed students were asked to explain to an imaginary student not only how to solve the tasks, but also to justify why their algorithms work.

We discuss here two tasks:

Task 4. Find *m* and *b* such that the points $(1, 4)$ and $(-2, 10)$ are situated on the graph of the function $y = mx + b$.

Task 5. Determine *b* and *c* such that the points $(1, -3)$ and $(2, 0)$ are situated on the graph of the function $y = x^2 + bx + c$.

Both tasks are applications of the Cartesian connection, but the mathematical object upon which the students had to operate was different, a linear function in Task 4, and a quadratic function in Task 5, respectively. In a Piagetian sense (Piaget, 1971), we wanted to see whether our students' schemes of action were dependent on the mathematical object upon which they had to apply the Cartesian connection. In other words, our question was whether the students who applied the Cartesian connection in the case of the linear function from Task 4, could apply the same theorem in the case of the quadratic function from Task 5. The analysis of this epistemological obstacle had to be nuanced, because another question arose: Did the students who correctly solved Task 4 and/or Task 5 apply the Cartesian connection? To frame the discussion theoretically, Piaget (1971) discriminates between three types of knowledge: empirical knowledge abstracted from objects, pseudo-empirical knowledge abstracted from individual actions on objects, and reflective knowledge abstracted from coordinated actions on objects. The latter type of knowledge constitutes the basis for mathematical knowledge. The key difference between empirical and reflective abstractions is that the latter implies a projection from a lower to a higher level of knowledge, together with a reorganization of the entire knowledge in hierarchically superior structures. Given these theoretical considerations, we rephrased our question: What type of abstractions were employed by the students who correctly solved both Task 4 and Task 5? We considered evidence that reflective abstractions were employed if the student could generalize the Cartesian connection (e.g., a point (x_0, y_0) is on the graph of the curve *C* if and only if its coordinates satisfy the equation of *C*), apply it regardless of the appearance of the mathematical object *C* (e.g., symbolic representations like, $y = mx + b$ or $y = x^2 + bx + c$), and state it as the mathematical justification behind Tasks 4 and 5.

Only 31 students correctly solved Task 4. Eleven of the students who correctly solved Task 4, failed to solve Task 5. All the students who correctly solved Task 4, as well as the students who correctly solved both Tasks 4 and 5, compared the tasks by referring to the mathematical objects involved (e.g. linear functions, lines), and the actions performed (e.g., "You must use the slopes"). None of the students referred to the underlying mathematical justifications behind the task sequence, i.e., none of the students could justify their algorithms from a mathematical point of view. The interviews, coded (Strauss & Corbin, 1998) and analyzed using Piaget's theory of reflective abstractions, confirmed the procedural (instrumental) nature of students' knowledge (Skemp, 1976) and the lack of justificatory principles. Since none of the students could justify their successful algorithms, we concluded that no reflective abstractions were employed by college students.

DISCUSSION

Analyses revealed that students had difficulty with the Cartesian connection. Notwithstanding that the content of the tasks referring to the Cartesian connection varied across studies, the findings of the present studies and other studies (Lobato and Siebert, 2002; Schoenfeld et al. 1993; Stump, 2001; Zaslavsky et al., 2002) point toward epistemological obstacles.

Identifying the slope of a line graphed in a system of Cartesian coordinates seems to be a more difficult task than, for example, identifying the rate of change of a linear function when two instances of the varying quantities, (x_1, y_1) and (x_2, y_2) are given.

One explanation may be that the graphing context for slope implies knowledge of mathematical conventions (e.g., if the oriented segment representing the rise is toward the positive sense of the *Y*-axis, its magnitude has a positive sign).Thus, the difficulty in Task 2 in identifying the slope of a graphed line lies with the connection with the graphic representation. Another explanation may be that the complexity of the problem increased for those students with a geometric perspective of slope. Indeed, calculating the slope of a graphed line as “rise over run,” implies knowledge of proportionality in the geometric context of similar triangles, or at minimum, coordination while identifying the rise and the run from a “slope triangle.” By contrast, students with an analytical perspective of slope, who calculated slope as “change in *y* over change in *x*,” or used the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, had less difficulty since the connection with the graphic representation was reduced to identifying the coordinates of the points, (x_1, y_1) and (x_2, y_2).

Another possible explanation for student difficulties with slope may lie in the way this topic is presented in the U.S. mathematics curricula. For example, in one Algebra I textbook used by the first study’s participants, the slope is introduced in physical context, followed by the geometric context:

> Some roofs are steeper than others. In mathematics, a number called *slope* is a measure of the steepness of a line. The *slope of a line* is the ratio of *rise* to *run* for any two points on the line (Rubenstein et al., 1995, p. 361).

Usually, the geometric perspective of slope as “rise over run” precedes the functional perspective of slope as rate of change, the latter being introduced in advanced mathematics classes, like Precalculus and Calculus. The results of this study showed that from the interviewed students, only those with a geometric perspective of slope had difficulties determining the slope of a graphed line in Task 2.

Insufficient presentation of the mathematical conventions behind the graphic representation of functions may be another explanation for some of the student difficulties in Task 3. For example, in another Algebra I textbook used by the participants in the first study, the graph of a function is not defined explicitly. Students are reminded in Chapter 1 about the procedure to graph a function, and an example is given, with no reference to the units of measurement for the axes (e.g., scale for the *Y*-axis):

> When you are given the rule for a function, you can prepare to graph the function by making a table showing numbers in the domain and their corresponding output values […] Let the horizontal axis represent the input *t* (in minutes). Label the axis from 0 to 5. Let the vertical axis represent the output *h* (in feet). Label the axis from 0 to 400. Plot the data points given in the table. Finally, connect the points […] (Larson et al., 2004, p. 49).

Difficulty with identifying the scale of the *Y*-axis in a non-homogeneous system of coordinates was evident in Task 3. Students attempted to calculate the “rise” by counting the tick marks "by ones" for the *Y*-axis, instead of recognizing that each interval between two consecutive tick marks was 30 units.

Students had difficulty connecting between the symbolic and graphic representations in Tasks 1, 4 and 5. Knuth (2000) argued that the American curriculum and instructional approach emphasize symbolic representations and manipulations, although symbolic representations encapsulate knowledge that students cannot successfully unpack (Kaput, Blanton, & Moreno, 2008). The graphic approach is often rejected by students and their teachers because graphic representations are not precise, and could introduce estimates and inaccuracies (Arcavi, 2003). The Cartesian connection necessary to solve Tasks 1, 4 and 5 can only add to the difficulty. But even when successful at solving Tasks 1, 4 and 5, students' knowledge was procedural, and mirrored their textbooks' knowledge - a collection of algorithmic steps for solving problems, lacking the mathematical foundations that justify the use of algorithms (Harel & Wilson, 2011; Postelnicu, 2011; Wu, 2011).

In short, the association between the epistemological obstacles and the geometric perspective of slope held by some of the students who encountered difficulties suggests that the precedence of the geometric perspective of slope, as well as the reconciliation between the two perspectives of slope, geometric and analytical, together with the mathematical conventions behind the graphic representation of functions, need to be addressed in our curriculum. The college students' difficulty to advance their knowledge by expanding their successful schemes, suggests that perhaps, even more important to be addressed is the need for active mathematical principles and coherence in our curriculum.

NOTES

1. In United States, linear functions are part of the mathematics curriculum for Grades 8-10. The usual high school course sequence is Algebra I (Grade 9), Geometry (Grade 10), Algebra II (Grade 11), followed by advanced mathematics courses like Precalculus and Calculus. The current trend is to have Grade 8 students enrolled in Algebra I.

2. Upon college admittance, students may be tested and placed in remedial mathematics courses like Intermediate Algebra, College Algebra or Precalculus.

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MENTAL MATHEMATICS & ALGEBRA EQUATION SOLVING

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The reported study on mental algebraic equation solving is part of a larger research programme, aimed at better understanding the potential of mental mathematics activities with objects other than numbers. Through outlining the study details, and the activities students engaged with, I report on the variety of interpretations given to what solving an algebraic equation is. Focusing as well on the nature of students' engagement, I discuss some implications/potential for algebra teaching and learning.

CONTEXT OF THE STUDY

To highlight the relevance and importance of teaching mental calculations, Thompson (1999) raises the following points: (1) most calculations in adult life are done mentally; (2) mental work develops insights into number system/number sense; (3) mental work develops problem-solving skills; (4) mental work promotes success in later written calculations. These aspects stress the non-local character of doing mental mathematics with numbers, where the skills being developed extend to wider mathematical abilities and understandings. Indeed diverse studies show the significant impact of mental mathematics practices with numbers: on students' problem solving skills (Butlen & Peizard, 1992; Schoen & Zweng, 1986), on the development of their number sense (Boule, 2008; Murphy, 2004; Heirdsfield & Cooper, 2004), on their paper-and-pencil skills (Butlen & Peizard, *ibid.*) and their estimation strategies (Heirdsfield & Cooper, *ibid.*; Schoen & Zweng, *ibid.*). For Butlen and Peizard (*ibid.*), the practice of mental calculations can enable students to develop new ways of doing mathematics and solving arithmetic problems that the traditional paper-and-pencil context rarely affords because it is often focused on techniques that are in themselves efficient and do not create the need for doing otherwise. There is thus an overall agreement, and across contexts, that the practice of mental mathematics with numbers enriches students' learning and mathematical written work about calculations and numbers: studies e.g. conducted in US (Schoen & Zweng, *ibid.*), France (Butlen & Peizard, *ibid.*; Douady, 1994), Japan (Reys & Nohda, 1994), and UK (Murphy, *ibid.*; Thompson, 1999; Threlfall, 2002).

This being so, as Rezat (2011) explains, most if not all studies on mental mathematics focus on numbers/arithmetic. However, mathematics taught in schools involves more than numbers. This triggers interest in knowing what teaching mental mathematics with mathematical objects other than numbers might contribute to students' mathematical reasoning. In this study, issues of algebra equation solving are probed into. Thus, what learning opportunities solving algebraic equations mentally offer students? What mathematical activity do students engage in? What mathematical strategies emerge? This paper reports on the nature of strategies used by a group of 12 students, and the varied meanings developed about "solving algebraic equations".

THEORETICAL GROUNDING OF THE STUDY: AN ENACTIVIST FRAME

Recent work in mental mathematics points to the need for a better understanding and conceptualizing of how students develop mental strategies. Faced with significant varieties of students' creative solutions and with dissatisfaction about their "classification" in known and precise categories, researchers have begun to criticize the notion that students "choose" from a toolbox of predetermined strategies in order to solve problems in mental mathematics. Threlfall (2002) insists rather on the organic emergence and contingency of strategies in relation to the tasks *and* the solver (e.g. what he understands, prefers, knows, has experienced with those tasks, is confident with; see also Butlen & Peizard, 1992; Rezat, 2011). This view on emergence of strategies is also outlined in Murphy (2004), who discusses Lave's situated cognition perspective on mental strategies as flexible emergent responses, adapted and linked to a specific context and situation.

In mathematics education, the enactivist theory of cognition has been concerned with issues of emergence, adaptation and contingency of learners' mathematical activity (from the work e.g. of Maturana & Varela, 1992; Varela, Thompson & Rosch, 1991). Therefore, aspects of the theory are used to ground this study in its intention to make sense of students' strategy development and mathematical activity. In particular, Varela's (Varela et al., 1991) distinction between *problem posing* and *problem solving* offers a preliminary answer to questions about emergence and the characterization of strategies generated for solving tasks.

For Varela, problem-solving implies that problems are already in the world, lying "out there" waiting to be solved, independent of us as knowers. In contrast, Varela explains that we specify, we pose, the problems that we encounter through the meanings we make of the world in which we live, which leads us to recognize things in specific ways. We do not "choose" or "take" problems as if they were lying "out there," objective and independent of our actions: we bring them forth. The problems that we encounter and the questions we ask are thus as much a part of us as they are a part of our environment: they emerge from our interaction with it, as we interpret events as issues to address, as problems to solve. Thus, we are not acting on preexisting situations: our interaction with the environment creates the possible situations for us to act upon. The problems that we solve, then, are implicitly relevant for us because we allow these to be problems for us.

Hence it is claimed that reactions to a task do not reside inside either the solver or the task: they emerge from the solver's *interaction* with the task, through posing the task. If one adheres to this perspective, one cannot assume, as René de Cotret (1999) explains, that instructional properties are present in the (mental mathematics) tasks offered and that these will determine solvers' reactions. Strategies emerge in the interaction of solver and task, influenced by the task but determined by the solver's experiences and understandings: in his solving habits for similar or different tasks, in his successes in mathematics with specific approaches, in his understanding of the tasks, etc. In this perspective, the solver does not choose from a group of predetermined strategies to solve the task, but engages with the problem in a specific way and develops a strategy tailored to the task he poses.

Thus, students *transform* the mathematical tasks for themselves, making them their own, which is often different from the designer's intentions (René de Cotret, 1999); Heirdsfield and Cooper (2004) and Rezat (2011) have indeed shown the occasional futility in mental mathematics of varying the type of problem or its didactical variables to encourage students to use specific strategies. When solving tasks, students generate a strategy tailored to the problem (they) posed, as acts of posing and solving are not predetermined but generated in interaction with the task:

> As a result of this interaction between noticing and knowledge each solution 'method' is in a sense unique to that case, and is invented in the context of the particular calculation – although clearly influenced by experience. It is not learned as a general approach and then applied to particular cases. […] The 'strategy' […] is not decided, it emerges. (Threlfall, 2002, p. 42)

Students are then seen to generate their strategies in order to solve their tasks. These are adapted responses, locally tailored to the tasks, emerging in interaction with them.

THE STUDY – DEFINING MENTAL MATHEMATICS

Because most work on mental mathematics is on numbers (often referred to as *mental arithmetic* or *mental calculations*) and defined accordingly, no formal comprehensive definition of mental mathematics appears in the literature. Based on the work on mental calculations, one tentative definition is: Mental mathematics is the solving of mathematical tasks without paper and pencil or other computational/material aids. This definition helps understanding the "constraints" to which the students are subjected to, the major issue being that students do not have access to any material aid, be it paper-and-pencil or other, to depend on for solving the problems offered to them. This study focuses on (any of) the strategies produced in this context.

METHODOLOGY, DATA COLLECTION AND ANALYSIS

One intention of the overarching research programme is to study the nature of the mathematical activity students engage in through working on mental mathematics. This is probed through (multiple) case studies, taking place in diverse educative contexts designed for the study (classroom settings/activities). The reported study is one of those case studies, conducted in one university mathematics education course, aimed for because these students are not novice solvers in algebra, enabling a focus on their solving of algebra tasks (and less on their difficulties with algebra itself).

Classroom activities were designed to offer algebraic equations for students to solve mentally. A variety of usual algebraic equations of the form $Ax+B=C$, $Ax+B=Cx+D$, $Ax/B=C/D$, $Ax^2+Bx+C=0$ and their variants were presented. The classroom organization took the following structure: (1) an equation is offered orally or in writing on a transparency to the group; (2) students solve the equation mentally (without paper-and-pencil or material aids to solve or leave traces); (3) at the signal they write their answer on a piece of paper; (4) answers and strategies are orally shared, noted on the transparency. The data collected comes from the strategies orally explained (and noted on transparencies), as well as notes taken after the session.

The data was first looked at, analyzed, in relation to the nature of the strategies generated by students for solving the tasks. Because this analysis is dependent on the type of mathematical objects worked with through the classroom activities, available theoretical concepts found in the literature to guide and enhance the data analysis were used. In algebra, unwinding/undo procedures (Nathan & Koedinger, 2000) or transformation of equations (Arcavi, 1994), to name but two, are examples of relevant dimensions used for the data analysis. This analysis rapidly led to considerations of the meanings given to solving an algebraic equation, the focus of this paper (other analyses regarding the strategy dimensions are to appear in another paper). Following Douady (1994), the goal of this paper is not to report on all learning that took place for students, nor to discuss the long-term outcomes for students in other contexts, but mainly to understand the meaning and functionality of the tools used (i.e. strategies for mentally solving algebraic equations) and explore their potential. The focus in this paper is thus placed on the problem posing aspects, that is, the nature of the mathematical strategies engaged in and its repercussions on the meanings afforded to what solving an algebraic equation is (see Bednarz, 2001).

FINDINGS – MEANINGS FOR ALGEBRAIC EQUATION SOLVING

Through solving the various tasks offered to them, students gave, implicitly, different meanings to what solving an algebraic equation represents. Those meanings are mathematically rich and contribute to deepen understandings of what solving an algebraic equation is. I outline below these various meanings.

***Meaning 1*: finding the value(s) that satisfy, make true, the equality**

Underneath this meaning is the notion of a conditional equality, where it is not only the idea of finding the answers/values that make the equation true, but also the fact that the equality *can* be true or untrue.

When students were given $5x+6+4x+3=-1+9x$ to solve, some rapidly asserted that there was no solution, because one can rapidly see $9x$ on both sides of the equation as well as the fact that the remaining numbers on each sides do not equate. It thus leads to the conclusion that there was no number that could satisfy the given equation, since no x, whatever it could be, could succeed in making different numbers equals. This strategy is related to what is often termed “global reading” of the equation (Bednarz & Janvier, 1992), that requires consideration of the equation as a whole prior to entering in algebraic manipulations, or what Arcavi (1994) calls *a priori* inspection of symbols, which is a sensitivity to analyze algebraic expressions before making a decision about their solution. (Arcavi gives the example of

$(2x+3)/(4x+6)=2$, which has no solution because whatever the value of x, the numerator is worth half the denominator, making futile undergoing additional steps.)

Another strategy students engaged in was one of "solving followed by validation". When having to solve $x^2–4=5$ one student rapidly transformed it into $x^2=9$, obtaining 3 as an answer. However, because he is in a mental mathematics context and is aware that his answers in this context are often rapidly enunciated and can lack precision, he decides to verify his answer. By mentally verifying if $(3)^2=9$, he realizes that $(–3)^2$ also gives 9 and then readjusts his solution. This manner of solving the equation gets close to the idea not only of finding one value that makes the equation true, but also of finding *all* values that make it true.

***Meaning 2*: deconstructing the operations applied to an unknown number**

This meaning requires reading the equation as a series of operations applied to a number (here x) and attempting to undo these operations to find that number.

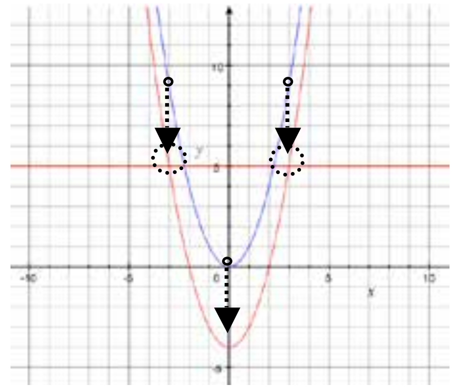
When having to solve equations like $x^2–4=5$, students would say: "My number was squared and then 4 was taken away, thus I need to add 4 and take the square root". Or, for $4x+2=10$, "What is my number which after having multiplied by 4 and added 2 to it gives me 10?" These are similar to inverse methods of solving found in Filloy and Rojano (1989) and Nathan and Koedinger's (2000) "unwinding", where operations are arithmetically "undone" to arrive at a value for x. As Filloy and Rojano explain, when using this method "it is not necessary to operate on or with the unknown" (p. 20), as it becomes a series of arithmetical operations performed on numbers. In this particular case, solving the algebraic equation is focused on finding a way to arrive at isolating x, in an arithmetic context.

Meaning 3*: operating identically on both sides to find *x

This meaning focuses on the idea that is often called "the balance" principle, where one operates identically on both sides of the equation to maintain the equality and obtain "x=something". For example, when solving $2x+3=5$, students would subtract 3 on each side and then divide by 2.

***Meaning 4*: finding points of intersection of a system of equations**

This is about seeing each sides of the equality as representing two functions, and thus attempting to solve them as a system of equations to find intersecting points, if any. For example, when solving $x^2–4=5$, some students attempted to depict the equation as the comparison of two equations in a system of equations ($y=x^2–4$ and $y=5$) and finding the intersecting point of those two equations in the graph. To do so, one student represented the line $y=5$ in the graph and then also positioned $y=x^2–4$. The latter was referred to the quadratic function $y=x^2$, which crosses $y=5$ at $x=\sqrt{5}$. In the case of $y=x^2–4$, the function is translated of 4 downwards in the graph, and then the 5 of the line $y=5$ becomes a 9 in terms of distances. Hence, how does one obtain an image of 9 with the function $y=x^2$? With an $x=\pm3$, where the function $y=x^2–4$ cuts the line $y=5$. The following graph offers an illustration of what the student did, mentally.



Solving an algebraic equation in this case is not about finding the values that make the equation true, but about finding the x that satisfies both equations for the same y, about finding the x coordinate that, for the same y, is part of each function

***Meaning 5*: finding the values that nullifies the equation**

This meaning focuses on the equal sign as giving an answer (see e.g. Davis, 1975), but where operations are conducted so that all the "information" ends up being on one side of the equation to obtain 0 on the other side. The intention then becomes to find the value of x that nullifies that equation, that is, that makes it equal to 0.

One example of a strategy engaged in was again about seeing the equation in a function view as in meaning 4, but here for finding the values of x that give a null y-value, or what is commonly called finding the zeros of the function where the function intersects the x-axis at y=0. For x^2–4=5, transformed in x^2–9=0, the student aimed mentally at solving (x+3)(x–3)=0, leading to ±3. The quest was mainly finding the values that nullify the function $y=x^2-9$, which gave point(s) for which the image of the function was zero. Another way of doing it, less in a function-orientation, is to use "binomial expansion" (what is called in French *identités remarquables*) for seeing that for the product to be null it requires that one of the two factors be null. This said, one needs to use neither a function nor binomial expansion to find what nullifies the equation. For example, if x+4=3 is transformed in x+1=0, one finds that –1 is what makes the left side of the equation equal to 0.

***Meaning 6*: finding the missing value in a proportion**

This meaning was engaged with for equation written in fractional form (e.g. Ax/B=C or Ax/B=C/D). In these cases, the equation was conceived as a proportion, where the ratio between numerators and denominators was seen as the same or consistent. In this case, the equality is not seen as conditional but is taken for granted, to be true, leading at conserving the ratio between numerator and denominator in the proportion.

For example, for $\frac{6}{x}=\frac{3}{5}$, reversed to $\frac{x}{6}=\frac{5}{3}$, students solved by saying "If my number is 6 times bigger than x/6, then it is 6 times bigger than 5/3". Another way offered was to analyze the ratio between each numerators and apply it to denominators which had, in order to maintain the equality, to be of the same ratio: "If 6 is the double of 3, then x is the double of 5 which is worth 10".

Meaning 7: finding equivalent equations

This meaning for solving the equation is oriented toward obtaining other *equivalent* equations to the first one offered, in order to advance toward an equation of the form "*x*=something". This is related to Arcavi's (1994) notion of knowing that through transforming an algebraic expression to an equivalent one, it becomes possible to "read" information that was not visible in the original expression. Through these transformations, the intention is not directly to isolate *x*, but to find other equations, easier ones to read or make sense of, in order to find the value of *x*.

An example of such was done when solving $\frac{2}{5}x=\frac{1}{2}$, where some students doubled the equation, obtaining $\frac{4}{5}x=1$, which was simpler to read and then multiplied by 5/4 to arrive at *x*=5/4. This is an avenue also reminiscent of arithmetical divisions, where equivalences are established: e.g. 5.08÷2.54 is equivalent to 508÷254, because 254 divides into 508 the same number of times as 2.54 into 5.08.

Similarities and differences in meanings attributed

Albeit treated separately, these varied meanings are not all different and some share attributes. Therefore, in addition to the variety of meanings, significant links can be traced between those, links that can deepen understandings about algebraic equation solving. For example, meanings 2 and 3 share an explicit orientation toward isolating *x*, where others do not have this salient preoccupation and focus on other aspects (satisfying the equality, finding points of intersection, etc.). Meanings 4 and 5 share a function orientation in their way of treating the equation, emphasizing each part of the equation as representing an image (or simply the value of the function).

Many meanings also focus implicitly on conditional orientations, be it concerning the satisfaction of the equality or simply the possibility of finding a value for *x*. For example, in meaning 2 and 3, it is possible that no value of *x* is found and the same can be said for meaning 4, where it is possible that there be no point of intersection of the two equations or for meaning 5, where possibly no value of *x* could nullify the left side of the equation (e.g. $x^2+\sqrt{2}=0$). Without being explicit about it, these orientations represent a quest for finding a possible value, a quest that can be unsuccessful. This contrasts heavily with meaning 6, because treating the equation as a ratio assumes or implies that a value of *x* exists. Meanings 1 and 6 however do share something in common, which is related to an examination of relations between the algebraic unknown and the numbers in order to deduce the value of the algebraic unknown. Both do not opt for a sequence of steps to undertake, but mainly for working with the equation as a whole (in global reading for meaning 1, in ratios for meaning 6). Meanings 3 and 7 share the fact that operations are conducted on the equation as a whole, be it through affecting both sides in the same way to keep the "balance" intact or to obtain new equivalent equations.

Finally, meanings 1, 2, 5 and 6 share the fact that they explicitly look for a number, where the algebraic unknown is conceived as an unknown number that needs to be found; a significant issue to understand when solving algebraic equations (Bednarz & Janvier, 1992; Davis, 1975). Hence, be it through looking at which number could satisfy the equation (meaning 1), which number could nullify a part of it (meaning 5), which number satisfies the proportion (meaning 6) or which is the number for which operations were conducted (meaning 2), all of them focus on *x* as being a number.

DISCUSSION OF FINDINGS

On the emergence of mental mathematics strategies

The variety of meanings brought forth through students' solving illustrates well how the various "posing" of the problems led to various strategies for solving and thus to various meanings attributed to algebraic equation solving. Each equation provoked numerous strategies for solving it, leading to numerous meanings attributed to algebraic equation solving. Thus, the same equation made emerge a variety of posings, of strategies, of meanings. This supports the view that strategies for solving emerge in the interaction of solver and task, where the solver plays an important role as he poses the tasks, and where the nature of the task plays a role as well, with a strategy tailored to it (see e.g. the impact of fractional or second degree forms on the nature of the strategies). Strategies emerge contingently where, as Davis (1995) explains, they are inseparable from the solver and the task, emerging from their interaction. Thus, building on Simmt's (2000) work, one could begin conjecturing that the tasks given were not tasks but mainly prompts for solvers to create tasks with: prompts were offered to students, not tasks. Tasks became tasks when students engaged with them. Students *made* the equations ones about system of equations, about functions, about ratio, etc., allowing a variety of meanings for algebraic equation solving to emerge along the way. This however merits more precise attention and further research investigating the processes of solving in a mental mathematics contexts are definitely needed (see e.g. my arguments about these processes and their emergence in Proulx, in press).

On the potential of mental mathematics for algebra

This variety of meanings, emerging with/in students' posing, shows promise for algebra teaching and learning. These meanings are significant, because they offer different entry paths into the tasks of solving algebraic equations and do not restrict a single view of how this can be done. Numerous authors have outlined difficulties experienced by solvers (from school to university) in solving algebraic equations (see Bednarz, 2001; Filloy & Rojano, 1989; Nathan & Koedinger, 2000). The emergence of this variety of meanings offers significant reinvestment opportunities for pushing further the understanding of algebraic equation solving in mathematics teaching. It opened spaces of exploration that can be taken advantage of in teaching, in order for example to unearth the various meanings given to solving algebraic equations or similarities and differences between those. Issues of conditional equations, of deconstructing an equation regarding operations done on a number, of maintaining the balance, of finding equivalent equations, of seeing an equation as a system of equations, and so forth, offered varied ways of conceiving an equation and of solving it. It opened various paths of understanding. However, more research is needed to investigate how these reinvestments can be done in classroom contexts (with students or teachers), as well as the sorts of understandings students/teachers gain from this.

For example, and without making paper-and-pencil a straw-men for criticism, the mental mathematics context can be seen to have provoked some strategies and meanings different than ones used in the usual written context for solving equations in algebra. An example is the transforming of equations into equivalent ones (e.g. $\frac{2}{5}x=\frac{1}{2}$ to $\frac{4}{5}x=1$). This reasoning is at the basis of solving equations in writing. But, it was here quite different than the usual transformations applied to equations, since the equation was not transformed in order to isolate x, but mainly to obtain other equations that were easier to read. In fact, even if many strategies were reminiscent of paper-and-pencil work, the major difference is that there was no paper-and-pencil work, leading to a dialogue taking place between the student and the task, *while* solving. Because students could not leave written traces or transform the equations in writing, and thus could not interact with what was obtained after each written step through manipulating the equation, the monitoring of the solution was done in real-time, through personal dialogue, through a story told in which the solver engaged through telling the story, to keep track of the operations being conducted and their adequacy. In short, students had to invent (to pose) their stories about the problem, to interpret the equation in their own terms, in order to find a way to solve it. These "steps", oriented by the task at hand, oriented in return the next steps. Solving was done through the action of solving, and not through applying a known procedure, making it quite a different solving experience. Important questions that can be asked at this point is: What can be gained through this experience of solving algebraic equations mentally? What avenues of solving can be developed through these experiences, be them similar or different than the ones in paper-and-pencil contexts? More research is needed along those lines, but already this emerging variety of meanings shows important promise of mental mathematics for enriching algebraic experiences of students and teachers.

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THE EFFECTS OF A TEACHING METHOD SUPPORTING METACOGNITION ON 7TH GRADE STUDENTS' CONCEPTUAL AND PROCEDURAL KNOWLEDGE ON ALGEBRAIC EXPRESSIONS AND EQUATIONS

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The purpose of this study was to investigate the effects of a teaching method supporting metacognitive strategy use on improving seventh grade students' conceptual and procedural knowledge on algebraic expressions and equations. The study was conducted in two seventh grade classes from a public school in the 2010–2011 academic year. A quasi-experimental design with pretest - posttest control group was utilized for the study. A Conceptual Knowledge Scale and Procedural Knowledge Scale were conducted with control and experimental groups before and after the implementation of the teaching methods. Data were analyzed using t-tests and a Mann Whitney U test. Results showed that there was a significant difference between experimental and control groups in terms of gain scores on conceptual and procedural knowledge in favor of experimental group.

Key Words: *Elementary Mathematics Education, Conceptual Algebra Knowledge, Procedural Algebra Knowledge, Metacognition*

INTRODUCTION AND RELATED LITERATURE

Algebra is a branch of mathematics, which turns relations examined by using symbols and numbers to generalized equations. Not only does it represent letters and quantities, it also allows calculations using these symbols at the same time (Kieran, 1992). Ideally, algebra lessons lead students to develop a profound understanding of algebraic concepts and the ability to see algebra as a central and connected branch of mathematics and the ability to apply algebra to a wide range of topics. If this happens, then students can be said to have a high algebraic competency (Oldenburg, 2009).

There are two concepts underlying algebraic expressions and equations. These concepts are "variable" and "equality". A variable concept is usually expressed through literal symbols. According to Philipp (1992), literal symbols have many different uses as label, constants, unknowns, generalized numbers, varying quantities, parameters, and abstract symbols. Equal sign is used for various meanings in algebra. Research demonstrated that children perceive the equal sign as a symbol indicating action instead of a relational symbol (Kiearan, 1992; Yaman et al., 2003). Küchemann (1981) determined that children have 6 different perceptions regarding letters. These are; 1) letters have numerical values, 2) letters can be ignored in a particular task, 3) letters are abbreviations of concrete objects, 4) letters are unknown numbers and they have only one value, 5) letters are generalized numbers and have only one value, and 6) letters are variables.

Dede et al. (2002) proposed reasons for the difficulties students experience in learning algebra as follows: not knowing about different uses of variables, not knowing about the role of variables in making generalizations, not being able to interpret variables, and failure to perform operations with variables. Baki (1998) listed students' misconceptions as errors in inclusion in parentheses and using operators, carelessness, and turning non-numerical expressions into algebraic expressions. Perso (1992) grouped the misconceptions in algebra under three main headings: the location of the letters, use of variables, and algebraic rules.

In order to address misconceptions in learning algebra, students need to understand concepts like variable, equation, and have preliminary knowledge such as arithmetical operation knowledge. Algebraic comprehension depends not on knowledge by the students of the formulas and understanding the calculations correctly, but instead, understanding of the concepts and operations, and development of mathematical thinking. Therefore, importance should be attached to concepts and relations instead of the procedural means of solution, and learning should be realized through conceptual learning that involves the knowledge of operations and concepts in a balanced manner (Baki & Kartal, 2004).

In sum, research reviewed above shows that students have difficulties in understanding the concepts of variable and equation, forming and solving algebraic equations, using algebraic expressions, and in algebraic problem solving (Baki & Kartal, 2004; Dede & Peker, 2007; Herscovics, 1989; Kieran, 1992; MacGregor & Stacey, 1993).

CONCEPTUAL AND PROCEDURAL KNOWLEDGE

Researchers describe two kinds of knowledge in learning mathematics; conceptual knowledge and procedural knowledge (Van de Walle, 2004). Conceptual knowledge can be defined as any concept, rule, generalization and the relation between them (Hiebert & Lefevre, 1986; Rittle- Johnson & Alibali, 1999). Procedural knowledge consists of two parts. The first comprises the symbols and language of mathematics. The second relates to the rules, algorithms or procedures used to solve mathematical tasks (Hiebert & Lefevre, 1986).

When the algebraic knowledge of students is examined in the context of conceptual and procedural knowledge, it can be seen that it is not based on conceptual learning where conceptual and procedural knowledge are balanced (Baki & Kartal, 2004).

METACOGNITIVE INSTRUCTION

A constructivist learning approach advocates that knowledge is not independent from the learner, and that the individual constructs knowledge him/herself in his/her mind (Olkun & Toluk, 2003). In orser to construct mathematical concepts and thoughts in their mind in a meaningful manner, students should have skills such as monitoring and regulating their own thought processes and mental activities as well as self-control of learning. These abilities are defined as metacognitive skills. Metacognition means one's awareness of one's own thought processes, and one's ability to control those (Beauford, 1996; Brown, 1978; Flavell, 1979).

Instructional practices such as writing, thinking aloud, using behavior cards (Demircioğlu, 2008; Pugalee, 2004; Özsoy, 2007); promoting learning environments that are conducive to the construction and use of metacognition (Schraw, 1998); supporting interactive problem solving (Kramarski, Mevarech, Liebermann, 2001; Schraw, 1997); asking reflective questions (Mayer, 1998; Schoenfeld, 1985); using control lists (Schraw, 1998) are used to improve students' metacognitive skills.

It has been shown that children who are given the opportunity to develop metacognitive skills experience positive and meaningful increases in mathematical success (Naglieri & Johnson, 2000; Özsoy, 2007; Teong, 2002). Even though research has shown a positive impact of metacognitive skills (prediction, planning, monitoring, and evaluation) on mathematical success, little is known about the effect of a metacognitive instruction on students' conceptual and procedural knowledge of algebra, particularly on algebraic expressions and equations.

PURPOSE OF THE STUDY

This research aims to examine the effect of a teaching method supporting the use of metacognitive strategies on conceptual and procedural knowledge of students in the area of algebraic expressions and equations. Moreover, we were interesting in finding out whether a focus on conceptual or procedural knowledge had greater effect.

METHODS

A quasi experimental design with pretest-posttest control groups was used in the study. Participants consist of 80 seventh grade students attending a public school in one of the low socio-economic level districts of the Istanbul province. Among the four seventh grade classes taught by the first researcher, two of them were arbitrarily selected. One of these classes was randomly assigned as the experimental group while the other group was assigned as control group. The experimental and control groups showed a balanced distribution in terms of gender. Both groups also showed no meaningful difference in terms of preliminary test scores assessing their conceptual and procedural knowledge on algebra.

The implementation was held in the fall semester of the 2010-2011 educational year, and lasted for a total of 6 weeks (24 hours) for both groups. The first researcher, who is the teacher of the experimental and control groups, taught the classes of both groups throughout the research. Before starting the process, the experimental group students were informed about metacognition over the course of 2 lesson hours, and were asked to implement metacognitive strategies throughout the process. In order for the students to develop these strategies, pre-implementation lessons were taught to the experimental group for two weeks using metacognitive strategies and with writing exercises.

Instruction supporting the use of metacognitive strategies was applied to the experimental group. This process included teaching through structured applications based on metacognition together with problem-based learning activities used and suggested in the research (Goldberg & Bush, 2003; Kramarski et al., 2001; Özsoy, 2007; Schraw, 1998; Schoenfeld, 1985; Wilburne, 1997). The lessons involved the exercises of thinking aloud, solving problems with groups of two, class discussions, writing, and keeping learning diaries. Students were expected to use "Problem Solving Metacognitive Behavior List", adapted from Goos et al. (2000) while solving problems and "List of The Explanation of Thinking Process", adapted from Beyer (1988) while explaining their thinking processes. The mathematical content of the activities in the experimental group and the control group was the same. No different content was provided to the experimental group than to the control group. The only difference was that students in the experimental group engaged in practices around revealing thought processes and supporting metacognitive skills.

In thinking aloud exercises, students were requested to think (regarding targets, plans, strategies, etc.) and make decisions aloud. In solving problems (with two exercises), one of the students talked about his/her problem solution (what he/she understood from the problem, solution plan, etc.) and his/her friend asked questions that would clarify his/her thinking process. They summarized and reviewed the process from time to time and checked their comprehension. When the thinking processes of the group were shared through class discussion, alternative solution methods were shared and the most suitable solution method for the problem was discussed. Writing studies involved predictions regarding easiness or difficulty of the problem and the time needed to solve it, planning regarding solution of the problem, operations during the solution, views regarding the decisions taken and evaluation of the solution process. The teacher's task was to make sure that the students used metacognitive steps and explained their thinking processes. The teacher asked several questions in order to improve students' predictions, planning, monitoring and evaluation skills such as "Can you solve this problem? How long does it take you to solve the problem? How are you going to solve the problem? What are you doing now? Can you summarize what you have done so far? What you should do after this point? How are you doing it? Is this method going to work? Do you think you have done everything correctly? Do you think another method can be tried? What is the best solution method?" An example of a writing study related to acquisition of "Explains linear equations" is shown below:

"Please solve the problem below, writing down your thinking processes. Construction equipment, which has 250 litres of diesel in its tank, consumes 10 litres of diesel per hour;

a) Please write down the equation of the relation between the amount of diesel and working time.

b) How many litres of diesel are left in the tank of this construction equipment?
c) For how many hours has this construction equipment worked when there are 140 litres of diesel in this construction equipment's tank?
d) How many hours does this construction equipment need to work to finish all the diesel in its fuel tank?
e) Since this construction equipment can work 5 hours per day at the most, how many days are needed to finish all the diesel in its fuel tank?"

In the control group, student-focused, ordinary instruction was implemented in line with the educational program. All the problems solved throughout the activities in the experimental group were also solved in this group. Active participation of the students was ensured and methods like question-answer, discussion were used. However, metacognitive activities such as writing, thinking aloud, keeping a learning diary and solving problems with groups of two were not used in this group. Teachers and students solved the questions without explicitly expressing their own thinking processes. Since much time was spent in clarification of the decisions made during the solution process and exploring alternative thinking processes in the experimental group, there remained problems that could not be solved during activities and these were given as homework to the students. In the control group, however, there was no such time problem and all of the problems planned for lessons were solved.

Two hour classes in the experimental and control group were observed by the other maths teacher of the school and notes were taken regarding the functioning of the classes. These notes were used for the purpose of collecting information on whether the teaching processes in the experimental and control groups were conducted as planned.

A Conceptual Knowledge Scale (CKS) formed using the body of literature was applied to the study for the purpose of measuring the conceptual knowledge of the students on algebraic expressions and equations (Akkuş, 2004; Hart, et al., 1985). The test involves different uses of the variable. Correct answers count as 1 point and wrong answers count as 0 point for every item. The KR-20 reliability coefficient of the scale consisting of 64 items with sub-problems was found to be 0.86. Some examples of the test items are shown below:

Item 3) Which one of the "$2n$" and "$n+2$" statements is greater in which case? (n is a natural number.)

Item 14) When is the expression $k+m+n=k+n+p$ correct?
a) Always b) Never c) Only when $m=n$ d) Only when $m=p$

A Procedural Knowledge Scale (PKS) formed using the body of literature was applied for the purpose of measuring the procedural knowledge of the students on algebra (Akkuş, 2004). The test consists of 17 free-response questions, which involves routine algebra problems requiring symbolic manipulation and calculations. A five-point rubric (0-4) was used to score the students' responses. Cronbach alpha coefficient of the scale was found to be 0.94. A t-test was administered and α value was taken as 0.05. Some examples of the test items are shown below;

Item 4) $\frac{3x}{2}+7=x+1$ Please solve the equation of the first degree with one unknown.

Item 10) Please find the value of y for x=0 on the line $3x+2=y$.

Both of the scales were arranged to measure conceptual and procedural knowledge on seventh grade algebraic expressions and equations in line with the educational program (addition, subtraction, multiplication; solution of 1st degree equations with one unknown; problem solving with equations, linear equations and charts).

RESULTS

A comparison of the experimental and control groups in terms of their mean scores from CKS shows that their pre test mean scores are low with low variability ($\bar{x}_{\text{Experimental}} = 10.73$, $\bar{x}_{\text{Control}} = 11.13$) but there is an evident difference in favor of the experimental group intheir posttest mean scores ($\bar{x}_{\text{Experimental}} = 28.63$, $\bar{x}_{\text{Control}} = 16.93$). However, considering the fact that the highest score achievable from CKS was 64, it was observed that the means of both groups were low.

Using a t-test, a comparison of the means of the gain scores ($\bar{x}_{\text{Experimental}} = 17.90$, $\bar{x}_{\text{control}} = 5.80$) of the two groups revealed a significant statistical difference in favor of the experimental group (t (78) = 5.701; p<0.05) (Table 1). Cohen d value 1.28 shows high effect size (Cohen, 1988, as cited in Gravetter & Wallnau, 2004).

Group	n	$\bar{X}$	s	df	t	p
Experimental Group	40	17.90	7.90	78	5.701	p<0.001
Control Group	40	5.80	10.85			

Table 1: CKS Gain Scores of the Experimental and Control Groups

A comparison of the experimental and control groups in terms of their mean scores from PKS shows, similar to the previous finding, that their pretest mean scores are low with low variability ($\bar{x}_{\text{Experimental}} = 2.80$; $\bar{x}_{\text{Control}} = 3.80$), while there is an evident difference in favor of the experimental group in their post-test mean scores ($\bar{x}_{\text{Experimental}} = 28.05$; $\bar{x}_{\text{Control}} = 14.25$). However, considering the fact that the highest score achievable from PKS was 68, it was observed that the means of the groups are low. Moreover, the standard deviation values are quite high in the distribution of the post test scores ($s_{\text{Experimental}} = 17.51$; $s_{\text{Control}} = 15.32$). This shows that the implementations might have had different effects on the procedural knowledge of the students with different personal characteristics.

A comparison of the means of the gain scores ($\bar{x}_{\text{Experimental}} = 25.25$, $\bar{x}_{\text{Control}} = 10.45$) of the groups revealed a significant statistical difference in favor of the experimental group (t (78) = 4.633; p<0.05) (Table 2). Cohen d value 1.04 shows high effect size.

Groups	n	$\bar{X}$	s	df	t	p
Experimental Group	40	25.25	15.55	78	4.633	P<0.001
Control Group	40	10.45	12.90			

Table 2: PKS Gain Scores of the Test and Control Groups

To examine whether or not the use of metacognitive strategies is more effective on the conceptual knowledge or procedural knowledge gains, the scores from the two scales were transformed to a 100-score scale, and the gain scores of the experimental group were recalculated (Table 3).

Group	Test	n	$\bar{X}$	s
Experimental Group	CKS	40	27.97	12.34
	PKS	40	37.13	22.87

Table 3: Gain Scores of the Experimental Group on CKS and PKS (Descriptive Statistics)

Because the variances of the distribution of test scores are not homogenous, Mann Whitney-U test was used for unrelated measurements, and no statistically significant difference was found between the conceptual and procedural knowledge scale scores of the experimental group (U = 610.50; p>.05).

Test	n	Mean Rank	Sum of Ranks	U	p
CKS	40	35.76	1430.50	610.50	.068
PKS	40	45.24	1809.50		

Table 4: Mann Whitney-U Test Results for CKS and PKS Gain Scores of the Experimental Group

CONCLUSION

The first two findings of this research showed that both the conceptual and procedural knowledge gains of the experimental group were significantly higher than the conceptual and procedural knowledge gains of the control group. Considering these findings, it can be said that the educational method supporting metacognition strategies is an effective method for developing the students' understanding of concepts of equation, unknown and equality concepts included in the subject of algebraic expressions and equations, and developing their skills such as forming and solving equations based on such concepts, and for problem solving via equations. These findings are consistent with the prior research findings supporting metacognitive instruction (Mevarech & Fridkin, 2006; Pilten, 2008).

The third finding of the research shows that there was no statistically significant difference between the average of the procedural gain scores and the conceptual gain scores in the method supporting the use of metacognitive strategies. It has been shown in research investigating the algebraic knowledge that students lack conceptual knowledge that their procedural knowledge is not supported by conceptual knowledge (Baki & Kartal, 2004; Bekdemir & Işık, 2007). The finding of the study can be interpreted as meaning that the conceptual algebraic knowledge of the students increases with their procedural algebraic knowledge, and that conceptual knowledge lays foundations for procedural knowledge.

The scores of the experimental and control groups in the conceptual and procedural knowledge scale before the implementation (pretest scores) were quite low, -particularly the pretest scores in the procedural knowledge scale. This might be attributed to the nature of the scales. Each item in the procedural knowledge scale relates directly to the concepts taught in seventh grade algebraic expressions and equations. The conceptual knowledge scale similarly involves questions containing different uses of the variable and measuring conceptual knowledge of fundamental algebra in addition to questions directly related to gains. The students may have had difficulty responding to questions directly related to content that they had not previously been taught.

It was found out that the average of both conceptual and procedural knowledge gain of the experimental group was meaningfully higher than the control group, and the effect of the implementation (Cohen d) was found to be high. Nevertheless, the success of the experimental group was low considering the final test scores. This may relate to low general success and insufficient knowledge and skills of the students respectively in mathematics and in fundamental mathematics. Besides, the study was limited in some aspects. Most importantly, the delayed test practice could not be held to provide greater evidence regarding the effectiveness of the implementation on algebra. Although the effect of the implementation was found to be high, no finding on the permanence of the acquired algebraic knowledge could be obtained.

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RELATIONAL UNDERSTANDING WHEN INTRODUCING EARLY ALGEBRA IN PORTUGUESE SCHOOLS

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In this paper we describe the findings of a research study about improvements and difficulties found by young Portugal children when using numerical equalities. The study was conducted at the time when a new curriculum was introduced in Portugal. We found that teachers introduced some general perspectives by using examples of algebraic properties, but that teachers generally do not appear to focus in depth on their relational meaning of equalities. The students' results were better than in previous international studies. This may partly be due to the introduction of relational examples in the new curriculum. Nevertheless, the results reveal that Portuguese students still encounter difficulties concerning relational understanding and the meaning of equivalence more generally.

INTRODUCTION

The theme of this work arose from the understanding of the importance of the equality relationship in the development of algebraic thinking, particularly in terms of approaching relational variability (Falkner, Levi and Carpenter (1999). For many years, it has been known that, when presented with the classic example to insert the missing number in the number sentence 18 + 27 = _ + 29, many students answer by writing 45 in the space. So, many children see the equal sign as an instruction 'to do', rather than as a symbol representing a relation (Carpenter, Franke and Levi 2003).

Taking into account this situation, and considering that there was a new curriculum in Portugal in which algebraic thinking was explicitly included in Primary schools for the first time, we conducted a pilot research study to analyze the possible positive influences of such a curricular decision, at the end of the first year of implementation. In fact this curriculum stresses relations between numbers and between numbers and operations (DGIDC, 2007). Learning frameworks and programs and aligned professional development initiatives now focus on patterning and structural relationships in mathematics, including equivalence, growing patterns and functional thinking.

Our main research question was *if, with the introduction of the new program, we can find some anticipatory results on questions involving equalities as well as identifying some trends of relational thinking among students.* Fundamentally, Years 2-6 are critically important for developing and consolidating the early understanding of structural relationships developed. Since we could not find any study in Portugal on this issue, prior to the introduction of the aforementioned program, we chose to compare results with studies in other countries.

EQUIVALENCE RELATIONS AND VARIABILITY THINKING

Researchers agree that early algebra involves more than the generalization of arithmetic structures. Some argue that, in addition, understanding of early algebra involves generalized arithmetic, functional thinking and modeling (Blanton & Kaput, 2004), whilst others emphasize three main aspects of algebraic thinking: equality, change and generalizations (Zevenbergen, Doole & Wright, 2004). Warren (2003) extends the understanding of early algebra by identifying four central aspects: (1) Relationships between quantities, (2) Group properties of operations, (3) Relationships between the operations, and, (4) Relationships across the quantities. Research in number properties and operations articulates the important connection between general structural principles underlying numerical relationships and study of invariant properties (Smith, 2011) that assume a special importance on the understanding of some more advanced concepts in Algebra. Schliemann, Carraher and Brizuela (2007) assert that students' difficulties can be derived from the way they were taught during the first years of schooling, because the equal sign is usually introduced as directional operative with the meaning of 'gives' or 'result'. Moreover, since most textbooks limit the tasks presented to students to an operational type (Smith, 2011), it is not surprising that students develop an operational, and not a relational, notion of equality.

We agree with Kieran (2004) that in the transition from arithmetic to algebra, students need to make adjustments in the way they think, including those students who are quite competent in arithmetic. In the initial levels there is a strong emphasis on obtaining the answer rather than on the representation of relations. Confronted by an equal sign, students tend to see it as a frontier between problem and solution with a left-right direction. To counter this tendency, Kieran suggested five ways to develop algebraic thinking and relational understanding: (1) Focus on relations, rather than on calculating an answer, (2) Focus on the inverse operations, not only on the operations themselves, and on the ideas of doing and undoing that are part of the process, (3) Focus as much on the representations as on problem solving, not just on solving. (4) Focus both on the letters and on the numbers, instead of only on the numbers, including working on letters that can be unknowns, variables and parameters, and, (5) Accept open literal expressions as answers and compare expressions for equivalence based on properties; refocus on the meanings of equal.

Following such a perspective, Molina, Castro and Ambrose (2006) conducted research with 3rd grade students aimed at promoting algebraic thinking. Their results indicate that all students showed an operational interpretation of the equal sign. However, when offered tasks of sufficient variety, students progressively evolved to a relational interpretation.

Our current research study was designed to examine whether similar improvements took place in Portugal following the introduction of a national program in which relational perspectives are specifically introduced..

EMPIRICAL STUDY

To observe the possible influences of the new early algebra perspectives to overcome operational perspectives, we have constructed a pilot task involving open numerical equalities. This task was given to a purposively chosen sample of students from the 1st to the 6th graders of a group of schools in July 2011, after the first year of the implementation of the new curriculum. The sample consisted of all students from 6 different schools that were engaged in extra curricular activities during the school holidays. It is usual in Portugal that, for some weeks after the end of the academic year, many schools are open to help some families with financial difficulties. Hence, the sample was selected not for cognitive reasons, but solely for socioeconomic reasons. No students from the third and fifth grades were in the sample because of this situation.

Our study was implemented in a group of schools that had already followed the new program and the follow-on 'Mathematics action plan' that was intended to support its implementation. We could not compare the results with previous ones, although we did consult a leader in the region, who informed us about the difficulties the students had encountered previously on equalities. Our final sample consisted of 24, 20, 24 and 27 students, respectively from the 1st, 2nd, 4th and 6th years of schooling.

The task analyzed here, was constituted by eight open numerical equalities with additions in which students had to place a number to make it true. Students were asked to explain how they found the result. Students had no time limitation to complete the task. The equalities were as follows:

(a)	$8 + 4 = \square + 5$	(e)	$6 + \square = 4 + 5$
(b)	$8 = 3 + \square$	(f)	$\square = 4 + 3$
(c)	$17 = \square + 17$	(g)	$\square + 4 = 5 + 7$
(d)	$3 + 5 = 2 + \square$	(h)	$9 + 7 = \square + 9$

The items were similar to those used in previous studies, and were particularly designed to encourage students to consider, as solving strategies, relational properties. Item (f) is the regular operation with the unknown at the beginning. Item (c) focuses on the neutral property of zero. Item (h) focuses on the commutative property. Items (e) and (b) are related to the use of lacunar examples of type "$\_\_ + a = b$" that are used regularly in schools. Items (a), (d) and (g) are related to the possible use of the property $a+b = (a+1) + (b-1)$ or $a+b = (b-1) + (a+1)$ used sometimes in mental computation exercises. Only the first item was exactly the same as in the studies of Falkner, Levi and Carpenter (1999) and Freiman and Lee (2004). We note that we did not investigate situations involving subtraction, although these were included in the new curriculum. In fact, all the teachers in the sample informed us that they also considered number sentences such as $39 - 15 = 41 - \square$, or $5 \times 18 = 6 \times \square$. However, only one teacher told us that she also analyzed the effects of changes to the numbers for number sentences involving subtraction or multiplication.

In our sample the results about item (a) were as follows (Table 1):

	Number of students	Answer: 8+4=17+5	Answer: 8+4=12+5	Other values	Answer: 8 +4 =7+5
1st Year	24	3 (13%)	13 (54%)	4 (17%)	4 (17%)
2nd Year	20	1 (5%)	8 (40%)	1 (5%)	10 (50%)
4th Year	24	1 (4%)	6 (25%)	3 (13%)	14 (58%)
6th Year	27	1 (4%)	5 (18%)	1 (4%)	20 (74%)

Table 1: Students results in the equality 8 + 4 = _ + 5

From Table 3, we can see that 13% of the first year students added all the numbers to obtain 17. In the other years, it happened too but in only one case per year. We observe that, in comparison to previous studies, a lower proportion of students gave the "operational" response of 12 disregarding the number 5. Similarly, the proportion of correct responses was greater. Given the somewhat opportunistic nature of our sample, these results need to be treated with caution. Moreover, we cannot be certain such answers indicate a fully relational understanding. Nevertheless, we were surprised by these results. In fact, 10% of students established written relational arguments, representing it by quantified arrows. (See Figure 1).

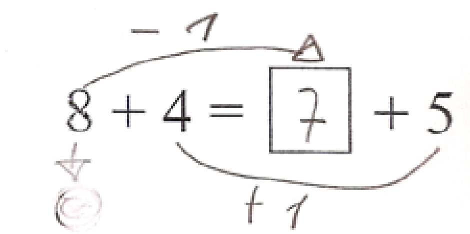

Figure 1: A typical relational argument given by students.

Although this can be considered as a relatively small proportion of students, we consider this a surprising and "unexpected" result.

For items (h), 9 + 7 = _ + 9, and (c), 17 = _ + 17, the presentation of the problem appears at first similar to the first, item (a). The empty space immediately follows the equal sign. Yet the items are quite different. For item (h), the commutative property can be used, whereas for item (c), the neutral property of zero is used. The results for item (h), 17% of the 1st year correct, 55% in the 2nd, 75% in the 4th and 82% in the 6th year of schooling, are slightly better than for item (a), with much better results in the 4th year. Similarly, the results for item (c) are quite different with much better performance in all years of schooling. Whilst we really cannot be absolutely sure that this is because of the use of neutral property, we consider that this is highly likely since the neutral property is introduced from the first year of school. When asked, some teachers supported this hypothesis, commenting “it’s sure, we did in the school the idea of 0+*a*= *a*”.

	1st Year	2nd Year	4th Year	6th Year
8 + 4 = _ + 5	17%	50%	58%	75%
9 + 7 = _ + 9	17%	55%	75%	82%
17 = _ + 17	63%	95%	88%	96%

Table 2: A comparisons of correct results for items (a), (c) and (h)

Although we there is not space to analyse the remaining equalities in detail, we note that in general the results were better than expected. However, only in two questions did all the students of a given year answer correctly: item (b), 8=3+□, in the 6th Year and item (f), □=4+3, for the 2nd year students.

The best performance of the 1st year students was in item (f), □=4+3, with 80%, whilst their worst performance was for item (a), 8+4=□+5, and item (h), 9+7=□+9 with 17%. The worst performance of the 2nd year students was for item (a), 8+4=□+5, with 50%. The best performance of the 4th year students was for item(b), 8=3+□, with 96% and the worst was for item (a), 8+4=□+5, with 58%. The worst performance of the 6th year students was for item (a), 8+4=□+5, with 74%.

In order to identify the possible curricular influences, after the summer, we asked the students; teachers about the students’ previous work. All of them told us that they introduced some examples of arithmetical relations, because, as one teacher put it, “*it’s prescribed in the new curriculum*” [1]. Three out of six teachers commented in similar terms the following teacher: “*it’s not enough to use the equal sign as a result of an operation, as it’s shown in the new curriculum*”. According to these teachers’ self reports, they used ***some*** examples of equalities in which the equal sign is ***not*** related to a result of an operation. None spontaneously volunteered the use of problems in an equation form. But when they were asked to show examples of missing value arithmetic problems, they did show us examples such as ___ + 9 = 11. In addition, a few examples like 4 + __ = 10 + 1 were given in which the missing value appears on on the left side, and both sides consist of sums.

When we asked the teachers about the use of arithmetical properties, they referred to commutativity, zero as the neutral element, and the distributive law. However, none mentioned other relational properties of equalities as “$a+b = (a+1) + (b-1)$” or “$a+b= 2a + (b-a)$”. We observe that the teachers did not perceive a need to make explicit the variability relations underlying mental computation exercises. For example, teachers commented that “*My dear, since the new curriculum, we use computation strategies as doubling one element and halving the other gives the same result*” and “*it’s difficult for the students to see that many properties appears for every pair of numbers*”. One teacher explained “*We did exercises with equalities, but we never insist on the fact of generalizing the property, as I know it is, for every number, if we* *add two to a one member of an addition, and decrease two, the result is the same*". Some teachers also explained that "*I always write the correct answer in the blackboard to correct their mistakes*". One of them asserts that "*I know it's better to discuss about the properties, but I tell them look if it's true or not*".

We also asked the teachers about the use of technological tools for studying arithmetic relations. All of them answered that "*we use variables in geometrical or functional tasks, but I never used technological tools for observing arithmetical properties… because I don't know how to do*".

CONCLUSION

First and foremost, we note that this was a pilot study and the results should be treated with some caution. Nevertheless, our study offers Portuguese data relating to an important aspect of the learning of algebra: the extent to which students understand equality operationally or relationally (Smith, 2011). From our study, we observe that, whilst some Portuguese teachers interpret the increased use of relational properties by the students as a very good result, they are not aware of the importance of the students verbalizing the relational meaning, even with the new curricular influences. Indeed, one of the teachers bemoaned "*the lack of a protagonist role of the students in the moment of exploratio*n".

It appears that, even though the results are better than for previous studies, it is of the utmost importance to promote further changes to teaching practices in Portugal. In our study, it appeared that "*although the new program explicitly includes algebraic thinking, students are still below what should be for the subsequent formal teaching of Algebra*". Many students possess an interpretation of the equal sign included in an operational view instead of relational. We concur with previous research that the students' difficulties are related to the way they were taught and not an expression of any inability to understand relations between quantities (Molina, Castro and Ambrose, 2006; Schliemann, Carraher and Brizuela, 2007; Haylock, 2006). So, for example, we infer from the results for items (a), (d) and (g) that many students are not aware of the general property $a+b = (a+1) + (b-1)$. Nevertheless less percentage of errors are due to the operational perspective.

In conclusion, although these results suggest there may have been some improvement, there is still a long way to go before we achieve acceptable student performances (Knuth et al., 2011). For that, more care is necessary in the teacher's practices to emphasize the use of numeric relations (and manipulative or technological representations. Such work should be linked to a certain development of algebraic thinking, because as Kieran (1992) points out, the distinction between arithmetic and algebraic thinking is the change from a procedural vision of operations towards a structural one.

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NOTES

1. Translations of teacher quotes are by the authors.

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CONCEPTUAL CHALLENGES FOR UNDERSTANDING THE EQUIVALENCE OF EXPRESSIONS – A CASE STUDY

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Whereas students' conceptual understanding of variables and equations has often been investigated, little is known about students' pathways to understanding the equivalence of expressions in generational instead of transformational activities. The case study reconstructs two conceptual challenges that must be overcome on these pathways to conceptual understanding of equivalence: (1) limited degrees of generality for variables and figures, and (2) operational versus relational perspectives on expressions.

THEORETICAL AND EMPIRICAL BACKGROUND

Understanding of equivalence as one meaning of the equal sign

Students' limited understanding of algebraic equality has often been problematized in empirical studies (e.g., Kieran, 1981; McNeil & Alibali, 2005). Many students only interpret the equal sign as a prompt to calculate the value, but algebraic thinking also necessitates the *relational* meaning as signifying symmetric structural relations between the left and right side of the equal sign. Whereas most research has focused on equations like $7x + 28 = x + 4$, the relational interpretation of the equal sign is not only addressed *(A) in algebraic equations*, but also in three other important aspects; *(B) arithmetical identities* like $7 \cdot (10+4) = 7\cdot10 + 7\cdot4$, *(C) equivalence of expressions* like $7 \cdot (x+4) = 7x + 7\cdot4$, being generalized from (*C*), and *(D) contextually bound identities* like "Right triangles with hypotenuse c and legs a, b satisfy $a^2+b^2=c^2$" (see Prediger, 2010 for these aspects). Aspect (B) refers to arithmetic; (A), (C) and (D) to algebra, but with different meanings of the variable. Whereas variables in equations (A) serve as unknown to be solved, variables in aspects, (C) and (D) serve as generalized numbers (Usiskin, 1988, p. 17). Hence, generalization is crucial for understanding algebra. Building an understanding of (A) and (B) has often been investigated, whereas (C) is considered more rarely (see next section). We defined the aspect of a relational understanding of equivalence (C) as the core subject of our design research project (see Prediger & Zwetzschler, 2013, for an overview).

In this paper, we present a small descriptive study within a larger design research project that focuses on the empirical specification of conceptual challenges that students encounter while developing conceptual understanding for the equivalence of expressions. We investigate these individual conceptions in a learning arrangement that promotes generational activities before transformational activities (Mason et al., 1985), as will be further explained in the next sections.

Three meanings for the equivalence of expressions

In our approach, we *generalize the operational-relational dichotomy from the equal sign to the equivalence of expressions*: How should students understand equivalences like a · b + 2 · b · h/2 = b · (a+h)? In line with the wrong priority attributed to operational meanings of the equal sign (Kieran, 1981; McNeil & Alibali, 2005) is the fact that many students (and some curricula) think about equivalence of expressions only in terms of *transformational activities*. But how to ground these transformation rules in conceptual understanding? Mathematically, they can be derived from the basic arithmetical laws (like commutativity and distributivity). But as Demby (1997) has pointed out, the general deduction from arithmetic to algebra is too complex and abstract for many learners (e.g., Lee & Wheeler, 1989). That is why a learning arrangement that fosters the development of conceptual understanding of equivalence of expressions should first involve its inclusion in *generational activities*, in which algebraic expressions are not only understood as a system of meaningless signs (being transformed according to arbitrary rules), but as pattern generalizers of arithmetical or geometrical pattern (Mason et al., 1985, p. 46 ff.). Within these activities, a *relational* understanding of the equivalence is achieved by comparing expressions with respect to equivalence (Kieran & Sfard, 1999) To sum up, three meanings of the equivalence of expressions are to be acquired, first, (a) and (b), then (c):

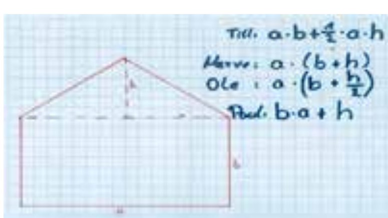
(a) *description equivalence*: …, if they describe the same phenomenon (same geometric pattern, same situation, same function, …);

(b) *insertion equivalence*: …, if they have the same value for all inserted numbers;

(c) *transformation equivalence*: …, if they can be transformed into each other according to the transformation rules. (Malle, 1993; Prediger, 2009)

For (a), description equivalence, a relational perspective on expressions, Kieran & Sfard (1999) compare functions where the table representation immediately leads to (b), the insertion equivalence. As an alternative approach to that Pilet (2013) focuses on an operational perspective (b) to build a conceptual understanding (see also, Solares & Kieran, 2012; Rittle-Johnson et al., 2011). Our tasks in Figure 1 (Prediger et al., 2011) follow Mason et al. (2005) and Malle (1993) by establishing description equivalence for areas of varying geometric shapes, complemented by establishing insertion equivalence.

(I) Which students calculate the same area? And which of the *expressions* calculate the area of the given geometric shape correctly?



(II) Insert different numbers for the variables. Check which of the expressions are equivalent.

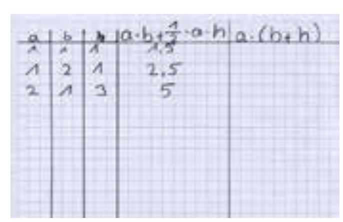


Fig. 1. Tasks (I) for experiencing description equivalence and (II) for insertion equivalence

In this paper, we reconstruct critical moments in the learning pathways towards description and insertion equivalence; the latter completion by transformation equivalence is not treated here. Generalizing the *operational-relational dichotomy from the equal sign to expressions*, we developed Task I to prioritize a relational perspectives on algebraic expressions against purely operational perspectives. That means, we do not emphasize the activity of *calculating* values of expressions, but that of *formulating, interpreting and structuring* expressions while relating them to geometric shapes. However, our empirical analysis will show (in Section 3) that students still adopt other variants of operational perspective which produces conceptual challenges for their pathway to description equivalence.

Additionally, we will show that the *degree of generality* is a relevant source of difficulties. The insertion equivalence is quite natural for students for one *specific* insertion, namely the specific side lengths in the given geometric shapes; this interpretation of variables is known as "letter as object" (Küchemann, 1981). However, two expressions are only equivalent if they have the same value for *all* inserted numbers. We will show how a limited degree of generality (being transferred from geometry to algebra) forms a second conceptual challenge for understanding general insertion equivalence.

METHODOLOGY

This study is embedded in a larger design research project (Prediger & Zwetzschler, 2013) that follows the methodology of Cobb & Gravemeijer (2006) with its dual aim of deepening the understanding of learning processes and designing learning arrangements. Therefore, it applies iterative cycles of (re)design and empirical investigation. Here, we concentrate on one step of empirical investigation with the following research questions: (1) Which conceptions do students activate or develop in a learning arrangement designed to foster the conceptual understanding of description and insertion equivalence? (2) How do the individual conceptions of variables, expressions and geometric shapes influence the learning pathways? Where do conceptual challenges appear?

Data gathering in design experiments

The tasks presented in Table 1 were part of the teaching-learning arrangement used for twelve design experiments in laboratory settings (Komorek & Duit, 2004). A teacher worked with 12 pairs of students in grade 7 to 9 in German comprehensive secondary schools), for three to five sessions of 45 to 60 minutes, the presented task lasted 20 to 50 minutes. All experiments were videotaped and partly transcribed.

Data analysis: Vergnaud's analytical model of concepts- and theorems-in-action

For the interpretative analysis of individual conceptions (research question Q1), we operationalized "conceptions" by adapting the theoretical constructs concepts- and theorems-in-action from Vergnaud's theory of conceptual fields, as this theory offers "a fruitful and comprehensive framework for studying complex cognitive competences and activities and their development" (Vergnaud 1996, p. 219).

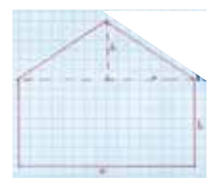
The *first step* of our analytic procedure allows to reconstruct, for each of students' visible activity or utterance, the underlying operational invariants: *Theorems-in-action* are defined as "proposition that is held to be true by the individual subject for a certain range of situation" (Vergnaud, 1996, p. 225). In order to adapt Vergnaud's construct to our specific needs, we symbolize theorems-in-action by <…> and always formulate the purpose and the means, e.g., <For calculating the value of an algebraic expression, I can replace the variable by the specific measures in the drawing>. These theorems-in-action are shaped by *concepts-in-action,* being defined as “categories (...) that enable the subject to cut the world into distinct (…) aspects and pick up the most adequate selection of information” (ibid.), e.g., ||Variable as unique hidden number||. In the *second step* of analysis, we categorize the reconstructed concepts-in-action according to their subject (variable, expression, connection between expression and geometric shape, …), their degree of generality and the underlying operational or relational perspective on the subjects. This allows us to identify connections that could be interpreted as sources for typical conceptual challenges (research question Q2).

RESULTS: RECONSTRUCTING CONCEPTUAL CHALLENGES

Without being able to provide wide empirical evidence from the case studies, we present short extracts of our analysis, show typical moments in the processes and discuss the connection between the reconstructed theorems- and concepts-in actions.

Episode 1 of Paula & Daniel: Degree of generality for variables and figures

Paula and Daniel (grade 9) collaborate on Task I (Fig. 1). Before Turn 62 they evaluate two given algebraic expressions as correct by relating sub-expressions to sub-areas of the figure, guided by the concept-in-action ||Relation between expression and shapes as corresponding by substructures||. For Till’s expression a· b + ½ · a · h, they don’t find structural correspondences and calculate instead:



62 Paula: So 0.5·a·h, you need values to calculate it. […]
66 Daniel: Therefore, we would need this height here.
67 Paula: Ah, just count it, don’t know. That’s 1 2 3 4. [...]
71 Paula: So *a* was 8, right?
72 Daniel: Yes.
73 Paula: 8 (*writes down a*=5), so the half would be, that would be 4 hm *(counts the units, then counts side lengths and calculates the area).*

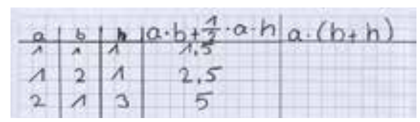
Paula’s activities in Turns 62-73 are guided by her individual theorem-in-action: <For finding out which expression is correct, I can calculate the value of the expressions>. Beyond it, we reconstruct her concept-in-action ||Relation between expression and shape as decidable by quantities||. Both students search for specific measures for calculating (Paula in Turn 62, Daniel in Turn 66). They solve their need by the theorem-in-action <For calculating the value of an algebraic expression, I can replace the variable by the side lengths in the drawing> (Turn 71ff), beyond which we reconstruct the concept-in-action ||Variable as hidden specific number||. Like many other students in our study, Paula und Daniel are guided by their individual focus on specific lengths. A short while later, they compare algebraic expressions by their values for a specific insertion:

93 Daniel: that was Till (*writes* Till = 48 Ole =)
94 Paula: mhm – Maybe we calculate about Ole, which result is the right one.

The theorem-in-action <For comparing two expressions, I can compare the results of the expressions> assists Paula to correctly evaluate Till’s and Ole’s expression as equivalent. However, the underlying concept-in-action ||Equivalence as equality of results|| is only partially correct, since it limits the insertion equivalence to specific numbers. Their limited degree of generality for the variables is connected to a well-known misconception about geometric shapes: Paula and Daniel do not apply the geometric concept ||Geometric shape as general figure|| in which changeable side lengths (and as a consequence the form of the shape) are considered, but instead they apply the individual concept-in-action ||Geometric shapes as specific drawings|| (see Parzysz, 1988) in which side lengths are fixed to the specific drawn measures.

Paula's and Daniel's restriction to ||Equivalence as specific insertion equivalence|| and ||Variable as hidden specific number|| becomes an evident obstacle for developing the concept of general insertion equivalence when working with Task II where the fictitious student Till inserts several numbers for comparing the expressions in the next scene.

27	Paula:	We filled in the right numbers and he took anyones?
…		
30	Daniel:	Huh? That's not possible.
31	Teacher:	Why is that impossible?
32	Daniel:	You just can't insert different numbers.



Due to their concept-in-action ||Variable as hidden specific number||, their theorem-in-action <For comparing two expressions, I can compare the results of the expressions> is limited to one insertion (the specific drawn lengths), so that they cannot get access to the general insertion equivalence. From this snapshot and comparable episodes from other case studies, we conclude that at this point in the learning process, the individual concepts-in-action on variables and geometric shapes, like those of Paula and Daniel,

Table 1. Degree of generality as a challenge on the pathway to insertion equivalence

	Specific $\longrightarrow$	**General**
Conceptions for variables	hidden specific number	changing / generalized numbers
Conceptions for geometric shapes	specific drawing with fixed side length	general figure with varying side lengths
Conceptions for Equivalence of expressions	specific insertion equivalence / specific description equivalence	general insertion equivalence / general description equivalence

provide a challenge for the development of conceptual algebraic understanding (see Table 1). Although in the later part of the design experiment, many students succeed in overcoming this challenge of limited degree of generality, we emphasize that in the first encounter, the geometric interpretation of expressions can become a source of a conceptual challenge if limited geometric understanding is activated.

Episode 2 of Jan & Niclas: Intermediate generality in operational perspectives

Jan and Niclas (grade 7) also work on Task I and start by finding an own way of calculating the area. Niclas struggles with Ole's expression a · (b + h/2). The understanding of generality in the tasks is a similar challenge for them.

56	Niclas:	Me, for example, I would know how to calculate the area, but the whole expression.
…	Niclas:	*(explains correctly how he would calculate the area of the drawing)*.
61	Teacher:	Mhm, just write it down anyway.
…		*(Jan wants to know, if he got right in understanding Niclas.)*
63	Niclas:	… can I just do it with units, that I count this *(he first touches the lower side and afterwards the height of the triangle)* so or just six units?
…		
66	Jan:	… there is nothing specified.
67	Niclas:	Yes
68	Jan:	There are none, so now that is, I mean, how many, let me say, that are 3 meters (*hints to side b)* that are 4 meters (*hints to side a).* It is only now that it is specified how long the sides are.
69	Teacher:	How long could they be, the sides?
70	Jan:	Different, as you can actually choose, x-variable.
71	Teacher:	mhm
72	Niclas:	Or maybe one unit as one meter, that are 16 meters *(hints to side a)* that are 9 meters *(hints to side b, gives a shrug)*, aren't they?
73	Jan:	Also possible.

Both boys operate with the individual theorem-in-action <For calculating the area of the given shape, I can insert values for the variables>, but, while negotiating which value to insert, divergent concepts-in-action appear. Whereas Jan emphasizes that different values can be inserted (Turn 68, 70) and thus activates a high degree of generality, Niclas first starts with the concept-in-action ||Variable as place holder for specific numbers|| in Turn 63. Reacting to Jan's objection in Turn 72, he broadens his theorem-in-action to <For calculating the area of a given shape, I can insert the side length with variable scales>. Thus, he changes his concept-in-action to ||Variable as a place holder for specific numbers but variable scales||. This concept-in-action is in line with the geometrical concept-in-action, ||Geometric shape as drawings with specific side length but variable scales||. Since this concept-in-action is still restricted to geometrically *similar* drawings, we classify Niclas' concepts-in-action as having an intermediate degree of generality (locating between the columns of Table 1). With these higher degrees of generality, their further pathway to *insertion equivalence* is smoother than that of Daniel and Paula.

However, their pathway *to description equivalence* is challenged by serious difficulties in connecting the shape and the expression. The problem first appears in Turn 56, when Niclas claims not to be able to formulate his own expression. His use of variables seems to be restricted to inserting and calculating, so we reconstruct the operational concepts-in-action ||Expression as prompt to calculate|| and beyond that ||Variable as place holders||, but not ||Expression as description for structures|| (see Table 2). In contrast to Daniel and Paula who can (sometimes) activate ||Relation between expression and shapes as corresponding by substructures||, Niclas and Jan only draw connections between expressions and shapes when the expressions are written with numbers instead of variables. Later, the teacher prompts them to find sub-expressions with variables in the figure:

406	Teacher:	Mhm and why did Till actually first multiply a times b and than a times h – and afterwards divide that by two? – Do you have an idea how he could have

		found that out?
407	Niclas:	Uff – well, maybe to make it easier or something like that.
408	Jan:	Well, actually he did a times h…
409	Niclas:	…Because he has – he has these lengths *[hints to a and b]* or this information *[hints to the expressions]* this is what he already has, that's why you can do this … *[interrupts himself, break 8 sec.]*
410	Jan:	Do you know how Ole works?
411	Niclas:	Hm – no idea *[laughs]* – how you can find it out

In Turn 409, Niclas explicitly refers to the algebraic expressions and the figure, but interrupts himself when trying to relate them to each other. The formerly used individual theorem-in-action <For connecting the shape and the expression, I can insert the side lengths> is explicitly excluded by the teacher's prompt to consider the sub-expressions with variables, but he does not find any other way to relate the shape and the expression. The challenge, therefore, is that they have a completely operational perspective of the expression, namely the concept-in-action ||Expressions as prompt to calculate|| which is directly connected to ||Variable as place holders|| (see Table 2).

Table 2. Operational – relational dichotomy as a challenge on the pathway to description equivalence

	Operational perspectives on variables and expressions ⟶	**Relational perspectives on variables and expressions**
Main activities	calculate	formulate, interpret, structure
Conception for algebraic expression	prompt to calculate, numbers must be inserted before dealing with expressions	description for structures (e.g., pattern) for unknown / general numbers
Conceptions for variables	place holder for numbers	specific numbers or changing / generalized numbers
Correspondence between algebraic expression and geometric shape	Relate only quantities (numbers, values <-> side length, areas)	Relate also structures (operations or subexpressions <-> substructures and parts of shape)
Conceptions for equivalence of expressions	only insertion equivalence	insertion and description equivalence

Turns 406-411 show how these concepts-in-action hinder the boys' capacity to relate the shape and the expression. Table 2 is the condensed outcome of a comparison of several cases and systematizes the observed problems and logical connections. It generalizes the well-known relational-operational dichotomy from the equal sign to variables, expressions and the understanding of equivalence. The restriction to the main activity in the operational perspective (calculating expressions) has consequences for the variable as well as for the correspondence between algebraic expression and geometric shape. For the pathway to description equivalence, operational perspectives on expressions must be complemented by relational ones that focus on own formulations of expressions, structures and interpretations. Unless the variable is considered only as place holder and the expression only as prompt to calculate, the correspondence between expressions and shapes can not be drawn by relating substructures. In this way, the concepts-in-action in the different lines of Table 2 are deeply connected and the transition from operational to relational perspectives is crucial.

CONCLUSION AND OUTLOOK

The empirical analysis of typical challenges showed two important dimensions in which students have to develop their initial conceptions on their pathways to a conceptual understanding of the equivalence of expressions. To understand a general insertion and description equivalence: (1) the degree of generality attributed to variables and geometric shapes (Table 1; vertical axis in Fig. 2), and (2) the operational versus relational perspectives on variables, expressions and - as a consequence - the relation between expressions and geometric shapes (Table 2; horizontal axis in Fig. 2) are the main challenges that have to be overcome. Although the four students finally succeeded in overcoming these challenges, our extracts of their processes show typical moments and intermediate states of the development in these two dimensions of challenges. The first episode with Paula and Daniel shows how a specific understanding of variables and geometric shapes limits students' conceptions for the equivalence of expressions.

In the second episode, Jan provides a higher degree of generality, and Niclas adopts an intermediate conception on variables as specified numbers with variable scales. Jan and Niclas additionally struggle with their purely operational interpretation of expressions, variables and the connection between geometric shapes and algebraic expressions. This interpretation hinders their pathway towards description equivalence.

General	general insertion equivalence *(Episode 2b: Jan)*	general insertion and description equivalence *(Intended Understanding)*
	intermediate state *(Episode 2a: Niclas)*	intermediate state
Specific	only specific insertion equivalence	only specific insertion and description equivalence *(Episode 1: Paula and Daniel)*
	Operational perspectives	**Relational perspectives**

Fig. 2. Overcoming conceptual challenges in two dimensions: Overview on the cases

In the larger design research project, these findings initiated the design of additional tasks that help to overcome these challenges (see Prediger & Zwetzschler, 2013). In those tasks, our purpose was to broaden students' perspectives from being purely operational to also becoming relational. Therefore, we integrated tasks that focus on structural connections between geometric shapes and (first arithmetic and later algebraic) expressions by making explicit the strategies for finding substructures in expression and shapes. One example is given in Task (III) in Fig. 3 (Prediger et al., 2011). As the figure shows, the students need to adopt a relational perspective in order to find out which elements belong together. The focus on substructures is strengthened by the verbalization of strategies as a third element that serves as conceptual bridge to overcome the gap between the drawing and the expressions.

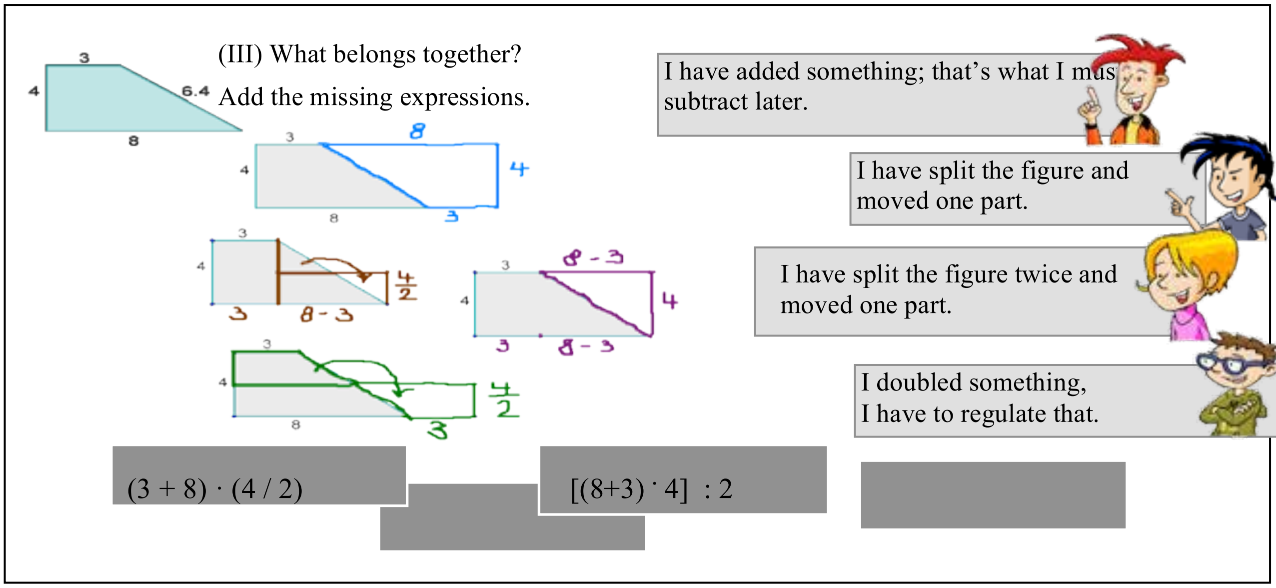


Fig. 3. Design of a task that focuses on drawing connections of substructures

The following design experiments showed that this task encourages students to draw connections and thus to gain access to the learning pathway towards understanding description equivalence. Further analysis of the generated learning processes are conducted in Zwetzschler (2013).

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HOW EXPERTS GRAPH FORMULAS

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Keywords: graphing formulas, heuristics, recognition.

Introduction

In secondary school, students learn to manipulate algebraic expressions, to solve equations and to graph formulas. In order to graph formulas easily, one has to read the formula. 'Grasping the structure of expressions' (Sfard & Linchevski, 1994) is an aspect of symbol sense (Arcavi, 1994), rarely addressed in textbooks. As a consequence, teachers have to develop their own teaching methods in this area, which requires pedagogical content knowledge including subject matter knowledge and teaching strategies (Hill, Ball & Schilling, 2008).

In our research we investigate Dutch teachers' pedagogical content knowledge in this context. As the first stage of the project we address the following questions: Which formulas can experts instantly visualize as a rough graph? And if instant visualization (recognition) is lacking, which strategies do they use to graph a formula? We analyse experts' performances in identifying knowledge involved and heuristics needed to complete these tasks. The experts' results will allow for a comparison of experienced and novice teachers and students on the same tasks. With these results we will get a clear view on the differences between experts and teachers and will be able to develop a professional development trajectory.

Theory

Chi, Feltovich & Glaser, (1979) and De Groot (1965) both claim that experts, compared to others, recognize more and deeper patterns and are more sensitive to critical features. They use larger hierarchically organized units of knowledge (chunks). In problem solving they perform qualitative analyses, consider potential actions and categorize problems. During the execution of such tasks they apply self-monitoring, evaluate result and processes, and look for alternative solutions. More specifically for mathematics, Polya (1945) formulates heuristics as strategies for problem solving in general. In line with this, Van Streun (1989) uses heuristics based on Polya's, but he explictly identifies recognition as the starting point of the problem solving process. On the basis of these general perspectives we designed a framework for specific strategies for graphing formulas (figure 1), in which we distinguish two steps, first recognition and then heuristic search.

Method

The experts involved in the pilot interviews each have more than 10 years' experience in their work, which often requires graphing formulas. We selected teachers at universities, a teacher educator, a textbook-author and a teacher who develops national mathematics exams. For this stage of the project we have developed a matching formula-graph task and a card sorting task in order to establish the experts' thinking units. A thinking unit is a formula which the expert can instantly visualize as a rough graph. In the third task we want experts to move beyond their recognition-zone and force them to use a heuristic search. We ask them to graph a more complex formula and work in the opposite direction: to find a formula that fits a given graph. During this task we ask experts to think aloud.

Task 3: Draw a rough graph for $y = 2x\sqrt{8-x} - 2x$ and write down a formula that fits the graph:

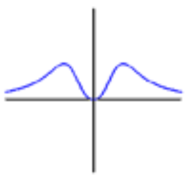
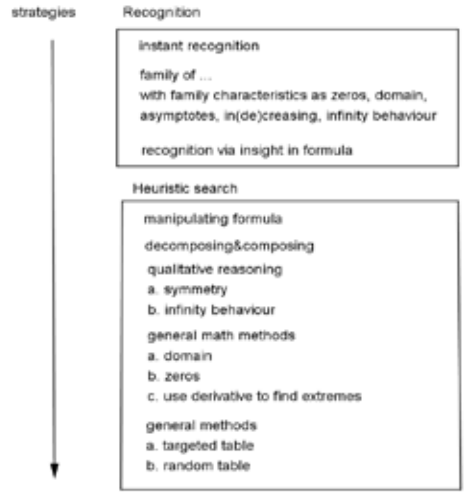


Figure 1: Framework



Results and Conclusions

Experts differ in the number of formulas they can instantly visualize as rough graphs. All experts show characteristics as formulated by Chi et al., especially regarding monitoring and evaluation. In general experts prefer qualitative reasoning and hesitate to start calculations. There seem to be two kinds of experts: those who try to decompose the formula into known components (thinking units) and those who start a standard program of strategies, formulated in our framework as general (math) methods. We hypothesized that experts first use recognition and then a heuristic search. The data suggest, however, that recognition guides the heuristic search. Gobet and Charness (2006) found for chess players a trade-off between knowledge and a heuristic search. We intend to adapt our framework (figure 1) according to these findings.

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GENERALIZING AND JUSTIFYING PROPERTIES OF REAL NUMBERS: A STUDY AT GRADE 9

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KEYWORDS: Reasoning, Generalization, Justification, Real Numbers

ALGEBRAIC REASONING PROCESSES

Developing mathematical reasoning is an essential condition for understanding mathematics and also for using mathematics in a proficient way. Also, as reasoning is recognized as inseparable from the representations and language through which it is expressed (Arzarello, Bazzini, & Chiappini, 2001), algebra, which forms the basis of the symbolic language of mathematics, is eminently suited to its development. To give some structure to the complexity of ideas, notations and activities involved in Algebra and algebraic reasoning, Kaput (2008) advocates a perspective of symbolism that emphasizes two central aspects of algebra: (i) algebra as symbolizing systematically generalizations of regularities and constraints, and (ii) algebra as syntactically guided reasoning and actions as generalizations expressed in conventional symbolic systems. One of the areas these aspects are embedded in is algebra which is the study of structures and systems of abstract relationships and procedures, including those from arithmetic (algebra as generalized arithmetic) (Kaput, 2008).

Reasoning ability is crucial to students. Developing students' algebraic reasoning involves far more than just memorizing concepts and routine procedures. Instead, a focus on memorization leads the students to develop a vision of mathematics as a disconnected set of rules rather than a logical and coherent science (ME, 2007). However, to develop reasoning is challenging for teachers, particularly in everyday classrooms, in all mathematical topics. In order to understand how to promote the development of students' algebraic reasoning, a fruitful step is to take a close look at two key mathematical reasoning processes – generalizing and justifying. Generalizing plays an essential role in the understanding of mathematics as this process of reasoning is one of the foundations of mathematics as a science. Also, justification is an essential process as it allows students to clarify their reasoning, contributing to the development of a deeper understanding of algebra.

AIMS AND METHODOLOGY

This study aims to analyse grade 9 students' algebraic reasoning processes while working on tasks involving real numbers and to know how those tasks contribute to develop students' mathematical reasoning in an algebraic context. Based on a collaboration with a grade 9 mathematics teacher, the study developed within a teaching unit supported by exploratory tasks involving real numbers. Following an interpretative and qualitative methodology, data collection includes videorecording of the lessons and students' written tasks. Data analysis is undertaken according to two main categories related to reasoning processes – generalization and justification. Two other transversal dimensions that go along with reasoning processes – representations and sense making – are also considered in the analysis. This poster focuses on two students working on a particularly significant task on the properties of operations with real numbers, involving a mixture of closed and open questions.

CONCLUSIONS

The results suggest that exploratory tasks with real numbers promote the development of students' conjecturing processes, particularly generalizations, but are not enough to make them to use justifying processes. While making generalizations, most students follow an inductive approach, generalizing the relations observed in particular cases to a larger class of objects. Sometimes these generalizations have a deductive nature. In contrast, justifying is not done spontaneously, but, in response to the teacher's questioning, students show that they are able to make justifications based on previous knowledge of properties or mathematical concepts and based on counterexamples that refute a statement. Therefore, we conclude that, alongside exploratory tasks, teaching practices such as questioning play a key role in promoting the use of reasoning processes. In these tasks, as students have ease on using more than one representation, the use of various representations seems not to limit the development of their mathematical reasoning. Sense making is intrinsically linked to the generalizations or justifications presented. Thus, when students' depict difficulties in making connections between concepts and properties required to solve the task, they also seem to have difficulties in generalizing or justifying. Exploratory tasks allied with teacher questioning practices, emerged as key elements to develop students' use of reasoning processes as well as to promote their deeper understanding of the properties of real numbers. Also, the use of a framework involving reasoning, representations and sense making showed to be useful to study the relationships between these three aspects.

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IMPROVING THE COVARIATIONAL THINKING ABILITY OF SECONDARY SCHOOL STUDENTS

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Keywords: Functions, Covariation, Forms of Representation

THEORETICAL BACKGROUND

Vollrath (1989) considers three aspects of the concept of function to be essential. First, in the point-wise view, every value of the domain corresponds to exactly one value of the range. Second, in the dynamic view, the aspect of covariation requires taking into account the neighborhood of the point: How does the function value vary if the input value is varied? Third, the global view contemplates the function as a whole. This is necessary if statements are made about symmetry, for example.

Researchers point out that the aspect of covariation is not sufficiently implemented in mathematics curricula (Malle, 2000; Thompson, 1994). Instead, school teaching mainly focuses on the point-wise aspect of functions even though it is hardly possible to construct adequate mental models of the concept of function without considering the covariational aspect. Despite researchers' consensus on the importance of this aspect, there is a lack of research on how to develop it in secondary education.

AIMS AND RESEARCH QUESTIONS

By developing empirically based training sessions in covariational thinking, we hope to better prepare students for the infinitesimal calculus and also to show a way of improving a neglected aspect of mathematical literacy in secondary grades. A research hypothesis is that covariational thinking is not restricted to upper secondary grades and that students from grade 5 onwards are able to make substantial improvements with regard to this aspect of functional thinking. On the one hand, we want to identify students' misconceptions when dealing with covariational tasks. On the other hand, we aim at identifying students' cognitive resources in the area of covariational thinking. As we conjecture that the forms of representation play a decisive role, we want to find out which form of representation is more intuitive when dealing with covariational tasks.

METHOD

A training session in seventh grade (n = 27) started with a material-based analysis of the covariation of different functional dependencies (linear vs. quadratic) with a discrete domain.Initially , the students explored the covariation in a qualitative and quantitative manner based on self-generated representations, namely tables of values and graphs, in a discovery learning environment for 20 minutes. Afterwards, the solutions were discussed in a teacher-class dialog for another 20 minutes. At the end of the lesson, the students were given a paper-and-pencil test consisting of six tasks, which were analyzed with mixed methods.

RESULTS AND PRACTICAL IMPLICATIONS

In the analysis of misconceptions we discovered that students confused the first and second difference. Many students incorrectly identified a function with constant second differences as a linear function instead of a quadratic function. This conflation forms an essential obstacle when dealing with covariation. Research on the development of teaching concepts to deal with this problem is still needed.

Furthermore, some students extrapolated linear growth as if it were proportional growth. These findings are consistent with the documented preference for proportional reasoning (De Bock, Van Dooren, Janssens, & Verschaffel, 2002). As a consequence, teaching in early secondary grades has to put more emphasis on non-proportional growth (e.g., exponential, quadratic, or logistic growth) to prevent an overgeneralization of proportional reasoning.

In a quantitative analysis, the role of the form of representation was investigated in a within-subjects design. The students performed significantly better at value table production than graph construction (sign test: $g = .35$, $p < .001$). This raises the question of whether the table of values activates students' cognitive resources in covariational thinking more than graphs. Accordingly, further research on this topic is necessary.

Moreover, students had more difficulty extrapolating quadratic functions than linear functions (sign test: $g = .24$, $p < .01$). This result underlines that school teaching should deal with different types of growth as early as possible.

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INTRODUCTION TO EQUATIONS: THREE CASES AS PART OF A VIDEO STUDY

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Keywords: comparative video study, introduction to equations, classroom interactions

In an international comparative video study called VIDEOMAT (Kilhamn & Röj-Lindberg, 2013) the aim has been to document mathematics lessons in Finland, Norway, Sweden and the USA on the introduction to algebra. A more specific teaching content, introduction of letters as variables, was determined through textbook analyses in each country. A series of consecutive lessons were video recorded in two to five schools in each country with the primary aim to make comparative analyses concerning algebra teaching and learning. As part of data collected within VIDEOMAT, the first recorded lesson in three primary schools in the Swedish speaking part of Finland has been analyzed. The analysis shows similarities on a structural level as well as remarkable differences in the classroom-specific approaches to the introduction of equations.

THEORETICAL BACKGROUND

The introduction of algebra is problematic for large groups of students, particularly when the numerical system is expanded to the use of letters to represent variables or unknowns. Concerning the introduction of equations, a distinction can be made between arithmetic and algebraic types. In arithmetic equations, letters appear as variables or unknowns on just one side of the equal sign, and hence the learner might approach the equal sign in a dynamic way. The adjustments from arithmetic to algebraic thinking include a focus on the equal sign as structural (Kieran, 2004), and on the possibilities for participating in public sense-making processes in the mathematics classroom (Schoenfeld, 2008). Furthermore, the nature of those processes needs attention. The problems connected with the introduction of equations in the mathematics classroom may be related to limitations in teaching approaches. Thus, an exploration of how the content of equations is introduced in classrooms becomes an interesting focus for research.

RESEARCH QUESTION AND METHODOLOGICAL FRAME

The research aims to address the question: What kinds of approaches to the introduction of equations are used in the three grade 6 classrooms, and how is each approach socially constituted? A sequence of four lessons on introductory algebra was videotaped in three grade 6 classrooms. The three schools followed the same national curriculum and the same textbook. Data were collected with three stationary video cameras, which recorded the class from up front, the interactions of a small group of students and the actions and voice of the teacher. In addition, information was obtained through teacher interviews, written work of the students and various lesson files. The content taught in the first recorded lesson in each class was similar: arithmetic equations (addition, subtraction) with one unknown term. A micro case study of these three lessons was performed, including a fine-grained analysis of the sections coded as 'teacher-led' (TL) or 'teacher-led with students' (TLS) (see Kilhamn & Röj-Lindberg, 2013). Similarities and differences between the three lessons were identified, with a particular focus on the roles students constructed through interaction, on the positioning of the students as evidenced in the use of pronouns in the TL and TLS sections (Rowland, 1999), and on the external representations used by the teacher.

RESULTS

The study shows similarities in the approaches to the introduction of equations at the level of lesson structure, as well as in the expectations on students to listen and answer questions, to learn procedures and to grasp the structural meaning of the equal sign. In each class, equality was represented as a structural relation between quantities. The clearest difference among the classes was found in the spaces for students to participate in the types of dialogue that may support shifts in the transition from arithmetic to algebra (e.g. reasoning and argumentation). In classes A and B, limited space was afforded for student initiatives: only 3-4% of the pronouns used during the teacher-led sections of the lessons appeared within student-talk. In class A, the teacher focused on recall and reproduction of steps in solving equations. In class B, the students were expected to solve problems, explain solutions and construct the corresponding equations. In class C, 28% of the pronouns appeared within student-talk as the students expressed their thinking, asked questions and examined statements of others. The study suggests that teachers should critically examine the interactions during their own lessons and in those of other teachers.

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INTRODUCTORY LESSONS ON ALGEBRA: A VIDEO STUDY (VIDEOMAT)

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Keywords: comparative video study, the introduction of algebra

VIDEOMAT is an ongoing collaborative research project that focuses on the contribution of video studies to the comparative analysis of mathematics lessons from Finland, Norway, Sweden, and the USA (California). The primary aim is to do comparative analyses concerning introductory algebra teaching and learning. The introduction of algebra has been defined in the scope of this project as the introduction of letters as variables. A second aim is to use the collected data for dialogue and professional development among teachers through analysis of their own practices and those of fellow teachers.

BACKGROUND

The VIDEOMAT project belongs within the framework of sociocultural studies (Säljö, 2006). In this approach, many concepts are used as analytical tools to make explicit the learning process in the classrooms, such as mediation (Carlsen, 2008) and dialogue (Bjuland, Cestari, & Borgersen, 2008). The analytical interests of the VIDEOMAT project are towards seeking similarities and differences in how algebra is introduced in the mathematical classroom, and discussed as well as analysed among teachers.

RESEARCH QUESTIONS

The following general questions will be addressed in this project:

1. Which teaching approaches to algebra are used and how are the lessons organized in the four countries?
2. How are the main components of an algebra lesson structured, implemented and discussed by the teachers?
3. How are the main components of algebra lessons discussed by teachers?

RESEARCH METHODS

Participants in the project were teachers and students. There were four or five teachers in each country from two to five schools. The specific grade level when the formal introduction of algebra takes place was not well defined in the curriculum from any of the countries. The research group decided to define this introduction through textbook analysis. In Norway data was collected in grade 7 and 8, where the students are 12 and 13 years of age, in Sweden and Finland in grade 6 and 7 where the students are the same age. In the USA the data was collected from grade 6 and 7 and here the students are one year younger, respectively 11 and 12 years.

Five consecutive lessons from each participating teacher were video-recorded. The timing was limited to the introduction of algebra. During the first four recorded lessons, the class followed the standard curriculum. During the fifth lesson all classes focused on three problems adapted from the TIMSS 2007 so as to obtain comparable material. Interviews were conducted after the fifth lesson with the teachers and comments were given by them at the end of each recorded lesson. Students' written work and teachers' lesson files (lesson plans, tasks, textbook, etc.) were collected as well. Lessons were videotaped by three stationary cameras. One camera was focusing on the students, one on the teacher and the third on a group of students.

Following the agreement between the participating countries to obtain comparable materials, the first lesson from each teacher and the interview have been transcribed. To make comparative analysis possible we have structured and organized a databank. The large sets of video data (60-70 hours from each country) were codified. The coding procedure was influenced by the manual for the 1999 TIMSS Video Study (Jacobs et al., 2003). The structuring of data has three different features: coverage codes, lesson graphs and key words (Kilhamn & Röj-Lindberg, 2013). Coverage codes serve the purpose of describing types of activity and interaction, lesson graphs give an overview of each lesson as a whole, and keywords highlight where and when specific aspects of algebraic content is treated (e.g., order of operations, variable, unknown, expression).

THE SECOND PHASE

In a second phase a virtual platform will be created to stimulate dialogues and professional development among the participating teachers through collaborative analysis of their own practices. The intention is to investigate how video recorded lessons could be used to mediate professional development. Teachers will select, share and discuss episodes from the recorded lessons.

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG4: GEOMETRICAL THINKING

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As with earlier CERME conferences, the working group on geometrical thinking was interested in reports of empirical or developmental research studies, and in theoretical essays, on the teaching and learning of geometry. One report has proved to be quite useful for recent meetings of the working group, especially since CERME5, and it served as inspiration to organize the functioning of the group at CERME8. It relates to the concept of geometrical work spaces developed by Kuzniak and his collaborators that received very detailed attention during the plenary lecture by the author (see references to the plenary talk entitled *Teaching and Learning Geometry and Beyond…*). By extending the idea of mathematical work, without standing in strict continuity with the approach of Kuzniak, the working group at CERME8 used the idea of four geometrical competencies (see Figure 1) as unifying elements to group the working group papers. Starting from the contributions of the participants, we identified as far as possible the dominant competency that emerged from each text in order to focus, in sub-groups, more specifically on certain challenges or consequences in the development of theses competencies.

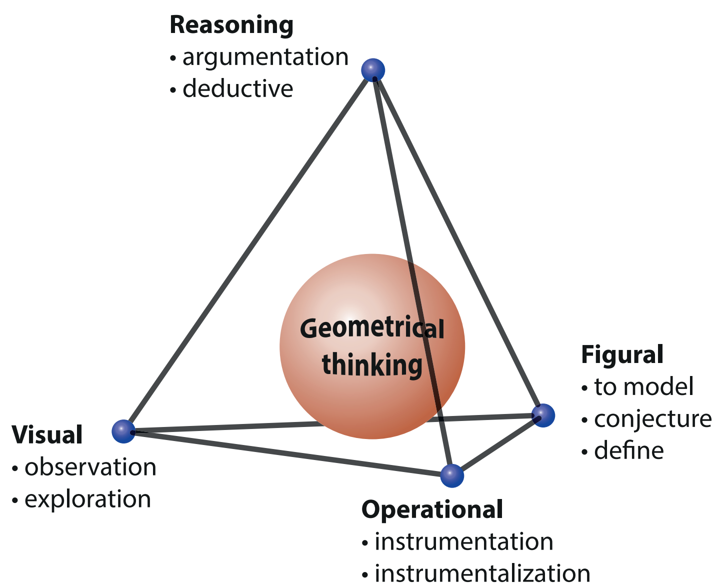


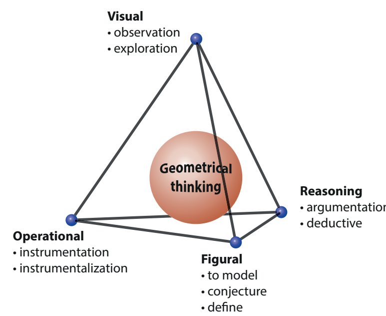
Figure 1: The geometrical competencies

Each geometrical competency acts as a pole of interest around geometrical thinking, so that if one focuses on the issues, or the effects, of a given competency (eg, argumentation and deduction in the reasoning competency), links are generated with the plane of the other three competencies (eg, visual-figural-operational) that also contribute to a better understanding of the competency in question. Because the geometrical competencies arise in relation to the mathematical work being undertaken, they address situations of training, education and learning. Although this is a practical notion that facilitates simultaneously the examining of geometrical thinking and the team working, the group has set up the following criteria for transversal comparison, in anticipation of the synthesis of the works:

1. Descriptions of the competencies we worked about;
2. Type of research questions;
3. What and how theoretical approaches are used to answer those problems;
4. On-going questions;
5. How the other competencies are involved / link between the questions.

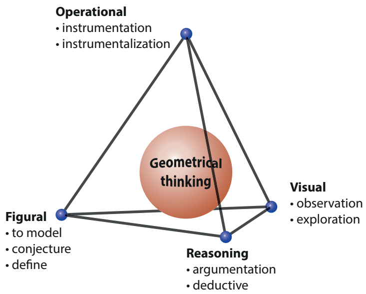
On the thread of these criteria, we summarize hereafter the results of the team and we conclude by projecting the results of the WG4 on the formation of geometrical thinking.

VISUAL POLE



In the visual competency sub-group, the role of visualization was considered in two distinctive ways; first, it compared with the entities of geometry that are not accessible materially and secondly, as the form or recoding and reporting information such a type of language. If we differentiate the habits of seeing from the field of experience, we underline that observation with making-sense entails some action – children start with this «observation», even if the geometrical thinking is more exigent and the observation process involves more than one competency. In the group, dynamic visualization appeared crucial to generate strong visualizing skills. Research has questioned how student capacity can be developed in order from them to become inventive visualizers, what are the relations maintained between visualization and the other ostensive modes of expression, what are the relations between the perception and the definition of geometrical concepts, and how students solve problems that require a topological introduction in the understanding of space, for example, by the movement or gesture.

To answer these questions, the works needed theoretical frameworks that required the use of constructs or competencies (like the visual and the reasoning) to better understand the behavioural diversity among the different habits/skills of the students and, when we need to observe more than one at the same time, to interrogate also the categories/competencies/constructs of reasoning, etc. We note that there is a tension between top-down theories that impose structure and constructs on that which they study, and those studies that look for emergent constructs in the activity of the geometrical thinking. The discussions led to a more thorough examination of the issues, especially to what extent do children take up the pictorial as a resemblance or as a symbolic rendering, how the visual competency is consistent with the other competencies in the development of the geometrical thinking, and what are the theories of perception that underlie our choices of research?



OPERATIONAL POLE

The wide variety of technological tools, jointly with the constant evolution of IT tools, provide a privileged role in the development of geometrical thinking. The competency on which we worked was oriented toward the re-organization of what is already present cognitively in the tools or with the use of the tools; more specifically:

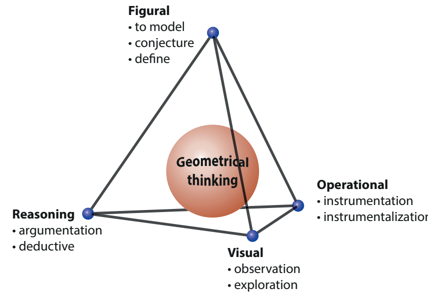
- to create a new reality through images of changes;
- to implement, operate with, and transform, strategies in order to create a new technical or cognitive scheme (not necessarily to achieve a goal);
- to internalize a tool to develop/to foster the conceptual aspects;
- to use a tool to answer new questions/to face new tasks;
- to use a tool to facilitate to reason at a higher level of abstraction;
- to facilitate to link geometrical thinking with other kinds of thinking.

From the transfer of learning, the use of tools and particularly, the gestures and visualization, the research presented asked in what ways do gestural-haptic modalities factor into students' spatial reasoning as they engage in problems through topological rather than Euclidean concepts, what is the role of gestures and manipulation in solving geometrical problems and what is the understanding of how the use of technology may help to face some of the problems that are related to seeing and the study of spatial geometry.

In addition to Fischbein's theory of figural concepts, Rabardel's instrumentation theory within cognitive ergonomics, and the van Hiele theory and its structuring stages of development, the problems raised the importance of gesture in geometry problem solving, the interactions between everyday thinking, idealized mathematical thinking and the «geometrician-physicist» approach – for example, when the use of a tool allows the exploration of a figure directly from the figural expression. The distinction between seeing and knowing was a constant element of discussion, as was the issue of the transfer of learning. Although several open problems were highlighted in the working group, the on-going questions on the uses of tools and on the relationship with proof that principally featured were the following:

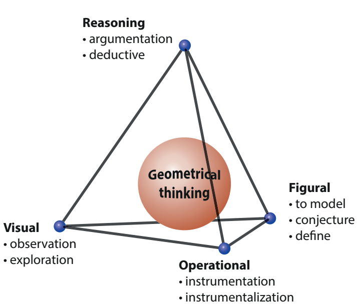
- Does the set square inhibit or delay the development of the interaction between students' figural and conceptual aspects of axial reflection by providing too much assistance? Is the set square a stumbling block for the internalisation of axial reflection?
- Which kinds of perception do teachers in initial training have with respect to the use of tools for geometrical construction?
- Why proving something that can be verified simply by measuring?

FIGURAL POLE



Traditionally, the figural and the visual are used in combination in the same process of representation. In our group, the meaning of the competency has converged on the interpretation of the drawing; that is to say a representation, with conceptions about the object recognizing its generic character, the status of the geometrical object, linking it to a theory, to properties, definitions, theorems, etc. First, there is a significant link between experimentation on drawings (with or without tools) and proof, especially related to the exploration on figures, the determination of what can be said about it and the motivation to proof, and secondly, the link between the construction and the definition of the geometrical objects, commensurate in the relationships with «the real world», the geometry as a model, and the figures as ideal objects. Two different perspectives were considered to tackle this competency: the students and the teacher, as well in the relations of teaching as of training.

In the contributions of the group, research approached the role of the figural aspects in a given task, at a given level, to determine its type and the kind of geometrical work involved. If the general research questions pose some links between the other competencies, the studies enhanced the way that technological tools influences the figural pole, particularly in regard to measurement, approximation, discretization and continuity, whether in terms of status of objects as their role in the geometrical work. Indeed in geometrical thinking, we recognize two-folded relations between the reasoning pole and the opposite face «operational-figural-visual». When the place allotted to the figural grows while working on this face, one might wonder how the reasoning pole is affected. But deductive activity may have consequences on the way students conceive the generic role of figures, and on the design of operations on drawings and the anticipated capacities of children.



REASONING POLE

Because the other competencies are clearly involved in all of the studies, we can say that the reasoning competency is a window in/on the geometrical thinking. The classical semiotic and cognitive theories participated at the definition of this window (e.g. Duval, Godino). A standpoint developed in the group is «reasoning in action», following an approach that complements the notion of discursive-graphic reasoning (Richard). Even if the operational competency is not so evident with traditional activities, the idea of reasoning in action was manifest in research comparing discovery and proof processes with origami, and solving unfolding problems with physical and mental images. If the knowledge model of Balacheff and Margolinas was helpful in comparing textbooks, the links between knowledge and behaviour shown in the model can be found useful in explaining reasoning in action.

In addition, to find the space in which conceptions are made possible to develop geometrical thinking from textbooks, the sub-group of reasoning competency focused its activity on the impact of manipulations on the generating of logically-correct deductive arguments, including comparisons in teaching with and without origami, and how the analytical and the visual thinking are cooperating (possible different objectives) and / or collaborating (same objective) in solving problem with unfolding. To answer the research questions in studies, theoretical approaches that we mentioned above were carried out by using a knowledge model to compare conceptions with textbooks, contrasting two teaching styles and implementing a pre-test/post-test tool to see the evolution, and applying a semiotic theory to analyse the synergy between two kinds of language. Although some research must continue – in some cases to complete the *a priori* studies from real experiments with pupils, student and/or teacher – we should consider to transfer the acquired competencies with/in other environments (paper-pencil, technological, informatics, ...) and to orchestrate different forms to teach in the development of the geometrical thinking.

During the discussions, or in the studies, all participants have pointed out the particular role of the reasoning pole. By way of summary, the reasoning competency helps to animate, structure and control the visual, figural and operational competencies. This is especially helpful in discovery, mathematization and validation steps. Moreover, from the geometrical work/working space point of view, this competency intervenes both in the video-figural genesis (visualisation-figuration), the instrumental genesis (instrumentation-intrumentalization) and the discursive-graphic genesis (devolution-institutionalisation). These geneses (Coutat & Richard) show how geometrical competencies are closely involved in the geometrical work and linked between the issues raised by the group.

CONCLUSION

At a time when visual information and the availability of technological tools have never been so present in our societies, the cultural role of geometry and its privileged place for the development of scientific attitudes should guarantee itself the future of the discipline in the curricula. However, during the exchanges between the members of the working group, we have raised many times a concern about the tendency, in several countries, to marginalize the geometry curriculum in favour of the study of probability and statistics. It seems that the common denominator is a logic that consists at substituting the considerate attention for the formation of the mind in favour of a utilitarian and technical preparation to market values. Research and the discussions of the working group have followed an opposite direction. Instead, they noted firstly the importance of «take for oneself» the universe of space and shapes, recalling that our societies are modelled by shapes and they belong to the world of space, to the point of shaping forms every day or naturally transform space by the action, often even unknowingly. Secondly, how the motivation for learning geometry begins with the understanding of the meaning of its objects, its formal structure and the challenges of the mathematical work and the geometrical competencies.

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USING ORIGAMI TO ENHANCE GEOMETRIC REASONING AND ACHIEVEMENT

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This paper was part of a broader study that examined the effect of the instruction on spatial visualization, geometric reasoning and geometry achievement. This paper focused on the geometric reasoning and geometric achievement. Participants were 167 high school students in Turkey. In a pre-test/post-test design, the experimental group was instructed with the origami-based instruction and the control group was instructed with the traditional instruction for four weeks. Geometric Reasoning Test and Geometry Achievement Test were prepared to measure geometric reasoning and geometry achievement, respectively. The results of the repeated measures Analysis of Variance on test scores showed that the origami-based instruction had a statistically significant effect on geometric reasoning and geometry achievement.

Key-words: origami, geometric reasoning, geometry achievement, high school geometry

INTRODUCTION

Reasoning, proving, creativity and problem solving are involved in Turkish high school geometry curriculum. These skills are expected to be developed through effective geometry instruction. For an effective geometry instruction, a closer look into the geometry learning may be necessary. Three theoretical frameworks may give insights in understanding geometry learning of students. Duval (1998) argues that geometric thinking combines three cognitive processes which are *visualization*, *construction*, and *reasoning*. Visualization is keystone for geometry instruction since students should be able to identify geometric figures in different dimensions in order to reach conclusions about geometric entities. Duval's (1998) cognitive processes formed the mainstay of the test design of our study. Besides, Smith (2010) asserts that geometric thinking is based on *proving*, *justifying*, and *argumentation*. Smith's (2010) categorization of geometric thinking was used in preparing the scoring rubric of the Geometric Reasoning Test (GRT). Furthermore, as a developmental model on geometric thinking, the van Hiele (1959/1985) theory states that geometric thinking progresses in hierarchical stages. Van Hiele stages were considered in the design of the instruction of the existing study. These three theoretical frameworks all agree on the use of manipulative materials in teaching geometry for effective learning of abstract concepts and relationships. The use of manipulatives in teaching geometry and mathematics is also suggested in the literature (e.g. Dorier, Gutiérrez, & Strässer, 2003; Sriraman & English, 2005). Manipulatives can be useful in facilitating students' progression to higher levels of geometric thinking. Thus, origami, the art of paper folding can be used in teaching geometry considering its manipulative nature.

Origami was remarked as a beneficial tool in geometry education in a wide age range of learners by many authors (e.g. Boakes, 2009; Çakmak, 2009; Golan, 2011; Hull, 2006; Pope & Lam, 2011; Sağsöz, 2008; Winckler, Wolf, & Bock, 2011). Origami is connected with many mathematical and geometric concepts and principles such as angle bisectors, fractions, division, ratio, triangles, polygons, congruence, and symmetry. Origami is also referred as a trigger for proof and strong mathematical arguments (Pope & Lam, 2011; Winckler et al., 2011) so that origami may be used to develop students' geometric reasoning and geometry knowledge.

Although origami is recommended by many authors as a useful instructional tool, research on the use of origami in schools is limited (Boakes, 2009; Çakmak, 2009; Sağsöz, 2008). Besides, the research involving origami is concentrated on primary and middle school students (e.g. Boakes, 2009; Çakmak, 2009). Therefore, there exists a need for investigating the effect of origami on high school geometry education. Thus, the focus of this paper was to investigate the effect of origami-based instruction on tenth-grade students' geometric reasoning and geometry achievement in a Turkish high school.

METHOD

Settings

The Turkish education system has kindergarten, primary, middle, secondary and higher education levels. The secondary school level consists of 4-year education (9th to 12th grade). There are seven types of high schools: Science High School, Anatolian Teacher High School, Anatolian High School, Social Sciences High School, Fine Arts and Sports High School, General High School, and Vocational High School. After middle school, there is an entrance exam to high schools except general and vocational high schools. Students who can enter to Science High Schools have the highest scores in the entrance exam. However, students in the General High Schools and Vocational High Schools have the lowest scores in the high school entrance exam. Thus, the academic achievement of Vocational High School and General High School students was generally low compared to other types of high schools.

A common curriculum, which is determined by the Turkish Ministry of Education (MEB), is implemented in all high school types. Students select their courses according to their interests and orientations. Science-Mathematics Orientation (SM), Turkish-Mathematics Orientation (TM), Turkish-Social Orientation (TS), and Foreign Language Orientation (FL) are the types of orientations that students select at the 10th grade. Students with SM orientation have courses like Mathematics, Geometry, Physics, Chemistry and Biology. Students with TM orientation also have Mathematics and Geometry lessons but do not have Science courses. Students with TS orientation do not have Science courses, either.

The study was conducted among 167 10th graders of a General High School in Turkey. The students were chosen from all three different academic orientations: SM, TM, and TS. There are 10 classrooms of 10th graders in the school. One TS classroom was excluded because the instructor was different than the instructor of other nine classrooms. Two classes of each orientation were randomly selected. Then, one of the two chosen classrooms with each orientation was randomly assigned to be in the experimental group and the remaining class was taken to be in the control group. So, the experimental group and the control group each have three classes with three different academic orientations. Students in the control and experimental groups had equivalent conditions. The same teacher instructed both groups. Besides, the same geometry topics were taught in both groups. Students' geometric achievement was similar at the beginning of the study when their previous exam grades were checked.

Treatment

Students were instructed on a geometry unit (triangles) for four weeks based on the curriculum objectives of the Ministry of National Education (MEB). The control group received traditional instruction, which followed the 10th grade MEB textbook. The traditional instruction is the instruction that teaches triangles without any origami activities. The traditional instruction is a non-origamic instruction. On the other side, the experimental group received origami-based instruction, which contained origami activities in addition to following the same textbook with the control group. The traditional instruction and origami-based instruction differed only in containing origami activities. Besides, the same teacher (the first author) instructed both the experimental and control groups. Each week, the control group had two lesson hours for the geometry instruction whereas the experimental group had three lesson hours since one extra lesson hour was used for instructions on folding. The topics for the instruction were the basics of triangles, angle and side relationship in triangle, angle bisectors, medians, perpendicular bisectors, and altitudes in triangles. The model of knowledge by Balacheff and Gaudin (2002) may help to enlighten the origami-based instruction. The folding is the operator in the origami-based instruction in order to reach logically deductive arguments in the concerned geometry topic. Besides, the representation system is the fold that represent line segment in geometry. The structure of the control is the deductive reasoning.

Week	Topic	Origami activity
1	Basics of triangles, classifying triangles	Dart; equilateral triangle
2	Angle and side relationship in a triangle	Swan
3	Angle bisectors in a triangle	Whale
4	Altitude, median, and angle bisector relationships, perpendicular bisectors	Folding a square paper

Table 1: Origami-based instruction

Table 1 shows the origami activities used in the origami-based instruction. The origami activities were developed considering the related literature (e.g. Hull, 2006). Lesson plans were prepared for both groups (Arıcı, 2012). In the origami-based instruction, students initially folded the origami models and then they were guided by the instructor to make the necessary geometric relationships and proofs. In the traditional instruction, students were also guided to make proofs about triangles but no origami activities were used in guiding students in the process of proving. In the traditional instruction, students used only geometrical statements in order to prove related conjectures. However, in the origami-based instruction, students were mediated through origami folds to deal with the aimed deductive arguments.

Figure 1 presents an example of origami activities in the origami-based instruction. Folding an equilateral triangle is based on Hull (2006). To fold an equilateral triangle, students were initially told to fold and unfold a square paper in half. Then, the teacher wanted students to fold a bottom corner (the D point) up to the fold line and mark the point on the line (the point C). After this, students were guided to fold the equilateral triangle (ACD). Folding the equilateral triangle lasted approximately 15 minutes. After folding, students were questioned about geometric relationships. They were asked why the folded triangle was an equilateral triangle (ACD). As a hint, students were asked to find the angle measures and side lengths of the triangles that were formed after folding. Students had to recognize the congruence of the triangles (ADG and ACG) in order to prove the triangle ACD to be an equilateral triangle (see Figure 2).

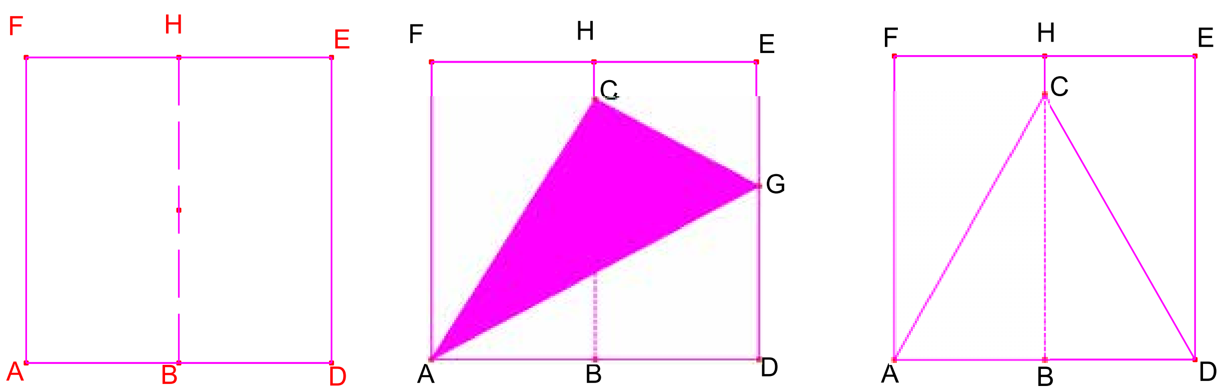


Figure 1: Folding an equilateral triangle

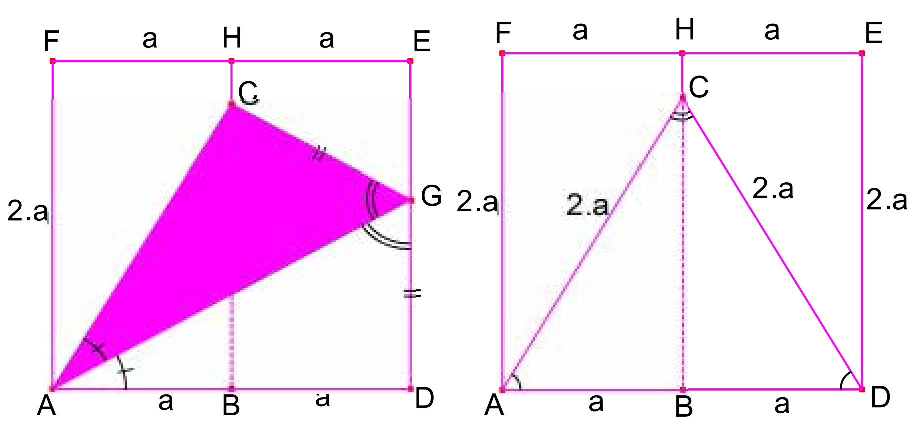
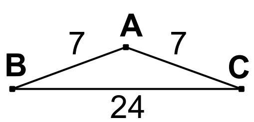


Figure 2: Steps to prove that the folded triangle was an equilateral triangle

Instruments

The pre-tests were administered to all students a week before the geometry instruction. After the instruction, post-tests were administered. The Geometric Reasoning Test (GRT) was prepared to assess students' geometric reasoning abilities related with triangles. Besides, the Geometry Achievement Test (GAT) was formed to assess students' geometry achievement concerning triangles that were instructed during the study. GRT and GAT were piloted before the administration of the tests. After the instruction, the GAT was administered and the GRT was administered one week later from the GAT.

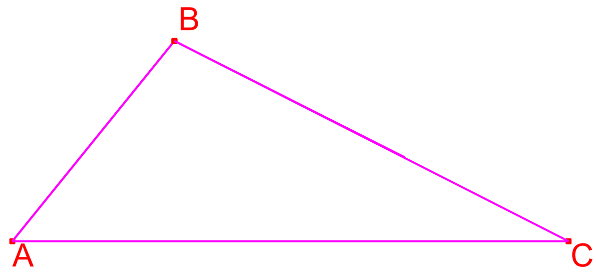
The GAT was developed based on the objectives of the curriculum concerning triangles. There were 11 open-ended items in the GAT. Parallel forms of the GAT were prepared as pre-test and post-test (Figure 3, a sample item from the post-test). The GAT was administered in 45 minutes (one lesson hour). A scoring rubric for the GAT was generated considering the solution steps of the questions. Items in the GAT contain at most four steps so that the maximum score for an item is given as 4. The Cronbach alpha for the GAT was 0.86.



Is it possible that the triangle ABC in the figure with length sides of 7, 7, and 24 units exist? Explain.

Figure 3: An item of the Geometry Achievement Post-test (translated from Turkish)

The GRT was prepared based on the literature about proof and construction questions including triangles (e.g. Jacobs, 2003). Duval's (1998) framework was used as a basis to form the items in the GRT. Duval's (1998) cognitive process of construction includes using tools and GRT has items based on construction. Items were about isosceles triangle, congruence of triangles, angle bisectors, inscribed and circumscribed circles of triangles, angle and side relationships. There were 13 open-ended items in the GRT (Figure 4, a sample item). The maximum score for an item in the GRT is 2 points because an item has at most two steps of the solution. Moreover, the Cronbach alpha for the GRT was 0.70. Scoring of the items in the GRT was based on the Smith's (2010) framework of reasoning. Responses which were aligned with the *argumentation* were given the lowest score in the GRT. Besides, responses which were aligned with the *proving* were given the highest score in the GRT. Proving contains a comprehensive explanation of geometrical expression. Justifying gives some reasons but these reasons are not sufficient to explain the geometric expression. Argumentation also involves explanation but this explanation is not necessarily accurate or geometrical.



How can you draw the incircle of a triangle ABC using lines? Draw the incircle and explain your answer.

Figure 4: An item of the GRT (translated from Turkish)

RESULTS

A repeated measures of Analysis of Variance (ANOVA) was done on each test. Data were analyzed to investigate whether there was any statistically significant difference between mean pre-test scores, between mean post-test scores, between mean pre-test and post-test scores of the experimental group and the control group. The significance level was kept on 0.05 during the analyses. The between-subjects factor was group (control or experimental) and the within-subjects factor was time (pre-test or post-test).

Descriptive statistics for the Geometry Achievement Test (GAT) and Geometric Reasoning Test (GRT) are shown in the Table 2. At the beginning, there were 90 students in the experimental group and 94 students in the experimental group. However, there were missing students during the test time so the missing ones were excluded from the analyses. GRT was administered after GAT so that the number of students who took the tests differed.

Test	Group	$\overline{X}$ (pre-test)	SD	$\overline{X}$ (post-test)	SD
GAT	Control (N=88)	1.52	2.54	5.95	6.79
	Experimental (N=79)	2.13	3.56	12.80	11.05
GRT	Control (N=90)	3.79	1.97	4.79	2.16
	Experimental (N=77)	3.14	2.38	6.42	3.78

Table 2: Means and standard deviations for the tests

According to the repeated measures ANOVA on the Geometry Achievement Test (GAT) scores, the interaction effect between time and group ($F(1,165)=25.02$, $p<0.001$, $\eta_p^2=0.13$), the main effect of group on GAT ($F(1,165)=19.53$, $p<0.001$, $\eta_p^2=0.11$), and the main effect of time on GAT were statistically significant ($F(1,165)=146.60$, $p<0.001$, $\eta_p^2=0.47$). Moreover, the pairwise comparisons for GAT by group and time indicated that there was not a statistically significant mean difference between pre-test scores for the experimental group and the control group ($F(1,165)=1.61$, $p=0.206$, $\eta_p^2=1.61$) but there was a statistically significant mean difference between post-test scores for the experimental group and the control group ($F(1,165)=23.75$, $p<0.001$, $\eta_p^2=23.75$). The results also revealed that the mean difference between post-test and pre-test was statistically significant for the experimental group ($F(1,165)=0.01$, $p<0.001$, $\eta_p^2=0.46$) and for the control group ($F(1,165)=26.68$, $p<0.001$, $\eta_p^2=0.14$). Besides, the mean difference between post-test and pre-test was higher in the experimental group than that in the control group.

According to the repeated measures ANOVA on the Geometric Reasoning Test (GRT) scores, the main effect of time on GRT ($F(1,165)=102.86$, $p<0.001$, $\eta_p^2=0.38$) and the interaction effect between time and group ($F(1,165)=29.10$, $p<0.001$, $\eta_p^2=0.15$) were statistically significant. However, the main effect of group on GRT was not statistically significant ($F(1,165)=1.97$, $p=0.162$, $\eta_p^2=0.01$). Furthermore, the pairwise comparisons for GRT by group and time indicated that there was not a statistically significant mean difference between pre-test scores for the experimental group and the control group ($F(1,165)=3.68$, $p=0.057$, $\eta_p^2=0.02$) but there was a statistically significant mean difference between post-test scores for the experimental group and the control group ($F(1,165)=12.06$, $p<0.005$, $\eta_p^2=0.07$). The results also showed that the mean difference between post-test and pre-test was statistically significant for the experimental group ($F(1,165)=0.01$, $p<0.001$, $\eta_p^2=0.40$) and for the control group ($F(1,165)=12.22$, $p<0.005$, $\eta_p^2=0.07$). Moreover, the mean difference between post-test and pre-test was higher in the experimental group than that in the control group.

CONCLUSION AND DISCUSSION

The results suggested that the origami-based instruction could have an effect on students' geometry achievement and geometric reasoning concerning triangles.

The results about geometry achievement revealed that there was a statistically significant change in geometry achievement scores of students, who received origami-based instruction, from pre-test to post-test time. The geometry achievement scores of students who received traditional instruction also showed a statistically significant change from pre-test time to post-test time. However, the average difference in geometry achievement scores from pre-test time to post-test time for students who received the origami-based instruction was more than that for those who received the traditional instruction.

Although there were some studies that found no significant effect of origami on students' geometry knowledge (e.g. Boakes, 2009), most teachers and authors recommended using origami in geometry teaching to enhance students' geometry knowledge (Golan, 2011; Pope & Lam, 2011). The difference of the test types might have affected the results concerning geometry achievement. For example, Boakes (2009) used a multiple-choice test to measure geometry achievement. But, our study used a geometry achievement test with open-ended items that were prepared in parallel with the curriculum objectives. Besides, the grade level might have affected the results concerning geometry achievement. Boakes' (2009) study involved middle-school students as the sample but our study used high-school students. The design of the instruction also had an effect on students' geometry achievement. The origami-based instruction combined Duval's (1998) cognitive processes of visualization, construction, and reasoning. Such an instruction might have facilitated students' knowledge about the related topic. Golan (2011) reported that using origami in geometry lessons aligned with van Hiele theory helped students to develop their geometry knowledge for higher levels of abstraction. The existing study also took into consideration the van Hiele theory in designing the geometry lessons using origami activities. Besides, Pope and Lam (2011) noted that origami was a good way to enrich school curriculum by providing opportunities for problem solving and creativity. Aligned with the literature, the results suggested that origami could be an additional source of instruction to enhance geometry knowledge of high school students.

The results about geometric reasoning were similar to those about geometry achievement. The results concerning geometric reasoning presented a statistically significant change in geometric reasoning scores of participants, who received origami-based instruction, from pre-test time to post-test time. The geometric reasoning scores of students who received traditional instruction also showed a statistically significant change from pre-test time to post-test time. However, the average difference in geometric reasoning scores from pre-test time to post-test time for students who received the origami-based instruction was more than that for those who received the traditional instruction.

Geometric reasoning involves proving (generating logically correct deductive arguments as Smith (2010) suggested) so that it requires a higher level of geometric thinking. Origami as a tool might have facilitated students to reason at a higher level of abstraction in geometry. The connection of origami with geometric reasoning was also stressed in many resources (e.g. Pope & Lam, 2011; Winckler et al., 2011). For instance, Pope and Lam (2011) presented proof examples that used origami to show that origami could be an important context to develop reasoning. Furthermore, Winckler and his colleagues (2011) stated that origami was an enjoyable way to teach geometric principles to high school students as a bridge between theory and practice. In parallel with previous publications related to origami and reasoning, the results of our study pointed out that origami-based instruction could promote high school students' geometric reasoning related with triangles.

There are certain limitations in our study. For example, classrooms of students were randomly chosen as to be in the experimental or the control group. However, students themselves were not randomly chosen because the school policy did not allow changing students' classrooms.

The effect of origami-based instruction on participants' geometry achievement and geometric reasoning mentioned in this paper implied that origami could be incorporated in geometry lessons. Teachers and curriculum planners should also take into consideration the benefits of integrating origami in high school geometry instruction. Origami is not just for fun but also may be a meaningful context for high-level thinking in geometry.

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LEARNING AND TEACHING GEOMETRY AT THE TRANSITION FROM PRIMARY TO SECONDARY SCHOOL IN FRANCE:

THE CASES OF AXIAL SYMMETRY AND ANGLE

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We are currently running a three year research project focusing on the teaching and learning of geometry at the transition from primary to secondary school in France. In a global approach, we take into account teaching contents, pupils and teaching practices at the same time. This research aims at identifying continuities and gaps in curriculum, textbooks but also in teachers' practices, in order to get a better understanding of the difficulties related to this transition. In this communication, we present the first results of the part of this project dedicated to axial symmetry and angle. Our analysis shows a large variety of choices concerning crucial aspects of the concepts, likely to facilitate the transition or make it more difficult.

INTRODUCTION

This paper presents an overview and the first results of a three year research project that started in January 2012. This study focuses on the teaching and learning of geometry at the transition from primary to secondary school in France. This transition is decisive for pupils future education but difficult for some of them, and may even lead to academic failure, in France as in most of the industrialized countries. Previous studies on mathematics teaching dealing with the questions of transition generally approach these questions from the angle of pupils' difficulties, or they compare the intended curricula showing for example the conceptual changes (see for example Salin, 2003). Colomb *et al*. (1987) take into account teachers too, but they are mostly focused on the teachers' representations of mathematics and mathematics teaching; however, later research (Robert, 2007) showed that representations are not the only factors which influence teachers' practices: Robert distinguishes institutional, social and individual determinants. More recently, Bednarz *et al*. (2009), aiming at developing connections between both orders of education (elementary school and middle school), also point out the necessity of considering teachers' point of view. Given that the involved processes are very complex, we choose to tackle the question with a global approach. We take into account several aspects of the problem (institutional, teachers' and pupils' points of view) and their connections. We thus study the following research questions: what is the nature of pupils' difficulties? In what extent are they connected to the notions? Can we identify, in curricula, textbooks and teaching practices of both levels, elements which can explain these difficulties or, on the contrary, facilitate the transition? Another specificity of this project is to focus on some particular concepts. Eventually, our aim is also to develop resources for teachers and devices for teachers' training in order to facilitate the transition. Datas include videos of class sessions, textbooks and the results of some tests pupils were submitted to.

In this communication we only expose the first results of the project part related to axial symmetry and angle. Those two mathematical subjects are chosen because of their ability to reveal various phenomena concerning the transition, as we will develop in the first part through a curriculum study. Using literature and our previous research, we then point out some crucial aspects of these two concepts as subjects of teaching and learning. In the third part, we study textbooks from both levels of education: we consider them both as resources for teachers to help them interpret the official instructions (OI) and as examples of classical tasks pupils are exposed to in classrooms. Finally, in the fourth part, we show some examples of ways teachers deal with these crucial points. Let us add that in the two last parts, we complete the analyses with the results of the tests.

CURRICULUM STUDY

In France OI for 5th grade (last grade in primary school, 9-10 y. o.) and 6th grade (1st grade in secondary school, 10-11 y. o.) show few differences on the subject of geometry. However, a change of status of the objects, from drawings to figural concepts (Fishbein, 1993) has to be initiated in 6th grade in order to prepare a transition from the paradigm of geometry 1 (G1) to geometry 2 (G2) (Houdement & Kuzniak, 1999). This change of status is explicited in the introduction of the OI (MEN, 2008b) but the way it has to be adapted for each subject is not detailed. This transition is supposed to be complete in 8th grade (Houdement, 2007).

Axial symmetry is one of the subjects usually chosen by textbooks and teachers to initiate this change (Chesnais, 2012). In primary school, pupils learn how to draw mirror images on graph paper, how to draw them on plain paper using tracing-paper and folding and how to identify axes of symmetry on a figure. The work is validated by perception or using tools such as tracing-paper. In 6th grade, pupils are supposed to extend their knowledge to the case where the axis crosses the figure and to learn how to construct mirror images on plain paper using geometrical tools (ruler, set square – triangle – and compass). The validity of the constructions is related to mathematical definitions and properties of the objects.

Concerning the angle concept, elementary school instructions (MEN, 2008a) introduce the right angle in grades 1-2 when pupils learn to distinguish between different geometric shapes (triangle, square …), but the general angle concept is introduced in grades 3-5. In both primary and secondary school, this concept is present in two parts of the instructions: “geometry”, where pupils study plane figures, and “attribute and measurement”. In grades 3-5, pupils learn how to compare angles and how to reproduce a given angle. They discover the different angles, and they must use a set square to validate their estimation of the acute, right or obtuse character of an angle, tasks that relate to paradigm G1. In 6th grade (MEN, 2008b) in the “attribute and measurement” part of the instructions, pupils are firstly supposed to compare angles without measuring them (using templates or tracing-paper); the main teaching objective is then learning how to measure angles with a protractor (using degrees), and how to construct an angle with a given measure, which are tasks related to G1. However the teaching of angle concept in 6th grade can also contribute to the transition from G1 to G2: some tasks may involve deductive reasoning based on the properties of the figures and not only measuring.

Some researchers suggest ways to deal with this transition problem. For example, Perrin-Glorian (2003) suggests a way to get primary school pupils used to considering figures according to their properties; she notably created a new kind of construction tasks called “figure restoration”: pupils are given a figure and the beginning of a reproduction of it; they have to complete it using geometrical tools. In another perspective, Houdement and Kuzniak (2003) question the relevance of teaching G2 within compulsory education. Without entering this debate, we consider that, if G2 is part of the curricula, besides special tasks, a specific work has to be done expliciting the “rules of the mathematical (geometrical) game”.

What we will seek for in textbooks and teacher’s practices are evidence of how they deal with this change of status of objects. In 5th grade, how do they prepare pupils to this change? In 6th grade, how do they deal for example with the apparent contradiction which is inherent to requiring measurement of angles in some tasks and considering it as non relevant when the measure of an angle has to be determined knowing the measure of its mirror image and the property claiming that symmetry preserves angle measures? We expect, for these two subjects (axial symmetry and angle), to observe phenomena of different nature, since they play different roles with regard to the transition from G1 to G2.

CRUCIAL ASPECTS OF AXIAL SYMMETRY AND ANGLE CONCEPTS

Mitchelmore and White (1998) underline that angle is a highly complex and multifaceted concept, which is constructed slowly and progressively. The construction process runs into numerous obstacles (see for example Lehrer *et al.*, 1998). The major one is that many pupils think that an angle's size depends on the length of its sides (Berthelot & Salin, 1994-95; Mitchelmore & White, 1998). Another element likely to hinder the conceptualization of angles and reported in several experimental studies (Baldy *et al.*, 2005, Lehrer *et al.*, 1998) is the prototypical conception of right angle (a right angle that opens on the right with arms parallel to the edges of the paper). The connection between those two obstacles was even recently pointed by Devichi & Munier (2013). Van Hiele (1986) considers that pupils must actively manipulate and experiment with geometric objects. In the same way, Mitchelmore and White (1998) point out the necessity of drawing upon children's informal knowledge to teach them geometric concepts like “angle” (White & Mitchelmore, 2010). In France, Berthelot and Salin (1998) stress a similar idea in saying that pupils should be taught geometric concepts using concrete activities, and they propose an adidactic introduction of angle, entitled Geometriscrabble (Berthelot

& Salin, 1994-95). These authors recommend also that pupils experiment in meso-space. Taking this approach, we proved the interest of using physical situations involving angles that pupils had to model (Munier & Merle, 2009). Mitchelmore and White (1998) suggest another way of invalidating side length: using situations involving both static and dynamic angles, which can be done using technology (Clements & Sarama, 1995). Other studies suggest using body movements to apprehend the angle (Wilson & Adams, 1992; Fyhn, 2008).

Using Vygotski's terms (Vygotski, 1962), we claim that axial symmetry is not only a mathematical concept but also an everyday concept (Chesnais, 2012). Moreover, we distinguish symmetry as an innate property of a figure (static aspect) and as a geometric transformation (dynamic aspect) (Chesnais, 2012). Mathematically speaking, the transformation precedes the property, since the last one results from the invariance of the figure under the transformation. In the everyday concept, the transformation is almost absent, except in the folding movement, but in this case a symmetrical figure is a figure such that half of it is the image of the other half: the symmetrical character of the figure is not associated to global invariance. We claim that linking both aspects is necessary to achieve the conceptualization of symmetry. This led us to point out one of the main misconceptions of symmetry, which is partly related to folding: symmetry as a transformation moving from one half-plane onto the other one. Overcoming this conception is necessary to conceptualize the fact of being symmetrical as global invariance. This misconception and the fact of not being able to link the static and dynamic aspects of symmetry don't prevent pupils to successfully perform classical tasks like constructing the mirror image of a figure located on one side of the axis or identifying axes of symmetry on a single figure. Some tasks may allow children's conceptions to change. For example, constructing the image of a figure crossed by the axis requires considering the symmetry not only as a one-way transformation. Also, having to complete a figure for it to become symmetric (task explicitly recommended in the 4th grade OI) requires to link the static and dynamic aspects of symmetry (Chesnais, 2012).

Curricula remain vague about what should be aimed at in terms of level of conceptualization concerning symmetry. In a G1 perspective, the static aspect of axial symmetry could be taught in a perceptive way or related to folding but it could not then be related to global invariance. However, global invariance can be considered in a G1 perspective, when associated to flipping tracing-paper over instead of folding it. In the perspective of the transition to G2, one must find a way to validate constructions or the existence of axes of symmetry without using instruments. It requires considering axial symmetry as a plane transformation acting on points. Let us add that previous instructions (until 2005) explicitly required teachers to distinguish between the transformation and the innate property and to define the last one as global invariance.

The above analysis shows the complexity of the concepts of angle and symmetry. Given that curricula don't contain details about how to handle these difficulties for each teaching level and how to handle the transition (particularly the change of status of geometrical objects), we can expect difficulties for both pupils and teachers. For instance, 6th grade teachers might overestimate pupils' knowledge about the status of the geometrical objects and figures or the meaning of measurement. They might also underestimate pupils' difficulties. In the following parts we analyze textbooks and teaching practices to identify how they deal with these questions.

TEXTBOOKS STUDY

In this part, we study lessons and exercises proposed by textbooks in order to analyze the way they deal with angle and symmetry; in particular, we will look for tasks designed to help pupils to overcome the above-mentioned misconceptions.

For axial symmetry, we studied eight elementary school's textbooks: *Cap maths* (Hatier), *Outils pour les maths* (Magnard), *Petit Phare* (Hachette), *La clé des maths* (Belin), *Au rythme des maths* (Bordas), *Euromaths* (Hatier), *J'apprends les maths* (Retz), *La tribu des maths* (Magnard). We also studied eight textbooks of 6th grade: *Multimath* (Hatier), *Dimathème* (Didier), *Bréal*, *Transmath* (Nathan), *Magnard*, *Diabolo* (Hachette éducation), *Triangle* (Hatier) and *Phare* (Hachette éducation). Concerning the angle concept, only a few of them were chosen as being representative of the variability of approaches: *Euromaths*, *La tribu des maths* and *J'apprends les maths* for elementary school and *Triangle*, *Phare* and *Sésamaths* (Generation 5) for 6th grade.

At both 5th and 6th grades levels, we could find some textbooks proposing exercises designed to make pupils overcome the sides' length misconception but also some which don't include any exercise of this type, the teacher's handbook sometimes not even mentioning this difficulty. One of the 5th grade textbooks (*Euromaths*) introduces angles using the Geometriscrabble situation designed by Berthelot and Salin.

At elementary school level, concerning the introduction of angle measurement, we could find one textbook which introduces angle measurement using 1° templates, anticipating the 6th grade OI: it even includes tasks in which pupils have to construct figures knowing the measure (in degrees) of some angles. Making other choices, *Euromaths* proposes tasks using fractions of a right angle (in compliance with the 5th grade instructions) which can help pupils to construct the meaning of angle measurement unit.

In 6th grade, the way the different textbooks introduce angle measurement is variable, from textbooks taking explicitly into account the introduction of measurement (using arbitrary units before degrees), to textbooks introducing it very quickly, directly using degrees (supposing obvious for pupils that the measure of the angle formed by two adjacent angles is the sum of their two measures).

Tasks related to G2 are present in all the textbooks: for example, exercises where pupils have to calculate measures of angles, reasoning on freehand drawings. One textbook clarifies explicitly the status of observation, measurement and demonstration but it remains implicit in all the other ones.

Concerning axial symmetry and elementary school, about the link between the static and dynamic aspects, only five out of eight textbooks contain at least one task consisting in completing a figure for it to become symmetric, although it is recommended in the OI; five of them contain a task consisting in identifying axes of symmetry on figures constituted of two different parts. Only two of them (*Petit Phare* and *Euromaths*) do both. These two textbooks are also the only ones trying to make pupils overcome the misconception of transformation from one half-plane onto another: the first one by adding elements on both sides of the axis when completing a figure for it to become symmetric; the second one proposes an exercise where the number of axes of symmetry of a figure is related to the number of ways it can be positioned to match its outline after flipping it over. The *Euromaths* teacher's handbook mentions the objective of enriching pupils' mental pictures. It also suggests teachers to experimentally bring up some properties of symmetric figures like the invariance of the axis' points or the fact that a segment joining a point and its image is perpendicular to the axis. In another perspective, *La Tribu des maths* mixes perceptive tasks and constructions of mirror images on plain paper using instruments, which should be dealt with only in 6th grade.

Half of the 6th grade textbooks separate what corresponds to the static and dynamic aspects of symmetry in two different chapters. Three of them show double figures in the chapter devoted to the axes of symmetry: this facilitates links between the two aspects. About the static aspect, three of them restrict the work to perceptive tasks; half of them mention global invariance but sometimes in an illogical way: *Transmath* introduces global invariance in the lesson right after exercises on folding and before bringing up the transformation; *Phare*, *Multimath* and *Bréal* link folding and global invariance but only *Multimath* mentions it in the lesson. All the textbooks contain constructions of figures crossed by the axis of symmetry.

Finally, we can say that the textbooks' approaches are very different. What we note is that textbooks handle the transition problem in various ways. Namely, some of 5th grade textbooks choose to introduce notions or tasks which correspond to 6th grade's curricula expectations whereas other ones make different choices: for example, *Euromaths* tries to prepare pupils to 6th grade precisely by working on the difficulties, the misconceptions and specifically on the meaning of the concepts (dynamic and static aspects of symmetry, meaning of measurement unit for angle etc.). Depending on the textbooks, it then appears that misconceptions are not necessarily handled in 5th grade. On the other hand, most of 6th grade textbooks consider that some aspects of the two concepts have already been grasped by pupils. Hence, the responsibility of dealing with certain aspects of the concepts and enabling pupils to overcome their misconceptions is devoted to teachers.

TEACHING PRACTICES

The methodology we use to study teaching practices focuses on the mathematical activity the teacher organizes for students during classroom sessions and the way he manages the relationship between students and mathematical tasks in two approaches: a didactical one and a psychological one (Robert & Rogalski, 2005). However we'll only mention here the didactical analysis. We worked on videos made during 'ordinary' lessons on axial symmetry and angle in six 6th grade classrooms. In each class, pupils were submitted to tests designed by the research team. For each teacher, we collected all the videos of the sessions concerning these notions, even if we will only mention in this paper some short extracts from four classrooms (the four teachers are named T1, T2, T3 and T4). Our examples aim at showing how some 6th grade teachers deal with the characteristics of the concepts we mentioned in the second part of this paper.

Example 1: Misconceptions about angles (T1, T2, T3)

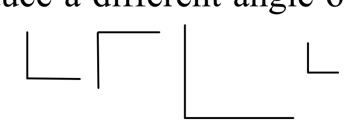
Concerning the sides' length misconception we have seen that instructions recommend starting with comparison activities independently of measurement, and that some textbooks propose such exercises which may facilitate the overcoming of this misconception. Yet the analyses of teaching practices show that some teachers use these exercises (for example T2 and T3), without seeming fully aware of what is at stake. Some of them seem to consider that it is not necessary to work further on this misconception. However, the misconception of the sides' length still appears frequently in both grades when pupils are asked to compare pairs of angles (for one of these pairs, the rate of success in the tests varies between 14 and 67 % in 5th grade and between 43 and 86 % for 6th grade).

As a second example, the following extract of the transcript shows that T1 underestimates this misconception and the salience of the prototypical right angle:

T1: What is an angle for you? […] Océane?
Océane: An angle it is like a right angle
T1: a right angle, OK, a right angle is an angle.

T1 then asks Océane to draw it and she draws a right angle. Then he asks her to draw another angle which should not be a right one. She draws again a right angle, modifying only orientation and sides' length [making it less recognizable]. T1 agrees and says that this angle is too small to be seen from the back of the classroom, then he removes all the drawings and draws two obtuse angles, totally different, with arbitrary orientation of the sides and he goes on with the lesson.

The results of the tests we ran show the salience of the prototypical right angle in both levels: for example when pupils are asked to draw an angle, then a different, a smaller and a larger one, numerous pupils are unable to produce a different angle or to change its size (see example below).



The rate of this answer varies between 0 to 57 % in 5th grade,
and between 0 and 28 % in 6th grade).

Example 2: Misconceptions about symmetry (T2, T4)

Both teachers propose to their pupils a task where they have to overcome the misconception of symmetry as a transformation from one half-plane onto another: T4's pupils have to construct the mirror image of a polygon crossed by the axis on graph paper; T2's pupils have to identify if two triangles crossed by a straight line are symmetrical of each other with regards to this line or not. In both classes numerous pupils seem unhinged by the task and ask questions to the teachers; for example, in T2's class: "Can we still talk about symmetry when the two figures cross each other?" T4's reaction reveals that he underestimates the difficulty, ("nothing makes it impossible", "*don't forget* to construct point C's image"). He mentions the rigor when using the techniques, instead of using a conceptual argument. T2 uses the folding of a piece of tracing-paper to make pupils check that the two parts of the figures coincide one-by-one and conclude that the two triangles coincide: she uses the fact that the transformation is working in the two directions.

Note that the results of T2's class are much better than T4's one for the task of the test where pupils have to construct the mirror image of a figure crossed by the axis.

Example 3: Transition from G1 to G2

Some items of the tests were designed to determine whether 6th grade pupils are able to mobilize the paradigm G2 or not. We ask them to draw an angle measuring 89°, then to say if this angle is a right one or not and to justify their answer. In the 6th grade classes a large part of the pupils (between 11 and 57 %) base their estimation on the validation with the instruments, protractor or set square. We identified moments of classroom sessions when problems related to this question arise.

CONCLUSION

Our analyses point that some of the choices done by textbooks' authors and teachers in 6th grade may cause difficulties for pupils during the transition from primary to secondary school. We have namely highlighted a lack of coherence between primary and secondary school textbooks. Indeed the results of the tests at both levels show that pupils haven't overcome the main misconceptions.

This research project has just started but the first results we expose here seem to confirm the validity of a global approach in order to understand pupils' difficulties in the transition from primary to secondary school. However, more data has to be collected and the analyses have to be completed in order to confirm our hypotheses and especially to study to what extent the observed teaching practices are representative of ordinary practices. Another objective is also to try to evaluate their effects on pupils' learning.

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WHICH GEOMETRICAL WORKING SPACES FOR THE PRIMARY SCHOOL PRESERVICE TEACHERS?

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Abstract: This paper is to describe the various GWS pre-service teachers could be working in, in connection with the different mathematics curricula implemented in primary and secondary schools in the province of Québec (Canada). It results from this study that the GWS[1] of reference in primary and secondary schools seem to be based on a parcelled out geometry GII (GII/GI) but pupils at both levels can succeed while working in a personal GWS being based on an assumed Geometry GI (GI/gII). It derives from this report that the primary school pre-service teachers work in a personal GWS near or identical to the GWS being used by the primary school pupil. Is this an ideal situation for pre-service teachers?

INTRODUCTION

In the Elementary and Secondary School Levels curricula in the province of Québec, Mathematics is presented as one of the best subject for training in reasoning and argumentation, the example given being a geometry task. And doing so, the curriculum refers to a description of different sorts of reasoning in mathematics such as "*deductive, inductive or creative*" reasoning (p. 140).

In the introduction to the Secondary School Level curriculum, one can read that, thanks to mathematics, "*students continue developing the rigour, reasoning ability, intuition, creativity and critical thinking skills they began acquiring in elementary school*" (p. 183). More precisely, Geometry is a mathematical subject particularly relevant for hypothetical-deductive reasoning as:

> in Geometry, students use reasoning when they learn to recognize the characteristics of common figures apply their properties and perform operations on plane figures by means of geometric transformations. […]. They learn the definitions and properties of the figures they use to solve problems involving simple deductions.
>
> [...]
>
> - Observations or measurements based on a drawing do not prove that a conjecture is true, but must be used to formulate a conjecture.(p. 201)

Since no example of geometrical work is given at both schools levels, it is difficult to foresee what kind of geometry pupils are expected to work in and what knowledge of geometry should derive from that. As a consequence one may wonder what kind of geometry will pre-service teachers master when entering their geometry course at

[1] The expressions ETG of reference, suitable ETG and personal ETG, are to be taken within the meaning of Kuzniak (2009)

university. To address these issues I will refer to the three geometrical paradigms as defined by Houdement-Kuzniak (2003) and the Geometrical Working Spaces as described by Kuzniak (2009).

To start with, I will give a very quick summary of the theoretical framework. Then I will analyze some tasks from Primary and Secondary textbooks along with the teacher's handbooks. This will lead to the issue I want to address: in which WGS are Primary School Pre-service teachers able to work?

THEORETICAL FRAMEWORK

Geometrical paradigms

According to Houdement-Kuzniak (2003), in the context of Euclidean Geometry as taught in Primary and Secondary Schools, one can distinguish three paradigms. These paradigms (see Table 1) can be characterized by their components: intuition, experience and deduction, the kind of space the pupil is working in, the status of the drawing and the privileged aspect of the drawing (or the object) for validation.

	Geometry I (Natural Geometry)	Geometry II (Natural Axiomatic Geometry)	Geometry III (Formalist Axiomatic Geometry)
Intuition	Sensible, linked to the perception, enriched by the experiment	Linked to the figures	Internal to mathematics
Experience	Linked to the measurable space	Linked to schemas of the reality	Logical
Deduction	Near to the Real, and linked to experiment	Demonstration based upon axioms	Demonstration based on a complete system of axioms
Kind of spaces	Intuitive and physical space	Physical and geometrical space	Abstract Euclidean Space
Status of the drawing	Object of the study and of validation	Support of reasoning and "figural concept"	Schema of a theoretical object, heuristic tool
Privileged aspect	Self-Evidence and construction	Properties and demonstration	Demonstration and links between the objects. Structure.

Table 1: Geometrical paradigms (Houdement-Kuzniak, 2003)

From different studies (Kuzniak, 2003; Kuzniak –Vivier, 2009; Kuzniak, 2010), it appears that in Primary School most curricula, textbooks and tasks in the classroom refer to GI. GII may appear from time to time in very specific situations. In Secondary School, it seems that either GI or GII can be referred to, the transition from one to the other being an important matter to address (Braconne-Michoux, 2008).

When considering a pupil or a mathematician working in Geometry, these paradigms were to lead Kuzniak to the idea of Geometrical Working Spaces. Moreover observations proved that the different paradigms are interlinked and the boundaries from one another are not clear cut.

Geometrical Working Space

In this paper the words Geometrical Working Space (GWS) will be understood in Kuzniak's meaning (2009):

> The Geometrical Working Space is the place organized to ensure the geometrical work. It makes networking the three following components: the real and local space as material support, the artefacts as drawing tools and computers (…) and a theoretical system of references possibly organized in a theoretical model depending on the geometrical paradigm.

Kuzniak (2009) refined the connections between the different paradigms according to their contributions to each other.

Assumed GI Geometry (GI/gII). In this geometry, the pupil is working on configurations from the real world, the validation relying on perception or measurements. But it may happen that theorems proved only in GII may be of use as "*technical tools avoiding measure or making calculations easier*".

Kuzniak presents an assumed GII Geometry referring clearly to Euclidean Geometry and its logical organisation. But in this geometry, theoretical properties emerge from intuition of space.

Parcelled out GII geometry (GII/GI). In this geometry, properties derive from GI experience. But here some hypothetical-deductive pockets can be developed using the properties already established.

Surreptitious GII Geometry (GI/GII). Adapting Kuzniak's definition of Surreptitious GIII Geometry, we may say that here, geometry teaching is drawn by more theoretical reasons aiming to GII, the pupil being left aside in the shift from one paradigm to the other. Examples of such geometry may be found only in textbooks.

In the province of Quebec only approved textbooks are to be published and are available in the classrooms. So we can assume that the interpretation of the curriculum and its expression in the textbooks are close. In other words, the referential WGS and the appropriate one are close to one another.

GWSs LIVING IN ELEMENTARY SCHOOL

If we are to quote the introduction to the curriculum, we may understand that during Primary School, a pupil works in GI then moves to GII:

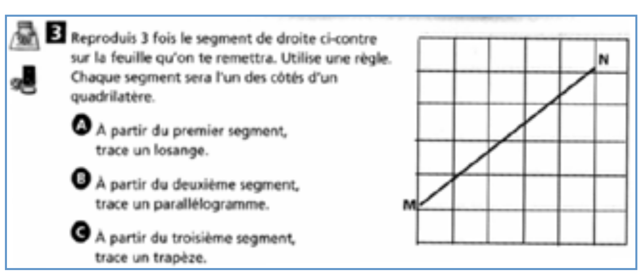
> Mathematics involves abstraction. Although it is always to the teacher's advantage to refer to real-world objects and situations, he/she must nevertheless set out to examine, in the abstract, relationships between the objects or between the elements of a given situation. For example, a triangular object becomes a geometric figure, and therefore a subject of interest to mathematicians, as soon as we begin to study the relationships between its sides, its vertices and its angles, for example. (p. 124)

If the teacher is to guide the pupils to abstraction, the meaning of "a geometric figure" has to be questioned. Is it as Laborde and Capponi (1994) meant it: a geometrical figure establishing the relationships between a theoretical object of geometry and the attached drawing? Or is it in a more general meaning where a precise drawing is made with geometrical instruments? This double-entendre is met further in the curriculum since the outcomes of year-2 are: "*the students* [...] *construct plane figures*..." (p. 147). And at the end of year-6, one can read: "[…] *the students* […] *can describe and classify plane figures,* […] *estimate, and measure or calculate lengths, surface areas …"* (p. 147). If we are to stick to the word "figure", we can suppose that there is a switch in the curriculum from the general meaning to Laborde and Capponi's meaning thus introducing a shift from GI to GII. But at the same time, the pupils can validate their answers mainly by measurement, and keep working in GI.

Furthermore according to tradition in teaching geometry in Quebec, students are working on instrumented drawings and the properties of the figures derive from their global aspect or measurements. So students mainly work in GI and gain some general knowledge from experience and "empirical generalizations". Such information is named "properties". What is the status of these properties? Theoretical or empirical? There is no clear answer to such a question. So if the WGS of reference at the end of Elementary School is probably a parcelled out GII Geometry (GII/GI) even a surreptitious GII one (GI/GII) insofar as proto-axiomatic considerations are to be taught whereas the supports on which the pupils work are drawings, are the pupils working in these very paradigms, the status of which being not clear and most of the validations being visual or instrumented as in GI?

To illustrate this interpretation of the WGS living in the classroom we will analyze some geometry tasks excerpt from two textbooks (year 4 and year 5) by the same authors.

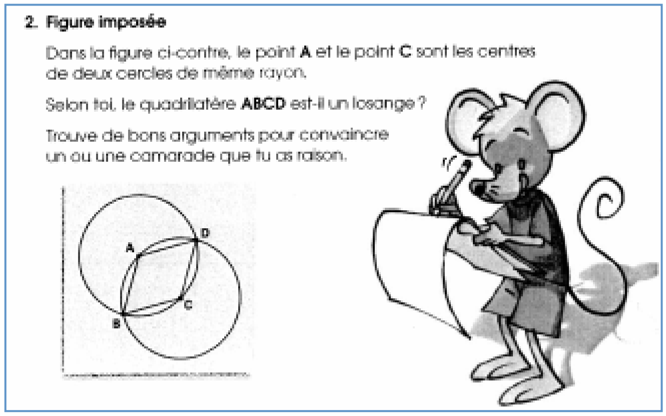
In figure 1 we give an example of a year-4 textbook (Clicmaths, 4[e] année). In task n°3, the questions can be answered in different geometries.



Figure[2] 1: Clicmath 4^{e} année (p. 65)

Here the pupil works on a squared paper with instruments. He/she can validate his/her drawings perceptively thus working in GI. If he/she uses general properties when counting the number of squares in the grid or relying on axes of symmetry, he/she works in an assumed GI (GI/gII).

The example in Figure 2 is to be found in the year-5 textbook by the same authors.



Figure[3] 2: Clicmath 5^{e} année (p. 53)

To answer this question, the pupil can say right away: “It’s a rhombus. I can see it”. But, because of the last sentence, he/she knows that such an answer is not allowed. So he/she can measure the lengths of the different sides and be convinced that it is rhombus. In both cases we can say that the pupil is working in GI. But as the last sentence suggests it, the pupil can guess that he/she must go deeper in his/her reasoning, reaching some part of GII, thus working in a parcelled out GI (GI/gII). The question which may arise then is: how the different reasons should be organized? Is there a first one and a second one or could they be worded in any order? According to the teacher’s handbook, the expected answer is a formal proof which is then clearly in an assumed GII (GII/gI): “*Points B and D are radiuses of the circle of centre A so AB = AD. ...*”.

We can see there a shift from GI to some parcelled out GII Geometry and even an assumed GII but we cannot be sure that the pupils will be aware of this new paradigm. How pupils can make the distinction between the drawings they observe or they produce and the theoretical objects represented? We cannot be sure of that. This exemplifies how difficult it is to manage the shift from GI to GII (Braconne-Michoux, 2008). One could say that the different WGS living in an elementary

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[2] Free translation: “N°3: Copy this line 3 times on the (squared) sheet of paper you’ll be given. Each line will be the side of a quadrilateral.

A. Starting from the first line, trace a rhombus.
B. Starting from the second line, draw a parallelogram.
C. Starting from the third line, trace a trapezoid.

[3] Free translation: “In the figure opposite, point **A** and point **C** are the centers of the two circles having the same radius. In your opinion, is the quadrilateral **ABCD** a rhombus? Find good reasons to convince a friend that you are right.”

classroom are: a WGS of reference aiming to a parcelled out GII geometry or a surreptitious one, a suitable WGS taking into account the WGS of reference but tending to GI and a personal WGS clearly in GI. But the shift from GI to GII is not clear: the drawing often has the status of a figure, without the pupil's knowledge, though it seems that this shift might be the pupil's responsibility.

WGSS LIVING IN SECONDARY SCHOOL

The Secondary School Level curricula are in continuation of the previous ones.

> In geometry, the students make the transition from the observation to the reasoning. They state and use properties, definitions and relations to analyze and solve a situational problem. They construct figures if necessary, using a geometry set or dynamic geometry software. (p. 198)

In this curriculum, in a footnote related the contents of learning , one can read: "*In a geometric space of a given dimension (0, 1, 2 or 3), a geometric figure is a set of points being representing a geometric object such as a point, line, curve, polygon or polyhedron*. " (p. 216). Is a geometric figure to be understood as a theoretical object? Perhaps … The geometric properties the students must know at the end of each cycle are presented as "Principles of Euclidean Geometry" and are worded as geometrical facts: "*All the perpendicular bisectors of the cords of a circle meet in the centre of the circle*" (Cycle One, p. 219) or "*The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.*" (Cycle Two, p. 127). So it seems that the WGS of reference is based on GII. But what sort of GII: an assumed one or a parcelled out one? As we did before we will work on examples, excerpt from a textbook "Perspective Secondaire 1", by the same authors as "Clicmaths".

In activity n°1 (see figure 4), the student is asked to draw specific quadrilaterals.

Situations d'application

À l'aide de tes instruments de géométrie, trace :

a) un parallélogramme ayant un côté de 5 cm ;

b) un losange comprenant un angle intérieur de 70° ;

c) un trapèze n'ayant que deux côtés isométriques ;

d) un parallélogramme ayant des diagonales perpendiculaires ;

e) un losange ayant des diagonales isométriques ;

f) un rectangle dont les diagonales sont perpendiculaires.

Figure[4] 4: activité 1 p 89 (Perspective Secondaire 1 vol. A1)

From questions d) to f) the quadrilaterals have different names (i.e.: in f) a square is drawn). But these particular quadrilaterals are not questioned. Moreover, according to the teacher's handbook, the objective of this activity is to get more dexterity with the

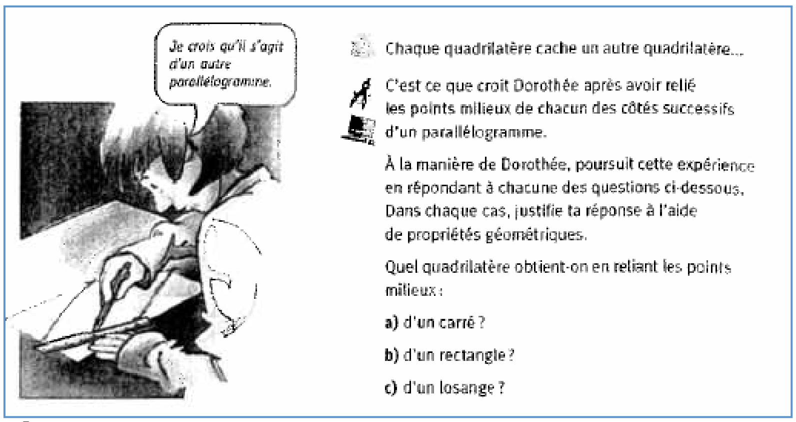
[4] Free translation : "Using your instruments of geometry, trace:

a) A parallelogram with a side 5 cm long; b) A rhombus with an interior angle of 70°; c) A trapezoid having only two equal sides; d) A parallelogram having perpendicular diagonals; e) A rhombus having isometric diagonals; f) A rectangle whose diagonals are perpendicular."

handling of the instruments. So this task has to be worked in GI. In our opinion, an opportunity to move from GI to GII is missed.

Further in the textbook, the student is asked to build different quadrilaterals using two toothpicks as diagonals, draw the figure he/she gets and justify the nature of the quadrilateral. In doing so the student goes back to real objects and his/her validations are mainly perceptive. But as a justification is required, students know they must quote the properties of the diagonals of the quadrilaterals. Here we can consider that the students work in GI using tools of GII, and in doing so,being in some parcelled out GII geometry. Are they reasoning or just giving the expected theoretical answer?

Exercise n°9 (see figure 6), is very interesting since the property of the midpoints in a triangle is unknown at this school level, the pupil cannot give a formal proof as a justification. So one can wonder in which paradigm the pupils may work.



Figure[5] 6: exercice n°9 p. 92 (Perspective Secondaire 1 vol. A1)

They can make a conjecture out of their detailed drawings. The best justification they can give is to report on the position of the diagonals of the new quadrilateral and derive its nature from that. So even if the question seems to be worded in GII, the answer the students can give is in GI: the validation is instrumented or perceptive. The justification given in the teacher's handbook refers to the lengths or the positions of the diagonals ("vertical" or "horizontal" as in their prototypic positions). So it is clearly worded as an assumed GI (GI/gII) task would be. We have here another example of the difficult transition from GI to GII.

In other textbooks in order to justify their conjectures students have to refer to a list of theoretical properties (referring to GII). But they are not asked to organize their reasoning nor have to search for theoretical reasons in order to build a convincing argumentation, along with a deductive reasoning. We can conclude that, though they have been approved by the government, the textbooks offer very rare opportunities to

[5] Free translation: "*Every quadrilateral hides another quadrilateral…*
That's what Dorothée believes after having connected the midpoints of every consecutive sides of a parallelogram.
As Dorothée did it, carry on with this experiment answering every question below. In each case, justify your answer using geometric properties. What quadrilateral do we get when connecting the midpoints
a) of a square? b) of a rectangle? c) of a rhombus?"

work in GII and the suitable WGS rests at most on a parcelled out geometry GII, while pupils keep on working in a personal WGS resting on GI. Since, as far as geometry is concerned, the following years of Secondary School are dedicated to geometrical calculations, many students may leave Secondary School with no experiment of an assumed GII (no formal proof or hypothetico-deductive reasoning).

WGSs FOR PRESERVICE TEACHERS

Most pre-service teachers, now University students, have been trained according to these curricula. So we can assume that their personal WGS rests on an assumed GI geometry or at best a parcelled out GII one, very few of them being able to work in an assumed geometry GII. In other words, many students have a personal WGS which is not different from that of an elementary school pupil. We can fear that this WGS is likely not to be adapted to the teaching of geometry at Elementary School Level, pre-service teachers sharing their pupils' difficulties and their misconceptions.

In order to illustrate our words, we report two class experiments. In the first one, we asked students to do the activity illustrated in Figure 7, excerpt from a year-5 textbook.

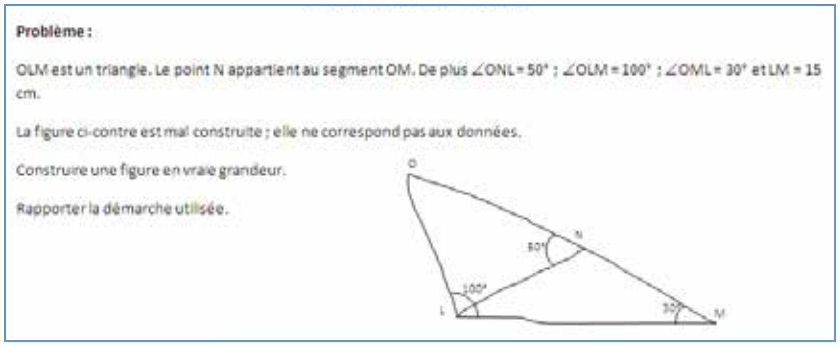
> Minh s'amuse à former des polygones à l'aide de cure-dents.
> Comment doit-il disposer 4 cure-dents pour former un quadrilatère qui n'est pas un carré ?

Figure[6] 7: Presto 5e année

To start with, as no toothpick was available, several students found it difficult to switch from toothpicks to pens. They kept on forming squares with the 4 pens in a prototypic position on the table. We had to help them to get to rhombuses. None of them tried to draw a schema of the situation. After discussion it appeared that this activity had highlighted the fact that several students were facing great difficulties in moving from real objects to represented ones and theoretical ones. These students' conceptions were still GI connected.

In the second one, we asked them to draw the triangle as asked in figure 8, excerpt from Repères-IREM (1993).

[6] Free translation :
Minh is enjoying forming polygons with toothpicks. How should he arrange 4 toothpicks to form a quadrilateral which is not a square?



Figure[7] 8: construction of triangles

For many students, the process consisted in being very careful while using their instruments (protractor and ruler), keeping in mind that ∠ONL should measure 50°. Though all the students knew that the sum of the angles of a triangle is 180°, very few of them were able to use it in order to set up a reliable drawing process. Very few students noticed that the triangle OLN is isosceles in L and none put forward the idea that point N could be drawn using compasses. Most students had been working in an assumed GI (GI/gII), not an assumed GII (GII/gI), just as elementary pupils might do. Then one can wonder how as teachers they will look at their pupils' work and how they will be able to help them.

The origin of the situation is probably to be searched in the different WGSs living at Secondary School level. As we saw in the previous section they are not different enough from the WGSs living in Elementary School: tasks to be answered in an assumed GII (GII/gI) are very rare or non existent in some Secondary School textbooks.

CONCLUSION

In conclusion, we can say that in either Elementary or Secondary Schools, the WGSs in use are based on an assumed GI geometry (GI/gII) tending to a parcelled out GII (GII/GI). The shift from GI to GII being difficult to deal with, more than often, the tasks can be answered successfully by students working in GI. If a stress were to be on a real move to an assumed GII at Secondary School Level, we could hope that this would help our future pre-service teachers in geometry. How can we make students who work only in GI move to GII and make familiar with a new WGS close to the WGS of reference? The concern is that the contents of the handbooks are not always a relevant support and the duration of training is probably too short to let the students adapt to this new situation. We have to set up geometrical questions which cannot be solved in GI and where the students' misconceptions will be revealed. This is a great challenge.

[7] Free translation :
OLM is a triangle. Point N is on the line OM. Moreover ∠ONL = 50°, ∠OLM = 100°, ∠OML = 30° and LM = 15 cm. The figure opposite is badly drawn; it does not fit with the given. Draw a figure at real size. Tell the process you followed.

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(DIS)ORIENTATION AND SPATIAL SENSE: TOPOLOGICAL THINKING IN THE MIDDLE GRADES

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In this paper, we focus on topological approaches to space, and we argue that experiences with topology allow middle school students to develop a more robust understanding of orientation and dimension. We frame our argument in terms of the phenomenological literature on perception and corporeal space. We discuss findings from a quasi-experimental study engaging 9 grades 5-8 students in a 6-week series of school-based workshops focused on knot theory. We discuss video data that shows how students engage with the intrinsic disorientation of mathematical knots through the use of gesture and movement.

Spatial sense, topology, corporeal space, embodiment, knots

INTRODUCTION

Fielker (2011) suggests that we need to broaden our conception of geometry and recognize that "geometries" are diverse kinds of approaches to space, some more suitable than others to the study of movement, transformation, connectedness, dimensionality and orientation. Taking this more expansive approach by inviting students to experiment with spatial reasoning creates an opportunity for students to attend to the corporeal and material aspects of mathematics. We focus on topological approaches to space, and we frame our analysis in terms of the literature on perception and corporeal space. We discuss data that suggests students' gestures often function as devices for orienting a knot in relation to a moving frame of reference. Our analysis suggests that the students are operationalizing gesture and vision and voice to literally dislocate themselves in relation to the knot so as to better explore it, and that the experience provides an opportunity for them to develop the topological concept of dimensionality, which we define as degrees of freedom of movement.

METHODS & DATA

Nine students (grades 5-8), three female and six male, were recruited from a New York State elementary school to participate in six two hour afterschool topology workshops over a three month period. These workshops led by the two principal investigators were offered as non-credit extra-curricular opportunities. The participants had not studied topology previously and their school math curriculum had not provided many opportunities for exploring spatial reasoning. The following questions guided our research: How do students solve problems that entail topological approaches to space? In what ways do gestural-haptic modalities factor into students' spatial reasoning as they engage in problems through topological rather than Euclidean concepts? How does the concept of dimension factor into students' spatial reasoning, and in what ways can problem solving with knots and knot diagrams develop a robust topological concept of dimension? We focused on knot theoretic activities that involved identifying, creating, modifying, comparing, sorting, decomposing and diagramming knots. Various media were used, such as strings, ropes, pipe cleaners, sculpie clay, ipads, and paper and pencils. The software Knot Plots generated dynamic images of knots which could be spun around and modified by the students. The focus of the activities was on how students moved back and forth between image, object and diagram, and how this entailed particular working concepts of dimensionality, directionality and orientation. Although the concepts of invariance and non-rigid transformations were introduced and explored at the first workshop, the PIs gave no other direct instruction, allowing students to develop collaboratively various knot theoretic tools as they problem solved, such as the (un)crossing number, the Reidemeister moves and the colorability constraints. Activities that shed significant light on the questions guiding the research were tasks that were: (1) open-ended and invited inventive diagramming practices for representing 3-D objects, (2) involved working with orientable and non-orientable surfaces, (3) entailed identifying discrete Reidemeister moves in continuously unravelling mathematical knots displayed in video. Project data consists of video and audio recordings of (a) workshops and (b) performance-based interview tasks completed 6 weeks and 6 months after completion of the workshops, as well as (c) drawn artifacts produced by students during both workshops and interviews. In order to focus carefully on the students' entire corporeal space as they worked, we coded the video data by tracking (a) student use of spatial language to describe their activity, and (b) hand gesture as a form of embodied orientation (in relation to a set of coordinate axes).

THE VESTIBULAR LINE & THE VISUAL LINE

Recent research on the kinesthetic perception of physical space and students' deployment of gestural/haptic modalities in problem solving and other mathematical activity has begun to shed light on the complex facets of spatial sense (Nemirovsky & Ferrera, 2009; Núñez, Edwards, Matos, 1999). Hostetter and Alibali (2008), for instance, draw on Gibson (1979) and others to argue that perception and action are mutually determining, and that knowledge emerges through these co-adaptive processes. This "tight coupling of motor and perceptual processes" underscores the ways in which we activate sensorimotor processes when working with concepts (Hostetter and Alibali, 2008, p. 497).

The mathematician Bernard Teissier describes "mathematical intuition" as being a fusion of two modes of perception, the *visual continuum* and the *continuum of motion*. Drawing on recent work in neurophyisiology (i.e Berthoz, 2005), Teissier (2011) puts forth what he calls the "Poincaré-Berthoz isomorphism" that links these two modes, suggesting that their fusion is at the source of mathematical invention "For instance, when we perceive a mathematical line or curve, we actually perceive two fundamentally different things: One, a "vestibular line" that is dynamic, seemingly flowing, "parametrized by time and Rythmed by the steps" (p.237) and two, a "visual line" associated more with boundaries and ambient spatial coordinates. In the case of the *straight* line, we either perceive it in terms of its intrinsic mobility (constant velocity) or as a "curve having everywhere the same orientation" in relation to a frame of reference (p. 238). The cognitive research of Berthoz (2005) suggests that the human mind makes strong links between the vestibular line and visual line, which accords with Poincaré's insights, noted by Teissier, that "the position of an object in space is related to the set of muscular tensions corresponding to the movement we must make to capture it by the equivalent of a coordinate change" (p. 238). The fusion of these two perceptions generates a "protomathematical object" which then lends itself to all sorts of mathematizing.

In order to study the fusion of these two modes of perception, we focus here on topology defined as the study of those properties of geometric objects that remain unchanged under bi-uniform and bi-continuous transformations (Debnath, 2010). Informally referred to as "rubber sheet geometry", topology is concerned with bending, stretching, twisting, or compressing elastic objects (O'Shea, 2007; Richeson, 2008; Stahl, 2005). Despite thousands of years of studying the metric relationships of polyhedra, no one prior to Euler had studied the non-metric relationships of connectedness.[1] Through the work of Gauss, Klein, Riemann and Poincaré, topology became the qualitative study of surfaces, manifolds, boundary relationships and curvature. Topology shifts our attention away from concepts of measure and rigid transformation, and focuses on the stretching and distortion of continuously connected lines and regions. In *Geometry and the Imagination* (1932/1952), David Hilbert claimed that "in topology we are concerned with geometrical facts that do not even involve concepts of straight line or plane but

[1] We know that Liebniz was already familiar with the formula, and historians speculate that Descartes was aware of a similar one (Richeson, 2008). Although Euler's solution to the bridges of Konigsberg problem (1736) comes prior to his letter to Goldbach about polyhedra, the significance of the latter is noted here due to the way it breaks with prior mathematical treatments of polyhredra.

only the continuous connectiveness between points of a figure". In this paper, we are concerned with how particular aspects of topological thinking allow for a better fusion of the two kinds of perception mentioned above. For instance, topology deploys a more robust concept of dimension than what we normally find in the geometry curriculum –rather than the 'size' of a space, topologists refer to dimension in terms of degrees of freedom of movement. Also, the concept of orientation in topology is defined in terms of the capacity to move around space and return to a particular point "oriented" in the same direction as when the motion began, rather than in terms of a fixed Euclidean frame of reference. Finally, the topological focus on distortion and stretching more generally fuses the two kinds of perception, since it transforms the static line into a mobile continuously varying object.

According to Smith (2006), "Euclidean geometry defines the essence of the line in purely static terms that eliminate any reference to the curvilinear ('a line which lies evenly with the points on itself)." He contrasts this rectilinear concept with the "operative geometry" of Archimedes, in which the straight line was characterized dynamically as 'the shortest distance between two points.'" (p. 148). Smith suggests this definition marks the line as a continuous operation and a "process of alignment" pursuing its own inherent variability.[2] A more elastic definition of the line attributed to Heron is "A straight line is a line stretched to the utmost" (Metrica, 4. Gray, 1979, p.128).

Student experiences with knots afford opportunities for developing an understanding of orientation and dimension as operative concepts rather than attributes. Instead of defining the line solely in terms of Euclidean measure and planar existence, knots embody a twisting stretching multi-dimensional line of flight that breaks through the plane. Knot theory emerged in the eighteenth and nineteenth century and developed within the field of topology. Mathematical knots are closed multi-dimensional (knotted) curves that are deemed equivalent if one can be deformed into another (note that this deformation is not a Euclidean rigid transformation). Mathematical knots take up the line as a non-linear and non-planar process of becoming (without end or origin), a process of actualization whereby new dimensions and new entanglements unfold. The curve or line in n-space is a multi-dimensional entity, suddenly possessing perspective and depth. One can relax the knot and loosen its crossings, and then imagine crawling along the rope, following its path into the depth of the page. The knot has no interior or exterior; it is all line, or all outside. Recent interest in including knot theory in the middle school curriculum points to how it helps students develop spatial reasoning

[2] According to Netz & Noel (2007), Archimedes treated diagrams as physical models while attending more to the "broader, topological features of a geometrical object" (Netz & Noel, 2007, p. 105).

through visual and tactile exploration of both quantitative and qualitative invariants (Adams, 2004; Handa & Mattman, 2008). Knot theory offers students a creative and generative approach to 'spatial reasoning'. It captures the dynamic multiplicity of space as a process of dimensional unfolding. One is always in the middle of a knot, pursuing its lines of flight.

KNOTS AND KNOT DIAGRAMS

From 1988-1991, Carol Strohecker ran a "knot lab" in an urban elementary school where 20 fifth graders explored topological thinking through the study of knots. Students used string and other media to compose knots, developing and discussing strategies for doing so. Using the snake method, which involved placing the string on the table, identifying a starting point and fixing one end of the string while moving the other end, students were able to perceive the relationships between strands differently, conceptually integrating the more entangled parts with the various loops that fed into them. The use of this method also seemed to occasion students' efforts at re-orienting their knots (rotating them on the desk), which further developed their "body syntonicity" (Papert, 1980) in that they began to decenter themselves as a perceiver located at a fixed position, and instead identified with the mobility and disorientation embodied in the knot. Strohecker suggests that the medium itself (the pliable string) afforded students an opportunity to develop their spatial reasoning in this way, describing how "many of the children involved their bodies in expressing their conceptions of knots and knot-tying, often relying on their arms or legs to represent ends of string moving into the form of a knot." (p.6). She also indicates that student language use while describing their knots, for instance expressions such as "up/down, top/bottom, above/below, over/under, in/out", revealed how they worked the string as though it were a boundary, dividing space into neighbourhoods that were either in or outside of the knot. Her research clearly shows that students dealt explicitly with topological concepts.

As Kuechler (2001) suggests, the capacity of the knot "to fashion decentred spatial cognition" (p.82) explains in part our fascination with knots in textiles and symbolic forms. She offers an "ethnography of knots" pointing to the prevalence and power of knotted effigies and knotted patterns across various cultures and times. Knots seem to refuse to be seen from one particular point of view or perspective. Knots are all movement along a curvilinear line, evoking fluid spatial relationships. "Each knot is, in a sense, its own universe, which invites contemplation of its topology both as it is being formed and as a completed object" (Strohecker, 1991, p. 215).

For middle school students, one of the biggest challenges – as well as being one of the richest areas for developing spatial reasoning – involves tasks of creating and decoding *diagrams* of knots. Knot diagrams introduce depth into the plane, conjuring a virtual dimension within the two-dimensional surface. The crossings in knot diagrams create a multi-dimensional effect, suggesting a layering precisely where Cartesian geometry would have imposed an intersection. Making sense of knot diagrams demands that one construe an "over/under" relationship in two dimensions and that one follow the continuity of a line as it seems to leap off the surface of the page. Moreover, knots are defiantly without orientation, and yet diagrams are attempts to capture and orient knots on the plane. Students have to decide what is in the foreground/background (and in some cases how that relationship might be evoked) and decide on a perspective and an orientation. Because of the absence of axes and other straight lines to structure one's vision, and because students are learning to pay attention to topological relationships rather than Euclidean ones, drawing knot diagrams often entails positioning oneself (as the observer) in multiple and diverse locations. Châtelet (2006) argues that knots and knot diagrams disrupt fundamental Euclidean spatial practices. They introduce a new manner of intervention and a new way of making mathematical images. They express both entanglement and rupture in the way they disobey the plane. Thus the knot and its diagram contest the usual epistemological barrier between geometric space and corporeal space. Thus any attempt to locate the knot within a mathematical frame of reference is complicated by its 'tied' nature, its folds and twists, connectedness and relationality.

ORIENTATION, MOVEMENT AND CORPOREAL SPACE

We see from the literature that (dis)orientation and dimensionality figures prominently in our engagements with knots. Ahmed (2010) argues that orientation is an essential aspect of spatial sense making. Put simply, "orientations are about the direction we take that puts some things and not others in our reach." (Ahmed, 2010, p. 245). Body orientations thus shape and map space by generating operative axes around which we define our movements. Thus orientation marks a "here" and a "now" from which we proceed. One might even suggest that orientation and motion are mutually implicated. With reference to the phenomenology of Husserl, Ahmed (2010) suggests that orientation marks a "zero-point" or starting point "from which the world unfolds" (p. 236). This implicit structuring of a zero-point for the body entails a sense of movement or potential movement, and this in turn conditions our perception of what is foreground and what is background. Indeed, the perception of depth (and dimensionality) is not as simple as one might first imagine, as it depends on a tactile-kinesthetic fusion of sensory impressions. The body is thus "something that I move with, not something I move, i.e., it has the characteristic of direct motility – I do not have to place my body in order to move it." (Rush, 2008, p. 18).

CASE STUDY

In this section, we discuss data collected during the post-intervention interviews with the student participants of our study. We focus on one task where the students were shown a 2-D depiction of a knot (figure #1), asked to explain if and how they might simplify the knot, and then asked to identify key moves in a video of the same knot unravelling into the unknot. We discuss below how Maya moves back and forth between the vestibular line and the visual line as she engages with this picture of a knot. The image demands a great deal of depth perception since there are loops under loops under loops. This layering of the rope makes for a complex perceptual task. Each time one focuses on one crossing and 'sees' a particular relationship of over/under on a plane of reference, one is then forced to dislocate that plane of reference when either the same two strands reverse the relationship of over/under at the adjacent crossing or a third strand appears beneath the other two and the student has to penetrate the imagined plane to incorporate this third strand in making spatial sense of the relationships. Despite the seeming complexity of the knot, it is reducible to the unknot after a series of moves.

Figure #1

Figure #2

Maya is first asked to count the crossings. She tilts her head and body as her fingers trace the path of the knot (figure #2). She names the crossings as either over or under *because* her finger follows the vestibular line and thus there is a *relative* experience of over or under. Then she stops and looks up and says, "but if you look at it from the other perspective wouldn't it be over or under?" (0:15) When asked to say more, she removes her hands, and explains, "You can see this one going under, this one going under, but you have to like focus on one and the one that you focus would go over or under. (0:52)

In so doing, Maya is shifting from perceiving the vestibular line to perceiving the visual line. With the visual line, the concepts of over and under cannot be assigned to a crossing unambiguously. In other words, at each crossing there are two strands, so there is no sense that a *crossing* has a definitive over or under designation. Such a designation only makes sense if one imagines oneself actually moving along one of the strands on the vestibular line.

Asked where she would start in order to simplify the knot, she picks a strand (lower right for her, labelled A on figure #3 below) and gestures as though she had used her pointer finger to stretch and pin the strand (Figure #4). She affixes this finger to the page, as though she were holding down the rope, while her other hand takes on the gesture of a pincher or grabber, hovering and rocking slightly back and forth in the air above the knot.

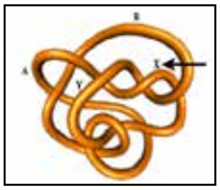
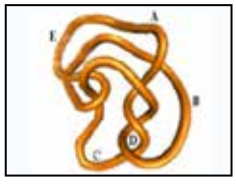


Figure #3 Figure #4 Figure #5

She then places both hands on the table and she taps her fingers rhythmically and in unison while she thinks. She switches her first answer and picks a second strand (labelled X on figure #3), and then she gestures to pull and stretch the strand out from under strand B. She flips her hand palm up (figure #5) to show that it would “go like that”. This is not a pointing gesture, even though it looks like one. It is a flipping gesture that is meant to embody the inversion required as the loop moves from background to foreground. The hand is working as a proxy here as she embodies the new sense of orientation. She could have kept her hand oriented as in figure #3 and simply indicated the grabbing or pincering of the strand and its being stretched to a new location, but in order to capture the new relative relationships between over and under, she inverts her hand so that it’s palm up. This is significant. It shows how she is following the vestibular line using the orientation of her hands (palms up or down or other). What was under is now over, what was up is now down – she is back to following the vestibular line and enacting – through her hands – the changing relationships. In this sense, the hands are proxies for the disorientation embodied in the knot. In relation to the space of the room, Maya is upside down and looking up at the backside of the knot, as though she were on the other side of the paper. Maya is shifting her projected perspective on the knot as she engages with the task. In other words, she is moving around the knot – within, behind, beside, on top – in ways that speak to her embodied engagement. It is usually her left hand that performs the flipping while the right hand performs the stretching. She then pauses saying “Oh, this one’s complicated” and spins the sheet of paper around, until it is oriented as in Figure #6.



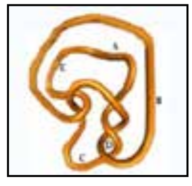


Figure #6 Figure#7 First Move Figure#8 Next Move Figure #9 Next Move

Rotating the paper shows that she is engaged with the disorientation of the knot and that she is aware that the image isn't locked into a particular perceptual grid of 'proper' up/down orientation. She then confidently suggests a series of moves: pulling strand C out, flipping strand C over D and towards B, pulling strand A out, and then flipping strand E over the body of the knot. Although the first two moves don't seem to simplify the knot, the combination of these last two moves is indeed a move that will begin the unravelling (notice that strand E is linked to strand A in such a way that the two are part of a loop that is buried beneath the knot, and three of the adjacent crossings are under the rest of the knot). One can grab and stretch the loop towards the left, eliminating these three crossings (Figures #7).

One of the challenges in this task is 'seeing' into and behind the knot, shifting one's imagined perspective, and noticing these kinds of patterns related to depth and adjacency. This entails shrinking and stretching lines, which we found to be associated with student gesturing of both hands simultaneously. One hand was consistently used to gesture the pulling and stretching of a particular strand, but both hands were used – as though at either end of an elastic – when the students wanted to eliminate a crossing using Redeimester moves #2 and #3.

CONCLUSION

Studying students' experiences with topology revealed how orientation is a complex component of spatial sense. Analysis of video data showed how students' corporeal space entailed a moving rather than fixed perspective, and that gestures embodied this implicit mobility. Rather than seeing the gestures as iconic or indexical, we analyze them as embodiments of perspective. Students' gestures reveal how they follow both the vestibular line and the visual line as they make sense of knots. We see here how the two modes of perception – the visual continuum and motion continuum – were taken up in her gestures as she pursued the shifting orientation entailed in the diagram.

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CLOSE YOUR EYES AND SEE… AN APPROACH TO SPATIAL GEOMETRY

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In this paper we present an experiment that we conducted last spring (May 2012) in a high school in Sicily. The activities concerned some properties of quadrilateral and tetrahedron and aimed to introduce spatial geometry through a correspondence that can be established between the two figures, by using suitable definitions. The study is part of a wide research centred on the investigation of visual challenges involved in doing spatial geometry and the role of technology to address these challenges.

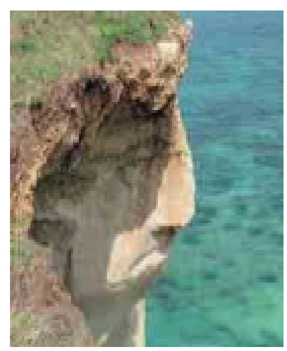
Keywords: visualization, spatial geometry, dynamic geometry software

INTRODUCTION

1. "Good night mum. Please, leave the light on!", used to say little James every single night before falling asleep.

"Good night mum" said little James few nights ago… He was scared of dark. But all of a sudden, he realized that in the dark –well not dark dark!– *he could see*. In the dark, everything was clear. He did not need the night light on anymore.

2. It was June 2012 when Giorgia Florenza, a seventeen-year-old student, taking pictures during an excursion with her classmates near Siracuse, noted that there was *a face* in the rock she was taking a picture of: "With a simple click can come out prodigious things that sometime may escape to naked eye (from: La «scoperta» durante una gita del Liceo Galilei alla riserva del Plemmirio, *La Sicilia*, 11 Giugno 2012, p. 47, our translation).



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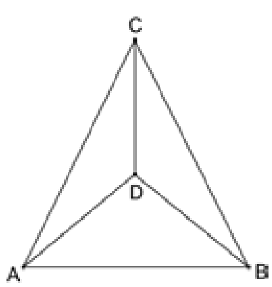


Teacher: And *what do you see* in the figure?

Student: A quadrilateral and/or a tetrahedron.

This is what a student said last May during a classroom activity in a scientifically oriented high school, in Giarre (Catania).

What one –being little James, Giorgia, or the student– sees or is able to see is *not* established *a priori* or universal, but it *may* depend on many heterogeneous factors in many heterogeneous contexts and experiences. It is situated and relative.

This paper presents a teaching experiment on Euclidean Geometry that we carried out in May 2012 in a high school in Sicily. The experiment dealt with some properties of quadrilateral and tetrahedron, and aimed to introduce the study of 3D geometry through a correspondence established between the two figures, by using suitable definitions. This study is part of a wide research whose focus is on aspects related to visualization and visibility, which are crucial for students when dealing with the shape of space and three-dimensional objects.

THEORETICAL CONSIDERATIONS

> The fifteen-year-old students most repulsive subjects in mathematics were spatial geometry and statistics. Only ten percent of teachers taught spatial geometry. They said that they did not have enough time to teach it, but the real reason is that the students 'cannot see in 3D'. (Bakó, 2003, p. 1)

This quotation regards a study that was conducted in France, but it marks a delicate situation that is still recognized as widespread at school, no matter what country is taken. In fact, the teaching of spatial geometry is often neglected and pushed in the background and not given great priority (Mariotti, 2005; Villani, 2006; Oldknown & Tetlow, 2008), partly due to the poor knowledge and confidence of teachers about the topic, and partly to the general reputation that spatial geometry is difficult because it is difficult *to "see"* (among teachers as well as among students).

Nevertheless, 3D geometry plays an essential role in all scientific disciplines, from physics to astronomy and chemistry, from engineering to figurative arts, as well as in everyday life. So, it seems very important to regard the study of spatial geometry at school and the development of spatial sense as crucial for shaping the mathematical literacy required to future citizens aware and able to make decisions. After all, the National syllabus endeavours to promote it as an integral part of the mathematics teaching and learning since early grades. We agree with Villani (2006) that "the cultural and operational-technical worthiness of spatial geometry must prevail over the difficulties of its teaching in all school grades" (p. 66, our translation).

The question of *seeing* is certainly primary. 3D geometry involves visual challenges, mainly depending on the fact that we have to do with representations of geometrical entities, these being bi-dimensional representations of three-dimensional objects. In so doing, the perception of third dimension is not easy, and strongly implies issues of visibility and visualization (see e.g. Hershkowitz, Parzysz & Van Dormolen, 1996). Based on the double aspect of geometrical figures, that is, the distinction between their figural and conceptual properties (Fischbein & Nachlieli, 1998), the transition from the perceptual to the conceptual (from the drawing to the geometric figure) is even more difficult. As Rojano (2002) pointed out, in fact, a

> problem source for geometry students in their transition to the conceptual is the lack of previous visual education that can aid the systematization of their visual experiences, for instance, in the search for patterns or in the distinction between the role of drawings as geometric objects or as diagrammatic models of these objects. (Rojano, 2002, p. 153)

In a study of the late eighties, Parzysz (1988) already highlighted the presence of a "knowing vs seeing" conflict in the teaching of space geometry, which entails that

> The problems of coding a 3D geometrical figure into a single drawing have their origin in the impossibility of giving a close representation of it, and in the subsequent obligation of 'falling back' on a distant representation […] an insoluble dilemma, due to the fact that what one *knows* of a 3D object comes into conflict with what one *sees* of it. (pp. 83-84)

The relationship between vision and perception of space had also been discussed by Jules-Henri Poincaré (1905), who, in his famous book *Science and Hypothesis*, was taking care of the problem of conceptualizing space. He was arguing that space is not a pre-existing concept, our knowledge of it being instead determined by our way of being and staying in the world, that is, by our ordinary experience with space and three-dimensional objects. Our eyes form bi-dimensional images of the 3D world but

> sight enables us to appreciate distance, and therefore to perceive a third dimension. But everyone knows that this perception of the third dimension reduces to a sense of the effort of accommodation which must be made, and to a sense of the convergence of the two eyes, that must take place in order to perceive an object distinctly. (p. 53)

The effort of accommodation of the eye is very important in the play of working with a drawing of a three-dimensional figure and recognizing in it the figure, as well as regarding the capacity to shift from the figural to the conceptual and back. The use of technologies may be relevant with this respect, since

> Computer software for the teaching of 3D geometry should allow students to see a solid represented in several possible ways on the screen and to transform it, helping them to acquire and develop abilities of visualization in the context of 3D geometry. (Christou *et al.*, 2007, pp. 3-4)

Our research interest in this paper is mainly the question of seeing: what one does see when looking at a three-dimensional figure; whether one is able to see a drawing as a three-dimensional figure, and its properties. At the same time, we want to understand how the use of technology may help to face some of the problems that are related to seeing and the study of spatial geometry. Visibility concerns the first face of the coin, meant as making things visible through images, or "thinking in terms of images" as outlined by Calvino (1988). Visualization instead regards the second face of the coin, in which means intervene in supporting making of images. The two faces are both part of thinking processes, neither a final result nor a static part.

In the literature, very few studies have regarded 3D geometry teaching and learning with technology. Some focused on students' exploration of the relationships between geometric figures in solid geometry (Accascina and Rogora, 2006; Baki, Kösa, and

Karakuş 2008; Oldknow and Tetlow 2008), on students' perceptions (Bakó 2003), on the use of virtual reality microworlds (Yeh and Nason 2004; Dalgarno and Lee 2010).

With the activity we consider here, we intend to open a debate about how the study of spatial geometry can be approached at high school, starting from known figures of the plane (quadrilaterals) and simple Euclidean properties of those figures that can be transferred in space through suitable definitions (holding for tetrahedra).

TEACHING EXPERIMENT AND ACTIVITY

The teaching experiment is part of a wider research, in which we became interested last year, with the main purpose to investigate *visual challenges* involved in doing spatial geometry and the role of technology to address these challenges. To this aim, we started from the consideration of previous research that, given suitable definitions, a correspondence can be established between simple figures like quadrilaterals and tetrahedra (Mammana, Micale & Pennisi, 2009). The correspondence entails a move from plane to space that we think of as a possible basis to introduce the study of 3D geometry at high school and that can be realised through the use of Dynamic Geometry Software. The potentiality of the DGS is to allow for visibility and visualization of properties of the figures, bridging the gap between what can be seen and what can be learnt. The cognitive potential of the correspondence as a powerful didactical means is thus of interest for our research.

In particular, this paper regards a classroom activity that we carried out last May in a grade 11 class of a scientifically oriented high school in Giarre (Sicily). The students did not have any formal instruction about spatial geometry. Primary aim of the activity was to introduce them to the discovery that some properties that hold for quadrilaterals in the plane are preserved for tetrahedra in space, when the former are suitably defined. In this way, students are shown two different kinds of figures that share definitions and properties, and are stimulated not only processes of exploration in space but also the need for arguing the validity of certain properties. Here, attention will be drawn only to high school work.

Definitions and properties. To establish the correspondence, we gave the following definitions, using four non-collinear points and the six segments they identify:

1. A *convex Quadrilateral F* is determined by four coplanar points, A, B, C and D, any three of which are non-collinear, called *vertices*, and by the six segments determined by these vertices, AB, BC, CD, DA, AC and BD, called *edges*. The triangles identified by any three vertices are called *faces* of the quadrilateral.
2. A *Tetrahedron F* is determined by four non-coplanar points, A, B, C and D, called *vertices*, and by the six segments determined by these vertices, AB, BC, CD, DA, AC and BD, called *edges*. The triangles identified by any three vertices are called *faces* of the tetrahedron.

Both figures have four vertices, six edges and four faces (letter *F* is expressly used in both cases, as well as same name for corresponding objects). No matter what *F* is, the couples of opposite edges and opposite face and vertex are equally defined. In addition, the segment joining the midpoints of two opposite edges is a *bimedian* of F, the segment joining one vertex with the centroid of the opposite face is a *median* of F. Given these definitions, the properties below are satisfied for F, again irrespective of its being a quadrilateral or a tetrahedron:

Property A. i) The three bimedians of F all pass through one point (*centroid*).

ii) The centroid of F bisects each bimedian.

Property B. i) The four medians of F meet in its centroid.

ii) The centroid of F divides each median in the ratio 1:3, the longer segment being on the side of the vertex of F.

The activities in which our students participated essentially focused on the discovery of properties A and B, and on their preservation in the passage from the quadrilateral to the tetrahedron. From the exploration of the quadrilateral in the plane, one can think of *moving* one vertex off the plane to transform the initial figure and to obtain a polyhedron, that is, a tetrahedron. So, the passage from plane to space can occur by means of the movement of a point, even encouraging to see changes and invariants.

Tasks and methodology. The whole experiment was based on the use of a DGS, Cabri Géomètre. We chose to use Cabri II Plus for the tasks about quadrilaterals and Cabri 3D for those regarding tetrahedra. Even if the students never met it before, Cabri 3D is the only DGS for 3D geometry released up to date. Providing learners with opportunities to redefine points, it furnishes a means to realize the move from quadrilateral to tetrahedron: the *Redefinition* tool. One way this tool works is to change a point into a free point in space, requiring an action that involves the uppercase key of the keyboard (entailing a certain awareness of the action).

The activities were carried out in a computer laboratory, in which the students were divided into groups (of three/four people). Each group had two computer at disposal, to use Cabri II Plus on the one side and, on the other, Cabri 3D. The group work was alternated with class discussions guided by one of the authors, while the other author filmed one group and the collective moments. The teacher of the class was present as an observer of the activity of the students.

The tasks given to the students are shown in what follows, every voice corresponding to an assigned worksheet:

Q1. Definition and identification of the quadrilateral Q and of its main elements: vertices, edges, opposite edges, opposite face and vertex.

Q2. Definition and identification of the bimedians of Q; their properties.

Q3. Definition and identification of the medians of Q; their properties.

T1. Construction of the tetrahedron T; definition and identification of its main elements: vertices, edges, opposite edges, opposite face and vertex.

T2. Definition and identification of the bimedians of *T*; their properties.

T3. Definition and identification of the medians of *T*; their properties.

The worksheets on quadrilaterals had parallel worksheets on tetrahedra. In general, the tasks *Qx* presented three sections: Definition, Construction and Exploration, while those of the kind *Tx* contained sections about Definition and Exploration, except for the first task that asked first to construct the tetrahedron using the Redefinition tool.

The activities centred on conjecturing and discovering, being that of Exploration their main section. Formal proofs were not involved, even if reflections on ways to test conjectures with the DGS were made by the students for the properties on bimedians and medians.

ANALYSIS AND DISCUSSION

In this section, we present short episodes from the activity in Giarre that show the interplay between seeing and the use of the DGS. Attention is drawn to the way seeing in space is encouraged by the use of the software. In particular, we discuss how the discovery of properties in space is prompted and determined by processes of both seeing in space and knowing what happens in the plane. The role of the DGS is crucial in this phase, which leaves room for proof of the validity of properties.

The episodes involve three girls, Chiara, Cristina and Gabriella, that work together in front of two computers, one with Cabri II Plus and the other with Cabri 3D. We are at the second meeting with the class. In the previous meeting, the students have used Cabri II Plus to define the main elements of a convex quadrilateral *F* in the plane and to construct them, completing the requests of the first two worksheets, essentially bound to identifying the defined elements (*Q1*) and to exploring what happens with the bimedians when dragging the quadrilateral from its vertices (*Q2*). They have also learnt the concept of medians of a quadrilateral and how to construct them. During the second meeting, the students first summarised their discoveries in a class discussion ("*the bimedians all pass through one point*", "*we conjectured that the intersection point of the bimedians is the middle point of each bimedian*", from written texts). Then, they started the group work again, looking for properties of the medians of the quadrilateral (*Q3*), finding that they all pass through the centroid. After that, the class comes to deal for the first time with Cabri 3D and the passage from plane to space through the use of the tool *Redefiniton* (*T1*). The groups are first asked to construct a quadrilateral *F* in the visible (grey) part of the base plane, given by default by the DGS (Figure 1a). Then they have to move one vertex of *F* off the plane by redefining it as a point in space and to reflect on the new figure *T* (Figure 1b shows the arrows active during redefinition of vertex D).

First episode (May 21, *T1*). Chiara, Cristina and Gabriella are rotating the figure they obtained with the redefinition and they are watching how it changes. The task asks them to identify its elements, given their definitions:

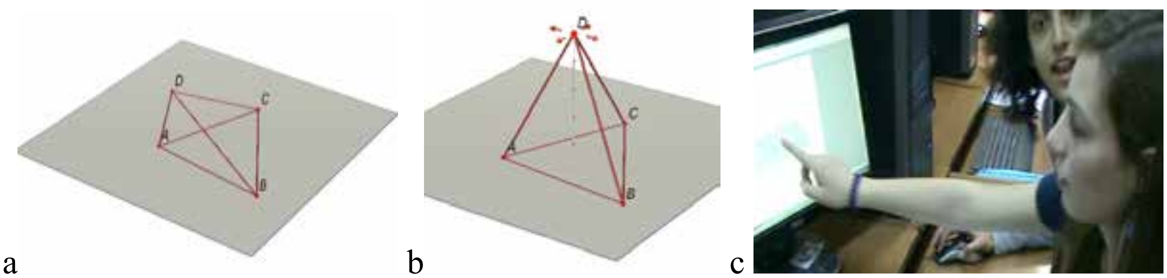
Chiara: There are four vertices A, B, C, D and four edges, that is, DC, BC (looking at the computer screen)…

Gabriella: One, two, three, four, five, six (counting the edges and pointing to them on the screen), yeah

Chiara: The four faces, one, two, three, four (pointing to the faces on the screen)… are (reading the worksheet) the vertices, the edges and the faces of *T*

Gabriella: Here it is, the face opposite to this vertex (pointing to one vertex) is this one (tracing the corresponding triangular face on the screen; Figure 1c)

Chiara: (suddenly, covering Gabriella’s voice while she is still acting on the screen) Yeah! The faces, the faces of a pyramid! Ahhh, they became the faces of a pyramid! (aloud and astonished) Wow, wonderful! (satisfied; Figure 1c). Before they were the faces of a quadrilateral, then with this [meaning the Redefinition] they became those of the pyramid. Good! (convinced)



a b c

Figure 1. a) and b) Quadrilateral and redefinition of vertex D; c) Chiara astonished, and Gabriella acting on the screen

This brief episode shows the surprise of the students when they recognize and are able to see the faces of the pyramid/tetrahedron as previous faces of the quadrilateral (see Chiara’s words “Wow, Wonderful!” and her facial expression in Figure 1c). The recognition occurs through the rotational movement enabled by the DGS that sustains the process of counting and identifying edges and faces of the tetrahedron. By means of this process the students start to discover the correspondence between the two figures (“Before they were the faces of a quadrilateral, then with this they became those of the pyramid”). The use of the verb “to become” is significant: it gives the idea that the passage from plane to space is occurred, as even the final use of the word “Good” accompanied by a convinced face points out. This action of becoming hides the change entailed by the redefinition of point D from a point of the base plane to a point off the plane.

Second episode (May 21, *T2*). The groups are requested to explore and conjecture what happens for the bimedians of a tetrahedron, again using the Redefinition (*T2*). To do so, they are given a Cabri 3D file that already contains a quadrilateral on the base plane together with its bimedians. Chiara, Cristina and Gabriella are looking at the quadrilateral on the screen, when Cristina begins to trace the three bimedians with her fingers.

Chiara: (taking the mouse) Do we have to rise point D? (trying to redefine it)

(The observer suggests rotating the figure to see if the point lies on or off the plane)

Chiara: (realizing that the point is on the plane) Ops, it's on the plane! So, we didn't redefine it (Figure 2a)

Observer: You have to get the arrows to be able to redefine it

(Gabriella takes the mouse and redefines the point correctly)

Chiara: Here it is, now it's ok

Gabriella: Ok (going on to rotate the figure)

Chiara: Arrange it (to see it better). Further up, turn, there... What does it happen to bimedians? (taking the mouse and starting again to rotate the figure very quickly) Hmmm, that they meet in a point in space (Figure 2b)

Gabriella: The same happening for a simple quadrilateral!

Chiara: That the middle point… Wait

Gabriella: That the middle point of the…

Chiara: These are opposite edges (pointing to them on the screen; Figure 2c), so it's the middle point of the opposite edges

Gabriella: No, the point of intersection is the middle point of each bimedian, because the bimedian is the segment that passes through the two middle points of two opposite edges

Figure 2. a) Point D on the base plane; b) and c) seeing the meeting point of bimedians

As soon as Redefinition is applied in the right manner and vertex *D* is raised off the base plane, the three girls do not have troubles to realise that the bimedians "meet in a point in space" (Figure 2). The rotational movement given by the DGS again supports seeing, revealing this property. Through it, the students are constantly changing their visual perspective just as if they were looking at the figure from many different points of view, physically moving around it. Gabriella, in a clever way, underlines the preservation of the property in space, evidencing the correspondence between the quadrilateral and the tetrahedron ("the same happening for a simple quadrilateral!"). This also entails the final conjecture that "the point of intersection is the middle point of each bimedian". The initial explanation given by Gabriella, concerned with the definition of bimedian ("the segment that passes through the two middle points of two opposite edges"), is not sufficient. The conjecture is instead verified using the distance tools of the DGS to measure suitable segments and to compare their lengths.

Once more, the students are able to reason in space transferring what they already know from the previous situation in the plane. Looking at the written texts produced by the groups, we can say that this is not an isolate case. For example, searching for properties satisfied by the medians of a tetrahedron, some of the students write: "Like for the medians of the quadrilateral, the point where they meet divides the medians so that their ratio is 3" (an expression not exactly correct, but that gives the idea).

The role of the DGS is fundamental here. The visualization capacities guaranteed by the actions of rotating and redefinition allow discovering invariances and changes in the move from plane to space, from the quadrilateral to the tetrahedron (and back). At the end of the activities, we had the feeling that the students were keen on shifting in a natural manner from plane to space. This is not only due to an increased familiarity with the DGS. The rotation and the redefinition effected a different kind of vision, one that strongly encompassed embodied engagement. The way to see in space was effectively changed and refined, much that the students became able to treat the object of the introduction both as quadrilateral and tetrahedron, that is, to "see" in a single drawing two geometrical figures (what we may interpret like *to close their eyes and see*). We might say to imagine one or the other, according to what they want to see for the purpose of the task. In this sense, our students learned things that are crucial in thinking about spatial geometry. Especially, they learned to accommodate the eye to make it work as a more expert eye that is able to discern relevant elements of an object, to change perspective, to take on multiple perspectives, and to make visible features that are not present at a first glance. All these factors entail positions from which to view the object, being physical or imagined, and shifts of attention, being overt or covert. This kind of engagement marks that the students enhanced their visual skills in relation to problem solving.

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ARE MATHEMATICS STUDENTS THINKING AS KEPLER? CONICS AND MATHEMATICAL MACHINES

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Our interest is the analysis of the thinking processes of some university students who worked on the design of a machine that uses a tightened thread to draw a hyperbola. Previously, the students worked with other machines for conics. We focus on the way past experience becomes part of a new experience, in which making of the machine is the end point of the task. This implies the presence of technological and scientific aspects, whose interplay is fundamental to shape thinking.

Keywords: instrumental genesis, mathematical machine, transfer.

INTRODUCTION

This paper centres on an activity that asked some university students to think of the way to design a machine using a tightened thread to draw hyperbolas. The activity is part of a course on Elementary Mathematics from an Advanced Standpoint (in the tradition of Klein; Bartolini Bussi *et al*., 2010) that the students attend at the second year of their Master's Degree in Mathematics. The topic of the course considers conic sections and their properties, starting from Greek Mathematics. During the course, the students worked through laboratory activities with some mathematical machines for drawing conics. A mathematical machine is defined as a tool that forces a point to follow a trajectory or to be transformed according to a given law (Maschietto, 2005; Maschietto & Bartolini Bussi, 2011). The students dealt with the use of machines in two manners: first, they explored some curve drawers for ellipse and parabola, and then they were given the task to construct a machine to trace hyperbolas.

Our interest in this paper is focused on how aspects and elements coming from the previous activities with mathematical machines are transferred in the new situation, and on how they imply and originate new ways of writing, new ways of drawing, new ways of thinking.

To this aim, we will cite Kepler's thought about the construction of a new machine. We will also refer to theoretical elements relative to transfer of learning and the utilization schemes that are concerned with the use of an artefact. In the analysis, attention will be drawn to how acquired schemes shape new schemes for the new machine. Additionally, we will address other matters that intervene in the task.

KEPLER, ANALOGY, AND TRANSFER

In his *Ad Vitellionem paralipomena*, Kepler (1604) considers the way to draw conics:

> Analogy also helped me a lot to draw conic sections. From reading Propositions 51 and 52 [*concerning the metric properties involving the foci*] from Apollonius's Third Book,

> one can easily see how to trace ellipses and hyperbolas: these tracings can be made with a thread (Figures 1A and 1B). […] I regretted that for long I wasn't able to describe the parabola in the same way. At the end, the analogy revealed to me that to trace this curve is not much more difficult (and the geometric theory does confirm it) (Figure 1C). (Kepler, 1604, Italian version, pp. 4-5; our translation)

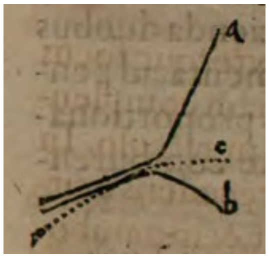
A  B  C 

Figure 1. Kepler's drawings: A) Hyperbola; B) Ellipse; C) Parabola.

Kepler's text (1604) is an example of the use of *analogy* in mathematics. We were already in the final part of our university course when we reencountered Kepler's text. We were surprised of the relevance given to analogy in geometrical reasoning:

> I love analogies a lot, considering them as my very reliable masters, experts of all the mysteries of nature; in geometry, one has to pay attention to them, especially when they enclose –even if with expressions that seem absurd– infinite cases intermediate between their extremes (and a centre), and thus put before our eyes, in full light, the true essence of an object. (Kepler, 1604, Italian version, pp. 3-4; our translation)

Mathematics Education research had studied analogy and analogical reasoning a lot (English, 1997), so Kepler's use was strikingly meaningful. But our aim is not to adopt a specific sense for analogy over the many that have been pointed out in the literature. Instead, here we want to refer to analogy in a naïve manner, following Kepler. He explains how, starting from reading Apollonius, he could trace hyperbolas and ellipses using a tightened thread, but not describe a parabola as easily. Analogy (the main form of reasoning in mathematics at that time) helped him in the case of the parabola. We can think of analogy as "continuity" and extension of thinking, say, transfer of knowledge, since knowledge acquired about the other conics is used in order to describe the parabola. Kepler's analogy as if he were thinking of a machine with tightened thread to obtain the tracing is significant to us.

The type of reasoning adopted by Kepler is interesting with respect to the kind of task given to our students (a kind of task that was chosen with the aim to study the effect of previous tasks). In a manner similar to Kepler, our students can recall aspects and elements of their past experiences with the other machines in order to face the task at hand about the new machine. In these terms, we can think of reasoning by analogy as a sort of 'transfer' of knowledge, of learning. Nemirovsky (2011) speaks of *transfer of learning* "in the context of common and experiential phenomena of learning":

> I see transfer as part of the study of how one experience becomes part of another. People can all sense that experiences do become part of other experiences. It is also clear, I

think, that such participation can be lived in numerous ways, some of which I suggest calling "transfer". (p. 309)

Nemirovsky suggests the importance of developing case studies describing learning experiences as "instances in which an experience clearly comes to be part of another in the view of the subject and/or the authors of the case study." (p. 310).

We present a case study investigating the question: How does previous experience with a parabola drawer and an ellipsograph with tightened thread become part of a new experience when students are asked to think of a machine to draw hyperbolas?

In past tasks, the students were required to study the functioning of a machine. In the new task they are challenged to make a machine to draw hyperbolas. An artefact is being built, so the ways to use it, the subject's utilization schemes, are also decided. Following Koyré (1967), the construction of the new machine implies a "creation of scientific thought or, better yet, the conscious realization of a theory" (p. 106).

UTILIZATION SCHEMES AND MATHEMATICS LABORATORY

Knowledge of a machine involves knowledge of the *utilization schemes* that can be activated with it, for reaching the task's goals. A machine is an artefact, according to the instrumental approach (Rabardel, 1995; Rabardel & Samurçay, 2001). An artefact is a material or abstract object produced by human activity and aimed to sustain new human activity for facing tasks. It is designed and constructed with a purpose and given to a subject. In the hands of the subject, as part of an educational task, the artefact becomes an instrument, a mixed entity composed of both the artefact (object) and the utilization schemes developed by the subject to reach the specific goal of the task. So, the instrument has a subjective and cognitive character. The development of the instrument, that is, the *instrumental genesis*, is composed of complex processes, instrumentalization and instrumentation, linked to the artefact's potentialities and constraints, and to the contextual activity, background and knowledge of the subject. In addition, Guin *et al*. (2005) asserted that instrumental geneses are conceptual geneses, and stressed then their importance for learning, in particular in mathematics. Furthermore, the recognition of the importance of the teacher's action in encouraging and guiding instrumental geneses, gave rise to various directions of study within the instrumental approach (e.g. Trouche, 2004; Trouche & Drijvers, 2010).

From the methodological point of view, activities with artefacts are typical of the *mathematics laboratory* deeply rooted in the Italian teaching tradition, in research studies about innovation in mathematics education, and in the mathematical tradition concerning the use of tools (Maschietto & Trouche, 2010). The laboratory is not meant as a physical space with equipment, but as a structured set of activities aimed to the construction of meanings for mathematical objects (Anichini *et al*., 2004). In laboratory activities are essential the task(s) to be addressed, the presence of tools that one can use and manipulate, the work methodology that affects relationships and interactions (students and teacher), and the presence and role of the teacher.

In this paper, the perspectives of the instrumental genesis and mathematics laboratory are relevant at least at two levels. First, regarding utilization schemes that a student activates to perform a given task with a given mathematical machine. Second, in terms of knowledge that a student uses when a mathematical machine is involved in the task. In the first case, the specific purpose of the activity guides the choice of certain schemes so action is aimed to reach a purpose. In the second case, previous knowledge is used to create new knowledge and meanings so that utilization schemes acquired in the past may be purposefully adapted to the new situation.

The mathematical machines studied by the students make use of a tightened thread and a pencil attached to draw curves for particular conic sections on a flat surface: ellipse, hyperbola, and parabola. These machines work in accordance to determined rules that are connected to the definitions of the curves under consideration.

The laboratory use of a mathematical machine, to study the way utilization schemes previously met are transposed (transferred) in the new situation and how this originates new ways of drawing and thinking, is interesting. To this aim, we need first to look at the kind of tasks given to the university students.

MATHEMATICAL MACHINES AND TASKS

In the course on Elementary Mathematics from an Advanced Standpoint, the students worked with curve drawers for conics [1] in the context of mathematics laboratory, after studying Apollonius's theory of conics (Heath, 1931). In five laboratory sessions we proposed five machines (we just describe in details drawers with tightened thread).

D1: Cavalieri's drawer for parabola [2]. It concerns Menaechmus's definition in which the parabola is identified by a proportion among segments.

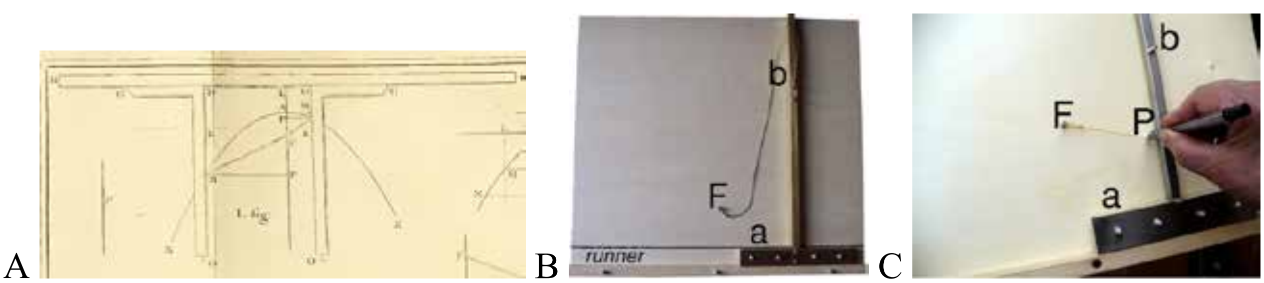
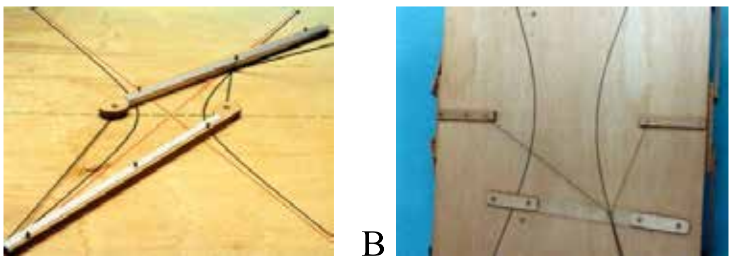


Figure 2. Parabola drawer with tightened thread

D2: Parabola drawer with tightened thread [2] (Figures 2B and 2C). It is composed of two perpendicular rods shaping a T. With respect to Figure 2B, the shortest rod (a) slides on a runner fixed to a wooden flat surface, a thread is linked to the free end of the longest rod (b) and to a pin (F) on the surface. In order to trace a parabola, a pen always has to stretch the thread near the rod (b) as the rod (a) is pushed and moves (Figure 2C). F is the focus of the parabola. When the thread's length is the same as the rod's (b), the runner is the directrix. In his *Traité analytique des sections coniques*, de l'Hôpital (1720) defined conic sections as curves drawn in the plane by particular tools with tightened thread (Figure 2A).

D3: Ellipsograph with tightened thread (gardener's method to draw elliptical flower beds) [2]. Two pins (foci of the ellipse) are fixed to a wooden flat surface. A pen stretches a thread linked to the pins and traces a curve when moving around them.

D4: Hyperbola drawer with tightened thread [2]. From an historical point of view, two kinds of machines were created. One of them (Figure 3B) mainly corresponds to Kepler's description. In both models, two pins (foci of the curve) are fixed to a wooden flat surface. The model in Figure3A shows two foci, two rods constrained each to turn around one focus, and a thread fixed to the other focus and to the free end of the rod. For every rod, a pen stretches the thread near it and traces a branch of the curve when moving around a pin.



A B

Figure 3. Hyperbola drawers

D5: Ellipse drawer with articulated crossed parallelogram [2]. It is composed of a crossed parallelogram with two couples of unequal rods.

In each session, the students worked following the methodology of the mathematics laboratory: group work, collective discussion and individual work. The teacher proposed the exploration of the drawers D1, D2, D3 and D5 through three general questions (in line with other teaching experiments that used mathematical machines). The first question "how it is made" aims to describe the physical structure of the machine and to detect parts and spatial relations. This task supports processes of instrumentalization. The second question "what it traces" focuses on the product of the machine. After putting a lead into a hole or keeping the thread well tightened by the means of a pen (see Figure 2C), the students can trace a curve (they can act on the machine) and analyze it. The third question "why" encourages the students to produce conjectures, to argue them, and to construct proofs.

The task was changed in the case of the drawer D4. The students were asked to imagine a mathematical machine with tightened thread to draw hyperbolas. Different from the previous tasks, the students were faced with a problem solving situation, in which they knew the final curve to obtain, but they did not have any machine to trace it. This specific activity constitutes the core of the paper.

At the end of the group exploration, the students were asked to write an individual report. A collective discussion followed, in which the teacher could share elements of the students' exploration processes and ask about the relationships between the explored machine and the drawn curve.

From the methodological point of view, the eight students attending the course were divided into two groups (A and B). One student per group had the role of observer. Our analysis is based on the students' reports and the notes by the observer.

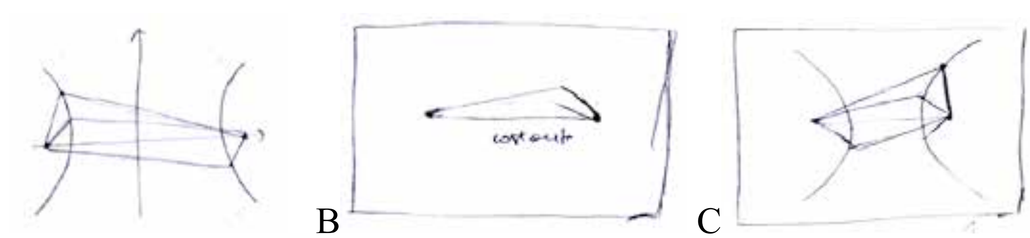
TRANSFER OF LEARNING AND UTILIZATION SCHEMES

In this section, we present sketches of the work of group B, through different phases. The situation can be considered as a learning situation: the students were dealing with a new experience in which they were implicitly required by the nature of the task to use their previous experience with drawers. In fact, the task specifically demanded for thinking of the construction of a new machine. The novelty of the machine lies in the request to draw a new curve: a hyperbola, not in the kind of machine, which still has to use a tightened thread.

We focus on the strategy that group B adopts to face the task, using written texts and drawings produced, as well as the notes taken by a student-observer that followed the entire work of the group. The students are referred to as B1, B2 (observer), B3 and B4. Four phases can be distinguished in their work, the first three depending on the tools with which the students work (paper and pencil; a wooden plan with two pins and a thread; a rod).

1) The students began by stating the metric definition of the hyperbola, writing down the relationship between foci and the generic point, or the equation. From the observer, we know that they made some considerations, like: "Surely, the machine draws only one branch", "The distance between the foci is constant, so the pins will be fixed". Since the beginning, the group thinks of the functioning of a machine that keeps the difference constant, what is present in the metric definition of hyperbola, and considers some components of the machine, like the pins.

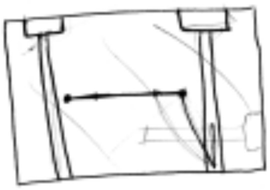
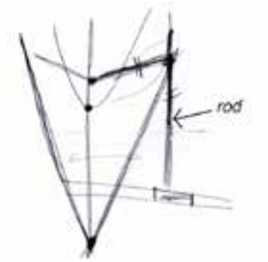
This is where the previous experience with other machines comes into play. First, the students look for "a machine initially similar to the ellipsograph, but that keeps the difference constant" (B3). In so doing, they first join the point with the foci and draw a certain configuration depending on the definition of hyperbola (Figure 4A). Then, they explore the situation recalling the tightened thread of the ellypsograph (Figure 4B). The final tracing of the hyperbola by means of the potential machine (Figure 4C) does not suggest any way to keep the difference constant, so the configuration is abandoned (the rectangles mark the flat surface of the machine, as in Figure 4B).



A B C

Figure 4. B4's sketches

The students go back to previous experience again, recalling the parabola drawer and the rod. Many sketches are produced in which the position of the rod is varied or two rods are considered, one for each focus, in the search for the suitable model for the machine (see Figure 5A).

A  B 

C 

Figure 5. A) B3's sketch; B and C) B1's sketches

This is a phase of instrumental genesis for the rod (a new utilization scheme, a rod added with respect to the parabola drawer) and for the runner (for instance, different positions with respect to the case of the parabola drawer), aimed to the solution of the task, that is, to drawing the hyperbola. The students transfer a component of the artefact (the rod) from experience with the parabola drawer, and activate utilization schemes consistent with the functioning of that machine. For example, the tightened thread with respect to a point of the curve. Figure 5A instead marks the side of the machine as the runner with a rod for each focus. In Figure 5B, the rod is sketched together with the runner it is supposed to move. This constraint changes when the second focus is considered (Figure 5C). Unfortunately, this phase is not successful.

2) The teacher asked the students if they needed some material components, recognizing their presence in the students' discussion and drawings (for instance, pins and thread are also there; see Figure 5A). A wooden plan with two pins and a thread were given to each group. Once the length of the thread is chosen, student B2 tries to put it around the pins, falling in the configuration of the ellipsograph. The trial is soon abandoned since "the shape of the thread does not trace the hyperbola. It is like in the parabola drawer". This reflection is crucial: even if the drawing of the metric definition of the hyperbola is the same as that of the ellipse, the students confirms in a material way that the use of the thread and its related movement has to change. So, the students first try to tie the tread to the pins with the help of a pencil as a rod, then they experiment new movements. Seeking to create a machine that can draw a hyperbola, after a while, they transform the parabola drawer into a new artefact: "Observed that the two foci ordered the distance to keep constant, we removed the runner and put a second point below" (Figure 6).

Figure 6. B1's report

The arrow between the two graphical representations in Figure 6 emphasizes the passage, while the removal of the runner is marked by the dotted line.

In the group discussion, the students highlight that it is not relevant to have rods in a chosen configuration, but the rods must be able to turn. Thus, using pencils as rods, the students can explore new movements.

3) Students' drawings and speech referred to other components of the machine (like the rods in Figure 5A, 5B and Figure 6). Aware of this, the teacher asked student if they need some other materials. As soon as a rod is furnished, two issues became relevant: 1) the way the thread has to be tied, and 2) the position of the pencil. For the latter, the teacher recalls the relationship between the positions of the pen to tighten the thread and the points of the curve. Then, the students are able to overcome troubles and to find a way to proceed when the rod is based on one of the two pins with the ends of the thread "anchored to the two pins, which we have interpreted as foci of the hyperbola, and keeping the thread tightened along the rod, we have tried to represent a figure that looks like a hyperbola" (from B3's written report).

In this way, at the end, the students are able to trace a branch's arc of hyperbola.

4) The curve traced, the students want to justify that it is effectively a hyperbola.

Figure 7. B4's sketch

After some doubts, they first try to use algebraic calculation and then they recover on the machine all the useful parameters, from the length of the rod and of the thread and the constant difference (see Figure 7). However, the students do not make any reference to the fundamental elements of the curve drawer, especially to the position of the pencil.

CONCLUDING REMARKS

Our interests in this paper are on the way previous experience with a parabola drawer and an ellipsograph with tightened thread becomes part of a new experience. This experience in particular asks for the creation of a hyperbola drawer. The task differs from the previous ones, in which a machine was the starting point not the final one. In particular, we have observed the work of one group, B, and we could see that the process of constructing the new drawer implies both technological and scientific aspects (following Weisser, 2005). Technological aspects concern having a machine and the ways it can be used (like in "so the pins will be fixed" or "it has to be able to turn", and in drawings the sketch of a rectangle, rods, etc.). Scientific aspects mainly regard the mathematical constraints that have to be satisfied for drawing a precise curve (like in "the distance between the foci is constant" or "it keeps the difference constant", and in drawings reference to constant difference). The teacher is aware of these two dimensions and of their connections. Sometimes, in fact, she furnishes material objects to the students, so that they can test possibilities of functioning for the machine that before they only could imagine and reproduce by diagrams.

At least for group B, the interplay of technological and scientific aspects is affected by past experience with the other drawers. A lived experience that enters in the new situation bringing components of those machines that the students already used. The interesting thing is that these tools are part of the process in two manners. First, the new machine is being based on variations of the previous machines, recalled at different times (that is, instrumentalizations of those machines). Second, the "ways of use" have to furnish different configurations. In fact, it is when the students discard a configuration that the utilization schemes of the previous drawers clearly appear.

We may possibly speak of *gestures of usage* (or *gestures of ways of use*) that entail movements of the components, these movements recalling or not past experiences (and, in the second case, being similar to thought experiments). The image of gesture implies action. In effect, the instrumented activity is fundamental here; it is the basis for movement and dynamicity, not only of the machine, but also of the students' thinking processes. It is in this sense that we see both continuity and extension of analogous thinking in a way similar to Kepler's. So, when the students say, for example, "like in the parabola drawer", we can interpret that they are thinking as Kepler. In the same sense, we look at the entire situation as an example of transfer of learning. First, it is a learning situation. Second, learning is not acquired by the automatic activation of previous schemes. Instead, it is shaped by the continuous back and forth between technological and scientific aspects entailed by the task that constrains the actors (learners) to constantly review and vary their ideas.

NOTES

1. In Italian secondary school, the conics sections are mainly studied in analytical geometry, starting from their definitions as loci of points.

2. http://www.macchinematematiche.org/index.php?option=com_content&view=article&id=76&Itemid=153

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SYNERGY BETWEEN VISUAL AND ANALYTICAL LANGUAGES IN MATHEMATICAL THINKING

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Visualization is a research field of growing importance in mathematics education. However, the study of its nature and relationship with other forms of recording and reporting information continues to be subject of reflection. In this paper we propose a way of understanding the language and the visual thinking, and their relationship with the language and analytical thinking, using the theoretical tools of the "onto-semiotic approach" of mathematical knowledge. By analyzing the mathematical activity deployed in solving a task, we show cooperative relations between the visual and analytic languages.

INTRODUCTION

Visualization has received much attention as a research topic in mathematics education, especially in the area of geometry (Bishop, 1989, Clement and Battista, 1992, Hershkowitz, and Van Dormolen Parzysz, 1996, Gutiérrez, 1996). Presmeg's work (2006) provides a comprehensive perspective of research on the role of visualization in teaching and learning mathematics in the International Group of PME. This survey concludes by stating 13 big research questions on this research field, and on which we focus on the following, in this paper: "What is the structure and what are the components of an overarching theory of visualization for mathematics education" (Presmeg, 2006, p. 227).

In this paper we are interested in advancing an answer to the problem of devising a theory to clarify the nature and components of visualization and its relationship to other processes involved in mathematical activity, their teaching and learning. A key aspect of developing a theory of visualization in mathematics education should include studying the relations of this form of perception with other ostensive modes of expression (in particular, sequential analytical languages), and especially its relation to non-ostensive mathematical objects (usually considered as mental, formal, or ideal objects).

As background on the problem of theoretical clarification of visualization in mathematics education we found Presmeg (2008), who expands the initial taxonomy suggested by Marcou and Gagatsis (2003) in terms of Peirce's triadic semiotics. Rivera (2011) analyzes the visual root of mathematical symbols and mathematical reasoning, and the implications of visualization to mathematics instruction.

THEORETICAL FRAMEWORK

As it can be read in Phillips, Norris & Macnab (2010) there is no clear consensus on how to define visualization, both in its role of process as object, and in its internal (mental) and external (perceptive) version. In our case we assume Arcavi's (2003, p. 217) proposal, which describes visualization in very general terms: "Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings". However, we consider necessary to deepen the distinction between visual and non-visual objects and processes, in order to study the necessary coordination between both types of objects in the construction and communication of mathematics.

We will analyze the notion of visualization by applying some tools of the "onto-semiotic approach" of mathematical knowledge (OSA) (Godino, Batanero, and Font, 2007). In this framework it is considered that the analysis of mathematical activity, the objects and processes involved in it, first should focus the attention on the practices carried out by people involved in solving certain mathematical problem situations. Applying this approach to visualization leads us to distinguish between "visual practices" and "non-visual practices" (symbolic/analytical practices), and to study the relationships between them. To this end we fix our attention on the kinds of languages and artifacts involved in a practice, which will be considered as visual if they put into play iconic, indexical or diagrammatic signs (Peirce, CP, 2.299).

Although symbolic representations (natural language or formal languages) consist of *visible inscriptions,* those inscriptions will not be considered as strictly visual, but as analytical or sentential. Sequential languages (e.g., symbolic logic, natural language) use only the relation of concatenation to represent relationships between objects. On the contrary, the diagrams use spatial relationships to represent the objects and relationships. "The idea is that sentential languages are based on acoustic signals which are sequential in nature, and so must have a compensating complex syntax in order to express certain relationships - whereas diagrams, being two-dimensional, are able to display some relationships without the intervention of a complex syntax" (Shin and Lemon, 2008, p.10).

Visual objects and visualization processes from which they come, form configurations or semiotic systems constituted by "the intervening and emerging objects in a system of practices, along with the interpretation processes that are established between the same (that is to say, including the network of semiotic functions that relate the constituent objects of the configuration)" (Godino et al, 2011, 255).

In the OSA it is highlighted the essential role of the ostensive dimension in mathematical practice when postulating that every mathematical object (abstract ideal, generally immaterial, not ostensive) has an ostensive facet, that is, publicly, visually, perceptually or otherwise demonstrable. This ostensive facet may consist of symbolic inscriptions, needed to represent the objects, understood as a unitary whole, and to be able to "operate" with them in progressive levels of generality, or using iconic or diagrammatic means that show the structure of the object, understood in a systemic way.

In the next section we show, by analyzing the solving of a task usually considered “of visualization" and using some onto-semiotic tools, that in mathematical activity participate, next to visual ostensive objects, other ostensive objects non visual (textual or analytic) to refer the non-ostensive objects involved (concepts, propositions, procedures, arguments). Both types of ostensive objects (visual and non-visual) play a role in the performance of mathematical activity, so that mathematics teaching should pay attention to the relationship between the two forms of semiotic representation of mathematical objects. Mathematical activity is analyzed theoretically and by using an example as being based on a variety of representational and linguistic or more general semiotic means. The goal is to highlight the intricate interplay of those means.

A MATHEMATICAL TASK AS A CONTEXT OF REFLECTION

In this section we discuss a mathematical task that uses visualization processes. The analysis of the proposed solution shows the network of visual and non-visual objects used, and the relationships established between them, namely, the semiotic system that forms. Briefly, the goal is to reveal the knowledge involved in the resolution and the synergy that exists between visual and analytical objects.

Statement: Write which of these figures represent the unfolding of a cube.

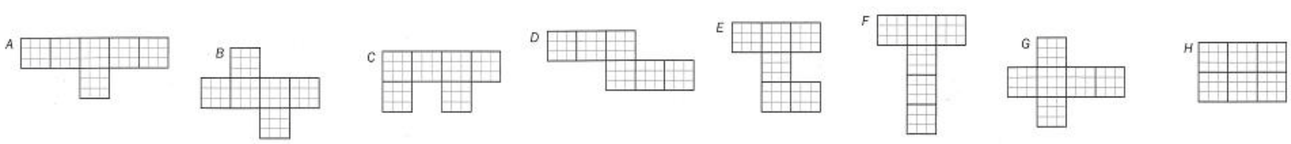


Figure 1: Potential unfolding of a cube

Solution:

The hexamine B, D, F and G correspond to a cube; in fact if we match (paste) the sides marked in figure 2 with the same symbols we get a cube.

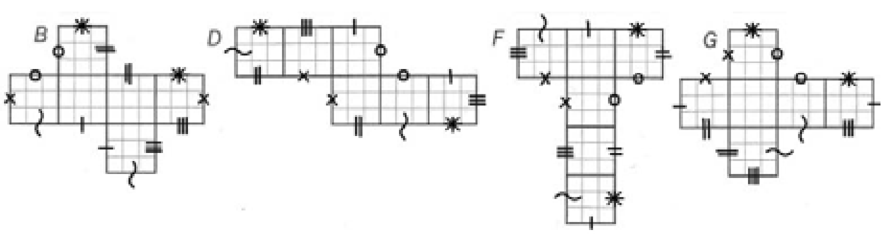


Figure 2: Unfolding of a cube

The remaining hexamine do not represent the unfolding of a cube. Indeed, the hexamine A, C and E do not correspond to a cube because of the overlap of two cube faces to fold the unfolding: in figure 3 we marked with the letters *a* and *b* the faces that overlap.

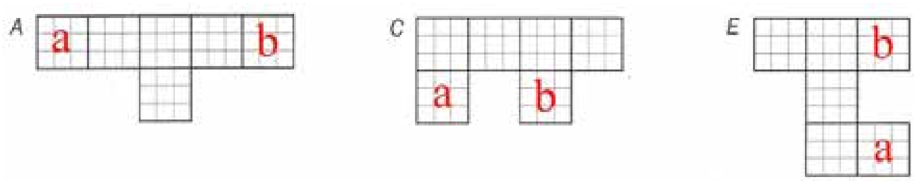


Figure 3: Faces that overlap

It can be visually tested (either empirically or mentally) the impossibility of closing the unfolding.

Onto-semiotic analysis of the solution

This is a visual mathematics task according to the characterization previously presented, but in justifying the answer, different analytical elements that are needed to prove its validity, emerge. To "see" the solution in figures 2 and 3 it is necessary to perform various visual operations and semiotic interpretations that may not be immediately perceived by the students. Such visual operations are supported by analytical elements that define the objects involved. For example, to justify that the sides marked with the same symbol overlap after the folding operation it must be made explicit the knowledge that the adjacent faces of a cube, by definition of this mathematical object, form a dihedral angle of 90°. Therefore, to construct the cube, the contiguous faces of the unfolding must rotate in the space 90º around the segments (which coincide with imaginary exes *x* and *y*). The direction of rotations varies as the considered unfolding.

The wording of the task is made up of linguistic elements (words) and visual elements that interact between them. The term "figure" refers to the drawings in figure 1: visual ostensive objects that have to be put in relation to the non-ostensive object "cube" regulated by a definition. To solve the task, the student must know the intuitive meaning of unfolding a polyhedron and recognize that the surface of the cube is developable. Intuitively, a surface is developable if it can be made from a Euclidean plane by "folding", which is "visually" manifested when it is possible to make appropriate models from a sheet of paper or flat cardboard. The development of the surface would be the plane figure obtained by the whole unfolding in the plane. This intuitive definition has to be applied to the particular case of the surface of a cube. The process of "unfolding" the surface of the cube in the plane can be represented visually (figure 4) and/or interpreted analytically.

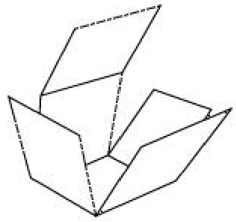
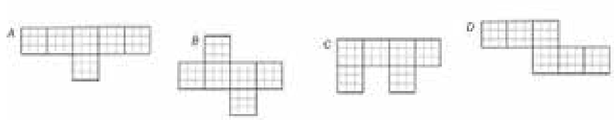


Figure 4: Possible way of unfolding a cube

Since the cube is a polyhedron with six square congruent faces, its unfolding is a set of six squares connected by one and only one side, so that: a) Each face of the cube corresponds to a single square of the development; b) It is possible to match all the sides of the squares that belong to the edge of the unfolding so that each pair corresponds to one and only one edge of the cube. Based on this last (analytical) condition it is argued (a visual way, figure 2) that the B, D, F and G hexamine represent unfolding of a cube, marking with the same symbol equivalent sides, that is, the pairs of external sides of the squares of the unfolding that are joined to form an edge in the cube.

In figure 3 it is visually argued that the hexamine A, C and E do not correspond to a cube due to superposition of a face when folding the unfolding. In fact the superposition of a face contradicts the (analytical) condition that each face of the cube corresponds to a single square of the unfolding. As regards the H hexamine it is showed that it does not represent an unfolding of a cube. This quite intuitive (and visual) statement can be justified asserting the impossibility of constructing a trihedral angle from four faces which sharing one vertex (fairly easy to visualize), or seeing that the representation contradicts the cube unfolding definition.

This example shows the synergistic relationships between visual and non-visual objects in mathematical activity carried out to solve a problem of visual type. In particular, we can observe that the visual explanation of the task solution is supported by analytical elements related to the conceptual properties of the cube development. Some key semiotic functions involved in the visual justification of the task are summarized in Table 1.

Visual expression	Analytical content
"Cube", as mental image (non-ostensive visual object) that the person internally represents.	Concept of "cube" (Definition 1): polyhedron of six square congruent faces.
Figure 1: 	Concept of "unfolding of a cube" (Definition 2): Set of six squares connected by one and only one side, such that: a) Each face of the cube corresponds to a unique square of the unfolding

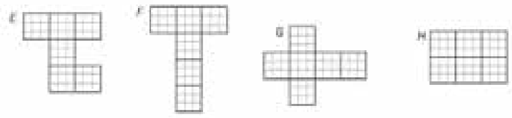
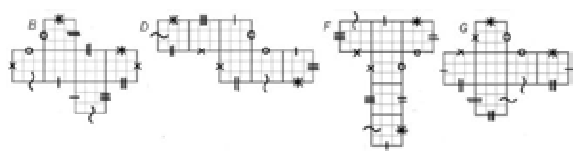
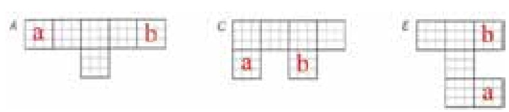
 They are visual ostensive objects referring to potential unfolding of a cube. The recognition of the unfolding of a cube can be accomplished through the visual operation "*fold / unfold*". The procedure may consist of the mental simulation of the physical action (non-ostensive visual process), be physically carried out (cut and paste) (ostensive visual procedure), or illustrated through visual language, as shown in Figure 4.	(property1). b) It is possible to match all the square sides belonging to the unfolding edge, so that each pair corresponds to one and only one edge of the cube (property 2).
If we fold the hexamine B, D, F and G along the sides and join together (physically or mentally) the sides marked with the same symbols, we obtain a cube (Figure 2):  There are not overlap of faces and the unfolding closes. (Visual checking, ostensive or mental)	The hexamine B, D, F and G represent the unfolding of the cube, since it fulfills definition 2. In particular, Figure 2 illustrates the analytical property 2.
In Figure 3 the faces marked with letters *a* and *b* overlap each other to perform the folding and one face would remain uncovered:  (Visual checking, ostensive or mental)	The hexamine A, C and E do not represent the unfolding of the cube because they do not respect the property 1.
It is visually verified (either ostensive or mentally) the failure of closing the unfolding.	The hexamine H is not an unfolding of a cube because the impossibility of making a trihedral angle starting from four faces sharing a vertex.

Table1: Semiotic functions implicated in visual justification

It is not always necessary to deploy the explicit analytical long speech that explains all the rules (concepts and propositions) which effectively support the justifications of the solutions. In this case, visual representations are revealed as a resource of effective expression to convince the reader that, indeed the hexamine B, D, F, and G correspond to the cube development, while this is not the case with hexamine A, C, E and H. But in any case, the rules defining the concepts and properties are still latent.

Fischbein (1993) notes that the mental transformations of three-dimensional objects are not only visual in nature (figural in the author's terminology): it is because we work with faces of a cube which has edges of equal size, faces which are square, angles which are right, and so on. "This is *tacit* knowledge that is involved in mental operations. Without this tacit conceptual control, the whole operation would have no meaning "(p. 159).

The delicate network of visual and non-visual ostensive objects, to refer to non-ostensive objects, always present in mathematical activity (cube concept, face, edge, vertex, and the related properties), and also for the effective realization of procedures and justifications, is put into effect not only with geometric tasks, but also with other mathematical contents. Godino, Gonzato, Cajaraville and Fernández (2012) analyze an algebraic task (proving that the sum of the first n odd numbers is n^2) with the support of visual representation, thus showing the same cooperative relations between the visual and analytical languages.

FINAL REMARKS AND IMPLICATIONS FOR MATHEMATICS EDUCATION

As conclusions of the analysis performed in this paper on visualization, we can say that the configuration of objects and processes used when carrying out a mathematical practice are the following:

(1) It always involves analytical languages in greater or lesser extent, although the task refers to situations on the perceptible world. This is essentially due to the regulatory-sentential nature of concepts, propositions and mathematical procedures.

(2) A non-visual task can be addressed, at least partially, through visual languages which enable to effectively express the organization or structure of the configuration of objects and processes used, especially with diagrams or with metaphorical use of icons and indexes.

Consequently, the configuration of objects and processes associated with mathematical practice will usually consist of two components, one visual and another analytic, which synergistically cooperate on the solution of the corresponding task (figure 5). The visual component can play a key role in understanding the nature of the task and at the time of making conjectures, while the analytical component will be in the moment of generalization and justification of solutions. The degree of visualization used in solving a task depends on the visual or non-visual character of the task and also on the subject's particular cognitive styles that resolved the task, as has been emphasized by several studies (Krutestkii, 1976; Presmeg, 1986; Pitta-Pantazi and Christou, 2009).

The analysis of visualization we have carried out, using some of the OSA tools, provides a complementary view regarding to other perspectives more focused on the description of visual/analytical cognitive styles and its influence on problem solving. Our goal has been to deepen into the nature of visualization and its relation to analytic-sequential forms of mathematical thinking. We sought to characterize mathematical practice in tasks involving visualization, whether performed by an individual (subjective knowledge), or shared in an institutional framework (objective knowledge), identifying the types of objects and processes involved in the performance of the practice.

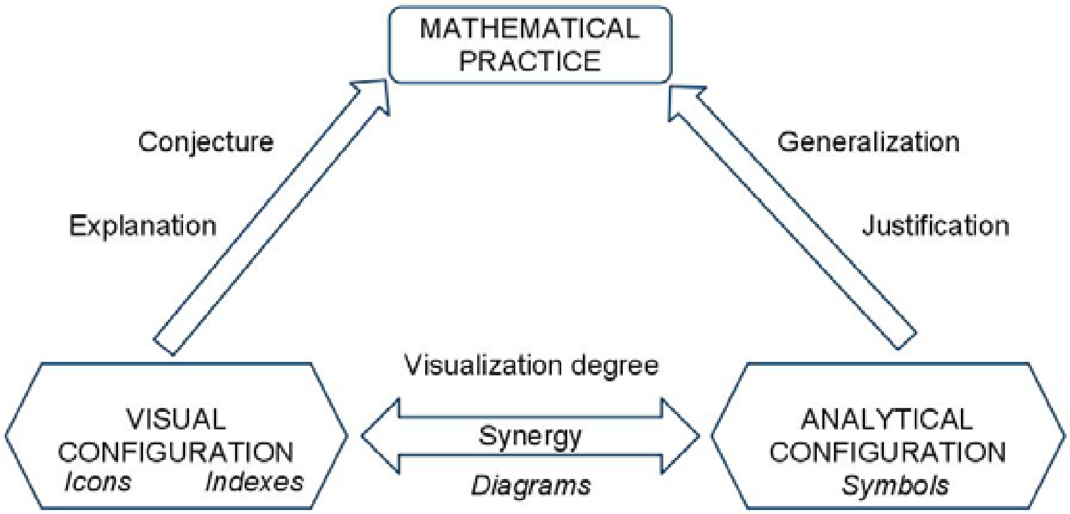


Figure 5: Synergy between visual and analytical configurations

A visual task can be tackled with analytical tools and vice versa, a non-visual task can be approached analytically with visual tools. Moreover, in conducting a visual practice non-visual objects are actually involved, and in the implementation of an analytical practice, visual objects, particularly diagrams, may be involved. This is a result of the implementation of the ostensive non-ostensive duality (Godino et al, 2011) to different types of mathematical objects, which carries out the dialectic between the visual and analytical. For any mathematical object the presence or intervention in its emergence and operation of an ostensive aspect (public, visible, symbolic or visual) and other non-ostensive aspect (rule, logic, ideal, mental) which interact in a synergistic way is postulated, as was shown in the analysis of the example in section 3.

An educational implication of our analysis is that subjects whose cognitive style is basically analytic (respectively, visual) should be instructed to develop visual skills (respectively, analytical), because both skills are useful for mathematical practice at different stages of their execution. Hence, it would be necessary to favor the development of the harmonic cognitive style described by Krutestkii (1976), which combines visual and analytical features.

It seems clear that visualization penetrates in all branches of mathematics, not only in geometry, in coordination with other forms of expression, especially analytical/ sequential languages. It is also present in the various levels of mathematical study, as well as in elementary as in higher education, or even professional. However, the analysis of the relative effectiveness of visual modes of reasoning regarding analytical modes, depending on the types of tasks and phases of study, is a subject that requires investigation. The interest of using iconic and diagrammatic representations has been generated by the assumption that somehow they are considered more effective than traditional logical representations for certain tasks. However, although there are some psychological advantages in using diagrams, they are often ineffective as representations of objects and abstract relations (Lemon and Shin, 2008).

The role of visualization in school or professional mathematical work is complex because it is often interwoven with the use of symbolic inscriptions, which although "are visible", their meaning is purely conventional. The problem is relevant even when the visualization referred to the use of visual objects, which interact not only with symbolic inscriptions, but also and mainly with the network of conceptual, procedural, and propositional objects that necessarily intervene in mathematical practice.

Teachers, curriculum developers and teacher educators should be aware of the role of visualization in building and communicating mathematics. On the other hand, one should not confuse the mathematical object with its ostensive representations, whether visual or otherwise. It is necessary to take into account the non-ostensive immaterial nature of mathematical objects and the complex dialectical relationships that are established between these objects and their material representations.

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CHARACTERISING TRIANGLE CONGRUENCY IN LOWER SECONDARY SCHOOL: THE CASE OF JAPAN

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Congruence, and triangle congruence in particular, is generally taken to be a key topic in school geometry. This is because the three conditions of congruent triangles can be used when proving geometrical theorems and also because triangle congruency relates to the idea of mathematical similarity via similar triangles. Despite the role of congruency, and of congruent triangles in particular, there appears to be limited research on the topic. In this paper we report on the approach to triangle congruency in textbooks used in lower secondary school in Japan, specifically in Grade 8 (students aged 13-14). We found that the modicum of practical tasks entailed various conceptions (measuring, transformations, etc.) whereas the proof problems which predominated expected students to utilise corresponding parts and known facts.

Keywords: geometry, congruency, textbooks, Japan, secondary school

INTRODUCTION

The thirteen books of Euclid, compiled over 2000 years ago, are universally acknowledged as a ground-breaking example of the use of the deductive method in proving a collection of mathematical theorems. Covering mostly plane and solid geometry, but also some aspects of elementary number theory and algebraic methods, the set of books subsequently became what Boyer (1991, p. 119) has called "the most influential textbook of all times". Many of the geometrical theorems that are proved in the books of Euclid use the three conditions for triangle congruency: the side-angle-side (SAS) condition (proved as Proposition 4), the side-side-side (SSS) condition (proved as Proposition 8), and the side and two angles (SAA) condition (proved as Proposition 26). These three conditions for congruency (SAS, SSS, SAA) are subsequently used in the books of Euclid to prove many subsequent propositions.

The power of the three conditions for triangle congruency for proofs in geometry, exemplified by the books of Euclid, together with the link from triangle congruency to the even more powerful mathematical idea of similarity (via the topic of similar triangles), means that triangle congruency generally constitutes a key topic in school geometry (exceptions occur when a transformations approach is used in which isometries are transformations that are equivalent to congruence). As an example of the position of congruence, the curriculum framework for TIMSS (Trends in International Mathematics and Science Study) specifies geometry within two components; "position, visualization, and shape" and "symmetry, congruence and similarity" (Robitaille et al., 1993, appendix C).

Despite the importance attributed to congruence, there is, as far as we have been able to ascertain, limited research on the teaching of congruency (at least, limited research published in English). Exceptions include González and Herbst (2009) and some papers in Japanese; the latter including Moriya, Kondo and Kunimune (2005), which provides ideas for daily lessons, and Shimizu (1979), where the mathematical and pedagogical values of the congruency conditions are discussed.

The approach of González and Herbst (2009) was to utilise the notion of 'conception' from Balacheff (see Balacheff & Gaudin, 2003, 2004; plus see Mesa, 2004), a notion derived from the epistemological position that "A knowing is characterised as the state of dynamical equilibrium of an action/feedback loop between a subject and a milieu" (p. 8). Referring to this framework, González and Herbst define four "conceptions of congruency" (ibid p. 155) and show how when using dynamic geometry software "There was evidence for students' learning in their shift from a visual perception conception of congruency to a measure-preserving conception of congruency" (ibid p. 179).

This raises the issue of how triangle congruency is characterised in other situations. As part of a larger-scale project examining how geometry is presented in school mathematics textbooks (see, for example, Fujita & Jones, submitted; Jones & Fujita, in press), here we report on a component of the project that addressed the following research questions: what characterises the approach to triangle congruency in textbooks used in lower secondary school in Japan? In what follows, we first expand on our focus on the geometry curriculum and school mathematics textbooks. Next, we outline the theoretical framework of four "conceptions of congruency" as introduced by González and Herbst (2009, p. 155). Following an explanation of the context for our study, and an outline of our methodology, we then provide our analysis of sample chapters from a Grade 8 textbook commonly used in Japan for students aged 13-14.

THE GEOMETRY CURRICULUM AND SCHOOL TEXTBOOKS

While geometry is an indisputably-important component of the school mathematics curriculum, arguments about the structure of the geometry curriculum have been going on for at least one hundred years (see, for example, Sinclair 2008; Usiskin 1987). As one well-known curriculum team from the early 1970s commented "Of all the decisions one must make in a curriculum development project with respect to choice of content, usually the most controversial and the least defensible is the decision about geometry" (The Chicago School Mathematics Project staff, 1971, p. 281). Such controversy means that the geometry provides an intriguing focus for studying how the mathematics curriculum is enacted through the medium of school mathematics textbooks.

In terms of textbooks, evidence from TIMSS (such as that documented in Valverde et al. 2002) has revealed the extent to which such textbooks provide the link between the *intended curriculum* that is contained within National Curriculum documentation and the *attained curriculum* that is learnt by students. This is done through the *implemented curriculum* that is taught by teachers. Textbooks lay out what Foxman (1999) has called the ‘potentially-implemented’ mathematics curriculum; that is, the curriculum that the teacher might implement in actual classroom practice. According to Thompson, Senk and Johnson (2012, p. 254) textbooks are a “particularly critical link between the intended and attained curriculum in school mathematics” because “they help teachers identify content to be taught, instructional strategies appropriate for a particular age level, and possible assignments to be made for reinforcing classroom activities”. It is how triangle congruency is enacted in school geometry textbooks in Japan that is the focus of what we present in this paper. It is to that context that we turn next.

TRIANGLE CONGRUENCY IN SCHOOL MATHEMATICS IN JAPAN

The specification of the mathematics curriculum for Japan is given in the ‘Course of Study’ (MEXT, 2008). Mathematical content is divided into ‘Numbers and Algebraic Expressions’, ‘Functions’, ‘Geometrical Figures’ and ‘Making Use of Data’. Our focus is ‘Geometrical Figures’ in Grade 8; Table 1 gives the detail of this topic.

(1) Through activities like observation, manipulation and experimentation, to be able to find out the properties of basic plane figures and verify them based on the properties of parallel lines.
(a) To understand the properties of parallel lines and angles and basing on it, to verify and explain the properties of geometrical figures.
(b) To know how to find out the properties of angles of polygons based on the properties of parallel lines and angles of triangle.
(2) To understand the congruence of geometrical figures and deepen the way of viewing geometrical figures, to verify the properties of geometrical figures based on the facts like the conditions for congruence of triangles, and to foster the ability to think and represent logically.
(a) To understand the meaning of congruence of plane figures and the conditions for congruence of triangles.
(b) To understand the necessity, meaning and methods of proof.
(c) To verify logically the basic properties of triangles and parallelograms based on the facts like the conditions for congruence of triangles, and to find out new properties by reading proofs of the properties of geometrical figures.

Table 1 ‘Geometrical figures’ in course of study for Grade 8 in Japan

From Table 1 it can be seen that the understanding of congruent figures is one of the main objectives. Also it can be seen that the conditions of congruent triangles are expected to be used in verifying various properties of geometrical figures. Our interest is what characterizes the approach to congruency presented in textbooks as this mediates between the intended curriculum (in Table 1) and classroom practice.

ANALYTIC FRAMEWORK AND METHOD

As we note above, in this paper we are following Balacheff and Gaudin, (2003) and González and Herbst (2009, p. 154) in taking a ‘conception’ as being “the interaction between the cognizant subject and the milieu – those features of the environment that relate to the knowledge at stake". By this framework, a conception comprises the following quadruplet (P, R, L, ∑) (Balacheff & Gaudin, 2003; González & Herbst, 2009):

- P: a set of problems or tasks in which the conception is operational,
- R: a set of operations that the agent could use to solve problems in that set,
- L: a representation system within which those problems are posed and their solution expressed, and
- ∑: a control structure (for example, a set of statements accepted as true)

In their paper, González and Herbst (2009, pp. 155-156) propose the following four conceptions of congruency.

- The *perceptual conception of congruency* (PERC) "relies on visual perception to control the correctness of a solution to the problem of determining if two objects (or more) are congruent".
- The *measure-preserving conception of congruency* (MeaP) "describes the sphere of practice in which a student establishes that two objects (e.g. segments or angles) are congruent by way of checking that they have the same measure (as attested by a measurement instrument)".
- The *correspondence conception of congruency* (CORR) is such that "two objects (segments or angles) are congruent if they are corresponding parts in two triangles that are known to be congruent".
- The *transformation conception of congruency* (TRANS) "establishes that two objects are congruent if there is a geometric transformation, mapping one to the other, which preserves metric invariants".

For our analysis we selected *Mathematics 2* because it is widely used in Japanese lower secondary schools (Grades 7-9). Our first step was to identify the geometry lessons in the Grade 8 textbook. We found that there were 34 geometry lessons, with the idea of congruent triangles introduced in lesson 8. Following this lesson that introduced the idea of congruent triangles, a further 22 lessons referred to congruent triangles. In terms of proof, of the 23 lessons that referred to congruent triangles, 19 included some proving opportunities (for additional analysis, see Fujita & Jones, in preparation; Jones & Fujita, in press).

Our next step was to look closely at the first eight lessons on congruent figures (presented in Chapter 4 Section 2 of the textbook). The lesson details are given in Table 2. From these lessons, we analysed tasks from lessons 8, 9, 10, 11 and 13 (we excluded Lesson 12 from our analysis because this lesson aimed at consolidating students' knowledge and understanding of elements of proofs such as suppositions, conclusions, and so on).

Lesson No.	Description
8	Introduction to congruent figures with 1 activity & 2 problems
9	Introduction of conditions of congruent triangles 1 activity & 1 problems
10	How to use the conditions of congruent triangles with 3 problems
11	Proofs using the conditions of congruent triangles with 1 activity
12	How to write proofs with 1 activity and 2 problems
13	Two proof problems with section summary

Table 2: Lesson details in chapter 4 section 2 congruent figures

Our next steps were to undertake an *a priori* analysis of the tasks in the textbooks:

- identify the quadruplet (Problems; Operations; Representation system; Control structure) in the selected lessons in the Grade 8 textbook;
- use the information from our analysis to characterise the approach to triangle congruency utilised in the textbook

• P: Problem	9Pa: The use of given data.
R: Operation	9Ra: To measure given data by using measurement tools. 9Rb: To draw triangles by using given data. 9Rc: To observe the figure resulting from movement. This involves either physical or mental manipulations. 9Rd: To measure sides and angles of triangles. 9Re: To overlap two constructed triangles. 9Rf: To observe appearance of triangles.
L: Representation system	9La: The constructed diagram is the medium for the solution of the problem. 9Lb: Measurement tools are the medium for lengths of sides and sizes of angles. 9Lc: Numerical values of angles and sides.
∑: Control structure	9∑a: If two constructed triangles look the same/different 9∑b: If all sides and angles measured are the same then two triangles are congruent. 9∑c: If two triangles are just overlapped then these two triangles are congruent.

Table 3 Conception of congruency in lesson 9

As an example of how we analysed the textbook lessons, we take ‘Problem 1’ in Lesson 9 (noting that, at this stage, the conditions of congruent triangles, SSS, SAS, ASA, have yet to be formally introduced); “Will each of the following conditions result in only one possible triangle? Investigate by actually drawing triangles. (1) A triangle with three sides: 4cm, 5cm, 6cm. (2) A triangle with a side of 6cm and two angles measuring 30° and 45°”. We coded the above problem in terms of the quadruplet (P, R, L, ∑) as set out in Table 3.

For 'P', this problem focuses on the number of triangles (whether only one or more than one) by using the given data. Students are expected to *operate* various actions to tackle this problem, e.g. drawing various triangles by using given data, observing whether two triangles are identical, and so on. Also, students might measure angles, they might overlap triangles to verify whether two triangles made by given data are identical (congruent) or not. *Representations* mediate operations and control systems, and in this problem they are mainly graphical, e.g. the constructed diagrams as the medium of solutions, measurement tools as the medium for lengths of sides and sizes of angles, numerical values of angles and sides as the medium of solutions and so on. The operations are validated by *control structure*, i.e. overlapping two figures are validated if they exactly overlap, two triangles can be the same if all sides and angles measured are equal, and so on.

FINDINGS

In what follows, in addition to the analysis of Lesson 9 Problem 1, we focus on analysing Lesson 8 and Lesson 13. Lesson 8 is the first lesson in which students in Grade 8 study congruent figures. The result of our coding is given in Table 4. The first task in the lesson (as shown in Table 4) is a practical task that can be completed in several ways, either physically (e.g. moving triangles, measuring sides and angle, and so on) and/or mentally. Some students might rely on visual appearance to judge whether two particular triangles overlap completely or not.

Following this lesson, the conditions of congruent triangles are formally introduced in Lesson 9 (as analysed in Table 3). The subsequent lesson, Lesson 10, starts from the following narrative "It can be determined if two triangles are congruent by checking to see if they satisfy any of the congruence conditions without having to check to see if they can completely overlap". In this way, after Lesson 10 the conceptions of congruency intended in the activities and problems relate to the correspondence conception of congruency (CORR). After Lesson 10, students are expected to use the congruency conditions to prove various statements. For example, in Lesson 13, students are required to undertake reasoning problems involving two steps, as illustrated in Table 5.

Using González and Herbst's four conceptions of congruency, the conceptions identified in the activities and problems in Lessons 8 and 9 are characterized as 'PERC', 'MeaP', and 'TRANS'. For example, in Lesson 8,

- PERC (P, R, L, ∑) as ('8Pa: To use given triangles', '8Rd: To observe appearance of triangles', '8La: Actual paper shapes are the medium for the presentation of the problem', '8∑c: If two constructed triangles look the same/different)
- MeaP (P, R, L, ∑) as ('8Pa: To use given triangles', '8Rc: To measure lengths of sides and sizes of angles of each triangle', '8Lc: Measurement tools are the

medium for lengths of sides and sizes of angles', '8∑b: If all sides and angles measured are the same then two triangles are congruent')

- TRANS (P, R, L, ∑) as ('8Pa: To use given triangles', '8Ra: To move triangles and overlap them', 8La: Actual paper shapes are the medium for the presentation of the problem, '8∑a: If two triangles are completely overlapped then these two triangles are congruent)

In Lesson 10 and subsequently, the activities and problems are mainly proofs and are characterized as 'CORR' because the tasks rely on identifying corresponding parts that are the same. Table 6 summarises the characterization of the lessons on congruency in the Japanese Grade 8 textbook.

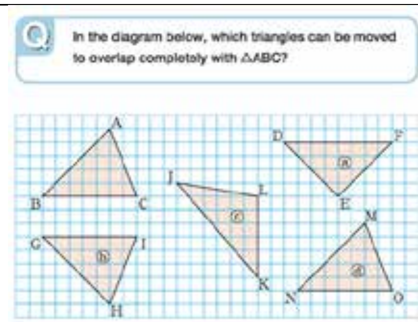
P: Problem	 8Pa: To use given triangles.
R: Operation	8Ra: To move (rotating and translating) triangles and overlap them. 8Rb: To observe the figure resulting from movement. This involves either physical or mental manipulations. 8Rc: To measure lengths of sides and sizes of angles of each triangle. This involves either physical or mental counting for length as squared paper is used. 8Rd: To observe appearance of triangles.
L: Representation system	8La: Actual paper shapes are the medium for the presentation of the problem, but some simple cases can be done mentally by diagrams on paper. 8Lb: Squared papers are the medium for measuring lengths of sides. 8Lc: Measurement tools are the medium for lengths of sides and sizes of angles. 8Ld: Numerical values of angles and sides.
∑: Control structure	8∑a: If two triangles are completely overlapped then these two triangles are congruent (no specific functions or vectors are mentioned) 8∑b: If all sides and angles measured are the same then two triangles are congruent 8∑c: If two constructed triangles look the same/different

Table 4 Conception of congruency in lesson 8

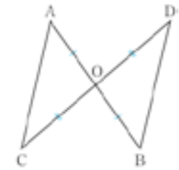
P: Problem	☐ Check 2 In the diagram shown on the right, if O is the midpoint of line segments AB and CD, then $\angle OAC = \angle OBD$.  13Pa: To use the conditions of congruent triangles.
R: Operation	13Ra: To identify what assumptions and conclusions are. 13Rb: To identify congruent triangles. 13Rc: To apply known facts to identify equal angles and sides. 13Rd: To apply the conditions of congruent triangles. 13Re: To deduce and identify equal angles.
L: Representation system	13La: The diagram is the medium for the presentation of the 13Lb: problem. 13Lc: The symbols are the registers of equal sides and angles. 13Ld: Already known fact such as the conditions and properties of congruent triangles mediate for the solution and reasoning.
∑: Control structure	13∑a: If we can find three equal components of triangles (SSS, ASA, SAS). 13∑b: If one of the conditions of congruent triangles is applied to two triangles. 13∑c: If two triangles are congruent, then the corresponding sides and angles are equal.

Table 5 Conception of congruency in lesson 13

Conception	P	R	L	∑
PERC	8Pa 9Pa	8Rd 9Rc, 9Rf	8La 9La	8∑c 9∑a
MeaP	8Pa 9Pa	8Rc 9Ra, 9Rb, 9R d	8La-8Ld 9La-9Lc	8∑b 9∑b
TRANS	8Pa 9Pa	8Ra-8Rb 9Rc	8La 9La	8∑a 9∑c
CORR	10Pa 11Pa 13Pa	10Ra-10Rd 11Ra-11Re 13Ra-13Re	10La-13Le 11La-11Lc 13La-13Ld	10∑a-10∑b 11∑a-11∑c 13∑a-13∑c

Table 6 Conceptions of congruency in grade 8 textbooks in Japan

DISCUSION AND CONCLUSION

Our aim of this paper is to characterise the approach to triangle congruency in textbooks in Japan. By using congruency conceptions as our analytical framework, we identify various intended conceptions in lessons related to congruency in textbooks. What we also found was that practical tasks contain various conceptions whereas proof problems expect students to have CORR as the main conception to solve problems. It seems that the Japanese textbook we analysed assumes, by designing the lesson progression, there will be conception changes from MeaP or

PERC to CORR. An issue with this is that it is still uncertain that students who have just finished Lesson 10 have fully developed CORR after several practical activities over two lessons and a statement that "It can be determined if two triangles are congruent by checking to see if they satisfy any of the congruence conditions without having to check to see if they can completely overlap."

In fact, Japanese Grade 8 still struggle to solve geometrical problems involving congruency. For example, a recent national survey in Japan reported that the proportion of Grade 9 students who could identify the pair of equal angles known to be equal by the SAS condition in a given proof was 48.8% (National Institute for Educational Policy Research, 2010). This indicates that many students in Japan have not fully developed their CORR conception of congruency despite studying congruent triangles and related proofs during Grade 8.

When we consider the "interaction between the cognizant subject and the milieu" (González & Herbst, 2009, p. 154), we should not simply conclude that these students are not capable to do proofs, but they might still be utilizing their MeaP or PERC when they face these proof problems instead of utilising control structure in CORR '$\sum$: If two triangles are congruent, then the corresponding sides and angles are equal'.

The conceptions we focused in this paper are those in textbooks, i.e. the intended conceptions. Our next task is to characterise actual students' conceptions when they interact with various congruency problems. Also it is necessary to design learning progressions and activities which enable students to develop rich conceptions of congruency as well as how to control their conceptions when they face various problem situations. One way to do the latter might be to use technology, such as that reported by Miyazaki et al. (2011).

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AN INVESTIGATION ON STUDENTS' DEGREE OF ACQUISITION RELATED TO VAN HIELE LEVEL OF GEOMETRIC REASONING: A CASE OF 6-8TH GRADERS IN TURKEY

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The aim of this study was to examine 6-8th grade students' performances when they were asked to identify, name, and draw geometrical objects. In order to investigate students' geometrical structures, van Hiele theory was employed with the help of a model developed by Gutierrez et al. (1991, 1998). The data was collected from 809 6th to 8th grade middle school students. The analysis of the data revealed that most of the students had difficulty to think 3 dimensionally, and so to reach 1st van Hiele level with complete acquisition. The findings are discussed and implications for educational field are presented.

Key Words: Geometric reasoning, van Hiele theory, degree of acquisition

It is a well-known fact that understanding students' thinking improves the quality of instruction (Anderson, 2000; Carpenter, Franke & Levi, 1998). Studies investigating students' geometrical understanding (Battista & Clements, 1996; Ben-Chaim, Lappan, & Houand, 1985) emphasize that to design and implement effective instruction for meaningful learning, students' thinking structures should be understood. Since developing three-dimensional understanding is important, many researches have focused on the ways of improving students' abilities on visualizing, drawing, naming, and constructing geometric solids (Ben-Chaim, Lappan, & Houang, 1988; Meng & Idris, 2012; Pittalis & Christou, 2010). While most of those researchers investigated the effect of using manipulatives (Meng & Idris, 2012), 3-D simulations, games, and virtual environments (Dalgarno & Lee, 2010); some others focused on drawing solids (Ben-Chaim, Lappan, & Houang, 1985, 1988). Ben-Chaim and colleagues (1985) found that 5-8th grade students had difficulties in relating isometric type drawings to the rectangular solids which those drawings represent. Another study explored students' thinking in 3D geometry, and argued that there is a close relation between the representation reasoning and the mathematical properties reasoning, and so between the figural and the conceptual aspects in 3D

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geometry (Pittalis & Christou, 2010). The authors also suggested to curriculum developers to enrich the activities by addressing different types of reasoning. Moreover, Duval (1999) distinguished four types of apprehensions for a "geometrical figure"; namely, perceptual, sequential, discursive and operative. Deliyianni and colleagues (2009) used Duval's framework to understand primary and secondary students' geometrical understanding and found that perceptual and recognition abilities appeared as first order effect on developing better geometric understanding. The authors suggested giving sufficient emphasis on geometrical figure apprehension in both primary and secondary school levels (Deliyianni, Elia, Gagatsis, Monoyiou, & Panaoura, 2009).

All those studies in the literature highlighted the importance of investigating how students construct geometrical understanding and how understanding students' geometric thinking informs teachers to improve their instruction (Panaoura, & Gagatsis, 2009). The present study contributes to the literature by investigating Turkish 6-8th grade students' level of naming, identifying and drawing 3D geometric solids. Specifically, this study intended to understand at which van Hiele level Turkish 6-8th grade students identify a geometric solid and to what degree they acquired this level of reasoning.

Van Hiele Theory

The van Hiele theory was developed by Dina van Hiele Geldof and Pierre van Hiele in 50s in order to examine children' geometrical thinking structures (Pegg et al., 1998). In this theory, children's geometrical thinking structures are classified into 5 different levels (visual-level 0, descriptive-level 1, theoretical-level 2, formal logic-level 3, rigor-level 4 (van Hiele, 1984, 1986). While this theory is a strong model to analyze students' geometrical thinking structures, it has also some limitations. For example, multiple choice tests may not accurately assess students van Hiele levels because such tests do not provide any space for students to explain their answers or reflect their ideas. Thus, another test developed by Gutierrez, Jaime, and Fortuny (1991) was adapted in this study.

Gutierrez, Jaime, and Fortuny (1991) suggested a way to assess the level of reasoning of students via creating an instrument with open-ended questions and providing explanation on evaluation of student responses through utilizing van Hiele model of reasoning. In the present study, one of the tasks adapted from Gutierrez et al. (1991) was employed to determine both van Hiele level and reasoning/acquisition level of the students.

The purpose of this study was to investigate students' geometrical structures, and so to examine 6-8th grade students' performances when they were asked to identify, name, and draw geometrical objects on a dot paper. The research questions were: 1) At which van Hiele levels 6-8th grade students responded to the questions in Gutierrez Test? and 2) What was the students' degree of acquisition related to van Hiele levels in Gutierrez Test?

METHOD

Participants

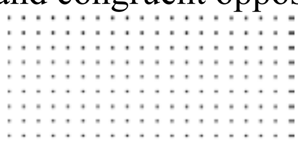
In this study, 809 6-8th grade students (283 6th, 259 7th, 267 8th grade students) during two class hours in 2009-2010 academic year. Considering the grade level differences, students are varied in terms of the previous knowledge about 3-dimensional geometry. Moreover, no training was provided to students before administration of the test because this study intended to understand students' performances in detail and how those performances vary in terms of grade level.

Data Collection Tool and Procedure

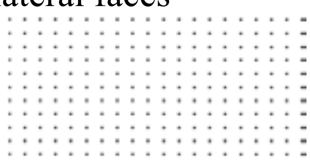
To collect data, an adapted version of an item from Gutierrez et al. (1991) was given to 6th-8th grade Turkish students. The item was piloted with 75 sixth to eight grade students. The final form of the item is provided below.

Please draw the objects on the dot paper provided below according to the given properties, and write the names of the objects into the boxes.

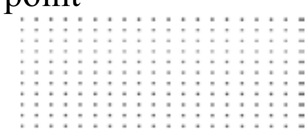
1) A vertical geometric object with rectangular lateral faces, and parallel and congruent opposing faces



2) A vertical geometric object with parallel triangle bases, and rectangular lateral faces



3) An object with a polygonal base and lateral faces which meet in one point



The item was composed of three sub-items. Since each sub-item has the potential to display whether students considered only one or more than one property to draw and/or to name the geometric object, it serves well to understand students' performances on identifying, naming, and drawing geometrical objects on a dot paper. Moreover, the item measures students' performance in two ways: (1) determining van Hiele level of students, and (2) determining reasoning processes that students went through (Gutierrez & Jaime, 1998). The reasoning processes and brief descriptions are presented below (Table 1).

Table 1: Reasoning processes and descriptions

Reasoning Process	**Descriptions**
Recognition	Identification of types, attributes and properties of geometric shapes
Definition	Defining geometric concepts. This level of reasoning includes two different aspect: (1) **Formulation** of definitions of the concepts learned, and (2) **Utilizations** of definitions which are either read in the book or learned from teacher and peers
Classification	Classifying geometric shapes and concepts into different groups
Proof	Proofing properties and statements

Gutierrez and Jaime (1998) also determined which van Hiele levels were associated with the item and which reasoning process were included. The item might be in different levels since students may produce answers in different levels with different reasoning processes. Students may produce answers to this item at level 0 and level 1 of van Hiele theory by using *Definition* as a reasoning process. (Gutierrez & Jaime, 1998).

Data Analysis

Researchers conducted a series of meetings to form a coding process following the van Hiele Model and test the reliability of scoring. The quality of students' responses indicated which of the van Hiele levels were attained. Then, students' responses to each sub-item were carefully read and classified according to van Hiele levels. Additionally, students' degree of acquisition of the levels 0 to 3 was analyzed by using the "percentage intervals" developed by Gutierrez and Jaime (1998) (Table 2).

Table 2: Acquisition levels and their intervals

Percentage Value of The Interval	**Acquisition Level**
0%-15%	No Acquisition
15%-40%	Low Acquisition
40%-60%	Intermediate Acquisition
60%-85%	High Acquisition
85%-100%	Complete Acquisition

Each sub-item was analyzed separately since van Hiele level of students' responses varied in each sub-item. Students who could not answer the item or produced unrelated answers were removed from the data set.

FINDINGS

Below, selected findings are presented for the first sub-item. Due to space limitations, the results of second and third sub-items are only summarized briefly.

It should be noted that in the case that students did not answer the item or answered it incorrectly, their van Hiele level could not be determined. If students drew or named the geometric object by considering only one property, then these students were assigned to the level 0. Students who correctly drew and named the geometric object by considering all of the properties given were grouped into the van Hiele level 1.

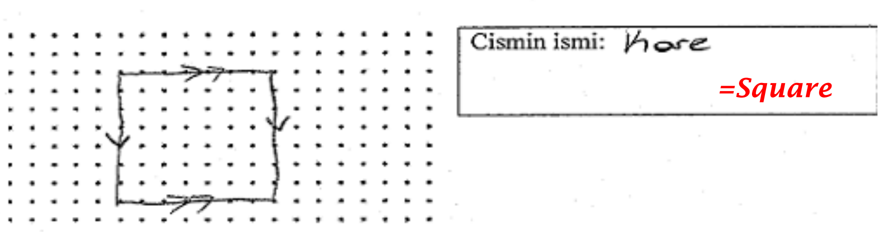
Sub-item-1

The first sub-item described a geometric object which is vertical and has rectangular lateral faces, and parallel and equal opposing faces; and asked students to write the name of the geometric object and draw it to the given dot paper.

Sub-item-1: Van Hiele level not determined

Considering the evaluation described above, van Hiele level of 171 (20.5%) students could not be determined. A student response resulted with no interpretation is illustrated below.

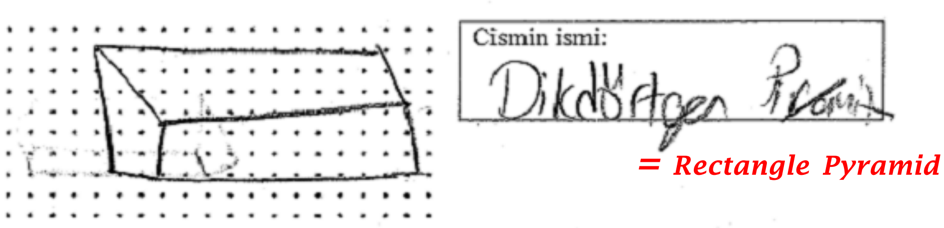
Sample student response #1:



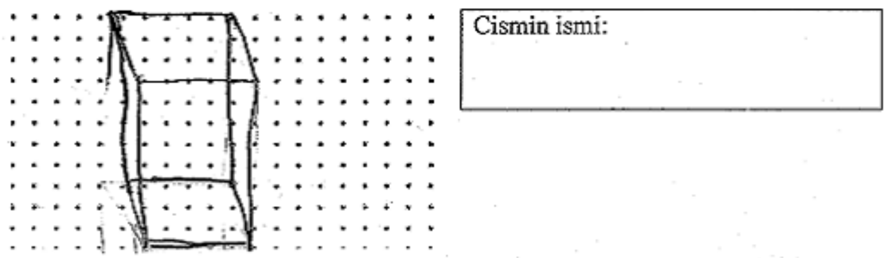
Sub-item-1: van Hiele Level 0

316 (38.1%) students answered this sub-item in the level 0 of van Hiele theory (112 sixth, 115 seventh, 89 eight graders).

Sample student response #1:



Sample student response #2:



As seen in student responses above, students either correctly wrote the name of the geometric object but could not draw it properly, or drew the object correctly but could not name it or named it wrongly.

When the results were analyzed according to degree of acquisition (Table 3), it was found that majority of 8th graders (88.8%) had *no acquisition* of the level. Half of the 6th and 7th graders had *no acquisition* of the level while the other half of them possessed the level with *low acquisition*. In this sub-item, there were no 6th graders, but a few 7th and 8th graders in *intermediate level of acquisition*. Again none of the 6th and 8th graders acquired the level completely. Only one 7th grade student reached *complete acquisition* of the level.

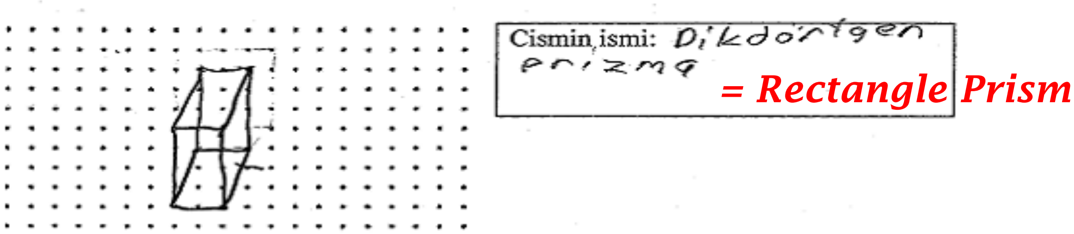
Table 3: Distribution of students' responses on sub-item-1 at van Hiele Level 0 across grade levels

Level of Acquisition	**6th grade**		**7th grade**		**8th grade**	
	N	**%**	**N**	**%**	**N**	**%**
No Acquisition	56	50,0	59	51,3	79	88,8
Low Acquisition	56	50,0	53	46,1	9	10,1
Intermediate Acquisition	0	0,0	2	1,7	1	1,1
Complete Acquisition	0	0,0	1	0,9	0	0
TOTAL	112	100	115	100	89	100

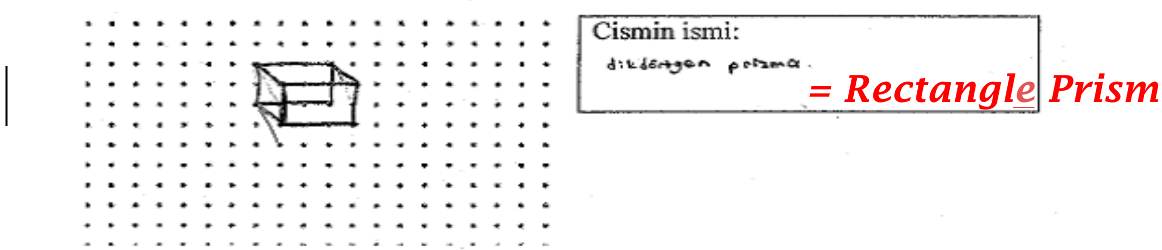
Sub-item-1: van Hiele level 1

346 (41.4%) students produced responses in the first van Hiele level (111 sixth, 115 seventh, 120 eight graders).

Sample student response #1:



Sample student response #2:



The sample student answers reveal that some of the students were able to identify the object, draw it, and name it correctly.

With respect to the students' degree of acquisition (Table 4), majority of the students in all three grade levels had *complete acquisition* of the level. In other words, 68.4% of 6th graders, 73.9 % of 7th graders and 67.5% of 8th graders attained the second van Hiele level completely. As seen in Table 4, approximately a quarter of students in each grade level had *intermediate acquisition* of the level for sub-item-1. Even though there were few students having the level with *low acquisition* and *no acquisition*, the students' responses reflected 1st van Hiele level with a high degree.

Table 4: Distribution of students' responses on sub-item-1 at van Hiele level 1 across grade levels

Level of Acquisition	**6th grade**		**7th grade**		**8th grade**	
	N	**%**	**N**	**%**	**N**	**%**
No Acquisition	3	2,7	1	0,9	4	3,3
Low Acquisition	1	0,9	3	2,6	5	4,2
Intermediate Acquisition	31	28,0	26	22,6	30	25,0
Complete Acquisition	76	68,4	85	73,9	81	67,5
TOTAL	111	100	115	100	120	100

Sub-item-2: Van Hiele level not determined, and van Hiele levels 0 and 1

In this sub-item, students were given properties such that "*A perpendicular object with parallel triangle bases, and rectangular lateral faces*", and asked to name and draw this geometric object. Similar to Sub-item-1, students produced answers in van Hiele levels 0 and 1. While 296 (35.7%; 116 sixth, 95 seventh, 85 eight graders) students answered the sub-item in the level 0 mostly with *low* and *no acquisition level*, 283 students (33.9%; 81 sixth, 108 seventh, 97 eight graders) were in van Hiele level 1 with *complete* and *intermediate level of acquisition*. Also, there were 254 (30.4%) students the answers of whom could not be determined.

Sub-item-3: Van Hiele level not determined, and van Hiele levels 0 and 1

This sub-item asked students to name and draw a 3D geometric object which had a polygonal base and lateral faces meeting in one point. The results were similar to that of sub-item 1 and 2. In other words, there were 288 (34.5%) students of whom van Hiele level could not be identified; there were 291 (35.1%; 100 sixth, 91 seventh, 100 eight grade) students in van Hiele level 0; 254 (30.4%; 85 sixth, 96 seventh, 73 eight grade) students in van Hiele level 1 with *complete* and *intermediate level of acquisition*.

DISCUSSION AND CONCLUSION

The findings revealed that 36% of the students in average drew 3D object properly and named it correctly, and reached to van Hiele level 1. Most of the students attained the 1st van Hiele level 1 with *complete acquisition*. The number of 6-8th grade students slightly differed from each other. This finding contradicts to Wu and Ma (2006)'s study with 5581 randomly selected 1-6th graders in Taiwan that their conclusion was that the higher the grades the higher the van Hiele levels. At this point, the fact that the methodologies, cultural aspects, and students' experiences with the topic were different in these two studies should be taken into account.

In short, more than 25% of the 6-8th graders' van Hiele level could not be determined; 36% of them had van Hiele level 0 mostly with *no and low acquisition;* and other 36% was in van Hiele level 1 mostly with *complete and intermediate level of acquisition.* This finding reveals that 64% of the students could not reach the 1st van Hiele level supporting that students' geometrical understanding and reasoning is low (Duval, 1998; Gutierrez et al., 1991). This might be attributed to the fact that the teaching strategies might be based on memorization (Battista, 2001).

Van Hiele (1986) stated that if students can not reach even to the descriptive level of geometry, it might be because they had no chance to experience geometric problems before. Instruction is believed to be the way for development through the van Hiele levels (Koehler & Grouws, 1992), and quality instruction is one of the most effective ways to improve students' understanding of geometry (Usiskin, 1982). We suggest curriculum developers and teachers to provide more opportunities for students to deal with three dimensional geometry. Moreover, even though 36% of students were found in the level 0 of van Hiele theory, they had either low acquisition or no acquisition at all. This might be another evidence of their need of geometrically rich experiences like the item utilized in this study or the original item developed by Gutierrez et al. (1991) in order to develop three dimensional thinking in geometry. Further studies are suggested to be conducted to understand how the classrooms could be transformed to support students' 3D reasoning. It is also recommended to analyze reasons of low or no acquisition in a particular van Hiele level so as to inform teachers and curriculum developers about the needs of students.

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CONSTRUCTIONS WITH VARIOUS TOOLS IN TWO GEOMETRY DIDACTICS COURSES IN THE UNITED STATES AND GERMANY

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In this article, I report on a study on experiences and perceptions from two culturally different cohorts of preservice teachers about teaching geometric constructions using different construction tools (compass and ruler, set square, patty paper, Mira, dynamic geometry software). The analysis of written reports indicated that the teaching experiment challenged participants' existing perceptions about teaching constructions and enabled them to reexamine their future practices. Both groups reflected on the experience both through the lenses of themselves as learners and as future practitioners arguing for and against benefits of using different tools when teaching constructions, and its effect on the development of geometric thinking.

Keywords: geometric constructions, geometric reasoning, construction tools, preservice teachers, reflection

INTRODUCTION

The topic of geometric constructions played an important role in the antiquity; likewise, nowadays, it has an important role in the mathematics curriculum. Euclid's *"Elements"* has been the most enduring and widely used mathematical work in the history of mathematics. In addition, it contains many geometric construction problems that had to be carried out using very specific tools: a collapsible compass and a straightedge, which are nowadays somewhat synonymous for constructions in school. Even though many attribute the teaching of geometric constructions to its long tradition, this topic is tightly connected to the central goals of teaching geometry, such as discovery learning, proving, concept learning and problem solving (Weigand & Ludwig, 2009) and, for these reasons, is of immense importance.

Though compass and straightedge have a long history as construction tools in geometry and in teaching geometry, many other tools are suitable for a constructive access to geometry. These include, but are not limited to, dynamic geometry software, Mira [1], patty paper [2], a protractor, a set square, TESE (a two- edged straightedge) and inch cards (Pandiscio, 2002; Serra, 2003; Weigand & Ludwig, 2009). According to NCTM (2000), construction tasks can encourage students to "draw and construct representations of two- and three-dimensional geometric objects using a variety of tools" (p. 308) and to recognize and connect mathematical ideas as a way to "develop robust understandings of problems" (p. 354). Use of these construction alternatives provide both a fresh view on the classical geometry topic and, together with classical tools, fosters different mathematical ideas promoting in-depth understanding of geometric concepts and highlights the connections among different ideas (Pandiscio, 2002; Serra, 2003). However, the pitfall of teaching this topic is that teachers often deliver mere construction steps to their students and emphasize the importance of precision (Schoenfeld, 1988). He asserted that such instruction, however, inhibits the students' possibility to develop problem-solving competencies, understanding of geometry, and development of mathematical power.

THEORETICAL CONSIDERATIONS AND AIMS OF THE STUDY

Perception is a central issue in epistemology. They are views or opinions held by an individual resulting from experience and external factors acting on the individual. Individual's perception is a broader look at the construct of beliefs encompassing beliefs, meaning, preferences, concepts, and mental images (Philipp, 2007). Our perceptions are influenced by our past experiences, beliefs and expectations. Researchers often apply conceptual lenses of teachers to interpret teachers' knowledge, beliefs and practices. Beliefs are often robust, permeable mental structures that evolve with experience (Thompson, 1992) and they influence the ways in which teachers perceive and deploy tasks. Perceptions and beliefs are formed very early in life whereas our thoughts and behavior patterns are governed by these. Teachers' individual perceptions, beliefs about the sense and nature of mathematics education and preparation are becoming increasingly recognized as fundamental contributors influencing the way they teach and in which innovative teaching concepts are implemented into day-to-day mathematical lessons (Philipp, 2007; Thompson, 1992). Consequently one's perception of reality is different and beliefs need to be challenged in order for a change to occur (Wilson & Cooney, 2002). Researchers (Thompson, 1992; Wilson & Cooney, 2002) place reflection as the agent of change where as a consequence teachers can consolidate their knowledge, shape the forms of knowledge produced, question their existing perceptions and values about teaching, influence their future classroom practices and so on.

The integration of tools and technologies is an important and actual theme today in teaching mathematics. New and emerging tools and technologies are introduced into the mathematics classroom that continually transform the mathematics classroom and redefine ways mathematics can be taught (Barzel, Drijvers, Maschietto, & Trouche, 2005). Several different perspectives have been developed, in particular instrumental approaches, which help explain the use of tool from a cognitive point of view (Rabardel, 2001). The main tenet in the instrumental approach is the difference drawn between an artifact and an instrument; the artifact is the object that is used as a tool, whereas tool is a material artifact that has a purpose to perform a task. However, the instrument involves techniques and schemes developed by the user during its use (instrumentalisation) that then guide the way the tool is being used and the user's thinking (instrumentation). When integrating different tools in the mathematics classroom different dimensions have to be taken into account: the relation between the use of tool and learning, the role of the teacher in technology-rich mathematics education, and the characteristics of technological tools (Barzel et al., 2005).

In light of these considerations, this was designed to seek an understanding about how learners from two different classrooms and learning cultures appreciate and make sense out of construction endeavors using different construction tools. The question arises: Do such opportunities allow development of a deeper understanding why and how can geometry meaningfully be taught? Hence, I explore how different conscious perceptions about different construction tools may influence the intention to implement innovations into their future classroom as suggested by a course initiative. The following research questions guided the study: Which elements of reflection can be identified in preservice teacher communication of experiences on using different construction tools when solving standard construction tasks? How do preservice teachers perceive the importance of using different construction tools with respect to teaching the topic of constructions?

METHODOLOGY

This was an exploratory study with participants from two cohorts of preservice middle and junior high school teachers from the United States and Germany. The research design differed for both countries because of the different classroom and learning culture. However, the similarity of items within the two designs allowed for a comparison study. The following is a discussion of two research designs.

Teaching experiment in the United States

This research was conducted during a mathematics education course "Teaching Geometry and Measurement in the Middle School" in Spring 2011, which is a part of the middle school teacher education program (grades 5-8) at a large southern university. The course focused on teaching methods, curriculum materials, and topics and psychological factors for developing geometric and measurement structures in grades 5-8. The class met twice per week for 75 minutes, where the professor and the teaching assistant (author of the paper) led the discussion and organized the lesson plans, and supported by two additional teaching assistants. In this study, a cohort of 36 prospective mathematics middle school teachers students participated. Data collection took place in the 3rd week of the course when topic of construction began. The class was divided into three groups in which a different manipulative was used: a compass and a ruler, the Mira, and patty paper. The students were instructed to solve standard Euclidean construction problems (duplicating a segment, duplicating an angle, constructing a perpendicular bisector, construction of an angle bisector, constructing a line parallel to a given line and a point not lying on the given line) with the guidance of the instructor. After 20 minutes they rotated to a different room where they engaged in the same activities but using a different manipulative. During the next class meeting time was spent on learning how to work in a dynamic geometry environment, namely in the Geometer's Sketchpad (GSP), performing afterwards the same construction tasks as in the previous class meeting.

Data collection methods for this part of the study consisted of the following: mathematical autobiography, written reports, and field notes. The purpose of the mathematical autobiography was for the students to reflect on their experiences as a mathematics learner in geometry thus far. Written reports were 2 to 3 page reflection papers that students submitted every 2 to 3 weeks designed for students to relate what they were learning in class to their own practice or experience. They needed to choose one aspect of class that was of interest to them, and discuss it in depth both through the lenses of a student and future practitioner. I analyzed 20 of those papers that reflected on the events in the above described class-meeting. All participants reflected on the four construction tools. Additional data were my own field notes I took while observing students performing the above described tasks.

Teaching experiment in Germany

This research was conducted during a mathematics education course "Didactics of geometry for grades 7-10" in the spring 2012 semester which is a part of the teacher education program for grades 5-10 at a large university in Germany. The course focused on developing and consolidating content and didactical competencies relevant to grades 7-10. The class met twice peer week for 90 minutes in a form of a lecture and discussion, respectively. In this study, a group of 20 prospective teachers participated who attended one of the discussion sessions. Data collection took place in the 4th week of the course where the students learned about different tools that can be used when teaching constructions (a compass and ruler, a set square, GeoGebra) as suggested in the literature (Weigand & Ludwig, 2009). In the discussion the students had the opportunity to have a practical experience with constructions using compass and ruler and patty paper. Similar to the previously discussed case, the students were given a sheet of paper with list of standard constructions they were to perform using first the patty paper and afterwards using a compass and a ruler. The students had 20 minutes to engage in these activities with a particular manipulative. As a part of their homework they performed the same activities using GeoGebra and set square.

Data collection methods for this part of the study consisted of the following: questionnaire with open-ended questions, written reports, and field notes. The questionnaire was divided into two sections. The purpose of the first sections was to learn more about the general background of the participants and their school experiences with respect to the topic of constructions. The second section asked the participants to circle if a given tool (compass and ruler, set square, patty paper, technology) should be used in teaching constructions where Likert scale (1–5) was used followed by an open question to elaborate on their decision. The study included a survey yielding quantitative data, however, those data were not intended for statistical analysis but as a way to help the participants reveal their thinking about aspects of their experiences, teaching, and perceptions. In addition, two open questions were posed so that the students could reflect on the benefits of these events on their current professional learning and future teaching, which allowed study comparison. Additional data were my own field notes.

Data analysis

Once data collection was completed, the textual analysis (Bernard & Ryan, 2010) of all written data reports, where conceptions and reflections about different construction tools were articulated, was used. For the purpose of this study, two stages of analysis, the within-case analysis and the cross-case analysis (Bernard & Ryan, 2010) were carried out. Within-case analysis focused on an analysis of different written reports. During this phase of the analysis, I read all documents and analyzed participant responses according to topics such as background and perceptions that allowed me to situate the participants according to their experiences as well as to ascertain the soundness of the development of the geometric thinking when using different construction manipulative. Once I identified these themes from the reports, I reexamined the data, taking note of any new evidence that came to light in support of these major ideas. The cross-case analysis was used to create a theory offering general explanations of perspectives on the experience of using manipulative for both groups.

RESULTS

Elements of reflection on the experiences of using different construction tools when solving standard construction tasks

The mathematical autobiography written by the American students revealed previous experience with the topic and the different tools; almost half of the students perceived having experienced a teacher-centered middle- and high-school geometry classroom rather than a student-centered one. Other students reported having ample opportunities to engage in discovery learning and individual or group problem solving using different manipulative. With respect to different construction tools, all participants had instrumented a compass and a ruler when solving construction tasks "I have used a compass and straight edge in the past", but three people reported their teachers also used different technology software "but she [teacher] used it only to demonstrate the construction steps. We didn't get the chance to use it". They perceived geometry as "an exciting and interesting part of mathematics". On the other hand, a questionnaire was administered to German students for the same reasons. The analysis of the questionnaire revealed that, though the participants had a positive experience with geometry in their previous educational experiences, the participants' beliefs about teaching constructions varied. While 11 participants were comfortable with the topic, 8 were rather anxious about the topic. The construction problems they had experienced were easy for most of the participants, while only 3 participants experienced difficulties. All of the participants instrumented the compass and ruler together with a set square in school, while one participant reported having instrumented the former two and technology. The topic was mainly taught in a teacher-centered classroom where the teacher explained specific construction steps and then they would practice the steps, which is similar to previous results (Schoenfeld, 1988). For the rest of this section, I discuss students' reflections on the experiences using different construction tools.

Mira – The Mira was only used with the American students, as this tool was not available for use in the didactics course in Germany. For most students this was their first encounter with the Mira, that is, the tool has not been instrumented thus far by them. Though a teaching assistant shortly introduced them to the tool, the participants struggled with its use, for instance, when duplicating a segment:

> But I have never used the Mira…I had the most difficulty in trying to replicate the dots on the other side of the Mira so that I could connect them with a straight edge. It was after exploring the tool I figured out I need to place Mira adjacent to the segment, look through the Mira to identify the endpoints on the other side of it. The Mira was the hardest tool for me to understand and use correctly.

Hence, after they instrumented the Mira taking into account affordances of the tool, they were able to use it as a construction tool. This experience prompted them to stress the importance of letting the students have an opportunity to "play with the tool before using it". Moreover, all of the participants instrumentalized Mira as a reflection tool in addition that can then be used when introducing the concepts of reflection, congruence and symmetry "As I played with the Mira, I noticed more about symmetry and reflections by moving the Mira…I would definitely use the Mira to teach sections regarding reflection." However, three students did not believe that the Mira should be used as a construction tool because "it may not be very useful in applying and relating constructions and properties to students' day-to-day experiences as easily as other tool."

Patty paper – For both group of students, this was their first encounter with the patty paper and similarly to the experience with the Mira the tool needed to have been instrumented by the participants. This process by German students was very interesting. First they tried to adapt their previous compass and ruler strategies. They used the compass and ruler on the patty paper similar as when constructing on a DIN-A4 paper before getting instructed by the teaching assistant (the author of the paper) that they are only allowed to use the patty paper and a ruler. Only after having considered the affordances of the tool their thinking guided how the tool got used. The use of the patty paper as a construction tool was very well received from the American students. The participants perceived that the tool is "very easy to manipulate", "its translucent appearance helps with copying lines on multiple patty paper sheets as well as visualizing angles and intersecting lines", "inexpensive, and it has so many possibilities" and promotes geometric reasoning

> By using the patty paper I got clarity to many geometric relationships. As a teacher, I will use patty paper because there are many ways that I can use the paper to help my students learn mathematics…not only to visualize, but also to experiment and discover the geometric principles for themselves.

Five students after having instrumented all of the tools in addition offered a trajectory with regards to using different construction tools in the classroom

> I really feel these three activities go hand-in-hand, and feel it might be very beneficial to go through them in the order discussed [patty paper, Mira, compass and ruler]. I would use the patty paper as a opening activity with younger students who might not be necessarily able to correctly use Mira or compass and ruler. After discovering concepts and basic relationship the progression to Mira and then to compass and ruler would come naturally.

The other participant added that technology would be used at the end to "solidify their earlier findings or help with questions that may arise from the other teaching methods". Hence, such experience allowed the participants to reflect on their current student role and experiences allowing to reexamine their beliefs about the teaching of the construction and shift in their future teaching practices. Perception of German students with respect to patty paper was divided; 7 participants were against its use, 9 were neutral and 4 were for its use when teaching constructions. The participants who were against its use included the following reasons: "it is imprecise", "does not support understanding of the topic and geometric relationships", "praxis-irrelevant" and "not usable in later education". Some participants were extremely vocal and believed that "That is not mathematics!" Opposite to those statements, the rest indicated that it supports "a student-centered classroom where student can learn new concepts and discover new relationships" while also helping students "illustrate and visualize figures".

Compass and ruler – All of the participants had previously instrumented compass and ruler. Only one student from the United States shared his belief that this tool should be used because it is "a classic approach to constructing and examining shapes, angles, and segments" and part of "old school", while others mentioned reasons, such as focus on mathematical reasoning, making connections between the geometric relationships, and its ease of use.

> I was forced to recall properties of whatever I was attempting to construct or translate in order to make it work. The mathematical reasoning behind this activity is very deliberate in forcing students to make connections. Further, the tools used are ones that they can more easily use in their daily lives and relate to.

Hence, having reflected as learners and future practitioners on their own activity and reasoning they developed their perception towards the tool and consequently recognized how the tool might promote their students' reasoning, respectively. On the other hand, 19 German participants (strongly) believed that compass and ruler should be used in teaching construction for the following reasons: "compass and ruler are the foundation of constructions", usability in praxis, ease of use, effective and precise construction tool, promotes discovering geometric relationships, and teaches precision and finger-abilities. Students in both cohorts recognized the historical value of the use of compass by "viewing" the tool as a part of "mathematics culture". Hence, beliefs towards the use of the tool in the future was influenced by the cultural, historical value assigned to the tool. The question that remains open is if the shift in belief did not occur because it was not challenged sufficiently or because the value is extremely instilled and therefore was not able to have been changed.

Set square – Set square was only used with the German students, as this tool is not a part of the American mathematical culture. The participants have had instrumented this tool throughout their education yielding homogeneous beliefs about its use when teaching construction; seventeen of them argued for its use because it was instrumentalized both as a construction and a measurement tool. They also believed that it is "more precise than compass and ruler", "integral part of geometry", "time efficient", "self-explanatory" and "supports development of finger ability".

Technology – Participants' opinions to using technology as a construction tool was similar for both groups. For the first group, some argued against its use because it does not require much thought to make connections and experiment compares to other tools "I really loved the hands-on engagement that this activity requires of the student, rather than simple dumb luck." The majority of the students were thrilled by the vast opportunities GSP offers such as visualization, illustration, drawing locus, and as an aid tool for lower achieving students

> I also think we would be doing students a disservice if we did not allow them to explore programs like GSP because it makes the relationship much easier to see, understand and to solidify their earlier findings. Technology makes all of this easier.

On the other hand, with respect to using technology when teaching construction, the answers from German students were not as homogenous: in total 14 students were for its use, 3 were neutral and 3 were against its use. Some reasons against its use, but not limited to, were: "not easy to use", "by a click of the mouse a figure can be constructed failing to promote understanding of the geometric relationships", "not usable in every day situations", "lack of availability to students", and "hinders the need for a proof". Hence, the participants had a limited view of the affordances of technology a perceived technology as a "product tool". Reinforcement of proof and visualization of geometric relationships between figures that technological tools allow was not recognized because they did not reflect as future practitioners. The rest argued for its use because of the affordances technology brings in general: visualization, speed, and accuracy.

Preservice teachers' perception on the importance of using different construction tools with respect to teaching the topic of constructions.

At the end of their report, the participants reflected about the influence of the experience of using different construction tools on their future teaching. All American students and eight German students experienced a shift in their beliefs towards the topic of construction "using different tools gives the students a tangible experience where use of the tool promotes problem solving and proving through the use of reasoning". However, some did not fully grasp the value of the experience writing that they learnt that "figures can be constructed with different tools", "construction with software is precise", and "different tools make teaching fun without connecting it to students' learning". In addition, fourteen German students said they will use the tools they used when they were in school. This apprenticeship of observation (Lortie, 1975) provides preservice teachers a limited understanding of teaching, which should not be underestimated. Such preconceptions that preservice teachers hold about teaching should be challenged to allow for a new geometry classroom. The rest remarked they would like to use them in their future career to differentiate instruction and support students' independent learning. On the other hand all of the American participants expressed that they plan on implementing these activities in their classroom, as these tools will deepen students' understanding of geometry explaining, "those [hands-on activities] are what students get the most out of in class." One of the participants summed up her thoughts by using her high-school teacher's saying: "Tell me and I will forget, show me and I may not remember, involve me and I will understand." Hence, by focusing on hands-on work students can expand their outlook on different tools helping them to broaden and deepen their understanding of mathematics (Pandiscio, 2002).

In summary, both groups of students reflected through the lenses of current learners and future practitioners. The American students reported on elements portraying broadening of their mathematical perspective, such as consolidating their knowledge about constructions and geometry and the mathematics behind each tool. The latter was communicated likewise by the German students. The participants' perceptions to the importance of different tools were centered around these qualities: development and promoting of geometric thinking, problem solving activities and processes, hands-on experience, and visualization. The analysis yielded rather remarkable results with respect to their future profession and with no doubt broadened their didactical perspective. Both group of students recognized that by using different construction tools, they could arrange a student-centered classroom focusing on the students' independent learning and strengthening their conceptual understanding of different topics. On the other hand, stressing the historical and cultural value of a tool, namely the compass and the ruler, was given by the German students.

FINAL THOUGHTS

Construction tasks are of immense didactical meaning in teaching geometry for many reasons, such as development of deductive reasoning, problem solving, argumentation and proving abilities, connections within geometric topics, and creativity (Weigand & Ludwig, 2009) but this topic does not get the necessary attention in mathematics (Pandiscio, 2002; Weigand & Ludwig, 2009). The results of this study show that a short lesson with carefully designed activities with well-chosen tools may enable prospective teachers' to learn about new instructional strategies, shift in their future teaching practices towards more student-centered pedagogies and broaden their content and pedagogical perspective targeted towards promoting geometrical reasoning. Hence, it is clear that beliefs and practices are linked; teachers' perceptions are critical factors in better understanding the development of quality mathematics teaching that we as educators need to confront during their education (Wilson & Cooney, 2002). Preservice teachers should have experience in solving construction tasks using different tools as well as opportunities to discuss curricular, pedagogical, and learning issues in variety of contexts before becoming in-service teachers and taking those responsibilities on themselves. Such experience with new tools has a profound effect on preservice teachers' knowledge of geometry and without a doubt is a powerful medium for the transformation of teaching and learning geometry.

NOTES

1. The Mira is a device constructed of red or green semitransparent plastic with a beveled lower edge that generates reflected images of geometric objects.

2. Patty papers are the waxed square papers often used for baking. They are translucent but also show creases when folded.

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INVESTIGATING MANIPULATIONS IN THE COURSE OF CREATING SYMMETRICAL PATTERN BY 4-6 YEAR OLD CHILDREN

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Summary. In this paper the role of gestures and manipulation in solving geometrical problems is investigated. Children aged 4–6 were subject to a series of observations during an experiment, aimed at finding a special placement for the figures in the symmetrical pattern. Results show, that rotation was taken as the first, most intuitive movement for them. Manipulation with rotation was taken independently on visual recognition of the relation of axis symmetry. It suggests that such approach can have a great impact on „tacit knowledge" used in further learning about geometrical transformations, and as consequence the dynamic imagination of rotation could be closer to acquaintance than other rigid movements on the plane.

Key words: geometrical intuition, tacit knowledge, rotation

IMPORTANCE OF GESTURES AND MANIPULATION IN EDUCATIONAL RESEARCH

In recent years, there have been a number of works dealing with the role of gestures and manipulation in the process of solving mathematical problems (Edwards, 2005; Radford, 2005; Freitas and Sinclar, 2012). The basis for such study is a convince that the construction of mathematical knowledge, at all school levels, should develop through activities that favor perceptuo-motor learning and involve interactions with body experiences (Arzarello *at al*., 2009). Generally, the relationship between the language, gestures and mathematical reasoning is considered. Sometimes, the distinction between different kind of gestures is made, for example between pointing and movements along the axis of the coordinate system (Bjuland *at al*., 2005) or gestures indicating the relations between some parameters (Steinbring, 2005) According to some opinion, the gestures related to linguistic expressions stimulate dynamic thinking in real time among his subjects (Nunes 2004).

It seems that gestures, movements and any other manipulations can play a crucial role in geometrical thinking. They can replace a verbal utterance (using language at the lower level of geometrical reasoning is quite difficult). But first of all they can represent various geometrical relations and transformations. For this reason, investigating gestures in the course of solving geometric problems can have a high educational value. This is confirmed by research carried out in different age groups (Bazzini, *at al.,* 2010, Freitas and McCarthy 2013, Ferrara and Maschietto, 2013).

THE ROLE OF MANIPULATIONS IN THE DEVELOPMENT OF GEOMETRICAL THINKING

Theories that describe the development of geometrical concepts indicate how the process of geometrical reasoning functions. There is not much said about the very beginnings of the geometrical cognition. A Czech mathematician and educator M. Hejny and philosopher of mathematics – P. Vopenka are the few ones who examine the very beginnings of the development of geometrical concepts and include it into the whole theory. According to their views, geometrical concepts ”emerge” from the surrounding world through a specific “geometrical sensitivity”, a kind of a sixth sense. “To notice something” is he first condition for the consciousness to focus on the geometrical phenomena. This first cognition is passive and static. Such an attitude is a mathematical specification of this, which developmental psychology defines as a place of visual thinking in the development of intelligence: The use of the images is considered to be one of the basic characteristics of the thinking of preschool children (Jagodzińska, 1991). Epistemology of geometrical reasoning requires the transition to further levels, in which an imagining of dynamic changes is desirable. Geometrical reasoning is consistent with operational thinking: while solving problems, we create a new reality; passage to the new reality requires the use of dynamic images of changes. Thus, finding and describing the mechanism of creation of operational geometrical concepts, based on dynamic transformation of imaginations, becomes an important research issue.

My previous investigations lead to the conclusion that 4- 6 year old children are able to act in the geometrical pattern environment (Swoboda, 2006). Children spontaneously arrange the plane, creating such relations between figures which may be described with the language of geometric isometric transformations. However, it is an activity which take place on the visual level, and child is interested only in the final arrangement of objects and the results of the action is verified visually.

It is worth to investigate very young (up to 7-year-old) children’s dynamic actions in such geometrical environment. At this level we couldn’t expect the children’s reflection on movement used, but intuitive movements and gesture can create a solid base for the further, more conscious activities. So it's worth to discern when and what actions children undertaken when they try to solve problems associated with the arrangement of one object in relation to the other.

RESEARCH – AIMS, RESEARCH TOOL

The experiment took place in March and April 2008. Children from a typical Polish kindergarten, aged 4 – 6, were subject to a series of observations (altogether – 60 children). This experiment was part of a broader study (Swoboda, *at al*., 2008; Swoboda and Tatsis, 2010), but for the purpose of this paper I will focus only on the aims described below. Children were tested individually and all session were videotapes and transcribed afterwards. As a research tool two types of tiles were used (fig. 1). The tiles were arranged separately on the table.

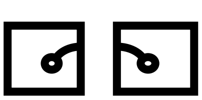

Fig. 1 – research tool

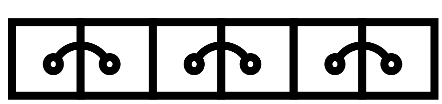

Fig.2 – a segment of the pattern prepared by a teacher

At the beginning a child was asked to continue the pattern prepared by the teacher (fig.2). When a child was not able to create the regularity, the teacher helps him in this.

Later on the teacher replaced one tile from a pattern by a different one in such a way that the regularity was distorted. The child was asked to show where a change was done and "to repair" the regularity [1].

The basic research aim was to investigate movements used by children. The research questions were as following:

1. How did the children recognize the symmetry in the tiles (visually, or by using any action)?
2. What kind of placement of the two congruent figures provoked the children to make any movements?
3. What kind of movement played the most important role in children's actions?

For creating a pattern, it was necessary to use two kinds of tiles - the motifs on the tiles were either "left" or "right". The left motif presented was a mirror reflection of the other one, thus, none of the types could have been obtained by rotating the other tile. Additionally, there was no motif on the back of the tiles, which made it impossible to correct the distorted pattern by making movement out of plane.

Building and correcting pattern required manipulations. Some of them were not interesting for me (browsing the tiles on the table, pointing the "wrong" place in the distorted pattern). I was interested in movements related to the verification of tiles placements. Such movements were observed in both parts of the experiment: during creating the pattern and during correcting it.

MOVEMENTS DURING CONSTRUCTING PATTERN

Two different strategies were observed in all children groups: 1. after visual recognition of difference between the tiles (placed on two separated piles) a child created pattern taking consciously two tile types, 2. a child starts by "blind searching", to connect tiles in order to obtain symmetrical configuration.

Example 1. Kacper (4-year-old boy).

> Teacher: *Kacper, look at this pattern and try to continue it*.
>
> Kacper: Takes one tile (the correct one) form the left pile, attaches it to the pattern but after a moment he starts to rotate it and later puts it back. Now takes a tile from the right pile, looks at this, says – no – and puts it back to the left pile. Takes another "right" one, visually compares it with the tiles at the table, put it over piles, but afterwards slides it under the right pile. After a moment he decides to take the right tile again and now he puts it in the pattern line, however without connecting it (keeping an empty place for one tile). In the second hand he takes another right tile and rotates it, trying to adjust to the gap between the pattern and non-connected tile. Next, he exchanges the right tile with the left one and constructs the whole symmetrical motif. He connects it with the pattern.

This boy stared from “blind searching”. The first movement, important for him, was the rotation. Although the first position he chose (and the tile) was correct, he felt the need to investigate its different placements. It is clear that he didn’t know how to use the visual information in a constructive way, he was only able to state, that some of his choices were not proper. In spite of this, he was very persevering in the work. Thanks to manipulation he gains some experiences, useful for solving his task.

Older children worked on the visual level. They successfully utilized the information, that two piles contains different types of tile. But also in such a situation some of them felt the need to investigate whether it was possible to obtain a symmetrical position, by using two tiles of the same type. It is clearly visible in the work described below:

Example 2 . Martynka (6-year-old girl).

> At the beginning she takes one tile from the right pile, later on a second tile from the left pile. She joins them and puts together as the pattern’ continuation. After that, she takes simultaneously two tiles from two different piles and using them creates another motif. From that moment she works very fast, sometimes taking simultaneously tiles, sometimes – taking one tile at time, but even in that situation she is aware where the tile should be put (fig.3) She uses almost all the tiles form the table. Two last tile were of the same type. She takes them into two hands and starts to manipulate – for a long time (23 sec.). She rotated them, trying to connect (fig.4). After that she looks at the teacher, by this informing that it is impossible to use these tiles for the pattern.

Fig. 3

Fig.4

Martynka’s work, described above, was not typical for older children. It seems that for the 6-years-old children the visual identification of one type of tile stopped any actions on them. It is visible in the Example 3.

Example 3. Karolek (6-year-old boy).

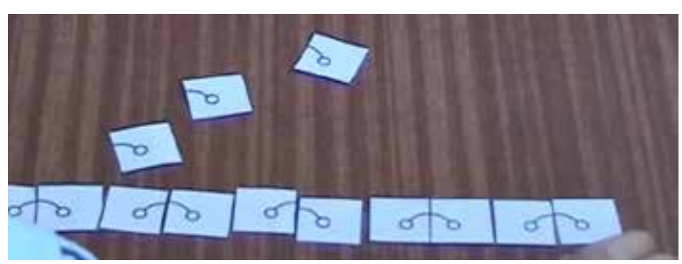

Fig 5

> K: At the beginning of his work he makes some trials and afterwards he works very fast, taking successively the correct tiles form the table without any doubt. In this way he builds a very long pattern, extending it on the right and left side. At the end, there are three identical tiles on the table left (fig.5).
> The boy sits (18 sec.), looks at the tiles.
> T: *Do you still want to work*?
> K: *no*.

This boy preferred to make a visual analysis than a manipulation – supposedly, the visual information was more important to him. In addition – he knew how to use these information. Perception was the foundation for each of his decisions, manipulations only supported and verified the undertaken actions. At the end of his work no action was needed – it was clear for him, that tiles are 'the same'. Those tiles were placed 'almost' parallel, the child didn't feel obligate to make movements to check anything.

MOVEMENTS DURING CORRECTING PATTERN

Again, two different strategies emerged here. The first one – 'replaced strategy', when the child exchanged the tiles, taking the proper one from the table. The other one – 'manipulative strategy', when child tried to obtain a correction by manipulation of tiles lying in the pattern.

Almost all 4-year-old children started their work from manipulation, making rotations. This way could be independent from the previous stage (creating pattern), where they differentiated 'right' and 'left' tiles.

Example 4. Zuzia (4-year-old girl)

> Teacher: *You built a very long pattern, it is enough for us. Now, please close your eyes, I will change something* (she distorted the regularity). *Open the eyes and say if there is something wrong.*
>
> Zuzia: (7s.) *here* (she shows by her hand, pointing a place in the pattern)
>
> T: *why?*
>
> Z: *Because here is in this direction and here in this one* (showing).
>
> T: *so, please, correct it.*
>
> Z: immediately starts to rotate - firstly by one tile, than by both tiles. Later she moves two tiles close to her, still making rotations. Movements starts to be slower and slower. Finally she takes one tile from the table and finalizes her work.
>
> T: *Perfect! But – did you notice, that the tiles were different in two piles?*
>
> Z: *Yes*.

At the first part of the experiment (creating the pattern) Zuzia muddled up all the tiles on the table. Teacher, wanted to help her, decided to tidy up and put tiles into the proper piles. After that the girl benefits from this. We may make a conclusion, that this 'teaching episode' was too weak for Zuzia to take an advantage of in the next stage of the experiment. Therefore it seems that maybe we should focus on children who in own way distinguished two types of tiles. Ola (4-year-old), described below, is one of these children. But, while correcting the pattern, she started from rotations, too.

Example 5. Ola (4-year-old).

> O: (2s.) she takes one 'left' tile from the pile, puts it in some distance from the patter (to keep a place for the 'right' tile). She moves her hand to the same pile, but immediately

recognizes, that this is not what she needed, than takes the 'right' tile and makes the whole motif. Next motives she builds very fast, creating long patterns (through the whole table).

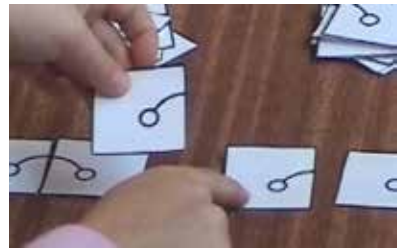

Fig 6

T: *Fantastic! And now I will give you a riddle (*she changes the tile*).Tell me where something is wrong?*

O: (immediately) *here* (she removes one of the double tiles, takes it in the hand – fig. 6).

T: *any why*?

O: *because it should be differently*

She moves one of the 'left' tiles to the right place in the pattern, the second one rotates by 180^0. Comes back to the first tile and rotates it many times (fig. 7). After some time she changes the action – she starts to manipulate with the second tile (fig.8), and in a moment she turns it on the blank side (fig. 9). She makes some other rotations, one by one with different files. Sometimes she changes an order of tiles (making shifts). Finally she stops to manipulate, keeping the tiles in her hands.

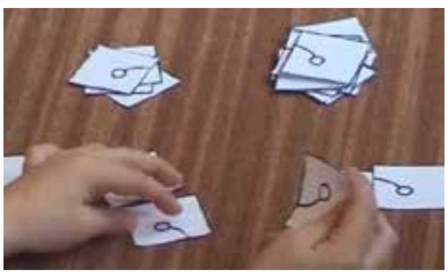

Fig 7

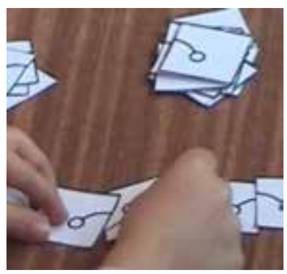

Fig 8

Fig 9

T*: is it possible to correct the pattern using those tiles?*

O: *no (with determination).*

T: *Do you know what I did? I replaced one tile by the other, from the table.*

O: she immediately takes the correct tile from the table and finalizes the work.

Ola showed a great awareness of how to build pattern. In spite of this, she starts its correctness from manipulations. She makes lot of movements, which have different meaning. First movement – parallel shift – is used for the convenience only. Movements used for searching for solution are rotations. She started from rotations, after that an idea of the mirror reflection emerged (when she turned a tile to the back side). If the tile would have been printed on both sides, she would have been successful. In the present situation Ola came back to the rotation, trying to compose the rotation with translation. After an investigation she states that it is impossible to solve this task.

Approach to the ways of repairing patterns vary in subsequent research groups – older children used replaced strategy more often. It is visible in the Table 1

Age	Numer of children	Manipulative strategy	Replaced strategy	Helpless
4	18	13 (72%)	4	1
5	25	9 (36%)	14	2
6	17	6 (35%)	10	1

Table 1: Approach to the ways of repairing patterns

DOES A LACK OF SUCCESS IN REPAIRING PATTERN BY USING ROTATION FORCES A REFLECTION ON PERFORMING MOVEMENT?

It seems that the answer for this question should be positive. It is supported by children's reactions, described in examples presented in this paper: after some trials with manipulations, children draw conclusion, that such actions are pointless. Although their attention was focused on the visual recognition of the motif, performing many repetitive rotations without obtaining expected result caused some awareness about its features. Children were conscious that it is impossible to achieve some placements of the figure by using it.

In order to illustrate the opinion that children can differentiate various types of movements, I use an example taken from the following stages of this experiment. The session took place some days later. In this session, the child who participated in the first session took a role of a teacher and tried to lead an experiment with his/her colleague from kindergarten [2].

Example 6. Michał (6-year-old boy).

Michał's task is to repair the pattern distorted by his colleague, Ola. (fig. 10)

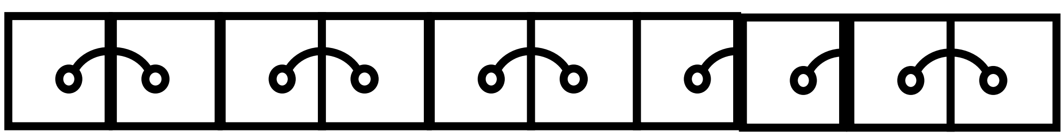

Fig 10

He gazes at the pattern motionlessly (4 seconds), locates the error – two tiles of the same type lying next to each other. Ejects one tile (lying on the right side), moves it over the first one (fig. 11). By 6 seconds he compares the two tiles, making almost unnoticeable rotations with the upper tile. For the moment he stops making any movements, then slightly rotates the bottom tile. Then, he shoves it below the pattern line and inserts the upper tile at this place (fig. 12). He puts his right hand down and for 7 seconds he asses the relationship between the tiles, than again he makes the minimum turns of the bottom tile (fig. 13) Suddenly he puts the tile back on the pile on the table, and with his right hand he reaches for the proper tile (fig, 14), fits it into the pattern and sits straight on a chair informing by this that his work has been done.

Fig 11

Fig 12

Fig 13

Fig 14

It seems that all the work on a solution of the problem was done in the mind. Diverse manipulations were aimed at obtaining different aims. The first movement - a parallel offset of one tile over another - was an auxiliary movement only, to help to assess the difference between tiles. The boy saw that both tiles are of the same type, but still wondered whether this fact excludes the possibility of axially-symmetrical arrangement. Minimum swivel movements firstly by one tile, and then the by other one testify that the student made the thought rotations.

SUMMARY

Observing children's work it was clear that the possibility of manipulation played a great role for them. Rotation was used as the main movement for investigating of changes in figure's placement. The reader can rightly state that children's behavior was provoked by the research tools. It doesn't change the fact that the children presented many behaviors (mainly during repairing pattern) that require deeper interpretation. Manipulation indicate for a need and a manner of such discovering, going beyond on visual recognition of geometrical phenomena.

Referring the results of observations to basic research questions, I conclude that:

1. Deciphering visual information concerning mutual position of two congruent figures clearly falls into two levels. The first one can be defined as 'I know that' and the second as 'I know how' On the first level, each possible arrangement is perceived on the level of impression and some arrangements are aesthetically preferred. Axis symmetry and parallel shift belonged to these preferred ones. But it does not give a sufficient basis to understand what actions can lead to an expected position of one figure in relation to another. Such understanding, knowledge 'know how' develops with age, what should be associated with gaining more and more experiences (both visual and manipulative). Younger children started from 'blind' discoveries, using manipulations and checking their effects visually. For older children, visual information was frequently sufficient enough.
2. Former information (visual and manipulative), that for axis-symmetrical position two different tiles are needed, does not have to prevent further study of this position through movement. It could be observed in children's behaviour while arranging the pattern and it was even more explicitly marked while improving the band. The fact that children from their own wouldn't

decide to exchange (replace) the tile can be interpreted twofold. One of the possible explanation is “didactical contract”, evoking the tacit conviction that if the teacher asks to repair the pattern that it is possible to do with the use of such tiles which are already used only. Even if so, children showed what are the sources of their actions which appear to be effective for such tasks. Other explanation – (quite obvious) is the lack of knowledge of the properties of isometric transformations. They could have known, that the tiles are right and left, but not, that by using “right” tile they are not able to receive the “left” one by making the physical manipulation on a plane. In mathematical terminology children could assume that the composition of rotation and translation would lead to a mirror symmetry.

3. When a child *searches for* an appropriate arrangement of two congruent figures relative to each other it starts from rotation. Rotations as treated as the basic way of object transforming. They are like an elementary tool used in solving problems with placement of figures on plane.

Children's behaviours related to the use of rotations are easy to explain: In the rotation and mirror reflection pieces looks differently than the model, unlike in the parallel transformation where pieces looks the same as model. In translation an identification of figure and its image is immediate and doesn’t require any conscious action. In rotation and mirror reflection the identification is easier after manipulation. Therefore such placements forces making manipulations. The manipulation of flat figures by rotations is more natural (and therefore more spontaneous) than mirror reflection.

However, that is not the explanation that I consider as the most important effect of the carried out research. Much more crucial to me is the statement that children *need* manipulations even in a situation where the former visual cognition suggests senselessness of such actions. This indicates child’s need to examine various solutions through movement, the visual information proves to be insufficient.

In psychology, an archetype is defined as reflection or instinctive reaction to the particular situation. Intuition can be treated as thinking on visual level. The results, presented here, clearly show that when child look how to compare one figure with another one placed differently, he/she starts from rotation. The child's attention in such action is still directed by the arrangement of two pieces in relation to each other and not to the movement as such, but these results can suggests that rotation can be used as the first tool for turning children’s attention on movements on the plane.

NOTES

1. Here, only one part of the research’ scenario is presented. More detailed description can be found in Swoboda, E; *Natural differentiation in a pattern environment (4 year old children make patterns),* Proceedings of CERME6,

2. In Poland this research method is quite popular. It is taken for observing the resistance of children’s behaviors.

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AN ACTIVITY ENTAILING EXACTNESS AND APPROXIMATION OF ANGLE MEASUREMENT IN A DGS

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We describe here an activity seeking a coordination between geometry and arithmetic. It starts from a geometric manipulation with GeoGebra – the iterate rotation of an isosceles triangle forming a regular polygon – and gradually leads students to consider the divisors of 360 and to reflect upon approximations and exact representations of rational numbers related to angle measurements. Through the issue of measurement in geometry, the activity takes place in the 'geometer-physicist' paradigm proposed by Tanguay & Geeraerts (2012) and instilled by Jahnke (2007). We propose an a priori *analysis of the activity, set within the theoretical frame of Kuzniak (2010, 2013) on* Espace de Travail Mathématique (ETM).

Keywords : linking arithmetic and geometry, exactness, approximation, measurement, dynamic geometry software, geometer-physicist's paradigm

In his most recent contribution to the symposium *Espace de Travail Mathématique* (ETM 3, Université de Montréal, October 2012), A. Kuzniak (2013) reflects, from a teaching and learning standpoint, on the role expected from bridging different mathematical fields when students are involved in a problem-solving process requiring a back and forth between two fields [1]. As a typical example, we may think of these problems in which one must optimize the area of a polygon subject to inscriptibility constraints, and where geometry is bridged to the algebraic/functional field. In the present article, we propose the *a priori* analysis (Artigue, 1988) of an activity mingling geometrical work and arithmetical work with GeoGebra. We will set it in a theoretical framing based on the work of Kuzniak (2010, 2013) on ETMs. By doing so, some aspects of the frame will be examined.

PHYSICS, BETWEEN EVERYDAY THINKING AND MATH THINKING

In a 2007 article, H. N. Jahnke compares generally valid statements depending on whether they come from mathematics, from physics or from everyday life. General statements from everyday life and physics share their empirical basis and the fact that the set of all conditions limiting their scope of validity is virtually unattainable. In everyday life, searching for completeness as regards these conditions is more often irrelevant. For instance in a given specific context, one may examine some precise conditions invalidating the statement "every evening, Johnny comes back from work around 18:00", but trying to figure out all possible misfortune in the world would be foolish. By contrast in physics, we try to relate each general statement to the most accurate domain of validity, even if we know that the theory will always remain subject to falsification by new observed phenomena (e.g. Popper, 1991). In established mathematics, determining the domain of validity of a general statement is not only possible but essential: it is indeed what accounts for the way mathematics operate, according to which the set of conditions is fixed and closed by the building of a (preferably axiomatized) theory.

In this respect, physics could be situated at the passage between the two others: as in mathematics, general statements in physics are connected by hypothetico-deductive developments that integrate them into a network and build them up as a theory. Yet the empirical bases of physics are not disqualified but rather enriched: any experimental verification about a statement not only corroborates it, but also increases the conviction that all other statements connected to it in the network are true. These considerations bring Jahnke (2007) to advance that in the classroom, dealing with mathematics as in physics would provide a more harmonious transition between everyday life thinking and mathematical thinking, and would also lay out a stronger epistemological foundation for teaching proof, with respect to its aim and to the type of certainty it brings.

A TRANSITION PARADIGM RECONCILING MEASURES AND PROOF

Following Jahnke, we propose in Tanguay & Geeraerts (2012, 2013), for the first years of secondary school, an approach of synthetic geometry [2] modelled after experimental physics. We speak of it in terms of *paradigm*, namely the *geometer-physicist paradigm*. We use the term 'paradigm' first to put forward its role as an articulation between two paradigms, the GI paradigm referring to the geometry of perception and intuition (Houdement & Kuzniak, 2006) and the GII paradigm referring to classical euclidean geometry. But also to stress the importance, regarding this approach, of considering the class (the school group) as a scientific community (Wenger, 1998) where are decided and assumed the motivations, the premises, the (didactical) contracts and prescriptions at the basis of such an experimental practice: it appears to us an essential condition for engaging students into progressively moving this practice towards the building up of a theory. Finally, because according to the historical/epistemological analysis of Jahnke (2010), such an approach would have been the one followed by the Pre-Socratic geometers, thus being at the very source of the developments that led to Euclid's *Elements* as a culmination.

With this paradigm, experimentation and empirical validations are brought back into the fold. In geometry, experimentation is mainly conveyed through construction and measurement: with a ruler, a protractor, a compass, or with specific functionalities of dynamic geometry software packages (DGSs) such as *Cabri-Géomètre* or *GeoGebra*. Researchers and teachers have often blamed measurement for engendering a hindrance to proof and proving : why prove something that can be verified simply by measuring? The measurement tools of DGSs magnify this effect (e.g. Boclé, 2008) because of their precision, but also because the dragging functionality allows an efficient investigation of examples. In the approach that we propose, each tackled statement is written and diagrammatically illustrated on a card (24 cm × 16 cm). Statements verified experimentally and statements proved deductively are systematically distinguished, and the two different statuses are made apparent on the cards. These are classified in a ring binder that each student has always at hand in his working space. The binder concretely accounts for the theoretical frame of reference. For more details, see Tanguay & Geeraerts (2012, §3). Empirical verification does not come into conflict with proving to the extent that every experimentally validated result keeps an hypothetical status which is explicitly stated and exhibited, and that the certainty of any deductively proved result remain dependent on results the proof uses as 'rules of inferences' (Duval, 1991 or Tanguay, 2007), in particular on the hypothetical results among them :

> The epistemological motivation of proof is not to be founded on the idea that proofs in contrast to measurement provide absolute certainty, but on the idea that proofs open new and more complex possibilities of empirical corroboration. In short, in an empirical environment *proofs do not replace measurements but make them more intelligent*. (Jahnke, 2007, p. 83, italics from the original text)

MEASUREMENTS, APPROXIMATIONS AND NUMBERS

To consider experimentation with measurement as would do a physicist in his lab, one must assume that measuring, even with DGSs, provides nothing more than approximations (see Tanguay & Geeraerts, 2013). In our opinion, neither textbooks nor ministerial programs deal adequately with this issue, leaving in limbo the epistemological status of measurement in geometry : « It is for us symptomatic that institutional teaching resources rebuke equalities such as $\frac{4}{3} = 1{,}33$ or $\sqrt{2} = 1{,}414$, but in the same time agree without a murmur with equalities such as $mes[AB] = 5$ cm, in contexts where inferred measures and measures obtained with the geometry tools are blithely combined » (Tanguay & Geeraerts, 2012, p. 21; our translation).

Besides, discussing about measurements as approximations allows discussions about the ideal character of the measured objects : points without dimension, line segments and lines without thickness, angles sprawling at infinity, with edges of zero measurement… These discussions then lead to institutionalizations whose content is no more the sole responsibility of the teacher. For instance, we may well imagine a class-discussion about the following GeoGebra display :

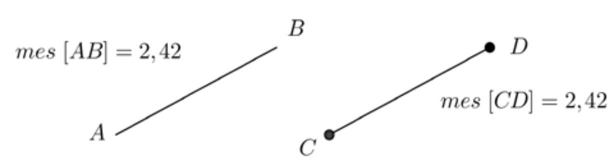

Fig 1 : what is the 'thickness' of point *A* ? of point *C* ?

If we now analyse how measurement can be situated into the *mathematical working space* (Kuzniak, 2010, 2013), using Kuzniak's two planes model, we observe that measurement contributes to, and is part of, both *figural genesis* and *instrumental genesis*. The latter is of course related to the measuring tools. As for figural genesis, the outcome follows from the fact that while putting a ruler down on a line segment with the aim of measuring it, the student tracks it down and isolates it as the side of a given figure, whose *dimensional deconstruction* (Duval, 2005) is thus sparked off : the sides must be located as the boundaries of the shape, the « corners » as endpoints of the sides, the endpoints being matched with the graduations of the ruler. This dimensional deconstruction is certainly resulting from a form of visualization but also refers to a set-theoretical modelling of the plane, with points as (atomic) elements and the sides as subsets of the figure. In this sense, through the cognitive activity of measuring, there is indeed a projection into the theoretical frame of reference, here the one pertaining to GII. Recall that the theoretical frame of reference is in Kuzniak or in Coutat & Richard (2011) one of the two poles of *discursive genesis*.

Besides, the effort of visualization on the measured figure – the figure as a '*representamen*' or 'signifier' in the *epistemological plane* of Kuzniak (2013) – is built on an effort of coordination with other signifiers, namely the numbers resulting from measurement : these numbers constitute a 'property' of the measured objects but *are not* these objects themselves. So, it is not a coordination between registers of representation in Duval's sense (1993) because the *signified* are not the same. Moreover, they belong to two different fields, the field of (synthetic) geometry and the field of arithmetic. In the perspective of Kuzniak's theoretical framing, it is as if the epistemological plane had been split in two, with a plane in each field and round trips between the two via the cognitive plane. Through these considerations, one can evaluate the complexity of what is involved in a measuring activity, be it conducted for construction, validation or experimentation purposes.

Regarding complexity, the subject is indeed not exhausted. In a geometric problem-solving context, it happens frequently that the task also deals with *computed* measures, for example when some measures are inferred from Pythagoras' or Thales' theorems, thus giving rise to irrational numbers or rational numbers with infinite decimal expansion. The representations of all the related numbers may then come from several registers [3] : the register of finite decimal or repeating decimal (with the period overlined), the register of quotients written in the form p/q, the register of representations using the root symbol, etc. There is then a need for a *coordination between registers*, as the one brought up by Duval. But in order for this coordination to be 'scientifically coherent', the non exactness of the decimal numbers resulting from measurement must be fully taken into account. For an example of a problem lacking in such a coherence, see the problem *Marie et Charlotte* in Kuzniak & Rauscher (2011) or in Kuzniak (2013).

In sum, if measuring in geometry belongs mainly to mathematical activities relevant to figures, it also resorts to numbers, so to a field that is not synthetic geometry and has its own representations and theoretical references. We insist that in that instance, the duality exactness-approximation should be a teaching goal and issue. Then almost inevitably, the issue will echo with the representations of rational and irrational numbers and their approximation by decimal numbers, and in parallel with the problem of (visually) representing ideal geometrical objects. In the proposed activity, the measurements are not carried out with the usual geometrical tools but rather, are obtained from the ad hoc functionalities of GeoGebra. Then, students' relationship to the displayed numbers gets more intricate (Tanguay & Geeraerts, 2013). The issue of exactness, for both the measurements and the associated numerical representations, is directly linked to the possibility of 'closing' (or not) the regular polygon to be produced. We put forward the hypothesis that regarding these issues, the discussions and reflections thus prompted will be rich and significant.

A SITUATION PLACING AT ITS CENTRE EXACTNESS OF MEASUREMENT AND OF NUMERICAL REPRESENTATIONS

Description

The following teaching situation has been designed by a group of researchers from Quebec and Mexico, working in collaboration. We started from a situation designed by L. Guerrero, in which working on regular polygons with GeoGebra was planned. The situation was intended for students from the beginning of secondary school (12-14 years old). Revising it brought us to extend the task towards arithmetic, via the measures of the angles involved. The arithmetic topics singled out – mainly divisors and divisibility – and the study of regular polygons are prescribed subjects from the secondary curricula of Quebec and Mexico.

The activity provokes a back and forth between geometry and arithmetic by considering the GeoGebra displays of the measures related to a well chosen angle. It can easily fill up two lessons. The first lesson involves the decomposition of the *n*-sided regular polygon in *n* isosceles triangles grouped around the centre, the decomposition being linked to the divisibility of 360 by the degree measurement of the central angle. The students are thus brought to examine, within a geometrical context, the list of divisors of 360. They work in teams of two, one team per each computer terminal. The instructions are open :

> The triangle ΔABC visible at the screen is isosceles, with [AC] ≅ [BC]. Form all the regular polygons you can by rotating ΔABC around point C. You may vary ∠ACB either with the slider α or by typing directly α, the measure of ∠ACB, in the given box. The slider *n* allows you to change the number of triangles obtained as images of ΔABC by repeated rotations around centre C and of angle α. Do you know the name of each polygon you formed ? As you go, fill in the following table. You can add as many lines as you want.

Measure α of angle ACB	Number of triangles, images obtained by rotating ΔABC	Name of the regular polygon you formed
	n =	
	n =	
	n =	
	n =	

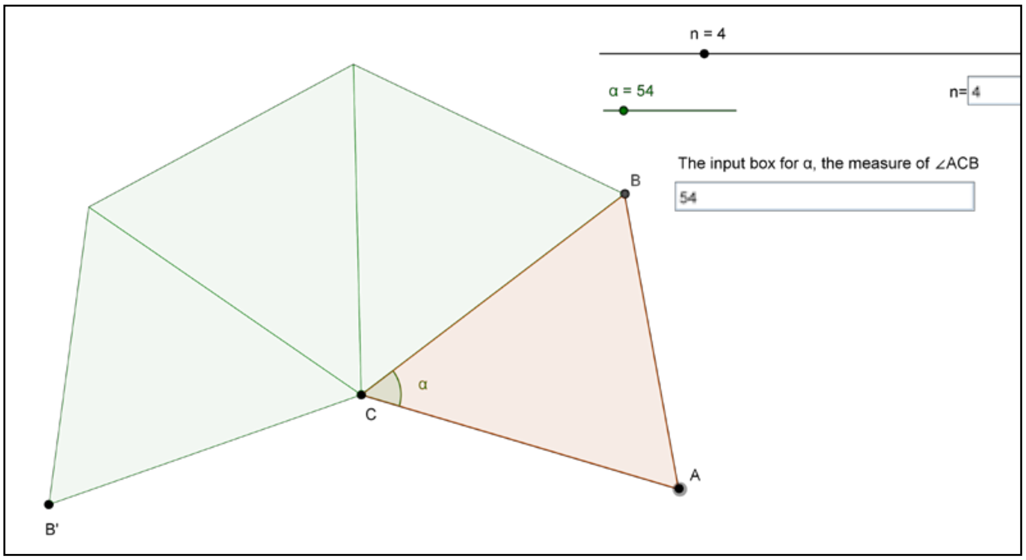

Fig 2 : The GeoGebra screen given from the start

To do the task, students lean on the fact that any regular polygon can be decomposed in as many isosceles triangles as there are sides. These triangles share a common vertex, the centre of the circumscribed circle. Some knowledge elements are thus activated here, that the students could have previously met or as well, that they may be discovering through the activity.

A priori analysis : the divisors of 360

The students can either fix a value for α and modify *n* afterwards, trying then to 'close' the figure; or keep *n* fixed and modify the angle, by exploring with the slider or by entering directly a value for α. We assume that exploration by trial and error with the two sliders will spontaneously be initiated, but will remain relatively ineffective : because of the imprecise control of the angle with the slider, students will probably obtain nothing more than the usual polygons met in elementary school, with standard values for α : the square, the hexagon, the octagon, possibly also the equilateral triangle. It should be noticed that the colour of ΔABC differs from the colour of its images, enabling an easy perception of overlaps. In this respect, it is important to provide the students with a slider for *n* [4] and not only a box where the number is entered. Indeed, seeing the triangles unfurl like a fan when dragging the slider allows to realize that the value for n is not the right one when ΔABC and its images coincide perfectly, e.g. when $n = 5$ and $\alpha = 90°$.

We assume that through this trial and error process, the link between the correct value of n for a given α, and the fact that $n\alpha$ must be equal to 360°, will emerge. By seeking for a better control of their trials, students will explore this link in a more systematic way, by considering the divisors of 360 and by using the input box for α.

We put forth the hypothesis that while engaging a whole-group discussion at about two-thirds of the first lesson, some teams will invoke the divisors of 360 (of course in their own words). In his/her institutionalization, the teacher clarifies the underlying reasoning and establishes, with the whole class, the list of all divisors of 360. Complying with the strategies he/she observed from his/her students, he/she can construct this list by considering the possible values for n (along an increasing list starting at 3) or by considering the values for α along a decreasing list, beginning with $\alpha = 120$. For each divisor, he/she ask the class what is the associated value, of α or n, respectively. He/she shows onto the screen the corresponding polygon and gives its name. He/she supplies the list of divisors with 1, 2, 180 and 360. He/she asks the class if an associated regular polygon could be considered for each of these four values. It appears to us important that the teacher carries the discussion to its ending, namely that depending on the adopted strategy, we won't keep the same divisors to construct the regular polygons : either we keep 1 and 2 as the degree measure of the central angle and we set aside 180° and 360°, or we set aside 1 and 2 as the number of sides and we construct polygons with 180 and 360 sides. At this point, the teacher may suggest that fractional angle measures are possible, for example by considering the regular polygon with 48 sides, and central angle measuring $15° \div 2 = 7{,}5°$. The status of these 'special cases' must then be explicitly clarified with respect to the notion of *divisor*.

A priori analysis : the heptagon

The goal of the following lesson is to tackle the notion of approximation, both from a numerical standpoint (the written representation, exact or not, of a rational number) and a geometrical standpoint (the status of the figure).

The starting point is brought up by the teacher, who comes back to the cases where the division of 360 by n does not result in an exact decimal value. He/she asks students to investigate these cases further. In principle, the case of the heptagon ($n = 7$) should be the first to turn up. Research (e.g. Krikorian, 1996) shows that spontaneously, secondary students don't make use of fractional notation. We anticipate that students will enter in the α box the approximation of 360/7 they'll get from their pocket calculator. GeoGebra then displays 51.43, its standard round up. The teacher must watch for teams who got to that point, first to ask them for a verification that the polygon is well closed, using the zoom [5]; and once they acknowledge that the polygon *is not* well closed, to show them how to get the maximum of 15 decimals into the menu bar (→ Option → Rounding). Then the teacher gives the following instruction : “add one by one the decimals of 360/7 in the input box for α, and at each step, zoom in to check if the figure is well closed”. The students must then find a way to obtain the decimals beyond the scope of their pocket calculator.

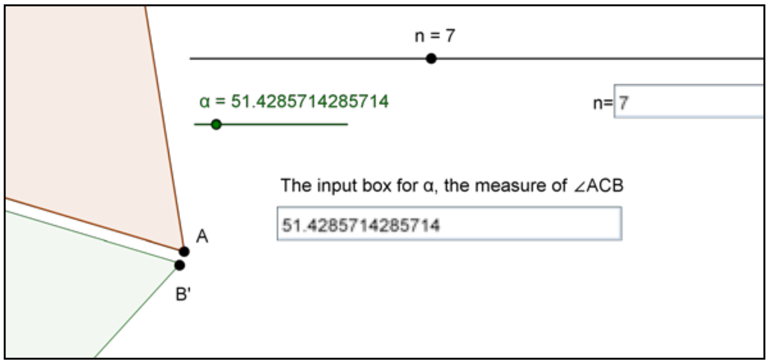

Fig 3 : a close zoom, for an approximation of 360/7 with 13 decimals

Even with 13 decimals, a sufficiently close zoom shows a figure which is not well closed. Entering 15 decimals or more, GeoGebra (version 4.0.41.0) rounds up at 14 decimals and displays 51,42857142857143. Then as close as we get with the zoom, we see a polygon *seemingly* closed. When the teams have reached that point, a whole-class discussion is called upon. The issue of the exact value of 360/7 is raised : “what is it ? Is it 51,42857142857143 ? If I multiply this number by 7, do I get 360 back ? Compute the product by hand.” In the same time, the teacher enter the multiplication into the input bar (at the bottom of the screen), and 360 is displayed in the *Algebra View*. “Are these computations by GeoGebra exact ? The polygon appears to be closed but can we rely on GeoGebra here ? It seems that we have reached the limit of the software ! So, this polygon with seven sides, we can close it or not ?”

In accordance with our theoretical framing, the idea is here to bring students to reflect on the ideal character of the heptagon regardless of what is produced and seen on the screen, and then to extend this ideal character to the already produced polygons and ultimately, to any geometrical object. Going back to arithmetic, students reflect upon what exact angle should be produced (and hence measured) to be able to construct this ideal heptagon. This brings them to consider in parallel the notion of exact representation of a rational number.

It may be an opportunity for asking students to do the long division of 360 by 7 by hand. The teacher may then explain the period and insists that the exact value of 360/7 needs, to be written exactly into a decimal form, an infinity of digits or else, the representation with a bar over the period to account for this infinity. “Under what form can we propose this exact value to the software ? Is it but possible ? And what about entering 360/7 into the box, as a value for α ?” Even with “360/7” entered in the input box, GeoGebra displays 51,42857142857143. The teacher may then confirm that the numbers handled by the software are approximations. He/she moves on to say that the regular polygon with 7 sides does exist (in theory), and that the exact measure (in degrees) of its central angle does not admit a finite decimal expansion, but can nevertheless be exactly represented by the notations $\frac{360}{7}$ or $51,\overline{428571}$. He/she concludes by stating that there exists a regular polygon with n sides for each integer n greater than 2 and that for each one, the central angle measures $\frac{360}{n}$ degrees.

CONCLUSION

So in a meaningful context, the students become aware of the dualities *ideal object – visual representation*, *exact measurements – approximations* in geometry, and link these to the representations of rational numbers. They know that some fractions don't have a finite decimal expansion and they now understand that one must not make use of the equality sign between such a fraction written in the form $\frac{p}{q}$ and any notation referring to a finite decimal expansion.

The activity has been the object of a pre-experimentation outside school with five children (aged 11 to 13), and our hypotheses have then been largely confirmed. More systematic classroom experimentations, in Quebec and Mexico, are to come. The activity is part of a larger research programme about the geometer-physicist paradigm (Tanguay et Geeraerts, 2012, 2013), in which more issues pertaining to measure and measurements will be explored. This programme is also in its early stages.

NOTES

1. The French word for *field* used by Kuzniak is *domaine*. It is closely related to the word and concept *cadre* considered by Douady (1986). Here, *field* should be understood according to a meaning referring to school mathematics, rather than to advanced mathematics.
2. By *synthetic geometry*, we mean geometry without coordinates, as opposed to *analytic geometry*.
3. They are indeed distinct registers in Duval's sense (1993), since for example one cannot add or multiply 5,6 with $4\sqrt{2}$ or with 360/7 without changing the representation of one of the two numbers.
4. Hence from the teacher and a teaching perspective, providing a slider for n is a form of *instrumentalization* of GeoGebra, in Rabardel's sense (e.g. Vérillon & Rabardel, 1995).
5. The zoom is easy to use with GeoGebra : use the mouse wheel ! To avoid the figure being pushed off the zone we want to zoom in – in this instance the neighbourhood of point A – one must just insert the mouse cursor in this neighbourhood.

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MATHEMATICS TEACHERS' PERCEPTIONS OF QUADRILATERALS AND UNDERSTANDING THE INCLUSION RELATIONS

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Abstract: *The aim of this research is to investigate how mathematics teachers comprehend the properties of specific quadrilaterals, how they construct the relations and classify quadrilaterals. This research was conducted on 9 mathematics teachers. 7 problems were designed. Semi structured interview and content analysis was used. The results indicated that properties of quadrilaterals which are best known by participating teachers were square and rectangles. The another results of the study was that teachers correctly defined properties related angles and sides quadrilaterals but have problems with properties related diagonals. In addition to this results some teachers were not classify quadrilaterals, and the teachers who use hierarchical classification were not fully established family relations.*

Key Words: Perceptions of Quadrilaterals, Classification of Quadrilaterals

THEORETıCAL FRAMEWORK

Concepts are quite important in mathematics. In order to express the entire cognitive structure about concept, Tall and Vinner (1981) have used the term "concept image" and stated that concept image includes mental image about the concept, properties and processes about the concept. When considered in the sense of geometry, there are 3 different situations about geometrical concepts. These are definition, image and properties of the shape of geometrical concept. Definitions are structured in a way that would features minimum information of the properties of concept. It was aimed for definitions as the basic principle that they would be economically short and understandable (De Villiers, 1998; Fujita, 2012). Apart from the definitions of concepts, every geometrical concept has a visual image. Visual image might be one step further than the concept. In this sense, typical (prototype) sample are the key factor. Properties of concepts are key factors in the context of making relations between concepts, differentiating concepts, making generalizations of concepts and finally making classification.

Classification of quadrilaterals is regarded important in making relations between quadrilaterals and therefore in solving problems and proof studies in geometry. Because if a quadrilateral is in the same family with another quadrilateral than solutions, proofs, properties etc. raised for this quadrilateral would be valid for the other one as well. In this sense, De Villiers (1994) points out that individuals can make two types of classification of quadrilaterals. The first one is hierarchical classification which done by relating quadrilaterals under subset according to the properties they have. The other one is partition classification which means classification of quadrilaterals in separate sets individually according to the properties they have. According to de Villiers (1994), hierarchical classification was defined as a type of classification which makes family relations of quadrilaterals more understandable. As various researchers have explained, making hierarchical classification of individuals and dominating education in this sense should be supported (De Villiers, 1994; De Villiers 1998; Fujita, 2012).

When the literature is studied, there are various studies about perception of quadrilateral nearly in every age group. In these studies it is seen that students are addressed questions about quadrilaterals, listing their properties, making separation and relation between them and making classification.

In the studies which are carried out for understanding of quadrilaterals by naming the drawn quadrilaterals or drawing the named quadrilaterals; different types of square, rectangle, rhombus and trapezium were asked to students in different age group who participated in the study. As a result of these studies, it was determined that quadrilaterals which have prototypical drawing in general were to a large extent marked or named correctly (Okazaki and Fujita, 2007; Fujita and Jones, 2007; Fujita, 2012; Clements, Swaminathan, Hanibal and Sarama, 1999; Monaghan, 2000).

The role of definition is very important for perception of quadrilateral. In the literature studies, personal definitions which were made free from academic definition were considered in understanding the perception so, the studies were focused on how the quadrilaterals were defined individually. In the study of Fujita and Jones (2007), it was found that the percentage of correct answer of prospective class teachers about quadrilaterals is quite low.

In the studies in which were asked to classify quadrilaterals, it was detected that great majority of the students make partition classification (Erez and Yerushalmy, 2006; Monaghan, 2000; Fujita and Jones, 2006; Berkün, 2011). The reason was stated as the images of quadrilaterals and the effect of these images on inclusion relations of quadrilaterals.

In the study of Okazaki and Fujita (2007) (with students aged between 15 and 18 and prospective class teachers) in which it was analyzed how the students make relations among quadrilaterals, it was determined that around 50% students gave correct answers for the question of whether rhombus and rectangle are parallelogram or not. Moreover, they have answered the view that square is rectangle and rhombus with quite a low percentage such as 35%. There are different studies which states that the relation between rectangles are defined correctly in low rates (Okazaki and Fujita, 2007; Elia, Gagatsis, Deliyianni at al., 2009; Heinze and Ossietzky, 2002; Fujita and Jones, 2007; De Villiers, 1998). In these studies it was found that not being able to make relation between square-rhombus and parallelogram-rectangle result from the fact that their angles are not 90^0. Similarly, the relation between square-rhombus and parallelogram-rectangle could not be done since their edge lengths are not same and they were included in different family.

Perception and classification of geometrical shapes contribute solution of problems both in real life and in different fields of mathematics (NCTM, 2004; Martin and

Strutchens, 2000). As it's understood from the mentioned studies, it was observed that individuals belonging to different age groups don't have difficulty with the recognition of quadrilaterals, yet it was observed that they have difficulty with the classification of quadrilaterals and with the understanding of relations between them. However, the maths teachers and primary school teachers have great responsibility in the correct description of quadrilaterals by the individuals, and relating them to each other and classification of them by the individuals. Maths teachers have an important role in especially perceiving the relations of quadrilaterals and classification of them, which necessitates higher cognitive abilities. That's why, the knowledge of maths teachers about quadrilaterals must be checked in detail. Moreover, when the literature is investigated, there is no study showing the knowledge of maths teachers about quadrilaterals. For this reason, it is expected that this study will compensate for the loss in the literature.

THE PURPOSE OF THE RESEARCH

In this study it was aimed at determine how mathematics teachers comprehend the properties of quadrilaterals, how they construct the relationship between quadrilaterals and classify them.

METHOD

In this study, semi-structured interview was done with the method of qualitative research. In the interviews participants are asked seven questions five of them are open-ended and two of them are multiple choice questions. The interview questions were prepared based on the related literature. Questions were composed of three sections. In the first section teachers were asked how they define basic features of quadrilaterals, in the second section how they relate quadrilaterals in pairs, in the third section questions were about the family relations between quadrilaterals. Sample questions about each section are in the Appendix 1. Interviews were done with 9 middle school mathematics teachers who work at 8 different schools at a city in Turkey. Three of these teacher's teaching experience is less than five years, of the two between five and ten years, of the other two between twenty and ten and of the last two have more than twenty years of experience. According to the year of teaching experience the ages of the teachers range from twenty-five to fifty. The two teachers who participated in the research graduated from training institute, which had a two-year education and the others graduated from an education faculty, which has a four-year education years. All the interviews were recorded with audio recorder. Before the data was analyzed, each audio record was fully transcribed into verbal data. Content analysis technique was used in the analysis of obtained data.

FINDINGS

In this study, data were obtained by middle school mathematics teachers, analyzed under three main titles such as perceptions of quadrilaterals and understanding of the relations of quadrilaterals and classification of quadrilaterals.

Mathematics Teachers' Perceptions of Quadrilaterals

In the study, teachers' knowledge about the properties of square, rectangle, parallelogram, rhombus and trapezium was questioned.

It was determined that teachers correctly define properties of parallelism and diagonal of square and rectangle as well as properties of angle and vertices. Only one teacher was not sure in defining diagonal property of rectangle.

In data about parallelogram, the first perception of teachers was the parallelism of vertices. As it is understood from the statement of a teacher, *"if it is parallelogram, then their opposite vertices are parallel"* that since the name includes the term parallel, the first perception was parallelism.

It was observed that teachers' knowledge about properties of angle and vertices of parallelogram are mostly correct but they have doubts about properties of diagonals. This irresolution is related with the equivalence of lengths and angle bisector.

The first perception of teacher about rhombus is the equivalence of vertices due to the name of quadrilateral. The view of a teacher suitable about this condition was:

"We teach it as a quadrilateral whose all vertices are equal. Even when defining this quadrilateral, we reflect as it diamond shape colloquially." (T7)

As it is understood from the statement of T7, teachers identify rhombus with diamond shape. Similarly, it was observed that other teachers use same statements.

Moreover, similar to parallelogram, it was observed that teachers' knowledge about properties of angle and vertices of rhombus are mostly correct but they have doubts about properties of diagonals. This irresolution is related with the equivalence of lengths and angle bisector.

It was determined that teachers have two different views about the shape of trapezium. It was found that this diversity of views results from "parallelism of sides" and it was defined as the parallelism of "at least two sides" or "only 2 sides". Views of two different teachers:

"Only two sides of trapezium can be parallel" (T6)

"According to the property of trapezium (hesitates), at least two opposite sides should be parallel." (T8)

It was observed that teachers have problem with angle properties. However most of the teachers (7 teachers) stated that "side angles which are not parallel are supplementary angles" others (2 teachers) stated that "consecutive angles are supplementary angles".

It was determined that some of the teachers both know the properties of trapezium and have different perception about formal status of trapezium. Statements about this view are:

"Trapezium is the form of rectangle whose sides have different cut" (T2)

"Considering it as the divided form of triangle but right and left sides are not parallel like the sides below and above." (T7)

When the explanations of Ö2 and Ö7 are analyzed, it is observed that they relate trapezium with the images of other quadrilaterals in the sense of form.

Mathematics Teachers' Understanding of the Relations of Quadrilaterals

Teachers who participated in the study were asked to make bilateral relations among quadrilaterals. Teachers making relations of between square-rectangle, rectangle-parallelogram, square-parallelogram, square-rhombus were determined. According to data, it was observed that most of the teachers focus on common properties and differences rather than classification.

For the common and different properties of square and rectangle, it was observed that they focus on angle, vertices and diagonal properties. Only one teacher make the relation that square is a special kind of rectangle. The statement of this teacher is as such:

"For example students are asked. Is square a rectangle or not? Most of the students say it is not. But I emphasize that square is a rectangle, specially." (T6)

The statement of a teacher among 8 others who regard the relation between square and rectangle in the sense of common and different properties is as such:

"… there are short sides and long sides in rectangle as well. But all the sides of square are equal, yet opposite sides of rectangle are equal. Diagonals of rectangle do not intersect perpendicularly but diagonals of square do so. All the sides of rectangle are not equal but all the sides of square are."(T3)

As it is understood from the statement of T3, they make relation between square and rectangle by focusing on the differences of sides and diagonals. Moreover, from this statement it can be concluded that teacher first of all visualizes rectangle as a quadrilateral which has short side and long side.

Properties of angle and diagonal were mentioned for the common and different properties of rectangle and parallelogram. However, properties of diagonal were stated wrong by most of the teachers. For example, a teacher pointed out a wrong property by saying **"***diagonal length of both shapes is equal." (T4)*

Apart from the answers which are based on common and different properties of rectangle and parallelogram, there are teachers who make family relation as well. One of these teachers gave this answer.

"… while teaching rectangle, we mention that it is a special kind of parallelogram. We make such a relation while teaching rectangle in order to show that we can obtain it out of a parallelogram." (T8)

In addition to these relations, a teacher made a relation based on the image rather than the properties of rectangle and parallelogram. The answer of this teacher was:

"... think that we make a rectangle pattern from modeling clay, when we pull from opposite corners equally we obtain a parallelogram, so we can make a parallelogram out of a rectangle" (T3)

A teacher who participated in the study stated there would be no relation between rectangle and parallelogram with this sentence.

"All the properties of rectangle are different from parallelogram, they do not match with each other." (T5)

Angle and side properties were mentioned for the common and different properties of square and parallelogram. There is one teacher who makes family relation for square and parallelogram. This teacher gave such an answer:

"... it is valid for parallelograms. Because it is parallelogram. Rectangle, square are parallelograms. We should give this message." (T6)

3 teachers who participated in the study made no relation between square and parallelogram. One of these teacher stated that:

"All the properties of square cannot be valid for parallelograms, we cannot make a relation because there are too many incompatible properties." (T3)

Angle and diagonal properties were mentioned for the common and different properties of square and rhombus.

For the relation based on angle properties, it was stated that if all the angles are right angle that the rectangle is square, if not then it is rhombus. There is one teacher who makes family relation (T6). In addition to these relations, two teachers made a relation based on the image rather than the properties of square and rhombus. Answers of these teachers are:

"the rectangle when you press one side of the square is rhombus." (T1)

"If we pull the sides of square to one side equally then it is rhombus. We should pull the sides equally because opposite sides are equal." (T3)

Mathematics Teachers' Classification Of Quadrilaterals

Teachers were asked to classify quadrilaterals in the study. According to data obtained from teachers' answers, there are 3 situations for quadrilaterals classification. These are:

1. Not making any relation among quadrilaterals (T1, T2, T5)

2. Partition classification of quadrilaterals: Making a relation with a table only according to common and different properties. (T4, T7)

3. Hierarchical classification of quadrilaterals: Making relation with specific family relations. There are 3 different schemes for this situation.

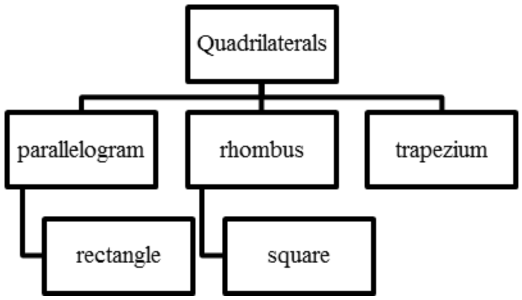

Figure 1: (T3, T8)

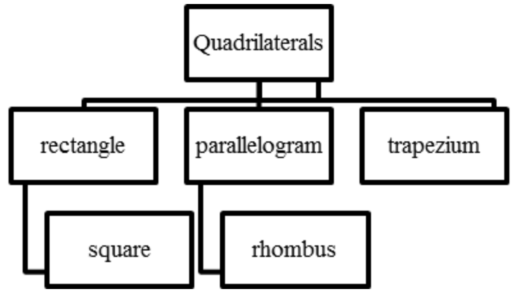

Figure 2: (T9)

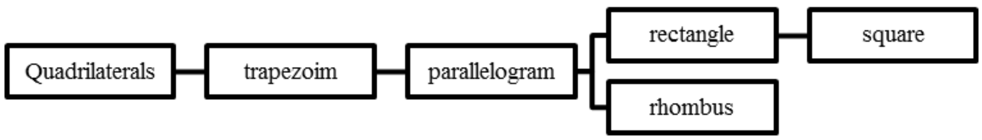

Figure 3: (T6)

In Figure 1, it is seen that teachers interpret square as a special kind of rhombus, but not a special kind of rectangle; interpret rectangle as a special kind of parallelogram but cannot interpret rhombus as a special kind of parallelogram. This results from the fact that teachers make interpretation based on side lengths. In this way, they could make relation between square and rhombus whose side lengths are equal and between rectangle and parallelogram whose only opposite side lengths are equal.

In Figure 2, it is seen that teacher interpret square as a special kind of rectangle and also interpret rhombus as a special kind of parallelogram but cannot interpret square as a special kind of rhombus. This condition results from the fact that teacher makes interpretation based on angles. In this way, they make relations between square and rectangle whose angles are 90º and between parallelogram and rhombus whose angles are not 90º.

When the figures for the third situation were analyzed, it is seen that teachers cannot make family relations totally. It was seen that teachers who have made Figure 1 and Figure 2 separates trapezium from quadrilaterals but teacher who made Figure 3 includes trapezium into the classification with other quadrilaterals. Since teachers generally do not consider property of parallelism, this causes them to ignore the fact that square and rectangle are parallelogram. It was stated that only 3 teachers interpret square or rectangle as parallelogram.

CONCLUSION

Generalization was not done in this study which was carried out with 9 mathematics teacher; yet in the light of data, specific profiles for the teachers' perception of quadrilaterals were put forward. It was determined that teachers who participated in the study correctly define properties related angles and sides of quadrilaterals but have problems with properties related diagonals. It was observed that although teachers have problems about properties related diagonals of quadrilaterals, they focus rather on diagonal properties while establish relationship between quadrilaterals. However, it was seen that they define properties of diagonal wrongly. In Monaghan's (2000) study, in which he presented children's views (aged 11-12 years) of the differences between some quadrilaterals; he concluded that the images, properties related sides and the angles are focused on for differences between quadrilaterals, but not properties related diagonals.

The geometrical figures among mentioned quadrilaterals which is least known by participating teachers was "trapezium". It was observed that teachers have difficulty in identifying the image of trapezium. This condition was observed in the study of Berkün (2011) which was carried out with earlier ages.

While parallelism is not prominent for square, rectangle and rhombus in the findings, it was stated among the properties of parallelogram and trapezium. Similar findings were observed in the studies of Fujita (2012), Fujita and Jones (2007), Okazaki and Fujita(2007), Heinze and Ossietzky (2002).

Three of the teachers who participated in the study could not make classification. Contrary to most of the studies (Berkün, 2011; Monaghan, 2000; De Villiers, 1994), the some of the teachers (4 teachers) could make hierarchical classification. However, they could not correctly and fully establish family relations. The reason of this was that teachers focused on only angle and side properties, could not interpret and relate all the properties together.

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APPENDIX 1.

1) Question Aiming at Description of Quadrilaterals:

A teacher asked his students the features of a trapezium. The answers of five students are as follows:

Student 1: The two opposing sides are parallel.

Student 2: The consecutive angles are complimentary.

Student 3: There are two types of trapezium. Right trapezium and isosceles trapezium.

Student 4: The high of a right trapezium is the one which is perpendicular to the paralel sides.

Student 5: The domain(alan) of each trapezium is the multiplication its height with the half of the total lenght of lower base and upper base.

When you look into the answers, which ones are correct and which ones are wrong? Why?

2) Question Aiming at Relations of Quadrilaterals:

Answer the questions below:

a) What is difference between a sguare and a rectangle?

b) What is difference between a rectangle and a parallelogram?

c) What is difference between a sguare and a rhombus?

d) What is difference between a trapezium and a parallelogram?

3) Question Aiming At Classification:

Below are five items about quadrilaterals.Which one is correct?

a) All the features of a rectangle are the same for all squares.

b) All the features of squares are the same for all rectangles.

c) All the features of a rectangle is the same for all parallelogram.

d) All the features of squares are the same for all rhombus.

e) None of the choices above is correct.

PLANE GEOMETRY: DIAGNOSTICS AND INDIVIDUAL SUPPORT OF CHILDREN THROUGH GUIDED INTERVIEWS – A PRELIMINARY STUDY ON THE CASE OF LINE SYMMETRY AND AXIAL REFLECTION

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Based on a detailed literature review, previous studies related to diagnosing and supporting (DS) children with mathematical learning disabilities are analysed with respect to the role of (plane) geometry. The results indicate that existing diagnosis and support instruments strongly focus on arithmetical topics while geometric contents are underrepresented or even non-existent. Then, first items DS line symmetry and axial reflection are implemented in 6th grades in the form of a test in order to specify children's understanding of these concepts and to find suitable items for the DS instrument. Finally, the test results are compared with findings through an individual interview conducted with a 5th grader beforehand.

Key words: diagnosis in geometry, axial reflection, line symmetry, figural concepts.

INTRODUCTION

Diagnosis and individual support has proved itself valuable to learn more about children's ways of mathematical thinking and to screen their misconceptions[1] and lacking mental models in order to contribute to a change for the better. A number of publications provide descriptions and lists of pupils' difficulties, error types and misconceptions, as well as suggestions how to foster particular mathematical issues. However, existing studies and commercial instruments for DS focus mainly on arithmetical issues. The field of geometry is either not being covered or only being minimally addressed. Consequently, it's even more important to develop a diagnostic instrument to assess children's geometric misconceptions and fields of difficulties in order to gain support points for the individual fostering. Thus, it is our aim to develop item-based interview guidelines suitable for detecting children's misconceptions and solution strategies regarding geometric issues and problems as well as to provide basic individual support rudiments. Our first investigations focus on line symmetry and axial reflection since there are a number of studies that describe pupils' misconceptions and error types (Küchemann, 1993; Xistouri, 2007; Bell, 1993) and are conductive to the development of DS items and interview guidelines. The purpose of this paper is to present the first outcomes of our study concerning line symmetry and axial reflection. It presents the items implemented in 6th grades in the form of a test and its analysis results regarding children's understanding and misconceptions of axial reflection. The overall aim is to develop task-based interview guidelines for the diagnosis and support of lower secondary students' figural concepts of line symmetry and axial reflection. This process will draw upon the theory of figural concepts by Fischbein.

THEORETICAL FRAMEWORK

The Concept of Line Symmetry and Axial Reflection

Hoyles & Healy (1997) investigated the processes through which pupils come to negotiate mathematical meanings for reflective symmetry by describing a micro-world, Turtle Mirrors. It is used "to help students focus simultaneously on actions, visual relationships and symbolic representations regarding reflective symmetry" (Panaoura, Elia, Stamboulides & Spyrou, 2009, p. 46). They also describe students' primitive and intuitional variety of strategies for solving paper and pencil tasks on reflective symmetry. The reflection of objects in horizontal or vertical axis was easier for the students than in slanted axis. Hoyles & Healy (1997) describe how pupils use an approximate strategy derived from paper folding while reflecting in slanted axis – called 'the strategy of imagining a vertical axis (IVA)' in this paper.

Pupils face different problems when constructing plane reflections in a line and identifying the lines of symmetry in plane figures. Schultz (1978), Grenier (1985) and Küchemann (1993) identified the following factors as relevant: (a) direction of the axis (vertical, horizontal, slanted (45°), other); (b) complexity of the object being reflected; (c) presence or absence of a grid; (d) slope of the object and (e) size of the objects and distance from the axis of reflection. The first four were incorporated by Küchemann into a structured sequence of questions and the levels of response were identified as global, semi-analytic, analytic and analytic-synthetic. In *global* responses the object is considered and reflected as a whole with no reference to particular parts, angles, or distances. In *semi-analytic* responses, a part of the object is reflected first and the rest drawn from its matching the original shape and size. In *fully analytic* responses, the object is reduced to key-points, each reflected individually. These are connected and the result is accepted even though sometimes the image looks wrong. In analytic-synthetic responses, the global and analytic responses are coordinated so that the image is precise and also looks correct (Bell, 1993, p. 130). Küchemann indicated that students have some informal understanding of geometric transformations such as reflection and rotation. However, children experience difficulties when working on shapes which involve these transformations.

Bell's study (1993) consisted of interviews and a diagnostic teaching experiment with students aged 11-12 years. This, too, revealed misconceptions. We take one example: children believe that horizontal/vertical objects have horizontal/vertical images or that horizontal objects have vertical images and vice versa.

There are several studies which deal with students' conceptions of reflective symmetry and their difficulties in understanding the concept; however, there is more to be uncovered and explored. As an example, "[l]imited attention has been given to interrelations among students' concept image of reflective symmetry and the use of different representations of the mathematical concept" (Panaoura, Elia, Stamboulides & Spyrou, 2009, pp. 47-48).

The Theory of Figural Concepts

Fischbein (1993) introduced the notion of *figural concepts*. He made an attempt to interpret geometrical figures as mental entities possessing simultaneously both conceptual and figural properties. According to the theory of figural concepts, the main objective is the development of the interaction between the figural and the conceptual aspect. Fischbein (1993, p. 160) stated that

> [a]lthough a figural concept consists of a unitary entity (a concept expressed figurally) it potentially remains under the double and sometimes contradictory influence of the two systems to which it may be related – the conceptual and the figural one. Ideally, it is the conceptual system which should absolutely control the meanings, the relationships and the properties of the figure.

Various students' difficulties in geometrical reasoning can be interpreted in terms of such a rupture between figural and conceptual aspects of figural concepts – possibly even their difficulties in answering questions referring to axial reflection. The theory of figural concepts provides a powerful tool and offers a theoretical framework for our analysis of students' understanding of axial reflection and other geometric topics.

METHODOLOGY

The purpose of this study is to compile task-based interview guidelines for the diagnosis and support of lower secondary students' understanding of axial reflection, and thus for the development of the interaction between their figural and conceptual aspects of axial reflection. To achieve this, a pre-test was conducted and evaluated with Küchemann's methodology (Küchemann, 1993). Its results, that is, the solution rate of the items and emerging students' typical misconceptions while working with those items, will be examined in order to select suitable interview-items. The subjects were 195 6th graders who were chosen from nine classes and three schools. The students were given two invented tasks on line symmetry (including three items) and four tasks on axial reflection (including 13 items). The latter were adapted from the CSMS transformation geometry test restricted to an investigation of reflection and rotation (Küchemann, 1993). However, in order to focus on basic knowledge and save working time, both difficult items (e.g., object intersects the axis) and equivalent items of the CSMS transformation geometry test were excluded from the pre-test.

The items for line symmetry will give an overview about students' concepts of symmetry. The first item of the first task pertained to their associations with line symmetry and axial reflection; the second asked the students to draw images with one, two and no axis of symmetry. The second task provided eight flags of European countries for which the children should decide by marking with a cross if the given flag is symmetric or not, state and draw the right number of axis of symmetry.

The 13 items for axial reflection (tasks 3 to 6) included working on squared and blank paper with and without the use of the set square. Items of tasks 3 and 6 involved the drawing of the image. Task 4 (see fig.1) requires sketching the axis of reflection between pairs of figures (4.1) or stating that and why this was not possible (4.2). Task 5 provided several points as possible reflections of point A. The students had to choose which of the given points was the image of A and explain their choice (see fig.1). Explanations encourage students to think analytically about the properties of reflection, even if they had originally made an intuitive choice.

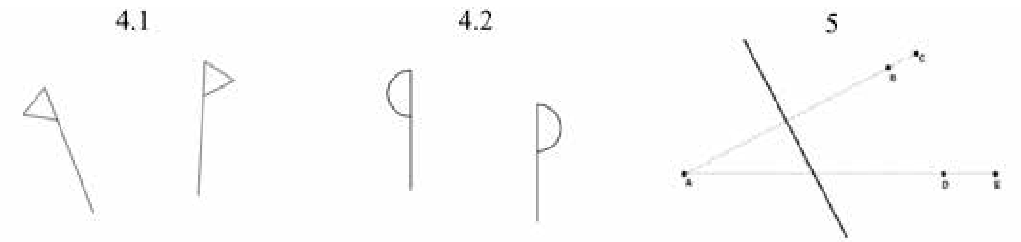

Figure 1: Items to tasks 4 and 5

In task 6 (see fig.2) students were allowed to use the set square; the remaining items required working without the use of the set square and other tools as well. In comparison, the equipment needed for the CSMS transformation geometry test was a ruler marked in centimetres; protractors and set squares should not be used for that test. The reason for our method lay in the fact that an individual interview with a 5th grader has shown that working with the set square provides wide support for the students. Many students have tendency to rely on procedural aspects of axial reflection, which is, e.g. positioning the set square in the taught way, measuring the distance of the object to the axis and finally transferring it to the distance of the image and the axis. This also became evident in the students' responses, especially in task 5. Since the students weren't allowed to use any tool for the test items – task 6 apart – and many of the questions involved drawing, regions had to be defined to delimit what was to be regarded as correct. For this, we used the marking scheme of Küchemann and the printed acetate sheets which he provides. He, too, defined regions by devising rules that took into account such factors as distance from the axis, the slope of the axis and whether or not a ruler was to be used. Thus, five codes were allocated to children's responses according to Küchemann's marking scheme. Code 1 includes responses which are correct and Code 2 those that are adequate, that is, not precise enough to be regarded as correct but also not obviously wrong. Code 3 refers to overt errors, e.g. reflecting horizontally or vertically when the axis is slanting or drawing the image parallel to the object. Code 4 refers to all other responses and Code 5 to unworked items. Two raters were involved in marking the students's responses and in assigning Codes to them. In order to perform reliability testing the Cohen's kappa coefficient was calculated. With $\kappa \approx 0{,}84$, it's safe to assume that a good inter-rater-agreement is present in this case.

Küchemann (1993) classified his reflection items into two types instead trying to discuss the effect on children's performance of the various combinations of the above mentioned features (a)-(d). These two types differ, generally, in terms of the strategies needed to produce Code 1 and Code 2 responses. Type A covers items where the axis is vertical (or horizontal) or the object is a single point (3.1, 3.2, 3.4, 3.5, 3.6). Type B includes items where the axis is slanting and the object is a line or a flag (3.3, 3.7, 6.1, 6.2) (see fig.2). So, type A items can be regarded as involving only one slope (of the object *or* the axis) because, having a vertical axis, the ability to reflect in a direction perpendicular to the axis requires merely the knowledge that the reflection takes the object to the other side. Type B items involve both the slope of the object *and* the axis.

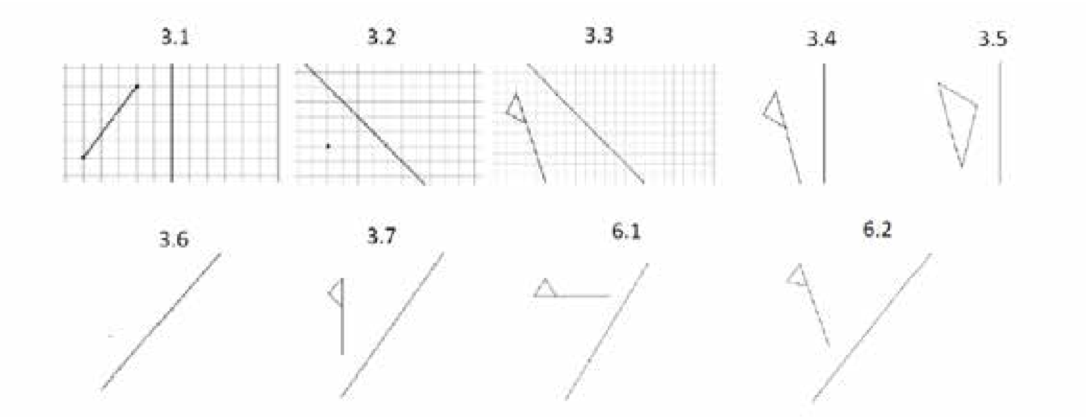

Figure 2: Items to tasks 3 and 6

Referring to the 13 items for axial reflection, each response to an item in each of the three reflection tasks was assigned one of the five codes. The percentages of success (Code 1) were calculated for each item. The items were grouped according to the percentages of success and item characteristics. The percentages were used inter alia for comparing the difficulties students experienced, both in Küchemann's test and in our sample, while working with the items, that is, for sorting the items by levels of difficulty. This is important because interviewed students will be provided with items also sorted by levels of difficulty. In order to intervene as a qualified professional, the challenging function of the interviewer will be to find out as much as possible about the student's figural concept of axial reflection or rather about the conflict between the conceptual and figural aspects generated by the given items. For this purpose, students' main strategies or error types in each item are worked out, which is useful for the interviewer to shed light on critical issues while interviewing the child.

RESULTS

The following tables 2 and 3 present the percentages of students' success in solving the reflection items. Since the items are grouped according to the percentages of success (Code 1), these tables also illustrate with which items students could cope more easily. The gradations in the tables are correlated to the item-levels of Küchemann (1993) with a factor of 0,90. A t-Test ($p>0.86$) shows that there is no statistical difference among Küchemann's results of 1980 and ours of 2012. This implies, considering our sample, that there is no change in students' misconceptions and understandings of axial reflection since 1980.

The solution rates of task 4 reveal that students coped more easily with stating that it is not possible to draw a mirror-line between pairs of figures (4.2) then with finding the mirror-line when it was possible (4.1) (see fig.1). A percentage of 66% of the students stated correctly in 4.2 that a mirror-line is non-existent. As for the results of table 1, 51% justified this statement by saying that the two figures are shifted or don't have the same height and 11% by stating that both figures are not parallel or don't face each other; 28% of the students drew vertical or diagonal axis between the two figures, which is again similar to Küchemann's results where 26% did so (Küchemann, 1993).

TASK	PERCENTAGE OF STUDENTS' (IN)CORECT (–/+) & NO (/) RESPONSES			PERCENTAGE OF GEOMETRIC NOTIONS IN STUDENTS‘ RESPONSES				
	+	-	/	A is faced by B/D	distance	slanted axis	direction	⊥
5	78	15	7	13	52	15	9	11
				shifted	face-	parallel-	other or no responses	
4.2	66	28	6	51	5	6	38	
4.1	53	42	5	no reasoning demanded				

Table 1: Items grouped according to the % of success & item characteristic 'reasoning'

Task 5 was evaluated as correct if point B was chosen as the image of point A – independent of the explanations. In task 5 (see fig.1), 78% of the children chose B as the image of A whereas 9% chose D. 13% justified their choice with 'A is faced by B(/D)', 15% chose B 'because the axis is slanted' and only 11% used the terms like 'perpendicular', 'right angle' or '90°'. 52% of the students gave explanations that referred to the distance and 9% to the direction. 21% of the responses referred to both the distance *and* the direction (e.g. "B/D is on the same line as A") or the distance *and* the perpendicularity. However, only 8% explained their choice correctly referring to the distance *and* the perpendicularity. In comparison, in the CSMS transformation geometry test, 21% gave explanations that referred to "the distance *and* the direction of the point in relation to the axis" (Küchemann, 1993, p. 141).

Considering the students' associations with the two topics in task 1, their tendency to rely on procedural aspects of reflective symmetry becomes obvious. 42% associated with axial reflection and line symmetry tools, with 24% mentioning the set square, 6% the ruler, 4% the compass and 8% the pencil. The terms 'right angle' and 'perpendicular' occurred only in 5% of the responses and 'sameness' in 9%. It is obvious that the properties are slightly associated with symmetry and reflection tools. Especially the set square (see fig.4), which assists students in solving symmetry and reflection items, seem to play a more prominent role in the students' associations than the conceptual aspects. Regarding students' explanations in task 5, the aspect of distance seems to be more internalized than that of perpendicularity. One root cause may possibly be the broad use of the set square whereby students just have to measure the distances to the axis; the aspect of perpendicularity is hidden behind the center line of the set square (like in a black box) which is to be positioned onto the axis in order to draw perpendicular lines. Thus, many students do not realize which geometrical action is hidden behind this ‘positioning’.

The supporting role of the set square is also striking in a videotaped interview with a 5th grader who is struggling with drawing the image of a ‘modified’ triangle in a slanting axis without the support of any tool. After a couple of minutes the child produces the reflection of the given object –nevertheless the result is the image of a reflection in a vertical axis because the child ignored the slope of the axis and imagined a vertical axis (IVA) (see fig.3).

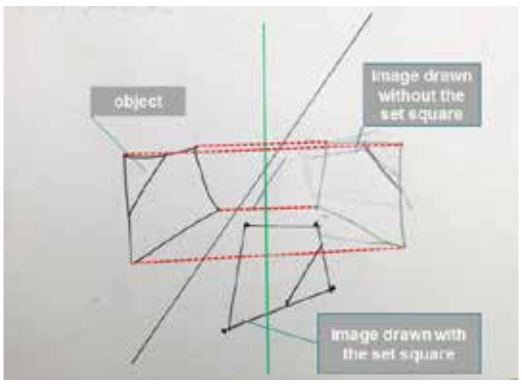

Figure 3: Students’ strategy of imagining a vertical axis (IVA)

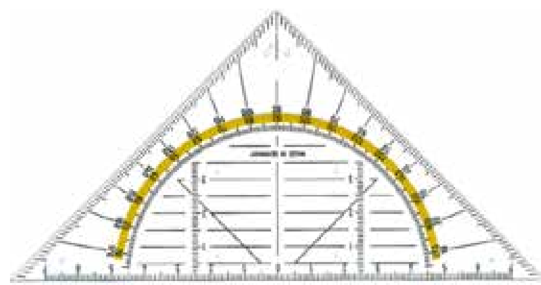

Figure 4: The set square

The child, still not aware of the answer’s incorrectness, is asked by the interviewer to check her result with the set square. Within a few minutes the child produces the precise correct answer on the same sheet by relying only on procedural aspects. In this case study, the support of the set square is enormous and it is our future aim to investigate other cases concerning this matter through guided interviews. Does the set square inhibit or delay the development of the interaction between students’ figural and conceptual aspects of axial reflection by providing too much assistance? Is the set square even a stumbling block for the internalisation of axial reflection?

The results of tasks 3 and 6 depict students’ typical misconceptions. 3.1, 3.2 and 3.6 are items where the axis is vertical or the object is a single point. It is noticeable that 8% of the pupils who could cope with 3.1 couldn’t do so with 3.2. Items 3.1 and 3.2 caused least difficulties for the students. In the former, 2% of the students’ responses were assigned to Code 3, including 67% reflecting horizontally (RH) and 22% drawing the image parallel to the object (IPO). In the latter, 7% of the students’ responses were allocated to Code 3, with all 7% RH. For the items 3.2 and 3.6 the percentage making overt errors (e.g. RH) was virtually the same. 8% of the responses to item 3.6 were Code 3 responses including 78% RH (see table 2). The presence of a grid in 3.2 seems not to be a simplification for the students. The presence of a slanting axis in both 3.2 and 3.6 and of a slanting object in 3.1 seems to cause RH.

ITEM	PERCENTAGE OF CODE					PERCENTAGE OF ERROR TYPES (CODE 3&4)							
	1	2	3	4	5	RH/ RV	IPO	IVA	D-	S-	T	ROT	Others
3.1	91	7	2	0	0	67	22	0	0	0	0	0	11
3.2	86	4	7	2	1	78	0	0	0	0	0	0	22
3.6	63	19	8	6	4	78	0	0	11	0	0	0	11
3.4	61	0	0	36	3	0	22	0	50	3	5	3	17
3.5	55	0	0	41	4	0	0	0	54	29	7	0	10

Table 2: Items grouped according to % of success & item characteristic 'type A'

Responses to items 3.4 and 3.5 were assigned to Code 1, 4 and 5 (Küchemann, 1993). It is striking that in 3.4 50% of the error types related to the distance (D-). 22% drew the image parallel to the object (IPO). In 3.5 54% reflected failing the distance and 29% the size of the object. Both items involved a vertical axis and a slanting object – in 3.4 a flag and in 3.5 a triangle. Here, the complexity of the objects had effect on children's performance.

3.7 and 6.1 are items involving a slanting mirror-line whereas the object to be reflected is horizontal or vertical. Working with items 3.7 and 6.1, the students experienced most difficulties. Virtually the same percentage of children made overt errors (e.g. IPO, RH/RV) despite the fact that the students were allowed to use the set square in 6.1. Code 3 and 4 responses to item 3.7 involved 31% IPO and 21% IVA; 10% translated the object and 17% confused reflection with rotation (see table 3). The percentage of students using the strategy IVA was even greater in 6.1: 47% imagined a vertical axis and reflected then. 18% drew the image parallel to the object and 20% rotated the object.

ITEM	PERCENTAGE OF CODE					PERCENTAGE OF ERROR TYPES (CODE 3&4)							
	1	2	3	4	5	RH/ RV	IPO	IVA	IPA	O-	T	ROT	Others
3.3	45	18	20	8	8	18	26	7	12	14	5	0	18
6.2	40	29	15	4	13	19	15	8	0	0	0	35	23
6.1	39	20	35	2	4	0	18	47	0	0	0	20	15
3.7	25	29	32	4	10	13	31	21	0	0	10	17	8

Table 3: Items grouped according to % of success & item characteristic 'type B'

Items 3.3 and 6.2 involve two slopes (object and axis), which implies that the slope of the object has to be related to the axis. Many children seemed to find the coordination that is required difficult and either ignored the slope of the axis entirely or ignored its effect on the slope of the image or didn't work on the items. In 26% of the Code 3 and 4 responses to item 3.3 the image was drawn parallel to the object and in 18% the object was reflected horizontally. In 14% students confused the orientation (O-) and in 12% they draw the object parallel to the axis (IPA). Code 3 and 4 responses to item 6.2 included 19% RH and 8% IVA, 15% IPO and 35% ROT. In both 3.3 and 6.2, the percentage of IVA errors was low. The proportion of children answering both items correctly is virtually the same, also the proportion making overt errors even though in 6.2 the use of the set square was allowed. Here, the question which arises is why there is a low solution rate in 6.1 and 6.2 in spite of the fact that the set square was allowed. Did the work without the use of a tool in the first 11 items generate a conflict between the figural and conceptual aspects of students' figural concepts of axial reflection? And is this related to the fact that the students were confused when they had to use the set square in task 6? Additionally, about 45% of the students' total responses in the pre-test involved IVA more than once. However, there's no perceptible pattern since the majority didn't make this mistake more than twice. Even so, 45% of the students applied IVA systematically. Is this the controlling figural aspect of axial reflection? A prototype? A visual Gestalt?

CONCLUSION AND DISCUSSION

Appropriate test-items and the pre-test's results will be used not only to compile item-based interview guidelines aiming to *diagnose* lower secondary students' understanding of axial reflection, that is, to develop their figural concepts of axial reflection, but also to *support* children to revise them. Certainly, one has to consider that harmonizing the two components of figural concepts is neither spontaneous nor simple (Mariotti, 1995). Hence, in a next step, individual interviews will be conducted with 6th graders in order to help students to develop the interaction between the figural and conceptual aspects of axial reflection. An intervention or an attempt to do so could be made by generating a conflict between the two aspects. Destabilizing repeatedly a student's figural concepts and making him/her adapt conceptual aspects can possibly have positive effects in the development of *his or her* figural concept of axial reflection.

Based on the above analysis and results, several questions have been raised and need to be answered through further analysis and investigation. Future interviews may clarify the results of the quantitative part of this study. Among other issues, we will focus on questions regarding the role of the set square in the process of learning and understanding axial reflection as well as regarding the development of students' figural concepts of axial reflection: firstly, does the set square inhibit or delay the development of the interaction between students' figural and conceptual aspects of axial reflection by providing too much assistance? Secondly, is the set square even a stumbling block for the internalisation of axial reflection? Lastly, did the work without the use of a tool in the first 11 items generate a conflict between the figural and conceptual aspects of students' figural concepts of axial reflection? At this point, the study of Son (2006) must be considered. It reveals that a large portion of pre-service teachers has misconception and limited understanding of reflective symmetry.

Furthermore, “they have tendency to rely on procedural aspects of reflective symmetry when helping a student understand reflective symmetry correctly although they recognized a student’s misconceptions in terms of conceptual aspects” (Panaoura, Elia, Stamboulides & Spyrou, 2009, p. 47). Therefore, it is important that we attempt to answer the above questions since this has implications not only for the students, but also for teacher educators and teachers.

NOTES

1. The word “misconception” has been used instead of “conception”. This is because the German notion *Schülerfehlvorstellungen* (= students’ misconceptions) is common in the field of diagnostics in Germany. That is why we followed the perspective of (Son, 2006) and (Bell, 1993).

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DEVELOPING SPATIAL SENSE: A SUGGESTION OF ACTIVITIES

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This poster reports about two activities we tested with pupils in order to develop their spatial abilities. These activities are analysed within a specific framework: Berthelot-Salin's ideas about the development of spatial sense, Parzysz's "Knowing vs Seeing" and a generative structure of activities in construction (Marchand, 2006).

PROBLEMATICS

In some curricula (France, Québec), one can read that learning geometry and particularly 3D-geometry should enhance pupils' spatial sense. But the activities related to such objectives are rare or lacking. Here we present two activities we tested with students in order to develop their spatial sense and spatial knowledge.

THEORETICAL FRAMEWORK

Our research is based on

- Berthelot-Salin's ideas about the development of spatial knowledge: to recognize, to describe, to make or to transform real objects; to move, to find or to communicate the location (position) of an object; to recognize, to describe, to build , to transform a living space or a moving space.
- Parzysz's "Knowing vs Seeing": when representing a 3D-object, one has to deal with a loss of information, and to learn the rules for drawing.
- A generative structure of activities (Marchand, 2006) elaborated around five criteria: development, space, object, task and mental image. The key elements essential to all activities are: deprive of seeing, touching or intervene on the object during the first specific phase.

FIRST ACTIVITY: "TOWERS IN A ROW"

In the first phase of this game[1], a pupil has to organize a row of 5 towers aligned in such a way that the number of visible towers is given at each end of the row.

In the second phase, the same rule applies on a 3x3 grid and a 4x4 grid, with a new condition: two towers of the same height are not allowed in the same row.

This game was originally devised for kindergarten. We used it with secondary school students with learning disabilities. It proved to be quite challenging and we could

[1] Adapted from Valentin D. (2005) *Découvrir le monde avec les mathématiques ; Situations pour la grande section*, Cycle 2, éditions Hatier, Paris

witness an improvement as far as the strategies were concerned. In the end the students were able to anticipate correctly the location of a tower without any trial and error. The hidden towers were clearly identified. This activity proved to be an introduction to the drawing of plane views of a 3D-object.

Such an activity is at the same time a game but an opportunity to develop a pupil's spatial sense in Berthelot-Salin's meaning. Moreover, according to Marchand's framework, it has characteristics of a good learning activity.

SECOND ACTIVITY: "INSIDE THE BOX"

In this kindergarten activity ("boîte à image"), pupils are invited to build a landscape inside a shoe box and to look at it by a small hole in one side of the box. Different tasks can be performed from that point on. This activity was experimented with first and second grade pupils. We chose to make them represent their scenery on paper, then, to imagine what they would see if we were to make a hole on the tight hand side of the box. We also asked them to describe their box so that others could identify it (ex.: there is a cat at the front, a tree on the left…) and played "guess who?".

Choices were made according to our framework and this activity proved to be a rich one to include in a sequence of others aiming to the development of spatial knowledge. At first, we saw kids having a hard time building their boxes so that they could see all the objects through the hole. Afterwards they struggled with the drawings (perspective, orientation, position of the different objects), with the significant words to describe the position of each object in the box and in relation with the others, and finally when anticipating the view from another point of view.

CONCLUSION

This project exemplifies how we can create rich activities to develop spatial knowledge, from Kindergarten to High School students with learning disabilities. But, is the development of spatial knowledge enhanced in the curricula and for what purposes? Moreover how can a student benefit of the spatial knowledge developed outside of the school in the mathematical class?

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ROLE OF SYMMETRY AXES ; UNDERGRADUATE STUDENTS' EXPERıENCE OF ıMPOSSıBLE FıGURES AS PLANE SYMMETRY GROUPS

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Geometry teaching in the secondary grades asks for reasoning with various diagrams. Impossible figures are such kind of diagrams that enables students to see reflection with respect to one and many symmetry axes at once. This adds a new type of complexity to the geometrical thinking and reasoning. In this study, five types of impossible figures were used to gather data on students' number of corrections and the time asked for completing this. Used figure types were hexagonal symmetry I, II, III(p6), line symmetry(pm) and triangle symmetry-II(p3). The symbols in the parentheses were the classifications of plane symmetry groups by Schattschneider. One way MANOVA analysis of variance concluded many interesting associations regarding the model and the role of the symmetry axis on students' answers.

Introduction

Reasoning with various transformational geometry diagrams is a challenging issue for students. Concepts like translation, reflection and glide reflection flourish within impossible figure diagrams. Within these diagrams, reflection and rotation with many axes is a possibility.

Procedure

For the study, students redrew three type of impossible figures with variations; line symmetry (pm), triangle symmetry II (p3), hexagonal symmetry I, II, III(p6). They were classified with respect to the plane symmetry groups' classification of Schattschneider (1978). Students' number of corrections, as can be seen from the paper was counted as first dependent variable. The perceived time was second dependent variable. Research model included the effect of figure type on dependent variables; corrections and time. Some descriptive analysis on means and frequencies as well as one way MANOVA were tested on the model.

Results and Discussion

As a result, line symmetry figure required less time and fewer corrections than other figures. This was supported with the difference mean time of the figures with reflection axis or not. Among the figures, only line symmetry figure had the reflection axis. This was explained by conceived difficulty level among figure types. Rotational order and translational reflection with respect to rotation made the task more difficult. Existence of symmetry axis helped to the understanding of it.

As with the number of corrections, hex sym I and III required the highest mean corrections (4,00). Tri-sym II and hex sym II mean was around 3,00. The difference between hex sym II and I /III can be explained with the existence of height feature in the former and nonexistence in the others.

In addition, dimension facet was added to the study by asking students an open ended question of what the figure made them to think of the dimension concept. Students' answers varied from many dimensions to fractal dimensions.

Hence, these kinds of figures may make the visual competencies of students on geometrical thinking visible but they may be facilitating for some students and inhibiting for others. How to understand which for whom is an issue with many facets as the WG4 group discussions suggest (WG4, Cerme 2013).

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GEOMETRICAL ASPECTS OF GENERALIZATION

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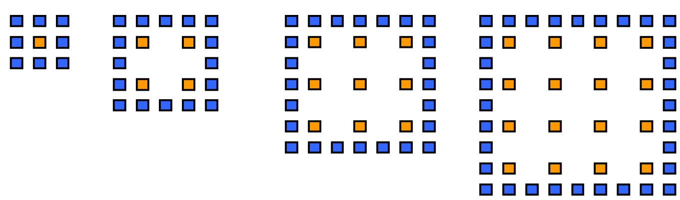
Generalization is considered one of the most import ant processes involved in mathematics. As Mason says: generalization is the heartbeat of mathematics (Mason, 1996). Whether it is viewed as part of higher level process, like abstraction (Dreyfus, 1991) or as the core process involved in a particular mathematics field, like algebra (Mason, 1996), there seems to be an agreement on its significant role in advanced mathematical thinking.

Discovering regularity and generalization are the kind of bridge between arithmetic and algebra. Looking on the historical development of algebra we do not omit the fact, that geometry have given many merits on this field. How could the geometrical aspects of generalization look like?

Methodology

The research, which was conducted among fifth-class students from primary school, aimed to find the answer to the following questions: How do the pupils form primary schools deal with a task involving discovering and spotting regularities? Are they able to make generalizations within some noticed rules?

The task that served as our research tool was an arithmetic-geometric and allowed a lot of different interpretations: there was given the sequence of four figures, each of them was built with two kind of elements. One is supposed to take how many elements we need for particular figures – not only those from the task, but also the next ones. The aim of the task constructed in this way was to check whether the students are able to discover the regularities occurring in the task in order to use them in their later work.



The research was conducted in school conditions, during normal classes. 38 students took part in this research. The students were working in pairs and their work was recorded. The atomic analysis of students' work and the atomic analysis of the video mentioned above were the research method that we used. Some additional information was obtained through the conversation with the teacher who was present during the whole experiment. The conversation took place not only after the finished work but as well during performing the task by the students.

Description of students' work

students started their work by counting the elements located in the pictures and then they filled in the table with the results of their work. Filling the first four lines took them comparatively little time. At this stage of solving the task any trials of discovering regularities were spotted. They did not find any rules until they reached the fifth line of the table (to which no illustration was attached) which made them think about the task and looking for the proper solution. The quest followed different ways, and its analysis let us distinguish the following three types of reasoning:

- *arithmetic*: ignoring the pictures and paying attention to the numeric data taken from the table; finding and discovering arithmetic connections between number given in the table; students' work rely on an analysis the number column from the table filled to the fourth line
- *geometric*: paying attention mainly to the pictures, spotting geometric connections; strong visual aspect; students' work rely on an analysis figures' pictures and drawing the fifth figure and the next ones, work connected with drawing the next pictures of figures
- *arithmetic-geometric*: both the table and the pictures were taken into account at the same extent, finding geometric connections and combining them with arithmetic ones; replacing geometric connections with arithmetic ones; students' work rely on an analysis existing pictures of figures and then filling the table.

Results

The most frequently ways of work were generalizations within the limits of arithmetic dependences, which based on the number sequences. But some students started their work direct the figural property. Sometimes this approach did not end with success. In geometrical aspect student have to take into consideration much more factors. In this particular case the figural aspect have to be connected with the length of sides or an area.

The analysis of existing figures run in different ways: some students paid attention on the mutual position elements in the figure; some of them analyzed particular parts of the figures and the others analyzed connections between next figures Take as a whole. In poster I presented the examples of students work classified to emphasis category. Those example one could interpret in the light of geometrical paradigm or van Hiele levels of understanding geometry.

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG 5: STOCHASTIC THINKING REPORT

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Keywords: *Concept formation; context; probability education; statistics education; teacher knowledge; technology; statistical literacy; stochastic; uncertainty; variability*

OVERVIEW

Stochastic thinking refers to the combination of probabilistic and statistical thinking. This type of thinking is non-deterministic and leaves room for uncertainty.

WG5 was attended by 30 delegates; 19 papers and 5 posters were presented in this working group. To remind the audience, each paper was presented in 5 minutes, after which it was discussed for about 15-20 minutes. At the end of each session common themes were discussed.

There were three main themes: students at the school level, teachers, and tertiary education and theoretical issues. The discussion about each theme was instigated by a discussant (Per Nilsson, Andreas Eichler and Corinne Hahn).

Students

Andreas Eichler & Markus Vogel	Principles Of Tasks' Construction Regarding Mental Models Of Statistical Situations
Ayse Yolcu & Cigdem Haser	8th Grade Students' Statistical Literacy of Average and Variation Concepts
Susanne Schnell	Coping With Patterns and Variability – Reconstructing Learning Pathways Towards Chance
Ute Sproesser & Sebastian Kuntze	Statistical Literacy and Language – A Qualitative Analysis Of Needs Study
Arthur Bakker, Xaviera van Mierlo & Sanne Akkerman	Integrating School and Work Perspectives On Statistics In Vocational Laboratory Education
Bridgette Jacob & Helen Doerr	Students' Informal Inferential Reasoning When Working With The Sampling Distribution
Per Nilsson	Reflection on student learning

Teachers

Raquel Santos & João Pedro da Ponte	Prospective Elementary School Teachers' Interpretation Of Central Tendency Measures During A Statistical Investigation
Eugenia Koleza & Aristoula Kontogianni	Assessing Statistical Literacy: What Do Freshmen Know?
Assumpta Estrada, Maria Nascimento, & José Alexandre Martins	Using Applets For Traning Statistics With Future Primary Teachers
Sandra Quintas, Hélia Oliveira, & Rosa Thomas Ferreira	The Didactical Knowledge of One Secondary Mathematics Teacher Of Variation
Orlando González	Conceptualizing And Assessing Secondary Mathematics Teachers' Professional Competencies: For Effective Teaching Of Variability-Related Ideas
Pedro Arteaga, Batanero, Cañadas, & Contreras	Prospective Primary School Teachers' Errors In Building Statistical Graphs
Daniel Frischemeier & Rolf Biehler	Design and exploratory evaluation of a learning trajectory leading to do randomization tests facilitated by TinkerPlots
Andreas Eckert & Per Nilsson	Contexualizing Sampling – Teaching Challenges And Possibilities
Andreas Eichler	Discussion

Tertiary education and broader theoretical issues

Chiara Andrà & Judith Stanja	What Does It Mean To Learn Stochastics? Ideas, Symbols And Procedures
Catarina Primi & Francesca Chiesi	Evaluating The Efficacy Of A Training For Improving Probability And Statistics Learning In Introductory Statistics Courses
Carmen Batanero, Gustavo Cañadas, Pedro Arteaga & Maria Gea	Psychology Students' Strategies and Semiotic Conflicts When Assessing Independence
Jorge Soto-Andrade	Metaphorical Random Walks: A Royal Road To Stochastic Thinking
Kellrun Hiis Hauge	Bridging Policy Debates On Risk Assessments And Mathematical Literacy
Corinne Hahn	Discussion

Posters

Maria Nascimento; Miguel Ribeiro; J. Alexandre Martins; Fernando Martins; Manuel Vara Pires; Cristina Martins; Margarida Rodrigues; Joana Castro; & Ana Caseiro	Giving sense to student's productions – a way to improve (future) teachers' knowledge and training
Ana Henriques & Ana Michele Cruz	Developing statistical literacy in primary level: Results of a teaching unit
Per Blomberg	Using a modeling perspective for learning probability
Christine Plicht	Diagrams, graphs and charts in biological courses A system of categories in the overlap of mathematics and biology
Corinne Hahn	School statistics and managerial statistics: Representations and boundary objects

STUDENTS

Some of the papers reinforced the image that we know from the literature: student knowledge of probability and statistics is disappointing in many countries. Eichler and Vogel developed a framework for analysing tasks' potential to diagnose young students' intuitions or understandings. Sproesser and Kuntze emphasized the importance of language as a mediating tool in learning statistics. Their research suggests that students may have good intuitions but often not the statistical language to express these.

The discussant asked for more prescriptive research that would help us to improve probability and statistics education. There were indeed several papers and posters that presented promising ideas or evaluated interventions (Bakker et al.; Plicht; Schnell; Soto-Andrade).

TEACHERS

Compared to previous working groups on stochastic thinking, this one had a large set of presentations on teachers' knowledge and learning. We consider this encouraging, because the field has produced a lot of insight on student learning, instructional materials and innovative computer software but we still know too little about teaching. The discussant even flippantly wondered if teachers were ready to teach the probability and statistics curricula in most countries.

Again, many presentations underlined the existing image from the literature (which is mostly Anglo-Saxon), that teacher knowledge about probability and statistics is poor. Yet there are directions of research that are encouraging, for example the use of applets (Nascimento et al.) or *TinkerPlots* (Frischemeier and Biehler) in teacher education. However, Arteaga et al. noted that teachers made more mistakes in graphig with Excel spreadsheets than without; the cause of this needs to be investigated further.

One striking observation is that there are many frameworks on teacher knowledge around. As a group we could hardly remember all the abbreviations concerning pedagogical content knowledge, mathematical and statistical knowledge for teaching. Apparently there is still a long way to go to understand what teachers need to know in order to teach the domain of stochastic thinking. It would be helpful to clarify the different emphases in the different frameworks and over time reach some convergence in terminology.

TERTIARY EDUCATION AND THEORETICAL ISSUES

Cañadas and colleagues highlighted students' problems with association – a topic that has so far received relatively little attention despite its importance in research.

The paper by Primi and Chiesi showed the importance of knowing mathematics for students self-efficacy in statistics education. The relation between mathematics and statistics is indeed one to investigate further.

Andra and Stanja addressed the thorny question of what characterizes stochastic thinking in terms of ideas, symbols and procedures.

GENERAL ISSUES

As commonly necessary in any working group, some time had to be devoted to the discussion of what we mean by particular *concepts*. Variability, statistical thinking, thinking and literacy are a few that returned in our discussions. It also struck us that we do not have conventional language to talk precisely about students' concept formation in flux. Depending on delegates' theoretical backgrounds, they preferred to talk about constructs or conceptions. We also discussed the difference between semiotic and cognitive conflicts.

Because of its applied and non-deterministic nature of stochastics, its link with *context* is crucial. Eckert and Nilsson showed how challenging it can be for a teacher to focus students' thinking on statistical ideas when tackling a contextual problem. Bakker et al. addressed vocational education, where the main focus seems work tasks rather than the statistical ideas behind them. Hauge proposed a more holistic approach to real-life problems that involve risk, which in itself combines probabilistic and contextual aspects. The latter two papers stress the interdisciplinary nature of stochastic thinking.

It was occasionally noted that statistics education is a younger field of research than mathematics education. Many of the issues raised have already been investigated in some related way in mathematics education. However, because of the differences between mathematics and statistics, we cannot always assume that findings from mathematics education research apply equally in statistics education.

FINAL COMMENTS

The group work was much appreciated. Delegates could make themselves well understood in English, even if they normally talked Turkish, Greek, Italian, Spanish, Portuguese, German, Dutch, Norwegian or Swedish. The size of the group was good and participation was not too skewed.

In the last sessions ideas were expressed for a European project and some joint effort in collecting data.

As a group we decided to change our name to *Probability and Statistics Education*. The main reasons are:

1. Though in German *Stochastik* refers to the combination of probability and statistics, stochastics has a rather narrow meaning in most other languages.
2. The new name better captures the broader issues addressed in the working group, not only thinking about also what is involved more generally in realizing better probability and statistics education.

WHAT DOES IT MEAN TO DO PROBABILITY? IDEAS, SYMBOLS AND PROCEDURES

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We propose to adopt Rotman's terms of idea, symbol and procedure as a frame for characterizing probabilistic thinking. We present a qualitative analysis of a questionnaire given to a pre-service teacher and show how the apparent difficulties might be explained by the use of Rotman's frame. The questionnaire addressed the interpretation of artifacts that may be part of a probability course at primary school level. The result of the analysis suggests that the interpretation and usage of signs in probability is not self-evident and that probability requires a specific usage and interpretation of artifacts and signs. This usage and interpretation may be interfered with experience with the same artifacts or signs in other domains of mathematics.

Keywords: elementary stochastic, cultural semiotics, stochastic thinking

INTRODUCTION

In the last decades, the increasing importance of the role of probability in many areas (economy, business, science, politics, and so on) led several countries all around the world to include it in their curricula for early years at school. As a consequence, teachers' knowledge and understanding of probability has become a crucial point in the agenda of many primary school teachers' educators (Kvatinsky and Even, 2002). This had raised new issues concerning both the nature and the learning/teaching of probability. Several models were elaborated that *describe* probabilistic thinking (for an overview see Jones and Thornton, 2005) by taking particular conceptions into account and identifying existing or missing (partial) conceptions in students. Learners difficulties were *explained* in terms of misconceptions and/or missing or incomplete conceptions. The findings of these studies are bound to particular concepts. For example, Bar-Hillel and Wagenaar (1991) study intuitive notions of randomness by judgment tasks, such as "is this series likely for the rolling of a fair die?", and production tasks, such as "give me an example of a series which is likely for the rolling of a die". Falk, Falk and Ayton (2009) characterize such intuitive notions of randomness as "equal total frequencies" and "excess of alternation" (*ibd.*, p.203).

In this paper, we adopt a cultural (semiotic) perspective, and we question whether and how this perspective may contribute to research on probabilistic thinking. This is a huge endeavor and, in this contribution, we can only start. From a cultural (semiotic) perspective, we consider probability as a cultural product. We understand it as part of mathematics and therefore historically and culturally determined.[1] For

[1]For a historical examination of the development of probability we refer the reader e.g. to Barbin and Lamarche (2004).

mathematics, Rotman (a philosopher and mathematician), proposed a characterization of mathematics "as a practice, as an ongoing cultural endeavor" (Rotman, 2000, p.3). In the next section we will present his ideas about mathematical activity and propose to adapt them for probability. Following his ideas, we will see that artifacts and signs play an important role in the mediation and communication of probability. In later sections, we will explore the potential of *Rotman's frame*. We will demonstrate, how Rotman's frame might be used to find new explanations for difficulties that learners face when dealing with tasks in probability.

ROTMAN'S FRAME: IDEAS, SYMBOLS AND PROCEDURES

In his work, Rotman (2003) investigates what it means *to do mathematics*, and states that "behind the various construals of mathematics as an activity […] lie three distinct, fundamental theoretical discourses that enter into the subject, namely: idea, symbol, and procedure." (Rotman, 2003, p. 1676). According to Rotman, for thinking mathematically all three – ideas, symbols and procedures - have to be coordinated, and there's not a hierarchy among them. We give a brief description of the three components, connect them to other works, and discuss coordination of ideas, symbols and procedures.

Idea stands for the domain of human thought, as delineated by the individual's narratives in natural language. Within the domain of probability, *Idea* can be understood as the domain of intuitive approaches to uncertain situations and fundamental ideas such as variability. According to Andrà and Santi (2011), a person intuits mathematical concepts when the access to the distinctive features of the mathematical object is self-evident, coercive, and global. Referring to Radford's (2008) perspective, Andrà and Santi claim that self-evidence and immediacy can be traced back to "the spatial-temporal, sensorimotor and perceptive activity that semiotic means of objectification accomplish, support, foster" (p. 115). The domain of *Idea* concerning probability is situated in space and time, where intuitions arise and are accounted in narrative ways. The development of personal and collective practices in mathematics is closely connected to the productive use of artifacts as instruments in knowledge-building activities, which brings us to the *Symbol* domain. This is the domain of signs, as well as communication and semiotic practices from notational devices to entire linguistic systems. The importance of symbols in mathematics learning processes is expressed by Duval (2008) who states that there is no *noesis* without *semiosis*, namely that any meaning making process needs a system of signs to take place. This is in accordance with Sfard (2000), who states the crucial role of symbols in knowledge construction. Within probability, *Symbol* is the set of signs and symbols that are commonly used in probability (Venn diagrams, formulas, tables, histograms, and so on). A sign can be meant as an artifact that is connected with a meaning (or used as representation for a mathematical idea). In order to see an artifact as a mathematical sign, it needs to be related to a mathematical *idea* (see also Stanja, 2012). For elementary probability, we have to consider the translation from the domain of artifacts (in space and time) to the domain of numbers and symbols. When an artifact such as a spinner is used, two cultural elements may emerge: the notion of probability as a ratio between favorable and possible outcomes, and the one of probability as frequency. The latter points out that "probability can be assigned only to an event that can be repeated" (Kvatinsky and Even, 2002, p.2), as it is the case with many, common artifacts (dies, cards, coins, spinners). Finally, *Procedure* is the domain of actions, processes, and operations on and with artifacts and signs. We can relate this to the ideas of Duval (2008), who identifies learning as becoming able to perform correct actions on mathematical signs. Actions in mathematics are transformations, which can take place within the same semiotic register (a *treatment*, e.g. transforming a formula into another formula), or between two different registers (a *conversion* e.g. from a formula to the graph). Procedures may be explained or justified by mathematical ideas, but they may be performed without reference to them.

In the following, we elaborate on an example of a pre-service teacher to see what it means to integrate idea, symbol and procedure in probability.

MARTA'S EXAMPLE

Background Information

Marta, a pre-service primary school teacher, received a questionnaire (figure 1) at the very beginning of a course in probability in the Academic year 2011/12 at the University of Torino.

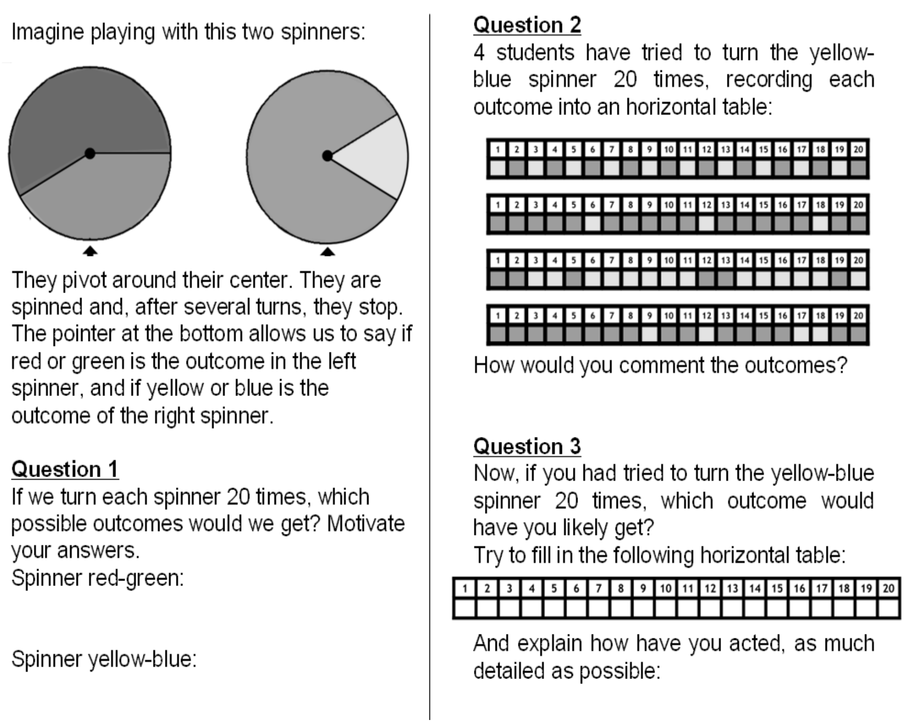

Figure 1: the English translation of the test given to the students.

Participants of the course were attending the third year of the undergraduate course in Primary School Education and had no previous school experience in probability (this is a quite common situation in Italy, since the introduction of probability in the curriculum is very recent). We were interested in how the pre-service teachers work on this kind of problems and how the domains of Rotman's idea, symbol and procedure emerge and interact with each other. We will describe how a pre-service teacher enters the world of random experiments with a spinner by dealing with the given problems.

The questionnaire was designed by Andrà and is based on interviews that had been carried out by Stanja in a different project in the school year 2010/11 with grade-3 students in Germany. In the first task (figure 1), the pre-service teachers were given an artifact, a figure showing a spinner, they were asked to give a prediction about the possible outcomes when turning the spinner 20 times and to give a justification. The students may have seen or played with a spinner before, but we stress that they had never engaged in a learning activity involving the meaning of the spinner in probability. The second task is a *judgment* task according to Bar-Hillel's and Wagenaar's (1991) taxonomy. There were four different sequences of 20 trials shown and the pre-service teachers were asked to comment on the outcomes. In the third task they were asked to imagine and draw a sequence that could be obtained by turning the spinner 20 times - a *production* task according to Bar-Hillel's and Wagenaar (1991), and to give an explanation of their action. We agree with Bar-Hillel and Wagenaar, that pure production tasks may hide the cases in which a student has "biased reflections of accurate notions of randomness" (*ibd.*, p.431). Therefore, it is important to ask the students to give a justification. All tasks had in common that they demanded the usage/coordination and interpretation of given socially constructed means the pre-service teachers were not familiar with, but that could be part of an elementary stochastics course (see e.g. Lopes and de Moura, 2002). The questions were open concerning the way of representing the outcomes and allowed to answer in more or less details. A similar task has been considered by Batanero, Godino and Roa (2004), in a study aimed at describing the knowledge needed for teaching probability at school. Specifically, the task pivoted around the concept of randomness, and the researchers observed that different students paid attention to different properties of the sequences (analogous to question 2 here).

We now present an interpretative analysis of Marta's answers to the questionnaire that focuses on her usage and interpretation of the given semiotic means. The analysis intends to understand how Marta made sense of the given means. Thus, it reconstructs a possible and plausible way in which Marta may have interpreted the given and her introduced semiotic means. We then discuss Marta's answers in terms of Rotman's Idea, Symbol, and Procedure. As an *a priori* analysis, we distinguish three cases. (1) The students carry out the task by means of symbolic operations and transformations, but their actions have poor spatial-temporal meaning, which is conveyed by the natural language in the domain of the Idea, as well as poor probabilistic significance, given by the domain of Symbol. In a sense, we can trace this tendency back to the empirical vision of stochastic knowledge, as argued in Batanero, Godino and Roa (2004). (2) The students intuit the general sense of the activity, but they have neither the ability to perform operations, nor the possibility to access the inter-subjective meaning of it in the realm of probability. Given such a relationship with the mathematical theory, this case can be part of a formal vision of stochastic thinking (Batanero, Godino and Roa, 2004). (3) The students can access the inter-subjective, cultural meaning of the activity, but hardly perform actions and refer to the concrete artifact.

Analysis

Marta's answers to question 1 and 2 can be seen in figure 2.

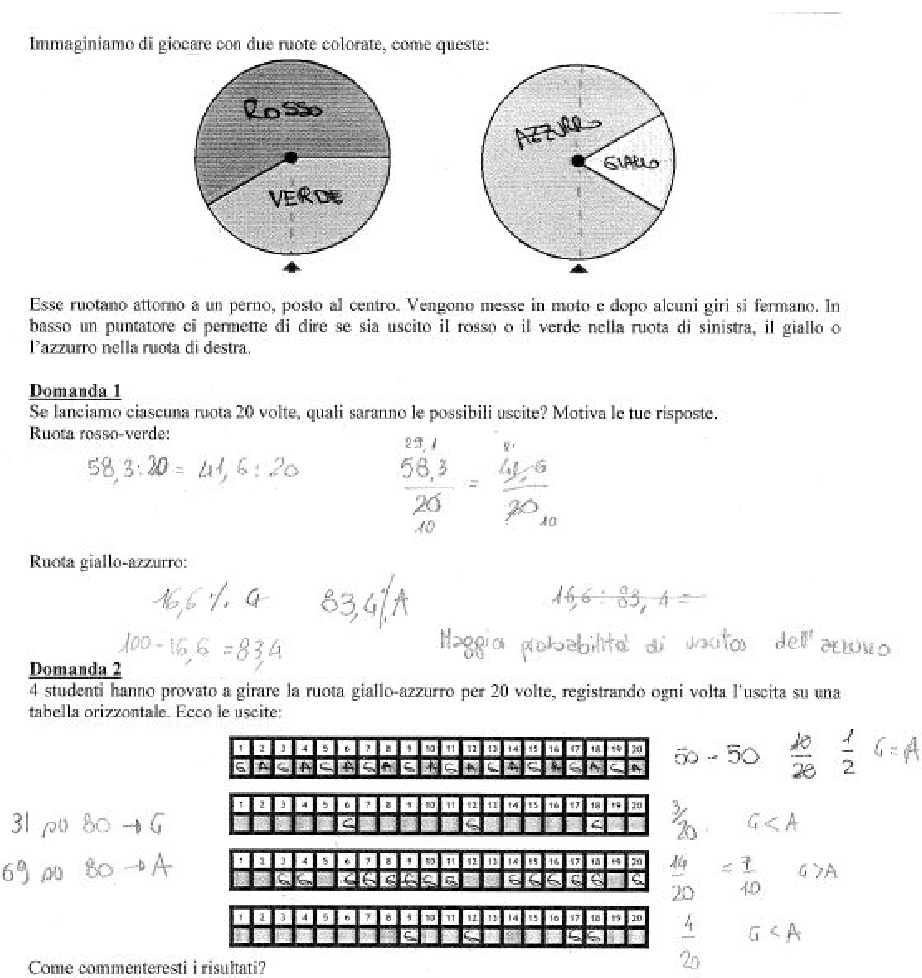

Figure 2: Marta's responses to question 1 and 2.

For the first task we observe that she changed the representations of the spinners by including markers (dotted lines) and that she wrote arithmetic expressions for both spinners. For the second spinner she also wrote a sentence (“Maggior probabilità di uscita dell’azzurro”, that is “Greater probability of blue to come out”). In order to reconstruct how Marta was possibly dealing with the tasks and how the domains symbol, procedure and idea interact, we have to interpret her writings and possible relations between them. Concerning the *procedure* component of Rotman's frame we can reconstruct from Marta's answers that she performs a *coordination* - an action - of the representations of the spinners and the symbolic system of percentages, which is supported by the marker lines: the components of the spinner representation (area sizes with the labels of the colors) are mapped in a 1-1 manner into the percentage system. The proportions expressed by percentages belong to the realm of *symbols* Marta uses. It is not clear whether she estimated/calculated the percentages or simply used a calculator (since there is no documentation of how she arrived at the percentages compared to her detailed writings elsewhere). Our interpretation, that Marta performed a spinner-percentages coordination is also supported by the writing of "G" (yellow) and "A" (blue) next to the percentages "16.6%" and "83.4%". The "16.6%" is obtained or justified by the treatment (in Duval's terminology a transformation within the same system of signs) in the symbolic system of percentages – the calculation "100-16.6=83.4". Here, Marta used the complement in the percentage system. Besides this coordination, Marta's way of searching for an answer differs for both spinners. For the first spinner, the writing "58.3:20=41.6:20" supports the interpretation that Marta wanted to relate the proportions to the 20 trials (":") and to compare the two expressions. Here, the "=" sign is used in an unconventional way. The "=" sign can be seen as a statement about the action (comparison) that has to be carried out and not as the result of a comparison. This is particularly interesting since it gives insights about the initial *ideas* that Marta develops, namely that she wants to compare the spinners by using fractions and percentages and connecting this to the trials. Marta then performed a conversion from the system of percentages to the system of fractions ("58.3/20=41.6/20") and started another treatment - canceling - that she did not finish. For this spinner she did not write anything else. Why did Marta stop here? One reason could be that the "=" sign takes on a different meaning now and Marta realizes that both sides are not equal. Another possibility is that she realizes that her calculations did not lead her to answer the question. Either case, Marta is concentrating on the procedure to be carried out. For the second spinner Marta proceeded in a different way. She did not relate the proportions expressed by percentages to the 20 trials but she tried to compare them directly. This is supported by her writing "16.6:83.4=" which could be understood as a *procedure* that has to be done, with a conversion to the language system "Greater probability for the outcome of blue" (translation).

For the second task, Marta performs a conversion from the first list to the symbolic system of percentages (equal frequency of yellow and blue → "50 - 50") and then to the symbolic system of fractions ("50 – 50" → "10/20"), where she performs a treatment (canceling: "10/20" → "1/2") and then converts this congruently to some idiosyncratic symbolic system of letters ("1/2" → "G=A"). For the other three lists, she performs similar conversions from the lists to the symbolic system of fractions regarding the frequencies of yellow and then to her letter system. Here, Marta seems to have only limited access to the realm of *symbol* usually used in probability. She uses fractions but it is not clear what she associates with them. It is not clear what the unconventional writings in her letter system stand for. Possible interpretations are that they describe the relation of the frequencies of yellow and blue, or represent frequencies that for Marta could be seen as the same as probabilities or as estimates of the relation of probabilities from the frequencies that may refer to different spinners with different proportions. It becomes not clear at this point that Marta is aware of the difference between frequencies and probabilities and if she knows about the relations of them. When we look at the writing left of the lists the first interpretation gets supported since Marta added the absolute frequencies for yellow from all lists, related them to 80 trials, and connected this to the letter system ("→G"). In her writing, the "69" for blue may be a result of a reference to "100%" as the whole and the usage of the complement idea. There are no further comments or interpretations given by Marta.

The inconsistencies in Marta's answers may be an indicator for her trying to make sense of the given new situation by resorting to known (mathematical) actions. The use of symbolic expressions may come from her experience in other domains in mathematics. Marta related percentages, proportions of the spinner and fractions to each other and the word "probability" by performing *congruent* coordination and she performed procedures on the system of fractions. However, there is no comment from her on the possible outcomes and there is no interpretation of "probability". The idea of variability also is not present in her writings. The intended meaning is accessed neither in intuitive terms nor in cultural and symbolic terms. However, Marta's answer to question 3 shows a leap: "I expect a majority of blue" ("Mi aspetto una maggioranza di azzuro."). Here, she might start to access the cultural form of reasoning about the likelihood of a sequence of outcomes: she uses the verb "to expect", which is historically related to the mathematical expectation, and the term "majority", which is related to proportions. However, there is no explicit justification as a reference to the spinner and it is not clear whether these terms have the conventional meaning for Marta. Moreover, we did not find any evidence in Marta's written answers that she has used any out-of-school experience with random situations.

WHAT DOES IT MEAN TO COORDINATE IDEAS, SYMBOLS AND PROCEDURES IN PROBABILITY?

What does happen if idea, symbol and procedure are not coordinated, and only one is predominant? If someone, like Marta, relies mainly on procedures, then there is only a weak or pointless reference to the domains of Idea and Symbol. The student carries out the task by means of symbolic operations and transformations. The actions have poor spatial-temporal meaning, which is conveyed by the natural language in the domain of the Idea, as well as poor probabilistic significance, given by the domain of

Symbol. Sometimes it is possible to carry out a procedure without accessing the intended meaning and significance of it. In probability, this occurs when the student simply manipulates signs or when already known systems of symbols from different domains of mathematics are used which may lead to *substituting strategies* in solving probability problems (see also Stanja, 2012). This risk can be felt as very cogent, if we consider that the students generally tend to import their beliefs about the deterministic nature of mathematics into probability (Meletiou-Mavrotheris and Stylianou, 2003). When considering the coordinations that Marta did, we noticed that she performed only congruent coordinations (1-1-mappings). This would not be a problem, if the task would be to determine fractions or percentages from circular representations (a typical task when working with fractions or percentages in Italy). In the domain of probability, this is not sufficient: the interpretation and usage of the given means differs from other domains and is more complex in probability. In answering the tasks in the questionnaire, it was not only required to perform good conversions, but also to consider the spatial-temporal and sensorimotor domain, the intuitive world of ideas, and the culturally-given and socially-shaped world of the symbols, where the spinner changes from a real world object to a mathematical sign referring to probability. In fact, there is an underlying intended meaning when the students are asked to comment a given sequence of outcomes: the notion of likelihood, a historical concept that had been developed in order to model the extent to which experimental evidence (from one or many trials) supports the expected “ideational” outcome. The intended meaning of the activity is to shape a shift from the artifact of the spinner towards the signs embedded into the table. When the students are asked to predict an outcome at their turn, another underlying intended meaning is present: the notion of variability. Variability can be connected in this case to the notion of model Fit (Konold and Kazak, 2008). We refer the reader to Prodromou (2011) for an insightful study on the students’ ways of relating experimental and theoretical perspectives on probability. From Prodromou’s research it emerges that students have difficulties in coordinating the actual data and the ideational model. To know about this difference and to understand the relation between frequencies and probabilities, in fact, would mean to be able to perform non-congruent coordinations between represented probabilities and represented frequencies. As it has also been pointed out by Batanero, Godino and Roa (2004), the classical and the frequentist approach to probability are complementary in nature. In random experiments different outcomes may be obtained at each trial. Unlike arithmetic or geometry, random experiments cannot be reversed.

CONCLUSIONS AND DISCUSSION

This contribution is a first step to explore Rotman's frame to study probabilistic thinking. The novelty of this contribution consists in giving insights on Rotman’s frame and showing how it might be used to explain the difficulties that a learner might face when dealing with probabilistic tasks. Using this frame, we argue that the coordination of semiotic means in probability differs from other domains and that there might be some interference between several domains of mathematics. We remark that our approach is different from Bar-Hillel and Wagenaar (1991), as well as from Falk, Falk and Ayton (2009), since we are not interested in how "bad" intuitions may impede the learning process, but our aim is to see what happens when intuitions/ideas, culture-symbol and procedure(s) are not coordinated, and thus which kind of difficulties the students may face.

Moreover, Marta's example illustrates that the signs used in probability are cultural products used by a community and whose usage is not self-evident and self-explaining. The inconsistencies in her usage of the means indicate that she did not answer the questions only with some pre-existing ideas in mind but that she was trying to make sense of the situation in front of her while interacting with it. It became clear that also the meaning and usage of known means (from arithmetic) need to be changed: it should be addressed deliberately and explicitly and therefore should be part of stochastics teaching. To our knowledge, this is not the case at the present, and there is a need for the design of suitable learning activities. Stochastics as cultural product implicates a different perspective on learning/the contents to be learned. Speaking in Rotman's terms this would mean that the links between the domains of Idea, Symbol and Procedure need to be supported. The meaning of interaction with cultural artifacts and signs, and with other persons is emphasized. Related to this is teachers' training. Following Batanero, Godino and Roa (2004), and with Rotman (2003), we maintain the importance for the teachers to integrate theoretical and empirical experiences in order to make sense of the activity, fostering the idea, the symbol, and the procedure in a coherent whole.

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PROSPECTIVE PRIMARY SCHOOL TEACHERS' ERRORS IN BUILDING STATISTICAL GRAPHS

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We analyze the graphs produced by 207 prospective primary school teachers when comparing data collected by them in a statistical project. These graphs are classified according to their correctness, and the type of error in case of incorrect graph. The influence of using computers on the errors produced is also analyzed. Results show the prospective teachers' difficulties in building statistical graphs, in spite that they will have to teach these graphs, since many of them shared errors reported in previous researches with children.

Keywords: statistical graphs, errors, graph construction, primary school teachers' education.

INTRODUCTION

Nowadays we are surrounded by statistical information that is often presented through statistical graphs, and building and interpreting statistical graphs is an important part of statistical literacy (Watson, 2006; Ridgway, Nicholson & McCusker, 2008). Taking into account the relevance of the topic, this research was aimed to evaluate the formative needs of Spanish prospective school mathematics teachers in relation to their competence in building elementary statistical graphs. Below we describe the research rationale and background, method, results and implications for teacher education.

STUDY RATIONALE

In Spain, curricular guidelines (MEC, 2006) include statistical graphs since first level of primary school education (6 year-olds children). The success of this curriculum depends on the correct preparation of teachers, that, until now did not include statistics education.

Moreover, few studies have focused on teachers' knowledge about statistical graphs and most of them are related to prospective teachers (González, Espinel, & Ainley, 2011). In this study we continue our previous research (Arteaga & Batanero, 2011) where we analyzed the graphs produced by 207 prospective primary school teachers in an open statistical project where they had to compare three pairs of distributions. In that paper we classified the graphs built by the prospective teachers according to its semiotic complexity and the participants' reading levels in Curcio's (1989) categorization. Results showed that, although most participants produced graphs with sufficient semiotic complexity to solve the task posed, only a minority reached an adequate reading level to get a correct conclusion from the graph. In this new study we re-analyse the same data with a different perspective: The focus now is the correctness of the graphs built by these teachers and the influence of using computers on possible errors.

STUDY BACKGROUND

Errors in building graphs

Several studies evaluated the competence in building statistical graphs by students (see Friel, Curcio, & Bright, 2001; Tiefenbruck, 2007 for a survey). Li and Shen (1992) found the following errors: selecting a graph that is inadequate to the type of variable or representing not related variables on the same graph; using inadequate scales; omitting the scales in at least one axe; not specifying the origin of coordinates and not using enough divisions on the scales. Wu (2004) found the following errors by secondary school students when building and reading statistical graphs: (a) errors related to scales, (b) errors in tittles, labs or specifiers (c) errors in pie charts, (d) difficulties with proportionality in a pictogram, (e) confusion between apparently similar graphs (for example, between histogram and bar chart), (f) confusing frequencies and variable values. Lee and Meletiou (2003) described four wrong reasoning when working with histograms: (a) Interpreting histograms as representation of discrete variables, assuming each rectangle refers to an isolate value instead to an interval of values; (b) Comparing frequencies in histograms using only the vertical axis (instead of areas) (c) Lack of appreciation of randomness in the data represented, and (d) Interpreting histograms as a bivariate graph.

Technology does not facilitate the work with statistical graphs, because the students need to learn the software options in addition to the graphs features. Ben-Zvi and Friedlander (1997) analyzed the graphs produced by some students when working with computers and suggested that some of these students used the software in a non critical way, since they were unable to select the most adequate options of the software.

Teachers or prospective teachers' competence in building graphs

Bruno and Espinel (2005) described the following errors made by prospective primary school teachers when building a histogram: misrepresenting intervals, omitting null-frequency intervals, using non-attached rectangles with continue variables**.** In Burgess's (2002) study some teachers were unable to integrate the knowledge they got from the graphs produced in their reports with the problem context.

Tiefenbruck (2007) investigated fourteen primary school teachers' understandings of graphical representations of categorical data with a questionnaire, where only a few questions asked the participants to build a graph. The teachers had a basic knowledge of graphical representations of categorical data. However, some of them incorrectly suggested that the histogram and stem and leaf plot were adequate for categorical variables. They also were confused when defining the scale and in describing how to create a scale from data. In this study we continue all this previous research in analysing the type of errors the prospective teachers made when building graphs. Contrary to Espinel and her colleagues or Tiefenbruk, instead of using questionnaires we gave the teachers an open-ended project; we also study the influence of computers on the possible correctness of the graph. Arteaga & Batanero (2011) is also a basis for this research.

THE STUDY AND METHOD

Participants in our study were 207 prospective primary school teachers in Spain who were enrolled in a mathematics education course; in total 6 different groups (35-40 participants per group). The participants studied statistics the previous academic year, including statistical graphs, which are an important component in the primary school curriculum (MEC, 2006). The data were collected as part of a practical activity where prospective teachers were encouraged to carry out an experiment to decide whether the group had good intuitions on randomness or not. The experiment consisted of trying to write down apparent random results of flipping a coin 20 times (without really throwing the coin, just inventing the results) in such a way that other people would think the coin was flipped at random (simulated sequence). Participants recorded the simulated sequence on a recording sheet. Afterwards participants were asked to flip a fair coin 20 times and write the results on the same recording sheet (real sequence). At the end of the session, in order to confront these future teachers with their misconceptions, participants were given the data collected in their classroom. These data contained six statistical variables: number of heads, number of runs and length of the longest run for each of real and simulated sequences from each student (part of the data are presented in Table 1; the total data consisted of 35-40 rows similar to those presented in Table 1. The prospective teachers were asked to analyse the data and produce a report with their conclusion (about the similitude or differences in the three pairs of distributions).

	Simulated sequence			Real sequence		
Student	N. of heads	N. of runs	Longest run	N. of heads	N. of runs	Longest run
1	10	14	4	11	9	4
2	12	9	4	11	16	2
3	11	12	4	11	16	2
4	10	9	4	8	9	4
5	11	11	3	7	11	4
…						

Table 1: Data collected by the students

RESULTS AND DISCUSSION

Although we did not asked the prospective teachers to use graphs in their analyses, 181 of them (87.4%) built some graphs to compare the number of heads 146 (70.5%) to compare the number of runs and 129 (62.3%) to compare the longest run in the real and simulated sequences. These high proportions of students who built graphs suggest they carried out a *transnumeration* process (Wild and Pfannkuch 1999) to obtain new information that was hidden in the raw data. These graphs were firstly classified into basically correct, partially correct and incorrect graphs and secondly, according the types of errors presented. Results are described below.

Categories of graphs

1. *Basically correct graphs.* Here we include *correct graphs* (correct title, axes, scales and labs) that are adequate to both, the problem posed and the variables being displayed (1.1). We also include unusual, but correct representations (1.2).

2. *Errors in the graph scale.* Watson (2006) warns about the need to be careful with the graph scales; however we found the following errors (Bruno & Espinel, 2005):

2.1. *Non proportional scales* where distances between different pairs of points that should be equal were different.

2.2. *Wrong representation of natural numbers on the number line, f*or example, omitting null-frequency variable values.

2.3. *Titles or scale values missing or confuse* Although the title, labels on the axes and scales are essential part of graphs (Curcio, 1987) because they provide the contextual information needed to interpret the variables represented and the relationships between the graph different elements, Bruno and Espinel (2005) found many prospective teachers with difficulties to include a correct and meaningful title. In our study many students provided imprecise titles or labels, but only a few built graphs with no tittles or labels.

2.4. *Not centred bars.* The variables in our study were discrete; however, many participants built histograms, which are used to represent continuous variables or when we need to group the variable values. Furthermore, some participants did not centre the rectangles in the class centre, although the variables only took integer values. This error was also reported by Bruno and Espinel (2005) and Espinel (2007)

2.5. *Wrong representation of intervals on the X axis.* Some participants made errors in representing intervals; for example they displayed intervals with a common point as disjoint. This error, also reported by Bruno and Espinel (2005) was more common between the participants who used Excel to produce their graphs.

2.6. *Inappropriate scales.* Li y Shen (1992) found some students who built a scale not wide enough to cover the range of variation of the variable represented. In our study we found this error as well as participants who built too wide scales, error described by Wu (2004).

3. *Incorrect graphs.* We found the following subcategories (See examples in figure 1):

3.1. *Lack of proportionality in the specifiers.* Participants did not take into account the conventions for each particular graph. For example in Figure 1a is difficult to read the frequency associated to each variable because the width of circular sectors is not proportional to frequencies.

3.2. *Confusing variable values and frequencies.* In the distribution, each variable value is associated to its frequency. Some participants confused both and exchanged variable values and their associated frequencies in the graph (Figure 1b).

3.3. *Representing variable values and frequencies together.* Some participants built attached bar graphs, representing each variable value with its frequency in two attached bars, as if they were two different variables; they usually used Excel. Figure 1c shows an example of this category where a prospective teacher shows a limited knowledge about the software options; therefore, he makes an uncritical use of the software (Ben-Zvi & Friedlander, 1997).

3.4. *Representing variable values multiplied by frequencies.* A few participants working with Excel displayed in an attached bar graph both the frequencies and the product of frequencies by the variable value. These participants also showed an uncritical use of the software and misunderstanding of statistical of distribution.

3.5. *Inadequate graph.* Some participants selected graphs that were inadequate to the problem they had to solve. For example some participants did not form the distribution and displayed bar graphs were a too high number of bars to be able to interpret the graph (Figure 1d).

3.6. *Representing non-related variables in the same graph.* Some participants plotted variables non comparable together; for example representing in the same graph the three variables under study (number of heads, number of runs and longest run) or their averages (Figure 1e).

3.7. *Non comparable statistics displayed in the same graph.* In this case some participants displayed averages and measures of spread in the same graphs and therefore they confused the meaning of central tendency and spread (Figure 1f).

3.8. *Several errors.* Some students made several of the errors described before.

In Table 2 we present a summary of results. Basically corrects graphs had the highest percentage in each variable, although the percentage of incorrect graphs or graphs with errors in scales (partially correct graphs) was also very high. About half participants constructed basically correct graphs in comparing the variables at the project: 47% (85 participants) built correct graphs for the number of heads, 43.8% (64 participants) for the number of runs and 45.7% (59 participants) for the longest run.

About 20% of errors were related to scales (non proportional scales, wrong representation of numbers on the number line, etc...) in spite of Watson's (2006) claim that special attention should be paid to scales, because through them people can display misleading graphs. Bruno and Espinel (2005) also indicated that these errors were widespread among prospective primary school teachers in their research.

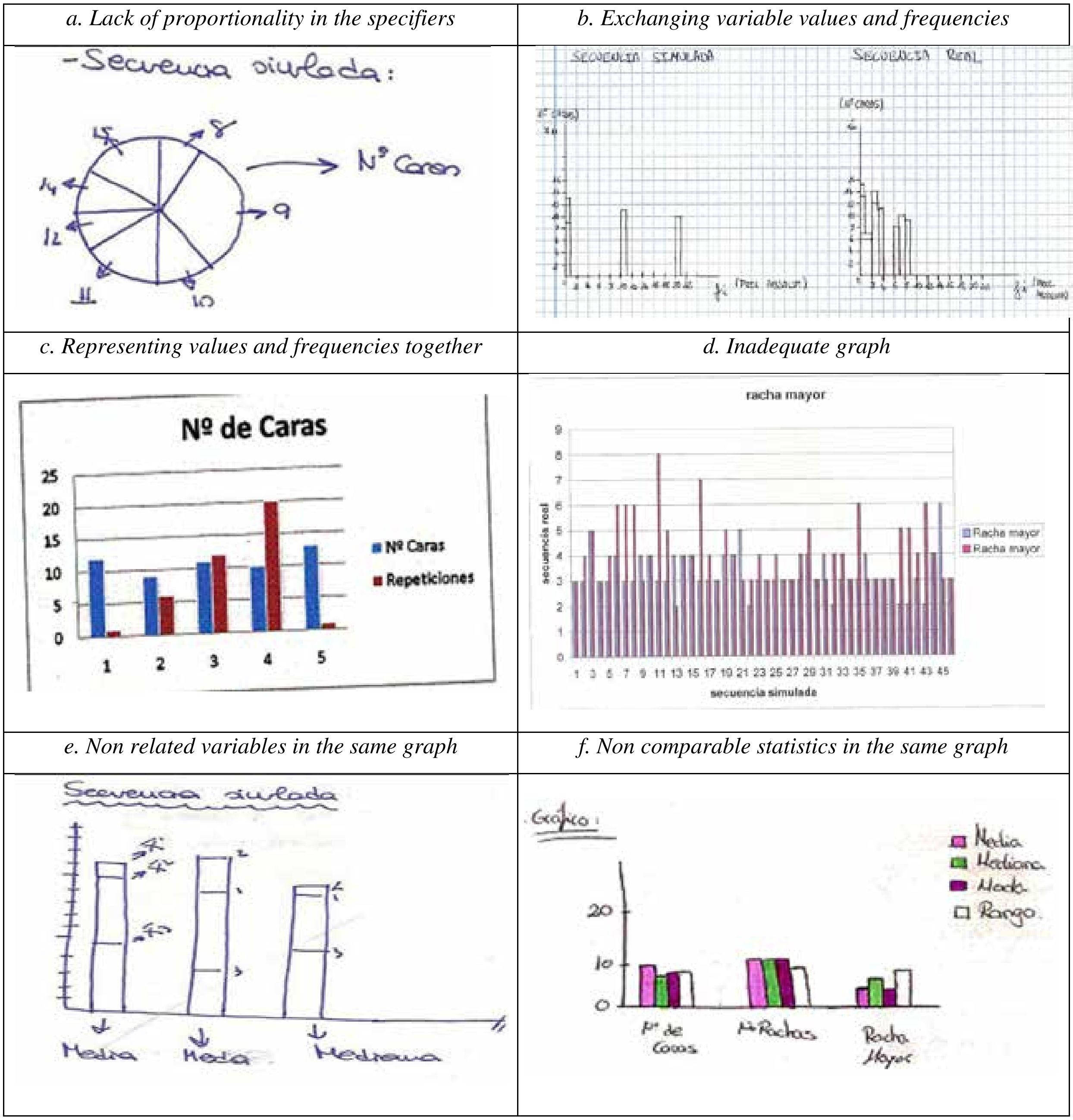

Figure 1. Incorrect graphs

The percentage of incorrect graphs was about 30%; adding the 20% errors in scales, this involves about half of the prospective teachers doing some kind of error. The differences in percentage of correct, mostly correct and incorrect graphs between the three variables posed in the project, were not statistically significant in the chi-square test of homogeneity for the distribution of errors (Chi = 1.03, 6 degrees of freedom, p = 0.9). This suggests that the errors in the graphs did not depend on the variable represented but are due to lack of the necessary competence related to graph construction.

		Number of Heads	Number of runs	Longest run
Basically correct	1.1. Correct	78(43.1)	57 (39.0)	53 (41.1)
	1.2. Correct and unusual	7(3.9)	7(4.8)	6(4.8)
Errors related to scales	2.1. Non proportional scales	0(0)	2(1.4)	3(2.4)
	2.2. Wrong representation of numbers in the number line	4(2.2)	10(6.8)	5(3.8)
	2.3. Confuse titles or scales values	6(3.3)	4(2.7)	4(3.1)
	2.4. Non centered bars	8(4.4)	6(4.1)	7(5.4)
	2.5. Errors in representing intervals	4(2.2)	3(2.1)	3(2.4)
	2.6. Inappropriate scale	13(7.2)	8(5.4)	8(6.3)
Incorrect graph	3.1. Lack of proportionality in the graph specifiers	3(1.7)	4(2.7)	1(0.7)
	3.2. Exchanging variable values and frequencies	2(1.1)	1(0.7)	1(0.7)
	3.3. Representing variable values and frequencies together	3(1.7)	2(1.4)	2(1.5)
	3.4. Representing variable values multiplied by frequencies	2(1.1)	2(1.4)	2(1.5)
	3.5. Inadequate representation	7(3.8)	3(2.1)	1(0.7)
	3.6. Representing non related variables in the same graph	14(7.7)	14(9.6)	13(10.1)
	3.7. Non comparable statistics displayed in the same graph	2(1.1)	2(1.4)	2(1.5)
	3.8. Several errors	28(15.5)	21(14.4)	18(14)
Total		181(100)	146(100)	129(100)

Table 2. Frequency and percentage of participants according to the graph correctness

Influence of computers

Participants in our study were free to use computers or not to solve the task proposed. There were 50 prospective primary school teachers who did their statistical graphs using computers (all of them with Excel); around a forth of the sample and less than a third part of those participants producing graphs.

An added problem when using computers is that, apart the statistical knowledge needed, it is necessary to know and manage the different software options. This provides an added difficulty to represent data using a graph; consequently many students simply accepted the output provided by the software without using the possibilities of changing the scale, graph type, etc.., i.e. they made an uncritical use of the software that was also observed in Ben-Zvi (2002). This happened in our study, where, although a minority of students used computers (Excel), in general, these students had more errors than those who made graphs with only pencil and paper. Table 3 presents the results obtained in our study.

	Number of heads		Number of runs		The longest run		Total	
	Computer	No	Computer	No	Computer	No	Computer	No
Correct	18(36)	67(51.1)	16(40)	48(45.3)	17(42.5)	42(47.2)	51 (39.2)	157 (48.2)
P. Correct	7(14)	28(21.4)	4(10)	29(27.4)	6(15)	24(27)	17 (13.1)	81 (24.8)
Incorrect	25(50)	36(27.5)	20(50)	29(27.4)	17(42.5)	23(25.8)	62 (47.7)	88 (27)
Total	50	131	40	106	40	89	130	326

Table 3. Frequency and percentage of prospective teachers according to the correctness of the graph and the use of computers

50 prospective teachers (27.5% of those who made graphs) used the computer to represent the number of heads in the real and simulated sequences, 40 (27.4% of those who made graphs) the number of runs and 40 (31%) the longest run. We note that the proportion of correct graphs was always higher if the participants did not use the computer. There were fewer errors in scales, possibly because the software automatically builds the scales, but there were much more significant errors when using the computer, such as choosing an inadequate graph or plotting frequencies and variables together.

When performing the chi-square test, to test homogeneity in the distribution of the three categories in the global data (last two columns of Table 3) among participants who performed the graph with or without a computer the test was statistically significant (Chi = 19.72, p = 0.0001 with 2 degrees of freedom). Therefore the use of software increased the participants' errors.

STUDY IMPLICATIONS

The strength of this study is that we provide specific information about the difficulties that prospective primary school teachers have when building statistical graphs. This is relevant, since graph construction is an important part of the graphical competence that a citizen should have (Wu, 2004; Watson, 2006) and because these teachers will have to teach this content in future. Consequently, we prove the need for these prospective teachers to have more education in working with statistical representations. In agreement with Bruno and Espinel (2009) and Monteiro and Ainley (2007) our research suggests that building graphs is a complex activity for prospective school teachers. We agree with these authors in the relevance of improving the prospective teachers' levels of competences in both building and interpreting graphs (Arteaga & Batanero, 2011), in order that they later can transmit these competences to their own students.

A limitation of the study is that we did not analyse the pedagogical content knowledge about statistical graphs in prospective teachers. Since Espinel, Bruno, and Plasencia (2008) observed lack of coherence between the graphs built by participants and their evaluation of tasks carried out by fictitious future students there is also need to evaluate this pedagogical knowledge

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PSYCHOLOGY STUDENTS' STRATEGIES AND SEMIOTIC CONFLICTS WHEN ASSESSING INDEPENDENCE

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In this paper we analyse the strategies used in assessing independence in a two-way contingency table in a sample of 414 psychology students in three different Spanish universities. Strategies are analyzed from the point of their complexity level, their correctness, and the semiotic conflicts involved in the student's reasoning. Although there was perfect independence in the data, most students provided a moderate-sized association coefficient and a positive judgment of association. Few strategies reached the highest complexity level and a number of semiotic conflicts were identified.

INTRODUCTION

Although contingency tables are common to present statistical information and association judgment is a priority learning issue in statistics courses (Zieffler, 2006), little attention is paid to its teaching, in assuming that its interpretation is easy.

This paper describes part of a wider research that was aimed to assess the students' understanding of association in contingency tables before teaching and compare this knowledge with that acquired by the students after a teaching sequence designed for this research (Cañadas, 2012). In the initial assessment three different items corresponding to direct association, inverse association and independence were used. In this paper we only analyze the students' strategies in the item corresponding to perfect independence, because this was the item where students were less accurate in estimating the association coefficient and moreover, the majority of students considered there was association in the data (Batanero, Cañadas, Estepa & Arteaga, 2011). Strategies are classified according to the levels defined by Pérez-Echeverría (1990), and according to its mathematical correctness. Results are compared with those described by Batanero, Estepa, Godino and Green (1996) and students' semiotic conflicts that may explain their wrong strategies are described.

PREVIOUS RESEARCH

Research on association was started by Inhelder and Piaget (1955), who described the strategies used at different ages when judging association in tasks that were formally equivalent to a 2x2 contingency table (see Table 1).

Later psychological studies were developed with adults, and showed that subjects perform poorly, when judging association in these tables. For example, Smedslund (1963) found that some adults base their judgment only on cell *a* in Table 1 or by comparing *a* with *b*. Allan and Jenkins (1983) showed the tendency to base the association judgments on the difference between cell *a* and *d* in Table 1. Although Allan and Jenkins (1983) found that comparing the diagonals *(a+d)* and *(c+b)* was common in adults. Jenkins and Ward (1965) remarked that this strategy of is only valid in tables that have rows with equal marginal frequencies. A correct strategy valid for all types of tables, according to Jenkins and Ward is comparing of the two conditional probabilities, *P(B|A)* and *P(B|NotA),* that is, comparing *a/(a+c)* with *b/(b+d).* Pérez Echevarría (1990) classified strategies that have been identified in psychological research until that time into 6 levels of performance. Levels 0 to 3 correspond to students who use 0 to 3 cells to perform the association judgment. In levels 4 and 5 the students use the four cells; the difference is that comparison between the cells are additive in Level 4 and multiplicative in Level 5.

	A	Not A	Total
B	a	b	a+b
Not B	c	d	c+d
Total	a+c	b+d	

Table 1: Format of a 2x2 contingency table

In a different perspective, Chapman (1967, pp. 151) reported a common bias that he called "illusory correlation": "*the report by observers of a correlation between two classes of events which, in reality, (a) are not correlated, (b) are correlated to a lesser extent than reported, or (c) are correlated in the opposite direction from that which is reported*". Many researchers have reported this bias and suggest that previous theories disturb the estimates of association (e.g., Wright & Murphy, 1984; Meiser & Hewstone, 2006). Batanero, Estepa, Godino, & Green (1996) analyzed the performance of 213 17 year-olds high school students and their strategies in association judgments and defined different conceptions of association: (a) *causal conception* according to which the subject only considers association between variables, when it can be explained by the presence of a cause - effect relationship; *unidirectional conception*, by which the student does not accept an inverse association, and *local conception*, where the association is deduced from only a part of the data.

Our research is aimed to go further in the analysis of strategies in the case where data show perfect independence, since this was the task in which students performed worse in Batanero et al. (1996) study, and in our previous study (Batanero, Cañadas & Arteaga, 2011). We also complement the type of analyses made in previous research in order to identify potential *semiotic conflicts* of students when assessing independence. This term is taken from, who adapted from Godino, Batanero, and Font (2007), who adapted from Eco (1979) the idea of semiotic function: "the correspondences (relations of dependence or function) between an antecedent (expression, signifier) and a consequent (content, signified or meaning), established by a subject (person or institution) according to certain criteria or a corresponding code" (p. 130). These authors also suggest that in mathematical practices different objects intervene: *problems, actions, concepts-definition, language properties and arguments*. For them "*the role of representation is not exclusively undertaken by language":* in accordance with Peirce's semiotic, they assume that "*the different types of objects can also be* *expression or content of the semiotic functions"* (p. 103). The authors term *semiotic conflict,* any "*disparities between the student's interpretation and the meaning in the mathematics institution"* (p. 133). This construct is weaker than that of conception, as stability is not required from the student, but only a misunderstanding or misinterpretation of a mathematical concept, property, language or procedure.

METHOD

The sample included 414 students in their first year of psychology studies from three Spanish universities: Almeria (115 students), Granada (237 students) and Huelva (62 students), all of them taking an introductory statistics course. The task given to the students (Figure 1) was adapted from Batanero et al. (1996), changing to a context of psychological diagnose, without varying the data. The samples included all the students enrolled in the course and attending the session; the difference is sample sizes was due to the size of the University. Though they had not yet studied association in the course they were following, the students had studied statistics in secondary education.

A researcher is studying the relationship between stress and insomnia. In a sample of 250 people he observed the following results:

	Stress disorders	No stress disorders
Insomnia	90	60
No insomnia	60	40

Looking to these data, do you think there is a relationship between stress and insomnia?

Please mark on the scale below a point between 0 (minimum strength) and 1 (maximum strength), according the strength of relationship you perceive in these data.

Figure 1: Task proposed to the students

In part (a) of each item, students are asked to provide an association judgment. There are two categories of responses: (a) the student considers that the variables in the item are related (judging association); (b) the student considers the variables to be not related (judging independence). The estimation for the association coefficient estimation is deduced by measuring the exact position of the point drawn by the student on the numerical scale (0-1) in the second part of the item. Finally, a qualitative analysis was used to identify the strategies used by the students and their semiotic conflicts. The classification of strategies was performed by two different members of the team; in case of disagreement, it was discussed with the other team members until an agreement was reached.

RESULTS AND DISCUSSION

Association judgment and estimation of association

In Table 2 we present the percentage of students who considered (or not) the existence of association between the variables and the mean value of their estimation for the association coefficient. Most students indicated the existence of association although data in the item correspond to perfect independence. This judgment is consistent with their estimation of the association coefficient, which is moderate-sized in average. The differences in mean estimate (in the ANOVA test) or in the percent of students judging association (in the chi-square test) between the three universities were no statistically significant. This suggests that students' responses were similar, despite the difference in educational context. Results may be explained by illusory correlation (Chapman, 1967) since in this item data contradicts the students' previous theories (that stress is related to insomnia), as well as by the causal conception of association, reported by Batanero et al. (1996), who found 55.4% of students judging association in an item with the same numerical data and where the data also contradicted previous theories.

	Almería (n=115)	Granada (n=237)	Huelva (n=62)	Total (n=414)
Mean estimate	0.51	0.47	0.44	0.47
Number (and percent) of students considering there is association in the data	87 (75.7)	194 (81.9)	42 (67.7)	323 (78.1)

Table 2: Mean estimates of association and association judgment

Level of strategies

In order to explain the above results we analyzed the students' strategies, and the mathematical practices involved in these strategies. Firstly, the students' strategies were classified according to their correctness, in three groups: (a) correct strategies (that always produce a correct association judgement); for example, comparing the proportion of people with and without stress disorder in both groups of people with and without insomnia; (b) partly correct strategies that produces a correct response only for specific tables and involves some correct (although partial) ideas about association; for example comparing the sums of diagonals in the table; is only valid when the marginal frequencies in the table are identical; (c) incorrect strategies, when students use a procedure that is a priori wrong in all type of tables; such as for example, using only cell *a* to solve the task. This classification was crossed with the levels of difficulty proposed by Pérez Echeverría (1990), in the following way:

Level 0 Strategy. The student used no information from the table and only took into account his/her own previous theories about the association; the illusory correlation (Chapman & Chapman, 1969; Murphy & Medin, 1985) is shown in this case, for example: *Since when you do not sleep this cause some stress* (Strategy 0, Student 5).

Level 1 Strategy. When the student only used one cell in the table; usually cell *a* because this is the cell when both characters are present and has a higher impact on our attention (Smedlund, 1963; Beyth & Maron, 1982; Shaklee & Mins, 1982, Yates & Curley, 1986): *"there is association, since 40% of the sample have insomnia and stress"* (Strategy 11; Student 111); other students used one of the cells *b* or *c* that contradicts the association: *"there is no relation since there are 60 people with stress and no insomnia and this is a big percentage"* (Strategy 12, Student 51).

Level 2 Strategy. Some students used two cells; for example, they compared *a* with *b* or *a* with *c*, so that they deduced association from only one conditional distribution, which is incorrect: *"If you look to the people with insomnia, there are more people with stress (90) than without stress (60)"* (Strategy 21, Student 21). Other students compared the cells with maximum and minimum frequency (Batanero et al., 1996): *"there are 90 people with stress and insomnia and 40 without stress and without insomnia; 90>40, but the relation is not too strong"* (Strategy 22, Student 61).

Level 3 Strategy: In this strategy students used three cells; for example, they compared cell *a* with *b* and *c*. In general, these students discarded cell *d* that correspond to the absence of both characters (Batanero et al., 1996): *"there is relationship as there are more people with stress and insomnia (90 people) and exactly the same number (60) with either stress and no insomnia or insomnia and no stress"* (Strategy 3, Student 153). All level 1 to 3 strategies are incorrect as the students only use part of the data and then show a local conception of association (Batanero et al., 1996), while part of the strategies in levels 4 and 5 are partly correct or correct. In Level 4 and 5 strategies students use all the cells; for example when they compare rows or columns. The difference is performing additive or multiplicative comparisons.

Level 4 strategies are based on additive comparisons of the four cells. One example is comparing the sum of diagonals *(a+d)* with *(b+c): "there are 130 people with both stress and insomnia or no stress and no insomnia, while there is only 120 with one of these symptoms.* (Strategy 41, Student 176). This strategy was found by Allan and Jenkins (1983). This strategy could provide a good solution when the marginal frequencies (number of people with and without insomnia) were equal, according to Shaklee (1983); for this reason we considered the strategy to be partly correct. In another example, students compared two conditional distributions in additive way: *In people with insomnia there is a difference of 30 having stress, while the difference in people without insomnia is smaller (20)"* (Strategy 42, Student 267) or else compared all the absolute frequencies among them: "*There are many with stress and insomnia (90) but the relationships is not strong, since having stress and no insomnia or insomnia and no stress (69) is also high, much higher than no insomnia and no stress (40)* (Strategy 43, Student 156).

Level 5 strategies use all the four cells with multiplicative comparisons, but still may be incorrect or partly correct. For example, a wrong strategy is to compute all the joint relative frequencies and compared them: *"I computed the percent of each data and compare the results:* $\frac{90}{250}100 = 36\%$; $\frac{60}{250}100 = 24\%$; $\frac{60}{250}100 = 24\%$; $\frac{40}{250}100 = 16\%$ (Strategy 51, Student 11). This procedure is incorrect, because the association should be deduced from conditional distributions and not from joint distributions. An example of partly correct strategy is assuming that all joint relative frequencies in the table should be identical (that is, 25% cases in each cell). We considered this strategy partly correct because the student computed some "expected" frequencies, compared them with the observed frequencies and deduced that there was no association because these two types of frequencies were different. The strategy could have worked (and then was partly correct) in case the computation of expected frequencies were correct: "*I divided 250 between 4 (25%) to see the number of people we should expect in each cell, in case of no relationship. However, although the number of people with no insomnia and stress and no stress and insomnia are close to 25% there is a big difference in the other cells; more than 25% people with both insomnia and stress and less than 25% people with none of them* (Strategy 52, Student 1).

Finally, among the level 5 correct strategies we find students who compared two conditional distributions; for example, *a/(a+b)* with *c/(c+d)* or else performed a similar procedure in comparing columns: "*When we observe the table, 60% of people with insomnia have stress and also 60% or people with no insomnia have stress; the percentage is the same"* (Strategy 53, Student 28). Another correct strategy is comparing possibilities in favour and against B for each value of A; which was described by Batanero et al. (1996): *There are 90 people with insomnia for every 60 with no insomnia when you have stress; that is the odds are 3/2; the same odds 60/40 apply when you do not have stress"* (Strategy 54, Student 21).

	Incorrect	Partly correct	Correct	Total
Level 0	13 (100.0)			13 (3.1)
Level 1	73 (100.0)			73 (17.6)
Level 2	108 (100.0)			108 (26.1)
Level 3	27 (100.0)			27 (6.5)
Level 4	27 (26.2)	76 (73.8)		103 (24.9)
Level 5	20 (26.3)	10 (13.2)	46 (60.5)	76 (18.4)
No response	14 (100.0)			14 (3.4)
Total	282 (68.1)	86 (20.8)	46 (11.1)	414 (100)

Table 3: Frequency (and percent of total) of strategies by level

In Table 3 we present the frequency of responses in the above categorization. Only 11.1% of students used correct strategies and 20.8 % of them partly correct strategies. Students tended to use either level 2 or level 4 strategies none of which are correct, and moreover there were a big percentage of students who did not use all the cells information, since their strategies were level 3 or lower. At level 4, about 30% students compared join frequencies among them, an incorrect strategy described by Batanero et al. (1996) and about 60% used the four cells with additive comparisons, a strategy described by Inhelder and Piaget (1955) in the concrete-operation level but that also was found by Batanero et al. (1996) in high school students. Finally most of level 5 strategies were correct as students either compared the odds ratios or compared conditional distributions a strategy proposed by Jenkins and Ward (1965) and also found in Batanero et al. (1996). The accuracy in the estimation of the association coefficient (true value is equal to zero) increases with the strategy correctness (Table 4) and the differences were statistically significant in the Anova test.

p value	Incorrect strategy		Partly correct strategy		Correct strategy	
(Anova)	Mean	Typical error	Mean	Typical error	Mean	Typical error
0.000	0.536	0.012	0.432	0.024	0.174	0.038

Table 4: Mean estimate of association coefficient in different type of strategy

Semiotic conflicts

The analysis of incorrect and partly correct strategies led to the identification of the several semiotic conflicts, which are classified below, according to whether they involve disparities in the meaning students assigned to association or independence (attributing wrong properties to these concepts) or other conflicts:

1. Incorrect properties assigned to association:

(a) Identification of causality and association, which was called *causal conception of association* by Batanero et al, (1996) was found in level 0 strategies. Since we could not observe the stability of this belief, we just considered it as a wrong interpretation of a property of association (a semiotic conflict), because although causality always involves association, association does not always involves causality, students misinterpreted that this relation was symmetrical.

(b) Assuming that association can be deduced from only a part of the data (which was called *local conception of association* by Batanero et al, 1996 and also appear in previous research, eg. Smedlund, 1963; Beyth & Maron, 1982; Shaklee & Mins, 1982; Yates & Curley, 1986; Pérez Echeverría, 1990). This conflict appears in all the strategies in levels 3 or below; again, we interpret this belief as a misinterpretation (conflict) as we could not observe its stability.

(c) Assuming that it is possible to deduce association from additive comparisons, a strategy that should have been overcome, according to Inhelder and Piaget (1951) at the formal operations stage, but which, however, appeared in our sample in all the level 4 strategies. Students using this procedure only took into account the favourable cases (and not all the possible cases) when comparing probabilities; and therefore this strategy involves a conflict in understanding the idea of probability.

(d) Assuming that association can be deduced from only one conditional distribution (Strategy 21), which was also described by Inhelder and Piaget (1951). Students here only used the conditional distribution of B given A and did not identify the relevance of the conditional distribution of B, given not A for the association.

(e) Assuming that the difference in absolute conditional frequencies is enough to support association, an error which was found in Smedlund (1963) and Shaklee and Mins (1982) and appear in Strategy 43. The conflict appears as students misunderstood the important of relative frequencies in the study of association.

(f) Assuming that the association can be deduced from the difference between the sums of diagonals in the table. That strategy was considered to be correct by Piaget and Inhelder, but Allan & Jenkins (1983) and Shaklee and Tucker (1980) suggested it does not work in the general case (for example, in the task given to the students). It appeared in strategy 53.

(g) Assuming that $a>d$ in case of association, which appeared in Strategy 22. These students considered that only these two cells influence the association and consequently they did not understood that cell *d,* have the same value than *a* on the association. We did not find this strategy in previous research.

2. Incorrect properties attributed to independence

(a) Expecting equal join frequencies in case of independence, which involves confusion between the ideas of independence and equiprobability. It appeared in strategy 52 and was not described in previous research.

3. Other conflicts: Basing the association judgment in their own opinion, instead of considering the data, which appear in level 0 strategies, where students were guided by the illusory correlation phenomenon described by Chapman & Chapman (1969).

DISCUSSION AND TEACHING IMPLICATIONS

Most psychology students in our study judged association in a task where there was perfect independence, due to the illusory correlation phenomenon and their previous theories, which affected their accuracy in estimating the association coefficient. Results in our study were worse, as compared with Batanero et al. (1996) since a higher percent judged association. These authors did not inform about the estimate of association by their students; in our study the estimation was consistent with the association judgment. Results were very close in all participating universities. Regarding the conceptions described by these authors, we observed the causal and the local conception. Since we could not check the stability of these conceptions, we used instead the idea of semiotic conflict, which only involves a mistaken interpretation of a mathematical expression, a concept, property or procedure. Our list of semiotic conflicts is wider than the list of conceptions described by the authors, as new conflicts related to the ideas of association and independence were identified in our study. For example, in addition to assume that association may be judged from only part of the data, another frequent conflict was assuming that association can be judged from absolute frequencies (instead of relative frequencies). Another new conflict is assuming that the cells in the four cells in the table should have equal frequency in case of independence. Since semiotic conflicts do not assume a strong conviction on the part of the students, it is possible to change them with adequate instruction and then the identification of these conflicts in the students is a first step in order to correct their wrong reasoning and improving their competence in judging association.

All these reasons and our results suggest the need for further research about teaching association, since the causal conception and the effect of illusory correlation does not seem to improve with traditional instruction (Batanero, Godino, & Estepa, 1998). Our purpose is to continue this work by designing an alternative teaching with activities that confront students with their conflicts and help them overcome them. This proposal will be tested and students will be assessed in order to compare their intuitive ideas with those acquired as a result of teaching.

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CONTEXTUALIZING SAMPLING – TEACHING CHALLENGES AND POSSIBILITIES

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The aim of the present paper is to shed light on mathematical knowledge for teaching probability. In particular we investigate critical instances when a teacher tries to keep track on the idea of sampling and random variation by allocating the discussion to an everyday context. The analysis is based on a certain episode of a longer teaching experiment. The analytical construct of contextualization was used as a means to provide structure to the qualitative analysis performed. Our analysis provides insight into the nature and role of teachers' knowledge of content and teaching. In particular, the study suggests the idea of a meta-contextual knowledge that teachers need to develop in order to keep track of the intended object of learning when allocating their teaching to an everyday context.

Keywords: Sampling, probability, descriptive statistics, mathematical knowledge for teaching, knowledge of content and teaching

INTRODUCTION AND AIM OF STUDY

This paper is part of a larger research project, which aims at investigating the nature and role of *Mathematical Knowledge for Teaching Probability* (MKTP).

Mathematical knowledge for teaching (MKT) (Ball, Thames, & Phelps, 2008) is a general knowledge framework which aims at structuring an understanding of teacher competences for teaching mathematics. The framework is divided in two main domains, *subject matter knowledge* (SMK) and *pedagogical content knowledge* (PCK*)*. Both domains are viewed equally important for teaching mathematics. On an overall level, the notion of SMK implies that all mathematical content matter is similar in nature, following a certain mathematical logic and structure of axioms. However, there are reasons to consider specifics of MKT in relation to probability theory as this is considered different in nature to other branches of mathematics.

Steinbring (1991) argues that not even the basic concepts of probability theory fit into a logical deductive way of acquiring mathematical knowledge. For example, the concept of probability is used to describe chance but what is unique with probability theory is that the inverse is also true. In this way, probability concepts are self-referential, which implies that teaching probability requires a dynamic outlook on knowledge development. Stohl (2005) also recognizes that probability-laden situations could not be structured in a purely logical and deterministic way as for instance algebra and geometry. Since probability deals with modeling random dependent situations it operates within a non-deterministic paradigm and it is not possible, by certainty, to derive and predict a certain result.

Connected to the specific nature of probability, there are reasons to take a closer look at the pedagogical content knowledge teachers should develop in trying to make sense of probability in teaching. However, there are few attempts made to study in detail different aspects of PCK for teaching probability. Against this background, the present paper aims to contribute with insights on *knowledge of content and teaching*, a certain sub-category of PCK in the MKT framework, in relation to the teaching of probability. Knowledge of content and teaching is described by Ball et al. (2008) as a repertoire of forms of representations, techniques and examples and the interaction between mathematical understanding and pedagogical assets in relation to students' learning. In particular the paper focuses on the challenges a teacher might meet when contextualizing *sampling* in an everyday context through classroom discussion.

Described by Gal (2005), there are two overall reasons for teaching probability in school. The first is internal in that probability is part of mathematics and should be developed in its own right. The second reason is external in the sense that "the learning of probability is essential to help to prepare students for life" (Gal, 2005, p. 39). Elaborating on this duality, Gal (2005) suggests that teaching should put sufficient emphasis on probability-laden situations in the real world in order to help students to develop *probability literacy*. However, even if Gal (2005) proposes that the teaching of probability should, to great extent, stress real world applications of probability discussions, little is known about how such teaching should be structured and the nature of MKTP required in such teaching, for making details of intended probability content available to the students' learning.

The teacher's role is seen crucial for orchestrating a classroom discussion in ways that allow students to make sense of the mathematics (McCrone, 2005). To make a story of a mathematic content available to the students and to support their meaning makings of that story, teachers should make details of the content explicit in the discussions, in both explanations and questions asked (Franke, Kazemi, & Battey, 2007). In the present paper we intend to look at a teacher's explications or lack of explications, when the teacher tries to make the story of sampling available for a class of Grade 5 students in a Swedish school by allocating the discussion to an everyday context.

Aim of Study

The aim of the present paper is to shed light on knowledge of content and teaching probability (KCTP) required for *contextualizing* a probabilistic content in an everyday story. In particular we intend to investigate critical instances when a teacher tries to make sense of the idea of sampling by allocating the discussion to an everyday context.

FOCAL PROJECT, CONTEXT AND CONTEXTUALIZATION

The analysis reported in the present paper was based of the analytical construct of contextualization (Halldén, 1999; Nilsson, 2009). The construct of contextualization can be considered a constructivist reaction to constructivists' purported neglect of contextual aspects of learning.

Context plays a certain role in contextualization. In a sociocultural perspective, context refers to stable physical and discursive elements of a setting in which a learning activity takes place (Resnick, 1989). However, based on constructivist assumptions, in the present paper context refers to the cognitive context shaped by the learner's personal interpretations of an activity (Cobb, 1986). To speak about students' processes of contextualization is to speak about how learners struggle to render a phenomenon or concept intelligible and plausible in personal contexts of interpretation (Caravita & Halldén, 1994). This idea rests on the principle that we always experience something in a certain way, from a certain set of premises and assumptions (Säljö, Riesbeck, & Wyndhamn, 2003). Related to that, contextualization gives sense to learning mechanisms of how and why different knowledge elements make the activation of others either more or less likely (cf. Shelton, 2003). In other words, talking about how teachers and students contextualize a phenomenon is a way of organizing and conceptualizing their views of the phenomenon and what these views imply for their understanding of the phenomenon and their way of communicating the phenomenon.

To clarify how different contextual elements serve as points of reference in processes of contextualisation, it has shown fruitful to structure between elements referring to a *conceptual, situational* and *cultural* context. The conceptual context refers to personal constructions of concepts and subject matter-structures. The situational context refers to interpretations made in the interaction between the individual and the immediate surroundings, including interpretations of figurative material, possible actions and directly observable sensations. Third is the *cultural* context, referring to constructions of discursive rules, conventions, patterns of behaviour and other social aspects of the environment (Halldén, 1999).

It has also shown to be fruitful to structure between the *focal project* (FP) of a reasoning process and the *context* in which the FP is treated or contextualized (Nilsson, 2009; Nilsson & Ryve, 2010). To talk about a FP means that we pay serious attention to what a teacher or a student is actually trying to make sense of, or trying to work out. So, what constitutes the FP in a pattern of contextualization is the problem, goal or intention that a teacher or a student engage in and interpret as being their obligation to solve or achieve (Nilsson, 2009). Hence, viewing meaning making as a process of contextualization an analyst strives to account for agents' FPs and the way they contextualize these projects in order to understand how and why content elements are brought into play within a reasoning.

METHOD

Shaughnessy (2003) emphasizes the role of data experimentation in the teaching and learning of probability. He claims that, when trying to make sense of data, students are encouraged to develop probability questions. However, from research we know that it is not an easy task for students to make sense of data. Makar and Rubin (2009) report that students often disregard frequency data and also are struggling to connect conclusions to the data they have collected regardless of sample size (Makar & Rubin, 2009). The authors point to the need for teachers to make this connection more explicit to the learners. Also, Pratt, Johnston-Wilder, Ainley, and Mason (2008) highlight students' difficulties interpreting information in a data sample as they challenged students (10-11 years old) to infer, from data, the unknown configuration of a virtual die. Similar to Stohl and Tarr (2002), Pratt et al. (2008), show that what became critical for the students was the role of sample size, to understand the difference between drawing conclusions on the basis of the global (long-term) and local (short-term) behaviour of a data sample.

The rationale of a teaching experiment

The episode that will be presented and analysed in the present paper concerns how a teacher introduces a discussion on sampling and sample size through an everyday contextualization. The episode emerged within the frame of a teaching experiment in which a class of fifth graders had first been planting pumpkins and sunflowers. Since this is the first time these students encounter probability theory in the school environment, we find it important to give a brief overview of the teaching that took place before the episode that is in focus of our analysis in order to better understand our interpretations of the teachers' and the students' acts.

A developmental team, consisting of Maria, the teacher of the class, the second author of the paper and Torsten, a specialist on outdoor education, stood behind the design of the experiment. Maria did all the teaching. Ten students participated in the study, which was stretched over two lessons. Each student was assigned an individual box of one square metre during the first lesson in which they were planting 18 pumpkin and 18 sunflower seeds. Maria held a probability lesson about three week after the seeds were planted, which was based on the germination of pumpkins and sunflower seeds. The teaching rationale of the activity was, at the first place, to challenge the students on the idea of sampling and, specifically, on the role of sample size for making probability predictions. The teacher tackled this issue by confronting the students with the variations in growing outcomes between the students' individual boxes. For instance, in one box only one of the 18 pumpkins grew. This was compared to boxes were up to eleven of the 18 pumpkins grew. Based on this, the teacher led a discussion about how we have to combine more and more observations in order to get a more valid result of the chance for a seed to grow. The first part of the teaching ended with putting together all student results and dividing the resulting number by the total number of planted seeds to get an estimation of the probability for a seed to grow. The same procedure was done for both the pumpkins and the sunflowers.

The rationale of the second part of the teaching episode was to represent the results of growing seeds in the structure of a two-way contingency table (Figure 1).

	Growing	Not growing	Total
Sunflowers	51	129	180
Pumpkins	67	113	180
Total	118	242	360

Figure 1: Two-way table of the growing of sunflowers and pumpkins

Not only the teaching but also the work of the developmental team were audio and video recorded. The developmental team made brief reflections on the teaching in immediate connection to the teaching. However, the detailed analysis presented in this paper was made by the authors of the present paper.

Data-analysis

This paper focuses on a certain episode of the entire teaching were Maria ends the lesson by telling an improvised story containing a sampling situation. The story is supposed to wrap up the discussions and introduce the coming lesson, in which the students were supposed to plan for and conduct their own probability-laden, statistical investigation. The third lesson was not actually supposed to be part of the teaching experiment. But implicitly it was, as the introduction of the lesson was included at the end of the second lesson.

In the analysis we are specifically looking for the participants FP and their contextualizations of their FPs. We account for this by attributing an intention to their behaviour. For example, a teacher does not assign a task to her students randomly. The teacher has an intention with this and by considering her behaviour as intentional, and continuously adjusting our interpretations of her intention, we create a model of her FP and the ways she is contextualizing the project.

RESULTS AND ANALYSIS

We enter the lesson when Maria portrays a picture where she waits outside a store for her husband to exit. While waiting, she keeps track of the number of each sex that visited the store. We have chosen to highlight certain transcripts from this episode as it helps us to illustrate and communicate our analytical points in relation to our research question.

The same context but competing focal projects

Maria portrays a situational context in which she is standing outside a store one early morning, reflecting on how many males that have visited the store. We ascribe to her the intended project, trying to contextualize sampling in a new way so the students get the opportunity to translate what has been discussed previously about samples.

Within this situational context Maria introduces a *genus context,* which is directed towards the distribution of men and women taking responsibility of shopping for groceries. Taking a close look at the way in which Maria is orchestrating the story, we note that she includes multiple FPs to her story, which we consider making it hard for the students to follow the storyline and the mathematical idea of it. We particularly distinguish two specific conflicting FP, dealt with within the same genus oriented context. First Maria has the intention of contextualizing sampling to the students and second she has the intention to convince her husband to more often do the shopping. Implicitly the story is also intended to create an interest for similar surveys since Maria is planning for an upcoming assignment on the topic of statistics and probability. She struggles to explicate the mathematical intention and give support to the students' meaning making process early in the story.

Maria:	Another male, that makes three men, and it continued. I wondered, are men such early birds? My husband usually doesn't go shopping this early. I'm going to tell him that I stood here and watched. Then when I've counted ten people, seven of them were male. So I thought I would tell him that seven out of ten were male.
	[…]
Maria:	The worst part was, he [the husband] didn't exit. So I kept on waiting, and then a lot of women started to appear
Student 1:	And then, when you looked out the window, you saw a girls' bus and a boys' bus.
Maria:	No, but there were women and women and women. Eventually 20 men and 20 women had exit the store.

Our interpretation is that Maria's mathematical intention has not been made plain for the students so far. Instead they are subjected to the intention to reduce the disparity between men and women in typical household chores like shopping for groceries. Since this is a mathematics lesson, there are reasons to believe that the students are searching for a mathematical intention in the story of which they should make sense. We interpret that one student has caught on such a meaning and that there exists an unofficial game between Maria and her husband. If there are more men shopping in the morning, Maria's husband must do so more often as well and she wins. So the student tries to challenge the rules and prerequisites with a statement.

Maria:	And do you know what I did then? I stopped counting.
Student 1:	Coward!
Maria:	You know, I didn't want there to be more girls than boys, right?
Student 2:	You didn't have to tell him [Maria's husband] that you counted.
Maria:	No, I actually didn't, since the outcome was what it was.
	[…]

Maria: If I had stood there and counted a hundred, maybe 70% would have been women, right?

[…]

Maria: And if I would of stood there even longer I would of gotten an even better result. What the probability was for a man or a woman entering the store, right?

The teacher does not show any signs of recognizing the fact that there are competing FPs in the discussion. We note that more students seem to try to make sense of the story in terms of a game. They sense that Maria is about to loose, calling her a coward and offering her a way out, and she is admitting that fact from their point of view by not telling her husband about the result. To think of the story in terms of a game rather than whether men shop for groceries or not, might fit better with the students' expectations of probability application and their attempt of meaning making. In other words, since Maria's competing FPs appear confusing, the students contextualize the intended learning object of the story in a way that makes more sense to them.

Maria keeps on pushing her genus perspective combined with fairness. She does not want it to be more girls than boys shopping for groceries in an equal society. There is also a preconception on the teacher's behalf that the students share this view. But randomness does not have a genus agenda, making the two FPs conflicting, so she steps in and manipulate the result by keeping the sample size small. Her intuition tells her that more women shop for groceries, since that mental picture is more available to her. The session illuminates that Maria has a sound understanding of sampling and the role of sample size. Because of her knowledge of sampling she acknowledge that it becomes a risk to continue the thought experiment. She anticipates that the result will probably not differ much from the assumed true value in the long run, opposing her gender equality agenda.

There are more references to Maria's mathematical intention towards the end of the story. She eventually manages to explicate the mathematical content right at the end in a statement formulated with probability wording relevant to sampling by posing the question "What the probability was for a man or a woman entering the store, right?". The situational and cultural context seems to be the focus of Maria, instead of the conceptual context, for most of the episode. Next we will analyse what motivates the story.

Competing driving questions

At the beginning of Maria's improvised investigation outside the store, the driving question is implicit and could be interpreted in different ways. We find two, equally likely, possible interpretations. It could be a question about how many men shop for groceries early in the morning where the FP would be to contextualize aspects related to descriptive statistics. Or a more probability oriented question about the probability for a man going shopping for groceries early in the morning, where the FP would be to contextualize aspects related to probability. Since the driving question is left unspoken, it is left up to the students to formulate a question that fits their individual meaning making of the story. We suggest that the close relation between statistics and probability when handling relative frequencies becomes an obstacle for teachers. It places great demand on teachers' subject content knowledge to be explicit about the driving question and how it corresponds with the teacher's FP with the activity, e.g., if the intention is to offer a description of a phenomenon or if it is to make a prediction of future behaviour of the phenomenon. More concretely, is the intention to describe differences between how many women and men that are shopping for groceries or is the intention to estimate the probability that the next person out of the store is a man? We interpret that Maria her self struggles between the two as she continues the story and right at the end of the story, Maria says the following:

Maria: If I had been there a whole day, I would of known if there are more women or men that goes shopping that specific day. And if I had been there for two days, I would of gotten an even better result. And if I would be staying there even longer I would have had an even better result: what the probability was for a man or a woman entering the store, right?

One way of interpreting Maria's words is to question her subject matter knowledge, that she is not sure her self about the difference between descriptive statistics and probability theory or even if there are any difference. But viewed from a perspective of mathematical knowledge for teaching we suggest that the classroom dialogue itself provides, probably unintentionally, the means for Maria to clarify the difference between a descriptive statistical and a probability-laden question formulation. The last turn of the last citation is actually the first time when she develops her contextualization of sampling in the story by including a question that is explicitly directed to probability to her context. It appears at the end of the discussion and we claim that it has not been explicit neither to the students, nor to the teacher from the beginning of the story. Hence, by contextualizing sampling as she did, we claim that also Maria developed her way of making sense of probability through the dialogue and the way she contextualized the question of sampling.

DISCUSSION

By taking this close look at a certain segment of the data gathered from the entire teaching activity, we are not in the position to make claims about the quality of the students' understanding of sampling and the role of sample size for making probability predictions. However, to make claims about the students' actual understanding has not been the issue of the present paper. The aim of the paper was to investigate aspects of teacher competencies in relation to the challenges a teacher might reach when trying to orchestrate the idea of sampling by allocating the discussion to an everyday context. Based on this we can reflect on the enacted object of learning (Marton & Tsui, 2004) in terms of how a teacher use everyday oriented contextualizations to offer students opportunities to discern and learn concepts and ideas of probability theory.

Gal (2005) suggests that the teaching of probability should orient towards everyday situations in order to support the development of probability literacy. Even if we adhere to such a view of teaching, our analysis shows that this might not be an easy enterprise for teachers. Moreover, our analysis suggest that the problem of connecting to everyday situations should not only be considered as an issue of teacher's insufficient mathematical content SMK (Ball et al., 2008). We suggest that, what became crucial to Maria relates to her knowledge of content and teaching. What seem to be crucial for Maria in her attempt to create certain learning opportunities for her students by allocating the teaching to an everyday situational context, is to be aware of what contexts and FPs she is actually communicating. In that sense, our study extends the notion of KCTP to not only being a matter of a teacher possessing a repertoire of forms of representations, techniques and examples (Ball et al. (2008).

Based on the last lines of reasoning, our initial observations indicate that teacher education should challenge teachers to develop a kind of meta-contextual knowledge in order to learn to be explicit about their agenda in addition to developing SMK of probability. If a teacher uses an everyday contextualizing to make sense of probabilistic ideas and concepts, the teachers should be aware of that such a context might involve many items and variables that can obscure the intended ideas and concepts and sometimes even inhibit them from coming to the fore in the teacher's and students' reasoning. This implies the need to challenge teachers to be more aware of the implicit assumptions on which a reasoning rests and how explicit they are communicating the intended object of learning in a whole-class discussions.

Makar and Rubin (2009) emphasize the need of a driving question in the teaching of stochastic. In our analysis we note that the teacher communicates several different FPs in the discussion, mediated by different driving questions. So, there is no lack of driving questions as such. However, what we consider problematic is that there are several FPs and questions in play simultaneously, which the teacher seems to be unconscious of. We consider the teacher's difficulty to include a clearly defined probability oriented driving question as an important aspect. The students are supposed to become challenged to think about and propose predictions that contain degrees of uncertainty and that are based on frequency information. As claimed above, the analysis indicates that most of what Maria says stay within the context of descriptive statistics. As we can see in the last part of the last quote above, and from what we hear when we listen to the activity in its whole, we consider her to have sufficient subject knowledge to formulate probability oriented questions. Again, we believe that the critical element relates to a meta-level. She have to be aware of what FPs she mediates to the students and the ways in which the questions she poses direct the communication.

Our main conclusion from the analysed episode is the need for a meta-contextual knowledge amongst teachers that have the intention to allocate the meaning of probability concepts to an everyday story. We also conclude that there is much left to do to define what a meta-contextual knowledge in probability constitutes of and what its consequences are for the teaching of probability. A more exhaustive teaching experiment, combined with qualitative analysis of interviews or focus group discussions with teachers would help to get further insight to these issues.

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PRINCIPLES OF TASKS' CONSTRUCTION REGARDING MENTAL MODELS OF STATISTICAL SITUATIONS

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In this paper we focus on the construction of tasks which is a specific topic within our on-going research into the development of a diagnostic tool for analysing young students' mental models when they act within simple statistical situations. For this, we discuss tasks from a theoretical perspective concerning statistical content, statistical thinking processes and – as the main aspect in this paper – the representation of tasks. We deduce principles of the tasks' development partly from statistics education research literature and partly from our previous research.

INTRODUCTION

The focus of this theoretical paper is on the development of tasks including simple realistic situations of uncertainty aiming to diagnose young students' perception of these situations. The development of tasks is part of a larger project (Eichler & Vogel, 2012), which aims to develop a diagnostic instrument for describing statistical situations' perception of students not having been taught in statistics and, further, to evaluate this perception of these students after a systematic schooling in statistics. Since in the mentioned research project a student sample was investigated using a first set of tasks involving simple statistical situations (ibid.), for a follow-up study, the existing set of tasks has to be extended. For this reason, we discuss theoretically both several principles of the tasks' development that can be deduced from theoretical constructs of psychology, and some empirical results of our previous studies. Thus, we firstly make some comments on well-known diagnostic tasks in stochastics education and analyse one well elaborated task of an intervention study of Bakker & Gravemeijer (2004). Afterwards we change this task step by step according to theoretical and empirical based prerequisites of a diagnostic task aiming to investigate non-schooled children's thinking in statistical situations.

DIAGNOSTIC TASKS IN STOCHASTICS EDUCATION

In their seminal work referring to children's development of thinking with probabilities, Piaget and Inhelder (1975; original 1951) used tasks that could be treated as diagnostic tasks to analyse the children's initial position of learning probability (Hußmann et al., 2007). The set of tasks which Piaget and Inhelder had used aimed to analyse the thinking of children of different ages, who were not schooled in probability. After Piaget and Inhelder, subsequent research also adopted diagnostic tasks to investigate young children's probabilistic thinking resulting in both replicating and modifying the findings of Piaget and Inhelder (Jones & Thornton, 2005). Fischbein and his colleagues diagnosed children's so called primary intuitions when working with probabilistic situations (e.g., Fischbein, 1975). They investigated instructional conditions that facilitate children to come from primary intuitions concerning probabilistic situations to secondary intuitions (ibid.). Nowadays, in tradition of Fischbein's work most research on children's probabilistic thinking is based on an intervention design and, further, on students that are schooled in stochastics (Jones & Thornton, 2005). This is even the case for the research in statistics education that has increased considerably in recent years when statistics became a crucial part of the stochastics curriculum in many countries (Batanero et al., 2011). According to this research development we have a lot of knowledge about non-schooled children's thinking in probabilistic situations, but little knowledge about non-schooled children's thinking in statistical situations focusing on data (Mokros & Russel, 1995). This research gap motivates our research aiming to analyse the decision making of young students (without schooling) in realistic statistical situations (Eichler & Vogel, 2012). However, in the educational literature only very few tasks can be found that allow for analysing unschooled students' thinking or decision making concerning statistical situations. For this reason, we started to develop tasks representing statistical situations to provoke non-schooled young students' thinking about and decision making in statistical situations. After we had used these tasks in pilot studies we modified the tasks, used the modified tasks in further pilot studies again and so on. In this way our task development is taking place until now. We analyse this development in what follows.

FIRST STEP OF THE DEVELOPMENT OF A DIAGNOSTIC TASK CONSIDERING CONTENT AND THINKING PROCESSES

Using existing tasks as starting point for the development

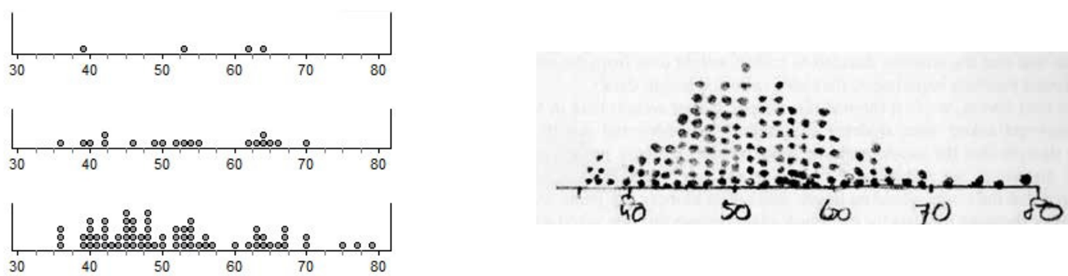


Figure 1: Collections of students' weight and students' prediction of a larger sample

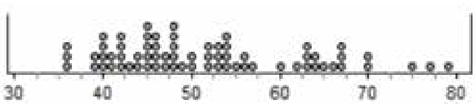
We begin with a task from Bakker and Gravemeijer (2004, p. 158). The specific task aims to provoke students' thinking about variation, sampling and representativeness:

> In a certain hot air balloon basket, eight adults are allowed [in addition to the driver]. Assume you are going to take a ride with a group of seventh-graders. How many seventh-graders could safely go into that balloon basket if you only consider weight?

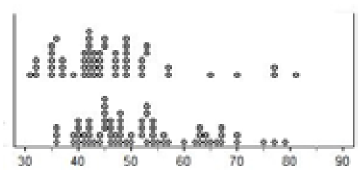
Firstly, the students tried to estimate the average of the weight of both students and adults. Afterwards they collect data to students' weight in their class as basis to make a prediction for a larger sample. It seems obvious that non-schooled children would not be able to think about centre and variation in this task in a way using statistical concepts like range, average, median, and so on since they do not have systematic knowledge of these concepts. However they might be able to think about variation and centre in an informal way. Potentially these children might also be able to think about the sample in an intuitive way that a bigger sample would give more certainty than a smaller sample. However, we restrict our considerations to an informal way of thinking about centre and variation representing two main aspects of statistical thinking (Wild & Pfannkuch, 1999).

Provoking students' thinking

Given that non-schooled children are able to read the iconic representation of the data, they could be potentially able to describe the distribution in an informal way. However, research in stochastics education give a lot of evidence that a task involving a decision making in terms of both a comparison and a prediction would particularly provoke children's comments about a situation (e.g. Yost et al., 1962; Fischbein, 1975). Thus, a task involving decision making in terms of making a prediction – Bakker and Gravemeijer (2004) included a prediction in their task – could foster the students to think about a statistical situation (see figure 2, left side). As well, it is possible to enhance the given task leading to decision making process in terms of a comparison.



Another seventh-grader of your school will be weighted. Estimate his weight. Give a rationale for your estimation.



There are the sample of your school and a sample from another school. Which students are more heavy-weight? Give a rationale for your answer.

Figure 2: Variation of the task including decision making

Tasks in this way could serve as a diagnostic task to provoke students' informal thinking about centre and variation. However these tasks presuppose the students' ability to read the iconic representation of the data, i.e. the transnumeration of data into a graph (Wild & Pfannkuch, 1999), which is obviously not common for non-schooled children (e.g., Shaughnessy, 2007). At this stage of task development we expand our theoretical framework with regard to mental model theory. This theory allows for describing the perception and processing of statistical situations that were not represented by elaborated statistical methods like the dot plot.

THEORETICAL ESSENTIALS OF MENTAL MODEL THEORY

We reduce our theoretical focus on those aspects of mental models, which were fundamental for characterizing our task development.

Johnson-Laird (1983, p. 156) states: "A mental model […] plays a direct representational role since it is analogous to the structure of the corresponding state of affairs in the world – as we perceive or conceive it." From the perspective of information processing Schnotz and Bannert (1999) describe mental models as being constructed individually according to a task and its requirements within a specific situation. By this, they conclude that a mental model represents *the structure* as well as *the function* of the modelled object in an *analogous relationship*. These are three essential characteristics of mental model theory which were important of being pointed out with regard to the task development:

- *Structure:* An essential process of mentally modelling a situation's structure is recognising the physical objects of the situation, e.g. a die and its characteristics (or a student and his weight), as well as the relationship of these objects and their characteristics. Given data are also to be seen as being part of a situation's structure because they represent results of a process having passed.
- *Function:* Concerning the dynamic aspect of mental models, i.e. the function, Seel (2001) suggests that, when coping with demands of a specific situation, an individual constructs a mental model in order to simulate relevant aspects of the situation to be cognitively mastered. Thus, here the term function is defined as mental simulation of a situation (in contrast by the mathematical construct of function). The function of mental models allows for deriving answers via mental simulation of systems by anticipating possible results given for example by throwing dice (or students becoming the weight they have when they were weighed). Mental simulations do not result in quantitatively exact conclusions but in qualitative ideas about the expected outcomes of such simulations (De Kleer & Brown, 1983). These "qualitative simulations" (De Kleer & Brown, 1983, p. 155) require sense making about the system or process that should be simulated, its constituent components and their relationships.
- *Analogous relationship:* This means, a mental model and the corresponding modelled situational object or process coincide structurally at least in some constituting elements. Thus, mental models can principally be inferred from observable information which represents mental modelling of a situation or task, the conditions of a students' specific situation (experience, pre-knowledge), and students' outcomes after working with tasks (written responses, videotapes).

These three characteristics become crucial when we analyse the demands on mental modelling of tasks concerning decision making in simple statistical situations, when we construct a hierarchical model of tasks' complexity and finally, when we construct new tasks on base of this model.

SECOND STEP OF THE DEVELOPMENT OF A DIAGNOSTIC TASK CONSIDERING MENTAL MODEL THEORY

Mental modelling of a situation represented in a task includes perceiving its *structure,* which is a precondition to conduct mental simulation adequately (*function* of the situation). This means that disclosing those elements which impact on a statistics problem situation in respect of its structure and its function would reduce the complexity of a situation represented in a task, especially for young statistics unschooled students. For this reason, the development of tasks has to treat the visibility of a situation's structure as a matter of principle of tasks' representation. The visibility of those elements which constitute the *structure* as well as the *function* of a situation impacts on mental simulation. Such a mental simulation equals the process of data generation when a statistical situation is regarded – it could be seen as a qualitative mental data generation.

Changing the task according to demands of mental modelling

In the task of students' weight the result of data generation (*function* of the situation) is represented by the given data. Thus, the *function* of this situation itself is hidden because having passed as well as the situation's structure including those elements that impacted on the students' weight. Similar tasks, in which both the structure and the function are potentially visible, are the following:

> Two students try to throw coins as close to the wall as possible. Who is the better player? Give a rationale for your answer.
>
> Andrea lets a paper frog jump several times. Estimate, where the next paper frog will end. Give a rationale for your answer.

The two situations include a request for decision making as defined above. Further, the data as result of the situation's function (i.e. process of data generation) could be given in the same way as shown in figure 2. Taken into account that young students were potentially not able to understand the construction of a dot plot, we used a pictogram to represent the data, e.g. the frog jumps (figure 3; Eichler & Vogel, 2011) and changed the situation a little.

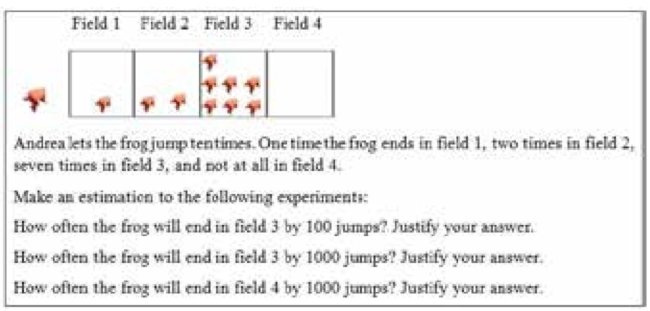


Figure 3: The frog task (early version)

However, a lot of young students confronted with this task in questionnaires and in interviews seem to misinterpret the situation. In particular, the equal arrangement of the frogs in the fields 1, 2 or 3 (fig. 3) led some students to the interpretation that those frogs being visible do not represent several jump results in reality. Those students were not able to match the descriptional representation (the text) and the depictional representation (the pictogram) to make inferences from the given information on base of an adequate mental model of the situation (Schnotz & Bannert, 1999).

An auxiliary change of the situation's representation in the task was to use a realistic picture of the situation in contrast to the more abstract representation using a pictogram (Vogel, 2006). Further, according to the findings in research into multiple external representations (Ainsworth, 1999; Mayer, 2001; Seufert, 2003), we combine for each step of the data generation a text with a picture. According to the cognitive load theory (Sweller et al., 1998), the text itself is formulated as simple as possible. Finally, we construct photos as pictures of the situation including all relevant objects representing the structure of the situation (cf. Schnotz & Bannert, 1999). We show in figure 4 only the changed representation of the situation since the task is the same as shown in figure 3.

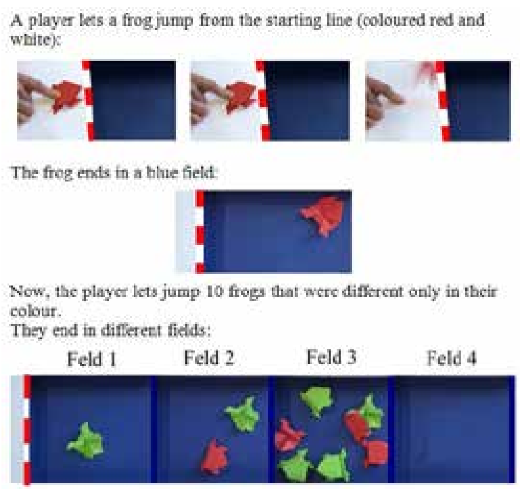


Figure 4: The situation of the frog task (final version)

Findings concerning the changed tasks

Using these special kind of "stochastics problem-storyboards" in an interview design (Eichler & Vogel, 2012) gave some evidence that even non-schooled students (referring to stochastics) were able to understand the situation in the intended way. Thus, after students have worked on the task given in the form described above, they were asked to conduct the experiments with real frogs and fields. In any case, the students were directly able to cope with both the structure and the data generation represented in the task.

Analysing the rationales that students gave in a paper and pencil test (students worked alone on several tasks like the frog task) yield amongst other results (ibd.) that students tend to identify different objects of a situation to be mainly relevant for the data generation. The consequence of this finding was to re-analyse the tasks in terms of characteristics of mental model theory using the terms *data*, *objects* and *mental simulation*. Regarding the frog task we can state with regard:

- to the *structure* of the problem situation that *data* representing the situation are given by the visualised results of 10 different frog jumps as well as the relevant *objects* of the situation. The objects include *human* objects, i.e. the jumper, and

non-human objects, i.e. the frogs, the fields, and the interplay of both sorts of objects;
- to the *function* of the problem situation that the process of *data generation* impacted by the objects and their interplay is visible and a prediction of future results of the data generation necessitates a *mental simulation* of the situation.

The distinction of given information in a task and a possible request to make a prediction to future data by mental simulation yield a model of a hierarchy of tasks' complexity that we could back up in a study based on eight tasks that are based on the same principles of construction as the frog task (Eichler & Vogel, 2012):

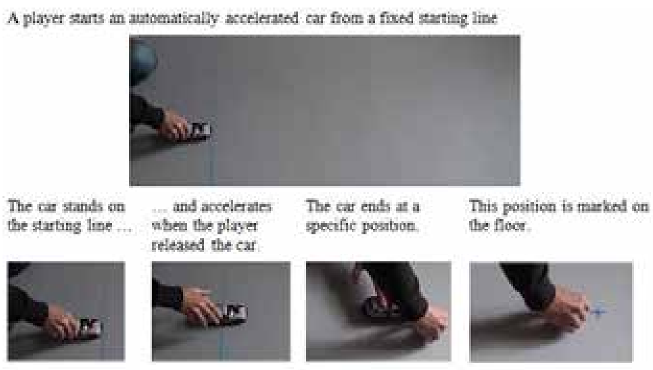
1. Lowest level: data given, human and non-human objects visible, mental simulation not requested (e.g. a task in which the results of coin throws have to be compared without making a prediction to future coin throws, see above);
2. Low level: data given, human and non-human objects visible, mental simulation requested (e.g. the frog task);
3. High level: data given, only non-human objects visible, mental simulation requested (this would be the case in the task concerning the weight of students, if it would be possible to develop a picture of the situation as described above);
4. Highest level: data not given, only non-human objects visible, mental simulation requested (e.g. a die task in which the students have to make a decision between two dice with different shapes to get a specific number).

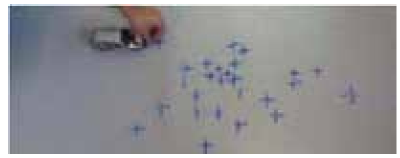
Although we finished the development of tasks with identifying the levels of tasks' complexity, and used them with a bigger sample of non-schooled students, there were other aspects of the tasks that could be taken into account to develop a valid diagnostic tool for investigating young students' mental modelling of statistical situations.

FURTHER TASK DEVELOPMENTS

Our findings gave evidence that disclosing the impact of a human object on the data generation complicate a task for the students. However, our set of tasks includes one counterexample in which the visibility of a human object hinders the students' understanding of the situation, i.e. the so called car task (figure 5). According to the results of the interview study the car task is the only task with which situation not all students were able to cope when they were asked to conduct the experiment shown in the task.

Some of the students' responses gave evidence that they make inferences (Schnotz & Bannert, 1999) of both, the formulation "an automatically accelerated car" and the pictures being obviously contradictory in respect to the intended meaning. In terms of mental model theory underlying the principles of our tasks' construction: For these students, the objects of the car task might be given (human impact: starter holding the car; non-human impact: car with a spiral spring that accelerates the car automatically) but their interplay might not be captured clearly enough by the description. For example, some students seem to overestimate the impact of the human object on the data generation: Some of the students' answers give reason for assuming that these students might have thought that the displayed hand causes at least partly accelerating because they cannot see the accelerating spiral spring inside the car. Thus, those students were not able to make inference from the text to one object (the spiral spring) that is highly relevant for the situation's structure and function.





Mark the position(s) where the next car (the next two cars) could stop. Give a rationale for your positions.

Figure 5: The car task

Based on these theoretical and empirical based considerations it is possible to change the situation's representation according to the underlying theory of mental models. Since the disclosed, but relevant non-human object (i.e. the spiral spring) could be the main reason for students' difficulties, we changed the situation's structure concerning the acceleration of the car: Instead of using an automatically accelerating car we use curves as accelerating distance of a toy car racing track (figure 6).

Figure 6: Use curves as accelerating distance of a toy car racing track in the car task

Thus, the assumed to be disturbing interplay between hand of the starter and car seems to be more clearly. Of course, this is a hypothesis which has to be proved.

According to our theoretical framework (founded by the theory of mental models and enriched by relevant aspects of theories of learning from multiple representations as well as of cognitive load theory) the principles of constructing new tasks in form of stochastics problem-storyboards could be applied concerning a considerable amount of existing tasks that potentially deal with centre and variation. Taking for example the question of design of paper helicopters (Ainley & Pratt, 2010) we could design a task, represent it comparably to the other tasks in a stochastics problem-storyboard and characterize it along our principles of task construction (figure 7).

situation

A player starts a paper helicopter several times beginning from a fixed starting height. Each time the helicopters' end position is marked on the floor.

request

"Mark the position where the next helicopter could land. Give a rationale for the marks."

Figure 7: Paper helicopter task represented in a stochastics problem-storyboard

CONCLUSION

In this paper we proposed principles of diagnostic tasks' construction aiming to investigate non-schooled young students' thinking within statistical situations. The aim of the paper was to discuss the interplay between theoretical models and empirical findings as basis for the tasks' development. The development of valid diagnostic tasks representing statistical situations is a crucial point of our on-going research program into young students' mental modelling within simple situations of decision making under uncertainty.

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USING APPLETS FOR TRAINING STATISTICS WITH FUTURE PRIMARY TEACHERS

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In recent years, statistics has been recognised as a basic component of citizenship education and its incorporation into school curricula in various countries confirms the importance of learning statistics. Nowadays, several resources are available online, but their uses within classes may vary depending on the use a teacher devises for them, therefore a critical analysis of these tools is needed. In this paper, we present the Applemat model for the analysis of statistical applets that might promote training statistics among primary teachers, thus emphasising their didactical purposes. During this past school year (2011/2012), an applets selection – as statistical teaching tools – was used for training primary teachers and the model was introduced in their classes. We will only present the model and the report from one of the student groups in this paper.

INTRODUCTION

Primary teachers (teachers of pupils aged from 6 to 12) must be prepared to teach statistics from the official programmes and adequately educate their students. Realistically, this would create the need to include statistics education in the curriculum of future teachers, including contents and statistical literacy, in order to allow the development of students as fully literate citizens. In our own work (Estrada, Batanero, & Fortuny, 2004), we have already found conceptual errors in elementary concepts such as mean, median and mode, outliers, standard deviation and sampling. This brought to our attention the need to rethink teaching methods. Given that statistics is one of the subjects where technologies has a major impact (Contreras, Martins, Estrada, & Batanero, 2011) we thought that we may use some of the available Internet resources – the applets – in our classes. It was thought that, due to their characteristics, they may enable us to develop a different approach to the statistical concepts. Thus, we agree with Anderson-Cook and Dorai-Raj (2003) who state: “We believe that the applets will be an easily accessible tool (…) to help students gain a better working understanding of the concepts.” In our previous work (Nascimento & Martins, 2008), we have already used statistical applets with Portuguese university students, as homework, in order to make them reflect on the statistical use of their (mis)concepts and help them gain a better understanding of how to work with them. Nevertheless, future primary teachers should have the opportunity to learn how to use applets as technological resources in statistical contexts, and attention should also be paid to the appropriate use of applets in the classroom.

In this work, we present the Applemat model version developed in the "Applemat Project" that enables future teachers to conduct a careful didactical analysis, with special emphasis on its utility in their future elementary classrooms. We also present an example of its use, devised by one group of students, as well as our own analysis of their work.

APPLETS AND TRAINING STATISTICS WITH FUTURE PRIMARY TEACHERS

In the professional development of teachers, ICT acts as a semiotic mediator that may change the epistemic configuration of the mathematics learning process (Font & Godino, 2006). However, Giménez (2004) states that teachers do not usually use these resources because they do not know their possibilities and limitations. The technological resources, namely the applets, also possess the conditions for a didactical suitability (Godino, Wilhelmi, & Bencomo (2005) later clarified by Godino, Batanero, & Font (2007) *apud* Godino, Batanero, Roa, & Wilhelmi, 2008) that defined it as the articulation of the following six types of suitability:

- Epistemic suitability: extent to which the statistical content is representative of the curricular content for a specific teaching level and whether its inclusion in the teaching is justified.
- Cognitive suitability: whether the content is adequate for the students' previous knowledge and the extent to which the instructional goals can be achieved.
- Media/resources suitability: sound use of technical tools, resources and time.
- Emotional suitability: whether the teaching/learning process takes into account the students' motivations, attitudes, affects and beliefs.
- Interactional suitability: whether the interactions between the teacher and the students and among the students themselves favour overcoming learning difficulties.
- Ecological suitability: degree to which the teaching/learning process is adapted to the social environment; possibility of establishing interdisciplinary connections.

Due to the availability of different statistical resources on the Internet, we think that teachers' training should introduce and promote the use of these ICT resources, specifically the applets, to help future teachers in recognising their value and applicability in elementary school classrooms. More generally, Tishkovskaya and Lancaster (2012) discuss the following:

> Probably, the most common way to use information technology to enhance teaching materials in mathematics and statistics has been to add statistics applet illustrations letting students experiment with mathematical statements. Some of these illustrations are very sophisticated and valuable new elements in instruction (…) which can be accessed over the Web and used for the purpose of statistics education.

Most of the statistical applets found are devised to show contents. But Romero, Berger, Healy, and Aberson (2000) describe that "In [WISE] earlier applets (…), students interacted by pushing on-screen buttons with their cursor. In [WISE] most recent applets, students are also able to act directly on the distributions. (…)".

Anderson-Cook and Dorai-Raj (2003) also reinforce that with the applets "[t]o answer the questions, students must interact with the applet, interpret and integrate findings, and explain and apply the concepts that they have learned".

More recently DePaolo (2010) for her courses used the "four A's for applet selection": "Appropriate, Accessible, Attributes of high quality, Appealing visually".

> 1. Applets had to (…) address topics and concepts commonly taught in these courses at introductory levels. They also had to have functionality (…); 2. Applets had to be accessible to undergraduate non-majors. They had to have an intuitive interface, explanation of purpose, and brief but helpful documentation on usage (…); 3. The applets' attributes had to be of high quality. They had to have reasonable loading and running speed and be free of errors and misleading output. Applets chosen were clear in their purpose and successfully performed their intended function or demonstrated their intended concept; 4. Applets had to be appealing visually. They had to have effective (though not necessarily fancy) graphics and produce output that is easy to understand and interpret.

In our view this "four A's applet selection" criterion (4AASC) adapted to the statistics curriculum of the prospective teachers or even of the elementary school one is a valuable guideline for choosing a "good" applet. Connecting with the didactical suitability components, the epistemic or mathematical suitability denotes the future teacher choice of a "good applet" using 4AASC adapted to the statistical contents of those elementary grades and also to the class kind of task for the pupils (either introductory, reinforcement or even test tasks). On the other hand, when the future teacher includes its adequate didactical use in the classroom work for that task we are referring to the cognitive suitability. "[T]eachers who are able to use today's technology in the classroom will be prepared to learn and utilise tomorrow's technology" (Powers & Blubaugh, 2005) stresses media/resources suitability. The emotional suitability, as pointed out by Díaz and de la Fuente (2005), determines that using applets for teaching statistics and probability increases students' motivation for the subject because they present the concepts in a more attractive way and the interactional suitability also enables them to play a more active role in their own learning. Taking these aspects into account, it is important that the teacher considers how to use these resources effectively – emphasising epistemic and cognitive suitability. So, using the applets, as well as doing their didactical analysis, may help future teachers in the statistics learning process – ecological suitability. In prospective teachers' training, the methodological techniques should be implemented to give them the incentive to incorporate different types of practices into their training. In this paper we will only present the Applemat model and the analysis of one report from a group of students where we articulate the criteria for the applet's selection, its assessment and its adequacy for didactical analysis based on the six types of didactical suitability presented earlier. Further work with the other students' reports will be presented in the near future.

METHOD

The first approach to the model involved an answer to the question of how future teachers should conduct the didactical analysis of the applets – as a teaching tool. This approach was developed within the "Applemat Project" (2006–2007), as part of the "Teacher Innovation Group" (TIG). The TIG group was coordinated by J. Giménez from 2006 to 2007 and received funds from the Catalonia government. The TIG working group had experienced university teachers from different areas – analysis, geometry, statistics, etc. – that felt the need to improve the ICT use in the university classrooms of prospective teachers. The model was an attempt to standardise the didactical analysis for all the scientific areas that the university students are required to know. In the two years, TIG elements viewed the components described as conditions for didactic suitability and developed the model with three sections. In the first section, the resource is described and its possible use in elementary classes is discussed. In this section, the weight of the six components is higher regarding the epistemic suitability and the cognitive suitability. In the second section, the applet is devised as a teacher training tool in order to manage and use it during teachers' training university classes, so the weight of the six components is distributed amongst them. Finally, in the third section, the possible improvements and extensions of the applet are proposed and the weight of the six components is also distributed amongst them. The Applemat model's details are presented in Table 1.

Section 1. Applet analysis and possibilities	Section 2. Management and use in class for teachers'	Section 3. Applet improvements and

	training	extensions
1.1. Applet description 1.2. Analysis and uses in primary class a. How may we use it in the class? b. Previous knowledge and contents required c. Limitations d. Techniques of use e. Other contents that may be incorporated into the class	2.1. Goals 2.2. Developed professional skills 2.3. Transfer of learning a. Initial problematic situation b. Previous knowledge reinforcement c. Using the applet	3.1. Applet improvements or extensions 3.2. Other learning possibilities 3.3. Alternative materials' advantages and disadvantages

Table 1: Applemat model details

In our research, this model was used in the Primary Teachers degree classes (3rd year) in a Spanish university 1st cycle (Bologna Degree) attending the Probability and Statistics Curricular Unit (course) in the school year of 2011/2012 (six credits from the European Credits Transfer and accumulation System, ECTS). In this course syllabus, besides the introduction to data analysis, probability and statistics contents, the didactics of probability and statistics are also present. The 123 students enrolled in this course were divided into two classes (58 and 65 students) and they presented a total of 32 works that were carried out in groups of three or four students.

Students using the Applemat model were not familiar with the didactical analysis of applets or any other ICT resources. Thus, the university teacher presented and subsequently practised the central tendency measures with the students, before using the Illuminations mean and median applet (retrieved from the NCTM Illuminations http://illuminations.nctm.org/LessonDetail.aspx?ID=L452) as the learning objectives, materials, the instructional plan with the task, the discussion and the questions for students were already available on the site. The students' task was then to search the Internet for different applets and select one to analyse using the Applemat model (Table 1). The basis for choosing their own applet must be one or both of the subjects: their course (at the university) syllabus topics, or the elementary curriculum topics on data analysis or probability. Once they had selected it, students worked as a team to apply the Applemat model. Finally, they had to present a written report of their work. This analysis was delivered at the end of the curricular unit and was assessed by the teacher, representing 25% of the final grade.

The assessment criteria are important in any course both for teachers and students, so the university teacher designed a first guideline evaluation document that included the 4AASC. The presented assessment criteria considered four groups: following instructions, presentation and style, statistical content and knowledge for elementary curriculum, and thinking and analysis. Firstly, following the instructions group assessed the adequacy of the applet choice for the goals of this course, the prescribed parameters of the task, and those of the model, and of the learned statistical topics. Secondly, for the presentation and style criteria the graded items were the clarity of the report text (including spelling, grammar and punctuation), its visual presentation (also considering its creativity and illustrations that aid understanding, integrated within the text), and finally the key aspects of its precision and rigour using statistical terms, expressions and concepts and their meanings. Thirdly, in what concerns the statistical content and knowledge of the Applemat model, the reports were analysed considering the depth and awareness of the statistical and didactical knowledge focusing on: applet description; possible classroom uses and applications; previous and actual statistical knowledge contents required; techniques of the applet uses and their limitations; other possible applications, improvements or extensions; and alternative materials. Finally, the thinking and didactical analysis of the reports assessed the integration of a critical perspective along the report, as well as the significance of the proposed recommendations.

Lastly, this school year of 2011/2012 was the first year in which the guideline evaluation document was used. As written by Biehler (2005) in the students' project reports scope we consider having a similar situation with our guideline: "Our work on analysing students' projects, developing a new project guide, and developing an assessment scheme is still in progress."

AN EXAMPLE USING THE MODEL

From the 32 groups of future primary teachers, a didactical analysis using the model (Table 1) was chosen and is presented here as an example of its implementation. This work was selected based on the applet's characteristics – available for download to our own computers – and its good grade. We only translated the main lines of the students' analysis regarding the "Tables and Statistical Graphs" Spanish applet report (http://www.edu.xunta.es/espazoAbalar/espazo/repositorio/cont/tablas-y-graficos-estadisticos), and now we present it.

Section 1. Applet analysis and possibilities

1.1. Applet description: This applet is an attractive visual resource that allows students to use elementary statistical techniques to obtain information about children's daily contexts including data representations, graphics and numbers, and also allows for critical reflection of the results. This applet also helps to describe, extract and interpret the information presented in the tables or in the graphs (…) which is based on daily problem solving. Furthermore, this applet has six folders. Each one is about one statistical chapter and we enter it by selecting one of the six persons (characters). It also contains a glossary and a guide for teachers. During its use, there is a voice that explains the activities and these explanations are also written in the applet window. 1.2. Analysis and uses in elementary classes: a. How may we use it in the class? (…) this applet has six folders and each one is about one statistical chapter: data techniques of collecting and classification (4 activities); building absolute frequency tables (4 activities) and relative frequency tables (3 activities); frequency polygons (4 activities); pie charts (5 activities); reading and interpretation of statistical concepts; b. Previous knowledge and contents required: reading double entry tables; reading coordinate axes; circular sector, including the concept of central angle measure; per cent, and the per cent value of a fraction; ability to use a calculator; c. Limitations: the applet does not have the possibility of error because if the answer is wrong, the student cannot continue the activity; the activities are not connected to each other; there is no way to change data; not all activities are for all elementary school levels; d. Techniques of use: choosing a correct answer; putting data into frequency tables; checking data; arithmetic operations like addition and subtraction; reading graphs; comparing graphs or data; completing a frequency table taking into account the added data; building a table from graph data; writing a legend; writing the title; e. Other contents that may be incorporated into the class: computations and measure.

Section 2. Management and use of class for teachers' training

2.1. Goals: explore the use of ICT in mathematics learning; learning to learn; development of manipulation and visualisation as a didactical procedure; (…) understanding of frequency tables; potentiate the study and the understanding of data graphs; development of the study and its interpretations using data; learn how to teach. 2.2. Developed professional skills: analysis skills; self-learning ability; critical thinking before the methodological procedures of learning; promote cooperative work; using, applying and creating manipulative resources to learn (…). 2.3. Transfer of learning: a. Initial problematic situation: if the user chooses chapter six – the baby with the bear – data information may have different graphic types, graphics may be compared or problems with statistical graphics may be proposed. (…) a chain of events is triggered when each character is chosen; b. Previous knowledge reinforcement: data techniques of collecting and classification; building absolute and relative frequency tables; frequency polygons; pie charts; reading and interpretation of statistical graphics; c. Using the applet: using interactive learning applets as a didactical resource; different ways of representing data; valuing different ways of data presentation, reading and interpretation.

Section 3. Applet improvements and extensions

3.1. Applet improvements or extensions: if the answer is wrong, the student cannot continue the activity, so this option should be changed and the student should have the opportunity to know what their mistake was. Since the activity folders are not connected to each other, some kind of connection may be implemented in the future; devise a way to allow data changing and provide a dynamic update of the graphic. 3.2. Other learning possibilities: in these students' work, another activity folder was proposed in order to connect all the contents in the other six folders already available; create activities in the folders in order to include some with percentages; include pictograms; some more activities to reinforce students' training. 3.3. Alternative materials' advantages and disadvantages: 1. Building tables and graphics with a worksheet; Advantages: All computers have worksheets. Enables the discovery of how this survey tool helps students' use of statistics in their daily lives as well as in other subjects. Allows building any graphics type; Disadvantages: Activities must be guided by the teacher, at least in the beginning, since the software is not particularly accessible to younger students. The difficulty levels of learning are not easily controlled by the teacher; 2. Motivating element, Advantages: Activities are based on contents and are linked between them; Disadvantages: If the connection between pages is lost it is impossible to finish the activity.

From our analysis of these students' work, we think that the group had a good report since the students followed instructions, i.e., the proposed Applemat model and the main 4AASC. In our view, Section 1 of the model report shows that the students used/explored the applet in order to understand it fully. The report had a simple and clear presentation and style. The statistical content and knowledge for the elementary curriculum was adequate and without wrong definitions. Lastly, the item about thinking and analysis was good in this report as the group was able to present a first critical opinion about this applet, as well as considerations that allow a better understanding of its use in the classroom. Sections 2.1 and 2.2 were weaker, nevertheless the students attempted to offer a glimpse of the goals and a lighter view of the professional development. The topics of subsection 2.3 were good in describing the transfer of learning. In relation to Section 3 of the model, the report was good as it detailed improvements and extensions of the applet. Overall, the group report was good using the Applemat model for the applet's didactical analysis.

FINAL REMARKS

The availability of Internet resources enables statistical learning to be accomplished in a different way, using these resources as dynamic promoters. In this work, we reviewed some of the perspectives concerning the applet's uses and we presented a model that provides guidelines for the applet's didactical analysis, within future primary teachers' training scope. In our own view, applets – as technological resources – and the Applemat model with its three sections articulated the six components for the didactic suitability, as defined by Godino et al. (2005) and later clarified by Godino et al. (2007, *apud* Godino et al., 2008). Concerning the epistemic or mathematical suitability, the Applemat model described was suitable for the future teachers to use/explore the statistical concepts, for instance, in the tables and graphs presented in the students' example. They made use of their statistical knowledge choosing and exploring the chosen applet with this model. In general, the "statistical content in (…) primary school level (…) can be introduced and justified through [an applet task] understood by the student" (Godino et al., 2008). With regards to the cognitive suitability, the Applemat model showed potentialities since the future teachers had to report the applet's adequate didactical use in the classroom. Again we stress that using the applets, as well as doing their didactical analysis, may help future teachers in the statistics learning process – ecological suitability. According to the media/resources suitability, a single computer and an Internet connection for each group of students will be enough. Also, students may suggest different alternative materials as they did in Section 3 of their report. Since the interactional suitability mainly depends on how the teacher organises their work in the classroom, the future teachers were required to work in groups in order to encourage conflicts and verbalise their occurrence. A drawback of this year use of the Applemat model was the shortness of time to promote the class discussions. In this case, the emotional suitability was viewed as the students' involvement (interest, motivation ...) in the study process by means of the Applemat model use. We consider that this was the most appropriate one and was the one that triggered the present work. Using the Applemat model, future primary teachers were able to value applets as a didactical resource based on their own observations and manipulations – with the detail given through the presented report of the group. To conclude, this school year of 2011/2012 was the first year in which the guideline evaluation document was used, so the assessment guideline is still in an improving stage. We are convinced that the university students' work should continue to be carried out in small groups to promote opinions and discussion that will potentiate their critical thinking, which is essential in the building of an "almost real" didactical proposal. Following our analysis of the work done we propose a four-step approach in the future teachers' classroom. Firstly, we will present them this year's work so they will have a basis from which to explore the model themselves. Secondly, they will do this year's work. Thirdly, after receiving the teacher's feedback, the students will discuss their own work in order to clarify their own concepts. Finally, in order to enhance the future teachers' didactical training for the elementary levels, they will prepare and simulate – for instance, for their colleagues – the didactical sequence in order to test it and improve it. With this approach, the applets will act as the semiotic mediator in order to change the statistics learning process, as already underlined by Font and Godino (2006). Although in the teacher analysis of this school year 2011/2012 almost all the reports got good grades, we intend to further develop their analysis in a qualitative and comparative way, detailing the differences between them, as well as any written conceptual (of probability or statistics) errors made. Finally, similarly to Contreras et al. (2011), we think that to promote an improvement of the statistical learning in elementary schools, teachers should take all these resources into consideration, particularly the applets. Therefore, these resources should be implemented in future teachers' training in a similar way to the methods outlined in the current paper.

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DESIGN AND EXPLORATORY EVALUATION OF A LEARNING TRAJECTORY LEADING TO DO RANDOMIZATION TESTS FACILITATED BY TINKERPLOTS

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We investigate the reasoning of preservice teachers about uncertainty in the context of randomization tests. Last winter term (2011/2012), we developed a seminar about data analysis with TinkerPlots, where group comparisons have played a fundamental role. A typical task in this context was: "Is there a difference between two groups or could that difference have occurred at random due to the selection of our sample?" In the first part of this article we describe a possible method to answer such a question with the help of a randomization test (see Rossman, 2008) facilitated with TinkerPlots. In the second part we point out how the participants of our course conducted a randomization test in a statistical project work and which problems and (mis-)conceptions were observed.

Key words: *Preservice teacher education, statistical education, group comparisons, randomization tests, TinkerPlots*

INTRODUCTION

The preservice education of teachers of mathematics at the University of Paderborn consists of three domains: mathematics, didactics of mathematics and pedagogy. An obligatory course called "Elementary Statistics and Probability Theory" is part of the programme. In addition the student teachers can participate in a seminar which deepens the course "Elementary Statistics and Probability Theory". The authors of this article have designed a seminar for preservice teachers in mathematics called "Developing statistical reasoning with using the software TinkerPlots" (Frischemeier & Biehler, 2012). In this course, the participants go through the whole PPDAC-cycle (Wild & Pfannkuch, 1999) which includes analysing data with the software TinkerPlots 2.0 (Konold & Miller, 2011) and writing down findings in a statistical report. Alongside describing and interpreting single distributions and exploring differences between them we wanted the participants to make conclusions about a wider universe and try to make generalizations of their findings. A typical task in connection with group comparisons was: "Is there a difference regarding a variable between two groups or could that difference have occurred at random due to the selection of our sample?" At the end, a statistical project work concluded the course. In the next part of this article we describe a possible method to answer such a question posed above with the help of a randomization test (see for example Rossman, 2008) facilitated with TinkerPlots. We focus on subject matter and knowledge of our students and do not discuss pedagogical content knowledge.

THEORETICAL FRAMEWORK

In general it is reported that pupils, students and pre-service teachers have a lot of misconceptions in testing hypotheses and using p-values (see e.g., Garfield and Ben-Zvi 2008, p.270). There we also get to know that most of the students have problems with questions concerning generalizing of a result found in a sample. An opportunity for emergent inferential reasoning, especially in connection with results from group comparisons, is a so called randomization test. For a detailed and formal introduction in randomization tests see Ernst (2004). Rossman (2008) recommends starting inferential reasoning with randomization tests. He introduces randomization test with the example "dolphin therapy". For details see Rossman (2008). A big advantage according to Rossman (2008) and an important argument for a first step into informal inferential reasoning via randomization tests is that "…this procedure for introducing introductory students to the reasoning process of statistical inference is that it makes clear the connection between the random assignment in the design of the study and the inference procedure" (p.10). Rossman (2008) further points out that a randomization test "…also helps to emphasize the interpretation of a p-value as the longterm proportion of times that a result at least as extreme as in the actual data would have occurred by chance alone under the null model" (p.10). Furthermore, Rossman points to the problem of the "final" conclusion and the possibility that the null model is correct (although this is "unlikely" given a small p-value). Cobb (2007) emphasizes that, with randomization tests, students in introductory courses have a better opportunity to understand the "core logic of inference" converse to an approach based on calculations from normal-based probability distributions. This was also proposed by R.A. Fisher "but […] was not realistic in his day due to the absence of computers". Cobb (2007) also emphasizes his 3R´s: Randomize data production, Repeat by simulation to see what´s typical (and what´s not) and Reject any model that puts your data in its tail. The randomization of the data production (Cobb´s first "R") is an important condition. However, we have decided to use randomization tests also in the context of observational studies comparing data to the hypothetical situation that a random assignment process has produced the data (see also Konold, 1994 for such an approach and Konold & Pollatsek, 2002 for the "process approach"). A bootstrap method would have been preferable if we deal with random samples, however, resampling with replacement seemed us too difficult for explaining it to students. If the data were produced by such a process (null hypothesis), it has to be judged whether a difference is likely to be due to random variation under the null model. A differentiation of methods and arguments for using a randomization test also with observational data can be found in Zieffler, Harring, & Long (2011). Rossman (2008) claims that teachers could use randomization tests to connect the randomness that students perceive in the process of collecting data to the inference to be drawn. He provides examples of how such a randomization-based approach might be implemented at tertiary level, while Scheaffer and Tabor (2008) propose such an approach for the secondary curriculum and provide relevant examples. Which misconceptions of students occur when doing a hypothesis test? Vallecillos (1994) report that many students who thought (like in a deductive process) that the correct application of a test with a significant result implies the truth of the alternative hypothesis. Another misconception regarding p-values is that the p-value is supposed to be the probability that the null hypothesis is true, given the observed data (Garfield & Ben-Zvi, 2008, p. 270). In the following section we describe a method for conducting randomization tests with TinkerPlots as an instrument to make further conclusions beyond the data and try to antagonize the misconceptions described above. We assume randomization tests to be an adequate method for this respect and see this as a refinement of group comparisons in concrete informal terms being aware that our approach cannot solve all the problems students have with hypothesis testing.

RANDOMIZATION TESTS FACILITATED BY TINKERPLOTS

The sampler of TinkerPlots 2.0 can be used as a useful tool for simulations. At first we will describe a typical task handed out to the participants working with the so-called Muffins data (Biehler, Kombrink, & Schweynoch, 2003), which is a complex data set with 538 cases and about fifty variables of a questionnaire concerning media use and leisure time of eleven graders. “How much time do the girls read more than the boys on average?”. Exploring the Muffins data reveals that the girls on average read approx. 0.82 hours per week more than the boys. This motivates the question: “Is there a difference regarding the variable “Time_reading” between boys and girls or could that difference be due to the selection of our sample?” The 538 students are neither a random sample of a clear-cut population, nor do we do random “treatment assignments” However, we can just imagine a process, where reading time is independent from gender. If the data were produced by such a process (null hypothesis), we have to judge whether a difference of 0.82 hours can be due to random variation under the assumption of no difference. We imagine that we divide the group of 533 students randomly into a group of 301 pseudo-females and 232 pseudo-males. Then we can calculate the mean difference of reading time in these two random subgroups. When we repeat this process many times, we can estimate the probability to get a mean difference greater or equal to 0.82 just by random group selections. The advantage of Tinkerplots is that such a random selection model can directly be implemented in the software. We formulate our null hypothesis “The data were produced by a process where there is no difference regarding the variable *time reading* between boys and girls.“ We estimate the probability that the difference between boys and girls is 0.82 hours or even higher under the assumption that the null hypothesis is true. This can be done in the following way by a simulation in TP - Figure 1 shows a screenshot with several steps of the randomization test in TP. We place 533 balls labelled with the 533 times of the variable “Time_reading” in our group of respondents in box 1 (see figure 1, upper-left corner, left urn) and construct another box 2 (see figure 1, upper-left corner, right urn) with 232 balls labelled with (pseudo-) “male” and 301 balls labelled with (pseudo-) “female”. In the next step a ball from each box is drawn without replacement. This is repeated 533 times (because of 533 cases): The 533 values of the variable “gender” are randomly assigned to the 533 values of the variable “Time_reading”.

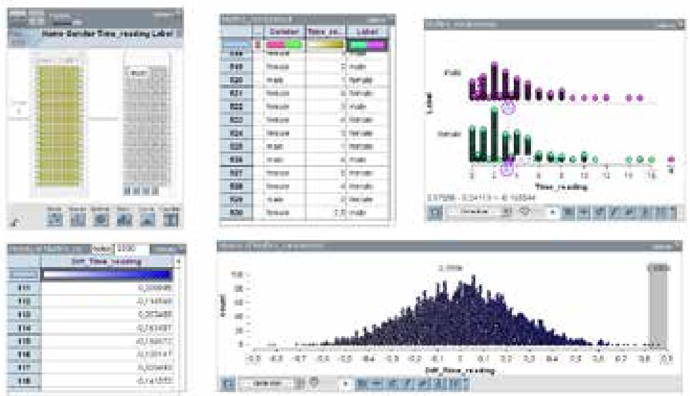

Figure 1: Screenshot of doing a randomization test with TinkerPlots

The results can be seen in the table and the plot “Muffins_randomised” (figure 1, middle and upper-right corner, see also the column “label” that contains the “pseudo-gender”). This whole process is repeated 5000 times and we collect the measure “difference of the means of the two groups” with the “History”-function in TinkerPlots. The table (see figure 1, bottom-left corner) contains the collected measures of 5000 simulated random assignments. In only three of the 5000 random assignments (see figure 1, bottom-right corner), the simulated result turns out to be as extreme or even more extreme than the observed difference in the Muffins data. With the divider-tool we can determine the number and the proportion of cases (measures) that are 0.82 hours and higher. This proportion is 0.0004, which is our p-value. The “result” of the randomization test is an estimated probability (estimated by relative frequencies) of 0.0004 that the difference between the means of the two groups equals 0.82 or is even higher under the assumption that there is no difference in the process producing gender and reading time. What can we conclude from these results? Due to this very small p-value there is a very strong evidence against the null hypothesis of no difference.

DESIGN OF THE LEARNING TRAJECTORY

In this paragraph we want to describe the learning trajectory we created for the introduction into randomization tests on the one hand and a so-called randomization test-plan we handed out to the participants while doing a randomization test on the other hand. The major goal was that the participants learn to perform randomization tests with TinkerPlots as we demonstrated in the paragraph above (although we have paid just few attention to teaching the logic of inference at this stage). At first the participants were introduced in the sampler of TinkerPlots by modelling and performing some simple chance-experiments. Our introduction into randomization tests began with the “Extrasensory perception (ESP)” - task (Rossman et al., 2001, pp. 376), which the students had to do on their own in teams of two. In the working phase the first and second author gave support and feedback when problems occurred. Afterwards the results were discussed in the whole group. In the next step, our intention was to draw parallels between the simulation of ESP-task and the performance of a randomization test regarding the Muffins task: "Is there a difference regarding the variable "Time_reading" between boys and girls or did that difference occur at random due to the selection of our sample?". The participants were handed out a randomization test-plan and then worked in groups of two on the muffins task described above. Before continuing with the description of the learning trajectory we want to give some arguments for handing out a randomization test-plan: Sweller (1999) have found out that the exploration of a complex learning trajectory such as the randomization test in our example, tasks the cognitive load of the learner very much. Due to the limitation of cognitive load he proposes to give the learner some help in form of structural aspects. With the randomization test-plan we give the participants a possibility to structure their thoughts and steps. With similar ideas to Biehler and Maxara (2007) we created the plan seen below (fig. 2).

Plan: How to do a Randomization Test

No	Step	ESP	Muffins
1	**Observation** Which difference do you observe between the means of the two groups in the dataset?	Number of correct answers = 20	*Mean of Time_Reading of boys = 2.685 Mean of Time_Reading of girls = 3.503 Difference = 0.818*
2	**Hypothesis H_0** As said in the task, the difference of the means of the two groups could have occurred at random. Generate an adequate N-hypothesis for your investigation.	The person does not have any extrasensory perception (ESP). He/she guesses with a success rate p = 0.25.	*The difference of the means of Time_Reading of boys and girls has occurred at random.*
3	**Simulation of H_0** How can you investigate the n-hypothesis with a simulation? Explain your procedure	Drawing 40times with replacement from an urn which is filled with 4 balls: 1 ball is labeled „r" (right) and 3 balls are labeled „f" (false)	*Place the 533 cases of Time_Reading in urn1. Construct urn2 with 232 balls labeled with "m" (male) and 301 balls labeled with "w" (female). Draw 533 times without replacement*
4	**Test statistic** Define the test statistic	X = Number of correct predictions	$X = \bar{x}_{Group1} - \bar{x}_{Group2}$
5	**p-value** Read of the p-value	P (X ≥ 20) = 0.0004 = 0.04%	*P (X ≥ 0.818) = 0.0006 = 0.06 %*
6	**Conclusions** Which conclusions can you make regarding your n-hypothesis?	The p-value (=0.0004) is very small, so we have a strong evidence against our n-hypothesis. We assume that the fortune teller has not guessed. Another possibility is: he could have guessed but that would have been very unlikely.	*The p-value (=0.0006) is very small. So we have a strong evidence against our hypothesis "The difference of the means of Time_Reading of boys and girls has occurred at random." Another possibility is: the difference occurred at random, but that is very unlikely.*

Figure 2: Randomization test-plan (with entries in "Muffins" – column, a task that our students had to do themselves)

In this "randomization test-plan", the participants have a structure for the simulation-process on the one hand and can write down their findings on the other hand. On the left side of the plan, the participants get an overview of the structure of the randomization test, short instructions and leading questions for each (the forth column is without entries when it is handed out to the participants). A special feature of the plan is the third column "ESP". The "ESP"-task is the exemplary task we used when introducing our students in randomization tests. To support the participants in step 6 to draw conclusions from p-values, we gave them further material in form of a hand-out which was supposed to give them hints how to evaluate possible p-values, as follows:

> Hand-out: *We have a very strong evidence against H_0, if $p < 0.1\%$. *We have a strong evidence against H_0, if $p < 1\%$. *We have a medium evidence against H_0, if $p < 5\%$. *We have a small evidence against H_0, if $p < 10\%$. (Hand-out for the participants)

So after doing the "ESP"-task on their own with feedback of the first and second author as we have described above, the participants were handed out the "Muffins"-task which they had to do in teams of two. The results were discussed in the whole group afterwards. In reflecting on our "randomization test" sessions we found that there were two neuralgic points when doing a randomization test: First the correct formulation of the null hypothesis, second an adequate conclusion drawing from the resulting p-value. Finally the participants had to do a randomization test in their statistical project work, which was a requirement for completing the course.

RESEARCH QUESTIONS

In this paper, we will focus on the final randomization tests in the students' statistical project reports. Three main research questions emerge: 1. How did the participants finally perform a randomization test with using of TinkerPlots in their project works? 2. Were they able to fulfil the six steps of the randomization-test plan? 3. At which stages/steps did problems (which?) occur?

DATA, PARTICIPANTS & METHODOLOGY

We have had a look at eleven statistical project reports that the participants had to do at the end of the course in teams of two. They were allowed to choose their own questions related to the data sets we provided. Due to this aspect it was possible that the level of difficulty differed from report to report and therefore from randomization test to randomization test. Doing at least one randomization test in their report was a requirement. 23 participants attended the course, 22 of them worked on the project reports in teams of two, so we have 11 project works (and 11 randomization tests) in total. The number of students' semesters varies from 4 to 11, most (11 from 23) students were in their fifth semester. For a deeper analysis, we analysed all written extracts of the statistical project reports that dealt with the randomization test-task while focussing on the successful execution of the six steps of the randomization test-plan and typical problems that occurred when going along these steps. So we have had a global view (cf. research question 1 and 2) and a local view (cf. research question 3) on the randomization tests of the project works. In the "global observation" we checked how well the participants performed the 6 steps of the simulation plan generally. If they accomplished a step as described in the example above, we called it "Step x successfully done". We analysed the several steps with the background of our theoretical framework. We defined categories of typical problems with a focus on step 5 & 6. Furthermore we will also give typical examples for the categories in form of written extracts of students´ project works.

RESULTS

Let us have a look (table 1) how well the teams did in the several steps when doing a randomization test with TinkerPlots (Note, that every team have had an different topic, so some of them were confronted with p-values larger than 10%, others with p-values smaller than 0.1%, for example).

Steps successfully done	**Number of teams (of 11)**	%
Step1	11	100.00
Step2	8	72.73
Step3	10	90.91
Step4	10	90.91
Step5	5	45.45
Step6	5	45.45

Table 1: Overview-Randomization tests in project works

Almost every team was able to conduct and fulfil the simulation of the randomization test with TinkerPlots in form of accomplishing steps 1,3 and 4. Step 2 (formulating null hypothesis), step 5 (identifying and reading of the p-value) and step 6 (drawing conclusions from p-value) seemed to be problematic points as we mentioned above. So we want to have a closer look on that now.

Step 1 – Reading of the difference between groups in the dataset

As seen in the table every team accomplished step 1 successfully.

Step 2 – Formulation of null hypothesis

Regarding step 2 we can say that the majority of the teams (8 of 11) gave a correct formulation of the null hypothesis when doing the randomization test. Two teams formulated the alternative hypothesis instead of the null hypothesis. For example when comparing the reading habits of boys and girls in the muffins data and investigating the variable "Time_reading" the hypothesis of one of the teams was:

H. & P.: The Girls tend to spend more time on reading than boys.

Two other teams (three teams in total) showed the same problem when formulating an adequate null hypothesis.

Step 3 & Step 4 – Modelling the simulation process in TinkerPlots

As seen in the table nearly every team (10 out of 11) managed to model the simulation process of the randomization test in TinkerPlots.

Step 5 – Reading of the p-value

A notable problem which occurred in step 5 was a false identification of p-value in form of the mean of the collected measures. Two out of eleven teams identified the mean of the measures as p-value. Let us have a look on the case of Laura & Sarah. They wrote, when investigating the hypothesis "the gender-difference on the means of the variable `Time_phone_20min` (Number of phone calls per week that last longer than 20 minutes) did happen by chance":

L. & S.: The mean of all means is approximately 0.000238873 after 5000 simulations und therefore smaller than 0.1%, which shows a strong evidence against the null hypothesis.

For them the mean of the 5000 collected differences is a very small value and seemed to turn out for them as a p-value.

Step 6 – Drawing conclusions from the observed p-value

In step 6 we found two phenomena: on the one hand "drawing premature conclusions from the p-value" such as "the p-value is smaller than 5%, therefore the null hypothesis can be rejected." and on the other hand "drawing false conclusions from an observed p-value". We will give an example for "premature conclusions" first. Alex and Kathrin concluded under the null hypothesis "the difference of the means of the variable "age" concerning the marital status of students is due to random effects":

A. & K.: The statement can be rejected with a p-value of 4% (which is smaller than 5%). Therefore the null hypothesis […] can be rejected.

We consider this as premature because we taught the students not to take a definite decision but express the uncertainty when a small p-value occurs as an amount of evidence. A more adequate statement in this case could be (relating to our hand-out which was supposed to support the learners when drawing conclusions from p-values): "Due to a p-value of 4% we have medium evidence against the null hypothesis". An example of drawing false conclusions from the observed p-value is the following: The null hypothesis is seen as true, because the p-value is significantly high (> 10%). Victoria and Corinna were investigating whether a gender-difference of time spent on working (in hours per week) occurred at random and concluded:

V. & C.: “The result of the randomization test shown here (0.1033) is a probability. Here we have a relative frequency of 0.1033 or 10.33%. This value corresponds to our p-value. [...] The p-value is bigger than 10% which means that the evidence is not so strong and therefore the null hypothesis must be true. “

They made a typical mistake, which is also reported in Garfield & Ben-Zvi (2008, p.270). Having a “large” p-value, they concluded, that the null hypothesis must be true. It is noticeable that this problem occurred at every (precisely: 3 out of 11) team, who conducted a randomization test in which the p-value turned out to be larger than 10%. The problem may have occurred due to paying not enough attention to a p-value larger than 10% in our learning trajectory. We can conclude that the participants have several problems to make conclusions on their own. Summary: two of eleven teams have identified the p-value as the mean of the collected measures. Four out of eleven teams rejected their null-hypothesis (particularly due to a small p-value) in form of “drawing premature conclusions from a given p-value”. Three of eleven teams fell into the category “drawing false conclusions from a given p-value” concluded that the null hypothesis must be true, because of a large p-value (> 10%). All in all we can say that almost every team was able to deal with the technical process of the simulation in TinkerPlots, but they had partly problems with steps 5 and 6. They have acquired procedural knowledge of performing randomization tests in TinkerPlots, but some still fail to formulate an adequate null hypothesis or to identify a p-value or fail to draw adequate conclusions from it. When evaluating a p-value in the project reports, the participants seem to have the attitude either to accept or to reject a null hypothesis, instead of saying something like “…there is a small/medium/strong/very strong evidence against the null hypothesis”. Living with uncertainty obviously is something uncomfortable.

LIMITATIONS AND FURTHER RESEARCH

For evaluating the learning trajectory in the sense of design research approaches (see Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) we plan a retrospective analysis and a redesign of the learning trajectory. Our aim was to give our students a method which allows them to make further conclusions from observational data. When using this method and conducting a randomization test, two crucial aspects arose: on the one hand the formulation of a correct null hypothesis, on the other hand drawing conclusions from a given p-value. Perhaps there is a need to revise our learning trajectory in order to improve students’ conceptual understanding of these aspects of randomization tests: how can they be better supported at major problems (for example the formulation of a correct null hypothesis; drawing conclusions from a given p-value)? We are currently (end of summer term 2012) conducting a qualitative interview study with the same participants, which is two-phased. In phase 1 they have to work on a group comparison (including a randomization test) exercise in teams of two. In phase 2 we interview them in form of a stimulated recall-method to elicit the thoughts and strategies of them when working on the task. This will hopefully give further insights into the cognitive processes of the students while working with randomization test with TinkerPlots. The interviews are informed by our previous analyses of project reports that we presented in this paper and are also directed towards the levels of conceptual understanding of the null model that they implemented with TinkerPlots sampler. The difficulties we observed with formulating null hypotheses may have a deeper origin in understanding a "null model" of no difference in a process approach with no random treatment assignments in the data production process.

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CONCEPTUALIZING AND ASSESSING SECONDARY MATHEMATICS TEACHERS' PROFESSIONAL COMPETENCIES FOR EFFECTIVE TEACHING OF VARIABILITY-RELATED IDEAS

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The importance of statistics education in secondary school has been emphasized in numerous mathematics curriculum reforms carried out recently in many countries, it being noticeable that variability may arise within all the statistical objects studied in such curricula. Despite this, there have been few attempts to conceptualize or assess empirically teachers' professional competencies (sensu Döhrmann, Kaiser & Blömeke, 2012) for teaching variability-related ideas. This article introduces a conceptual framework for examining teachers' statistical knowledge for teaching alongside teachers' beliefs and conceptions of variability, as well as a survey instrument developed based on it. Preliminary results of an ongoing exploratory study are reported, and implications for teaching and teacher training are discussed.

Keywords: Teachers' professional competencies, statistical knowledge for teaching, teachers' beliefs, teachers' conceptions of variability.

INTRODUCTION

In recent years, curricular reforms in many countries have brought into the secondary school mathematics curriculum topics related to statistics (e.g., NCTM, 2000), aiming towards statistical literacy. It is noticeable that variability—a property of an statistical object which accounts for its propensity to vary or change, which is considered by several researchers as a fundamental concept in statistics (e.g., Shaughnessy, 2007)—may arise in many different ways in such topics. Therefore, nowadays secondary mathematics teachers must instruct several variability-related ideas—such as the one of distribution, since through the lens of this idea statisticians examine data variability (cf. Pfannkuch & Ben-Zvi, 2011, p.326)—, and such work demands from them specific professional competencies, without which the aims of the mathematics curriculum regarding statistics education cannot be achieved.

Döhrmann, Kaiser and Blömeke (2012) point out that "successful teaching depends on professional knowledge and teacher beliefs" (ibid., p. 327), and, with this in mind, they framed mathematics teachers' professional competencies in terms of cognitive and affective-motivational facets (cf. Figure 1). In such framework—which is the theoretical basis of the international study Teacher Education and Development Study in Mathematics (TEDS-M)—, Döhrmann and her colleagues highlighted *subject matter knowledge* (SMK) and *pedagogical content*

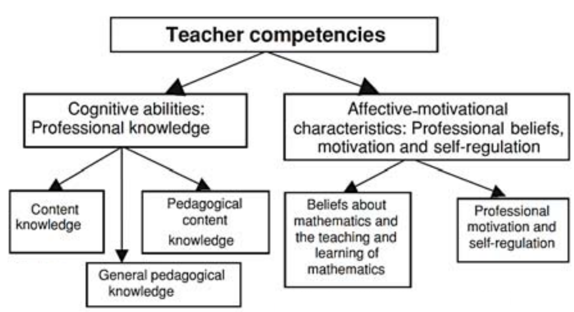


Figure 1: Conceptual model of teachers' professional competencies, according to Döhrmann et al. (2012)

knowledge (PCK) in the cognitive facet, and teachers' professional beliefs in the affective-motivational facet, as fundamental criteria for effective teacher education.

In the case of statistics education, scarce studies can be found in the literature focused on both the SMK and PCK entailed by teaching variability-related contents to help students achieve the aims of statistics education (cf. Shaughnessy, 2007), as well as on the beliefs held by in-service teachers on statistics teaching and learning of such contents. Hence, it is by no means surprising the urgent call for increasing research on these areas made by a number of concerned researchers, particularly for studies on teachers' professional knowledge and practices while teaching variability (e.g., Sánchez, da Silva & Coutinho, 2011, p.219), as well as for teachers' beliefs on statistics itself and on what aspects of statistics should be taught in schools and how (e.g., Pierce & Chick, 2011, p.160). Accordingly, the purpose of this paper is to respond to such calls by proposing a conceptual framework for secondary teachers' professional competencies to teach variability-related contents, which integrates *statistical knowledge for teaching*—henceforth SKT, the knowledge, skills, and habits of mind needed to carry out effectively the work of teaching statistics—, conceptions of variability, and statistics-related beliefs, aiming to identify indicators that could serve to examine such competencies, to help get a clearer picture about the level of competence to teach variability-related contents attained by secondary mathematics teachers, and also about the existence of any competence-related deficiency that might need to be improved.

THE MKT MODEL

Ball, Thames and Phelps (2008) developed the notion of *mathematical knowledge for teaching*—henceforth MKT—focusing on both what teachers do as they teach mathematics, and what knowledge and skills teachers need in order to be able to teach mathematics effectively. This model describes MKT as being made up of two domains—SMK and PCK—, each of them structured in a tripartite form (cf. Figure 2). Moreover, this model clarified the distinction between SMK and PCK, and refined their previous conceptualizations in the literature.

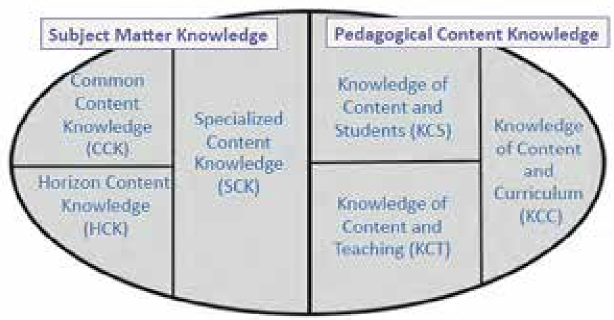


Figure 2: Domains of MKT (Ball et al., 2008)

According to Ball et al. (2008), SMK can be divided into common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge (HCK). Furthermore, Ball and her colleagues presented a refined division of PCK, comprised by knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC) (the interested reader should refer to the original article for a detailed discussion of these constructs).

Through this model, Ball and her colleagues made significant progress in identifying the relationship between teacher knowledge and student achievement in mathematics. However, as highlighted by some researchers (e.g., Petrou & Goulding, 2011, p.16), the MKT model does not acknowledge the role of either beliefs or conceptions about the subject matter in teachers' practices, which could be a drawback, since it is well documented in the literature that beliefs and conceptions are important factors affecting the work of teaching (cf. Philipp, 2007).

CONCEPTUALIZING TEACHERS' PROFESSIONAL COMPETENCIES FOR EFFECTIVE TEACHING OF VARIABILITY-RELATED TOPICS

While several models have been developed in the literature aiming to conceptualize MKT, few have been done on SKT. All those few conceptualizations of SKT proposed to date are cognitive-oriented models that have assimilated some of the categories present in the aforementioned model for MKT (cf. Groth, 2007; Burgess, 2011; Noll, 2011), taking into account neither all the six components identified by Ball et al. (2008) and the role of beliefs in teachers' professional practice, nor the conceptions of variability held by the teachers, which could result in an inadequate picture of teachers' preparedness to teach statistical contents related to variability.

In an effort to fill such gaps, a conceptual model for secondary mathematics teachers' professional competencies to teach variability-related contents is proposed. This model is a two-faceted one: it includes a cognitive as well as an affective facet. The cognitive facet is a sixfold conceptualization of SKT, comprised by all the knowledge subdomains identified by Ball et al. (2008) in their MKT model, with the construct CCK—defined as the mathematical knowledge and skills expected of any well-educated adult—being adapted to meet the case of teaching statistics. In this regard, statistical literacy will be seen as CCK, since the acquisition of its related skills—e.g., identifying examples or instances of a statistical concept; describing graphs, distributions, and relationships; rephrasing or translating statistical findings, acknowledging the omnipresence of variability in any statistical context, or interpreting the results of a statistical procedure—is expected from any individual after completing school education (cf. Gal, 2004; Pfannkuch & Ben-Zvi, 2011).

The affective facet of the model proposed in this article is comprised by two components: teachers' beliefs about statistics teaching and learning, and teachers' conceptions of variability, since both beliefs—defined by Philipp (2007, p. 259) as "psychologically held understandings, premises, or prepositions about the world that are thought to be true"—and conceptions—the set of internal representations and the corresponding associations that a concept evokes in the individual, often explained in the literature as "conscious beliefs"—, have been widely regarded in the literature as factors influencing every aspect of teaching (cf. Philipp, 2007). A detailed discussion of the conjectures that informed the development of this conceptualization can be found in González (2012).

ASSESSING TEACHERS' PROFESSIONAL COMPETENCIES FOR EFFECTIVE TEACHING OF VARIABILITY-RELATED TOPICS

The Survey Instrument.

Based on the conceptual model previously outlined, a pen-and-paper instrument, designed to be completed in one hour and comprised by tasks addressing variability-related concepts present in the secondary school mathematics curriculum, was developed, in order to elicit and assess each one of the eight components of teachers' competencies to teach variability-related contents identified by this study. Each question in the instrument was developed based on previous studies with similar aims reported in the literature (e.g., Ball et al., 2008; Isoda & González, 2012).

In order to provide a comprehensive framework for conceptualizing the cognitive aspects of teachers' competencies in the context of teaching variability-related ideas, twelve indicators were identified from the literature and selected for assessing SKT from teachers' answers to the designed instrument (see Table 1).

A: Indicators associated to Statistical Literacy (CCK):

1. Is the teacher able to give an appropriate and correct answer to the given task?
2. Does the teacher consistently identify and acknowledge variability and correctly interpret its meaning in the context of the given task?

B: Indicators associated to SCK:

1. Does the teacher show evidence of ability to determine the accuracy of common and non-standard arguments, methods and solutions that could be provided on a single question/task by students (especially while recognizing whether a student's answer is right or not)?
2. Does the teacher show evidence of ability to analyze right and wrong solutions that could be given by students, by providing explanations about what reasoning and/or mathematical/statistical steps likely produced such responses, and why, in a clear, accurate and appropriate way?

C: Indicators associated to HCK:

1. Does the teacher show evidence of having ability to identify whether a student comment or response is mathematically/statistically interesting or significant?
2. Is the teacher able to identify the mathematically/statistically significant notions that underlie and overlie the statistical ideas involved in the given task?

D: Indicators associated to KCS:

1. Is the teacher able to anticipate students' common responses, difficulties and misconceptions on the given task?
2. Does the teacher show evidence of knowing the most likely reasons for students' responses, misconceptions and difficulties in relation to the statistical ideas involved in the given task?

E: Indicators associated to KCT:

1. In design of teaching, does the teacher show evidence of knowing what tasks, activities and strategies could be used to set up a productive whole-class discussion aimed at developing students' understanding of the key statistical ideas involved in the given task, instead of focusing just in computation methods or general calculation techniques?
2. Does the teacher show evidence of knowing how to sequence such tasks, activities and strategies, in order to develop students' understanding of the key statistical ideas involved in the given task?

F: Indicators associated to KCC:

1. Does the teacher show evidence of knowing at what grade levels and content areas students are typically taught about the statistical ideas involved in the given task?
2. Does the designed lesson (or series of lessons) show evidence of teacher's understanding and support of the educational goals and the intentions of the official curriculum documents in relation to the teaching of the statistical contents present in the given problem, as well as statistics in general?

Table 1: Set of indicators proposed to assess SKT through the answers to the survey items

In regard to the affective facet of the conceptual model proposed here, the conceptions of variability that might be distinguished in teachers' answers will be classified using the eight types of such conceptions identified by Shaughnessy (2007, pp. 984-985). In the case of teachers' beliefs about statistics teaching and learning, the limited research on this issue (e.g., Pierce & Chick, 2011, p.159) suggests that they could be identified through examining the features of the lesson plans that teachers produce, such as the tasks chosen to consider a particular statistical idea, and the types of instructional strategies teachers planned to use during the lesson. What teachers planned to do—which is related to the construct KCT—will be analyzed using the four categories reflecting on teachers' beliefs developed by Eichler (2008)—i.e., traditionalists, application preparers, everyday life preparers, and structuralists—, which will provide valuable information on teachers' beliefs about the nature of statistics, as well as about the learning of statistics (cf. Tatto et al., 2012, pp.154–156).

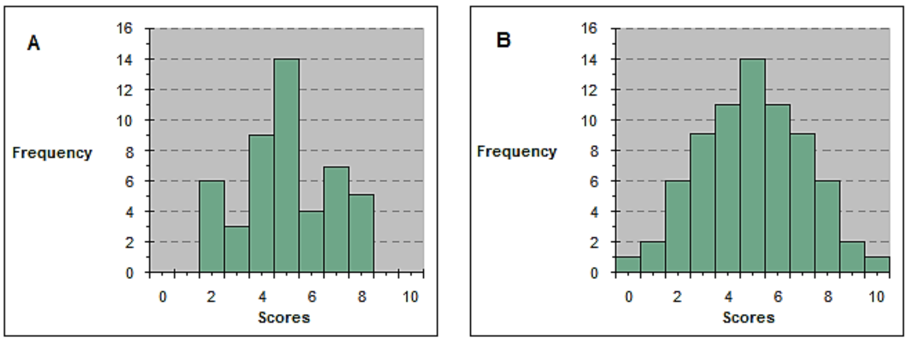
Profile of Item 1.

In a first stage of this study, a survey instrument comprised of one item—Item 1, which is depicted in Figure 3, and deals with several ideas of descriptive statistics—was designed, and then sent by postal mail to three public secondary schools in Hiroshima Prefecture, Japan. The fact that the majority of the statistical contents present in the Japanese mathematics curriculum are ideas related to descriptive statistics was crucial in the selection of the task in Item 1. Two more stages of this study are planned in the future, each of them using a one-itemed questionnaire dealing with the ideas of probability and sampling, respectively.

ITEM 1

Please, read carefully the following task and answer the questions below:

Choosing the distribution with more variability. Look at the histograms of the following two distributions:



Which distribution (A or B) do you think has more variability? Briefly describe why you think this.

(a) Answer this task in as many different ways as you can. Please, be sure to show every step of your solution process.

(b) What are the important ideas and concepts that might be used to answer this task?

(c) Suppose that, after posing this task to your students, three of them come up with the following answers:

Student 1: "Distribution A has more variability because it's not symmetrical."

Student 2: "Distribution A ranges from 3 to 14, while Distribution B ranges from 1 to 14. Then, Distribution B has more variability."

Student 3: "The bars in Distribution A are clumped closer to the central bar than they are in Distribution B. Then, Distribution B has more variability."

Dealing with each student separately, please comment briefly on each of these answers, focusing on whether the answer is correct or not, why you think so, and what reasoning might have led students to produce each answer.

(d) Suppose you pose this task to your students. What are the most likely responses (correct and incorrect), misconceptions and difficulties you would expect from them? Briefly explain why you think so. (Regarding to the most likely answers that you might get from the students, please do not include those mentioned in part (c).)

(e) Mathematically/statistically speaking, is any of the answers given by the students interesting or significant? If yes, briefly explain why and on what aspects. (Please, focus your response on whether there is a significant mathematical/statistical insight in the student's answer, and whether there are forthcoming contents in future classroom subjects connected to the notions/concepts being said or implied in such answer.)

(f) Briefly describe how the important ideas and concepts involved in the solving process of the given task are addressed in official curriculum documents across the different grade levels of schooling.

(g) Suppose you want to plan a lesson (or a series of lessons) to introduce the meaning of variability in the context of the given problem to your students. Briefly describe as many instructional strategies, activities and/or tasks as you can think of that would be appropriate to use for this purpose, and sequence them accordingly, explaining why you chose to put them in such a particular order.

Figure 3: Item 1 – "Choosing the distribution with more variability" task

The original version of the task in Item 1—developed by Garfield, delMas and Chance (1999), and reported in the literature as an effective means of investigating teachers' conceptions of variability in the context of histograms (e.g., Isoda & González, 2012)—was modified to facilitate the calculations that could be made while the respondent gives answer to the task, and was also enriched with questions aiming to elicit all the facets of teachers' professional competencies to teach variability-related contents identified by this framework. A mapping between the components of SKT that would be elicited by each question in Item 1, as well as the indicators associated to each cognitive aspect considered by this framework, can be appreciated in Table 2.

The context of the task posed in Item 1—comparing distributions—requires from teachers the mastery of several variability-related concepts—e.g., distribution, measures of variation, frequency distribution table, and histogram. Therefore, the task in Item 1 was selected in order to see, among others, (a) whether by looking at the histograms of two distributions of scores, teachers could figure out which one has more variability, and then use data-based arguments to defend their answer; and (b) how the respondents conceptualize variability in the context of the given task.

Elicited Knowledge Component of SKT	Associated Indicator of SKT	Question
Statistical Literacy (as **CCK**)	A1	(a)
	A2	(a)
Specialized Content Knowledge (**SCK**)	B1	(c)
	B2	(c)
Horizon Content Knowledge (**HCK**)	C1	(e)
	C2	(b)
Knowledge of Content and Students (**KCS**)	D1	(d)
	D2	(d)
Knowledge of Content and Teaching (**KCT**)	E1	(g)
	E2	(g)
Knowledge of Content and Curriculum (**KCC**)	F1	(f)
	F2	(g)

Table 2: Knowledge components of SKT elicited by each of the questions posed in Item 1

About the Participants in this Case Study.

At the time of writing this article (September 2012), the survey process—which began in July 2012 in Hiroshima Prefecture, Japan—was ongoing, and expected to be completed by the end of September 2012. In this paper, a preliminary analysis of the data gathered from one of the schools participating in this study, comprised of the written responses to Questions (a) and (g) on Item 1 given by four senior high school mathematics teachers working in such school, will be reported. The respondents were between 28 and 56 years old; they had between one and thirty-four years of teaching experience—with three of them with at least 13—, and were the first group of teachers that voluntarily and anonymously responded and mailed back the survey booklets.

Results and Findings regarding Question (a).

Three out of four teachers answered this question. From those who answered, two teachers—Teachers 1 and 2—used three different approaches: Teacher 1 answered the task by comparing the range, variance and interquartile range of both distributions; while Teacher 2 answered the task by comparing the range, the shape, and the mean absolute difference from the mean of both distributions. Teacher 4 answered using only one approach: by comparing the largest data span from the mean in both distributions.

It is quite surprising that all these teachers made computation errors in every approach that involved calculations. Among all the calculation errors done by them, one was recurrent: although both Teacher 1 and Teacher 2 identified correctly Distribution B as the one with more variability via comparing the ranges, when computing them they used as minimum and maximum values 2 and 8 in Distribution A, and 0 and 10 in Distribution B, respectively; that is, they used the class marks instead of the lower and upper class limits, which are 1.5 and 8.5 for Distribution A, and are –0.5 and 10.5 for Distribution B. This might be understood as an indication that these teachers considered the variable represented in both histograms as a discrete one.

In a similar way that Teacher 1 and Teacher 2 calculated the distribution ranges, Teacher 4 calculated the largest data span from the mean in both distributions. This teacher mistakenly argued that the largest data span from the mean is ± 3 and ± 5 units for Distribution A and B, respectively. Once again, it is noticeable that this teacher used for his calculations the class marks instead of the lower and upper class limits, possibly considering the variable represented in both histograms as a discrete one.

Regarding the conceptions of variability held by the respondents, the answer given by Teacher 4 indicates he holds the fifth conception of variability identified by Shaughnessy (2007)—"Variability as distance or difference from some fixed point"—, since this teacher points out deviations of the endpoints from some fixed value, such as the mean, when asked to consider the variability in both histograms. Reflecting a rather sophisticated recognition of variability, and at an even higher level, are the answers given by Teacher 1 and Teacher 2, which suggest that they hold the eight conception of variability identified by Shaughnessy (2007)—"Variation as distribution"—, since these teachers were able to use theoretical properties of the histograms to calculate numerically—although mistakenly—the measures of variation associated to each distribution in order to make their decision, exhibiting in such way an aggregate view of data and distribution, since they seem to be predominantly concerned with the variability of an entire data distribution from a center (cf. Shaughnessy, 2007, p.985). Unlike Teachers 1 and 2, Teacher 4 does not exhibit an aggregate view of data and distribution, since he is rather concerned with the variability of just the endpoints from a measure of central tendency.

Results and Findings regarding Question (g).

The purpose of this question is to elicit evidence of the indicators associated to KCT outlined in Table 1—namely E1 and E2. In order to determine the presence of such indicators in teachers' answers, a criterion-referenced assessment rubric was designed, based on the characteristics of effective classroom activities to promote students' understanding of variability compiled by Garfield and Ben-Zvi (2008).

All the four teachers answered this question (cf. Figure 4). In relation to Indicator E1, the answers given by Teacher 1 and Teacher 4 are the ones that seem to exhibit a higher level of knowledge on the key characteristics of effective activities that promote students' understanding of variability identified by Garfield and Ben-Zvi (2008), such as the implementation of tasks involving comparisons of data sets, aiming towards describing and representing variability with numerical measures when looking at the given data, and promoting whole-class discussions on how measures of central tendency and variation are revealed in data sets or graphical representations of data (ibid., pp. 207-209).

Regarding Indicator E2, the answers given by Teacher 1 and Teacher 4 are also the ones that seem to evidence more knowledge on how to sequence activities and strategies intended to promote students' understanding of variability. For example, in both answers is explicitly stated that the lesson must start by presenting students with some simple data, in order to interpret it (Garfield & Ben-Zvi, 2008, pp.135-137). However, Teacher 4's answer is at an even higher level compared to the others, since explicitly states that variability should be described and compared informally at first—e.g., by describing verbally how the data is spread out—, and then formally, through measures of variation (cf. ibid., p.208).

Teacher 1	Teacher 2
Task:「Among 2 distributions, which one do you think has more variability?」 **Activities:** ① Check different ways (range, variance, standard deviation, interquartile range) for examining variability. ② Place students in groups, asking to each group to use only one of the methods in ① to discuss about what things could be told about the variability of the given distributions. ③ Each group will share with the rest of the classroom what they considered in ②. ④ Depending on the method used, and while checking different considerations, think about how to look at variability. Students will experience personally the need of using several methods and finding out the appropriate one in order to consider data trends.	1. To make students think about which of 2 given histograms, A and B, has more variability. 2. To make students think about whether they can make their decision based only on the sample size. 3. To judge the variability using the variance. 4. To give practice problems to students.

Teacher 3	Teacher 4
In mathematics there are a large number of approaches in many directions concerning "variability": Introduction of the formulas related to variability. Studying variability through the use of computer technology. Based on the aforementioned approaches, bring up for discussion various topics in society and the corporate world, such as product development, among others, as well as their connections with practical applications.	– Give 2 histograms, A and B. – To make students think about in which histogram the variability is larger, and to make them expose about what they think. …At this stage, a detailed explanation about "variability" has not yet been provided. – After their presentations, to explain about "variability", and to make students think again about which histogram has more variability. – To explain, among other things, different terms besides "variability", provide different histograms, and practice.

Figure 4: Translation of answers to Question (g) given by the four surveyed teachers

In relation to the beliefs about instruction of statistics held by the surveyed teachers, the answers given by three of them—Teachers 1, 2 and 4—provide evidence that they might be *traditionalists*—i.e., teachers more concerned about students gaining algorithmic skills, and less about context and applications—, with only one teacher—Teacher 3—providing evidence of being an *application preparer*—i.e., a teacher focused on teaching theory and algorithms, so the students could use them to solve real-world problems. Moreover, it seems that Teachers 1, 2 and 4 see statistics as a process of inquiry; that is, as a means of answering questions and solving problems. Furthermore, Teachers 1 and 4 planned lessons in which they encourage students to find their own solutions to statistical problems, while fostering the development of statistical discourse and argumentation in the classroom, which provide evidence that they might believe that statistics learning should be carried out as active learning. The answers given by Teachers 2 and 3 give evidence they might believe that statistics learning is a teacher-centered individual work.

CONCLUSIONS

Based on teachers' performance in Question (a) of Item 1, some answer tendencies were identified; for example, calculating particular measures of variation considering the variable represented in both histograms as a discrete one. Only one teacher mistakenly used the shape of the histograms to answer, interpreting the variability in the given histograms as the differences in the heights of the bars, which is a common misconception in this kind of problems (cf. Meletiou & Lee, 2003; Isoda & González, 2012). Despite of this, evidence of two teachers in this group exhibiting an aggregate view of data and distribution—i.e., holding the conception of variability known as “Variation as distribution”—is noteworthy.

Regarding teachers’ performance in Question (g) of Item 1, the one of Teachers 1 and 4 stands out from the others. Some of the characteristics identified in their answers are consistent with those of effective classroom activities to promote students’ understanding of variability made by the specialists (cf. Garfield & Ben-Zvi, 2008). Nevertheless, except for the answer given by Teacher 3, all the lessons planned by the respondents lack consideration of an explicit daily-life context, which is vital to internalize in the students that statistics helps solve everyday problems and tasks (cf. Gattuso & Ottaviani, 2011, pp.122-123, 129). Due to this fact, the majority of the surveyed teachers might be considered as traditionalists.

The fact that teachers’ answers showed, among others, a lack of knowledge about how to relate the given task to different data representations, such as boxplots and frequency distribution tables, and due to the importance of making an appropriate interpretation of variability for statistics, courses where Japanese senior high school mathematics teachers could learn more about developing intuitive ideas of variability and the interrelationship among variability-related concepts; describing and representing variability; using variability to make comparisons; being able to map the characteristics of a given histogram to alternate representations; and so on, could be required.

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BRIDGING POLICY DEBATES ON RISK ASSESSMENTS AND MATHEMATICAL LITERACY

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The aim of this paper is to generate discussions on why to develop mathematical literacy and critical thinking related to risk assessments and how. The paper presents ideas and key points from the academic literature on science for policy related to uncertainty and risk assessments. These are elaborated and exemplified. I further show that these match aspects of mathematical literacy and related literature on research in mathematics education. Perspectives from the two academic communities are then combined, resulting in a framework for increasing competences useful for public participation. The theoretical foundation of the framework constitutes the starting point for a series of future research questions and research projects.

Key words: risk assessments, societal issues, critical democratic competence, mathematical modelling, statistical literacy

INTRODUCTION

We are facing tremendous challenges in society, covering scarcity on food and energy supplies and pollution of various kinds and sources. Science is given an important role in finding solutions to these problems, yet 'facts' are occasionally disputed with scientific advice at the centre of debate. The academic literature on science for policy has responded to this situation, and increased attention is given to the handling of uncertainty in science for policy in situations characterised by high stakes and diverging interests. Experts' common handling and communication of uncertainty through quantitative measures, as for example risk assessments, probability distributions, error bars or safety factors, are argued to be insufficient for sound decision making in such situations. In this paper, I will focus on risk assessments, often defined as the probability of an unfortunate event multiplied with its quantified impact.

From a democratic viewpoint, challenging policy issues benefit from an engaged public. Facts are often presented in numbers and thus require mathematical literacy. In the next section of the paper, I present some background information and research on mathematical literacy that is linked to active citizens and critical thinking. The proceeding section provides examples where risk assessments have been disputed. These are briefly discussed, demonstrating some key points from the academic literature on uncertainty in science for policy. The paper finally presents a framework for increasing mathematical literacy and critical thinking where perspectives from the two academic fields are combined. The ultimate aim of the framework is to enhance mathematical literacy to strengthen public participation where risk assessments are at the centre of policy making. The framework is quite general, targeting students at any level of educational institutions, from primary school to teacher education institutions.

CRITICAL DEMOCRATIC COMPETENCE

UN describes literacy, including mathematical literacy, as a condition for critical awareness and which can stimulate participation and ensure increased influence on society. PISA 2003 and 2006 (OECD, 2003; OECD, 2006) defines mathematical literacy as: " [...] an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned and reflective citizen" (OECD, 2006, p. 72).

The Norwegian National Curriculum reflects the societal needs in terms of mathematical literacy. It highlights the competence to "understand and critically evaluate information, statistical analyses […] to understand and impact processes in society" (Ministry of Education and Research, 2006). Further, the curriculum includes statistics, functions and graphic representation together with practical problem framing, of which all are relevant skills for risk assessments.

Mathematical literacy, including some definitions of statistical literacy (Ben-Zvi & Garfield, 2004), is closely linked to the development of critical democratic competence; the ability to criticize, evaluate and analyse applications of mathematics in society (Blomhøj, 1992, 2003). Blomhøj (1992, 2003) argues that mathematical modelling can increase the development of critical democratic competence. The modelling process can be described as consisting of the following cycle of phases: a) Framing the problem, b) defining the boundary (the choices of what to include and what not to include in the model), c) mathematization (developing the model by describing the system/situation with mathematical concepts and language) d) the mathematical analyses (the mathematical calculations or representation from applying the model), e) interpretation and evaluation of the model results, and f) evaluation of the model's validity (Blomhøj, 2003).

Risk assessments are usually based on statistical enquiry to calculate the probability, and mathematical modelling to calculate impacts. However, the latter often requires data as well, and the estimated probability may also be based on mathematical modelling. This means that we need to add the following to the process list above, between b) and c): i) decisions on how and what to measure/sample and ii) structuring and displaying data (Lehrer & Schauble, 2004). The added phases require statistical literacy, including Watson's third tier: *critical awareness* (Watson, 1997). All phases demand critical and skilful evaluation, as they all include some kind of choice with no officially "correct" answer. Mathematical modelling and statistical enquiry will often stand in contrast to the rather common perception that science and mathematics produce an answer which is exact and correct. The acknowledgement of this is in itself a step towards critical democratic competence.

Research on school children working with aspects of mathematical modelling or risk assessments reveals children's development of critical thinking and critical democratic competence (see for example Hansen, 2009, 2010; Alrø, Blomhøj, Bødker, Skovsmose, & Skånstrøm, 2006).

Citizens cannot be expected to understand complex mathematical calculations, but insights in the modelling or sampling process may help understand uncertainty aspects in the process. I now turn to the academic literature on science for policy and present key aspects related to uncertainty and risk assessments. The purpose is to discuss how these perspectives can be utilised in developing mathematical literacy and critical thinking related to risk assessments in schools and in teacher educations, which is addressed in the last section of the paper.

UNCERTAIN SCIENCE FOR POLICY

Science is often seen as a key factor in solving societal problems and often plays an advisory role in developing policies related to environmental risks, for example related to greenhouse gas emissions or fisheries. While science has long been seen as 'truth' seekers, where science feeds the policy makers, we increasingly experience that scientific 'facts' are disputed. The discussions around whether global warming is real and whether it is manmade are examples of this. It is argued that the nature of problems science is asked to provide advice for has changed considerably. Now scientists have to deal with complex systems, of which knowledge and data may be scarce, and where the issue at hand is controversial with great values at stake (Funtowicz & Ravetz, 1993). A consequence of these challenges is that some policy processes have become more open, where cross-sectoral approaches, stakeholder involvement or public hearings have influenced policy making.

Risk assessments often play an important role in decision making, for example in fisheries management, managing permissions for petroleum exploitation and managing permissions for nuclear power plants. In such cases, risk is often defined as the probability of an unfortunate event multiplied with its effect. I now briefly present some discussions on existing risk assessments to demonstrate potential challenges with risk assessments in general.

Wild capture fisheries management is often based on risk assessments, where a management principle is to keep the probability of depleting a fish stock low. In general there are too many fishing vessels compared to the amount of fish, so that fishing needs to be restricted. Fisheries scientists provide risk assessments for this purpose. These are often disputed, and lack of trust between advisors, managers and the fishing industry is common. Several problems with calculating risk have been addressed (Hauge, 2011). The probability calculations are based on uncertain assumptions, ranging from the ability to predict what species fishermen catch to assumptions on environmental conditions, which often vary from one year to another and may be impossible to predict so that it adds relevance. It is further argued that these uncertainties are downplayed by the science community, although this may indeed not be the intention (Hauge, 2011).

The question of opening Norway's northern offshore areas to petroleum production has been a long and heated political debate in Norway because the areas host some of the worlds' largest fish stocks and bird colonies. A central issue has been the development of risk assessments, which are defined by the probability of a major oil spill multiplied by its environmental impact. There has been disagreement between the petroleum sector on one side and the fisheries and environmental sector on the other side, on which of the previous blowouts are relevant for calculating the size of a major oil spill. Further, the relevance of these are questionable due to lack of data and the quite narrow scope of defining both the event and the impacts (focussing on birds, marine mammals and two fish species) (Hauge et al., 2012). The debates among experts and in the public have centred on the quality of the risk calculations rather than their scope and relevance. The existing risk assessments thus define what issues are plausible to discuss and criticize.

Similar problems are discussed regarding the risk calculations of meltdowns in nuclear power plants. The empirical basis for calculating the probability is argued to be weak since few power plants are based on comparable technologies. In addition, experience has shown that accidents are often initiated by a combination of two or more simultaneous events which in isolation are quite harmless. However, due to the complexity of power plants they have caused unpredictable occurrences, which have been challenging to interpret for the operators (Perrow, 1999).

IPCC (International Panel on Climate Change) uses the term 'likely' based on probability calculations: "Most of the observed increase in global average temperature since the mid-20th century is very likely due to the observed increase in anthropogenic greenhouse gas concentrations" (International Panel on Climate Change, 2012). 'Very likely' refers to a probability assessed to be above 90% (IPCC, 2012). Sceptical proclamations to climate change are not uncommon, and for a non-expert it is challenging to weigh 'evidence'. Yet, the uncertainty in, for example, IPCC's temperature predictions should be apparent for an interested reader, as IPCC can inform us that these are based on emission scenarios and a "hierarchy of models". The latter implies that several research groups around the world produce temperature predictions, and that a weighted average is chosen as the 'most likely' one.

There are several similarities between these examples: i) they all represent great values at stake: monetary value, nature, life style and identity, ii) reputable scientific communities produce advice iii) scientific advice has great influence on public debate and/or on policy making, iv) scientific advice is associated with substantial uncertainty, and v) advice or predictions are presented with hyper-precision, understood as numbers appearing much more certain than what the associated uncertainty would imply (Funtowicz & Ravetz, 1994).

Funtowicz and Ravetz argue that policy issues where stakes are high and 'facts' are uncertain require post-normal science, where quality in science for policy is defined as both high quality, peer-reviewed science and as fit for purpose. (Funtowicz & Ravetz, 1993). Although science may be 'truth seeking', traditional scientific questions and methods may not be relevant for a specific policy problem. Uncertainty should therefore be specifically addressed, and stakeholders, including the public, should evaluate the relevance of scientific input to policy and decision making (Funtowicz & Ravetz, 1993; Funtowicz & Strand, 2007). However, scientists and engineers are often not trained to address uncertainty in other ways than probability distributions, error bars or safety factors.

In response to this situation, several frameworks have been developed to enhance relevance in advice and assessments by articulating values and the qualitative aspects of uncertainties. The following list covers essential issues related to these:

a. Mapping values at stake (van der Sluijs et al. 2005a, 2005b, 2008).
b. Mapping potential risk bearers, which may depend on the final policy (van der Sluijs et al. 2005a, 2005b, 2008).
c. Uncertainty in the translation from a policy problem to a scientific question: whether the scientific problem answers the policy question or only part of it, and whether it is neutral to values and risk bearers (Walker et al. 2003).
d. The sources of uncertainty, referring to data, methodology, underlying assumptions, and the scope of investigation (Walker et al., 2003).
e. Characteristics of the uncertainties: for example whether uncertainty can be represented by measurement error or whether it involves unknown aspects, implying that uncertainty cannot fully be quantified. Another characteristic would be whether the uncertainty is reducible (Funtowicz & Ravetz, 1990; Wynne, 1992; Walker et al., 2003).
f. Extended peer-review (Funtowicz & Ravetz, 1993; van der Sluijs et al. 2005a, 2005b, 2008).

Uncertainty due to translating the policy problem to a scientific problem is a central issue. The relevance of scientific framings has been questioned in all four examples above: whether managing single stocks are possible when fishermen catch a mixture of species (Hauge, 2011), whether the developed risk assessments are adequate for giving permits for petroleum exploitation (Hauge et al., 2012), whether single event accidents are characteristic for nuclear power plant accidents (Perrow, 1999) and whether all the efforts of reducing uncertainty in IPCC's temperature predictions will be helpful for policy making.

The purpose of addressing characteristics of uncertainties is that numerical results accompanied by error bars, or other quantified uncertainty measures, do not necessarily convey the soundness of the knowledge base. Appropriate data or knowledge may not be available, demanding more or less qualified guesses. Such choices have been made, for example, in modelling global temperature development: assessing historic temperatures, deciding data grids, deciding which physical processes to be left out in the model, etc. Uncertainty aspects like these cannot be reflected in quantified uncertainty measures and need to be addressed in addition. Frameworks are developed for this purpose (see for example Walker et al., 2003).

Scientists and experts often need to limit the scope of their investigations and make unsupported assumptions, having access to limited data, as the case was in the four examples above. The implied choices influence policy debates in ways people may not be aware of, which again can have political consequences. From a democratic viewpoint, such policy issues benefit from an engaged public. The public may not grasp the complex mathematics or science behind predictions or advice, but the public may be competent in raising relevant questions to knowledge production and understand the nature of associated uncertainties and its societal consequences.

CRITICAL DEMOCRATIC COMPETENCE RELATED TO RISK

A question imperative to the paper is how educational institutions can work with the concept of probability, modelling and risk in order to improve mathematical and statistical literacy related to societal questions where risk assessments influence policy making and decisions. I now present a framework for developing critical democratic competences, but which also can serve as a starting point for theoretical discussions on planned research projects. The framework consists of three rather complementary approaches. The first is to conduct risk assessments, the second is to practice with statistical and modelling concepts through issues presented in the media, and the third is to discuss risk assessments in light of the above list of value and uncertainty questions.

Conducting risk assessments requires and develops both statistical and modelling competences. Through first-hand experience on estimating probabilities of unfortunate events and on developing and applying models for calculating impacts, the student is trained in developing mathematical arguments. She further experiences that choices and evaluations need to be made in every phase of the process, and that mathematics is not necessarily about searching the one and only correct answer. Such experiences are crucial for developing critical thinking related to the use of mathematics in social life and are highlighted in literature on critical mathematics education, on modelling competences and on statistical literacy (Ernest, 2001; Blomhøj, 1992, 2003; Barbosa, 2006; Gal 2002; Watson, 1997, 2004; Lehrer &

Schauble, 2004). The ambitions on which parts of this process are to be addressed, and how, will vary, depending on age of the students and on the teacher's goals.

Practice with statistical and modelling concepts through media issues is valuable for linking classroom concepts to concepts used in the media. Media issues can be used as a basis for simple problem solving (Watson, 1997, 2004), or they can be used for searching and understanding concepts. Internet offers a rich spectrum of issues where relevant concepts can be found, such as probability, event space, stochasticity, risk, variable, stochastic variable, graphical representations etc. The student thereby gets experience with, and the habit of, discussing issues presented in the media. This also enables recognition that mathematical skills are relevant for media issues.

Discussing risk assessments presented in the media is the third approach. The calculations in such assessments are normally much too advanced to study for any lay person. But the exercise of exploiting a calculated risk through posing questions related to the list in the above section on Uncertain Science for Policy can enhance critical thinking. The list can be supplemented by Gal's (2002) "worry questions" about statistical messages. The purpose of this activity is less about concluding on correct statements about the risk assessment. Rather, by suggesting and discussing what choices the experts may have faced, the student can gain insight in what questions might be relevant to ask for increasing understanding about values and uncertainties associated with the risk assessment. The student can thereby act as an extended peer-reviewer. The introduction of socially relevant, real-life problems in its full complexity has been requested in the literature (Ernest, 2001). It also prepares for the role as consumers of mathematical or statistical information, which Gal (2002) points out is a more common role for adults than being a producer of such information. This last framework approach requires teachers that are inquiry oriented and feel comfortable with exploring unknown ground.

These three approaches to increasing critical democratic competence of students, specifically related to risk assessments, are partly complementary. Although all approaches seek to enhance critical thinking, they do so from slightly different angles. An aim is that a student is able to transfer insights related to what Wild and Pfannkuch (1999) call enquiry contexts to reading contexts. This can be facilitated through reference to the other approaches when working within one approach. Taken together, the framework aims at increasing the critical democratic competences to enable future adults in participating in societal issues. The last decade there has been an increasing global demand for participatory policy processes. Skills in asking the right questions concerning quantified information are crucial in this regard.

As a final remark, I find it important to emphasize that we wish to be critical to risk assessments, not necessarily because we expect scientists and experts to cheat or to perform badly, but because they are often required to give answers to problems to which solid conclusions cannot be expected. The scientists and experts involved are often the best in their field, and they provide valuable knowledge, but you may need to be critical to the context in which this knowledge is used.

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STUDENTS' INFORMAL INFERENTIAL REASONING WHEN WORKING WITH THE SAMPLING DISTRIBUTION

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Introductory statistics students struggle with the complexities of the sampling distribution, particularly when asked to draw a conclusion based on a sample of data. This study investigates the informal statistical reasoning of seven pairs of secondary students as they collect a sample of data in an attempt to draw a conclusion based on the related sampling distribution. We present analysis of this task and the results of a pre/posttest assessment of their informal inferential reasoning with the sampling distribution.

INTRODUCTION

Recent research efforts in statistics education have focused on informal statistical inference to understand how students begin to reason about data they encounter. Informal inferential reasoning is thought to develop as it brings together the underlying concepts of descriptive statistics, probability, and the sampling distribution to infer about populations before the procedures of formal inference are introduced. Makar and Rubin (2009) defined informal inferential reasoning as generalizing about a population using sample data as evidence while recognizing the uncertainty that exists. This focus on informal inferential reasoning as students take part in informal statistical inference tasks has been under investigation for the last decade (Ben-Zvi, 2004; Pfannkuch, 2006; Pratt, Johnston-Wilder, Ainley, & Mason, 2008; Watson and Moritz, 1999). While researchers are building definitions of informal inferential reasoning and frameworks for researching its development (Makar & Rubin, 2009; Pfannkuch, 2006; Zieffler, Garfield, DelMas, & Reading, 2008), exactly how informal inferential reasoning develops and how students demonstrate such reasoning across a range of contexts is still under investigation. This research was designed to add to the understanding of that development by investigating how secondary students demonstrated informal inferential reasoning at the end of a year-long course in introductory statistics in the United States. In particular, we were interested in two questions: (1) what knowledge of the sampling distribution do students exhibit and (2) could students make an informal statistical inference by situating a single sample in relation to the related sampling distribution.

BACKGROUND

Even with instructional activities designed to help students develop an understanding of the complexities of the sampling distribution such as those implemented in the study by Saldanha and Thompson (2002), students are likely to experience difficulty in drawing informal conclusions with the sampling distribution. The majority of secondary statistics students in Saldanha and Thompson's study compared a single sample statistic to the population parameter rather than to the sampling distribution when asked to determine if it was unusual. Many of the difficulties students experience with the sampling distribution surround their misunderstandings of the effects of sample size and thus, the variability of the sampling distribution. Chance, delMas, and Garfield (2004) found this when using interactive software and paper-and-pencil tasks designed to assist students in making the distinctions between the population, samples, and the sampling distribution. For the introductory statistics students in their study, placing the sampling distribution in relation to the population and individual samples was problematic. The difficulties students have with the complexities of the sampling distribution and drawing conclusions with them have the potential to plague them further as they attempt to navigate formal statistical inference.

The definition proposed by Makar and Rubin (2009) was used to determine if students reported in this study were demonstrating informal inferential reasoning. The larger study, of which the sampling distribution task reported here is a part, was modified from a task framework for research on informal inferential reasoning developed by Zieffler, Garfield, delMas, and Reading (2008).

DESIGN AND METHODOLOGY

This research was part of a larger research project that examined the relationship between students' informal inferential reasoning and their formal inferential reasoning as this developed over a year-long course in introductory statistics at the upper secondary level. In this paper, we focus on students' reasoning about the sampling distribution as they completed a task designed to engage them in informal inferential reasoning. This task was the third in a sequence of four task-based interviews (Goldin, 2000) completed by seven pairs of students. We will also discuss their responses to the sampling distribution questions on a pre/posttest they completed as part of the larger study. The posttest included the same informal inferential reasoning questions as the pretest as well as formal inferential reasoning questions.

Setting and Participants

The students taking part in this study were enrolled in introductory statistics courses for college credit in their high schools. These students were either in their 11th or 12th grade year, 16 to 18 years of age, and had completed at least the first two courses of the three mathematics courses required for high school graduation. These students came from one of eight statistics classes taught by four different high school mathematics teachers from two high schools. These statistics classes met for approximately three and one-half hours each week for the 40-week school year beginning in September, 2011.

A pair of students from each of the eight classes was asked to take part in the task-based interviews; seven of the student pairs completed the study. The pairs of students represented a range of prior achievements in mathematics as judged by their classroom teachers.

Task-based Interviews

The structure of the task-based interviews followed the principles and techniques proposed by Goldin (2000). These included task-based interviews that (1) were designed specifically to answer the research questions, (2) included tasks with appropriate content for students' to grasp, (3) were structured based on key statistical concepts that gave students a variety of ways to demonstrate their understanding, (4) included an explicit interview protocol that allowed students to think about their responses without critiquing the correctness of their responses, and (5) involved students in free problem solving while they interacted with another student. The interview tasks were designed with multiple parts that increased in complexity.

Three classroom activities were implemented in each of the classes prior to the task-based interviews to provide the students with experiences in informal inferential reasoning. The three activities took place in the following order in conjunction with the progression of the class curriculum: (1) comparing distributions of data (Watson & Moritz, 1999); (2) a sampling and probability exploration (Konold et al., 2011); and (3) a sampling distribution activity.

Following each of these three classroom activities, the task-based interviews were conducted with the seven pairs of students. These interviews began with a recall of the classroom activity to gain insight into what the students learned from the activity. They were then asked to complete another problem in that same topic area to probe how their informal inferential reasoning was developing. A fourth task-based interview focused on formal statistical inference.

The focus of the research reported here involved the third task-based interview. This interview task was influenced by the work of Saldanha and Thompson (2002) who found that even after instruction on the sampling distribution, students tended to compare the results from a sample to the distribution of the original population rather than to the sampling distribution. The classroom activity and task in this study were designed to support what they called a "multiplicative" conception of sample where the distinction between the population distribution, the distribution of a single sample taken from the population, and the distribution of the sample statistics of many samples is understood. The interview tasks included a task used by Chance, delMas, and Garfield (2004) in which students identified sampling distributions based on a population distribution and answered questions about the variability of these sampling distributions. This interview task culminated with an informal inference task which required students to situate a single sample statistic in comparison to a related sampling distribution of many samples.

Pre/posttests

The study began with a pretest assessment to measure students' informal inferential reasoning. The study concluded with a posttest which contained both informal (identical to pretest) and formal statistical inference questions. There were two items that specifically addressed informal inferential reasoning with the sampling distribution. Students' responses to these items are included in this analysis.

DATA ANALYSIS

The task-based interviews were video-recorded, transcribed, and coded for common themes in students' understanding of key statistical concepts. For the Sampling Distribution task-based interviews, we analyzed students' responses to determine the extent to which they (1) distinguished between the sampling distribution and the population distribution, (2) recognized the effects of sample size on the variability of the sampling distribution, and (3) drew an informal conclusion by situating a single sample in relation to the corresponding sampling distribution.

Students' responses to the sampling distribution items on the pretest and posttest were compared to determine if students improved on these items. Each item had two parts and students' responses to both parts were analyzed together to determine if students displayed consistent reasoning.

RESULTS

We will first report on the findings from the task-based interview on sampling distributions. This is followed by the results of the four pre/posttest sampling distribution questions for the seven pairs of students.

Sampling Distribution Task-based Interview

Part 1 of the third task-based interview involving sampling distributions had students predicting what sampling distributions would look like for a given population distribution and considering the variability of those sampling distributions. Given a tri-modal distribution and its mean (Chance, delMas, & Garfield, 2004), students chose the distributions that represented 500 samples of size 4 and size 16 from five possible graphs.

All seven pairs of interviewees chose sampling distribution graphs that were approximately normal in shape. Six of them also correctly identified the effect of sample size on the variability of the sampling distributions. Only one pair incorrectly identified the variability of the sampling distribution for a sample of size four; however, they did correctly identify the variability as less for the sample size of 16. This was an indication that these students had a base knowledge of the sampling distribution and its characteristics.

To begin the second part of the task-based interview, students viewed the Random Rectangle simulation in *Fathom*, shown in Figure 1. The population of rectangles, labelled with their corresponding areas, is on the left and the graph of the areas of the total population of rectangles is in the upper middle. The Sample of Rectangles graph below that in the center displays a single random sample of rectangles. The Measures from Samples of Rectangles graph in the lower right displays the sampling distribution of the mean areas. Students were able to watch a demonstration that animated how each sample was taken from the population, graphed, and then the mean area from each sample was added to build the sampling distribution.

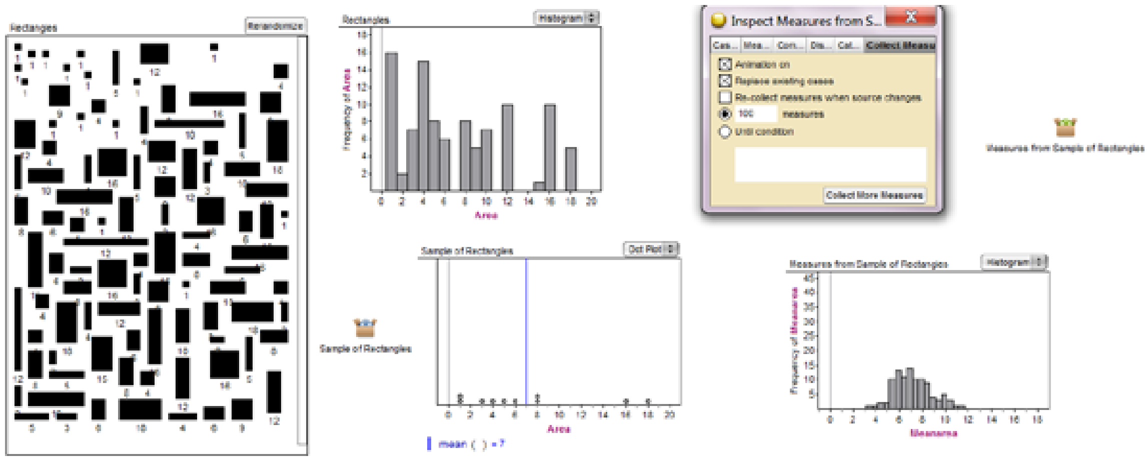

Figure 1: Screen shot of Random Rectangle simulation

Following this demonstration, the students were shown three sampling distributions generated from this simulation for 100 samples of sizes five, 10, and 25 rectangles. When asked about how all three distributions compared to one another, six of the pairs referred to these distributions as becoming more centered or having the same mean with five of them also referring to the decrease in variability as the sample size increased, as did this student:

Interviewer: So we went from a sample size of 5, then to 10, now to 25. So how about this one [of sample size 25]?

Student: This one's even more compact. The last one [of sample size 10] got all the way out to like 12. This one hasn't gone past 10 [referring to maximum mean area].

The remaining pair referred to the decrease in variability alone, mentioning the formula $\left(\frac{\sigma}{\sqrt{n}}\right)$ for standard deviation in support of this decrease.

Students were then asked what mean areas would be likely and which would be rare or unlikely for each of the sampling distributions. The students had no difficulty in identifying ranges of outcomes surrounding the peak of the approximately normal distributions as likely and those in the tails as rare. They demonstrated an understanding of the probabilities and variability associated with the normality of these sampling distributions of mean areas.

In an effort to bring the previous concepts of normality and variability related to the sampling distribution together to make an informal inference, the interview concluded with the sampling distribution in Figure 2. In the second interview of the larger study, students tossed small plastic houses to approximate the probability that a house would land upright when tossed. Students were shown this sampling distribution which was generated from 200 samples of 10 houses tossed, recording the proportion of houses landing upright. The interviewees were then asked if they could determine whether the probability that a hotel would land upright was the same as that for a house. The hotels were slightly larger with a rectangular rather than a square base like the houses; and both the houses and hotels were available for students to manipulate.

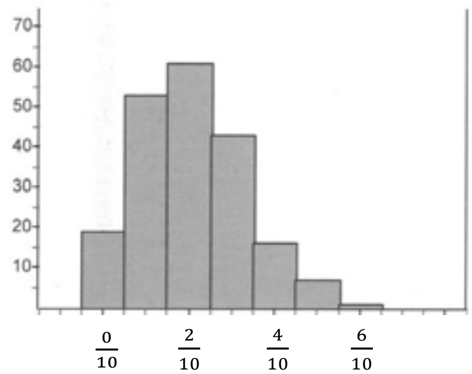

Figure 2: Sampling distribution for 200 tosses of 10 houses

Four of the seven pairs expressed that they would need to generate a sampling distribution exactly the same as the one they were shown for the houses with 200 tosses of 10 hotels. Another pair thought they would need to toss 32 hotels five times as they had with the houses in the second task-based interview.

Since time constraints did not allow for replicating the sampling distribution, all of the pairs tossed 10 hotels. Table 1 displays the number of tosses by each pair and some of their concluding remarks. Three of the pairs tossing 10 times averaged their 10 tosses to obtain a proportion for the hotels landing upright.

Number of Tosses	Students' Concluding Remarks
10 (averaged)	Like we had more two's and it looks like this one has more two's. So I feel like it would have the same probability as the house. I'm still doubtful. What we found was about 25%. So about a fourth of the time it'll land upright, if not a little bit more than that. And for this [the sampling distribution] we have like 20%, less than 20%, so that's just me doing math in my head and I just don't think it's likely. Plus we only did 10 trials.
2	I think you'd have to try probably more times, many more times, but, as it looks right now it's about the same. Maybe a little bit less.
1 hotel 10 times	Yeah, it was pretty similar. Between 1 and 2 [houses landing upright out of 10] so I think that [their results] still validates that that's relatively the same.
3	I'm figuring it's not going to be that far away. I think it's going to be roughly the same. It's maybe just a little bit less because it's weighted differently.
10 (averaged)	So it would probably look normal [sampling distribution for hotels] but not as variable as the other one [sampling distribution for houses]. I'd say they're very similar, the two, yeah.

10	We didn't do nearly enough. I mean you did this 200 times, we did this 10 times so like you can't really say like, oh look what we did really quick and that refutes that. Then I'd say it's different, but not by a lot.
10 (averaged)	You gotta take more samples. …but with one trial, I think regardless of the outcome, you can't really compare that to what you got from this population [referring to sampling distribution]. It may fit into what you have seen. Like right here, this value right here, like 1, 2, [referring to peak in sampling distribution] ours was close so we could say yeah, it does compare similarly but I'm not going to bet my life on it.

Table 1: Number of tosses of hotels and students' concluding remarks

All of the pairs' tosses resulted in proportions that were at or close to the peak of the sampling distribution; however, their remarks demonstrated a variety of conclusions. At least one student from four of the seven pairs stated that the probability was slightly different. They were not taking the natural variability of sample proportions into consideration even though they had just identified likely and rare outcomes with the sampling distributions of mean areas of rectangles. Four of these pairs expressed their skepticism in the accuracy of their results due to their small number of tosses. The majority of these students were not yet ready to draw a conclusion from a single sample of data based on the variability of the sampling distribution. However, their statements, many expressing a degree of certainty or uncertainty, provided evidence that they were at a point in their informal inferential reasoning when they might be able to consider this next level of reasoning. This could be seen when we asked the following question of the pair who did three trials:

Interviewer: So is it [the results] enough less to say, do you think, that the probability is different?

Student: How far away would it have to be? Like, I mean, I don't know, I think it would be a couple percentage points. You know just a little lower.

This students' question lays the foundation for formal statistical inference. This may provide an opportune time to explore more deeply what it means for these sample proportions to be relatively the same or slightly different.

Sampling Distribution Questions on the Pre/posttest

There were two items on the pre/posttests involving drawing conclusions from a single sample based on the corresponding sampling distribution. The first item asked students to draw conclusions directly from a graph of the sampling distribution. For the second item, students were shown the population distribution and given the population mean. They were then asked to draw a conclusion based on the results of a random sample of size 50.

The first pre/posttest item had students drawing informal inferences based on a sampling distribution for the proportion of heads expected when a fair coin is balanced on its edge 10 times. Marked on the sampling distribution in Figure 3 were the results of 0.7 heads and 0.9 heads from two different samples.

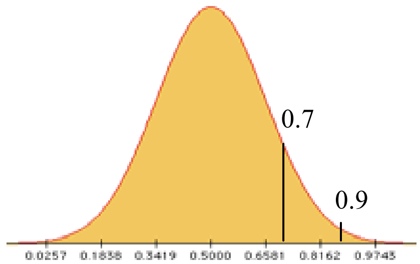

Figure 3: Sampling distribution for proportion of heads

In the first part of this item, students were asked if it was reasonable to conclude that the coin was fair with a sample proportion of 0.7 heads. Eight of the students answered correctly on the pretest that this result, which was between 1 and 1.5 standard deviations from the mean, was reasonable. This improved to 10 students answering correctly on the posttest; however, one student changed a correct answer on the pretest to incorrect on the posttest.

In the second part of the item, students were asked if it was reasonable to conclude that the coin was unfair with a sample proportion of 0.9 heads. Nine of the students answered correctly on the pretest and the posttest that this result, which was over 2 standard deviations from the mean, was reasonable. Two students changed their correct answers on the pretest to incorrect on the posttest.

Only six students answered both parts of this item correctly and one student answered both parts incorrectly. Of the seven other students, three concluded that both results indicated that the coin was unfair while the other four students concluded that both results indicated that the coin was fair. These responses are consistent with reasoning students displayed when working on the hotel task. This item, with the sample results of 0.7 and 0.9 clearly marked on the sampling distribution graph, provided further evidence that most of the students were not able to appropriately use that sample data as evidence as they were not fully considering the probabilities and variability associated with the sampling distribution.

For the second item on the pre/posttests shown in Figure 4, the students were shown a left-skewed distribution of exam scores for a particular exam. The average exam score for this population was 74 out of 100 points.

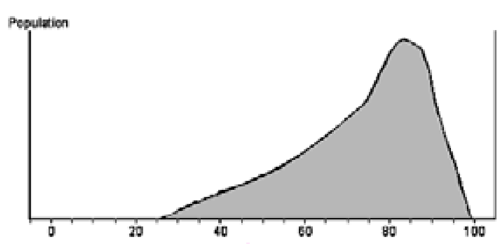

Figure 4: Population distribution of exam scores

In the first part of this item, the students were asked if a current group of 50 students with an average of 78 points did better on average than expected for this exam. Five students answered this part correctly on the pretest and seven answered correctly on the posttest. However, three students changed their responses from correct on the pretest to incorrect on the posttest.

The second part of this item asked the students if this higher sample average could just be due to chance. Nine students answered correctly that this higher score could be due to chance on the pretest and 10 answered correctly on the posttest. One student changed their answer from correct on the pretest to incorrect on the posttest.

Taking both parts of this item together, six of the students responded correctly and were consistent in their reasoning by answering that this score could not be considered better than what could be expected and that the higher sample average score was due to chance. Three other students were consistent in their incorrect reasoning by answering that this higher sample average could be considered better than what could be expected and that it was not due to chance. The remaining five students displayed inconsistencies in their reasoning. Four of these five answered incorrectly that this higher sample average could be considered better than what could be expected but also that it was due to chance. The incorrect and inconsistent responses to these two questions was further evidence that the majority of these students experienced difficulty in drawing conclusions based on the relationship between a population and the corresponding sampling distribution including the variability associated with it.

DISCUSSION AND CONCLUSIONS

Overall the students had general knowledge of the sampling distribution. They knew it took the shape of a normal distribution and they made references to the decrease in variability as the sample size increased. They also identified sample data values that would be considered likely and rare based on probabilities associated with the normality of sampling distributions. However, when it came to making a decision about the hotels in the last part of the interview, they were not completely prepared to reference what they knew. They were hesitant to compare a single sample or even a small number of samples to the sampling distribution. This was likely due to their knowledge of the Law of Large Numbers and their desire to eliminate variability in their sample(s) for comparison. This prevented the majority of the students from being able to use the probabilities and variability associated with the normal distribution in drawing their conclusions. Their responses to the posttest items provided evidence that these concepts were still creating difficulties for students at the completion of their introductory statistics course.

Referring to Makar and Rubin's (2009) definition of informal statistical inference, students were inferring informally; however, for the majority of them, the level of uncertainty they interpreted in the data they collected was preventing them from appropriately using their data as evidence. It would be worthwhile to investigate how students would reason about the need for more samples in a task, like the one with the houses and hotels, if they were able to generate another sampling distribution or take more samples for comparison as many of the pairs initially wanted. Allowing students to explore their notions about taking more samples and the sampling distribution may help them to understand that this will not necessarily provide them with more certainty in their conclusions. Discussions about efficiency as well as accuracy in data collection could take place. Introductory statistics students likely do not have practical experience working within budget or time constraints for using data to draw conclusions. Therefore, taking many samples or creating another sampling distribution for comparison to make a decision may have seemed like the only certain methods. Students need experiences that will help them to understand that one sample can be enough to draw an inference with the sampling distribution. Developing activities that allow introductory statistics students to explore their notions regarding samples and comparing them to the sampling distribution may support their informal inferential reasoning that will, in turn, support their formal inferential reasoning.

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ASSESSING STATISTICAL LITERACY: WHAT DO FRESHMEN KNOW?

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While there are many studies about the statistical literacy of students, those that concern the statistical literacy of pre-service teachers are relatively few. In the present study we attempt to investigate the level of statistical literacy of pre-service teachers in their first year at the university and after the end of schooling. For the purpose of this study we adapted the framework of Watson (1997, 2003) and Gal (2002) for statistical literacy, while for the assessment of the participants' responses we used a modified model of SOLO taxonomy. Our findings indicate the low level of statistical literacy among pre-service teachers in their first year at the university.

Keywords: Statistical literacy, pre-service teachers, statistical knowledge, freshmen.

INTRODUCTION

Many researchers (among others Watson & Callingham, 2003; Budgett & Pfankuch, 2007) in the field of statistics education have emphasized the importance of statistical literacy for the effective participation of students in the society after the end of school. As the definition of statistical literacy is still being refined (Rumsey, 2002), in this paper we refer to it as 'the ability to understand and critically evaluate statistical results that permeate daily life, coupled with the ability to appreciate the contributions that statistical thinking can make in public and private, professional and personal decisions' (Wallman, 1993). Statistical literacy is a combination of abilities expected from citizens in information-laden societies, and is often considered as an expected outcome of schooling and as a necessary component of adults' numeracy and literacy (Gal, 2002).

Several studies have shown that teachers' knowledge is connected to what and how students learn and depends on the context in which it is used (Ball & Bass, 2000; Cobb, 2000). Consequently, it is important to inquire into the cognitive level of pre-service teachers in order to modify accordingly the content of the courses, which are relevant to Mathematics, at higher education.

While there are many studies about the statistical literacy level of students (e.g. Watson & Callingham, 2003), there are relatively few about pre-service teachers. Focusing on the dimension 'as an expected outcome of schooling' of statistical literacy, we designed and conducted a research which aims to identify at what level the students, who have just started their university studies (freshmen) in the Department of Education at the University of Patras, were statistically literate after the end of the schooling and the start of their university studies. In particular, our research questions were the following.

a) What is the level and depth of knowledge about fundamental statistical concepts held by pre-service teachers in their first year at the university? b) How do they use such knowledge in order to perceive and criticize information about the world around them? c) In other words, at what level pre-service teachers are statistical literate?

LITERATURE REVIEW

Statistical literacy is a major goal of several curricula of mathematics around the world (e.g. NCTM Principles and Standards, 2000; ACARA, 2010). According to the Australian Curriculum (ACARA 2010, p.2):

> Students should develop an increasingly sophisticated ability to critically evaluate chance and data concepts and make reasoned judgments and decisions. They should develop an increasingly sophisticated ability to critically evaluate statistical information and build intuitions about data.

In the spirit of these ideas, several studies have been conducted in order to define statistical literacy (e.g. Watson, 1997; Watson & Callingham, 2003) and to investigate students' statistical literacy at different levels of education (e.g. Budgett & Pfankuch, 2007). Furthermore, the ARTIST Web site (https://app.gen.umn.edu/artist/) created by delMas and his colleagues provides and evaluates tools for the assessment of students' statistical literacy (for more details see delMas et al., 2007). In the field of adults' statistical literacy, the research by Gal (e.g. 2002) was a major contribution to the conceptualization of statistical literacy, while Moreno (2002) focused on the connection of statistical literacy with citizenship, and Shield (2006) in the W. M. Keck Statistical Literacy Project immersed statistical literacy in society. Recently, Kaplan & Thorpe (2010) applied to adults the framework of statistical literacy, proposed by Watson & Callingham (2003).

THEORETICAL FRAMEWORK

As "the research in statistical literacy has unveiled a very deep construct involving a myriad of types and skills and cognitive processes" (Shaugnessy, 2007, p.966), for the present study we have restricted ourselves to a framework based on a combination of the work of both Watson (1997) and Gal (2002) in relation to the kind of statistical knowledge students should have by the end of schooling.

Watson (1997) proposed a three-tiered Statistical Literacy Hierarchy, which is described as follows:

> 1. Understanding of basic statistical terminology. 2. Understanding of statistical language and concepts when they are embedded in the context of wider social discussion. 3. Ability to question claims that are made in context without proper statistical justification.

Respectively, Gal (2002) suggests that, for full participation in the society, students after the end of schooling should be able to:

(a) … interpret and critically evaluate statistical information, data-related arguments, or stochastic phenomena, which they may encounter in diverse contexts, and when relevant.

(b) … to discuss or communicate their reactions to such statistical information, such as their understanding of the meaning of the information, their opinions about the implications of this information, or their concerns regarding the acceptability of given conclusions. (Gal, 2002, p. 2-3).

For our research, we used a framework built on the combination of the above theories in order to define our assessment goals and to develop the respective items. A parameter also taken into consideration was the statistical contents found in the elementary mathematics curriculum that pre-service teachers will have to implement in the future. More precisely, we focused on: a. the average, b. the reading and the interpretation of tables and graphs and c. the critical questioning of claims embedded in a social context.

METHOD

In order to answer the research question, we designed and conducted a research project during the first semester of the academic year 2011-2012.

Participants

The participants were 166 students (pre-service teachers), 137 female and 29 male at their first year of their studies in the Department of Education, University of Patras. The students were taught Statistics in the 4th, 5th and 6th grade (ages 10-12) of primary education, the 2nd and 3rd grade (respectively 8th and 9th grade - ages 14-15) of Junior High School and the 3rd grade (12th grade - age 18) of High School. We will refer only to the content of the 3rd grade of High School (12th grade) as it was the most recent to the participants. Statistics is taught at the 3rd grade of High School as a part of the course “General Mathematics” for two hours per week and it is obligatory for all students regardless of their programs of study (Theoretical, Practical and Technological Direction) (Ghinis et al., 2009). The participants of our research were 134 (80.7%) of Theoretical Direction and 32 (19.2%) of Practical/Technological.

In their last year of High School (3rd grade), in the chapter of Statistics, students are taught how to process statistical data and interpret critically statistical conclusions. The Syllabus includes the following subjects (Pedagogical Institute of Greece, 2007):

Basic concepts: The students are taught basic statistical concepts such as population, variables (quantitative and qualitative), census and sample.

Presentation of Statistical data: The students are taught about frequency distributions and their graphical representations.

Location measures and measures of variation: The students are taught how to compute the arithmetic mean, the median, the mode (location measures) and the range, the variance, the standard deviation and the coefficient of variation (measures of variation) of discrete and continuous variables.

Questionnaire

The questionnaire items that we used for this study were either adapted from items used in previous researches (Aoyama & Stephens, 2003; Watson, & Callingham 2003; PISA, 2003; Harper, 2004; delMas et al., 2007) or developed by the researchers for the needs of the present study. The questionnaire included ten items, seven open-ended and three multiple-choice. The items were chosen and developed, according to our framework, in order to assess either the ability of the participants to understand fundamental statistical terminology or their ability to question statistical concepts embedded in real life problems.

Students (pre-service teachers) were requested to justify their answer for all items. Time given for response was about 1.5 hours. For the coding of the responses we adapted the SOLO model of Biggs & Collins (1982) as has been used by Watson & Moritz (2000). The complexity levels are described in the next table:

Code	Level	Description
4	**Relational**	**Correct answer with correct justification.**
3	**Multistructural**	**Correct answer with partial justification.**
2	**Unistructural**	**Not able to interpret correctly the data or irrelevant use of data.**
1	**Prestructural**	**No justification. Justification based on irrelevant data or personal estimation.**
0		**No response or Yes/No answer without justification.**

Table 1: Codes and description

Each item had a scoring rubric which was designed to identify increasing quality of response and these varied from 0-2 to 0-4, depending on the complexity of the item. The coding was done independently by two raters. An interrater reliability analysis using the Kappa statistic was performed to determine consistency among raters. It was found to be Kappa = 0.736 ($p < 0.001$), which indicates substantial agreement between the two raters (Landis & Koch, 1977).

RESULTS

Students' performance showed mixed results: on five- out of ten items - students' performance was rather low, while on the other five the majority of students' responses were at the relational level.

Due to space limitations we analyse only the low-performances' items.

Discussion on low performances' items

The next items involve contexts that may appear in the media, and their objective was to assess students' 'ability to question claims that are made in context without proper statistical justification' (Watson, 1997).

Item 3

In Zedland, opinion polls were conducted to find out the level of support for the President in the forthcoming election. Four newspaper publishers did separate nationwide polls. The results for the four newspaper polls are shown below.

- Newspaper 1: 36.5% (poll conducted on January 6, with a sample of 500 randomly selected citizens with voting rights.)
- Newspaper 2: 41.0% (poll conducted on January 20, with a sample of 500 randomly selected citizens with voting rights.)
- Newspaper 3: 39.0% (poll conducted on January 20, with a sample of 1000 randomly selected citizens with voting rights.)
- Newspaper 4: 44.5% (poll conducted on January 20, with a sample of 1000 citizens with voting rights who responded the phone call.)

Which newspaper's result is likely to be the best for predicting the level of support for the President if the election is held on January 25? Justify your answer.

Figure 1: Item 3

Code	Description	Examples
4	Choice of the third newspaper and correct justification based to the date and to the sample. Discriminating that the sampling that was based to phone calls isn't so accurate.	"The third newspaper because the date is closer to the elections' date and the sample is bigger. It is better than the fourth because the sample method is more reliable (examples why is more reliable)''
3	Correct answer with partial justification. (Referring only to the date or to the sample)	"The third newspaper because the sample is bigger"
2	Using arithmetic data in order to find how many people will vote for the president. Irrelevant justification.	By multiplying percentages with the number of the sample.
1	Choice without reasoning. Choice of the fourth newspaper considering the way of collecting data as more accurate.	" The fourth newspaper because it is more reliable the sampling method"
0	No response.	

Table 2: Codes and examples to the codes for Item 3

The third item (derived from PISA, 2003) concerned reasoning about samples and it was used in order to assess the participants' ability to question statistical claims embedded in a social context. Only the 40% of the participants' answers were classified at the relational level (code 4). The analysis of the students' responses revealed that they had some difficulty to understand why a sample cannot be representative of the population.

> Student 65: The fourth newspaper because they used phone calls in order to gather the data.

Also, several students confused the poll's percentage with the sample size.

> Student 18: Fourth newspaper because 44, 5 % of the voters is almost 500 people more than the others' newspapers population.

Item 7

The state based its decision on the redundancy on the data table below, according to which 50 to 66 civil servants who belong to the age group 62-64 want to go out on redundancy voluntarily. Do you agree with this conclusion? Justify your answer.

Reasons for retirement and retirement age			
	Retirement Age		
Reasons for retirement	Under 62	Between 62 and 64	65 or more
Age	10,5	21,6	64,6
Redundancy	10,5	50	14,6
Health problems	26,3	11,9	8,3
Close business	10,5	1,5	0
Privileges	10,5	3	0
Opportunity for the younger	2,6	1,5	6,0
Unhealthy labor	5,3	4,5	0
Family matters	7,9	0	2,9
In order to enjoy life	7,9	1,5	2,1
Other reason	7,9	4,5	2,1
	100%	100%	100%
Total	76	66	48

Figure 2: Item 7

Code	Description	Examples
4	Ability to discriminate and use correct the data of the table. Correct use of percentages.	"The conclusion is wrong because the 50% of 66 is 33".
3	Their reasoning is based on the complementary information. They realize that "the rest percentages" don't add to 50, but they cannot extend this reasoning to the rest 50%.	"If you add all the other reasons it is less than 50"
2	Not able to interpret correctly the table.	"I agree because it refers to 50 people in 66".
1	Choice without reasoning. No justification or justification based on irrelevant data.	"It is wrong, nobody would choose redundancy".
0	No response. Yes or No answer without justification.	

Table 3: Codes and examples to the codes for Item 7

This item was constructed and used in order to assess the participants' ability to question a misleading claim embedded in the context of wider social discussion. While this specific task was rather easy as it required only the reading of a table of data, only the 2% of the participants responded to the relational level. The majority of the participants' answers (43%) were at the prestructural level (code 1). Students justified their judgment by resorting only to their own beliefs (Budgett & Pfannkuch, 2007) rather than giving data-based justifications.

Student 11: It's a lie; nobody wants to go to redundancy.

Student 21: It is impossible so many people to be in redundancy.

The interpretations of this item indicates that, since Statistics is a social construct (Best, 2001), students' reaction is connected to their knowledge of real-life. This result is consistent with others researchers' (cf. Shaughnessy et al., 2004) who claim that when the context is familiar to students, they tend to rely on their personal opinions.

Student 11: It's a lie; nobody wants to go to redundancy.

Student 21: It is impossible so many people to be in redundancy.

Item 10

Two newspapers published articles about criminality and they used the next graphs. Which of these do you think that represents better the reality? Justify your claim.

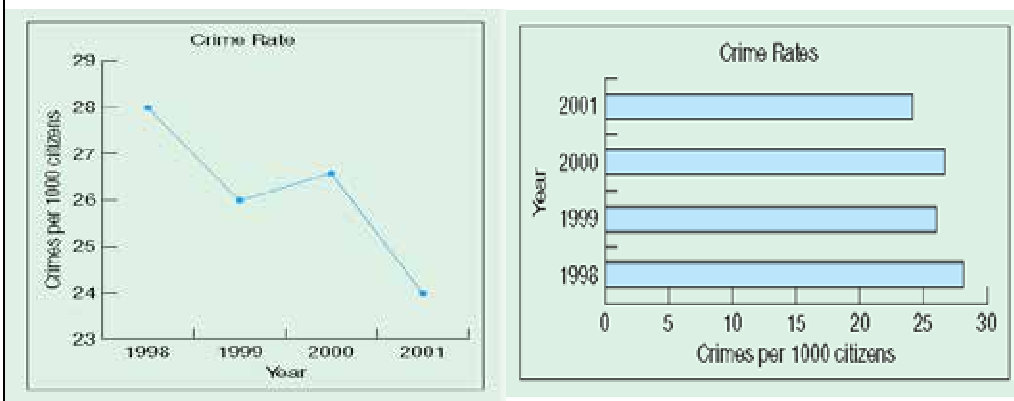

Figure 3: Item 10

Code	Description	Examples
4	Correct justification: ability to discriminate the wrong choice of the scale and the false affect that it creates.	"The bar-graph is better because the scale starts form 0". "The time-graph is not correct, it is like an enlargement"
3	Correct answer with partial justification.	"The bar-graph is better because it is more accurate"
2	Wrong justification based on personal estimations.	"The time-plot, because it is easier to read it, to understand it"
1	Response without justification	" The time plot is better''
0	No response.	

Table 4: Codes and examples to the codes for Item 10

This item, adapted from Harper (2004), refers to a misleading graph and its aim was to assess the ability of the students to discriminate the false effect that it creates. Only 1% of the participants were able to answer at the relational level, while the majority of them (59%) had chosen the time-plot as they considered it easier for interpretation.

Student 45: I believe that the 1^{st} (the time-plot) is better because through it, it is easier to understand the decline of crime rates during these year.

It is a fact that graph interpretation cannot be effective, if the reader does not possess basic graph reading skills. Among these skills we find the ability to recognize when the scale of a graph is truncated and the misleading impression that this fact creates, as in this specific item. While some students were able to recognize that the differentiation of the two graphs was due to the different scale, they could not understand why this fact created a wrong impression.

Student 41: As it is obvious at the 2nd graph the crime rates ranges from 23 to 29. So, the rest of the graph is useless. On the contrary at the 1st graph the given values are between 23 and 29, which make it more accurate.

DISCUSSION AND CONCLUSION

The goal of the present study was to assess the level of statistical literacy of pre-service teachers in their first academic year. The findings revealed that although students had been taught Statistics at all school levels and most recent in High School, their level of statistical literacy was rather low, especially when it comes to the questioning of statistical claims in a social context. These findings are consistent with those presented in other studies (Godino et al., 2008), which suggest that prospective primary school teachers in many countries enter the Departments of Education with a very limited statistical competence.

In sum, the participants were able to read and interpret simple graphs such as bar graphs, line-plots and pie-charts, and had knowledge of basic statistical notions such as the mean and the range. However, they had difficulties concerning sampling, graph evaluation and proportional thinking in a real life context. Also, they could not give proper justification to statistical claims in the context of a social discussion.

For the purpose of this study we used misleading tables and graphs and we asked the participants to question arguments based on them. For the answer of these specific items, most of the participants relied more on their experience of real world data than on their statistical knowledge. This fact indicates that even adults after the end of schooling, like students (Watson & Chick, 2004), are not able to detect the "unusual".

According to Gal (2002) there are five interrelated "knowledge bases" that must be used to exhibit statistical literacy: mathematical knowledge, statistical knowledge, knowledge of the context, literacy skills and critical questions. The results of the present study show that items with low performance demanded more than plain mathematical or statistical knowledge. We argue that items demanding one or more of the last three "knowledge bases" (context, literacy skills and critical questions), turned out to be difficult for the pre-service teachers.

Consequently we suggest that a content of a Mathematics Course for pre-service teachers should further focus on these aspects by incorporating in the teaching, strategies of misleading graphs' recognition and critical analysis and examples of real life statistical claims originating from the media.

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EVALUATING THE EFFICACY OF TRAINING ACTIVITIES FOR IMPROVING PROBABILITY AND STATISTICS LEARNING IN INTRODUCTORY STATISTICS COURSES

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The aim of the present paper is to ascertain the efficacy of a training acting on both cognitive and non-cognitive factors in improving probabilistic and statistics learning in psychology students enrolled in introductory statistics courses. We previously tested the relationships between performance and students' general and mathematical background (cognitive factors), maths self-efficacy and attitudes toward statistics (non-cognitive factors). Results stressed the role of both factors. Thus we developed training activities focused on competences and self-efficacy and we verify that it helped students in improving probability and statistics learning as well as their confidence.

Key words: Probability Learning; Statistics Learning; Training Activities; Introductory statistics

INTRODUCTION

Being able to provide good evidence-based arguments, and to critically evaluate data-based claims are important skills that all citizens should have, as stated by the European Parliament and Council in defining the key competences within the Lifelong Learning Program (2006). Thus, from an educational point of view, it is fundamental to develop students' statistical reasoning, and to provide them with tools and knowledge to understand and use quantitative information. The ability to think statistically about uncertain outcomes, and to make decisions on the basis of probabilistic information is relevant in many fields (e.g., for businesspeople, physicians, politicians, lawyers), and an inability to make optimal choices can be extremely costly, not only at the individual level, but also for society in general.

Then, statistics has been introduced as part of a wide range of curriculum programs in many countries. However, the discipline is viewed as a difficult and unpleasant topic. At the university level, students often perceive statistics courses as a burden, encounter difficulties, experience stress and anxiety, and, eventually, many of them fail to pass the exams. It is also common for students to have low expectations regarding statistics classes, and to have negative attitudes towards statistics. Finally, as passing exams in statistics is a requirement for many degree courses, failing to achieve this might result in students' abandoning their chosen professions.

Starting from these assumptions, several researches have focused on the identification of models in which the role of non-cognitive factors, such as beliefs and feelings, has been taken into account in explaining learning statistics. These factors appeared to be related to maths aptitude, previous maths knowledge, educational background and reasoning ability. Thus, statistics achievement might be the result of the interplay between cognitive and non-cognitive factors (Budé et al., 2007; Chiesi & Primi, 2010; Dempster & McCorry, 2009; Tremblay, Gardner & Heipel, 2000).

Starting from this premise, the present work aimed to assess the impact of both cognitive and non-cognitive factors on probability and statistics learning in psychology students enrolled in introductory statistics courses (Study 1). In light of the results of Study 1 we developed training activities and we verified its efficacy (Study 2).

STUDY 1

It was hypothesised that learning was related directly to students' general background and maths competence, as well as to their maths self-efficacy, beliefs about their own ability in dealing with statistics and feelings toward the discipline. Konold and Kazak (2008) suggested that some of the difficulties students have in learning basic data analysis stem from a lack of rudimentary idea in probability. For this reason, learning was operationalized including both probabilistic and statistical reasoning.

The causal paths among the mentioned variables were explored using structural equation modelling (SEM), a very powerful multivariate analysis technique. Among the strengths of SEM is the ability to construct latent variable: variables which are not measured directly, but are estimated in the model from several measured variables each of which is predicted to 'tap into' the latent variables. This allows the modeler to explicitly capture the unreliability of measurement in the model, which in theory allows the structural relations between latent variables to be accurately estimated. Thus, once the concepts used in the model have been operationalized and measured, the model is tested against the obtained measurement data to determine how well the model fits the data. In our model (Figure 1) cognitive and non-cognitive factors were considered as the latent exogenous variables (i.e., the independent variables) having an impact on learning considered the endogenous latent variable (i.e., the dependent variable). Each one was measured using respectively two, three and four observed variables (see details below).

Methods

Participants

Participants were 238 psychology students attending the University of Florence in Italy enrolled in an undergraduate introductory statistics course. Most of the participants were women (86%). This proportion reflects the gender distribution of the population of psychology students in Italy. All students participated on a voluntary basis after they were given information about the general aim of the research (i.e., collecting information in order to improve students' statistics achievement).

Description of the Course

The course covered the usual introductory topics of descriptive and inferential statistics (including basic concept of probability theory and calculus), and their application in psychological research. It was scheduled to take place over 10 weeks, and takes 6 hours per week (for a total amount of 60 hours). During each class some theoretical issues were introduced followed by exercises. Students were requested to solve exercises by paper-and-pencil procedure, and computer packages were not used.

Measures

General Background Test (GBT). This is a scholastic assessment test consisting of 100 multiple-choice questions (one correct out of five choices) divided into five sections: Maths, Biology, English comprehension, critical reading and reasoning. The time for the test was 85 minutes. A single composite score, based on the sum of correct answers less the wrong answers (the score for a wrong answer was -.25) was calculated.

Prerequisiti di Matematica per la Psicometria (PMP, Galli, Chiesi & Primi, 2011). The PMP measures maths abilities needed by psychology students enrolling in introductory statistics courses. The scale is a 30-problem test. Each problem presents a multiple choice question (one correct out of four alternatives). A single composite score, based on the sum of correct answers, was calculated.

Survey of Attitudes Toward Statistics (SATS-28©, Schau, Stevens, Dauphine & Del Vecchio, 1995; Italian version: Chiesi & Primi, 2009). The scale provides a multidimensional measure of attitude. It contains 28 Likert-type items using a 7-point scale ranging from strongly disagree to strongly agree, assessing four components: *Affect* measures positive and negative feelings concerning statistics (6 items); *Cognitive Competence* measures students' attitude about their intellectual knowledge and skills when applied to statistics (6 items); *Value* measures attitudes about the usefulness, relevance, and worth of statistics in personal and professional life (9 items); *Difficulty* measures students' attitudes about the difficulty of statistics as a subject (7 items).

Solution of Maths Problems (Kranzler & Pajares, 1997). This is a subscale of *Mathematics Self-Efficacy Scale-Revised* (MSES-R, Kranzler & Pajares, 1997; Italian version: Galli, Chiesi, & Primi, 2010). The subscale is composed of 18 problems with different levels of difficulty and it has been developed to assess the students' confidence to solve these problems.

Probabilistic Reasoning Questionnaire (PRQ, Chiesi, Primi & Morsanyi, 2011). This questionnaire contained 10 multiple-choice probabilistic reasoning tasks (one correct out of three alternatives). Each task was scored either 1 (correct) or 0 (incorrect). The scores on the probabilistic reasoning tasks were summed to form a composite score.

Introductory Statistics Inventory (ISI, Chiorri, Piattino, Primi, Chiesi & Galli, 2009). This test consists of 30 multiple-choice items (one correct out of four choices) to evaluate learning at the end of an introductory statistics course. Half of the problems refers to descriptive statistics and the other half to inferential statistics. Each task was scored either 1 (correct) or 0 (incorrect) and a composite score was obtained.

Procedure

The GBT was administered before the beginning of the course. The SATS was presented during the first day. The PMP was completed during the second day, and the PRQ and the ISI at the end of the course.

RESULTS

As shown in the table (Table 1), significant correlations were found between Probabilistic and Statistic Learning with the other variables. These results support the hypothesised relationships. Concerning Attitude towards Statistic, only two scales Cognitive Competence and Affect were significantly correlated with Probabilistic and Statistic Learning. For this reason we did not introduce Difficulty and Value in the model.

	M	*sd*	1	2	3	4	5	6	7	8	9
1. General Background	*44.43*	*10.32*									
2. Maths Basics	*22.70*	*4.20*	.41**								
3. Maths Self-Efficacy	*83.40*	*11.21*	.28**	.38**							
4. Cognitive Competence	*27.50*	*6.00*	.08	.29**	.44**						
5. Affect	*24.11*	*6.35*	.02	.24**	.32**	.77**					
6. Difficulty	*23.80*	*4.48*	.04	.16**	.36**	.55**	.59**				
7. Value	*45.51*	*7.88*	.08	.13*	.22**	.42**	.35**	.23**			
8. Probability Learning	*6.20*	*1.40*	.31**	.22**	.24**	.25**	.15**	.02	.15*		
9. Descriptive Statistics Learning	*11.14*	*2.24*	.54**	.25**	.24**	.23**	.19**	.08	.05	.34**	
10. Inferential Statistics Learning	*9.63*	*2.75*	.19*	.26**	.19**	.23**	.14*	.13	.02	.14*	.48**

* $p < .05$, ** $p < .01$

Table 1. Means, standard deviations (in brackets) and correlations among the measured variables.

The final model included three latent variables and nine observed variables: Cognitive Factors and Non-Cognitive Factors were the exogenous latent variable that influenced directly the Statistic and Probabilistic Learning. Cognitive Factors were measured through the GBT (General Background) and the PMP (Maths Basics). The two scores of the SATS subscales (Cognitive Competence and Affect) and the score of the MSES-R (Maths Self-Efficacy) were used as indicators of Non Cognitive Factors. A covariance path was traced between errors of subscales measuring the attitude dimensions since the high correlation (r = .77) between the two subscales. Another covariance path was traced between Maths Basic and Maths Self-Efficacy (r = .38). The endogenous latent variable (Probability and Statistics Learning) was measured through two scores of the PRQ scale - obtained by dividing the test randomly in two parts with 5 items for each (Probability 1 and Probability 2) - and two scores of the ISI scale - obtained considering separately the descriptive and the inferential items (Descriptive and Inferential). Covariance paths were traced between the two indicators derived from the PRQ and the two indicators derived from the ISI.

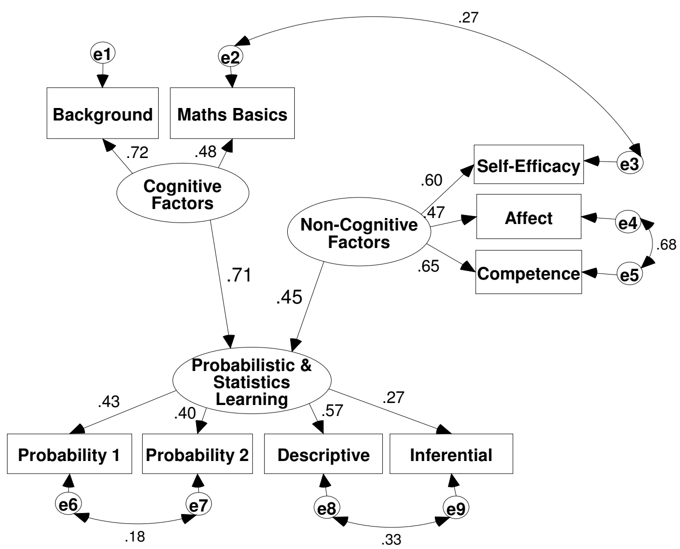

Figure 1. Final model with standardized parameters (paths are all significant at .05 level).

SEM analyses were conducted with AMOS 5.0 (Arbuckle, 2003) using maximum likelihood estimation on the variance-covariance matrix. Univariate distributions of all variables included in the model and their multivariate distribution were examined for assessment of normality. Skewness and kurtosis indices (ranging respectively from -.78 to .26 and .57 to .47) attested that the departures cannot be expected to lead to appreciable distortions (Marculides & Hershberger, 1997). The index of Multivariate Kurtosis (Mardia, 1970) (β = 1.86, $c.r.$= 1.02, p > .05) indicated that there was not a significant departure from multivariate normality. That is, data met the assumption of multivariate normal distribution required by SEM. The model showed a good fit to data ($\chi^2/df = 2.45$; $CFI = .94$; $TLI = .90$; $RMSEA = .07$) and all the estimated coefficients were statistically significant.

As expected Cognitive and Non Cognitive factors had a significant direct effect on Probability and Statistics Learning. However, the relationship with the Cognitive factors was stronger.

DISCUSSION

In line with previous research, general background and mathematical knowledge acquired during the high school had a direct and strong effect on learning probability and statistics. It does not mean that the statistics learning depends solely on these math knowledge but they constitute a necessary tool to keep in touch with statistics. Indeed, at least at the introductory level, a basic mathematic skill such as understanding ratio (i.e., being able to *interpret* the results of the computation) may be important in *understanding* probability. For this purpose it might be useful to arrange training activities aimed at mastering the basic mathematical skills required during the course. Moreover, since perceived competences and affect concurred in determining learning, to help students enhance their confidence it might be useful to give exercise that allow students to experience mastery of the topics, and to provide feedback about their results to allow them to monitor their progress.

STUDY 2

Starting from this premise, we developed training activities for consolidating the basic mathematical skills required during the course. The training involved students in individual and group activities. The contents are some math basics deemed necessary to successfully complete the introductory statistics courses. In detail, exercises included: a) addition, subtraction, multiplication, division with fractions, and exponentiation, required in descriptive and inferential procedures; b) fractions and decimal numbers from 0 to 1 necessary to deal with probability; c) first order equations necessary in the standardization procedure and in the regression analysis; d) relations between numbers included in the range from -1 to 1, and the meaning of absolute value necessary for drawing conclusions in hypothesis testing. At the beginning students should solve exercises individually and after they interact within group to compare solution and to prepare a final report.

The aim of this second study was to verify the efficacy of the training activities for improving probability and statistics learning. For this purpose we compared the training group with a control group, i.e. students who were not presented the training activities but solved typical introductory statistics exercises such as computing means and standard deviations.

Methods

Participants

Participants were psychology students enrolled in an undergraduate introductory statistics course. One hundred twenty-four students were randomly assigned to the training group and 55 to the control group.

Training activities

The training consisted in two didactic units lasting two hours each. Students were divided into groups (about ten students for each group). In each unit each student received a booklet with a series of exercises about some maths basics deemed necessary to successfully complete the introductory statistics courses.

Each student had to perform individually reporting how he/she solved it. Then the groups discussed about the correct solutions and the more common errors producing for each exercise a report of the group's activities. At the end of each unit, students might have other exercises to do at home.

As for the training group, the activities of the control group were organized in two didactic units. Students were divided into groups (about ten students for each group) and requested to solve exercises by paper-and-pencil procedure referring to frequency distributions, graphs, means and standard deviations. They solved them individually and then a group activity followed to discuss about exercises' correct resolution and errors, and to produce a report. At the end of each unit, students might have other exercises to do at home.

Measures and Procedure

At the end of the training all students were administered the *Probabilistic Reasoning Questionnaire* (PRQ) to measure the probabilistic learning and the descriptive problems of the *Introductory Statistics Inventory* (ISI - Descriptive) to measure the statistics learning. We added for each problem of the ISI a scale to measure how self confident they were to solve it.

Moreover, we took into account an achievement measure. Since some students were unsuccessful in the final examination and therefore they needed several attempts to attain their goal, the number of failed attempts was registered.

RESULTS

Furthermore to measure the association between training and achievement we created two groups: one with the students who never failed (no failure group) and the other one with students who fails one or more times (failure group). As we can see from the graph (Figure 2), we found a significant association between the training and the failure ($\chi_2(1, N = 174) = 3.93$, $p < .05$) with a small the effect size ($\phi = .15$). In particular there is a higher possibility to pass in the training group than in the no training group (64% *vs.* 48%).

Significant differences were found between the training group and the control group. In particular, the training group improved in probabilistic learning, statistic learning, and confidence (Table 2).

	Training		Control		t	d^a
	M	SD	M	SD		
Probability Learning	15.38	2.50	13.80	3.03	3.14**	.57
Statistics Learning	7.45	1.27	6.90	1.52	2.51*	.39
Statistics Learning Confidence	37.83	5.52	35.98	5.65	2.02*	.33

*p<.05,**p<.01. [a] d is a measure of effect size. Cut-offs are the following: small ≥ .20, medium ≥ .50, large ≥ .80.

Table 2. Mean scores compared with t test (and related effect sizes) for the Training and the Control groups.

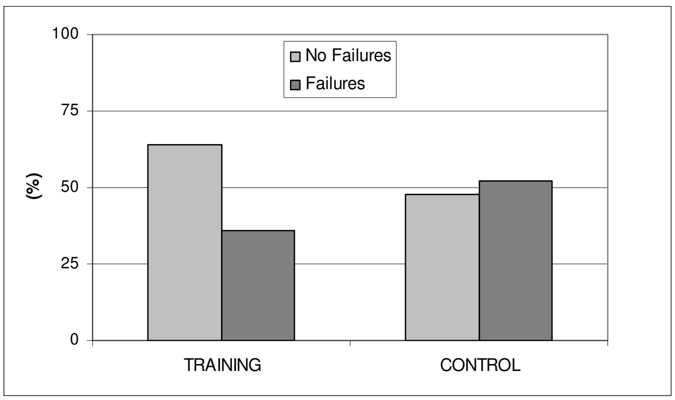

Figure 2. Percentage of failures in the Training and the Control groups.

DISCUSSION

The present study attested that students who participated to the training had positive changes not only in statistics and probabilistic learning but also in their confidence. Further the training seems to reduce the probability of failures. In sum, these results suggest that this training offers an example of an educational approach to introduce in introductory statistic course.

CONCLUSION

Teaching statistics with psychology students produces difficulties. Students are not primarily interested in statistics and dislike anything “mathematical”, often they do not have a strong background in Maths and they are not confident about their capabilities. The first aim of the present work was to investigate the impact of both cognitive and non-cognitive factors on psychology students’ probability and statistics learning. As expected, and in line with previous research (Leight Lunsford & Poplin, 2011; Tremblay et al., 2000) mathematical knowledge acquired during high school had a direct and strong effect on achievement. Additionally, mathematics self-efficacy and attitudes toward the discipline (i.e., perceived competences and affect) concurred in determining performance in statistics (Dempster & McCorry, 2009).

The second aim was to develop and testing the efficacy of a training to strengthen basic mathematical skills and improve students' confidence in learning statistics. Individual and working group activities were proposed in which the students could perceive that they handled the basic tools to deal with an introductory statistics course. In this way they both consolidate their abilities and acquired more confidence. Students in the control group who solved statistics exercises without reinforcing before their mathematical skills showed worse performance and less confidence when compared to the training group. Indeed, math skills help in understanding and interpret statistical measures.

In conclusion, our findings suggest that maths competences are a necessary tool to keep in touch with statistics and to help students to believe that they have the capacity to cope with the demands of an introductory statistics course.

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THE DIDACTICAL KNOWLEDGE OF ONE SECONDARY MATHEMATICS TEACHER ON STATISTICAL VARIATION[1]

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This study aims to contribute to the comprehension of the didactical knowledge of the teacher who teaches statistics at the secondary level, looking at the classroom practices of one teacher regarding statistics variation. Results stress the intertwined character of the various domains of the teacher's didactical knowledge, giving evidence of a strong relationship between the teacher's knowledge of statistics variation and the level of depth by which she approaches this concept in the classroom.

INTRODUCTION

Recent research points out the emergency of a set of fundamental big ideas for the teaching of statistics, such as data, variation, distribution and representation, among other concepts, which every student should know at the end of secondary school (Burrill & Biehler, 2011). In fact, there is an increasing recognition that variability is a fundamental idea for the teaching of statistics (Shaughnessy, 2006; Garfield & Ben-Zvi, 2008). Wild and Pfannkuch (1999) emphasized its importance considering that it is "the reason why people have had to develop sophisticated methods to filter out the messages in data from the surrounding noise" (p. 236). However, it is a difficult concept for students to reason about (Garfield & Ben-Zvi, 2008). According to these authors, such difficulty does not rely on computing formal measures of variability (e.g. range, inter-quartile range and standard deviation), but rather on understanding different representations of the concepts involved in those computations as well as their relationships with other statistical notions.

Teachers should help students to build on their intuitive notions of centre and variability and integrate these concepts when addressing data and doing data analysis as suggested by Shaughnessy (2006). Teaching statistics requires a profound content knowledge of this area; however, this is not enough to ensure effective teaching (Ponte & Chapman, 2006). The specific knowledge held by mathematics teachers, which Ponte (2012) calls *didactical knowledge*, is essentially oriented towards the practical activity of teaching mathematics and is based on "knowledge of theoretical nature" (Ponte, 2012, p. 86) about mathematics, mathematics teaching, and general education; in addition, didactical knowledge is founded on "knowledge of social and experiential nature" (p. 86), namely about students, actions in the classroom, professional, scholar and surrounding communities, among others. Sánchez, Silva and Coutinho (2011) suggest the study of teachers' professional knowledge while teaching variation as an urgent matter. Thus, it is important to analyse teachers' didactical knowledge in statistics, in particular of this topic, considering the ways it is evident in the practices of teaching variation in the secondary classroom. This study, which is part of a larger investigation, intends to make a contribution to this call.

BACKGROUND

Didactical Knowledge in Statistics

The model of teacher's didactical knowledge of Ponte (1999, 2012), particularly inspired in the works of Schön (1983) and Elbaz (1983), comprises four domains that orient directly the school practice. Each domain is here described in the realm of statistics, looking for enhancing its specificity.

The first domain is the *knowledge of the content* to be taught. It is essentially focused on teachers' interpretations of the discipline and it includes the understanding of concepts and procedures, statistical reasoning, argumentation and validation forms. It also includes the ability for epistemological reflection about the meaning of concepts and procedures and about the nature of statistical knowledge (Batanero & Godino, 2005). The teacher must be familiar with specific elements of this subject and develop insights toward a more formal understanding about them, namely concerning the fundamental aspects of statistical thinking, such as recognition of the need of data, attention to variation (Wild & Pfannkuch, 1999; Shaughnessy, 2006).

A second domain is the *knowledge of students and their learning processes*, which incorporates the knowledge of their difficulties, mistakes, obstacles and typical strategies in problem solving. Moreover, this domain includes knowledge of students as people, their interests and ways of acting and learning. In this domain, it is also important that teachers have a perception of the level of understanding that students have or will attain regarding different concepts (Ponte, 1999; Batanero & Godino, 2005). The *knowledge of curriculum* includes the knowledge of curricular goals and objectives, horizontal and vertical articulation/alignment of topics, and knowledge of materials and forms of assessment (Ponte, 1999; 2012). The current curricular framing enhances the active role of the students in building their mathematical and statistical knowledge (DES, 2001).

The fourth and last domain of teachers' didactical knowledge is the *knowledge of the instructional process*, which includes planning, teaching of lessons and assessment of practices. This knowledge is evident when teachers organize their own practice and give answers to the various situations of interaction with students (Ponte, 1999; 2012). It also encloses the capacity of adapting the subject matter to be taught along the various grade levels, of analyzing several methodological resources and of communicating with students (Ponte, 2012; Batanero & Godino, 2005). All domains of didactical knowledge are present, in some way, in the activity of the teachers when they teach (Ponte, 2012). Yet, the knowledge of the instructional process constitutes the fundamental core of the didactical knowledge model, since teachers' fundamental decisions that orient their practices and regulate their teaching processes are made in the scope of this knowledge (Ponte, 2012).

The Teaching and Learning of Statistical Variation

The fact that many mathematics teachers have a limited training in statistics and have not experienced data analysis makes it difficult to develop their statistical thinking, which has effects on their practices (Ben-Zvi & Garfield, 2004). Indeed, more often than not, teachers focus on computations and procedures rather than on data analysis and interpretation of results (Rossman, Chance & Medina, 2006). Sánchez and colleagues (2011) still add that teachers do not give enough attention to explanations and adequate use of terminology in statistics. Although acknowledging the usefulness of technical procedures, Scheaffer (2000) stresses the need for a genuine "understanding of analysis and communication of [statistical] results" (p. 158).

Wild and Pfannkuch (1999) assume that the *consideration of variation* is one of the main aspects of statistical thinking and describe it as the capacity of measuring and modelling variation and making decisions from data. It also involves looking for and describing patterns in the variation and trying to understand these with respect to the context. The term *variation* describes or measures the change in the variable, while the term variability refers to the phenomena of change (variable) (Reading & Shaughnessy, 2004). In this paper these terms are used indistinguishably.

Several studies state that students face serious difficulties in perceiving some measures of variability. In order to help them developing their reasoning about variability more efficiently, Garfield and Ben-Zvi (2008) suggest that their initial statistical activities should move from understanding informal ideas (e.g. differences in data values) to understanding and interpreting formal ideas of variability (e.g. standard deviation as measure of average distance from the mean; factors that can cause standard deviation to be larger or smaller). The concepts of distribution, mean and deviation from the mean are essential for reaching a meaningful notion of standard deviation (Sánchez et al., 2011) which, in turn, is important for understanding variation. It is crucial that teachers help students to build on their intuitive statistical concepts toward a more sophisticated understanding.

METHOD AND CONTEXT

This study is part of a larger ongoing qualitative research that focuses on Portuguese secondary mathematics teachers' didactical knowledge of statistics. This communication refers to Estela, one of the three case studies of the wider research. In 2010/2011, they taught a 10th grade class and demonstrated interest in participating in this investigation. Estela has a master degree in mathematics education and she has been teaching mathematics for 23 years.

The data that informed the present study were collected using several instruments, namely: (1) participant observation, with audio and video recordings of eight lessons on statistics taught by Estela to a class of 25 students; (2) three semi-structured interviews to Estela with audio recording; and (3) documental collection of the resources used by the teacher in the observed lessons, especially working sheets. The data analysis was accomplished in a descriptive and interpretative way, pondering and articulating the various collected elements. This analysis was guided by four assumed pre-categories that correspond to the four domains of the teacher's didactical knowledge of Ponte (2012, 1999).

The environment in Estela's class can be characterized as informal, with a good pace of work and a positive interaction between teacher and students and among students. The concepts or topics to be taught are often introduced through a task. Most of these tasks are selected by Estela from different textbooks and only a few are adjusted or created by her. Students usually work in pairs and the task's correction is mostly performed by the teacher in interaction with some students, once she realizes that most students have completed its resolution.

Estela followed the sequence of contents suggested in the secondary mathematics curriculum (DES, 2001), which is divided in three parts with the following order: (1) generalities about statistics (historical evolution, aim and utility; population, sample, sampling - some intuitive ideas; the statistical procedure: descriptive and inferential statistics); (2) organization of quantitative and qualitative data (process of summarizing data sets numerically and graphically); and (3) introduction to the study of two-dimensional distributions (graphical and intuitive approach).

RESULTS

According to Estela, the 10th grade mathematics curriculum regarding statistics focuses mainly on descriptive statistics, that is, on the summarization of quantitative and qualitative data through the use of multiple representations, giving little attention to inferential statistics and to interpreting results. Estela considers that the fact that statistics is not an object of assessment on national examinations and has no continuity throughout the remaining secondary grade levels contributes to a smaller importance given by teachers and students in comparison to other curricular topics.

Referring to the curriculum, Estela says that "There is almost no inductive statistics in here!", considering that it calls very little to making inferences and does not value the generalization over a population based on a sample. Furthermore, she says that:

> That's why students tell me "Teacher, I can't wait to get started with statistics". Because they already know most of it [the contents]. What is really new about it? Maybe the spread measures and the two-dimensional variables (…) Also [the boxplot], but now in the new [basic education] curriculum it's there. It makes me wonder. (…) When the new generation in the 8th grade gets to 9th, there will be a new 10th grade curriculum because the boxplot won't be taught if they've had it in the 8th. Will this be a factor for a change? Will there be room for working more on inferential statistics?.

Through this excerpt, Estela enhances the need of a new statistical curriculum at secondary level that takes into account its inferential side, since the majority of the issues approached for the first time at this level were recently enclosed in the mathematics curriculum for basic education.

In her practice, the concept of (arithmetic) mean was mentioned for the first time in class by a student who referred to it describing its computation formula. And it was only in this way that the concept of mean, often targeted in the observed lessons when solving several tasks, was approached by students and Estela. It is a value that was most of the time computed by a graphical calculator (TI-84 plus) after students inserted the list of numerical data and followed the instructions: *Menu Stat* - choose *Menu CALC* - choose *select option 1: 1-Var Stats* with the corresponding data list.

However, in one of the interviews, while reflecting on some informal ideas attained by students concerning the mean, Estela mentions its visual representation, considering that the mean may be more difficult to be visualized by the students in some graphical representations (histograms, bar diagrams) than the median:

> Since the mean is very influenced by outliers but I believe that they were able to more or less figure out if the mean was on the left or on the right of the median. (…) The intuitive idea is trying to approximate the mean and the median, but sometimes it is not [possible], just when the [data] distribution is symmetrical. This is the idea that I have about what they got [about the mean].

In this excerpt, when mentioning the visualization of the mean graphically, Estela seems to convey that the fact that there may exist outliers in the distribution may affect the symmetry, and that the visualization of the mean by the students, in this case, may not be so easy. However, she considers that students would not have difficulties in finding the mean in case the distribution of data was symmetrical.

Regarding the concept of standard deviation, students got used to see its symbol on the graphical calculator, when they obtained summary statistics for a numerical data set (discrete or continuous) before learning it in class. For Estela, knowing how to calculate the standard deviation by hand, besides being a lengthy procedure, is not a curricular goal. When talking about this concept in class, she shows its formula and mentions that it is “the positive square root of variance”. She defines it as a spread measure that “measures how far the data is from the mean” and exemplifies on the whiteboard how students should compute this measure with a set of numerical data. Yet, she also says that students must determine this value using the graphical calculator, following the instructions: *Menu STAT* - choose *Menu CALC:* choose *option 1: 1-Var Stats* and add *List*, whose corresponding value is found in S or in σ.

With the intention of showing a different way of interpreting standard deviation, Estela talks about the confidence interval “mean *plus* or *minus* standard deviation that contains approximately about 68% of data…”. She checks together with students the veracity of this statement, reaching the conclusion that in fact 71% of the data were included in the referred interval. Additionally, she alerts very quickly that this distribution of data should be of a “particular shape” in order for this property to hold, saying that it is something to be learned with much more detail in college.

In Estela’s opinion, the intuitive ideas that her students got about standard deviation could be summarized as: “given two sets of data values with the same mean… the one that has larger standard deviation value is the one where data is more spread out [with respect to mean]". This excerpt seems to indicate that this teacher considers that students take standard deviation as a spread measure. Her intuitive idea is correct, as long as none of the distributions have outliers.

Estela believes that statistics is a topic teachers do not fully master. For example, with respect to the concept of standard deviation, Estela only recently learned that after making the computation on the graphical calculator for obtaining the standard deviation value, she has to choose one of the two possible results. That is, this choice depends on whether the data set corresponds to a sample or a population. In her opinion, this issue is not highlighted in school textbooks and moreover is not part of teachers' usual content knowledge. Having in mind the computation of standard deviation on a graphical calculator, she mentions that

> Because sometimes it shows up in the graphical calculator… when I'm computing standard deviation S and Sigma appears on screen. You [may] remember that two or three years ago in a vocational mathematics exam there was a big issue because students were choosing the σ value [population standard deviation]… but since the task stated it was a sample then they [students] had to consider S [sample standard deviation]. And who were the teachers aware of it? None. (…) Do you know where I heard that? In my master degree, never in college [during undergraduate studies]! (…) And nowadays they [teachers] reach it [this distinction]. Textbooks didn't.

Since the graphical calculator is a mandatory resource at secondary level, Estela thinks that one must take as much advantage as possible from this device, particularly useful in students' learning, namely when working on a task with a lot of data, releasing them from the burden of extensive computations. In her view, this resource can also help to promote teachers' statistical knowledge for those who lack preparation in this field, allowing them to develop some skills in statistics, as it generates additional questions about results and representations:

> Since now they are not required to know how to compute those formulas with a lot of data, computing the mean is unthinkable, computing the median is unthinkable, the quartiles, the standard-deviation. (…) I'm still from a time when the graphical calculator wasn't mandatory in school and of making huge handmade tables on the blackboard, it doesn't make sense [anymore]… they [the students] found them very difficult and I thought it was the easiest part… I'd start with the mean, then would do x minus the mean… it was very easy and the calculator [nowadays] does it easily. And with the calculator we can have more data and even us teachers who didn't have this teacher preparation [in statistics]… for us to learn and develop more ideas and skills about statistics [with the obtained representations]… it may generate more discussion and one can do more than trivial things.

Estela thinks that the statistics in the secondary mathematics curriculum emphasizes, above all, its descriptive side, where computation and procedure repetition prevail, with little space for carrying out interpretations and reaching conclusions. Thus, in her practice, the tasks that she often asks students to work on contain several subsections, each of them very directed and with little focus on interpretation:

> The questions [in statistics] have to be asking more [specific] things (…) and are very oriented to answer something, in Mathematics it is just that something, it is not for discussing… Since when? ... Mathematics is exact and it's over! ... and statistics is the same… it is for reaching that something and nothing more!

In one class, Estela proposed a task that she designated by "Properties of the mean and standard deviation" (see figure 1 below):

In a company the wages in the year of 2005 were the following:

Wages (euros)	600	700	800	1000	1500	2000	4000
Absolute Frequencies	20	50	15	9	3	2	1

a) Compute the mean, mode, median, quartiles and the standard deviation of this distribution;
b) Supposing there was a 5% wage raise in 2006, what would be the wages in 2006?
c) Compute the mean, mode, median, quartiles and the standard deviation of the 2006 wage distribution with the 5% raise. What do you conclude, comparing with the results from a)?
d) Supposing there was a 50 euro raise in 2006, what would be the wages in 2006?
e) Compute the mean, mode, median, quartiles and the standard deviation of the 2006 wage distribution with the 50 euro raise. What do you conclude, comparing with the results from a)?

Figure 1: Task about the properties of the mean and standard deviation

Estela thinks that this task, proposed in a working sheet, was very successful in promoting the study of those two concepts starting from the very same problematic set up, as opposed to the tasks she found in textbooks which presented separated contexts for analyzing the proprieties of each concept.

After reading the task, Estela asked students to make all the requested computations at once on the graphical calculator. Next, she reminded them that they should insert two different lists of numerical data (one corresponding to the wages and the other to their respective absolute frequencies). Estela showed how to proceed with the calculator. Several students revealed difficulties in understanding why they had to multiply by 1.05, instead of multiplying by 0.05, for wages to increase by 5%. Estela spent some time trying to clarify this issue. After having made all the initially requested computations writing them on the whiteboard – simultaneously with students – and having pointed out the influence of making a specific percentage or constant growth in the data values and their effect on each statistical measure, Estela concluded the task in the following way:

> So, the story's moral: (…) if I multiply the values of the statistics variable by a number, all the statistical measures become increased by that number, increased…, I mean… multiplied. But if I add up a number to the statistics variable, all the values of the statistical measures become increased by that number, except for the standard deviation that remains unchanged. Is it understood?

The development of this task was quite centered on the computation of the statistical measures using the calculator and on the comparison of the different attained results, as the initial data were being changed. This task was not restricted to the proprieties of the mean and standard deviation since the mode and quartiles were also compared. Neither the teacher nor any student questioned any obtained result, in particular, why the behavior of the standard deviation was different from the other statistics measures when the initial data were increased by 50. This situation may suggest an overvaluation of the results from the graphical calculator, which may have hindered from wondering why the standard deviation remained unchanged. Teacher and students reached the right conclusions without interpreting and understanding deeply the formal ideas of variability of this task.

DISCUSSION

We now analyze the dimensions of Estela's didactical knowledge of statistics, with particular incidence in variation, emerging from her practice. Estela shows evidence of owning a *knowledge of the curriculum* restricted to some aspects with particular incidence on descriptive statistics. Such knowledge was noticed in her practice, for example, when working on the concept of mean, focusing mainly on its computational side (Rossman et al., 2006) and not mentioning explicitly others possible interpretations for the mean.

The *knowledge of the students and their learning processes* as a result of her statistics teaching experience gives her the conviction that students have more difficulties in seeing the mean graphically than the median. However, Estela thinks that in the presence of a symmetrical graph or an approximately one, students know that the values of the mean and the median are close. Also, concerning standard deviation, she considers that students are able to identify it as a spread measure, taking into account the residual values with respect to the mean value. Ponte (1999) and Batanero and Godino (2005) pointed out that teachers need to be aware of their students' reached level of comprehension of the concepts, as well as difficulties, in order to improve their teaching and assessment practices.

Estela's work in the classroom on an application of the standard deviation and the search for a confidence interval seem to indicate that, in her view, the topic of statistics should be expanded in terms of content, encompassing more inferential statistics. The same happens when she refers the need to develop the secondary curriculum given the recent curricular changes in basic education, which started to incorporate much statistics content currently addressed at the secondary level. However, in the task that she called "Properties of the mean and standard deviation", which had potential to help students to develop their reasoning about centre and variability, Estela ended up giving work orientation much centred on computations using the calculator, and in this interaction, neither the students nor the teacher questioned the obtained results. This situation suggests a certain overvaluation of the calculator results, since the need of reinforcing the interpretation and explanation of results was not observed. This is a commonly found phenomenon in mathematics teachers (Rossman et al., 2006; Sánchez et al., 2011) and resonates with Estela's interpretation of the written curriculum. Furthermore, this interaction did not bring along any conjecture to be pursued. In fact, there are other properties, regarding variation, that could have been object of reflection in this or other tasks, namely that the median is a value located between the extreme values of the distribution, or that the residual sum with respect to the mean is zero (Garfield & Ben-Zvi, 2008). Nevertheless, these situations did not occur in Estela's classroom, raising important questions concerning her *knowledge of the instructional process* in teaching variation.

In her instructional process, Estela seems to give some relevance to the deepening of the mean and standard deviation concepts, since she proposed a task in this direction; yet, the understanding of formal ideas of variation as suggested in the literature (e.g. Garfield & Ben-Zvi, 2008; Sánchez et al., 2011) does not seem to be totally achieved. This situation may be related to Estela's formal comprehension of the standard deviation concept as noticed while presenting its definition in class or when reflecting on the informal ideas that students may hold about standard deviation. For instance, in the latter situation, she did not contemplate all the hypotheses that would make it always valid. Her *knowledge of the content in statistics* seems to affect the level of depth in which these concepts were developed in the lessons.

This analysis of Estela's practice gives evidence that the various domains of the teacher's didactical knowledge are interrelated, and consequently they must be taken into account in the study of the teachers' practices and also in teachers' preparation and professional development programs.

NOTES

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PROSPECTIVE ELEMENTARY SCHOOL TEACHERS' INTERPRETATION OF CENTRAL TENDENCY MEASURES DURING A STATISTICAL INVESTIGATION

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The introduction of statistics as a topic of the elementary school curriculum makes it necessary to reinforce teachers' preparation in this area as well as to understand prospective elementary school teachers' statistical content knowledge. This paper aims to understand the meanings that prospective teachers give to central tendency measures during a statistical investigation. We observe innovative interpretations as well as interpretations based on the context of each question and showing real understanding of concepts. But we also notice that most groups do not interpret central tendency measures, just analyzing data by reading graphs and tables. For the future, we suggest that prospective teachers must work with tasks requiring the interpretation of different measures to understand the differences among them.

Keywords: Statistical investigation, Mean, Median, Mode, Teacher education.

INTRODUCTION

The Portuguese official curriculum documents for kindergarten and primary levels (ME, 1997, 2007) give emphasis to statistics. This topic has great importance nowadays since society depends more and more on results from statistical studies (Groth, 2006). With industry, medicine and other society sectors recurring to data to make decisions, statistics should be a focal part of the new information era (Wild & Pfannkuch, 1999). Statistical literacy, regarded as the "ability to interpret, critically evaluate, and communicate about statistical information and messages" (Gal, 2002, p. 1), is essential in the education of every citizen and naturally in the education of prospective teachers. Therefore, it is important to know how prospective teachers interpret and communicate statistical information.

Statistical investigations allow students to become active in the learning process. During these projects, students pick a theme of their interest, define goals, select instruments to collect data, choose samples, collect, analyze and interpret data to answer the proposed questions (Batanero & Godino, 2005). Additionally, during an investigation, they perform every step of the PPDAC cycle (Problem, Plan, Data, Analysis, Conclusions) described by Wild and Pfannkuch (1999), in an environment inciting meaningful learning (Ponte, 2007). Furthermore, students are able to appreciate the importance and the difficulty of the statistical work and the interest of statistics in solving real life problems (Batanero & Godino, 2005). Moreover, teaching through statistical investigations allows the identification of students' difficulties in their mathematical knowledge and, sometimes, even to detect concepts and ideas that seemed well consolidated but are not (Ponte, 2007).

This article shows part of an investigation that aims to understand prospective teachers' statistical and didactical knowledge. Our specific aim is to analyze the ways prospective teachers (for Pre-K to grade 6) interpret central tendency measures (mode, mean and median), based on their reports of statistical investigations.

CONCEPTUAL FRAMEWORK

Teachers are a key element in the educational process (Ponte, 1994). They need to know in depth the content they teach, as this is true even for teachers of early years (Ma, 1999). A solid mathematical knowledge is essential to promote a learning environment where students want and can learn mathematics, and this must be addressed since preservice teacher education. Heaton and Mickelson (2002) state that teachers' statistical knowledge encompasses the ability to conduct statistical investigations, describe information using different methods and form conclusions. To Mulekar (2007), prospective teachers must have a deep comprehension of statistical concepts in order to give coherent meaning to results. Central tendency measures (mode, mean and median) have a particular interest as they are frequently found in daily life (Groth, 2006). To Groth, understanding of these measures is an important component of statistical literacy. Nevertheless, according to Jacobbe (2008), even grade 1 exemplary teachers do not have conceptual knowledge of the two most basic statistical concepts – mode and mean.

In Portugal, the mode appears in the official curriculum documents in grades 3 and 4 (ME, 2007) and teachers tend to consider it an easy concept to understand. However, some studies suggest a more complex picture. For example, Fernandes (2009), researching difficulties and errors in statistics from prospective teachers (for grades 1 to 6), refers gaps on the comprehension of this concept, especially when they select the biggest frequency instead of the corresponding value of the variable. In a study with 40 prospective teachers (for Pre-K to grade 6) in their undergraduate program it was frequent to find answers like "the mode is 9, since there is a bigger number of students that see television" (Martins, Pires & Barros, 2009, p. 7). In the interpretation of this measure it is recurrent to associate the mode to the biggest number on the results table, the biggest absolute frequency, the value that appears more times and the biggest frequency category or interval (Martins et al., 2009). Some of these interpretations reveal confusion, since the value that appears more times could be seen as the number on the absolute frequencies column that is repeated more often and not as the value of the variable that is repeated more times.

The concept of mean is introduced in grades 5 and 6 (ME, 2007). Research about this concept is vast, since it is very used in statistical studies. Leavy and O'Loughlin (2006) indicate that there are two types of understanding – conceptual and procedural: "Computationally, the arithmetic average is the score around which deviations in one direction exactly equal deviations in another direction" (Leavy &

O'Loughlin, 2006, p. 55); conceptually, the mean may be seen as a balance point or center of gravity, representing the data set. To the authors, interpretations of the mean as the fair share (the value that represents the data set as if all data were equal), or as the balance point (where higher values compensate lower values) show conceptual understanding of the concept. When computing the mean, a frequent mistake is to determine the mean of the absolute frequencies in qualitative variables (Martins et al., 2009). Students use several interpretations of the mean, just restating the algorithm (Chatzivasileiou, Michalis & Tsaliki, 2010; Fernandes & Barros, 2005), indicating it as the "sum of all results divided by the existing values" or "sum of numbers" (Martins et al., 2009). Others associate the mean to the notion of balance, the average value, the value that balance the highest and lowest values (Martins et al., 2009), the fair share value, the typical expected value (Chatzivasileiou et al., 2010; Konold & Pollatsek, 2004), the location measure (a close but not exact value) (Chatzivasileiou et al., 2010), a signal in noise (where the mean of different observations is a close approximation to the actual value, ignoring the errors) or a data reduction value (a value to reduce the complexity of all data) (Konold & Pollatsek, 2004). As incorrect interpretations, there are answers based on the maximum value, the minimum value, a specific value, the median and the mode (Chatzivasileiou et al., 2010; Leavy & O'Loughlin, 2006). On the latter cases, students lack recognition of the data set as a whole and tend to focus on individual values (Chatzivasileiou et al., 2010).

Concerning the median, concept of grades 7, 8 and 9 (ME, 2007), the scenario is also problematic. Research shows that there are difficulties on the understanding and interpretation of this concept on grade 12 students (Fernandes, 2009) as well as on prospective teachers (grades 1 to 6) (Fernandes & Barros, 2005). Computing it, several prospective teachers indicate the central value of the absolute frequencies, others confuse this concept with the mode, and the most frequent mistake is to determine the central value without ordering the data (Martins et al., 2009). Even some grade 1 experienced teachers calculate the mean when they are asked for the median (Jacobbe, 2008). Interpretations of this concept include associations to the point where the number of values above equals the number of values below (Konold & Pollatsek, 2004), to the central value (although not always on the most correct way) as well as to "the value in the middle", the value that "divide in half", the value that is "somewhere in the middle", the second quartile (without more explanations), the value that "divide the sample in half and balance big values with small ones", the "average number of all results" and the "mean value" (Martins et al., 2009). The last three interpretations show some confusion between the concepts of median and mean.

METHODOLOGY

Participants in this study are prospective elementary (grades 1 to 6) and kindergarten teachers in an undergraduate program of a Portuguese school of education that took a course on Discrete Mathematics, Statistics and Probability during the 2nd semester of their 2nd year of studies, in 2010/11. This is the single course in the program dedicated to the development of statistical knowledge. During the course, prospective teachers worked statistical concepts through exploration and discovery, and strong emphasis was given to their interpretation in real contexts. Additionally, they worked with Excel to organize data and calculate statistical measures. Prospective teachers also carried out statistical investigations, individually or in groups (of 2-3), in themes chosen by them, with a look on their possible use with their future students. During such investigations, prospective teachers were asked to make records and do a written report to be presented to the class at the end of the semester, including data organization (indicating all relevant measures), analysis, interpretation, conclusion and a reflection about their work. Towards the end of the work each group received feedback from the teacher with questions to help them to reflect about what was done but never received simple corrections of it. After the presentation they received a final grade for all the process and final product. In this article we analyze these written reports to discuss their understanding of central tendency measures.

The 36 prospective teachers that attended the course were organized in 16 groups (6 groups attended day classes – D – and 10 groups after work classes – PL). 21 out of the 36 authorized the participation in this study. They were organized in 12 groups and chose themes such as recycling, food, water habits at home and routines. They undertook statistical investigations through questionnaires with 12 to 25 questions, with about 70% of the questions involving qualitative variables. When working the data, all groups used Excel, making use of the corresponding functions to determine the mean and the median. Regarding the mode, they realized it was more convenient to not use the Excel function "mode", since this does not work when analyzing qualitative variables and also when the variable has more than one mode. Only two groups made a mistake determining measures of a variable, giving an absurd value for the mean in the particular context. All the other groups found correct measures, showing that Excel can be a useful tool to prospective teachers. Given the use of technology for computations, the focus of this paper was on the interpretation of such measures in the context of each question. This means that, for a single concept, a group may give different interpretations, depending on the questions, and, therefore, a group may be taken into account in different interpretations. Analysis of data was exploratory and involved the categorization of interpretations observed in these reports. Categorization was made according to definitions and cases of doubt were sorted out between the two authors. Codes are used to identify the written reports of groups (G1 through G10) and the class (D/PL).

RESULTS

Mode

There are a large number of interpretations for the mode concept in the written reports of statistical investigations (Table 1). Four groups associated the mode to something that happens "more times" like happened in previous studies (Martins et al., 2009). Example of that is the analysis of the question "What is your profession?" where the group stated "the mode of the profession group is 'intellectual and scientific professions', since it is the profession group that is repeated more times in the sample" (G4D, p. 11). Interpretations made by five groups also associated mode to an answer that appears a large number of times, but made reference to "frequency" or related words, like "the number of accompanying people more frequent is 1 (mode)" (G3D, p. 23).

Appropriate interpretations	Number of groups
Value associated to the highest frequency	5
The majority of the sample…/most…	4
Value referred/repeated/verified/chosen/appears more times	4
Value associated to more/greater number of respondents	4
Other interpretation using "more"/"bigger" in context	2
Predominant value	1
Satisfactory value	1

Table 1: Appropriate interpretations of the mode

Four groups used expressions like "the majority" or similar that imply a correct idea: "the mode is 3, therefore the majority of students already went 3 times to the theatre" during the analysis of the question "how many times did you go to the theatre?" (G6D, p. 28). Also related to the size of the sample was the category of interpretations where the mode was associated to more or to a greater number of respondents. In that category were interpretations like "the greater number of respondents is from the feminine gender (mode)" (G3D, p. 6). A similar idea seems to be meant by a group that referred something that is predominant ("it is the feminine gender that predominates" – G6D, p. 23) and by another group that indicated "here we obtained a satisfactory answer of 'yes', being the mode of this qualitative variable 'yes'" (G8PL, p. 8). Interpretations of the mode made by two groups were connected to the context, making use of the expressions "more" or "bigger": "the television is the information method more used by the respondents (...). Hence, it can be concluded that the mode is television" (G2D, p. 17).

Table 2 summarizes the interpretation problems made by prospective teachers. Two groups did interpretations also associated to the size of the sample, but the expressions chosen were not as efficient as in the previous cases. For example, in the analysis of the question: "Since when do you recycle?", a group used the expression "we noted that 36% of the people started to recycle between 2006 and 2008, where, because this is the mode interval, we conclude that the greater part of the sample started to recycle by that time" (G2D, p. 9). This response is problematic due to the fact that if something is happening to a large part of the sample that does not mean that that "part" is the greater and, consequently, is the mode.

Additionally, a group, in the analysis of the question "Where do you eat breakfast?", refers "the mode (…) is home, since is the higher value in the graph and on the table"

(G1PL, p. 29). This statement, encountered also in previous studies (Martins et al., 2009), can be problematic when the variable is not qualitative, since in the quantitative case the highest value on the table could be from the variable and not from the frequency.

Problematic interpretations	Number of groups
A great part of the sample…	2
Highest value	1
Confusion	1
Without interpretation (quantitative variables)	7
Without interpretation (qualitative variables)	9

Table 2: Problematic interpretations of the mode

A group that made the following statement: “the ages to which we made more questionnaires were young people aged 19” (G1D, p. 8). We observe some confusion but it is unclear if it is in the comprehension of the mode or in the phrase construction in Portuguese. This statement conveys the idea that the questionnaires were made to ages and not to people, which can be confusing to who reads the reports.

It should be pointed out that, for the case of the mode, all groups who had problematic interpretations of this measure were also able to give correct interpretations. Therefore, the problems with some interpretations may be due to the fact that they tried to give an alternative description for the mode. Also important is the fact that in the case of qualitative variables, nine groups did not make any reference to the concept of mode and to what this measure represents and means when analyzing the data. The same happened to seven groups regarding quantitative variables. This means that, especially in the cases of variables where the mode is the only statistical measure that can be determined, some participants make only readings from graphs and tables, which seems to be easier to them, since they only need to extract data from the representations.

Mean

Table 3 summarizes the adequate interpretations of the concept of mean. A group argued the following expression “the average age is 3.4, which represents the age balance” (G1PL, p. 11). This response demonstrates that the group understands that the mean can be seen as a balance point (Leavy & O’Loughlin, 2006), but it is unclear the real understanding of the prospective teachers of the meaning of this measure. This interpretation may reveal conceptual comprehension (Leavy & O’Loughlin, 2006), but this is not evident, in this case.

Two groups used the fair share model when interpreting the mean, like the example of the analysis of the questions “How many favorite games do you have?” where the group wrote "it means that if every child had the same amount of favorite games, each one had 13, 33 games" (G2PL, p. 21). This type of interpretation also found by others researchers (Chatzivasileiou et al., 2010) reveals conceptual understanding of the mean concept as suggest by Leavy e O'Loughlin (2006).

Appropriate interpretations	Number of groups
Value that represents the equilibrium (balance point)	1
Fair share model	2

Table 3: Appropriate interpretations of the mean

Problematic interpretations of the mean are associated with misconceptions regarding the distinction between this concept and other statistical measures (Table 4). The members of a group, both times they tried to interpret the mean, confused mean and spread measures, using the mean (and the median) to indicate whether the data was spread or not: "the values of the mean is 7.6 and of the median is 8, then we can conclude that the values are not very spread" (G6D, p. 28). We note that, in some quantitative variables, all groups made statements like "the mean is…" or "the sample, in average…" that do not represent an interpretation of the mean.

Problematic interpretations	Number of groups
Confusion between the mean and spread measures	1
Without interpretation	12

Table 4: Problematic interpretations of the mean

Median

There were several adequate interpretations of the concept of median (Table 5).

Appropriate interpretations	Number of groups
Value that divides the ordered data	1
Up to the median there is 50% of the sample...	1
50% of the respondents… or less…	3
50% of the respondents… at maximum	1

Table 5: Appropriate interpretations of the median

We observed the following interpretation of one group: "up to the median there is 50% of the sample, and after the median, there also is" (G5D, p. 13). This shows that these participants comprehend that the median divides the sample in half, 50% to each side of the median. Nevertheless, this can be problematic because of the non-inclusion of the median on the second half of the sample ("after the median").

Another group made the interpretation "the median is 7.5, which is what divides the ordered data" (G4PL, p. 11), revealing that the group understands that the median is a number that divides the data when ordered, but do not reveal a real understanding of this concept in context.

Four groups make appropriate interpretations showing understanding of this concept in the context of the data, like: “50% of the people expect to return to the fair, in the same year, at maximum 2 times” (G3D, p. 31) or “50% of the children have 4 or less games” (G2PL, p. 21). This type of interpretation discloses understanding of the meaning of median in the context of the variable at study, making use of expressions that, in reality, make sense on data interpretation.

Nonetheless, not all interpretations were appropriate (Table 6). Three of the interpretations show that the median is a central value that divides data in 50% on each side, however without revealing the real comprehension of the concept. For example, one group wrote: “the Q2 is the median, which mean, the value that is in the center, equivalent to 50%” (G1D, p. 11). This interpretation shows confusion as it states that the median is equivalent to 50% of the data, which is incorrect. In the following example a group tried to make a statement like the one previously indicated as appropriate and very useful in context, however got confused, maybe because they do not really understand the concept: “50% of the people at maximum come to the fair with 3 accompanying persons” (G3D, p. 23).

Problematic interpretations	Number of groups
Value that is in the center	1
Value equivalent to 50%	1
Value to a maximum of 50% of respondents	1
Confusion between median and spread measures	1
Without interpretation	12

Table 6: Problematic interpretations of the median

Like in the case of the mean, one group used the median to make assumptions about the data spread, showing some confusion between the median and other statistical measures. In the case of the mean, contrarily to the previous measures, there are groups that only give adequate interpretations, groups that only give problematic interpretations, and also groups who are taken into account with both adequate and problematic interpretations. Moreover, it should be noted that none of the groups made the interpretation of the concept of median in all quantitative variables that they studied.

CONCLUSIONS AND IMPLICATIONS

The analysis of the written reports of the 12 groups highlights gaps on prospective teachers’ understanding of central tendency measures. Analyzing contextualized data, they interpret the measures in a way that demonstrates some confusion about their meaning, sometimes making no distinctions among statistical measures. Additionally, they use interpretations not mentioned in previous researches maybe probably because they had to write a report with the analysis of several variables and try to not repeat the same interpretation all over. Example of that are the interpretations of the mode like something that happens to “the majority” and to “the greater number” of respondents (appropriate interpretations) or as a “great part” or “higher value” (problematic interpretations). Moreover, some groups gave appropriate meanings, like indicating the mean as a balance point (Leavy & O’Loughlin, 2006) or referring to the median as the center of the ordered data, but without showing a real comprehension of the meaning of the concepts. Additionally, we observe concrete and contextual interpretations like in the case of the mode as the value related to the majority, the mean as the fair share model and the median in where 50% of the sample is associated to this value or less. We suggest that interpretations related to the context of data are those that demonstrate stronger understanding of the concepts. Furthermore, an interpretation may be categorized as appropriate or not, depending both on the type of variable (quantitative or qualitative) and on the associated information (for example, if it is stated the relative frequency in the case of mentioning “great part”). Lastly, most or all groups do not make interpretations of central tendency measures, giving only readings of graphs and tables.

These results are important since they show a large variety of interpretations of central tendency measures, some appropriate and others not, that go beyond those mentioned in previous researches (e.g., Chatzivasileiou et al., 2010; Fernandes & Barros, 2005; Martins et al., 2009). Prospective teachers should experience and discuss all of them, in order to realize which ones better convey the information provided by data. Additionally, it is essential to make prospective teachers to compare and contrast the three central tendency measures to recognize their differences. Finally, this study shows the importance of emphasizing concrete interpretations of contextualized data, to consolidate the understanding of central tendency measures and to learn to distinguish them and grasp their utility.

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COPING WITH PATTERNS AND VARIABILITY: RECONSTRUCTION OF LEARNING PATHWAYS TOWARDS CHANCE

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Dealing with the phenomenon of chance in a mathematically sustainable way requires not only the prediction of outcomes using probabilities, but also taking into account the omnipresent variability as a constituent property of chance experiments. This paper investigates how students aged 11 to 13 make sense of random data from chance experiments by relating patterns and variability to one other. The empirical findings from design experiments are analysed using an interpretative framework. The presented case of two students illustrates how they gradually move from investigating and explaining single deviations to developing a conception of variability as a phenomenon which is especially visible for short series of repetitions.

Key words: *variability, patterns, knowledge construction, design research*

Probability education is widely reported as quite challenging for students, as pre-instructional conceptions seem to persist through school and beyond, even if they oppose mathematically sustainable ones (overview in Shaughnessy, 2007). One explanation of the impact of these initial conceptions is a constructivist perspective on learning: individual, active constructions of mental structures always build upon the existing mental structures by accommodation to experiences with new phenomena. Thus, initial conceptions have a major influence on substantial teaching-learning processes, which seems especially true for stochastics education (cf. Prediger, 2008). The reason for this may be the proximity of the everyday phenomenon of chance (in which the unpredictability of outcomes is often emphasised) and the related, but different, mathematical one (in which probabilities make predictions for patterns in the long run). The nature of chance itself thus seems to be counter-intuitive: patterns and variability are seemingly counterparts but only together do they constitute what stochastics is about.

Building on a previous CERME paper (Prediger & Schnell, 2011), this article focuses on the question of which conceptions students construct concerning variability and patterns when they are working on the experiment-based learning environment “Betting King”.

PATTERNS AND VARIABILITY AS COUNTERPARTS

Moore (1990) uses the interplay of variability and patterns as a defining element for the phenomenon of chance itself: ”Phenomena having uncertain individual outcomes but a regular pattern of outcomes in many repetitions are called *random*. 'Random' is not a synonym for 'haphazard' but a description of a kind of order different from the deterministic one" (p. 98, emphasis in original). This definition gives way for different insights that are crucial for stochastics:

Patterns and variability: While especially frequentist approaches to probability are concerned with identifying *patterns* in many repetitions of chance experiments (as estimations for unknown probabilities, see below), it is the omnipresence of *variability* (i.e. "the propensity for something to change", Shaughnessy, 2007, p. 972), which is causing the uncertainty that stochastics deals with (cf. Moore, 1990, p. 135; Wild & Pfannkuch, 1999).

Short-term and long-term distinction: Even though variability is omnipresent, patterns (also called "signals", Shaughnessy, 2007, p. 973) can still be identified, especially in the long run. This is well known as the empirical law of large numbers: While relative frequencies vary a lot for only a *few repetitions* of an experiment (short-term perspective), the variation of relative frequencies is smaller for a *larger number of repetitions* (long-term perspective) (cf. Fig. 1). The patterns identified in the long-term perspective are on the one hand an estimation for the theoretical probabilities. On the other hand, theoretical probabilities of an event (determined *a priori* by the ratio of the number of favourable cases to the number of all cases of the event when each case is assumed to have the same probability) serve as prediction of the patterns which will be observable in series of many repetitions.

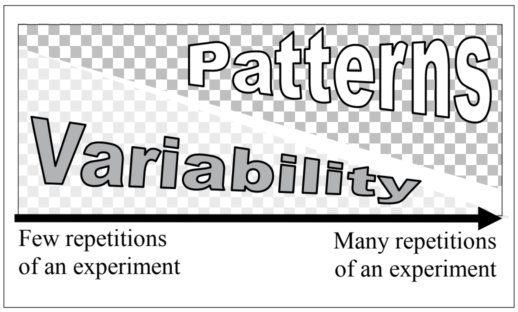

Fig. 1. Observability of patterns and variability in relation to the number of repetitions

Thus, when learning about probabilities in an experiment-based approach, students have to make sense of patterns and variability. To do so, the distinction between *short-* and *long-term contexts* is crucial (Prediger & Schnell, 2011).

PROCESSES OF LEARNING ABOUT VARIABILITY AND PATTERNS

There has been some research about how students cope with variability in statistics (especially in exploratory data analysis; overview in Shaughnessy, 2007). Results such as Ben-Zvi's (2004) show how students gradually develop a sense of how to take variability in statistic samples into account: They use frequency overviews, address outliers and compare variation within and between distributions.

For probability education though, research on learning pathways of students coping with variability and patterns is still limited (cf. Jones et al., 2007, p. 928): For instance, Pratt and Noss (2002) investigate how students (aged 10 and 11) can successfully progress from a short-term context to a long-term context when working in a supporting learning environment. While initially conceptions such as unpredictability of single events are dominant, new conceptions emerge in the learning process: Patterns are more stable for large numbers of experiments and expected or observed patterns can be linked to theoretical probabilities. These results give insights how students gradually develop a sense for the long-term context and sustainable patterns in the long run. The question remains how students build an integrated knowledge of both patterns and variability.

LEARNING ENVIRONMENT 'BETTING KING'

To help learners gain experience with the phenomenom of chance, they have to be provided with opportunities to investigate data from random experiments, allowing them "to become aware of the underlying differences between short-term and long-term perspectives" (Prediger, 2008, p. 151). For this purpose, the learning arrangement 'Betting King' (to be published in Prediger & Hußmann, 2014) was developed as part of a textbook, introducing probability in grade 7 (students of age 12 to 13). The main activity consists of predicting results of a chance experiment: The players bet on the outcomes of a race with four coloured animals, which make a step forward when their colour is rolled with a coloured die with asymmetric colour distribution (red ant: 7/20, green frog: 5/20, yellow snail: 5/20, blue hedgehog: 3/20). The term 'outcome' (or 'position') refers to how often each animal was rolled. The game is first played on a board, which is later replaced by a computer simulation (Fig. 2). The differentiation between the short-term and long-term contexts is realised by the possibility to choose the number of throws of the die, after which the results of the game are determined (between 1 and 10,000 in the simulation). Furthermore, record sheets and written tasks are used to support the learning processes.

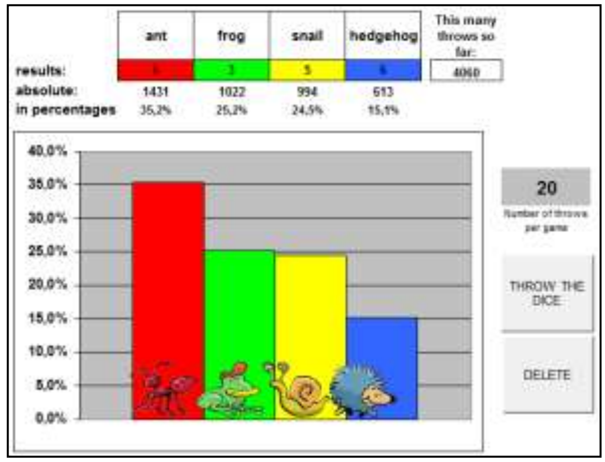

Fig. 2. Layout of the computer simulation at the end of the learning arrangement (translated and slightly rearranged for this paper)

There are two consecutive game settings used to motivate explorations of the generated game results: In the first setting, "Betting on Winning", students bet on the animal that will have the highest outcome and is therefore the winner of the game. Here, students usually soon discover the pattern that the red ant wins more often and can link it to the unfair colour distribution. Next, the security of this pattern is addressed in relation to the total number of throws: While for short games the winning animal varies a lot, but the red ant wins most often for large total numbers of throws such as 2,000.

The second game setting, "Betting on Positions", motivates to more closely examine the generated outcomes: Students play one-on-one against each other and predict for each animal on which position it will land in relation to a pre-set (large) total number of throws. The bet closer to the result wins one point per animal. Soon, the game itself is left and students are encouraged to investigate patterns and variations of positions presented in different representations offered by the simulation (numerical and graphical). Here, the focus of the tasks first lies on comparing random outcomes (i.e. absolute frequencies) for series of games with the same total number of throws (cf. table 1).

Game number:	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.
Position of red ant:	681	692	706	709	682	738	747	670	698	713

Table 1. Series of randomly generated absolute frequencies of the red ant in a series of games with 2000 throws

Students can – for example – compare the (absolute and relative) range of results (as a measure for the variation) with the range of outcomes for a series of games with a smaller total number of throws such as 20 and thus gain deeper insight into the empirical law of large numbers. At the end of working on this learning arrangement, a dynamic perspective is introduced, where relative frequencies (presented as percentages) can be investigated when the total number of throws is increased gradually.

RESEARCH QUESTIONS AND DESIGN OF STUDY

Based on the previously elaborated theoretical perspective and the intention of the designed learning environment 'Betting King', the following questions will be addressed in this paper:

> *Which constructs of variability do students develop over the course of working on the learning environment and how are the constructs linked to an according pattern?*
>
> *How do students develop an overall conception of variability over the course of the design experiments?*

As the empirical data shows, the repertoire of different patterns observed by students is so rich (cf. Schnell, 2013) that this paper will focus on the scenes when students explicitly notice variability in the data in contrast to a related patterns.

Data collection

The foundation for this paper is a study in a laboratory setting, which was conducted to research the individual learning pathways in more detail (cf. Schnell, 2013; Schnell & Prediger, 2012, for the broader design research study). Here, the above presented learning environment was used in design experiments (cf. Cobb et al., 2003) with nine pairs of students (age 11 to 13). Each design experiment lasted between four and six consecutive sessions of 90 minutes. They were semi-structured by an experiment manual that defined the sequence of tasks as well as interventions for anticipated problems in crucial learning. The data corpus of the study includes videos of all 40 experiment sessions, records of the computer simulations as well as all written products.

Data analysis

The *in-depth data analysis* is guided by an interpretative approach, identifying scenes in which new discoveries are made and integrated into the individual conceptions, here conceptualised as a network of *constructs* (cf. Schnell & Prediger, 2012 for the theoretical background of the analytical model). In these scenes, students' individual constructs are reconstructed with respect to four elements: the emerging insights (summarised by <*proposition*>), the *stochastic context* in which an insight occurs, the *situational context* to which it refers (such as the representation; due to page limitations this will not be addressed in this paper) and which *function* the construct has in the specific situation (e.g. explanations, descriptions of deviations). Due to the complexity of the "Betting King" game situation, the stochastic context was specified not only as short-term and long-term context: Students' constructs can relate to **O**ne or **M**any games (differentiation O-M), meaning single games or a series of (at least three) games in which the differences between the outcomes are addressed. Furthermore, the game(s) can have **S**mall or **L**arge total numbers of throws (differentiation S-L); in the learning environment, small numbers are usually represented by 1 to 40 throws and large numbers by 100 to 10,000. In the last phase of working on the given tasks, the continuity between these dimensions is investigated by looking at games with gradually increasing total numbers of throws. While patterns in accordance with the colour distribution of the die can be best observed for the context ML (Many & Long), variability is more visible for the context SM and single deviations (e.g. outliers of positions) in OS and occasionally also in OL.

In this article, the in-depth analysis of one pair of students (Ramona and Sarah) learning pathway is used to exemplify how learners cope with variability. For this, constructs with the function "description of deviations" were picked out and contrasted with constructs describing a related pattern. To show the development of how variability is experienced, the constructs are contrasted and differences regarding the stochastic context, the function and the influence of predictions are identified.

EMPIRICAL FINDINGS: RAMONA AND SARAH'S LEARNING PATHWAY

The case of Ramona and Sarah shows how students start with initial conceptions of variability which then get complemented when the focus is shifted from single games (OS context) to series of games (MS and ML context). Comparisons with other pairs of students will be made in the conclusion.

1. First encounters with variability: Constructs with function explanation (OS-context)

CONSTRUCTS EXPLAINING PATTERNS				CONSTRUCTS EXPLAINING VARIABILITY			
Short-title	**Proposition**	**Sto. con.**	**Time**	**Short-title**	**Proposition**	**Sto. con.**	**Time**
ANT-BEST	<Red ant is always best>	MS	1[1]; 21:15				
COLOUR-DISTRI-BUTION	<Red ant: 7, green frog: 5, yellow snail: 5, blue hedgehog: 3>	All	1; 28:05	HEDGE-HOG-LUCK	<When blue hedgehog is lucky, it will get the blue faces a lot>	OS	1; 30:17

Ramona and Sarah identify the red ant as the best animal (ANT-BEST) already at the end of the second game (with 25 and 38 throws in total). When roughly seven minutes and two more games later the asymmetric COLOUR-DISTRIBUTION of the die is discovered, this is backing up the ANT-BEST pattern-construct (for more detail see Prediger & Schnell, 2011). At this point though, the girls notice that the order of the animals on the board game (with a total number of 28 throws) does not fit the expected order from the colour distribution: blue hedgehog is third and yellow snail is last. Ramona thus constructs HEDGEHOG-LUCK in order to explain the deviation in this one single game from the expected pattern:

Session 1; beginning 30:17;
After having played a game of 28 throws, the colour distribution is discovered

483f. Ramona: Blue has good chances, too, because- (…) Blue has sometimes a lot of luck and then it gets the three faces sometimes very often.

In this scene, variability is noticed as deviation from the pattern ANT-BEST. The discrepancy between the order of animals according to the COLOUR-DISTRIBUTION and the empirical result on the board is solved by building the construct HEDGEHOG-LUCK. Its function is to explain *why* this deviation occurred (i.e. the supposed "luck" of the blue hedgehog). In several other instances, LUCK is used again to *explain single deviations* of expected patterns, for example when blue hedgehog finishes on an unexpectedly high position in 'Betting on Positions'.

2. Patterns versus absence of patterns (ML- and MS-context)

CONSTRUCTS ADDRESSING PATTERNS				CONSTRUCTS ADDRESSING VARIABILITY			
Short-title	**Proposition**	**Sto. con.**	**Time**	**Short-title**	**Proposition**	**Sto. con.**	**Time**
ANT $_{100,1000}$	<For series of 100 and 1000, red ant always wins >	ML	2; 52:20	DIFFER-ENT$_{1,10}$	<For series of 1 and 10 it is always different>	MS	2; 52:20
RANGES $_{2000}$	<For series of 2000, observed positions vary within certain ranges>	ML	3; 45:53	NO-RANGES$_{20}$	<For series of 20, positions are always messed up and no ranges are observable>	MS	3; 46:16
HIGH-BARS-	<The higher the total number of throws, the less	ML	3; 87:39	LOW-BARS-	<For series of low total numbers of throws, bars	MS	3; 88:04

1 The first number indicates the experiment session in which the construct was first verbalised.

LESS	move the bars in the diagrams >			MORE	move a lot>		
PERCENTAGES-LIMITS	<Percentages have limits beyond which they don’t change anymore>	ML dyn	4; 35:05	PERCENTAGES-CHANGE	<With steps of 20, percentages change a lot in the beginning>	MS dyn	4; 42:30

At the end of the first game type “Betting on Winning”, the students are asked to fill in an overview of all used record sheets, summarizing which animals won at least once for the total numbers of 1, 10, 100 and 1000 throws. After this, the girls are asked to verbalise their observation:

Session 2; beginning 52:15; after having filled in the overview table:

1079 Sarah: For 100 and 1000, [red] ant always wins. And for 10 and 1, it’s always different-

In this statement, Sarah contrasts the stable *pattern* of the winning red ant for high total numbers of throws ($\mathrm{ANT}_{100,1000}$; stochastic context ML) with the *absence of a pattern* of a specific winning animal for low numbers of throws ($\mathrm{DIFFERENT}_{1,10}$; MS). Here, the function of the variability-construct is not an explanation of a single deviation, but a description of an observation Sarah made for a series of many games with a small total number of throws.

This contrasting of small and large total numbers of throws becomes a conscious strategy in the following, where Sarah first verbalises the construct RANGES: <Positions stay in certain ranges> and Ramona immediately projects this on the differentiation between small and large total numbers of throws:

Session 3; beginning 45:3; Predicting positions for games of 2,000 throws

769 Sarah: I was throwing the dice just like that, then I saw that 400 and 500-*points to screen)* mostly this gets 500 and sometimes also 400, this *(points to screen)* gets 700 and 600 and there 200 and 300.

(…)

772 Ramona: But that’s only like that for <u>2,000</u>. Ah yes, of course, right? Yes. Type in 20.

(…)

790 Ramona: *(After having looked at a series of games of 20 throws)* Ah, no, for 20, it’s always mixed up.

Again, variability is constructed as the absence of stability ($\mathrm{NO\text{-}RANGES}_{20}$). The idea of contrasting long-term and short-term contexts in the search for the existence of patterns is then also used for the bar-charts (constructs HIGH-BARS-LESS and LOW-BARS-MORE).

Furthermore, the influence of variability and patterns for finding a secure bet is made explicit in relation to the stochastic context:

Session 4, 73:30, at the end of playing ‘Betting on Positions’

1057f.	Sarah:	[For 10,000] nothing moves, I think. No, not much. So it's easier to bet when you have these high numbers than the smaller ones.

Variability now seems to be a phenomenon which is actively addressed and influencing the security of bets.

This restriction, an absence of a pattern in games with small total numbers of throws, becomes a challenge when the girls encounter variability also for large total numbers of throws: When the game switches to percentages, a dynamic view of games' results is introduced and the results of every game are cumulated (which is discussed in detail with the interviewer at 33:15). While Ramona verbalises the construct <Percentages have limits beyond which they don't change anymore> (Session 4; 35:05) after a few games, the relation to the high total number of throws is not yet clear. Thus, the girls are asked to keep experimenting with the games.

Session 4; beginning 38:30; (Sarah keeps generating games of 20, so far 14 games of 20, i.e. a total number of 280 throws)

567	Ramona:	The first digit [of the percentage, which are shown with three decimals, e.g. 35.0; cf. Fig. 2] always stays the same. And the last digit doesn't – never.
568	Interviewer:	Hm.
569	Ramona:	And in the middle, it stays the same, too. *(Sarah is generating more games of 20 throws; now roughly 3000 throws in total)* Yes, sometimes. But not always. For [blue] hedgehog it does- no it does not. Oh man!

While Ramona's initial reaction to percentages was to expect a stable pattern (PERCENTAGES-LIMITS), she is now struggling as the percentages keep changing. Maybe as a solution for this, she splits the percentages into the individual digits, for which she can find a stable pattern for the first decimal, the absence of a pattern for the last decimals, but then also encounters problems with the digit in the middle. Aside from the individual approach to percentages, this shows that for Ramona, the absence of patterns (and thus variability) means it *is* changing. A notion of changes that occur *more or less frequently* seems not yet established.

It takes four more minutes and generating several series of growing numbers before the girls can construct the variability observation as counterpart for PERCENTAGE-LIMITS and relate both to their stochastic context.

CONCLUSION AND OUTLOOK ON OTHER STUDENTS

When Ramona and Sarah start working in the learning environment, they perceive variability as the *deviation of a single game* from an expected pattern. This leads to building a construct which provides an explanation as to why this occurred — which is later used for outliers of all kinds. This initial conception of variability gets complemented when variability also occurs in series of games with small total numbers of throws. Here, the girls not only notice the *absence of a pattern* but keep actively seeking for it when confronted with new representations. They are not giving explanations for these constructs but make use of the interplay of patterns and variability to evaluate the security of predictions in relation to a long-term and short-term context. Ramona's struggle with percentages shows, however, that the division of the two contexts (long term vs. short term) might be too rigid when it comes to a dynamic perspective. Since, up to this point, the students only addressed series of games with the same total number of throws, the contrast of patterns and no-patterns was sustainable. For the dynamic perspective though — and percentages specifically — the occurrence of occasional variations is posing a new challenge for Ramona.

The scenes presented here are snapshots from the learning pathways of two girls. Even though these pathways are highly individual, some common features of the pathways towards variability can be identified that appeared with all nine pairs of students: at the beginning of all design experiment series, constructs with the *function of explaining single deviations* could be identified. Their underlying conceptions are used less often over time when the *focus shifts to series of games* with a large total number of throws and are almost exclusively used for deviations of single games with small or large total numbers of throws.

While Ramona and Sarah find a meaningful way to deal with patterns and the absence of them by relating them to different stochastic contexts, the omnipresence of variability can also pose challenges of which patterns can be observed: Some students investigate how many of the four animals change their relative position to each other for series of small and large games (where green frog and yellow snail 'battle' for the second place) instead of focussing on the values of the differences of the outcomes.

Other students over-generalise the variability of outcomes per se: Even after having simulated many games with 10,000 throws in which only the red ant won, they explain this merely as an effect of "chance" and claim that another animal would win as often, if they played on another day or if someone else played the game. They deny the existence of (meaningful) patterns at all, before and even after discovering the asymmetric colour distribution.

Overall, most students from the empirical study managed to evaluate the security of predictions correctly in relation to short-term and long-term contexts. However, how they cope with encountering variability is highly individual. To understand these complex processes in more detail, further research focuses on a broader overview of the different conceptions of variability (cf. Schnell, 2013) and tries to identify types of processes of how conceptions develop when variation is encountered (cf. Schnell and Prediger, 2012; Schnell, 2013).

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METAPHORICAL RANDOM WALKS: A ROYAL ROAD TO STOCHASTIC THINKING?

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We are interested in finding means to facilitating the access to stochastic thinking, especially for non-mathematically oriented learners. We argue that random walks are such means, since they allow a friendly visualisation of randomness, are amenable to experimental and enactive metaphorical study and provide "universal models" for a broad spectrum of stochastic problems. An explicit example of a didactical situation is given (Brownie's walk) that has been tested with students and in-service teachers of various backgrounds since 2007. The findings suggest that this sort of approach significantly facilitates the access of non-mathematically oriented learners to otherwise cryptic mathematical notions, allowing them to construct them while solving situated concrete problems.

INTRODUCTION

We are concerned about facilitating the access to stochastic thinking. We are especially interested in approaches helpful for "general" non-mathematically oriented students in school, college and university. For instance, students majoring in humanities and social sciences or prospective or in-service elementary school teachers.

We claim that random walks provide such an access, indeed a "royal road" so to say.

Why random walks? Because random walks cross boundaries, appearing in the "natural world" as well as in the "cultural world". They are a visual embodiment of randomness, that can be easily enacted and simulated, from primary school onwards. We can approach them in manifold ways: statistically, metaphorically, probabilistically… They provide "universal models" and metaphors for sundry phenomena. Indeed they facilitate the access of non-mathematically oriented learners to stochastic thinking, enabling them even to construct probabilistic notions while solving situated concrete problems. They show the way to a bottom-up approach to probability and statistics.

From the didactical viewpoint, it is crucial that the study of simple random walks may be undertaken "bare handed", with practically no previous statistic or probabilistic tools or concepts. Students may tackle the paradigmatic question: *Where is the walker, after a given number of steps?* equipped with sheer common sense. Most important, they may realize that there are *several levels of answers* to this sort

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of question, the 0^{th} level being: Nobody knows! For subsequent levels see the example of Brownie's walk below. Random walks also provide a propitious soil for the emergence of metaphors that make possible the construction of the concept of probability: for instance, the *probability* of finding the walker at given position at a given time appears as the *portion* of the walker found at that position and time.

Historically random walks appear rather late in science, as Brownian motion of pollen grains (Powles, 1978), although it might be argued that they were "anticipated" by Lucretius in 60 AD, who observed dust particles "skirmishing" under sunlight and thought this to be caused by "motions of matter latent and unseen at the bottom" (loc.cit). Contemporary examples include Brownian motion in metallic alloys (Preuss, 2002), foreseen by Einstein in 1905 (Einstein, 1956), besides fluctuations of stock markets! Interestingly, we find also instances of "cultural random walks", quite motivating for humanists. Recall for example Saint Francis of Assisi's friars walking along the road network of medieval Italy to preach the Gospel, trying to be just an instrument of God's will. So they wanted to avoid choosing rationally their direction at each crossroad. Seen from the outside, their goal was to random walk in the road network, choosing at random at each crossroad. Saint Francis devised a clever method to implement this random choice: He told a friar to spin and whirl nonstop, in spite of dizziness or nausea. As a result (friars not being whirling dervishes) the friar finally collapsed and fell to the ground. Then the whole company would choose the road closest to the direction shown by the friar's head (Anonymous, ca. 1600).

In this paper, we describe a specific example of the use of random walks to introduce the students to stochastic thinking, that may be regarded as a first avatar of a fundamental didactical situation (Brousseau, 1998).

This approach has been tested with students and teachers with various backgrounds (ranging from scientific and humanistic students to in-service elementary school teachers and juvenile offenders). An a priori and a posteriori analyses were carried out, in the sense of didactical engineering. Finally, results obtained are discussed and some conclusions are drawn.

THEORETICAL FRAMEWORK: METAPHORS AND DIDACTICAL SITUATIONS.

Nature and Role of Metaphors in Mathematics Education.

It has been progressively recognized during the last decade (Araya, 2000; Bills, 2003; Chiu, 2000, 2001; English, 1997; Johnson & Lakoff, 2003; Lakoff & Núñez, 2000; Parzysz et al., 2007; Presmeg, 1997; Sfard, 1994, 1997, 2009, Soto-Andrade 2006, 2007, 2012, and many others) that metaphors are not just rhetorical devices, but powerful cognitive tools, that help us in building or grasping new concepts, as well as in solving problems in an efficient and friendly way: "metaphors we calculate by"

(Bills, 2003). See also Soto-Andrade (2012) for a recent survey. We make use of conceptual metaphors (Lakoff & Núñez, 2000), that appear as mappings from a "source domain" into a "target domain", carrying the inferential structure of the first domain into the one of the second, and enabling us to understand the latter, usually more abstract and opaque, in terms of the former, more down-to-earth and transparent. We notice than in the literature the terms representations, analogies or models are often loosely used as equivalent to metaphors.

Recall that Grundvorstellungen (better translated as "fundamental notions" than "basic ideas") for mathematical content, that are very often operationally equivalent to conceptual metaphors, have been developed for a couple of centuries in the German school of didactics of mathematics (vom Hofe, 1995, 1998). In the case of probability they are usually given in a more prescriptive way than we do here for metaphors (Malle & Malle, 2003).

Our approach regarding the role of metaphors in the teaching-learning of mathematics emphasizes their "poietic" role, that brings concepts into existence, described as "reification" by Sfard (2009).

Didactical Situations.

Instead of introducing first concept and tools that students will apply later to solving exercises and problems as in traditional teaching, in the spirit of Brousseau (1998), we aim at students constructing or discovering those tools and concepts, when trying to solve a problem or challenge, in the context of a didactical situation. The concepts or tools we intend to teach should emerge as "the" way to "save your life" in the given situation.

We describe the role of metaphors in didactical situations with the help of a "voltaic metaphor": Metaphors are likely to emerge, as sparking voltaic arcs, in and among the learners, when enough "didactical tension" is built up in a didactical situation (Brousseau, 1998) for them. This requires setting up a suitable didactical situation and succeeding in having the students sustain and bear the necessary didactical tension. The notion of didactical tension deserves here further study.

OUR DIDACTICAL SITUATION: BROWNIE'S RANDOM WALK

Perhaps the most natural visual example of randomness is Brownian motion. So we begin by presenting Brownian motion to the students in an interactive form (using real time videos and also simulating applets). Then, the task of studying this phenomenon being complex, we try to settle for some "baby version" of it. Usually the students themselves have the idea of "simplifying" the phenomenon that we want to study, as much as possible, but without "throwing away the baby with the bath's water".

We consider here a specific "baby avatar" of Brownian motion: A 2D random walk, whose protagonist is a puppy, suitably called Brownie (as sometimes suggested by the students themselves). Also an even simpler version, a 1D random walk may appear, that we do not consider here. We discuss below, somewhat halfway between “bricolage” (Gravemeijr, 1998) and didactical design (Artigue, 2009), how this random walk has been tackled by the students with the help of suitable metaphors and without a previous knowledge of probability or statistics.

Where is Brownie?

Brownie is a little puppy that escapes randomly from home, when she smells the shampoo her master intends to give her. At each street corner, confused by the traffic’s noise and smells, escaping barely from being overrun, she chooses equally likely any of the 4 cardinal direction and runs nonstop a whole block until the next corner. Exhausted, after 4 blocks, say, she lies at some corner. Her master would like to know where to look for Brownie and also to estimate how far she will end up from home…

Research background (the students)

a) primary school teachers enrolled in a 15 month professional development program, at the University of Chile, aiming at improving their mathematical competences, from 2007 to 2010 and in 2012.
b) 1st year University of Chile students majoring in social sciences and humanities, from 2007 to 2012 (1 semester mathematics course).
c) University of Chile undergraduates majoring in mathematics and physics (one semester probability and statistics course) in 2009-2011.
d) University of Chile prospective secondary school teachers (one semester probability and statistics course) in 2009 – 2011.
e) Secondary school teachers enrolled in various professional development programs, in Chile, in 2006-2008 and 2011-2012
f) elementary school students (6th to 8th grade), whose teachers were engaged in the professional development program described above (in progress).

Methodology

Learners were observed by the author and two assistants during work sessions (some of them videotaped), their written outputs were kept or scanned. They also answered questionnaires related to their use, appreciation and preferences regarding metaphors used in courses a) and b). Learners a) and e) did group work most of the time. Other learners participated in interactive lessons with a high degree of individual participation, horizontal interaction included.

We describe now the a priori and a posteriori analyses in the sense of didactical engineering or didactical design (Artigue, 2009) related to this experimentation.

A priori analysis.

The mathematical situation.

We have several approaches to the problem.

"Arid" theoretical approach: We calculate stepwise the probabilities of finding Brownie at the different crossings in the city map. We obtain in this way the probability distribution of the sequence of random variables X_n = "Brownie's position after n steps" and D_n = "distance of Brownie to the origin after n steps". To obtain a general expression for this distribution, a clever idea is to do harmonic analysis and synthesis of these processes with respect to the (non-commutative!) symmetry group of the random walk.

Statistical approach: To see what is going on, we can make a statistical simulation of the random walk, eventually using a worksheet.

Metaphorical approach: With the help of a "hydraulic metaphor" (Soto-Andrade, 2006, 2007), we replace the puppy's random walk on the city map by a fission or sharing process on a grid. There we may see, for instance, the grid as a system of ducts and imagine a litre of water at the origin that flows symmetrically and fairly to the 4 next neighbours, each time. We have then to calculate stepwise with a deterministic process, which is equivalent to the original stochastic process, but has the virtue to avoid probabilistic language.

The didactical situation.

Working in groups, the teachers or students should realize quickly that there are impossible corners (street crossings) for Brownie, even close to her home. There might be divergent opinions on whether returning home is possible after 1 or 3 blocks, for instance. Then, after having spotted the corners where it is possible to find Brownie (after a 4 block run), some (up to one half of the) students or teachers will believe that they are equally likely. Others would have the vague intuition that some corners are more likely because Brownie can get there by several different paths.

Quite late, some will have the idea of experimenting, by simulating a good number of puppies, to try to settle the question. The majority of them will now be convinced that some corners are more favorable to find Brownie; her home for instance. They will not have much trouble in setting up a corner *ranking.* But they will have some trouble in quantifying their feeling of bigger or smaller likelihood. How to assign "weights" to the different corners? Some will usually think of counting paths to quantify their vague feeling.

It is likely that the "Solomonic metaphor" (cut the puppy into four pieces) or the "hydraulic metaphor" (the puppy flows equitably to the four immediate neighbours) will not emerge spontaneously. But this could happen under some minimal prompting like: *What, more concretely, could you imagine instead of this (rather abstract) fair random choice between the 4 cardinal directions?* (using some discrete gesture language).

The students or teachers should not have much trouble quantifying the likelihood of presence of the puppy at the different corners, with the help of the hydraulic metaphor. They will realize quickly the conservation law of the puppy: putting together her pieces at each step you reconstruct the whole puppy. The idea might also emerge in the classroom to unleash a pack of puppies from home, that would spread out evenly, splitting into four equal groups at each corner (better begin with 16 puppies that will run 2 blocks each). We call this a “pedestrian metaphor”.

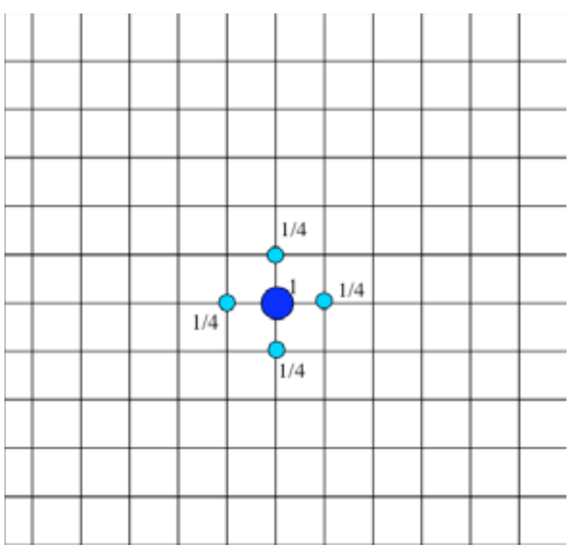
So we hypothesize that – eventually under some prompting – Solomonic, hydraulic and pedestrian metaphors may emerge, that will enable the students to solve the problem for a given (small) number of steps. Path counting will also emerge as a competing strategy to quantify likelihood, especially among students having being previously exposed to more mathematics.

A posteriori analysis.

Unexpectedly our primary school teachers had a strong tendency to see without hesitation the 9 possible corners for Brownie after a 2 block run as equally likely! Dissenting opinions emerged very slowly among them. On the contrary learners from groups b), c), d) quickly sensed that corners closer to her home were more likely.

We have witnessed the emergence in the classroom of metaphors like the Solomonic, hydraulic or pedestrian one (Soto-Andrade, 2006, 2007) right after the students or teachers were prompted to try to see the situation “otherwise”, while working in groups, with discrete support from the facilitator. Typically in a class of 30, one or two immediately “see” the puppy split into four pieces. These metaphors emerged more easily from learners in groups a) and b). Initially however many teachers seemed to have feeling that they were violating some (unspoken) didactical contract when allowing themselves to metaphorize.

In cases where this sort of visualization seemed out of reach, the hydraulic metaphor enabled us to act out the situation: 13 in-service elementary school teachers distribute themselves suitably on the nodes of a virtual grid on the floor, one of them holding a container with 1 litre water that she shares equitably with her immediate 4 neighbours, and so on… Each one of them could estimate the amount of water that



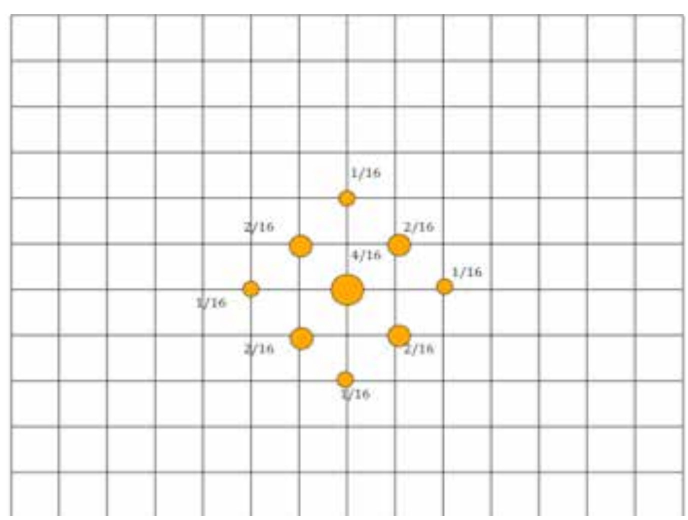


Fig. 1 Brownie’s splitting (0 and 1 block). **Fig. 2. Brownie’s splitting (2 blocks).**

she will have after a given number of steps. Eventually they simulate this (more easily although less dramatically) with a square of board that they partition into four pieces and so on. See figures 1 and 2 above.

Results

As hypothesized, Solomonic, hydraulic and pedestrian metaphors did indeed emerged among our students, eventually with the help of some prompting. Path counting appeared also, among more mathematically literate students, leading to an assignment of probabilistic weights based in Laplace rule. One obstacle to the emergence of metaphors was the current didactical contract saying that metaphorizing is not good mathematical manners. Enacting the walk with the help of a hydraulic metaphor was a big help for most learners, especially for elementary school teachers.

With this course of action, most students and teachers succeeded in constructing the concept of probability at the same time that they solved the problem in context, with the help of suitable conceptual metaphors, in the sense of Lakoff and Núñez (2000).

Discussion

Our voltaic metaphor seems suitable to describe what happened in the classroom, regarding the emergence of metaphors that enable the learners to solve the problem they are tackling. To build up the necessary didactical tension is here crucial but not easy, especially when working with big classes (more than 50 students).

On the other hand, we remarked that in our hydraulic acting out of the random walk, each participant does something very simple, but the outcome is the analogical solution of a non trivial problem. This could be seen as just one simple example of application of "swarm intelligence" to the didactics of mathematics.

Also the role of the teacher in this sort of "mise-en-scène" turns out to be quite delicate. Definitely better than the metaphor "a teacher is a technician" emerges here the metaphor "a teacher is a tightrope walker". In particular, experience shows that didactical micro-gestures of the teacher - tightrope walker may feed a butterfly effect in the classroom.

Notice that in our metaphoric approach the students are not given a "Grundvorstellung" (fundamental notion) for probability before they address the problem. On the contrary, they are prompted to tackle the problem "bare handed" first and eventually look for a friendly metaphor for the *concrete random walk* they want to study (e.g. "Brownie splits"). When trying to give pertinent answers to the questions asked, "poietic metaphors" may emerge that enable them to construct the abstract probability concept, like: "probabilities of finding Brownie at a given corner are pieces of Brownie". This fits the framework of Brousseau's didactical (better, adidactical) situations (Brousseau, 1998).

Brownie's walk seen from Google Earth (zooming out).

First, Brownie's walk can approximate efficiently Brownian motion: just make the grid denser and denser…If you zoom out, with the help of Google Earth, tracking Brownie with GPS, you will see the trajectories of pollen grains.

Second, the title of this section is also metaphoric: if we zoom out cognitively, we realize that Brownie's random walk is a paradigmatic example of a random walk (a metonymy, in fact) and that random walks play very often the role of universal models in probabilistic as well as statistical problems. For example:

1. The famous Italian unfinished tournament problem: Two players of equal strength compete in a tournament that consists of a series of games. The winner, who will get the one million euros prize, is the one who completes 10 wins first. Now, the tournament must be interrupted by "force majeure" when one player had won 8 times and the other 7 times. How should the prize be divided fairly between the two players?
This problem can solved very easily by modelling or metaphorizing it by a random walk (a 2D one!) and solving the random walk metaphorically. This is friendlier than Pascal classical solution (apparently he was not very fond of random walks). If we recall that "mathematics is the art of seeing the invisible" (Soto-Andrade, 2008) we would agree that students are doing real mathematics when they see the evolution of the tournament as a random walk on a 2D grid with absorbing barriers.

2. Evaluating screening tests (e.g. false positive problems) is a tough task for experienced physicians (Zhu & Gigerenzer, 2006; Gigerenzer, 2011). The typical question that physicians answer wrongly is: "I have got a positive HIV test, how likely is that I am really a carrier?"

Natural frequencies have been suggested (loc. cit.) as a means to get a correct answer without much toil, in a way a 10 year old could do. But if you look at this problem as a question about a (2 step) random walk, that you can solve with the help of a pedestrian metaphor, your recover exactly natural frequencies.

It is interesting to compare the relative popularity, among students in various levels and countries, of the hydraulic and pedestrian metaphors. According to Gigerenzer (loc. cit.), pedestrian metaphors (i.e. natural frequencies) should be much more popular, because you just manipulate whole integers and you compute a fraction only at the very end, or even you get your result in the form "*m* out of *n*", as in pre-fraction days, that is all the same efficient for practical purposes. In the hydraulic metaphor however you need to manipulate fractions all the way. We have found experimentally however, consistently year after year, that 1st year university students majoring in Social Sciences and Humanities (group a) above) tend to prefer the hydraulic metaphor (8 out of 10 approx.), even if computing with fractions is not all that smooth for them. Apparently one reason for that is the *conceptual* impact of seeing one litre of grapefruit juice (or probabilistic fluid, if you like) draining downwards through a graph of ducts, something that helps them tackle Zeno's paradox, for instance.

3. The classical fundamental didactical situation of Brousseau (1998) for statistics where 7th graders try to find out the composition of a bottle containing 5 marbles of 2 different colours by looking inside through a tiny opening (that lets them see just one marble at a time), can be modelled by a random walk in the plane that "ends up" engaging into *one* region out of four possible wedge-like regions (corresponding to the 1-4, 2-3, 3-2 and 4-1 compositions)

CONCLUSIONS.

We claimed that random walks are a means to facilitate access to stochastic thinking, especially for non-mathematically oriented learners. As grist to the mill of this claim, we described here an explicit didactical engineering for a concrete example of a random walk (Brownie's walk) whose bare handed study facilitates the construction of the concept of probability, thanks to the likely emergence of various helpful (mainly enactive) metaphors.

We have tested this didactical engineering with learners of several backgrounds, students as well as in service teachers across Chile. According to their performance (assessed by tests and group work) and answers to questionnaires and interviews, this approach (random walks metaphorically tackled) did help them to understand and even construct otherwise cryptic mathematical notions and to solve problems involving randomness in a friendly and efficient way (e.g. false positive problems). We noticed some initial resistance to metaphorizing, mainly among university students majoring in mathematics and secondary school teachers. University students majoring in humanities and social sciences and primary school teachers were more prone to metaphorizing, especially after being granted permission to do so ("you are not supposed to metaphorize in mathematics" seems to be the implicit current didactical contract). Among university students remarkable metaphorical capacities were detected in students coming from alternative schools, like Montessori or Waldorf. Otherwise it seems clear that traditional teaching in Chile, especially from grade 7th onwards tends to thwart metaphorizing. This suggests promoting this approach already at the beginning of elementary school and exploring related approaches emphasizing embodiment and enaction. Further research on this didactical phenomenon, quantitative as well as qualitative, seems commendable.

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STATISTICAL UNDERSTANDING AND LANGUAGE – A QUALITATIVE ANALYSIS

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This paper presents results of a qualitative study based on data from 83 German eight-graders on the role of language for the understanding of a statistical problem. As competencies in the domain of statistics are necessary for social participation, communication skills related to basic statistical models appear as crucial. Language thus comes into play at various levels when learners make sense of situational contexts statistically or when they present solutions. Misconceptions might be connected to a development need of domain-specific language. The results of this study suggest that the development of the understanding of statistical problems is framed, i.e. enhanced or restricted, by a corresponding development of domain-specific language. Language-related knowledge thus could benefit from focused support in corresponding learning environments.

INTRODUCTION

Language plays a role for statistical thinking—not only abilities of data-related reading (e.g. Curcio, 1987) appear as crucial, but it is also important to be able to express statistical observations or ideas in an adequate language. Whereas competency aspects of data-related reading can be assessed with test instruments and have been integrated in competency models for statistical literacy (cf. Watson & Callingham, 2003; Kuntze, Lindmeier & Reiss, 2008; Kuntze et al., 2010), the area of data-related speaking or writing calls for more focused empirical research.

Consequently, this study investigates the relationship between students' understanding of a given problem and their ability to express their ideas in a statistically adequate language. Through this focus and on the basis of this problem, we try to get insight how secondary students' statistical understanding could be supported. The students were given problems related to statistical contexts and asked to give written answers. An analysis of these answers focused on the way students expressed their ideas and in particular on the role of language for the statistical understanding shown in the answers. The results of a case-based analysis indicate that some students have difficulties that appear to stem from the area of language use rather than from their pre-formal statistical understanding. Such pre-formal understanding could even be further developed by learning opportunities targeted on language.

THEORETICAL BACKGROUND

In moderate constructivist approaches to learning, there is an epistemological consensus that building up knowledge is closely linked with the development and refinement of language: This is not only the case for learning in the early childhood but also up to the contexts of domain-specific academic knowledge (cf. Reinmann-Rothmeier & Mandl, 2001; Vygotsky, 1986; Radford, 2003; Sfard, 2013).

For instance, when making sense from situations in which the interpretation of data is required, learners draw on their prior experiences and knowledge. Such knowledge is often pre-formal, like Fishbein's (1975) primary intuitions. The same appears to apply for language, which also develops from pre-formal to a more formal use of domain-specific language (Vygotsky, 1986). However, both of these development processes have shown to be far from 'linear': Neubert and colleagues (2001) describe the learning process by the terms of deconstruction and reconstruction which continuously frame the interaction of individuals with the surrounding socially influenced perception world.

In the domain of statistics, there is still little empirical evidence about the interplay of pre-formal statistical knowledge and language. However, language is considered to come into play when students have to decode a complex statistical situation or when they have formulate questions or conlusions from data (Shaughnessy, 2007). When passing through PPDAC cycles (Wild and Pfannkuch, 1999), language may frame the corresponding statistical thinking process. Gal (2003) identified the ability to express someone's opinion concerning statistical information as crucial for statistical thinking and statistical literacy. Gal as well as Watson (1997) emphasised the importance of the adequate use of statistical terms as well as the ability to (critically) communicate one's reaction to statistical information, which makes reference to the two areas of language and domain-specific knowledge introduced above. Even though Watson and Gal underlined the importance to use standard statistical language and although Biehler (1997) points out that some problems in data-related communication are caused by a lack of formal language, we see the need to focus also on pre-formal language use. In line with Makar and Confrey (2005), this study hence examines students' abilities to express their ideas and their understanding of a statistical situation regardless the level of abstraction of the language used. By accepting all stages in the development of domain-specific language in the analyses of cases of students' answers, we strive to access their statistical understanding of the situation, focusing also on elements of pre-formal knowledge. This corresponds to a research need, as questions such as how statistical understanding develops together with language or how statistical understanding can be fostered in learning environments focusing on language have not be answered fully in prior research.

Concerning our research interest related to language use and statistical understanding, we would like to recall that we keep distinct the (potentially non-verbal) statistical understanding of learners from the form this understanding may be expressed by means of language. It may appear as relatively obvious that a developed statistical understanding may coexist with a high statistics-specific language mastery, or that difficulties in the statistical understanding such as inconsistent pre-formal knowledge can coincide with difficulties in statistics-related verbal expression. But there could also be the following cases: For example, a student might have a developed pre-formal statistical understanding of a phenomenon (in the sense of intuitions, cf. Fishbein, 1975) coupled with a low ability to express this understanding with domain-specific vocabulary. Conversely, a learner might have acquired a “language toolbox” related to statistics but as a consequence of a non-optimal statistical understanding, he or she might be unable to use this vocabulary adequately in a corresponding argumentation (cf. Makar & Confrey, 2005). Even if we assert that all four cases are basically possible, we expect based on Vygotsky’s (1986) general approach to thought and language that the development of specific language supports statistical understanding and that non-verbal statistical intuitions may promote the process of making sense of statistical vocabulary. Students’ abilities to express their statistical understanding as well as the role of specific language development for aspects of statistical thinking thus require in-depth empirical attention.

Consequently the study focuses on the following research questions:

How do students express their ideas and their understanding of statistical problems?

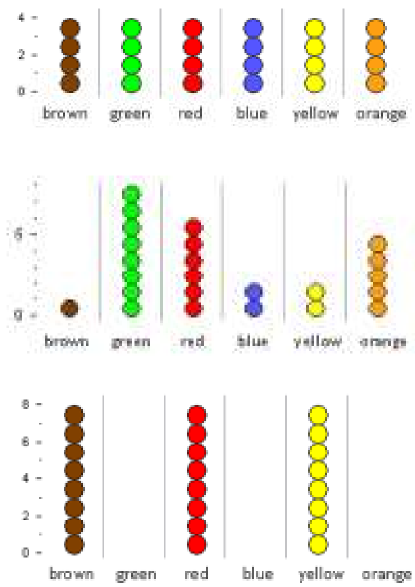
How developed is students’ statistical understanding and how adequate is their corresponding domain-specific language use?

DESIGN AND METHODS

This study was carried out as part of the project ReVa-Stat, (“Developing concepts of data-related reduction and statistical variation as a support for building up statistical literacy”). In the first phase of this research project, 83 students of grade 8 (39 girls and 44 boys, mostly 14 years old) were asked to work on tasks related to different statistical problems. These tasks addressed activities such as organizing and representing data as well as reflecting about variation or data. As the students had not attended any specific statistics course prior to participating in this study, they were asked to describe their ideas in their own words. We did not stimulate students to use a specific type of language. During four lessons, the students worked on this specific learner-centred material, which affords exploring their understanding, views and ways of expression. In the material, students were encouraged to first discuss the presented statistical problems in pairs and then to write down their answers. Data was gathered by collecting their written work. During the work phase of the students, a member of the research team observed the implementation and use of the material in the classrooms.

In the following, we focus on answers the students gave when working on tasks in a relatively early phase (first lesson) of their work with the learning material. The task (see Fig. 1) refers to the context of frequencies of colours in packages of chocolate lentils (cf. Engel & Vogel, 2005, and Eichler & Vogel, 2012, for the task context, Watson & Callingham, 2003, for the question format). Prior to this task, students had been informed that a sweets company produces the same number of chocolate lentils of each of the 6 colours, they had then been given a package of 24 chocolate lentils as well as data about 9 more packages. The students had also been asked to calculate the average for every colour within the 10 given packages. The purpose of this activity was to give them hands-on experience that the mean is less variable than the original data. The students were asked to comment on their decisions for the three given diagrams.

1. Some of Marie's classmates were asked to open several packages of Chocolate Lentils and to represent the numbers (of the different colours) in a bar graph.
Marie, however, is sure: "A few of my classmates cheated! They didn't really count, but they just invented the numbers!"
What do you think: which diagrams have been invented? Justify your answer!



a) ○ authentic ○ cheated
because ______________________

b) ○ authentic ○ cheated
because ______________________

c) ○ authentic ○ cheated
because ______________________

Figure 1: One of the "Chocolate Lentils" tasks

The written results of the work of the students were analysed using a qualitative bottom-up methodology based on Mayring's qualitative content analysis (2000). An interpretative analysis of the data was carried out by two researchers.

In this analysis, we were aware that also on the methodological level, examining whether a learner has a particular understanding of a specific situation is an interpretative and once again a construction process on the base of the learner's expressions, in which her or his prior pre-formal or formal knowledge may be reconstructed through an interpretive process (Vygotsky, 1986; cf. also Neubert, Reich & Voß, 2001).

In a first step of the analysis process, particular cases of students' answers were selected against the background of the research interest in order to identify types of answers inductively. In a second step, theses cases were analysed in more depth so as to further develop categories. In a third step, the developed categories were used to obtain an overview of the whole sample. For this purpose two researchers rated all cases and were able to classify them according to these categories in a top-down coding procedure. To deliver insight in what we conceive to be the categories of "relatively high/low language mastery" and "relatively high/low understanding", which were developed in the first and second steps of the analysis process, Table 1 shows the corresponding coding scheme for this third analysis step. Note that the term "understanding" is used to describe students' ability to get into the core problem in the sense of Fishbein's intuitions (1975) whereas the notion "language" describes the students' way to express the own thoughts concerning the given task.

Statistical understanding	
A: Relatively high statistical understanding / pre-formal knowledge	At least intuition visible that the core problem is to judge the regularity vs. the variation of the shown structure, Argumentation can be general or experience-driven.
B: Relatively low statistical understanding / pre-formal knowledge	The student did not realize that the regularity vs. the variation has to be judged; other irrelevant argumentation or lack of argumentation.
Language mastery	
1: Relatively high language mastery	Appropriate use of terms such as "distribution", "(ir)regular", "(non)uniform", "(un)systematic", "(un)equal", ... ; description of probability / frequency with terms like: "seldom/rare", "often", "probable", "randomly", ...
2: Relatively low language mastery	No or inappropriate use of the terms above; description of the situation without generalisation (e.g. "There are differences."); over-generalising way of expression (e.g. "In every package there are all colours.")

Table 1: Overview of coding scheme for third step of analysis (top-down coding)

RESULTS

In the following, we will discuss the answers of three students referring to the above-mentioned coding scheme.

A sample answer of a student is presented in Figure 2 („Peter"). In his answer, Peter uses the word "distribution". This notion is described as "uniform" and "non-uniform" in the case of a) and b). According to Peters answer in b), in "real", the distribution is "fairly non-uniform" in his view. This expression appears to describe his understanding of statistical variation in this situation. The ways he uses the notion "distribution" in a) and b) enables him to characterise the data shown in the diagrams and to develop short and relatively dense argumentations.

Interestingly, Peter's use of language in the answer to c) shows that he has made sense of the term "distribution" in an individual way: He argues that the distribution "is simply too big". In our analysis, two possible interpretations have emerged: Peter either uses the term "distribution" in a different way here than in a) and b), which may suggest that the connection of the word "distribution" to a statistical concept is still relatively loose, or he sees the word and concept of “distribution” like shifting balls between rows in the diagram, so that a perfectly uniform distribution would then be a ‘small distribution’ (because there would then be little concentration of balls in any of the categories), whereas an accumulation of all balls in one category would then be the ‘highest possible distribution’.

What do you think: which diagrams have been invented? Justify your answer!

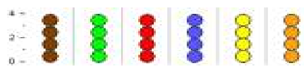
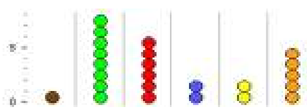
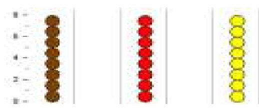
a)		[Geschummelt] weil „es sehr selten so [eine] gleichmäßige Verteilung“ [gibt]	[cheated] because “very rarely such a uniform distribution” [occurs]
b)		[Echt] weil „die Verteilung recht ungleichmäßig ist wie in echt.“	[authentic] because „the distribution is fairly non-uniform like in real”.
c)		[Geschummelt] weil „die Verteilung der M und Ms einfach zu groß ist.“	[cheated] because “the distribution of the M&Ms is simply too big”.

Figure 2: Peter’s answers

In the case of Peter’s answer to question c), it is very probable that he has a correct statistical understanding of the situation without however perfectly mastering the language side in the sense that he would be able to share his thoughts using the domain-specific vocabulary adequately. Both interpretation alternatives further suggest that the language use in his answer not only points to a need of some further refinement of conceptual understanding related to the notion of ‘distribution’, but also that knowledge about the use of language itself should be supported in order to enhance his understanding.

The example of Peter’s answer also gives insight into the development process of domain-specific language together with conceptual understanding. Whereas Peter uses the notions of uniform and non-uniform distribution showing and applying corresponding understanding, his answer to c) (“big” distribution) shows that his domain-specific language concerning the term “distribution” still needs to be refined. In equal measure, he might not be familiar with the underlying concept, which supports the assumption that the development of conceptual understanding goes together with a development of language.

Anne (see Figure 3) answers question a) and b) by referring to her own experience with packages of chocolate lentils. She gets into the problem in a very elementary way—we do not know whether she has the statistical understanding related to the concept of probability, i. e. whether she sees that it is quite improbable to encounter a package with an exact uniform distribution without any variation like in the case of a). She might have an intuitive and experience-driven pre-concept in this area but is not able to express her ideas through more specific terms. Comparable to this example, she appears to have recognised a similarity between her package and the distribution shown in case b) but she lacks describing this phenomenon. Very probably, she had not found identical frequencies in her package in the prior experiment, but the similarity she sees is a structural one, with frequencies showing statistical variation. Anne obviously has the language ability for describing her experience, however she might still not completely be able to express her thoughts beyond the specific experience.

What do you think: which diagrams have been invented? Justify your answer!

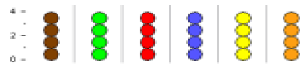
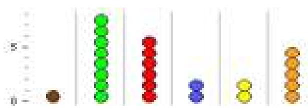
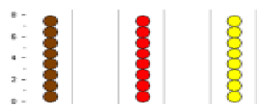
a)		[Geschummelt] weil „bei keinem von uns waren von allen Farben 4 M&Ms.“.	[cheated] because “nobody of my group had 4 M&Ms of every colour.”
b)		[Echt] weil „in meiner Packung es auch so aussah.”	[authentic] because “my package looked like this.“
c)		[Geschummelt] weil „von allen Farben müssen welche darin sein.”	[cheated] because “there have to be some [Chocolate Lentils] of every colour.”

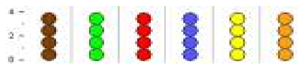
Figure 3: Anne’s answers

At the first sight, Anne’s answer to c) seems to reveal a lack of conceptual understanding, as she claims that all colours must be there in each package. In particular, prior to working on this task she had access to packages with missing colours. This makes it more likely to interpret her answer as an expression of irritation by the balanced distribution that might not have appeared “typical” to her. In contrast, in her written answer, she focuses on a different aspect. This utterance might hence be a case in which the statistical understanding diverges from the written answer, possibly as a consequence of a non-ability of expressing the statistical understanding in words. However, the analysis yielded this interpretation as one possibility among other possible interpretations.

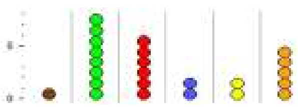
Anne’s answers reveal a certain understanding (at least in the sense of intuitions, cf. Fishbein, 1975) of the statistical situation. However, she is not able to communicate her ideas neither using standard language nor using more general terms.

Figure 4 shows Tim’s answers. In contrast to the previous examples, Tim might have misunderstood the problem or have used an inappropriate logical structure. In case a) he argues that the distribution is authentic because the number of chocolate lentils in the shown package is 24, as stated previously in the learning material. His emphasis that “this corresponds to the truth” shows that this observation predominates over other possible considerations, so that he appears not to seek for other possible criteria for his judgement. Although he explains in a) his decision by the number of chocolate lentils, in b) and c) he changes his justification. However, again one single criterion is dominant, and Tim does not question whether the aspect is relevant. Now, his reason for the authenticity of the distribution is that “there is at least one chocolate lentil of every colour”. The order of the words in Tim’s answer (“at least” appears to have been inserted in the wrong place) indicates that he had later seen the need of inserting the "at least", probably because he saw that the frequencies were not all equal to one. In this case, Tim has obviously tried to make the language of his answer more exact, without however developing a more deepened understanding for example of statistical variation.

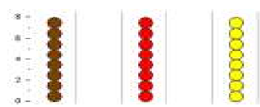
What do you think: which diagrams have been invented? Justify your answer!

a) 

[Echt] weil „es 24 M&M sind und das der Wahrheit entspricht.“ — [authentic] because "there are 24 M&M[s] and this corresponds to the truth."

b) 

[Echt] weil „von jeder mind. Farbe eins dabei ist.“ — [authentic] because „there is one [Chocolate Lentil] of every at least colour."

c) 

[Geschummelt] weil „es Farben fehlen.“ — [cheated] because "colours are missing".

Figure 4: Tim's answers

Finally, Tim justifies the decision that c) is invented with the statement that "some colours are missing". We do not know if he thinks that the author made a mistake and "forgot" these colours or if Tim considers such a distribution as atypical.

Both from the point of view of statistical understanding and language, Tim's answers suggest a need of further development. In this case, both the level of being able to communicate with statistical terms and the level of having corresponding conceptual understanding show deficits.

As a result of this case-based qualitative bottom-up analysis, we derived categories from the interpretation process of selected cases. Figure 5 gives an overview of these categories. In these categories, qualifiers like "low" and "high" refer to the sample of the cases in our study and hence will have to be adjusted in potential adaptations to other samples.

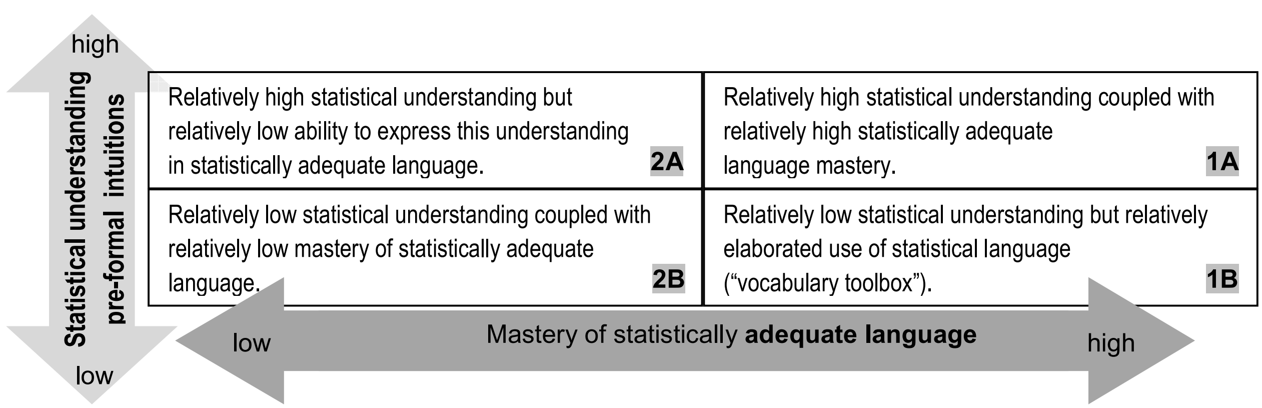


Figure 5: Bottom-up categories

In the third step of this study, a top-down coding procedure was done with all 83 cases of answers using the categories in Figure 5. The analysis was done by two raters independently (inter-rater reliability: κ=0.86). In all cases with different codes an a-posteriori agreement of the raters could be reached in a subsequent joint interpretative analysis. The analysis yielded that 52 out of 83 cases were assigned to category 2A, which means that they have at least an intuitive understanding of the statistical situation, being however unable to express this understanding in adequate language. Fewer students were classified into the categories 1A (13 out of 83), 1B (1 out of 83), and 2B (17 out of 83). Whereas Peter's answers (Fig. 2) can be seen as a case illustrating category 1A, Anne's (Fig. 3) and Tim's answers (Fig. 4) have been classified into categories 2A respectively 2B.

DISCUSSION AND CONCLUSIONS

Consistent with the findings of Makar and Confrey (2005) the examples presented above as well as the results of the top-down analysis of the whole sample show that the students frequently use every-day language to express their ideas related to the statistical problem considered. Often, the non-availability of adequate notions affected the quality of their statements. Moreover, the use of appropriate language and the statistical understanding appear to be interrelated and both appear to play an important role for the ways students approach statistical problems, supporting Shaughnessy's (2007) assertions. In the sample of this study, the language development tended rather to have lagged behind the development of pre-formal knowlege or understanding. In more general terms, both areas—the refinement of statistical understanding on the one hand and the domain-specific language development on the other—appear as almost two sides of a medal, for their complementary importance for conceptual learning; the evidence appears to be in line with Vygotsky's (1986) thoughts, reflecting them in the domain-specific context of the given statistics problem. Figure 6 gives a schematic overview of this domain-specific complementarity.

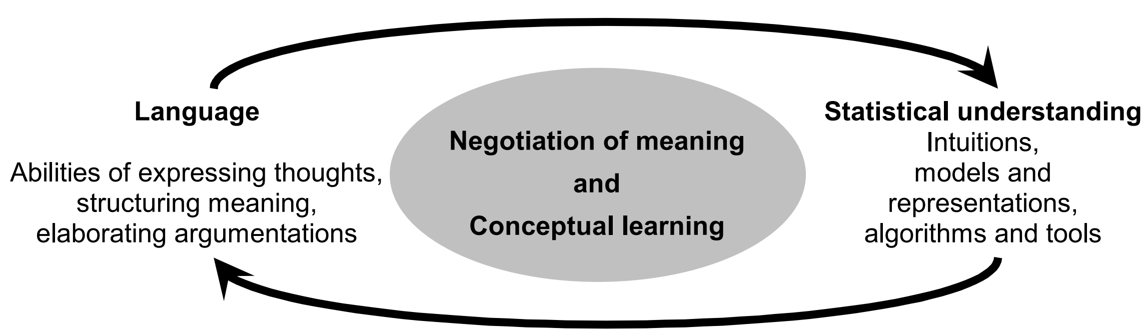


Figure 6: Language, statistical understanding and conceptual learning

Expressing statistical thoughts by means of written language often requires reorganising and deepening these thoughts—and it is hence a learning opportunity. In particular, misconceptions and incongruencies in statistical understanding often get apparent and can inform teachers and students about the stage of the ongoing learning process.

For building up statistical thinking in the classroom, the findings raise the question how a combined instructional focus on the development of statistical understanding and language can be implemented. We conclude that an explicit awareness of language can also foster the students' statistical understanding. In a follow-up study, effects of corresponding learning materials with an emphasis on language use will be examined.

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8^{TH} GRADE STUDENTS' STATISTICAL LITERACY OF AVERAGE AND VARIATION CONCEPTS

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The purpose of this study was to investigate 8th grade students statistical literacy in average and variation concepts through Watson's (1997) three tiered framework. A total of 1074 eight grade students were surveyed with an instrument developed by researchers about statistics content in the elementary mathematics. Students' responses were examined through nine major questions in this instrument for this study. Descriptive analysis of correct and incorrect responses indicated that majority of students understood average and variation concepts though measures of central tendency and spread. Reasons of results and educational implications were discussed.

Keywords: Statistical Literacy, Average, Variation, Elementary Students

INTRODUCTION

Statistical messages have extensively been in the media through several types of arguments, advertisements, or suggestions (Ben-Zvi & Garfield, 2004). Therefore, the ability to understand, interpret, and critically evaluate statistical messages in daily lives of individuals which have been addressed as "statistical literacy" (Watson, 1997), has become important in information societies. Gal (2004) provides a statistical literacy conceptualization and its elements in a model for adults or "future adults", in his term. In this model, communication with statistics, interpretation and judging of statistical claims are treated as the possible skills of statistically literate individuals. In addition statistical literacy plays a crucial role in public and private decision making of individuals where their daily life is full of data (Wallman, 1993).

Statistical literacy has also been a part of the school mathematics curriculum to prepare students to encounter the needs of society when they complete their compulsory education (Watson & Callingham, 2003). Understanding of average and variation is fundamental for statistical literacy as the words "average", "variable", or "vary" are a part of everyday language (Watson, 2006). Although there are studies in the literature examining statistical literacy from different aspects such as sampling (Watson & Moritz, 2000) or graphing (Aoyama & Stephens, 2003) in terms of grade level, research considering statistical literacy of average and variation concepts is scarce specifically in middle school context in Turkey.

Turkey has undergone through some revisions in the elementary mathematics curriculum considering the inclusion of statistics and probability as one of the five content areas of school mathematics in recent years. Statistics and probability domain of Turkish elementary mathematics curriculum consists of concepts such as sampling, measure of central tendency, graphs and tables, measure of spread, probability, and beginning inferences (MoNE, 2005). This study provided a brief reflection regarding implementation of current elementary mathematics curriculum in Turkey with respect to students' capabilities in statistical literacy in terms of average and variation concepts. The results could point out further research in the context of both mathematics curriculum development and teacher education. Therefore, this study investigated Turkish 8th grade students' statistical literacy of average and variation concepts towards the end of their compulsory education.

THEORETICAL FRAMEWORK

Watson's (1997) three tiered framework is at the core of present study and used as a main analysis of students' statistical literacy. Statistical literacy has been addressed by Watson (1997) as the ability of understanding, interpreting, and evaluating statistical messages in various contexts. She presented statistical literacy in a three tiered framework:

> **Tier 1:** Familiarity with and understanding of terminology used in statistical messages.
>
> **Tier 2:** Interpretations of these statistical terms where they are contextualized in statistical claims which appear in the media or elsewhere.
>
> **Tier 3:** The ability to question others' statistical reports critically; in other words, the critical evaluation of biased statistical information and posing possible critical questions to this statistical information.

More specifically, the first tier refers to the familiarity with terminology used in statistical messages in media. To illustrate, understanding the term "average" in context or defining "average" is a feature of Tier 1. For variation concept, the ability for Tier 1 includes expressing ideas of variation in daily life of individuals. The second tier includes the interpretations of these terms where they are contextualized in statistical claims. For example, interpreting or applying ideas of average in a variety of context is a characteristic of Tier 2. The last tier is the ability to question others' statistical reports critically; in other words, the critical evaluation of biased statistical information and posing possible critical questions to this information constitute the third tier of statistical literacy. For instance, examining whether mean or median is an appropriate average in a given statistical report or recognizing extreme values in distributions are basic characteristics of Tier 3 for statistical literacy of average and variation concepts (Watson, 1997; Watson & Moritz, 2000).

The compulsory education in Turkey addresses the elementary school period which comprises grades 1 to 8. It aims at developing informed citizens who possessed knowledge of statistics with an appreciation of the importance regarding the position of statistics in society (MoNE, 2005) through the National Elementary Mathematics Education curriculum. The elementary mathematics curriculum in Turkey is in spiral in nature and statistical topics including average and variation presented through measures of central tendency and spread across grades 6 to 8. At the end of the elementary school, students are expected to learn meanings of these concepts, how to measure them and where to use them (MoNE, 2005). Therefore, investigating the level of statistical literacy of average and variation concepts that students have developed at the end of the compulsory education becomes important in order to understand the effect of elementary mathematics education. The present study investigated eighth grade students' statistical literacy in the average and variation concepts in terms of Tier 1, Tier 2, and Tier 3 which Watson (1997) have addressed and contributed to this framework from Turkish context.

METHODOLOGY

Participants

A total of 1074 eighth grade students from 48 classes in 9 randomly selected public schools in a district of Ankara participated in the study. Data were collected by the first author (except 7 classes who were surveyed by their own teachers) in participating students' classrooms during regular class hours.

Data Collection and Analysis

The data collection tool used in this study was prepared to investigate 8th grade students' statistical literacy in sample, average, graph, inference, chance and variation in terms of three tiers. The instrument was developed by the researchers, piloted with 292 8th grade students, and revised through mathematics education researchers', mathematics teachers', and students' comments. The Cronbach's alpha reliability measure in the pilot study was .72 and in the implementation was .75. The analysis of four items related to average concept (A) and five items related to variation concept (V) are presented in this paper. There were both multiple choice and open ended questions for these two concepts. A holistic rubric was prepared in order to classify students' responses in open-ended items and eliminating subjectivity based on their responses in the pilot study and related literature on statistical literacy. Students' responses were coded as non-statistical/incorrect, pre-statistical, and statistical for open-ended items. In addition, explanations of terminology with arithmetic procedures were coded separately.The responses of Statistical Literacy Test were classified according to the codes in the rubric. These codes were summarized as frequencies and percentages. Then, for descriptive statistics; mean, standard deviation, percentages, and frequencies were calculated.

RESULTS

Statistical Literacy of Average Concept

The first tier of statistical literacy of average concept gives indication of students' understanding of average concept. The item presented below intended to measure students' understanding of average in context.

"Last year, an average of 20 people had died due to traffic accidents." What do you understand of the word **"average"** in this sentence?

The answers of students were classified through four categories which were blank or incorrect responses, pre-statistical responses, responses through measures of central tendency and statistical responses. The results indicated that majority of students either explained the term “average” through pre-statistical words (48.7%) or described through measures of central tendency (29.6%). The most notable response in pre-statistical responses was “almost” (34.8%) while “arithmetic mean” or “add them up and divide” algorithm were the most frequent descriptions (26.2%) for those who explained average through measures of central tendency. However, statistically correct responses constituted only 5.2% percent of total responses.

The analysis of students’ familiarity with methods for finding average or central tendency indicated that 36% of them labeled in a multiple choice item either “median” (18.1%) or “mode” (18.5%) as if these were not a method for finding average. This finding indicated that almost one third of the participants did not count median and mode as a measure of central tendency.

The second tier of statistical literacy of average concept requires students apply ideas related to average in context. The results showed that 40.2% of students correctly interpreted average in context whereas others (59.1%) chose the incorrect interpretations. It could be inferred that for average content in the second tier, only less than half of the participants had performed properly.

Evaluation of statistical claims which involves average concept constitute third tier group. There were two items related to this group. The findings indicated that majority of students gave incorrect responses or left this item blank (82.7%) which indicated that they were not able to critique a statistical claim in average context. The rest (16.4%) could correctly evaluate the appropriateness of this claim. The explanation for this evaluation was asked through another item. These items are presented below.

The number of problems solved in a math class is counted and represented in the following table.

Student	Number of problem
A	2
B	6
C	2
D	22
E	3
F	2
G	1
H	2

In order to summarize these data, the mean is calculated and found 5.

a) Do you agree with this?
b) Explain your answers with reasons.

The categorizations of explanations given by students for the second item are presented in Table 2.

Classification of Response	Students' Responses	Percentage
Blank/Wrong/Unrelated		93.1%
	Justification with arithmetic mean	52.4%
	Wrong explanations related to context	2.4%
	Other blank/wrong explanations	38.3%
Pre-Statistical		2.2%
	Notice the difference between numbers	1.3%
	Notice the outlier/extreme value	0.9%
Statistical		3.7%

Table 1: Descriptive Statistics for Item A4

Table 1 indicated that most of the participants provided wrong or unrelated responses (93.1%). These students accepted the statistical claim in average context without criticizing either providing wrong explanations related to context (2.4%) such as "Five questions can be solved in a class period" or justifying the results with arithmetic mean (52.4%). The rest of the participants gave pre-statistical (2.2%) or statistical (3.7%) responses. The statistical responses included either recognizing outlier in the data set or stating that getting average with median or mode is more appropriate. The difference between these statistical and pre-statistical responses was the appreciation of variability in the data set occurred in statistical explanations whereas recognizing outlier appeared in pre-statistical responses.

The detailed analysis of items revealed that majority of students had inadequate knowledge regarding average concept for statistical literacy. The most notable finding was that several students understood average which was a characteristic of the first tier behavior as "add them up and divide" algorithm which referred to the arithmetic mean and they did not consider median and mode as a way of finding average. In addition, only less than half of the participants were able to interpret average in context as a characteristic of second tier of statistical literacy. The majority of participants had failed to evaluate a statistical claim which was contextualized as third tier where they could not recognize outlier or justified this claim by providing evidence through arithmetic mean.

Statistical Literacy of Variation Concept

In the first tier of statistical literacy of variation concept, students were asked to select the data set which had more variability among others without context. Results revealed that majority of students (60.8%) were able to choose the data set with more variability. Students were additionally asked to provide explanations for their selections. These explanations were categorized as in the following table.

Classification	Codes based on Students' Responses	Percentage
Blank/Wrong/Unrelated		58.2%
	All numbers are same	6.5%
	Other blank/wrong explanations	51.8%
Pre-Statistical		12.1%
	Numbers are increasing	4.7%
	Numbers are different	7.4%
Descriptions via Measures of Spread		27.2%
	Range	23.1%
	Inter quartile range	0.1%
	Standard deviation	4%
Statistical		1.8%
	Larger variability	1.2%
	Away from average	0.6%

Table 2: Descriptive Statistics for Item V2

Table 2 indicated that majority of students (58.2%) either gave wrong responses or did not explain anything related to their answers in the first part. Of these, those who selected the data set which had the same numbers explained their responses through stating "all numbers are the same" (6.5%). Some of the participants (12.1%) provided pre-statistical explanations either stating that "numbers are increasing" (4.7%) or "numbers are different" (7.4%). A considerable percentage of students (27.2%) explained their responses through measures of spread. The most notable response in this category was "range" (23.1%) while "standard deviation" response was quite frequent (4%). Yet, very small percentage of participants (0.1%) explained their response through "inter quartile range". Statistically correct responses constituted only 1.8% percent of total responses where they either indicated the large variability in data set (1.2%) or distance from the average value (0.6%).

The second tier of statistical literacy of variation concept required students to interpret statistical claims involving variability. Results related to this ability indicated that majority of students (74%) were able to interpret statistical claims involving variability. It could be inferred that variability in the second tier was accomplished by most of the participants.

The third tier of statistical literacy of variation concept demanded students to evaluate the data sets and chose the one had more appropriate variability among others. The task for this tier is shown below.

A group of students noted the highest temperatures in Ankara during one year. They find the highest average temperature in Ankara as 16^{o}. Different from this group, three students predicted possible highest temperature for six different days in a year.

Students	**Predicted Temperature**
Seda	16, 35, 1, 5, 29, 10
Zeynep	16, 16, 16, 16, 16, 16
Umut	16, 15, 14, 26, 8, 17

a) Which students provide the data set regarding average temperatures with the most appropriate variability?

a) Seda
b) Zeynep
c) Umut

b) Explain your answer.

The results revealed that majority of students gave incorrect response where only 23.6% of the participants did choose the data set with more appropriate variability which was spreading around center (Umut). Of the incorrect responses, 20.7% of students did choose "Seda" which had greater variability whereas almost one third of the students labeled "Zeynep" which consisted of the same numbers. The classification of the explanations regarding their answers is given in Table 4 below.

Classification of Responses	**Codes based on Students' Responses**	**Percentage**
Blank/Wrong /Unrelated		73.1%
	Same numbers in the data set	14.1%
	Equal to the average	8.9%
	Other blank/wrong responses	50.2%
Pre-Statistical		11.6%
	More difference between numbers	10.3%
	Different numbers	1.3%
Statistical		13.9%
	Appropriate variation	3.5%
	Different numbers but closer	3.5%
	Around average value	6.8%

Table 3: Descriptive Statistics for Item V5

Table 3 indicated that a high percentage of students (73.1%) either gave wrong and unrelated responses or left the explanation part blank. Those who picked “Zeynep” as data set which had the most appropriate variation explained their answers either as “the numbers were equal to the average” (8.9%) or “numbers were the same” (14.1%). The pre-statistical explanations included either more difference between numbers (10.3%) or different numbers (1.3%). Still, there were statistical explanations (13.9%) which consisted of responses such as “appropriate variation” (3.5%), “different but closer numbers” (3.5%), and “around average value” (6.8%).

The detailed analysis of statistical literacy of variation concept indicated that students obviously performed differently in different tiers. For instance, although it was possible to say that there were inadequate knowledge in understanding and evaluating variability, almost 75% of participants correctly interpreted variation in context. One of the interesting finding was that some (6.5% and 14.1% for separate items) of students, indicated that more variation was involved where the data set consisted of same numbers. In addition, very small percentage of students (1.8%) gave statistically correct explanation regarding understanding of variation whereas most of them (27.2%) described variation through measures of spread.

DISCUSSION

The aim of this study was to investigate statistical literacy of average and variation concepts of 8th grade Turkish students. The present study confirmed the previous findings that Turkish students tended to consider the average as the arithmetic mean or “add them up and divide” algorithm (Toluk-Uçar & Akdoğan, 2009) as a characteristic of the first tier of statistical literacy of average concept. Most of the students did not consider median and mode as other ways of finding average of a given data set. Only less than half of the participants were able to interpret average in context, which is a second tier ability. This might be derived from students’ poor understanding of average concept. Their performance in evaluation of a statistical claim involving average as a representative value, which was a Tier 3 skill, was poor as they could not recognize extreme values, or they explained this claim by providing evidence through arithmetic mean. These results confirm previous findings in which students did not consider average as a representative value for the given data set (Mokros & Russell, 1995).

Students’ understanding of average as a summarizing or representative value in this study might be related to Turkish elementary mathematics curriculum. Although Turkish curriculum has addressed average concept through measures of central tendency (mean, median, and mode), students have started to learn average through arithmetic mean, which may result in understanding average as “add them up and divide” algorithm. In addition, while teaching average concept, teachers may not focus on its characteristic of representative value of a data set; instead they may devote majority of instructional time for computational skills.

Students' performance in the second tier of statistical literacy of variation concept was relatively higher than the first and third tier, which could be attributed to objectives in the curriculum and statistics instruction in Turkish schools. Almost one third of the 8th grade students explained the meaning of variation through measures of spread, particularly range. These responses might be due to the emphasis on the computational skills in statistics content. Turkish elementary curriculum represents variation concepts through measures of spread (standard deviation, range, and interquartile range) and students might have conceptualized variation concepts through range because it was easier to calculate. Although majority of participants were able to interpret variation concept in various contexts, their responses in other tiers indicated that they considered that there were more variation where the data set consisted of the same numbers. These kinds of responses might be regarded as a sign of possible misconception about variation concept of 8th grade students.

The analysis of students' statistical literacy of average and variation concepts indicated that students conceptualize these contents through arithmetic mean and range which are measures of central tendency and spread. Since these two concepts are fundamental for statistical literacy and further statistics outcomes, understanding, interpretation and critical evaluation of them in various contexts should be emphasized both in curriculum and instruction.

The reason for relatively lower performance in the first and third tier of statistical literacy of average and variation concepts compared to the second tier might be originated from the item formats. Tier 1 and 3 items required students to explain the reasons of their answers; hence, these questions were in open-ended form whereas questions in the Tier 2 were in multiple choice formats. Since 8th grade students were more familiar with multiple choice items due to national placement exams, they did well on the second tier questions.

The findings of this study revealed that Turkish 8th grade students had performed lower in first and third tier of statistical literacy of average and variation concepts compared to Tier 2 which was interpretation of statistical claims. Since statistical literacy is an important feature for building active and critical citizens, elementary mathematics curriculum should aim at developing statistical literacy within statistics and probability content area in each grade level. Furthermore, objectives might be modified in relation to support for statistical literacy. There was only one objective regarding evaluation statistical messages in the context of graph concept. Therefore, curriculum makers or planners should identify and include objectives regarding critical evaluation and questioning of statistical claims to promote the development of statistical literacy within elementary school students. For instance, evaluation of arithmetic mean as a representative value or understanding variance within or between data sets should take place as objectives in the curriculum so that there would be the possibility of instructing those objectives.

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UNDERSTANDING OF STATISTICAL GRAPHS IN PRIMARY: RESULTS OF A TEACHING UNIT[1]

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This poster reports a classroom based study involving exploration tasks focus on the construction, reading and interpretation of statistical data represented in tables and graphs aiming to analyze grade 3 students' understanding of statistical graphs during a teaching unit as well as to identify their difficulties.

Statistical literacy, primary students, understanding of statistical graphs

In Portugal, the new mathematics syllabus for basic education underlines the importance of developing students' statistical literacy due to its recognized role in the general education of citizens and presents more demanding learning objectives for statistics. The major aspect of curricular orientations at primary level is the development of "the ability to read and interpret data presented in tables and graphs as well as to collect and organize them in order to solve problems in a variety of contexts related to students' everyday life"(ME, 2007, p. 26). However, there is plenty of evidence in literature and from our own experience that many students can read tables and graphs but they miss the conceptual abilities to interpret and draw conclusions from graphs (Shaughnessy, 2007). It is therefore necessary to promote classroom opportunities for students to contact with a variety of statistical representations developing their understanding of graphs. This study aims to analyse grade 3 students' understanding of statistical graphs and tables, before, during and after a teaching unit, as well as to identify their difficulties, with the purpose of improving the teaching and the learning of statistics. The study followed a qualitative and interpretative methodology and involved 25 grade 3 students of a primary school in Portugal. The teaching unit was conducted during the academic year of 2011/2012 and included a sequence of 7 structured and articulated tasks focus on the construction, reading and interpretation of statistical graphs and tables, assuming that students' understanding of statistical graphs involves those processes (Wu, 2004). During the exploration of each task, in the classroom, the students worked in pairs or small groups. Upon finishing the exploration, the students presented their work orally to the class. These discussions provided opportunities to answer students' questions, to ask them to explain their reasoning, to progressively introduce new representations and probing the students' understanding of graphs. Data collection methods included participant observation and the written documents produced by students (task explorations and tests). Concerning the reading and interpretation of statistical graphs we took Curcio (1989) levels - reading the data, reading between the data and reading beyond the data - as a reference to analyse students' responses.

The preliminary results point to several difficulties regarding the construction of graphs which are consistent with those found in previous investigations (e.g. Lima & Selva, 2010). Major errors identified were related to title (absence or adequacy), labels and specifiers, scale selection, choice of an appropriate graphic to the situation presented. Concerning the interpretation of graphs it was found that students show few difficulties in level 1 questions (reading the data), but regarding levels 2 and 3 questions, requiring interpretation and inference, the students show they lack the capacity to explain their reasoning. As the teaching unit went on, the students evolve positively. Gradually they became aware of the several statistical representations that they could use to represent data, increased their level of responses to tasks involving the interpretation being able to better explain their reasoning and were attentive to all details of the graphs, developing, this way, their understanding of graphs.

The poster begins with a presentation of the study including the aims, framework, methodology and the teaching unit. The focus then is on examples of students' work in representing and interpreting statistical data to document the results of the study.

NOTES

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DIAGRAMS, GRAPHS AND CHARTS IN BIOLOGICAL COURSES A SYSTEM OF CATEGORIES IN THE OVERLAP OF MATHEMATICS AND BIOLOGY

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A main communication concept that can be found in everyday life and in different courses in school are representations like graphs, charts and diagrams. They are used for visualization, to show qualitative and quantitative connections between the data and its context. In our research program, we investigate statistic representations in the overlap of mathematics and biology. Statistical data always has a context and to read the data, it can require a basic knowledge of the applied topic. One main aspect of the project is to determine how students can read these representations and examine how they respond to the way they are designed.

Roth et al. (1999) investigated inscriptions in high school biology textbooks and scientific journals. Their research was mainly focused on graphs and scattered plots to determine out about their frequency and what practice is required to read them. A subsequent study about diagrams in a biological environment was presented by Lachmayer et. al. (2007). They proposed a model of cognitive abilities and their description of diagrams as depictional representations is based on the work of Schnotz (2001). To analyze how you can read statistical data in general, Curcio (1987) distinguished three levels how to read a graph. His model considers the context of the data: reading the data, reading between the data and reading behind the data. A fourth level that reaches further was added by Shaugyness (2007): reading beyond the data. In the current project these different competence levels are considered to include the biological component of the multiple representations.

To achieve an overview of statistical representation in biology courses, a sample of over 70 diagrams, charts and graphs was taken from schoolbooks designed for different types of secondary schools and levels. To acquire more multiplicity a textbook for university students was also investigated.

For a further and deeper analysis of these visualizations I developed a system of categories. The six categories are obtained on mathematical and biological motivation of the diagram, chart or graph. A commonly used distinction different from this is to categories reps in graphs, pie-charts, bar charts a.o. (Kattmann, 2006).

The poster suggested for presentation will focus on the system of categories and depict ways of presentation of data with biological context. Further analysis considered the different ways of the design and how the diversity of the mathematical correctness between the categories. Regarding also graphics in the representations based on the system of categories I found, a further classification can be made: diagrams that are mostly motivated by the biological content and diagrams that are motivated by a mathematical approach. There can a transfer to Curcios competence-model be achieved. To read the biologically motivated diagrams knowledge of the context is required. Therefore you need to *read beyond* the data in order to *read* the data. That means that you need to know the biological subject behind the data to get all the information that are provided in the diagram or graph.

I will present a variety of ways how such diagrams and graphs are designed and then you see that a main aspect belongs to the used graphics. There are several levels how the graphics are integrative part of the representation (e.g. the bar of the chart are presented as milk cans or a picture of a cow is next to the graph). A tendency is visible, that with an increased number of graphics included and part of the syntax of the representation (like bars as a milkcan), the accuracy of numerical value represented decreases (then the volume of the cans don't correlate with the represented kilogram of milk).

The next step in progress after this theoretical observation is to find out how students actually distinguish between the categories of the representations and if they respond to the different ways of the included graphic. Is there a connection between the categories and the ability to read the data or beyond the data? Therefore students were interviewed interpreting some diagrams, charts and graphs to see how they cope with the data and their biological context. The results of the interviews are required for an intervention study, to promote the reading competence of graphs, diagrams and charts with biology as applied topic.

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG 6: APPLICATIONS AND MODELLING

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Keywords: *Mathematical modelling;technology; theoretical perspectives*

The starting point of the working group was a presentation of all the present authors (not all authors were at CERME), of which many were newcomers to this group and to the CERME conference. After that presentation it stood clear that the WG 6 consisted of many different approaches to modelling and applications. We also explained that this time the WG 6 would be organized the following way: The papers would be grouped and presented in a 10 minute talk each. When the presentations were finished we (the listeners) would divide ourselves in groups around the authors of the papers for a constructive critique, thereby helping them to improve their paper as soon as possible and in good time before the proceedings. Since CERME welcomed an inclusive attitude this time, many of the papers were rather non-theoretical.

The following meeting gave room for presentation of empirical research on lower secondary level where Bock & Bracke reported on real-world or realistic problems given to lower secondary students, Greefrath discussed solution plans which can aid the processing of modelling problems, Rafiepour addressed the levels of competency of modelling in grade 9th and 10th Iranian students, Rafipour also presented a poster, Albarracin & Gorgorio showed how Spanish middle school students approach Fermi problems, and Spandaw presented that Dutch students prefer modelling themes such as 'electronic devices and computer games'.

The third meeting focused on empirical research on upper secondary level where Aerlebaeck described students' use of average rates of change to construct, interpret and describe representations, followed by Doerr who examined the characteristics of teaching in a classroom setting where students created and interpreted models of changing physical phenomena. Eley reported on a study where students were enabled to question models and "optima" declared by others, Hamacher & Kreussler talked about how to merge educational and applied mathematics by engaging students in the location of bus stops. Sundtjoenn suggested that her students showed an inquiry approach in the work place related task she gave them and Laval and Almeida & Kato talked about their posters.

The fourth session gave room for a first summary and for open questions regarding topics for further research but also for papers regarding in-service-courses and practising teachers such as papers by Bautista et. al. implied that educational backgrounds of middle school teachers influenced their ideas about mathematical models and about their teaching of modelling. Hoff Kjeldsen argued that in order to facilitate teachers' use of theory in modelling it is important to develop intermediate representations and modes of collaborations. Mousoulides talked about the importance of parents supporting modelling activities in compulsory school. A very good session with good discussions.

The fifth meeting continued with papers concerning In-service-courses / Practising teachers by Borromeo Ferri who talked about attitudes and opinions of primary teachers towards implementing mathematical modelling in their lessons. Vorhölter and Grünewald investigated what kinds of interventions by teachers are adequate when allowing their students to do modelling and Frejd talked about his study of how professional mathematicians and modellers view mathematical modelling.

The sixth session we discussed research on teacher education with papers by Barquero who talked about the conditions that facilitate the life of mathematical modelling at educational institutions and the constraints that hinder the same to be carried out. Salett Biembengut talked about how a general and common style of thought with connection to mathematical modelling could become alive among educational institutions in Brazil. Benacka described the techniques of Excel with respect to modelling activities without data. Carreira gave a well organized description of how the two representation modes in GeoGebra facilitate students' understanding of linear functions, while Maltempi argued that the plasticity of the cybernetic world allow the creation of a space where imaginative aspects can be actualized, thus representing a differential for the practice of mathematical modelling itself.

The seventh session concluded a summary of the work in the WG 6 by a joint debate concerning prospects for further research led by Lingefjärd, Carreira, Kaiser, and Wake. One major conclusion was that many of the presented papers lacked a scientific or at least a theoretical perspective. Many papers were more of the kind: "this is what I do with my students". So the sessions closed with the following questions, rather identical with the ones from CERME 7 and still actual because of the outspread stack of papers in our group:

How do we move forward in the scientific debate on applications and modelling?

How do we include applications and modelling in curriculum and school practice?

What kind of empirical research is necessary?

What kinds of measures are necessary and can be constructed? Is the development of teaching units or learning materials sufficient? What is the role of comprehensive learning environments?

What can we say about the role of culture in the teaching and learning of mathematical modelling? Is it culture-free or what do we need to consider as cultural effects in our debate?

Reference:

Blum, W., Galbraith, P., Henn, H.-W., & Niss, M. (2007). *Modelling and applications in mathematics education. 14th ICMI study*. New York: Springer. Applications and Modelling

Papers presented in the group:

Lluis Albarracín & Nuria Gorgorió. Fermi Problems Involving Big Numbers: Adapting a Model to Different Situations

Jonas B. Ärlebäck, Helen Doerr & AnnMarie O'Neil. Students' Emerging Models of Average Rates of Change in Context

Berta Barquero, Lidia Serrano & Vanessa Serrano. Creating Necessary Conditions for Mathematical Modelling at University Level

Alfredo Bautista, Michelle H. Wilkerson-Jerde, Roger Tobin & Barbara Brizuela. Diversity in Middle School Mathematics Teachers' Ideas About Mathematical Models: The Role of Educational Background

Jan Benacka & Sona Ceretkova. Excel Modelling in Upper Secondary Mathematics - A Few Tips for Learning Functions and Calculus

Maria Salett Biembengut & Emilia Melo Vieira. Mathematical Modelling In Teacher Education Courses: Style of Thought in The International Community - ICTMA

Morten Blomhøj & Tinne Hoff Kjeldsen. The Use of Theory in Teachers' Modelling Projects – Experiences From an In-Service Course

Wolfgang Bock & Martin Bracke. Project Teaching and Mathematical Modelling in Stem Subjects: A Design Based Research Study

Rita Borromeo Ferri & Werner Blum. Barriers and Motivations of Primary Teachers for Implementing Modelling in Mathematics Lessons

Cor Willem Buizert, Jeroen Spandaw, Martin Jacobs & Marc de Vries. Themes for Mathematical Modeling that Interest Dutch Students in Secondary Education

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Nicholas Mousoulides. Parental Engagement in Modeling-Based Learning in Mathematics: An Investigation of Teachers' and Parents' Beliefs

Abolfazl Rafiepour Gatabi & Kazem Abdolahpour. Investigating Students' Modeling Competency through Grade, Gender and Location

Trude Sundtjønn. Pupils Discussions While Working on a Workplace Related Task

Posters presented in the group

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Abolfazl Rafiepour Gatabi & Fereshteh Esmaili. The role of modelling on effects of Iranian students

Lourdes Maria Werle de Almeida & Lilian Akemi Kato. Different approaches to mathematical modelling possibilities for the deduction of models and the student's actions

FERMI PROBLEMS INVOLVING BIG NUMBERS: ADAPTING A MODEL TO DIFFERENT SITUATIONS

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1 ABSTRACT

In this paper we describe a classroom experience based on the sequencing of Fermi problems related to estimating large quantities. The models used by a group of Compulsory Secondary Education (16 year-old) students are described herein. The variations the students apply to the models in order to adapt them to similar problems formulated in different contexts are shown. In the conclusions section, we reflect on this didactic proposal and the possibilities it offers to the students, so that they assimilate and internalize the models worked on.

2 INTRODUCTION

In this study we introduce Fermi problems oriented towards the estimation of large quantities. We understand that the modelling processes which appear in the solution of this type of problems with a realistic context cannot replace real-life decision-making, however their use in the classroom promotes attitudes which can be useful for everyday life (Jurdak, 2006). The problems presented in our study are based on the estimation of large numbers. We herein describe the way students adapt their modelling strategies to different problems with similar mathematical structures but which are formulated in different contexts. Seldom found as class problems, solving them demands the students to create their own strategies, adapt them and use them to solve different problems. This leads them to identify with and include these strategies in their mathematical knowledge base, which according to Schoenfeld (1992) may help students become competent problem solvers.

3 THEORETICAL REFERENCES

3.1 Modelling

One of the most relevant scientific activities involves creating models which provide an abstract recreation of objects, phenomena or processes we wish to understand. In recent years, there has been a strong tendency to attempt to approach model creation to the classrooms.

Lesh & Harel (2003) define model as follows:

> Models are conceptual systems that generally tend to be expressed using a variety of interacting representational media, which may involve written symbols, spoken language, computer-based graphics, paper-based diagrams or graphs, or experience-based metaphors. Their purposes are to construct, describe or explain other system(s).
>
> Models include both: (a) a conceptual system for describing or explaining the relevant mathematical objects, relations, actions, patterns, and regularities that are attributed to the

> problem-solving situation; and (b) accompanying procedures for generating useful constructions, manipulations, or predictions for achieving clearly recognized goals. (p. 159)

According to this definition, a model can be understood as an abstract way of representing a particular phenomenon or reality. The way students elaborate models in order to solve problems is a matter of discussion and different views exist on this subject (Borromeo Ferri, 2006). However, in general terms, it is agreed to be a multi-cyclic process. According to Blum (2003), modelling processes can be structured into five main stages: i) Simplifying the real problem into a real model; ii) Mathematizing the real model into a mathematical model; iii) Searching for a solution from the mathematical model; iv) Interpreting the solution of the mathematical model and v) Validating the solution within the context of the real-life problem.

3.2 Estimation and Fermi Problems

When we intend to answer questions such as: how long would I take to get to the train station? How many 5kg paint cans do I need to paint the walls of my flat? Or, how many teaspoons of sugar do I need to cover the 250 grams indicated in the recipe? We need to make estimations. By estimation we mean a rough calculation or judgement of the value, number, quantity or extent of something. After revising the literature, three types of estimation can be found: numerosity, estimation of measurements and computational estimation (Hogan & Brezinski, 2003). This fact is due to the existence of a wide range of tasks which, although they don't share the numerical patterns which enable their execution, all require the concept of estimation (Booth & Siegler, 2006). Numerosity refers to the ability to visually estimate the number of objects arranged on a plane; measurement estimation is based on the perceptive ability to estimate length, surface area, time, weight or similar measurements of ordinary objects, while computational estimation refers to the process by which the value of a calculation, such as 13.2÷4.3+6.91, is approximated.

There are yet two types of activities that are referred to as estimation for which we haven't found any relevant studies in the field of Mathematical Education. On the one hand, in a branch of Statistical Inference, estimation is known as the group of techniques which allow calculating an approximate value for a population parameter from the data provided by a sample. On the other hand, another mathematical activity regarded as estimation is the calculation of values obtained either from predictive activities or from approximating a reality by using a model which represents a situation. A good example of this kind of situations is directly reflected in what are known as Fermi problems.

Ärlebäck (2009) offers the following definition of a Fermi problem:

> Open, non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations. (page. 331)

Ärlebäck (2011) states that working with Fermi problems can be useful to introduce modelling in the classrooms for several reasons. Indeed, we have confirmed that they don't require any specific type of previous mathematical knowledge, the students are obliged to estimate several quantities by themselves (since the problems don't provide numerical data) and are encouraged to discuss the issue with their peers.

4 THE GOALS OF THE STUDY

Based on a class activity carried out with students in their 4th year of Compulsory Secondary Education, using a sequence of Fermi problems aimed at the estimation of large quantities, we study the models the students create to solve the problems as well as how they adapt the models to other similar problems. Thus, the goals of the study are:

1. To identify the models students use to solve problems

2. To identify the modifications they introduce to previously applied models when facing new problems

5 METHODOLOGY

The work dealt with in this paper is part of a wider study presented in Albarracín & Gorgorió (2011) where we research individual strategies for solving Fermi problems aimed at the estimation of large quantities. These types of problems are not currently included in the Spanish curricula and are not usually worked on in class, which means that students aren't being taught specific methods by which to solve them. Thus, students are obliged to create their own resolution strategies which include models of the situations proposed. By definition, the resolution process of Fermi problems can be based on breaking them into smaller problems, which should be easier for the students to approach than the original problem. Dealing with large quantities doesn't allow for simplistic approaches to the solution, and thus we expect the students to come up with strategies richer in mathematical elements, from the abstract representation of the studied reality.

The experience presented hereafter was carried out on a group of 22 pupils in their 4th year of ESO (compulsory secondary education, 16 year-olds) with no previous instruction in modelling. This time we asked the students to solve a set of Fermi problems in teams, which required estimating the number of objects distributed over a surface area in contexts initially familiar to them. The first problem refers to the school itself, while the following four problems require information which isn't directly available to the students, and which they would have to obtain from an external source. The problems are the following:

• Problem A: How many people fit in the school playground?

• Problem B1: How many people fit in a concert at the Palau St. Jordi[1]?

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[1] Palau St. Jordi is a pavilion built for the Barcelona'92 Olympic Games

• Problem B2: How many people fit in a demonstration held in the town hall square of your city?

• Problem B3: How many people fit in a demonstration held in Plaça Catalunya (Barcelona)?

• Problem B4: How many trees are there in Central Park?

The problems were worked on in several sessions. In the first session, the students were set problem A and asked to write an individual resolution proposal. Afterwards, the students were arranged into work teams of 3 or 4 (6 groups in total) which had to come to an agreement on a group resolution proposal and determine the actions and resources required to estimate the number of people that fit in the school playground. In the second session the students executed the previously planned work and started to write their reports. In the third session, the reports were completed and they shared the results and methods used.

In the fourth session, the students were presented with the following four problems (B1, B2, B3 and B4) and were allowed to access the internet if necessary. The purpose of the fifth session was to complete the different resolutions and produce a second results report. The sixth session was carried out as a conclusive activity in which the results were compared with information obtained from external sources.

The data used in this study are the reports created by the students and the observations collected by the first author during the experience. We herein present some of the data collected. Fig. 1 shows the resolution of some students who propose estimating the amount of people that would fit in the high school playground by counting the number of them that would fit in it if arranged into rows and columns.

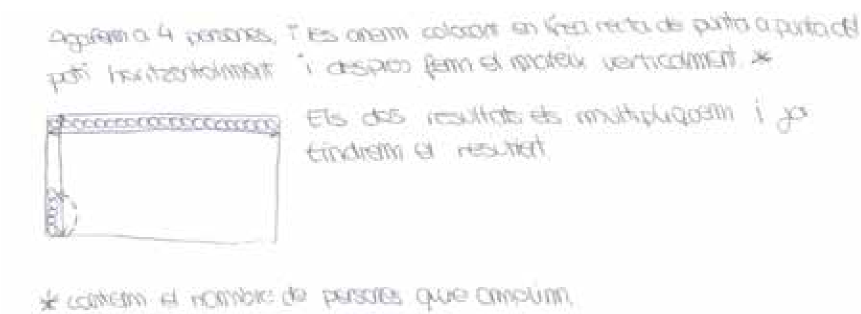

Fig. 1. An example of the resolution of problem A.

The abovementioned proposal reads “ We would take 4 people and place them in a straight line from one end to the other of the playground crosswise and lengthwise, counting the number of people that we fit into it. We would then multiply both results.” As displayed in Fig. 2, the students have made their calculations and obtained 26 and 82 people respectively for each dimension of the courtyard, for which they give a result of 2,132 people.

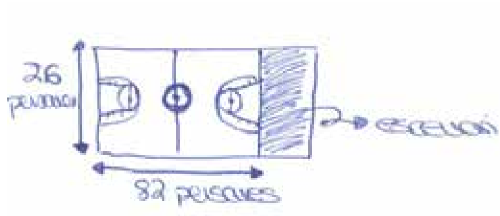

Fig. 2. Data collected for the resolution to problem A.

In Fig. 3 we present some students' resolution of problem B2, which requires estimating the maximum amount of people that would fit in the town hall square during a demonstration.

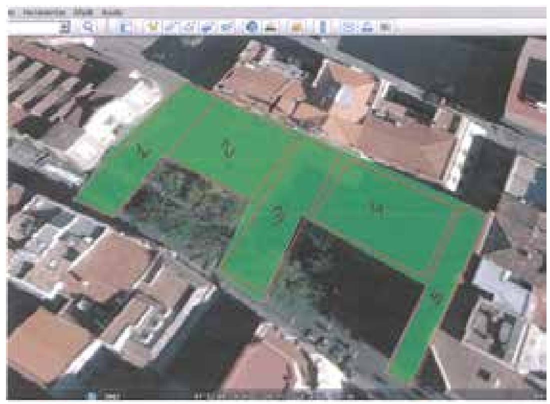

Fig. 3. Available space in the town hall square.

The caption of this picture is the following: "We searched for the town hall square on Google Earth and marked the areas where people could be in with the polygon drawing tool. We assumed they wouldn't tread on the green areas. We afterwards divided the shaded area into rectangles to make it easier to find out their surface areas. We assumed 3 people could fit in a square metre." After that, the students calculated the number of people that could stand in each of the separate areas with their previously-obtained surface data. Figure 4 shows the calculations for zones 4 and 5, as well as the final result.

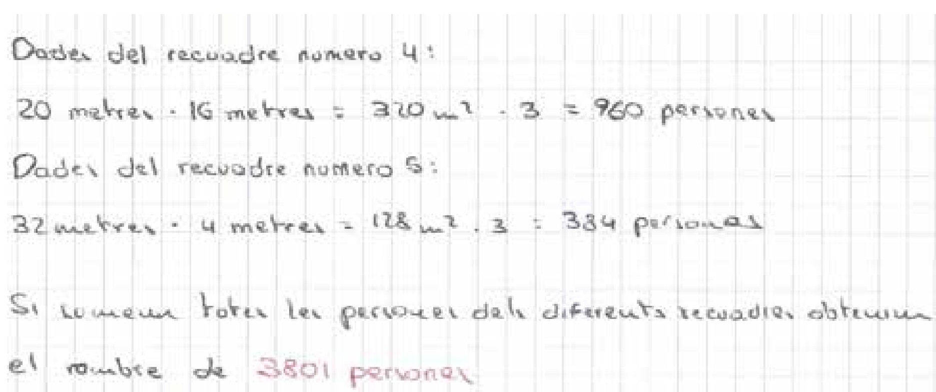

Fig. 4. Calculations done for the resolution of problem B2.

The data was analyzed following the model presented in Albarracín & Gorgorió (2012), which identifies the resolution strategies proposed by ESO students for several Fermi problems which require estimating large quantities. Within the scope of our research, we understand a resolution strategy as a plan of action or policy designed to achieve a major or overall aim.

The analysis is centred on describing the specific actions the students propose and placing them in more general settings. For instance, some students propose counting the number of people attending a demonstration one by one, by asking all of them to write down their name on a list, or suggest recording video footage of a leakage in order to count the number of drops falling throughout the recording by hand. Both of these proposals to diferent problems portray different plans of action but show the same intention, which is to carry out an exhaustive count of the entirety of objects in the problem. Therefore, these proposals show diferent actions with the same kind of plan, which we interpret as adaptations of different problems to the same type of strategy.

By using the quantitative data analysis software NVivo 8, we established different analysis categories corresponding to the strategies detected. These categories are: lack of strategy, exhaustive count, use of an external source of information, reduction of the problem to a smaller one, comparison with a real-life situation and breaking the problem into different parts to be solved separately.

The latter strategy contains elements of modelling. The way the students break up the main problem into different sub-problems is determined by how they represent the situation studied. Several models have been identified for each situation, which portray different ways into which the problem can be broken up. Some of these models coincide with those identified in this study, and will be explained in the following sections.

6 MODELS DETECTED FOR THE FIRST PROBLEM

We will firstly focus on the resolution of the problem of estimating the number of people that fit in the school playground (problem A).

Not all the individual proposals included a resolution scheme that enabled the required estimation to be made, but we however observed that the teamwork yielded suitable work plans for all groups. The students' proposals described the situation by means of mathematical concepts and their relationship to the studied reality as well as the procedures required to reach a solution, which means they modelled the problem following Lesh & Harel's (2003) definition.

It is worth noting that the high school playground has a rectangular plan view.

In the following we present the mathematical models created by the different teams to estimate the amount of people in the high school playground, which has a rectangular plan view.

Four of the teams used a strategy based on the idea of population density. The students measured the length and width of the playground in order to obtain its surface area. On the other hand they carried out experiments to determine the number of people that would fit on a small surface area. Using the experimental data, they obtained a value for the density of students that would take up one square metre and then multiplied it by the surface area of the playground in square metres.

One of the teams based its strategy on the iteration of a reference point. The students measured the length and width of the playground in order to calculate its surface area and carried out experiments to determine the area a single person would occupy. Using this value, they divided the total surface area of the playground in square metres by the surface taken up by one person. This process is equivalent to the iteration of a reference point, which is a length estimation strategy which consists of mentally counting the number of times and object (reference point) may be placed on top of the object to be measured (Joram, Gabriele, Bertheau, Gelman & Subrahmanyam, 2005).

One of the teams based its strategy on a grid distribution. The students in the team lined up one after the other. In order to move the line forward, the last person in the line would advance to the position in front of the first person in the line. While moving forward they counted how many people would be needed to occupy the length and width of the playground. To find out how many people would fit in the playground, they multiplied the two experimentally calculated values. This model responds to a similar idea to the product rule used to obtain the surface area of a rectangle.

We would like to stress that the previous three models are based on a type of resolution which establishes different sub problems that must be solved separately: What's the surface area of the playground? How many people fit in a square metre? Or: how many people can we line up along the length of the playground? This way of proceeding corresponds to Ärlebäck's (2009) definition of Fermi problems.

Once they completed the activity, the students reported their results, as well as the methodology used to obtain them. This idea-sharing session succeeded in getting the students to compare their methods and adopt the models their classmates had used. They also discussed their results, since the values obtained ranged between 1200 and 2200 people for a playground of 350 square metres. The students accepted the idea that there couldn't be just one single correct result, but a suitable interval including all possible solutions.

Afterwards the results were compared with capacity data from concert venues and the students realised their density values were rather high. This triggered discussions in which they tried to clarify the points in which they disagreed (surface area occupied by an adult versus that occupied by a teenager, comfort, safety rules). This allowed the students to connect the model used and the decisions taken with the reality being studied.

7 ADJUSTMENT OF MODELS TO THE FOLLOWING PROBLEMS

In the second session the students solved the remaining 4 problems. This time they weren't allowed to go to the locations referred to in the problems to take any measurements, and they therefore decided to search for the required information on the internet. Given that the formulations of the problems contextualized the estimation in public spaces, the students were faced with the difficulty of deciding how to adapt the models they had built for the previous problem to the new problems set. Given that the data we are working with is the students' output, this study cannot analyze the cyclical process of modelling for a specific problem. However, we can study the modifications made to adapt certain models for their use on different problems.

The first notable fact is that the teams generally used the population density model for the new problems, possibly because they considered this model more versatile and adaptable to any situation. Only two of the teams used the iteration of a unit, the space taken up by a tree, in order to solve the problem on Central Park. They considered that the density of trees expressed as units by square metre wasn't a manageable number.

After analysing the resolutions presented, we detected several variations in the models used by the students.

The students identified unavailable spaces when devising the resolution strategies for problems B2, B3 (squares in an urban centre) and B4 (trees in Central Park). Contrarily to the case of the school playground, it's not possible to cover the whole extent of a public space, since some of it taken up by street furniture or roadways. Considering this constraint, the students looked for an aerial photograph of the areas to be studied in Google Maps, delimited inoperative areas and marked them. Then they made some measurements with a tool from Google Maps in order to calculate surface areas.

The students identified spaces with different densities when embarking on the resolution of the problem which refers to the Palau St. Jordi, a large pavilion where concert attendants can either sit on the tiers or stand in the arena. They were unable to find an aerial photograph of the pavilion interior because it is indoors. They needed to approximate the surface area of each of the two parts by educated guesses. Once estimated the surface areas, they applied different population densities to them.

In problems B1 (population density of the steps), B2, B3 (population density in a demonstration) and B4 (density of trees) the students realised that the density of the objects may vary according to the circumstances. When estimating the population density of trees in Central Park, the students collected information regarding types of trees and their ages. They also determined that the density of people seated on one of the tiers in Palau St. Jordi is different from that of the people standing in the concert. They also distinguished between the population density in a concert from that of a demonstration. When the students completed their estimations using methods they had created themselves, the activity ended up in a discussion on the validity of the results obtained. They used information from different sources on the web (Wikipedia and several online newspapers) in order to contrast the results and even discarded some of the statements made by the media. Therefore, the students validated their results and established direct links between the studied reality and the models they had created and adapted to different situations.

8 CONCLUSIONS

Based on the collected material and data analysis, we can state that all the working teams in this study solved the problems presented to them as Fermi problems, by establishing small sub problems they solved separately by means of calculations or estimations. For the problem on the number of people that would fit in the playground, the students' proposals included different types of models. This suggests that working with different Fermi problems may help and encourage students to generate a wide range of strategies and models, which agrees with Ärlebäck's (2011) statements on the possibilities offered by Fermi problems.

Using an initial problem to be worked on at school followed by list of similar problems that require researching information in external sources has allowed the students to discuss how the previously created models adapted to the different problems (Ärlebäck, 2011). This discussion has driven the students to generate new models which adapt to the new situations to estimate the amounts required in each of them.

When working with problems which pose new difficulties, the students adapted their models to the represented reality, developing more complex versions of the proposed models. Since the work was carried out in teams, we ensured that all the students in class took part in the discussion on the different key aspects of decision-making for generating the models used. This allows the students to adopt these models and include them in their knowledge base (Schoenfeld, 1992). Therefore, the proposed sequence of problems has led to a higher level of understanding of the methods and concepts used; connecting those to different situations which have similar approaches but differs in some content details. It is worth noting that students started their project using the models suggested by their own team but easily adopted ideas proposed by other teams, which implies that the different idea-sharing sessions of methods and results were crucial to the whole process.

Finally, we observe that students are able to compare some of their results with information gathered from different sources. This comparison provides information which they may include when elaborating future models, such as the need to decrease the population density for safety reasons. On the other hand, some of the data found on the web did not agree with some of the results the students had obtained using their most perfected models, accepted by the whole class, which lead them to question the truthfulness of this information. Manifestly, the process reached its most relevant point when the students realised that their own analysis of a situation may disprove information provided by the media. In conclusion, by means of these resolution processes the students have given meaning to the concepts worked on in class and compared their output with real information, as described by Jurdak (2006).

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STUDENTS' EMERGING MODELS OF AVERAGE RATES OF CHANGE IN CONTEXT

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This paper describes a model application activity in the context of how light intensity varies depending on the distance from a point light source within a model development sequence focusing on developing students' understanding of average rate of change. Students' early models were found to be linearly decreasing functions and they did not distinguish between the light intensity and changes in light intensity. Most students were able to draw on several representations (most notably tables and graphs) to represent, describe and distinguish between the light intensity and changes in light intensity. Some students encountered difficulties when describing change when the rates were negative and understanding how the graphical representation of average rates of change depended on the interval width.

The difficulties encountered by students when reasoning about and interpreting rates of change are well documented in the research literature (Carlson et al., 2002; Monk, 1992; Thompson, 1994). The covariational reasoning students need to simultaneously attend to both the changing values of the outputs of a function and the changing values of the inputs to the function, is foundational for understanding average rates of change in pre-calculus and instantaneous rates of change in calculus (Johnson, 2012; Oehrtman, Carlson & Thompson, 2008). However, relatively little research has attended to the particular challenges students encounter when reasoning about negative rates of change, the role of context in the tasks involving change, and students' construction and application of representations. Many studies investigating aspects of average rates of change use different representations (graphs, tables, figures, symbols, written language, simulations, enactments) to create tasks on which students' work is then analyzed (e.g. Johnson, 2012; Herbert & Pierce, 2012). Often, the use of different contexts provides a background situating the representations, and students are asked to extract contextual meaning from a given representation.

Whitney (2010) investigated students' initial understandings of rate of change and their preference for using different representations (tables, graphs, equations and contexts) in the context of linear functions. The students tended to use tables as tools for organizing information and "graphs were not students' first choice in representing information" (p. 186). Along the same line, Herbert and Pierce (2012) found that the rate-related information understood by students through different representations was highly individual, and that "understandings of rate in one representation or context are not necessarily transferred to another representation or context" (p. 455). This is consistent with the findings of Ibrahim and Rebello (2012) comparing student work with different representations in the subject matter areas of kinematics and work.

In this study, we examined students' use of average rates of change to construct, interpret and describe representations of the relationship between measurements of light intensity and distances from a light source. We focused our attention on students' descriptions of how the light intensity varies with the distance from the light source and on the students' interpretations of the graphs that they created to represent the average rates of change. The questions guiding this investigation were: How do students interpret and describe their constructed representations of average rates of change? How do students attend to context when interpreting and describing their constructed representations of average rates of change?

THEORETICAL BACKGROUND

We used what Kaiser and Sriraman (2006) call a contextual modeling perspective on teaching and learning mathematics (cf. Lesh & Doerr, 2003) and a design-based research methodology (Cobb et al 2002; Kelly et al. 2008) to design and implement a sequence of modeling tasks aimed at supporting the development of students' understandings of average rates of change.

Model development sequences

Model development sequences (Doerr & English, 2003; Lesh et al., 2003) begin with model eliciting activities where students are confronted with the need to develop a model in order to make sense of a situation familiar to the students. When the model is elicited, the purpose of successive model exploration activities and/or model application activities is to help the students extend, revise and refine their emerging models. The focus of model exploration activities is on the underlying structure of the elicited model and especially on the strengths of various representations as well as on how to productively use these different representations. In model application activities the students apply their model to new situations, which in turn can lead to the model being further extended, revised and refined and to further understandings of the utility of the representations.

METHODOLOGY

The model development sequence designed to support the development of students' understandings of average rates of change formed the basis for a six-week course taught by the third author for students who were preparing to enter their university studies. Throughout this sequence, students were engaged in the processes of iteratively revising and improving their models of average rates of change while interacting with other students and participating in teacher-led class discussions. A total of 34 subjects (10 female and 24 male) participated in the study; 21 students had studied calculus in high school and 13 had not studied any calculus. The students worked in small groups to complete the different activities. The model eliciting activity used motion detectors to elicit the notion of negative velocity with motion along a straight path. To explore changing negative rates, multiple model exploration activities were used based on computer simulations (Kaput & Roschelle, 1996) and structured exercises (Watson & Mason, 2006) focusing on different representations of negative rates. Two model application activities in the sequence were carried out as lab projects where the students spent several days on the activities and wrote final reports. One model application activity focused on creating a model of the intensity of light with respect to the distance from a light source (the Light Lab), and the other aimed at modeling the rate at which a fully charged capacitor in a resistor-capacitor circuit discharged with respect to time. Each activity included class discussions focusing on the relationships among different representations of negative rates of change and students' interpretation and descriptions of change in different contexts and settings. The work reported on in this paper is the analysis of the students' work on the Light Lab, which took place at the beginning of the fourth week of the course.

The Model Application Activity Design

The Light Lab task was designed and carried out in six phases as illustrated in Figure 1. Prior to the Light Lab, the students' work on the model eliciting and exploration activities almost exclusively involved different representations of one-dimensional motion along a straight path. The aim of the Light Lab was manifold; it was originally designed as a model application activity in that it provided the students with an opportunity to apply their emerging models of average rate of change elicited in the context of linear motion in a new context. However, the iterative design process inherent in the design-based methodology led to the successive inclusion of elements of exploration focusing on the graphical representations of the average rate of change in non-linear contexts.

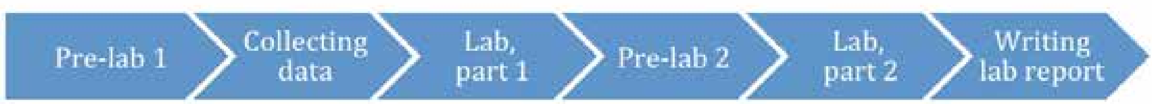


Figure 1. The six phases of the Light Lab

The first pre-lab task aimed at making explicit the students' intuitive and initial models about the relationship between the intensity measured at different distances from a point source. The students were asked to consider a one-dimensional scenario of an approaching car and to sketch a graph of how the intensity of the car's headlights varied depending on the distance you are from the car. The students were also asked to describe how light is emitted from a point source in terms of how the light rays emitted from it are dispersed, and what implications this might have for their emerging intensity model. The students' work on this task was collected and discussed in class.

In the second task, students collected light intensity data using a flashlight with the focusing cap removed, a meter stick, and a light sensor connected to their graphing calculator. They collected 15 measurements of the intensity at one cm intervals from the light source and transferred the data to their computers. The light sensor measured light intensity in a relative and arbitrary unit called "light intensity units" (LIU). Due to physical limitations of the light sensor, the first two or three data points gave a maximum reading of one LIU on the light sensor.

In part one of the lab, the students were asked to make a scatter plot of their data and to write descriptions of how the intensity of the light changed with respect to distance from the light source. They were encouraged to compare this relationship to their predictions from the first pre-lab. The students were also asked to calculate the average rates of change of the data in one cm intervals and to (1) describe the average rate of change of the data as distance from the light source increases; (2) to graph the average rates of change in a scatter plot; and (3) to create a separate rate graph of the calculated average rates of change.

In the second pre-lab task, the students were given four images representing light intensity at different distances. The number of dots per square inch indicated the intensity, and the task for the students was to determine the intensity in dots per square inch at a given distance from the light source. The second pre-lab introduced an inverse square relationship as a model for how the intensity varies with distance from the light source, and familiarized the students with finding such a functional expression fitting a given data set. This task was given as homework and subsequently discussed in class.

In the second part of the lab, the students were asked to determine a function fitting their collected data, explain their work, and analyze the average rates of change of their function using the difference quotient. Due to the limitations of the sensor, students needed to construct a piecewise function, consisting of a horizontal linear piece for the first two or three data points and an inverse squared piece for the remainder of the data set. More specifically, the student were asked (1) to calculate the average rates of change of the function at integer x values and an interval width of 0.5 cm; (2) to show these average rates of change in a function plot; (3) to create a rate graph for the average rates of change values; and (4) to describe how the average rate of change of the function changes as distance from the light source increase.

Data analysis

In this paper, we report on the analysis of the student data from pre-lab 1 (n=34; done individually) and the lab reports (n=18; all written by pairs of students with the exception of one student writing alone). The analysis was done in two phases following the principles of grounded theory (Strauss & Corbin, 1998). First, codes were developed to categorize the students' reasoning and answers on each of the questions for pre-lab 1. This initial analysis focused on capturing the students' initial models of how the light intensity varies with distance from the light source and how light disperses from a point source. In the second phase of the analysis, the students' final lab reports were read and coded, focusing on interpretations and descriptions of how the intensity varied with the distance from the light source and its average rates of change, the graphs the students produced, how they attended to the context throughout their writing, and how the students' used their ideas from the pre-labs.

RESULTS

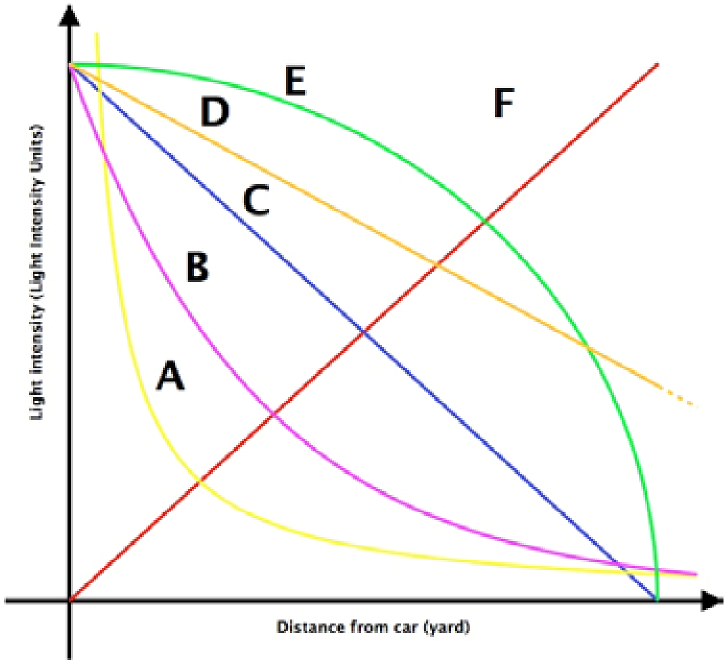
Our analysis yielded three sets of results. First, we found that most students' early model of changing light intensity was a linearly decreasing function and did not distinguish between the light intensity and changes in light intensity. Second, as students' models of light intensity with respect to distance developed, most students were able to draw on several representations (most notably tables and graphs) to represent, describe and distinguish between the light intensity and changes in light intensity. Third, some students encountered difficulties when describing change when the rates are negative and understanding how the graphical representation of average rates of change depends on the interval width.

Students' initial models of variability of light intensity at different distances

The students' initial models of the relationship between light intensity and the distance from the light source are shown in Figure 3, which illustrates the different types of graphs the students constructed to describe how the light intensity from a car's headlights varies with the distance to the car. The table displays the number of students' graphs in each category; the cut off between the C and the D graph was a slope of -0.7. Nearly all of the students (n=28, 83%) drew a linear relationship between the intensity of light and the distance from the source. All but one of these linear graphs (shown as C and D in Figure 3) correctly show the intensity decreasing, but incorrectly at a constant rate. This is likely due to the students assuming that the speed of the approaching car is constant (c.f., Doerr, Arleback & O'Neil, submitted) and confusing the constancy of speed of the car with the constancy of the decrease in light intensity. The one student who drew graph F incorrectly reasoned that the light intensity increased as the distance from the car increased. The four students who drew graph A, with its asymptotic behavior at the y-axis, were likely drawing on their formal physics and/or mathematical knowledge of the inverse square law for light intensity. We note that all students had taken a prior course in physics. Since graph B shows a maximum intensity at the instant the distance is zero, the one student who drew graph B may have attended to his perception of the context as well as his formal previous knowledge.

Nearly all students (n=26, 76%) provided a qualitative description of the intensity at 1000 yards compared to at 2000 yards, using vocabulary like "greater" and "less". The remaining 24% of the students, on the other hand, made a quantitative statement describing the intensity at 1000 yards to be exactly twice the intensity at 2000 yards. This was done exclusively by students drawing a C graph in Figure 3. While it is possible that this could be the case, this reasoning is more suggestive of exponential decay (whose change is multiplicative in structure) than a linear function, whose change is additive in structure.

When asked to "Compare the rate at which the intensity is changing at 1000 yards and 2000 yards", 17 of the 34 students described or calculated the rate of change their graphical representation. One student, referring to his C graph, wrote: "The rate at



Graph	Number of students
A	4 (12%)
B	1 (3%)
C	22 (65%)
D	5 (15%)
E	1 (3%)
F	1 (3%)

Figure 3. Students' initial models of intensity vs. distance from light source

which the intensity is changing at 1000 yards and 2000 yards is the same". However, 16 of the 34 students compared the values of the function at the two distances rather than the rate of change at those distances. For some of these students, their description did not mention the rate of change at all (for the C graph: "The intensity doubles from 2000 yards to 1000 yards"); other students explicitly confounded the function's values with the rate of change values (C graph: "The rate of intensity is doubled"). The remaining one of the 34 students simply gave an equation of a linear function that did not correspond to his graph.

When asked to "Draw some representative light rays leaving the light source" the students drew figures of light dispersing either as cones, perpendicular rays or waves

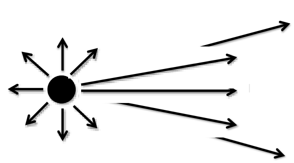
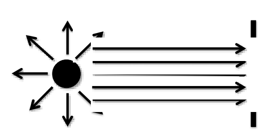
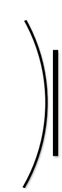
A. Light rays "cone-like"	B. Light rays perpendicular	C. Light disperse as waves
		
28 students (82%)	5 students (15%)	1 student (3%)

Figure 4. Students' initial models of intensity vs. distance from light source

(see Figure 4). Students drawing a cone-like model have a potential rationale for why the light intensity decreases when distance increases, whereas the perpendicular model implies a constant light intensity, independent of distance. One student, likely drawing on previous knowledge from prior coursework in physics, drew the light dispersing as waves. However, regardless of their model in Figure 4, all students concluded that the light intensity would decrease as distance increase. This illustrates the compartmentalization of conflicting ideas in the students' emerging models that the students have still not resolved and pulled together.

Students' descriptions of the light intensity and its average rate of change

All students used tables to organize and summarize their raw data and various calculated quantities. Scatter plots of the data were included in all reports, either as separate graphs, together with graphs where the average rates of changes were drawn in, or, in graphs showing how their symbolic piece-wise function fit the data. Nearly all (89%) of the reports included correct descriptions of the values of the data drawing on the context of light intensity. Some descriptions tended to be short and close to strict mathematical statements: "The intensity of the light source decreases as the distance increases." Other descriptions were more closely situated and connected to the lab setting: "As the distance from the light source increased, the light intensity decreased. The farther away the sensor was away from the light sensor the smaller the light intensity became." These two sentences convey the same information using slightly different language: the first is a general sentence about the behavior of how light intensity varies with distance from the light source, similar to the more mathematical description in the previous example. This relationship is then expressed in terms of the context of the lab. One can also note the use of the word "smaller" in the last sentence, which does not quite capture the notion of light intensity, pointing at the complexity of trying to make precise and meaningful interpretations of mathematical statements and relationships in context.

In some cases, the students made explicit references back to their data to support their descriptions. In the following example, the description accompanied a table listing the light intensity values and the calculated average rates of change values as well as a scatter plot of the data with the average rates of change drawn in, connecting all consecutive points (as shown in Figure 5A): "As the distance from the source increases the intensity of the light decreases. For example, at 3 cm the light intensity reads .832 LIU [Light Intensity Units]; at 4 cm the light intensity reads .637 LIU; and at 5 cm the light intensity reads .463 LIU. It is also noted that rate at which the data changes as the distance from the light source increases changes." This description also includes a correct conclusion about the long run behavior of the average rates of change of the light intensity. Over half (56%) of the students constructed and correctly interpreted a similar graph of the average rates of change. In this particular case, the students continued to elaborate their description of the behavior of the average rates of change: "Although the majority of the average rates of change are increasing, the absolute value of average rates of change are decreasing which tells us that as the distance from the light source increases the intensity of the light decreases quickly at first and then decreases more slowly." In this statement, the students argued from the graph of the average rates of change values (Figure 5B), and articulated that these values are increasing. But the students then considered the absolute values of the negative average rates of change to provide the qualitative statement about the rate that is "decreasing quickly at first and then decreases more slowly." All reports but two (94%) included the correct rate graphs of the average rate of change over the 1 cm intervals similar to the one in Figure 5B and 44% of the student made it explicitly clear how to see the average rates of change in the scatter plot (see Figure 5A) by referring to the slope of the line segment connecting two consecutive points.

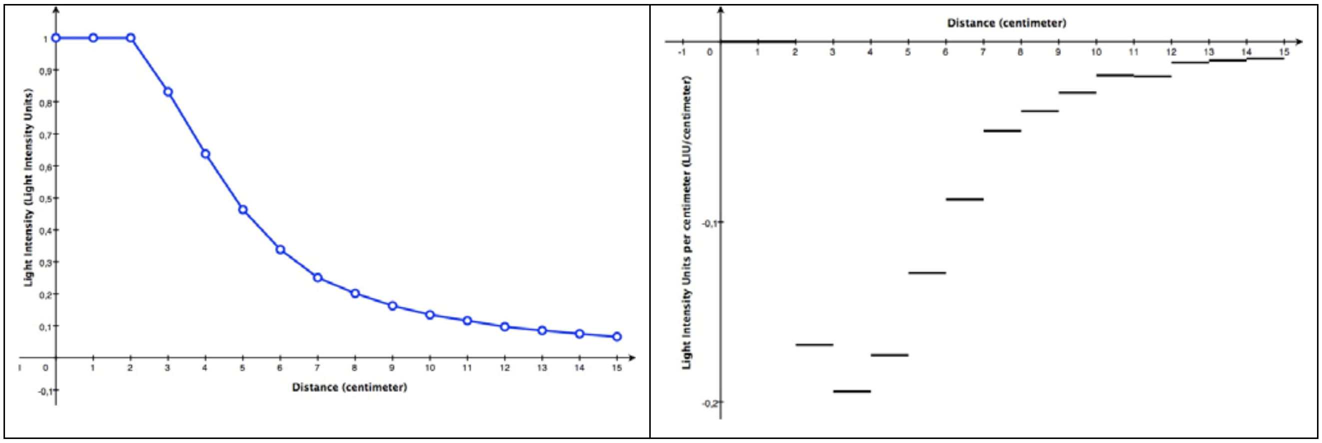


Figure 5. (A) shows the scatter plot and (B) the corresponding rate graph of typical data the students collected

Students' difficulties describing and representing change

For some students, their descriptions of the behavior of the light intensity contradicted their description of the average rate of change of the light intensity. For example: "The intensity of the light decreases at a decreasing rate with respect to the distance from the light source, as predicted earlier. The average rates of change increase at a decreasing rate as distance from the light source increases." This points to the complexity of the ideas involved in reasoning with negative rates and their different representations, especially when cast in an applied context.

Some of the mathematical demands of the model application activity were challenging for the students. For example, when using the difference quotient to create a rate graph for the average rates of change of the derived function at integer x values and interval length 0.5 cm, many (78%) of the students plotted the interval width as 1 cm. This indicates that the key idea that average rates of change are always calculated over an interval was not yet fully understood by the students. The students had broad guidelines as to what to include in their reports. Hence, there were aspects of their models that were discussed by some students, but simply not addressed by others. For example, only some students (56%) provided arguments for why the first few data points had a measurement of 1 LIU; some students (22%) discussed sources of error that might have distorted the data during the measurement.

DISCUSSION AND CONCLUSIONS

The analysis shows how the students' diverse emerging models of how intensity varies with the distance from the light source converged as the students worked through the phases of the Light Lab. In line with the results of Whitney (2010), the students extensively used tables, but they also created contextually based graphical representations, from which the majority could successfully describe the data they collected within the context of the flashlight and light sensor. More than half of the students could meaningfully describe how the average rates of change of the light intensity varied. Using a meaningful context and actively engaging the students in the model application activity seemed to support some of the students in developing their models of changing phenomena.

The results of this study also suggest a number of issues, such as the meaning and importance of the interval width in the difference quotient and the validation of developed models, need to be addressed to support the continued development of the students' emerging models of average rate of change. Some of these issues could be addressed through a re-design of the activity. However, it should be emphasized that this model application activity was not the end of the model development sequence. Rather, it was followed by another model application activity that engaged students in examining the rate at which a fully charged capacitor loses its charge. The next step in our analysis of the development of students' emerging models will be an examination of that data. This study points to the potential value of examining the development of students' emerging models over sequences of modeling activities.

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CREATING THE NECESSARY CONDITIONS FOR MATHEMATICAL MODELLING AT UNIVERSITY LEVEL

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This paper focuses on the study of the 'ecology' of mathematical modelling in the teaching of mathematics at university level. Using the framework of the Anthropological Theory of the Didactic, we introduce the notion of study and research courses as an 'ideal' didactic device for integrating mathematical modelling into current educational systems. We explore some of their essential characteristics or principles that can help with their design, testing and analysis in order to face and overcome some of the constraints that hinder the development of mathematical modelling activities.

1. RESEARCH BACKGROUND: THE ECOLOGICAL DIMENSION IN THE MATHEMATICAL MODELLING PROBLEM

The starting point of our research is the problem of teaching mathematics as a modelling tool in first-year university courses. More specifically, we focus on the intricate problem of studying the 'ecology' of mathematical modelling practices in these institutions, that is, the set of *conditions* that favour and the *constraints* that hinder (or prevent) its large-scale development as normalised activities in current educational institutions (Barquero, Bosch & Gascón 2008 and 2010, Artaud 2007, Blum et al. 2002 and Doerr & Lesh 2011). We postulate that the integration of modelling activity needs an in-depth analysis of this ecology.

We use the framework of the Anthropological Theory of the Didactic (ATD) and its conception of mathematical modelling. The ATD uses the notion of mathematical praxeology (MP) as a fundamental tool to describe and analyse mathematical activity. In accordance with García et al. (2006), we consider modelling as a *process of reconstruction and articulation of MP of increasing complexity*. This process can start with the study of a question arising in an extra-mathematical situation, that we call the 'initial system', and leads to the construction of a MP that can act as a 'model' of the considered initial system. This work usually creates new questions, which require the construction of new models, the previous model thus acting as a 'system' of this new modelling process (Serrano, Bosch & Gascón 2010). From this perspective, intra-mathematical modelling—that is, the process of modelling a mathematical system—appears as a particular case of mathematical modelling and allows considering it not only as a way to make the functionality of mathematics visible, but also as a key tool for the construction and connection of mathematical contents. Thus, *mathematical modelling cannot be considered only as an aspect or modality of mathematical activity but has to be placed at the core of it*. This integration constitutes an essential aspect of our research problem. We will refer to it as the *ecological problem* of mathematical modelling:

> What *limitations* and *constraints* in our current educational systems prevent mathematical modelling from being widely incorporated in daily classroom activities? What kind of *conditions* could help a large-scale integration of mathematical modelling at school? What kind of *didactic devices* would make large-scale integration of mathematical modelling possible?

In order to face this problem, from the ATD we propose to use the notion of *study and research courses* (SRC), introduced by Yves Chevallard (2006), as a didactic device to facilitate the inclusion of mathematical modelling in educational systems, and, more importantly, to explicitly situate mathematical modelling problems in the centre of teaching and learning processes (Barquero et al. 2008 and Winsløw et al. 2012). In this paper, we will introduce some of what we consider the main traits of SRC. They can be understood as working principles (or assumptions) that can be taken into account for a design, implementation and analysis of a SRC. We postulate that they are necessary to create appropriate *conditions* to face some of the most important *constraints* to the 'normal' life of mathematical modelling at university institutions (Barquero et al 2012).

2. TOWARDS A NEW DIDACTIC PARADIGM: RESEARCH AND STUDY COURSES

Chevallard introduced the notion of SRC (Chevallard 2006) as a general model for designing and analyzing study processes. Its main purpose arose from the need to introduce a new epistemology to replace the still dominant 'monumentalistic' epistemology (where mathematical contents appear as monuments to visit) for one which could (re)establish the 'raison d'être' and the functionality of mathematics at school. In Chevallard's words, a change of paradigm at school is completely necessary: from the paradigm of *visiting works and its shortcomings* towards the new didactic *paradigm of questioning the world* (Chevallard 2012).

2.1. General conditions and research methodology for the testing of the SRC

During the academic year 2006/07, our research group started implementing SRC with first-year university students of the business and administration degree (4-year programme) at the IQS School of Management of Universitat Ramon Llull in Barcelona (Spain). Since then, they have been implemented year after year with some variations and improvements. A special device, called the 'mathematical modelling workshop', was introduced in the general organisation of the mathematical course. It consisted in weekly sessions of 90 minutes in which one third of classroom time was devoted to students' work. More than half of their personal work was done outside the classroom. The instructor of the course was also responsible for the workshop sessions. Said sessions ran in parallel to the two-hour weekly lecture sessions, which included problem-solving activities. Its attendance was mandatory for the students and it constituted forty per cent of the final grade.

In the general organisation of a workshop the students (around 50 in the class) are organized in teams of 3 or 4. They work under the supervision of the instructor responsible for the course and, if possible, of a researcher who acts as an observer. In most of its implementations, the workshop focuses on a single initial problematic question Q, to which students have to provide a complete answer during the entire academic year. It can also consist in three linked questions, one for each term. Once the initial question is presented, two kinds of workshop sessions are combined every week: teamwork and presentations. In the first ones, each team has to look for 'temporary' answers to partial questions derived from Q and prepare a 'partial' report with these answers. Then, the reports are orally defended on the subsequent sessions by some selected working teams. A discussion follows to state what progress has been made and to agree on how to continue the study process. During the presentation sessions, one member of the class (named the 'secretary') prepares a report containing the main points in the discussion and the new questions put forward to be studied in the following sessions. At the end of each term, every student has to individually write a final report of the entire study (evolution of problematic questions, work with different models, relationship between them, etc.).

At this stage of the process, the collected data of the implemented SRC comprises the students' team and individual reports, the teacher's written description of the work carried out during each session, the worksheets given to the students a nd a brief questionnaire handed out to the students at the end of the each term. It constitutes the empirical base upon which the *analysis a posteriori* of the SRC rests.

2.2. Development of the SRC implementation: How does the population of users of a social network evolve over time?

We here focus on the most recent experimentation of the SRC, during the academic year 2010/11. On this occasion, the implementation of the SRC focused on the generative question (Q_0) about *the evolution of the number of users of a social network* called *Lunatic World* (see figure 1). The initial question led to consider different kinds of mathematical models depending on the assumptions made about the initial system.

This initial question was divided into three sub-questions, one of which was approached in each term. The division was made based on the necessary tools for their resolution. For instance, the initial and generative problematic question Q_0 was partially approached using discrete models and assuming independent generations of users during the first term of the course. It was then approached using functional models, so as to fit the best continuous function to real data during the second term. The third branch, developed during the third term, came from the use of discrete models, assuming that all users belonged to different groups. It led to reformulate Q_0 as Q_3 (see figure 2). We will refer to this case in the following section.

www.LunaticWorld.com

What is a social network? A social network is a social structure made up of a set of actors (such as individuals or organizations) and the dyadic ties between these actors. The study of these structures uses social network analysis to identify local and global patterns, locate influential entities, and examine network dynamics.

We will focus on studying the evolution of the number of user of a social network, which is called *LunaticWorld*. It was created in 2004 by 18 users. One of its main characteristics is that a person can become part of it only if another user invites him/her. In the following table you can find information about the evolution of the number of users of this network over some years. With this information, it is proposed to look for responses to the following initial question ($\boldsymbol{Q_0}$):

Year	Number of users
2004	18
2005	56
2006	151
2007	447
2008	1034
2009	3143

Given the size of population over some time period,

- Can we predict its size after n periods? Is it always possible to predict the long-term behaviour of the population size?
- What sort of assumptions on the population and its surroundings should be made?
- How can one create forecasts and test them?

Figure 1. Introductory worksheet to the workshop:

Presentation of the social network and of the initial question Q_0

MATHEMATICAL MODELLING WORKSHOP – 3rd term

The social network *Lunatic World* has recently introduced important changes. From now on, all users will be distributed in three different groups: ***Basic***, ***Medium*** and ***Premium***. After one month, a user can remain in the same group, be promoted to another group or leave the network. Moreover, the only type of users that are allowed to invite other users to join the following month are 'Medium' and 'Premium' groups.

CASE 1 [One of the 16 cases that were distributed to the different teams]:

- *15% of 'Basic' users remains as 'Basic' one month later and 75% of 'Basic' changes to 'Medium';*
- *25% of 'Medium' changes to 'Premium' the following month and 50% remains as 'Medium'. Each 'Medium' user invites on average 4 new users that enter as 'Basic' users the following month;*
- *90% of 'Premium' remains in the same group. Each 'Premium' user invites on average 3 new 'Basic' users.*

Figure 2. Introductory worksheet to the third branch of the SRC:

Discrete models with users distributed in different groups

The different phases of the design, application and analysis of this SRC were similar to those developed in a SRC on population dynamics (Barquero at al. 2008 and 2009)[1]. After the implementation of SRC with first-year university students for more than six academic years, we can talk more concretely about the main traits of SRC that seem important to create appropriate *conditions* for the 'real' development of mathematical modelling at university institutions. As mentioned earlier, they can be understood as working principles (or assumptions) for the design and carrying out of SRC.

[1] For more details see: http://webprofesores.iese.edu/valbeniz/bbarquero/BarqueroBoschGascon_app.pdf

3. CREATING CONDITIONS FOR MATHEMATICAL MODELLING

3.1. A generative question is the starting point of functional study processes

The starting point of a SRC should be a 'lively' question with a genuine interest in the community of study. We call it the *generative question* of the study process, and denote it by Q_0. It should not be a question imposed by the instructor to cover certain didactic needs fixed a priori. That is, obtaining answers to Q_0 has to become the main purpose and an end in itself. The study of Q_0, together with the derived questions that can appear along the study, is the *origin*, *engine* and *'raison d'être'* of all the study process. In this sense, Q_0 should be present during the entire study process and acts as its articulating axis.

The case of the SRC on '*How does the population of users of a social network evolve over time?*' provides a good example of the power of its generative question Q_0. With its implementation, we verified how the sequence of questions arising from Q_0 led the students and the teacher to consider most of the main contents of the entire mathematics course (see Barquero et al. 2008). In each term, different aspects that revolve around the initial situation were analyzed. They required the mobilization of various types of mathematical models: forecasting the number of users in the short and long term, considering time as a discrete variable (first-order sequences models, *1st term*), the same forecast considering time as a continuous variable (differential equations, *2nd term*), and the forecast in discrete time distinguishing three user groups with different privileges (models based on matrix algebra, *3rd term*). However, during the SRC, these contents appeared in a very different structure from the traditional organisation. Instead of the classical 'logic of mathematical concepts', the workshop was more guided by the 'logic of the problematic questions' and 'types of models' that appeared progressively.

Another important outcome consists in the necessity to break the rigidity of the classical structure 'lectures - problem sessions - exams', based on the sequence 'introducing new contents - applying the contents'. It can be considered as an important constraint to the integration of mathematical modelling. However, it was still important to ensure that both lectures and problem sessions were taken into account during the workshop. As a result, there had to be a bidirectional relationship between all these didactic devices. On the one hand, 'lectures and problem sessions' are used to provide students with some of the necessary tools to be able to follow the workshop. And, vice versa, the workshop motivates and shows the functionality of the main content of the course.

3.2. SRC have a tree structure, as a consequence of the search for answers to Q_0

During a SRC, the study of the generative question Q_0 evolves and opens up numerous other 'derived questions': Q_1, Q_2,…, Q_n. One must constantly question whether these derived questions are relevant. The fundamental criterion to decide whether they are indeed relevant is to ensure that they are capable of providing answers R_i that are helpful in elaborating a final answer $R^{\heartsuit}$ to Q_0.

As a result, the study of Q_0, and of its derived questions Q_i, leads to successive temporary answers R_i which would be tracing the possible 'routes' to be followed in the effective experimentation of the SRC. We claim that the work of production or construction of $R^\heartsuit$ can be described as a tree of questions Q_i and temporary answers ($R_i = \text{MP}_i$) related to each other during a modelling process that is both *progressive* and *recursive*. For instance, in the following diagram we can see, in terms of questions and their successive answers, the structure of the 3rd branch of the SRC concerning 'Discrete models with users distributed in different groups'. Its study led to the consideration of two MP: the first one is based on the construction of models based on Leslie matrices and its use in the short-, medium- and long-term forecast of users' distribution and the second one focuses on the study of powers of matrices, with Leslie matrices being a particular case.

$$\boldsymbol{Q}_0 \Rightarrow \boldsymbol{H}_1^{(3)} \text{ and } \boldsymbol{Q}_1^{(3)} \to \begin{cases} \left(\boldsymbol{Q}_{1.1}^{(3)}, \boldsymbol{R}_{1.1}^{(3)}\right) \\ \left(\boldsymbol{Q}_{1.2}^{(3)}, \boldsymbol{R}_{3.2}^{(3)}\right) \\ \left(\boldsymbol{Q}_{1.3}^{(3)}, \boldsymbol{R}_{1.3}^{(3)}\right) \end{cases} \Rightarrow \boldsymbol{Q}_2^{(3)} \to \begin{cases} \boldsymbol{Q}_{2.1}^{(3)} \to \boldsymbol{R}_{2.1}^{(3)} \\ \boldsymbol{Q}_{2.2}^{(3)} \begin{cases} \left(\boldsymbol{Q}_{2.2.1}^{(3)}, \boldsymbol{R}_{2.2.1}^{(3)}\right) \to \left(\boldsymbol{Q}_{2.3}^{(3)}, \boldsymbol{R}_{2.3}^{(3)}\right) \\ \left(\boldsymbol{Q}_{2.2.2}^{(3)}, \boldsymbol{R}_{2.2.2}^{(3)}\right) \to \left(\boldsymbol{Q}_{2.3}^{(3)}, \boldsymbol{R}_{2.3}^{(3)}\right) \end{cases} \end{cases}$$

MP Leslie matrices

MP Study of $\{M^n\}_{n\in\mathbb{N}}$

Some examples on the formulation of questions that appear in the structure of the third branch of the SRC:

$\boldsymbol{Q}_{1.1}^{(3)}$: How can we describe the evolution of the distribution of users in groups under the new conditions of *Lunatic World network* (see figure 2)?

$\boldsymbol{Q}_{1.2}^{(3)}$: Will it be always possible to forecast the future distribution of the users after some periods of time? Which will be the long-term distribution of users? **[…]**

$\boldsymbol{Q}_{2.2.2}^{(3)}$: In the case of Leslie matrices (L) of 2nd order, what are the main properties of L^n? What can we say about $\lim_{n\to\infty}\{L^n\}_{n\in\mathbb{N}}$? Can we generalise their properties to the case of a Leslie matrix of *n-order?*

Figure 3. Structure of the 3rd branch of the SRC in terms of questions and answers

Let us stress the importance and utility for students/teacher/researchers of what we have called the *mathematical a priori design* of the SRC to guarantee that the generative question is sufficiently 'fertile' to lead to many other derived questions. The design also gives a detailed description of the possible evolution of the study of Q_0 in terms of potentially derived questions and their successive answers (Q_i, R_i), which would be tracing the possible routes to be followed in the effective experimentation of the SRC.

3.3. Promoting the role of the study community: The dialectics between individuals and the community

A 'real' integration of mathematical modelling needs to *promote the role of the study community* along with that of the *director* of study. This study community has to be in charge of 'collectively' studying Q_0 and producing an appropriate answer $R^\heartsuit$. In contrast to the 'dominant pedagogy' where there is dominance of 'individual' work under the orders of teacher, the group of students and the teacher have to share the set of tasks and negotiate the responsibilities that each of them has to assume.

This displacement going from the individual to the community has numerous important consequences to enable the existence of mathematical modelling. On the one hand, the collective study of questions provides the opportunity of *defending answers* produced by the community, instead of accepting the *imposition of official answers*. On the other hand, this work required students to take on a lot of new responsibilities that the 'traditional didactic contract' exclusively assigns to the teacher, for instance: addressing new questions, creating hypotheses, searching for and discussing different ways of looking for an answer, comparing experimental data and reality, choosing the relevant mathematical tools, criticizing the scope of the models constructed, writing and defending reports with partial or final answers, etc. The teacher thus has to assume a new role of acting like the leader of the study process, instead of lecturing the students. And it soon appeared that the teaching culture at university level did not offer a variety of teaching strategies for this purpose.

3.4. The dialectics of questions and answers as the engine of the SRC

An important dialectic that is integrated in the SRC is the task of posing questions and that of the search for answers. In the 'traditional' didactic contract, the responsibility of posing questions generally falls on the teacher, while students only come up with doubts or questions that the teacher can answer quickly.

As we saw in the experimentation of the SRC, the mathematical modelling process required the entire community to focus on the study of a single question for a long period of time (the whole year!). This question had to remain 'alive' and 'open' session after session. Furthermore, the relevance of the derived questions and the opportunity of its consideration must appear as yet another gesture of the study process. It had to be negotiated between the teacher and the students.

This situation is rarely seen under the 'dominant' pedagogy. For instance, the teacher is only attributed with the ability to 'teach' certain contents, the value of which nobody questions. In order to overcome the constraints that appeared during the experimentation of the SRC (students' passiveness, their request for a close supervision by the teacher, etc.), the teacher introduced some relatively new didactic devices. For example, the teacher asked the students to pose at least one new question that arose from the work carried out when they handed in their weekly team report. Moreover, at the beginning of the following session those new questions were brought together and students–under the teacher's watchful eye–agreed on the way to continue. It was an excellent way to compare and discuss the work done during the entire process, and more particularly, a way for the study community to formalise all the questions approached as well as their successive temporary answers.

3.5. The dialectics of the diffusion and reception of answers

Against the temptation of imposing some answers that are acceptable within the educational institution, the group of students needs to be invited to *defend* the successive answers R_i they provide, although they may still be of a temporary nature.

In the case of our experimentation, as we mentioned before, we introduced a device named ‘Report of results’, which was relatively foreign to the mathematical teaching culture. Each week, in groups, the students had to elaborate a written text in which they gathered both the documents provided by the teacher, and the partial results of the work done in the workshop session. They complemented it with their personal comments and the information on the subject they were able to gather. They had to hand in the report to the teacher. These dossiers thus contained the answers that each group would defend in class at the beginning of each session. At the end of the workshop, each student had to hand in their own ‘final report’ that no longer contained the chronicle of the study process but focussed on presenting and defending a final answer to the question initially posed. Undoubtedly, the students did not easily accept elaborating, reading and defending the reports due to the difference compared to other study devices used in other subjects. Despite all the resistance put up by students to the changes introduced during the implementation of SRC–working in groups, scheduling the study on their own, formulating questions, selecting mathematical contents, using a computer and bibliographical resources, writing and defending temporary answers, etc.–all these responsibilities (traditionally assumed by teachers) were progressively accepted by them. This increasing autonomy assumed by the students during the SRC seems a necessary condition to carry out the activity of mathematical modelling.

3.6. The dialectics of ‘media’ and ‘milieu’

The implementation of a SRC can only be carried out if the students have some pre-established answers accessible through the different means of communication and diffusion (that is, the *media*), to elaborate the successive provisional answers R_i. These *media* are any source of information such as, for instance, textbooks, treatises, research articles, class notes, etc. However, the answers provided are constructions that have usually been elaborated to provide answers to questions that are different to the ones that may be put forward throughout the mathematical modelling process. They have to be ‘deconstructed’ and ‘reconstructed’ according to the new needs. Other types of *milieus* will therefore be necessary to put to the test and ‘check’ the validity of these answers.

In our experimentations with SRC, this dialectics was crucial for mathematical modelling. When constructing a model from a certain system, it was essential for the student to have access to answers that were not reduced to the ‘official’ answer of the teacher (or textbook), as well as to the means to validate them. Students were systematically asked to look for information in the *media* about the types of models they provided. In particular, they had to check whether said models already existed and whether they were important enough so as to be assigned a specific name, and so on. The validity of the models constructed or provided was carried out from data –

which in our case the teacher had provided – and through numerical simulation with Excel or the symbolic calculator Wiris (www.wiris.com). We find good examples of this in the 3rd branch of SRC based on Leslie' matrix models (see figure 2 and 3) where, for example, to be able to forecast the short- and long-term distribution of users into groups or to simulate the *n*-power of a Leslie matrix, the numerical simulation provided by Excel or Wiris could work as *media* to formulate some conjectures about their pattern but also as *milieu* to check or refuse the conjecture they could have formulated.

4. DISCUSSION AND FURTHER RESEARCH

Using the 'ecological' metaphor, we can say that for mathematical modelling to be able to 'live' normally in a teaching institution (more specifically at university level), an in-depth study of the *conditions* that facilitate and the *constraints* that hinder the type of mathematical activities has to be carried out. In our research, we have used the notion SRC as a reference model of didactic organisations, which are proposed to allow mathematical modelling to 'live' in educational systems. In this paper, we have introduced some of what we consider as their main characteristics that can be used as principles for their design, testing and analyses. Far from being closely-characterised, what it is important to highlight is that most of their traits appear to face some constraints that generally come from the dominant 'epistemological' and 'pedagogical' models and that make the life of mathematical modelling difficult (Barquero et al. 2010 and 2012). We have mentioned several of them throughout the paper: coming from 'monumentalistic' school epistemology, from the classical organization of mathematics following the 'logic of mathematic contents', from the 'traditional' didactic and pedagogic contract, from the rigidity of the classical structure 'lectures – problem sessions – exams', from students' passiveness, etc.

Given the fact that the origin of most of these constraints is located at the generic levels of 'school' and of 'society', it seems obvious that they are not to be directly modified only through changes introduced by the teacher in the classroom. We therefore propose a different way, which, in a sense, is the opposite approach. We suggest beginning by proposing and changing the *gestures of the study*, which requires the introduction of *new didactic devices* which make the carrying out of gestures possible. After more than six years of implementation of SRC at university level, we can say that SRC have become more and more consolidated as a normal didactic device. Despite their initial difficulties, our present research moves forward towards a progressive and generalized introduction of certain '*study gestures'* and the appropriate *'didactic devices'* that could make it possible to effectively transform the type of scientific activity carried out in university classrooms.

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DIVERSITY IN MIDDLE SCHOOL MATHEMATICS TEACHERS' IDEAS ABOUT MATHEMATICAL MODELS: THE ROLE OF EDUCATIONAL BACKGROUND[1]

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The purpose of this study was to explore the relationship between mathematics teachers' educational backgrounds and their ideas about 1) what constitutes a mathematical model of a real-world phenomenon, and 2) how models and empirical data relate. Participants were 56 United States (US) in-service mathematics teachers (grades 5-9). We analysed teachers' written responses to three open-ended questions through content analysis. Results show our participants do not hold a unitary understanding of mathematical models. Teachers with backgrounds in Mathematics Education and Science Disciplines especially stressed the usefulness of models to show general relationships, whereas those with backgrounds in Other Disciplines stressed the importance of producing exact results.

INTRODUCTION

In the United States, and elsewhere, mathematics teachers across grades K-12 are increasingly required to include modelling in their teaching. In fact, mathematical modelling is one of eight practices standards proposed by the Common Core State Standards for Mathematics (2012). Considerable research has been devoted to exploring how mathematics teachers from different grade levels solve modelling problems (Blum & Borromeo Ferri, 2009; Verschaffel, De Corte, & Borghart, 1997), and to describing their beliefs and conceptions of the role of modelling activities in the classroom (Kaiser & Maaß, 2007). However, little is known about how teachers understand what a "mathematical model" is, and about the relationships between models and real-world phenomena.

Defining modelling in the context of mathematics education is a complex task (Blum, 2002). Most researchers and policymakers agree that mathematical modelling involves using the tools of mathematics to distill key elements of a real-world situation and articulate the relationships between those elements. This distillation enables the learner (or the model creator, more generally) to further explore the situation using the tools of mathematics, with the ultimate purpose of mobilizing those findings toward accomplishing further goals in the original situational context (Lesh & Doerr, 2003). By definition, then, model creators themselves must decide what particular mathematical representations, tools, and methods are appropriate to

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use when modelling a given situation. What constitutes a mathematical model, therefore, varies across situations and contexts, as well as users and audiences.

This study explores how middle school mathematics teachers think of mathematical models, considered as the external *objects* produced during the modelling process. Several definitions of the term “mathematical model” have been proposed by educational researchers. For example, Niss (1989) defines mathematical model as a combination of one or more mathematical “entities,” whose relationships are chosen to represent aspects of a real-world situation. In a similar vein, Lesh and Doerr (2003) argue that models are conceptual systems expressed using external representations, serving as tools to construct, define, and explain other systems.

These definitions elaborate on models’ representational nature, and emphasize how models embody the decisions modellers make during modelling process (Janvier, 1987). Mathematical models consist of one or more representations purposely chosen and displayed in a way that allows modellers to highlight what they identify as the most important variables and relationships of the phenomenon under study. Deciding what is and is not important constitutes, in fact, one of the main tasks to be completed in modelling activities. Thus, mathematical models themselves may substantially vary depending on the final goals, preferences, and/or biases of their creators. It is perhaps for that reason that the above definitions do not specify which particular representations can be considered as the “components” of mathematical models. In a nutshell, mathematical model is a “slippery” concept, open to numerous interpretations.

In this study, we use the variable “Educational Background” to explore the diversity of teachers’ ideas about models as external objects. Existing literature has not yet focused on investigating the relationship between mathematics teachers’ educational background and their ideas about mathematical models. In the field of science education, the interview study conducted by Justi and Gilbert (2003) explored the “notions of model” held by 39 science teachers with different disciplinary backgrounds – holding degrees in Chemistry, Physics, Biology, or Primary Teaching Certificate. The ideas of these science teachers tended to differ according to their educational backgrounds. For example, most teachers holding a Primary Teaching Certificate strongly subscribed to everyday views of the notion of model, according to which a model is a reproduction of something or a standard to be followed. Teachers with a degree in Biology expressed similar ideas, although they referred to broader variety of *uses* of models. Finally, teachers with a background in Physics or Chemistry discussed the notion of model in more comprehensive ways, consistent to perspectives currently held by scientists and philosophers of science. Moreover, they emphasized the usefulness of models for making predictions. These results suggest that the use of mathematical models might in fact differ across different disciplines.

PURPOSE AND JUSTIFICATION

The purpose of this study is to provide evidence that middle school mathematics teachers with different educational backgrounds tend to have diverse sets of ideas regarding 1) what constitutes a mathematical model of a real-world phenomenon, and 2) the relationships between models and empirical data. Drawing on the assumption that teachers' ideas about mathematical content are crucial mediators for the way they teach such content to their students (Sánchez & Linares, 2003), the evidence presented in this paper can be taken as an indicator that teachers with different disciplinary backgrounds might be addressing the teaching of models and modelling activities in very different ways. A better awareness of this diversity of ideas can inform teacher educators as they design programs to prepare mathematics teachers (both pre- and in-service) for an increasingly modelling-focused curriculum. Our study is also relevant for cognitive researchers interested in teacher thinking, as it shed light on the role that educational background plays in teachers' ideas about subject matter.

METHOD

Participants

Data for this study were gathered during the third week of a professional development program carried out in the Northeast of the US. Participants were 56 grade 5 to 9 mathematics teachers from nine school districts. There were 49 female teachers and 7 male teachers, ranging from 26 to 63 years of age. When data were collected, their professional experience as mathematics teachers ranged from 2 months to 28 years. The teachers had a variety of educational backgrounds, which we grouped into four categories:

- Mathematics: when the teachers earned their bachelor's or master's degree in mathematics (13 teachers);
- Mathematics Education: when teachers' bachelor's or master's degree was in mathematics education and they did not hold a bachelor's or master's degree in mathematics (8 teachers);
- Science Disciplines: when teachers' bachelor's or master's degree was in any science discipline (such as physics, engineering, or chemistry) and they did not hold a degree in mathematics and/or mathematics education (8 teachers);
- Other Disciplines: when teachers' bachelor's or master's degree was in other disciplines (such as Special Education, History, English, or Literature) and they did not hold a degree in any of the above-mentioned disciplines (27 teachers).

Modelling Problem and Target Questions

The modelling problem used (see Figure 1) presents Dolbear's Law, which expresses the relationship between the rate of chirping of the snowy tree cricket (N) and air temperature (T). A linear relationship between these two variables is proposed by the model creator. To explain Dolbear's Law, different representations of this relationship were shown to the teachers. Notice that the problem explicitly characterized the equation (N = T - 39) as the *model.* The list of ordered pairs, tables, and graphs were characterized as *representations*. The nine data points presented in these representations were referred to as *data*.

The teachers were asked a set of 12 open-ended questions. Several questions focused on the advantages and disadvantages of some given representations over others. Other questions focused on the similarities and differences among these representations. For this study, we analysed only those questions involving the ideas of "model" and "data," which were the following:

- Question 1: *How would you characterize the relationship between the model and the data?*
- Question 2: *Could you extract the data from the model?*
- Question 3: *Do you think the model conveys more or less information than the data? Why?*

Figure 1: Problem used in the study

Famous Amos and his Cricket Thermometer: Using a Function to Model Data

Amos Dolbear (1837-1910) [...] Today Dolbear is remembered, if he's remembered at all, mainly for Dolbear's Law, which expresses the relationship between the rate of chirping of the snowy tree cricket (*oecanthus fultoni*) and the air temperature. When they congregate in large numbers, these insects chirp in unison at a rate that depends on the temperature. Here are some empirical data on the subject [J.S. Walker, *Physics*, 4th ed. (Addison-Wesley, 2010), p. 451] given in the form of ordered pairs (*T*,*N*), where *T* is the temperature in degrees Fahrenheit and *N* is the number of chirps in 13 seconds:

(69,28), (74,34), (60,19), (77,39), (80,45), (66,23), (71,30), (57,18), (63,22)

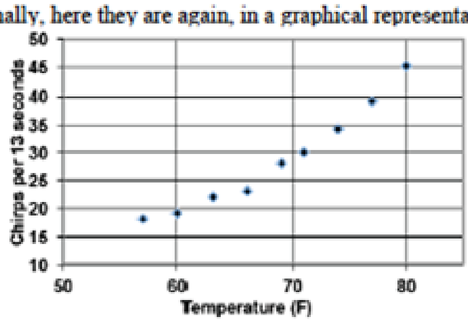
A) Here are the same data presented in tabular form:

Temperature in degrees Fahrenheit *T*	Chirps per 13 seconds *N*
69	28
74	34
60	19
77	39
80	45
66	23
71	30
57	18
63	22

B) and the same data again in tabular form, but this time in order of increasing temperature:

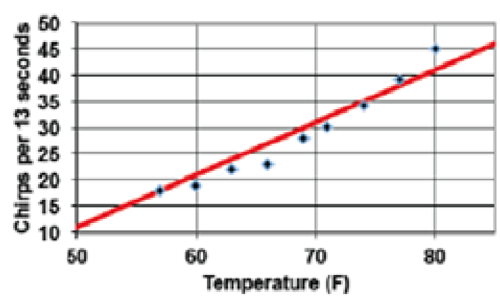
Temperature in degrees Fahrenheit *T*	Chirps per 13 seconds *N*
57	18
60	19
63	22
66	23
69	28
71	30
74	34
77	39
80	45

C) Finally, here they are again, in a graphical representation:



Based on these data, Dolbear proposed a *model* for the relationship between chirp rate and temperature, expressed in the following formula: $N = T - 39$ where, again, N is the number of chirps per 13 seconds, and T is the temperature in degrees Fahrenheit.

D) Here is a graphical representation of the model, plotted together with the actual data.



RESULTS

Teachers' responses to the three target questions were analysed using sets of non-mutually exclusive categories. In this section, we first focus on the content specific to each question. Then, we analyse some additional ideas that systematically appeared throughout the three questions. Tables containing the frequencies and percentages obtained by each group are presented.

Question 1: How would you characterize the relationship between the model and the data?

Teachers with different educational backgrounds tended to refer to different representations when characterizing the notion of "mathematical model" (Table 1). Recall that all sets of categories used in this study are non-mutually exclusive in nature. This is why counts in many columns of the tables presented go over 100%.

	EDUCATIONAL BACKGROUND			
	Mathemat N=13	Math Ed N= 8	Science N=8	Other Disc N=27
The model is referred to…				
As an Equation	**6 (46.1%)**	**5 (62.5%)**	1 (12.5%)	3 (11.1%)
As the Data Points	1 (7.7%)	1 (12.5%)	**3 (37.5%)**	**9 (33.3%)**
As the Line of Best Fit	**8 (61.5%)**	**4 (50%)**	**7 (87.5%)**	**17 (62.9%)**
Doesn't specify / Unclear	2 (15.3%)	0	0	0

Table 1. Frequencies and percentages obtained by Question 1 analytic categories*

* Values equal or higher than a third of each subsample (33.3%) are bold. Categories are non-mutually exclusive

Teachers from all educational backgrounds frequently referred to the line of best fit as the model, especially the group Science Disciplines. The groups Mathematics and Mathematics Education tended to express that the model was the equation provided ($N = T - 39$). They referred to it using the terms formula, algebraic expression, and/or function. In addition, a number of teachers from the groups Science Disciplines and Other Disciplines referred to the original data points as being part of the model. Interestingly, other representations provided such as the ordered pairs, the tables, and the written description of the scenario were never referred to as the model.

Question 2: Could you extract the data from the model?

We found opposite ideas regarding the viability of extracting the original data points from the model (Table 2). Whereas the groups Mathematics and Mathematics Education provided us mostly with negative answers (i.e., the data points *cannot* be extracted from the model), the groups Science Disciplines and Other Disciplines showed more diverse views, providing both affirmative and negative responses. These findings make perfect sense if we consider that the data points were regarded as part of the model by groups Science Disciplines and Other Disciplines, but not by the groups Mathematics and Mathematics Education.

	EDUCATIONAL BACKGROUND			
	Mathemat N=13	Math Educat N= 8	Science Disc N=8	Other Disc N=27
Affirmative answer	2 (15.3%)	1 (12.5%)	**4 (50%)**	**10 (37%)**
Negative answer	**11 (84.6%)**	**5 (62.5%)**	2 (25%)	**14 (51.8%)**
Elements of both	0	1 (12.5%)	0	0
Unclassifiable / Unclear	0	1 (12.5%)	2 (25%)	3 (11.1%)

Table 2. Frequencies and percentages obtained by Question 2 analytic categories*

Values equal or higher than a third of each subsample (33.3%) are bold. Categories are non-mutually exclusive.

Question 3: Do you think the model conveys more or less information than the data? Why?

This question also elicited different responses among our participating teachers. Most teachers from all backgrounds think that the model conveys more information than the data, although the justifications given present interesting differences (justifications are not presented here due to space limitations). In addition, the opposite idea (the model conveys less information) was also identified among some teachers (Table 3).

	EDUCATIONAL BACKGROUND			
	Mathemat N=13	Math Educat N= 8	Science Disc N=8	Other Disc N=27
More information	**10 (76.9%)**	**5 (62.5%)**	**5 (62.5%)**	**13 (48.1%)**
Less information	0	2 (25%)	0	5 (18.5%)
It depends, Different, More and Less	3 (23.1%)	1 (12.5%)	2 (25%)	5 (18.5%)
Unclassifiable / Unclear	0	0	1 (12.5%)	4 (14.8%)

Table 3. Frequencies and percentages obtained by Question 2 analytic categories*

* Values equal or higher than a third of each subsample (33.3%) are bold. Categories are non-mutually exclusive.

An analysis across Questions 1, 2, and 3: Exactness *vs.* Relationship

Content analysis led us identify the existence of two ideas that recurrently appeared in most teachers' responses throughout the three questions analysed here: the ideas of "Exactness" (or lack thereof) and "Relationship." This finding is interesting to us because the questions posed did not explicitly ask about these issues.

As can be seen in Table 4, most teachers from the groups Mathematics Education and Science Disciplines consistently referred to the usefulness of models to show relationships or general patterns in the data. This result was consistently found across the three questions analysed. Very few references to the idea of exactness were identified among these two groups. Teachers from the group Other Disciplines, in contrast, tended to stress the idea of exactness very often. For them, models should be able to provide exact/precise approximations to empirical data. Teachers from the group Mathematics did not show a clearly defined tendency. Interestingly, Question 3 triggered responses focused on both ideas (exactness and relationship), although specially that of relationship.

	EXACTNESS			RELATIONSHIP		
	Question 1	Question 2	Question 3	Question 1	Question 2	Question 3
Mathematics (N = 13)	5 (38.4%)	5 (38.4%)	6 (46.1%)	5 (38.4%)	3 (23%)	**8 (61.5%)**
Math Educat (N = 8)	0	2 (25%)	3 (37.5%)	**6 (75%)**	**5 (62.5%)**	**5 (62.5%)**
Science Disci (N = 8)	1 (25%)	3 (37.5%)	2 (25%)	**7 (87.5%)**	**6 (75%)**	**8 (100%)**
Other Discipli (N = 27)	**17 (62.9%)**	**16 (59.2%)**	**20 (74%)**	8 (29.6%)	11 (40.7%)	12 (44.4%)

Table 4. Exactness vs. Relationship: Frequencies and relative percentages*

* Values equal or higher than higher than a half of each subsample (50%) are highlighted. Categories are non-mutually exclusive.

DISCUSSION AND CONCLUSIONS

Researchers have proposed that mathematical models consist of one or several mathematical entities purposely chosen and displayed to describe, predict, and/or explain phenomena (e.g., Lesh & Doerr, 2003; Niss, 1989). Consistent with that idea, the first conclusion of this study is that middle school mathematics teachers also understand and think of mathematical models as external tools of representational nature (Janvier, 1987). In the context of “The Cricket Problem,” most teachers cited several external representations as constituting the model, including the line of best fit, algebraic notation (equation, formula, and/or function), and to a lesser extent, the raw data points. Teachers associated these representations with the model even though the text of the problem explicitly defined the equation as the model and characterized the others as representations. Other external representations available – such as the list of ordered pairs, the written description of the scenario, the unordered table, and the ordered table – were never explicitly referred to as part of the model, even though some of them contained precisely the same information as the data points.

Our second conclusion entails two ideas: a) middle school mathematics teachers in the United States do not hold a unitary understanding either of what a mathematical model *is* or of what such a model is *for*; and b) teachers with different educational backgrounds understand mathematical models in systematically different ways (Justi & Gilbert, 2003). This study illustrates the extent to which the notion “mathematical model” can be interpreted differently by different audiences. Even though all teachers were provided with the same materials, they spontaneously focused their attention on different representations and analysed the situation using different criteria. While teachers from all backgrounds tended to include the line of best fit in the model, teachers with backgrounds in Mathematics and Mathematics Education were more likely than those in the other two groups to mention algebraic expressions, and to state that the data could not be extracted from the model (those with Mathematics degrees were most emphatic on this point). In contrast, those with backgrounds in Science or Other Disciplines were more likely to consider the data points themselves as part of the model, and thus to consider it possible, if perhaps difficult, to extract the data from the model.

A somewhat different pattern emerges, however, when the teachers' reasoning is considered more closely, with attention to the recurring themes of "exactness" and "relationship." Teachers with backgrounds in Mathematics Education and in Science Disciplines strongly stressed the important role of the model in illustrating or clarifying the general relationship between the two quantities (temperature and number of chirps). For them, models are powerful tools to show general patterns in the data and to be able to generalize beyond the particular set of data at hand to the natural phenomenon under study. Teachers from a Mathematics background tended to take a more formal approach, viewing the data and model as mathematical objects to be compared, with less concern for the real-world phenomenon being described. Teachers with backgrounds in Other Disciplines were most concerned with exactness – whether the model could precisely reproduce the original data.

An educational background in Mathematics seems to be associated with understanding models as an "idealization" of the data, as abstract representational tools whose main work is to *predict*. In addition, teachers with this background are concerned with formal aspects of models. For example, they frequently expressed the importance of clarifying the domain and range of the function, and referred to the tensions between mathematics and the natural world (e.g., crickets would die under extreme temperature conditions). Finally, these teachers focused on the question of how the model and data are related as "abstract entities" (Kaiser & Maaß, 2007), with less reference to what the user might want the models for.

Teachers with educational backgrounds in Mathematics Education and Science Disciplines demonstrated some similar ideas to each other. They focused on what the model is good for, as well as what the user can do with models. In addition, they described models as powerful tools not only to *predict* but also to describe the general relationship between variables and to help us see patterns and generalize (Justi & Gilbert, 2003). Exactness of models is not an important issue for these teachers, as it is for the Other Disciplines group. Teachers with a Mathematics Education background extensively referred to the advantages of having a model (e.g., to visualize patterns, estimate unknown data, see trends in the data). However, they did not describe in detail the specific conceptual information of the problem at hand, as teachers from Science Disciplines did. Indeed, having a background in Science Disciplines is associated with focusing on the specific characteristics of the model (specific kind of functional relationship, strength of the relationship, presence of outliers, etc.). Moreover, it was primarily the Science Disciplines group who tended to see the model presented as just one of many possible models, and who discussed other possibilities (i.e., an exponential model).

Formal education received by teachers from the Other Disciplines group was focused neither on mathematics nor science content knowledge. The preoccupation of these teachers with the exactness of the model suggests that they might conceive of mathematics as an abstract, authoritarian discipline (Kaiser & Maaß, 2007). Like many college students, teachers from the Other Disciplines group tend to view models as either exactly right or else completely arbitrary, in which case the choice of a model becomes entirely subjective. The characterization of the "degree of exactness" of models along a *spectrum* is a more sophisticated idea than just dividing them into "exact" and "non-exact" (or "right" and "wrong"), as teachers in the Other Disciplines group did. Similar to the Mathematics Education group, these teachers did not pay much attention to the conceptual/contextual information of the modelling scenario at hand. In contrast, they tended to refer to the model in the abstract (Verschaffel et al., 1997).

EDUCATIONAL IMPLICATIONS

Given the diversity of potential educational backgrounds among middle school mathematics teachers, it is crucial to develop a better understanding of the ways in which these diverse backgrounds might influence their ideas about mathematical models, and subsequently their teaching of modelling. The findings of this study can inform the design of units on mathematical modelling for both pre-service and in-service mathematics teachers. For example, it would be enriching for teachers with backgrounds in Mathematics and Mathematics Education to deal with situations of exploration and analysis of the different every-day constraints that might affect mathematical models. Similarly, teachers with backgrounds in Other Disciplines would benefit from experiences in which the exactness of models is not an essential issue. This would allow them to explore the advantages of visualizing general trends in the data. More generally, the evidence presented here shows that there is room for all teachers – regardless their educational background – to expand the range of representations they consider as, or include in, mathematical models, and the goals and purposes of generating, analysing, and evaluating such models. Our findings further suggest that one way to do this might be to encourage teachers with different backgrounds to collaboratively engage in modelling activities, in order to better understand the role of perspective, available tools and skills, and sense-making play in modelling activity.

LIMITATIONS AND DIRECTIONS FOR FURTHER RESEARCH

This study is clearly exploratory and suffers from a number of limitations. First, the evidence comes from a single source of data —i.e., written responses to open-ended questions. The present analysis constitutes the first step in our research agenda on middle school mathematics teachers' modelling ideas and approaches. Other data sources should be included to validate our claims. This is precisely our next goal. We are currently in the process of interviewing a subset of the teachers who participated in this study. Among the goals of our interviews is to clarify some of the findings presented here. Another limitation is that the sample of participating teachers was uneven regarding the four educational background groups, and some of the groups were too small for robust statistical analyses. This imbalance roughly reflects the current backgrounds among middle school mathematics teachers in the United States, where teachers with a Mathematics or Mathematics Education background are outnumbered by teachers with backgrounds in Other Disciplines. The fact that this study was conducted in the context of a professional development program did not allow us to select the sample having the educational background criterion in mind. Therefore, further studies should be conducted to determine whether the differences identified here are also observed in other samples of mathematics teachers. It would be also necessary to study mathematics teachers' responses in other types of modelling situations (e.g., probabilistic simulation situations, theory-driven models).

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EXCEL MODELLING IN UPPER SECONDARY MATHEMATICS – A FEW TIPS FOR LEARNING FUNCTIONS AND CALCULUS

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The paper brings a few tips to using spreadsheets for teaching and learning functions and calculus at gymnasium (grammar school) in a constructionist way when the learner develops the application by himself. A questionnaire survey was conducted with 34 gymnasium students in four 90 minute lessons to find out if they found the lessons interesting, understood the mathematics and learned new spreadsheet skills.

INTRODUCTION

Mathematical modelling and applications has been in the focus of research in mathematics education for several decades (see Blum & Niss, 1991; Beare, 1993; Blum, 2002; Borromeo Ferri & Blum, 2011; Lingefjärd, 2011). Connection of mathematics to other sciences, its relevance to the outside world, learning concepts in context and connecting them through applications, teaching conceptually through helping students construct their own meanings grounded in real-life experiences – these are some of the basic ideas that underpinned this new vision of mathematics education at the pre-college level (Abramovich, 2003).

A powerful tool that enables easy access to ideas and concepts through a computational experiment are spreadsheets. The tool represents a most popular general-purpose software used by educators to promote the spirit of exploration and discovery by integrating experiment in teaching.

Educators discovered the educational potential of spreadsheet in mathematics and sciences in the early 1980s. Spreadsheets allow using problem-solving and heuristic methods that are close to the talented pupils. Only applications developed in special programming environments offer such ability of analyzing scientific problems.

Baker and Sugden (2003) gave a detailed account on spreadsheets in education from 1979 till 2003. They prove on a wide range of research papers about spreadsheets in teaching mathematics, physics and computer science that

"… there is no longer a need to question the potential for spreadsheets to enhance the quality and experience of learning that is offered to students. Traditional barriers (…) need to be removed, either by ensuring that access to computers is improved or by changing assessment methods. Further expansion is needed of the types of topics that can be effectively covered by spreadsheet examples ..." (also see (Sugden, 2007)).

The paper was written on the occasion of launching the electronic journal Spreadsheets in Education. The goal of the journal is to create a forum for scholarly research into the use of spreadsheets at all levels of education, in which ideas on the use of spreadsheets can be exposed, explored, and reported to a practising audience to enable to adopt a technology for life to which all students should be exposed. Many articles has been published in the journal on spreadsheets in upper secondary mathematics (Abramovich, 2003, 2012), (Sugden, 2005), physics (Benacka, 2009), informatics (Lovászová & Hvorecký, 2003), (Benacka, 2008), etc. There are many articles in other journals and books on spreadsheet modelling in upper secondary mathematics and sciences (Neuwirth & Arganbright, 2004), (Benacka, 2011).

While developing the spreadsheet application, students project their mathematical knowledge into a tangible form and gain another view of mathematics. Spreadsheets offer to the students the power of dynamical discovering, which is an important element of inquiry based learning (IBL) in mathematics and science. Spreadsheets enable the students to develop applications that satisfy his/her specific requirements (the cells can be formatted conditionally; the parameters can be controlled by scrollbars and other components, etc.). These all are powerful motivating factors. When the application is ready, it can be saved and used again; if necessary, it can be modified or improved. A great advantage of Excel® is that it is a common equipment of computers at school and home; it is accessible almost everywhere.

In this paper, we bring a few tips to using spreadsheets for teaching and learning functions and calculus at gymnasium (grammar school; age 15–19) level. The aim is to learn the concept in a constructionist way when the learner develops the application from "nothing". The topics are graphing functions, finding extremes, solving systems of linear equations, and calculating area by the idea of the Riemann integral and Monte Carlo method. Some of the applications can be implemented in a short time provided the user is skilled in spreadsheets; however, as the authors' teaching practice has shown, most of graduates of secondary schools and even upper secondary teachers are not familiar with the skills used. A questionnaire survey was conducted with 34 gymnasium students. The research questions were if the students found the lessons interesting, understood the mathematics and learned new spreadsheet skills. The results are discussed.

GRAPHING FUNCTIONS

Functions are a significant part of mathematic curriculum at upper secondary school. Students study the properties of the graphs, the relations between the parameters and the graph, etc. The knowledge is essential for studying STEM (see Michelsen, 2006).

The application on the left side of Fig. 1 graphs functions if the definition domain is R. The graph is of xy line type made over 100 points in range B17:C117. Cell C13 contains the formula =(C12-C11)/100. Cell B17 contains =C11, Cell B18 contains the formula =B17+$C$13. Cell C17 contains =$C$5*(B17-$C$6)^2+$C$7. The formulas are copied down as far as row 117. The application on the right side of Fig. 1 graphs functions if the definition domain is not R. The graph is made over 10000 points in range B17:B10017, M17:M10017. Cell C13 contains the formula =(C12-C11)/10000. Cell C17 contains =($C$5*B17+$C$6)/($C$7*B17+$C$8). Cell M17 contains =IF(ISERROR(C17),NA(),C17). The points that are out of the definition domain are skipped due to function NA().

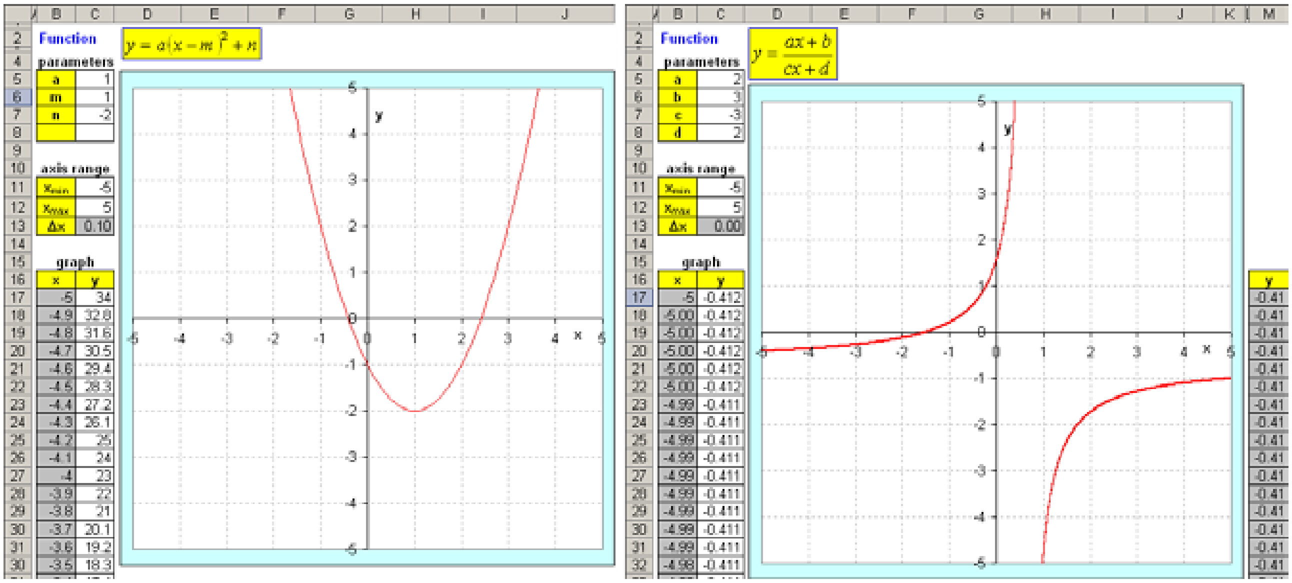

Figure 1. Graph of a function with D = R (left) and D ≠ R (right)

The trajectory of a projectile moving in a vacuum is depicted in Fig. 2 as a practical application of graphing functions. The topic is taught at Slovak gymnasium in the optional subject Physics seminar. The projectile moves in accordance with the physics formulas after clicking and holding down the spinbutton next to cell Q4. The application is an example of modelling in interdisciplinary teaching (see Maass & Mikelskis-Seifert, 2011).The details can be found in Benacka (2009).

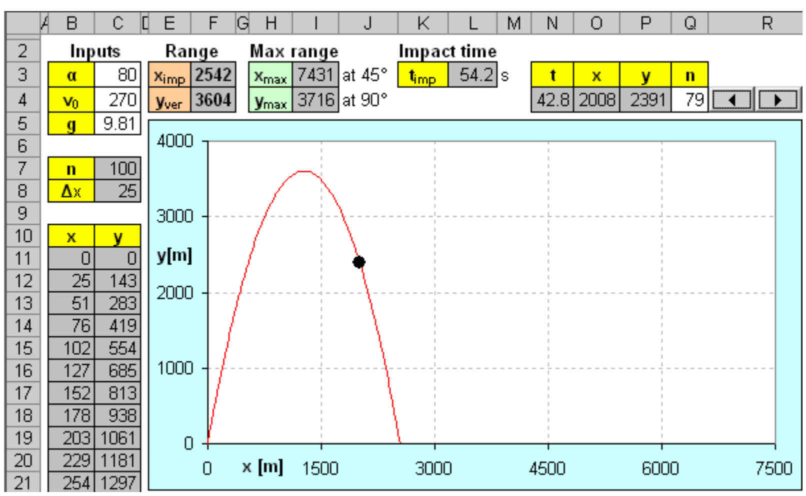

Figure 2. Projectile moving in a vacuum

FINDING EXTREMES

Tasks with extremes are traditionally solved within calculus. However, they can be solved in Excel® without using derivatives. The problem of finding the maximum volume of a box made of a square paperboard of 1m side length is solved in Fig. 3. After inputting the function $V(x) = x(1-2x)^2$ in range C17:C117 and adjusting the ranges of the axes, the maximum clearly appears on the graph. It can be found by enlarging the axes ranges (Fig. 3, right side) or by using Solver (range J4:K4).

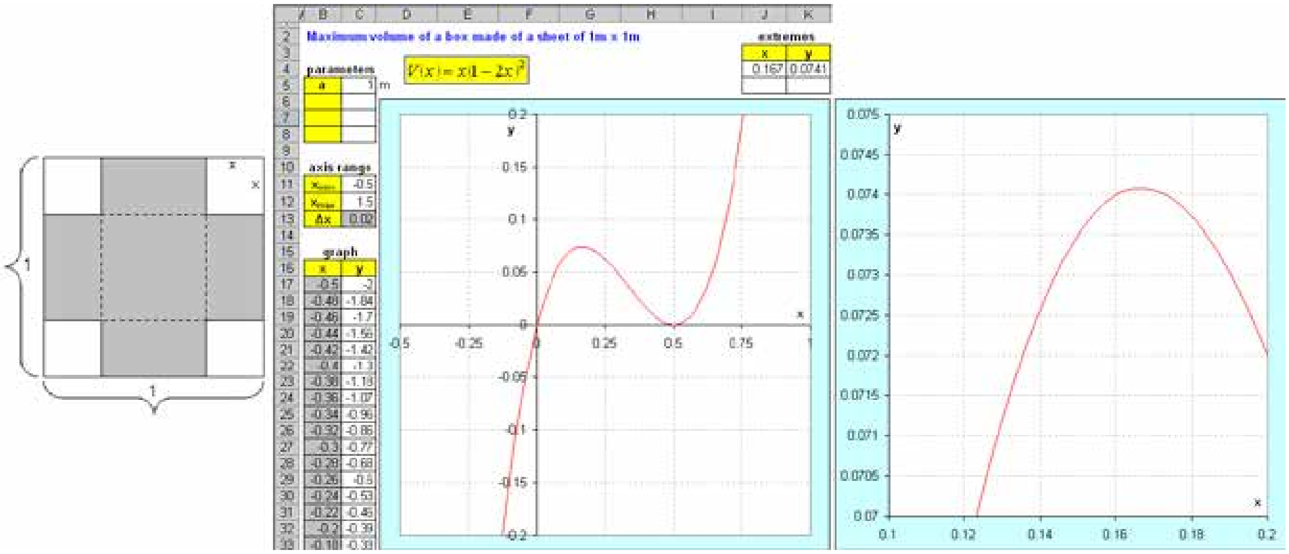

Figure 3. Maximum volume of a box made of a square paperboard

SOLVING SYSTEMS OF LINEAR EQUATIONS

Another possibility of using Solver is to solve systems of linear equations. The problem of finding the parabola that is going through 3 given non-collinear points is solved in Fig. 4. Point A, B, C are given in range L6:M8 and added in blue into the chart. The system is solved in range L12:P15. Cells L12, M12 and N12 contain the formulas =L6^2, =L6 and =1. They are copied down into rows 13 and 14. Cell O12 contains the formula =M6, which is copied down, alike. Cells L15, M15 and N15 contain the estimated values of *a*, *b*, and *c;* it is enough to input 1, 1, 1 at the beginning. Cell P12 contains the formula =L12*$L$15+M12*$M$15+N12*$N$15, which is copied down into rows 13 and 14. Cells O14 and P14 contain the formulas =SUM(O12:O14) and =SUM(P12:P14). They are necessary for Solver. The Solver constrains are $P$12 = $O$12, $P$13 = $O$13 and $P$14 = $O$14. After clicking button "Solve", the values of *a*, *b* and *c* appear in cells L15, M15 and N15. Then, the formulas =L15, =M15 and =N15 are inserted in cells C5, C6 and C7.

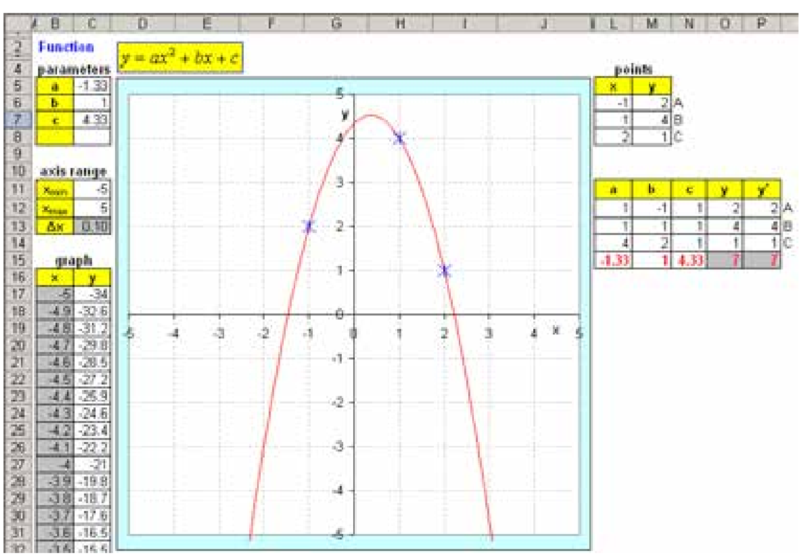

Figure 4. Finding the parabola that is going through 3 given non-collinear points

CALCULATING AREA – THE RIEMANN INTEGRAL

The application is in Fig. 5. The maximum number of subintervals is 1000. Cell N6 contains =COUNTA(L11:L1010). Cell N8 contains =(N5-N4)/N6. Cell M10 contains =N4. Cell M11 contains =M10+$N$8, which is copied down as far as cell M20. The equation of the function is copied from cell C15 into cell N10. Cell N10 is copied down as far as cell N20. Cell P11 contains =IF(N10<N11,N10,N11), cell Q11 contains =IF(N10>=N11,N10,N11); the formulas are copied down as far as row 20. Cell P10 and Q10 contain =SUM(P11:P1010)*N8 and =SUM(Q11:Q1010)*N8. Cell Q4 contains =(P10+Q10)/2, which gives the result – the average of the lower and upper integral sum. Cell Q5 contains =(Q10-P10)/2, which is the maximum possible error. To get a more accurate result, it is enough to select range L20:Q20 and pull down, e.g. as far as row 20 (Fig. 5, middle). To reduce the number of subintervals, e.g. to 5, it is enough to select the redundant range L16:Q30 and delete (Fig. 5, right).

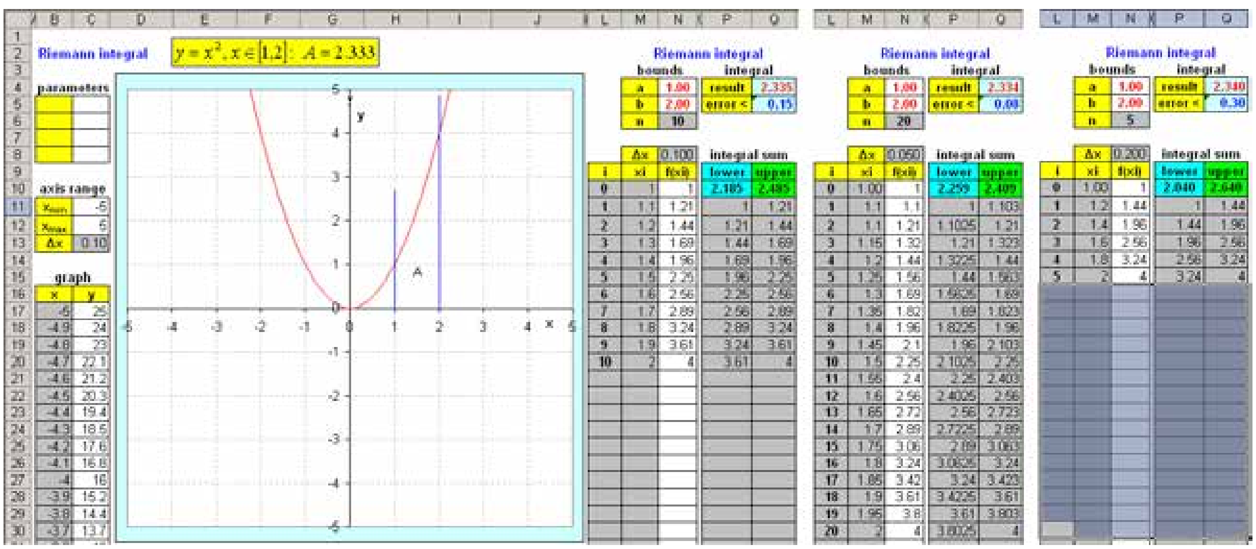

Figure 5. Calculating area by the Riemann integral

CALCULATING AREA - THE MONTE CARLO METHOD

The application is in Fig. 6. Bounds *a* and *b* are inputted in cells C3, C4. Maximum *M* of the function on the interval is inserted in cell C5. The area *S* of the rectangle between *a*, *b*, *M* and axis x is computed in cell C6. Cell F2 contains the number of the generated numbers. Cell C9 contains =$C$3+($C$4-$C$3)*RAND(). Cell D9 contains =1/C9. Cell E9 contains =$C$5*RAND(). Cell F9 contains the formula that gives 1 if y_i is on the graph or below it otherwise it gives 0. The formulas from range C9:F9 are copied down as far as row 1008. The (x_i, y_i) pairs from ranges C9:C1008, E9:E1008 are graphed as xy point graph (the red points). Cell F3 gives the number of the points on the graph or below, which are the "good" ones. Cell F4 gives the result. The calculation is based on the fact that the ratio of the number of the "good" and "all" points equals the ratio of the area below the graph and the rectangle that comprises it. Cell C2 contains =LN(2). Cells F5 and F6 contain =C2-F4 and

=F5/C2*100. If 10000 points are generated instead of 1000, then the error is smaller about ten times (Fig. 6, right side).

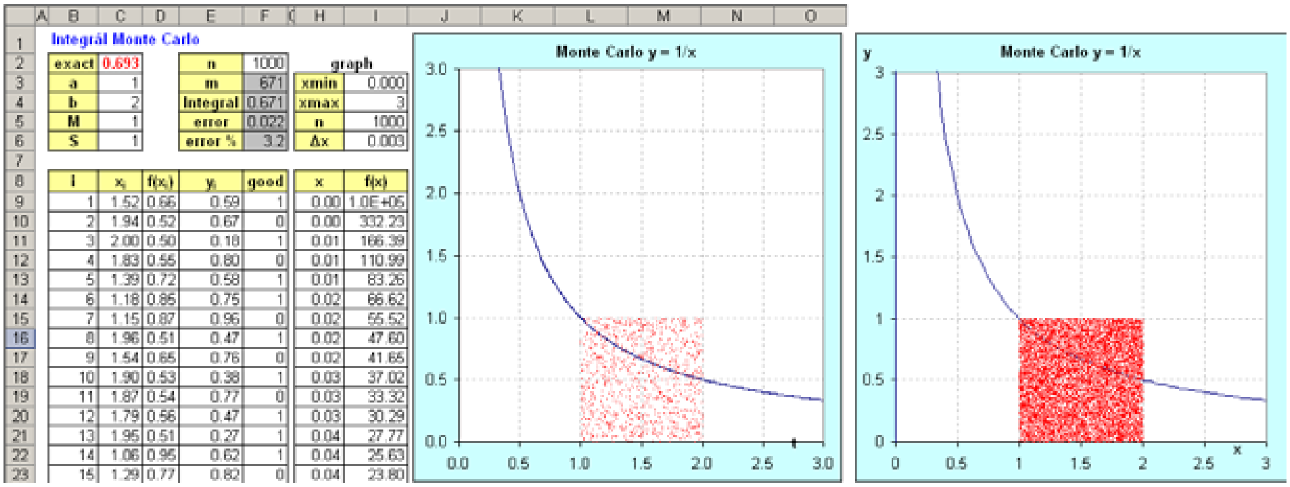

Figure 6. Calculating area by Monte Carlo method; 1000 points (a), 10000 points (b)

THE SURVEY

Applications identical or similar to the presented once were developed with four groups of gymnasium students, 34 altogether. They were familiar with writing formulas, relative and absolute addressing and making bar and pie graphs. They had never graphed a function, heard of Goal Seek and Solver, and used Active X or Form components. A questionnaire survey was conducted with them to find out if they found the lessons interesting, understood the mathematics and learned new skills.

Group NZ: Applications identical to those in Fig. 1 were developed as an introduction to graphing functions in Delphi with eight students of age 17 – 19 at Gymnazium in Nove Zamky in a 90 minute lesson within Computer Modelling Club that the first author runs at the school. It was emphasized at the end that "Now, you are able to graph any function given by a formula", however, no real background was given to the functions graphed, e.g., what problem they origin from, etc. Then, the students were asked to answer the following three most subjective questions: A) The lessons were (1:very; 2:quite; 3: little; 4:not) interesting; B) I understood (1:all; 2:majority; 3:minority; 4:nothing) of the mathematics; C) I learned (1:very much; 2:quite much; 3: little; 4:no) new in Excel. The average answers are in Tab. 1 in row NZ.

Group 2A: Applications similar to that in Fig. 2 were developed with ten students of class 2A (age 16 – 17) of Gymnazium Parovska in Nitra in two 45 minute lessons (20 min break). They simulated free fall and projectile motion in a vacuum. The theory was presented, which was new to the students. Despite it was emphasized that the nearest place in which the theory holds is the Moon due to the vacuum, the students were much engaged – putting the ball in motion was clearly a hit. The students were asked the same questions as Group NZ just the word "mathematics" was replaced with "physics" in question B. The answers are in Tab. 1 in row 2A.

Group 2B: Application similar to those in Figs. 3 and 4 were developed by nine students of class 2B of Gymnazium Parovska in Nitra in two 45 minute lessons (10 min break). The first one was introduced as follows: "You are the owner of a small factory that produces pins. You pack the pins in boxes that you make from square sheets of 100 cm sides that the paper factory delivers to you. The method is shown in the figure. The bigger is the volume of the box, the more money you save on the costs. Find the x at which the volume of the box is maximum". To make the solution closer to everyday life, side a was inputted in centimetres and the volume was calculated and graphed in litres. It was found that x can only go from 0 to $a/2$. The maximum $V_{max} = 74.1\,\ell$ at $x = 16.7\,\text{cm}$ was found by Solver. Then, the problem was solved of finding the x at which the volume is 50ℓ, first by Solver and then by Goal Seek. After showing finding the first root ($x = 6.65\,\text{cm}$), the students quickly found the second one ($x = 29.4\,\text{cm}$). The second application was introduced as follows: "Imagine a battle going on the Moon. Cannon shell was fired by the enemy. We can monitor its motion by radar. Three positions are given by points A, B and C in cells L6:M8. In a vacuum, the trajectory of a projectile is a parabola. Find the points from which the shell was fired and in which it will hit the ground. Find the maximum altitude reached". The application can be solved without creating the system in range L12:O14. Then, y' is computed in cells N6:N8 by formulas referring to cells C5:C7, and Solver constrains refer to cells M6:M8 and N6:N8. Showing the points with negative y coordinate has no sense. At the end of the second lesson, the students were asked the same questions as Group NZ. The answers are in Tab. 1 in row 2B.

Group 2C: Applications similar to those in Figs. 5 and 6 were developed with seven students of class 2C of Gymnazium Parovska in Nitra in two 45 minute lessons (10 min break). It was emphasized at the end that "Now, you are able to calculate any area bounded by the x axis and the graph of a given function", however, no real background was given to the tasks. The students were asked the same questions as Group NZ. The answers are in Tab. 1 in row 2C.

Group	Number	Topic	A	B	C
NZ	8	Graphing functions	1.6	1.3	1.9
2A	10	Free fall and projectile motion	1.1	1.5	1.5
2B	9	Finding extremes and roots, interpolating	1.1	1.8	1.6
2C	7	Calculating area	2.0	2.0	1.9
Altogether	**34**	**Weighted average**	**1.3**	**1.6**	**1.7**

Table 1. Average answers to the questions

The most important answer to the authors was that to the first question. It can be seen in Tab. 1 that even if there was no real background to the task, the students found the lessons quite interesting (groups NZ and 2C); if there was a more or less real background, they found the lessons very interesting (groups 2A and 2B). The relative frequencies of the answers for each group are graphed in % in Fig. 7. There were no answers of 3 or 4. The relative frequencies of the answers for all students, which are the main outcomes of the questionnaire, are graphed in Fig. 8.

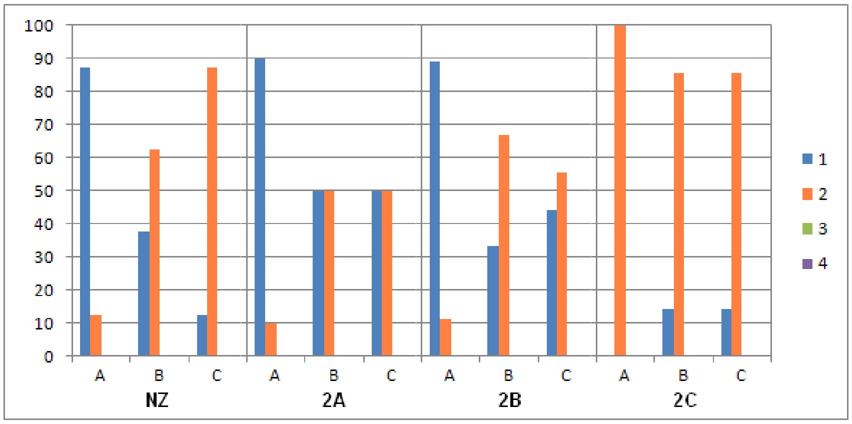

Figure 7. Relative frequency of the answers for each group in %

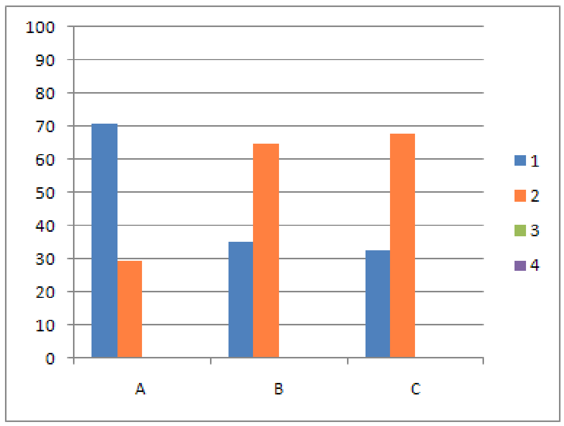

Figure 8. Relative frequency of the answers for all students in %

It holds that: (A) 100% of the students found the lesson interesting (71% very, 29% quite); (B) 35% understood everything and 65% understood the majority of the mathematics; (C) 100% of the students learned much new in Excel (32% very, 68% quite). There is the question if the samples were representative enough. While the students in group NZ are supposed to have positive attitude to maths and sciences, the students in groups 2A, 2B and 2C were chosen randomly. In Slovakia, classes in gymnasium comprise 30 students at most. In the school that the experiment was carried out in, the number of the computers in the computer rooms is 10 so the classes are divided to thirds following the alphabetical list of the students. Thus, each group was a third of the class. The lessons were regular once according to the timetable.

Remark: Actually, four questions were given to the students in Group 2A. The fourth one was if they would like to model fall and projectile motion in the air, which are more realistic but much more complicated than those in a vacuum. They all answered "yes". The question is whether they really wanted to make the model or just kill another lesson. However, when the first author was finishing the lessons with group 2B next day, two of the 2A group students came to him and said that they would come even after the school to develop the models in the air. That was a hit!

CONCLUSIONS

The paper brought a few tips to using spreadsheets for teaching and learning functions and calculus at gymnasium in a constructionist way when the learner develops the application by himself. A questionnaire survey was conducted with 34 gymnasium students to find out whether they found the lessons interesting, understood the mathematics and learned new skills in Excel. The outcome is that 100% of the students found the lesson interesting (71% very, 29% quite), 35% understood all and 65% understood the majority of the mathematics and 100% learned much new in Excel (32% very, 68% quite). Relaying on these facts it can be concluded that:

1) The combination of the topics and "learning by doing" with spreadsheets resulted in successful lessons.
2) Despite being rather unknown, mathematical modelling with spreadsheet is of great interest to students and has a big potential in promoting mathematics and sciences to them.
3) The capability for developing, without programming, applications that enable prompt verifying ideas and concepts and promote exploration and discovery through giving immediate feedback to changing data and formulas make Excel a convenient modelling tool at upper secondary level.

ACKNOWLEDGEMENT

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MATHEMATICAL MODELLING IN TEACHER EDUCATION COURSES: STYLE OF THOUGHT IN THE INTERNATIONAL COMMUNITY - ICTMA

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This paper presents the mapping of mathematical modelling (MM) pedagogical practices in teacher education courses from publications available in the books organized by the International Conference on the Teaching of Mathematical Modelling and Applications – ICTMA. Mapping refers to the identification, survey, organization, classification, and data analysis. Forty-two papers published in the ICTMA books were identified about teacher education and the data were analysed through the lenses of style of thought in MM, according to the theory of Ludwik Fleck (1979). This study aimed at understanding how knowledge was built by society and became common part of the group, and not at analysing the quality of knowledge departing from immutable criteria. The analysed productions revealed that these researchers constitute a collective of thought indicating the concern with and the support to the official documents and to the issues related to teacher education. The collective of thought about modelling in initial and continuous teacher education courses follows institutionalized standards.

Key-words: Mathematical Modelling, ICTMA, Style of Thought, Mapping.

INTRODUCTION

Mathematical modelling (MM) refers to the process toward the elaboration or the creation of a mathematical model about a problem-situation of some knowledge area, by making use of mathematical theories not only for the solution of a particular situation, but as support to other areas. This process, applied in all sciences, has contributed extraordinarily to the evolution of human knowledge. In modelling, the procedures are basically the same employed in scientific research: problem delimitation; familiarization; hypothesis; model formulation; problem resolution; interpretation; and model validation. According to Biembengut (2003), such procedures require from the modeller: mathematical knowledge and problem-situation knowledge, skills to read, describe and refine the data obtained from the phenomenon under the light of math, and creative and critical senses in the formation, the resolution and the evaluation of the model elaborated. Modelling is rooted in the human creative process and it encompasses scientific research.

In the last four decades in several countries, the defence for this process or method is increasing in the teaching and learning of math, in all levels of schooling. The main justification relies in offering students the opportunity to make use of math in order to comprehend a problem-situation of other knowledge area and know how to solve it. It means that teachers are able to provide students with opportunities to integrate math to other knowledge area, in particular, an area that students display interest in knowing more about. Osawa (2007) claims that the studied problem-situations give rise to knowledge learnt from experience, to comprehension reached through mathematical proofs, judgment, thought and foundation. According to Herget (2007), MM still facilitates students' comprehension of what is abstract (symbolic), by changing their perceptions in relation to the entities that surround them and in different ways, by stimulating their interests and, as a consequence, their learning.

MM for Education has emerged in the 1970s, impelled by the dissatisfaction and criticisms professors and businessmen interwoven in relation to the education of undergraduate students. The situation instigated many professors to find ways to justify math in the schools' curricular program. Among them, David Burghes in 1978, while working with high school teachers, in Cranfield University, produced papers and books about modelling, offered teachers workshops, and organized the first two International Conferences on the Teaching of Mathematical Modelling and Applications – ICTMA, in Exeter University, the United Kingdom, in 1983 and 1985. Since then, every two years, there is an expressive conference with a great number of participants and representatives worldwide (Biembengut, 2003).

This interest in the ICTMA international conferences gave rise to an organized community that, besides promoting the conferences, is part of the International Congress on Mathematical Education – ICME. In the period 1983-2011, there were 15 conferences, and also, the Study Group 14 Conference of the ICME on modelling and applications (M&A). The studies presented in these conferences are published in printed books. There are 15 books published until 2011; 443 papers on M&A with different focuses and from these, 42 were identified by being pertinent to pedagogical practices in MM in teacher education courses. In this corpus, there are applied and theoretical studies. From the applied ones, the focus is on the different schooling phases: from the early years of Elementary School to the end of graduation, and on the initial and continued education of teachers. According to Blum, Niss and Galbraith (2007), in the period 1965-1975, research suggests that M&A promoted arguments in favour of the inclusion of Mathematical Education; in the period 1975-1990, studies are characterized by the development of curricula and instructional materials to encompass the components of M&A; and from 1990 on, empirical studies on the teaching and learning of M&A have been added to the theoretical emphasis of research of the previous phases.

MM for Education has been stimulated and sustained by the gradual establishment of math teachers' communities, as well as the study groups and research groups that followed the proposals for the teaching of math. Worries about what, how, how much, and to whom teach math have contributed to the strengthening of the studies in MM for Education. These studies lead to research whose results imply proposals for teaching and learning that, in a cyclical process, promote new studies. The studies presented in conferences bring a style of thought that when shared, make ideas circulate and produce a collective of thought.

According to Ludwik Fleck'[1] theory (Compared Epistemology), style of thought "consists of a determined attitude and a type of accomplishment that completes it". And it is characterized by "common problem features that interest the collective of thought, by the judgments the collective thought considers evident and by the methods employed as means to know" (Fleck, 1979, p.145). Consistent with this theory, each study expresses the researcher's style of thought from the type of accomplishment and/or the collective of people that have a similar style of thought constituted by knowledge and/or shared practices.

Fleck proposed a theory about knowledge focused on heuristics. His theory allows for a historical and epistemological analysis to distinct areas of knowledge. It is centred in the analysis of academic productions, oriented by socio-historical studies to comprehend the interaction between the scientific practice and the contexts in which they occur. It is an "interactive model of the knowledge process, connected to the social, historical, anthropological and cultural presuppositions and conditionings, that as is processed, transforms reality" (Delizoicov et al., 2002, p.56).

Following Salles (2007, p.32), "throughout time, the action of researchers and the social factors that interfere with the constitution of science constructs a specific trajectory, with no determined beginning or end". It means that the most diverse historical, social and epistemological factors and entities interfere in the generation of scientific knowledge, that in turn requires reflection upon the facts and entities involved, careful and accurate historical analysis so that the common traces in the process of constitution of scientific concepts are identified.

The educational effort to provide better teaching of math culminated with the development of research on MM for education worldwide. As this defence for M&A in the ICTMAs has been on for about three decades, it is considered that styles of thought about MM for education are formed due to the idea circulation from a collective of thought and, this way, understandings change or vary permeating discussions about teacher education in several countries. It is feasible to inquire: *what are the styles of thought about mathematical modelling in courses of initial and continued teacher education in the international scene*?

In the ICTMAs there is circulation of ideas from styles of thought. Each person comes from a social and historical context (active connections) and each person perceives the reality in such a subjective way that nurtures his /her research (passive connections). Therefore, people's interactions allow for the establishment of styles,

[1] Ludwik Fleck (1896-1961), who was a Polish doctor, published, among other works, Genesis and development of a scientific fact, in 1935 by an editor in Switzerland, and in 1979, in English, with the foreword written by Thomas Kuhn.

and by recurrence, collectives of thought about M&A for Mathematical Education. This study aimed at analysing the productions published in the ICTMA's books about pedagogical practices on MM in teacher education courses. These productions, as shared by the scientific community, as ICTMA's, consolidate scientific knowledge.

METHODOLOGICAL PROCEDURES

This study is bibliographical since the data base consists of 42 papers about the pedagogical practices in MM in teacher education courses. From these, 35 were published in nine ICTMAs conferences (1995 to 2011) and 7 were published in the Study Group[2] 14 book (2007). Although the ICTMAs began in 1983, research focused in teacher education began to be presented from 1993 conferences on, whose publications were released from 1995 on. For this reason, the sources come from the period 1995-2011. This kind of scientific production brings a set of studies conducted by M&A researchers. These studies generated knowledge accepted by the scientific community. According to Biembengut (2008), studies based on bibliographical documents may offer a map about the theme of the problem or hypotheses to conduct the verification by other means.

Thus, the papers, written by whom participated in one or more ICTMAs, constitute a natural source that reveal different contexts (institutions and countries) and enable different interpretations and analyses. In this research phase, the focus resides in identifying the style of thought about modelling the authors of these papers display. We expect in the following phase, interview these authors to better understand their MM practices. This study was developed in two stages, named identification map and recognition map, as follows. These stages did not occur separately.

2.1 *Identification map*

This phase consisted of identifying the field in which the object is inserted in. The 42 papers analysed belong to 32 "first authors"[3] of 12 countries of five continents. Following, we identified the countries, the number of papers and the number of same authors(s) and different authors (d): Australia (4: 2s, 2d), Brazil (6: 2s, 3s, 1d), Canada (2d), China (1), Denmark (1), Germany (5: 2s, 3d), Mexico (2d), Portugal (1), South Africa (2s), Sweden (5s), United Kingdom (1), USA (12: 2s, 10d).

In the first moment, we opted to assemble the papers into three groups, published between the years 1995-2011: the first (1995, 1997 & 1999), the second (2001, 2003 & 2005) and the third (2007, 2009, 2010 & 2011). We hypothesized there would be some changes throughout the years. But, departing from a careful study of all papers, we could not identify significant differences. In each paper, the researchers identified

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[2]MM was the 14[th] theme for the study of the International Commission on Mathematical Instruction (ICMI), initiated in 2002 and published in 2007.

[3]We use the expression "first authors" due to the fact that we counted 'one' for the papers that have more than one author.

and classified the questions, sources and methods upon which the data were obtained. A summary of each paper was done and common topics were highlighted. As all the papers present: (1) justification, (2) process, (3) possibilities and (4) difficulties in making use of modelling for Education, we considered these topics as categories of analysis. These data allowed us to have a system for recognition. Due to space limits, the summaries of the papers and the resulting maps will not be presented in this article.

2.2 *Recognition map*

We sought to get acquainted with Fleck's Comparative Epistemology (1979) to have a better glimpse of the styles of thought about the pedagogical practices in MM in teacher education courses worldwide. According to him, a style of thought is understood as shared practice and knowledge. For him, when this style is shared, the group displays a collective of thought read the papers attentively and identified similarities among them and could delineate styles of thought in MM. We identified possible traces that were recognized and valued by the community. In addition to that, we sought to comprehend the studies departing from confluent and/or indicative places that suggest styles of thought in a community of modelling for Mathematical Education. In the following section of this work, we sought to make the data explicit such that an illustrative image of the studies and the results could be produced, based on Fleck's epistemological theory.

This study did not aim at analysing the quality of knowledge produced departing from immutable criteria. Instead, it aimed at comprehending how knowledge was constructed by the community and how it was integrated in the common pile of the group in the period 1995-2011. We admit the coexistence of different knowledge models and educational development so that we are able to recognize the similar points among these studies and the possible factors conditioned by the scientific communities. We assume that these processes do not express people's neutrality. We understand that scientific knowledge does not emerge in the methodological order of observation/experimentation in a distinct perspective from the empiricist conception. It is not neutral from the conception of who studies, of who prescribes results.

RESULTS & DISCUSSION

The analysed 42 papers on pedagogical practices in MM in teacher education courses bring the historical, social and educational context in which they were produced. All of them present applied research. They describe the process of modelling and use a group of students, research collaborators, future teachers or teachers in continued education as participants to collect data. These studies depart from specific experiences lived in the classroom and contribute to understand the different issues involved. The authors defend the primacy of MM in the development of Basic Education teachers. According to Sales (2007), these activities contribute in such a way that the researcher learns to identify stable elements in the research object and, establish facts tacitly accepted by the collective of thought that permeates the international community.

Although curricular programs are similar in the 12 countries from the 5 continents that the authors of the studies analysed belong to, it is not possible to deny that the studies occurred under the cultural reference and own meaning that these curricular contents have to the respective population; ways that guide us to derive the styles of thought. Meanwhile, when organizing the text statements into categories, it was possible to identify that some occurrences and reflections are common in the publications, even considering the elapsed time between one conference and the other and the participants of different countries. These similar reflections suggest the circulation of ideas, allowing for a collective of thought.

According to item 2.1, the established categories were: (1) justification, (2) process, (3) possibilities and (4) difficulties in making use of modelling for Education. In what follows, we turn to the reflections upon each category, considering the styles of thought present in the productions and that commune with the collective of thought.

(1) The authors' *justification* is based on the understanding that MM for Education allows the student in each schooling phase: (a) to learn the mathematical concepts better; (b) to interpret the meanings of the mathematical concepts; (c) to use technological resources to solve problems; and (d) to make students aware of social and environmental issues. This justification, in the 42 papers, is endorsed by the criticism to the style of teaching still in vigour in almost all the countries. The justification of the majority of authors - departing from the criticism to the current teaching and departing from the defence for modelling - suggests a style of thought in this community. This style shows that *MM for Education allows teachers and future teachers to become aware of the various issues of society and of the results the current schooling structure produces*. This consciousness can display another style of thought about the way to teach; guiding them to commit themselves to make use of pedagogical practices that promote better formation of students in Basic Education. This understanding requires knowing how and when to approach curricular contents, but also knowing how to make the students from Basic Education, in particular, draw on this knowledge in moments beyond the school limits. It is a style of thought legitimized by the community of thought that works with modelling for Education.

(2) The *process* of MM defended by the authors is that the teacher should: (a) depart from a subject or problem-situation of any area of knowledge that is interesting to the group of students; (b) ask the group to look for data that give rise to issues and then, seek for the problem solution; (c) orient the students to formulate these data making use of any mathematical structures (concepts, definitions, properties); and (d) guide them to solve the issue and evaluate the results. This style of thought about how to do modelling in sciences is maintained in the process of school teaching; it is how students are prepared for searching. It is characterized by the common traces of the problem-situations that the majority of papers bring as examples. The proposed

problem-situations conduct students to a collective of thought about MM as a process or method that enables knowing and developing competence to deal with issues of their surrounding environment.

The main argument is that modelling provides students with opportunities to make connections between the language present in their surroundings and the mathematical language. During this process, students improve their conceptual structures, their understandings of the mathematical concepts, and more, their critical and creative senses are improved in the formulation of data and in the evaluation of results. This argument indicates a style of thought that considers *MM for Education a way to surpass the linear process of teaching that is decontextualized from students' experience.* It is considered that each student has his/her own cultural spectrum that is revealed in his/her doings, outside the school context. When s/he experiences the modelling process in a triad: research-school-reality, the student can perceive his/her talent, his/her interests and even the importance of the school in this process of literacy to scientific and professional issues or issues related to companionship.

(3) About *possibilities*, the authors argue for MM as a means so that students may (a) construct their knowledge by understanding the concepts involved; (b) choose significant problems to the context; (c) become capable of explaining their reasoning with the correct use of mathematical language; (d) have better performance in mathematical modelling activities; and (e) know how to use it in their pedagogical practices, since these students are future math teachers. This defence, present in all papers, suggests that when doing, people get to know and when people get to know, they do. It is a cyclical process. Going from the simplest to the most complex issues and continuously revising can make modelling a more effective process to deal with the various issues that involve the living context. *MM for Education is a dynamic process: it can be modified whenever necessary so that students' knowledge can be improved.* This instance reveals another style of thought.

(4) The *difficulties* in the implementation of MM reside in the current educational structure: available timings and schedules, curricular programs divided into various subjects. Difficulties emerge for future teachers and in-service teachers. For future teachers the difficulties reside in knowing how to use the mathematical language to describe the problem-situations and, thus, resist to changes. And for in-service teachers, difficulties reside in the available time they have to get acquainted with the themes chose by the students to orient them.

The curricular model in teacher education courses is similar in the various countries. It consists of several disciplines, each one under the responsibility of a teacher and with a restricted number of class hours. Math, although present in all schooling years of Basic Education, follows, in general, the same process of teaching, without connection to the other disciplines of the curricular structure. This model contributes to the fact that the students from these teacher education courses, who experienced a 'traditional' way of teaching for more than 12 years, have difficulty in interpreting the context and data from a subject of any knowledge area; in recognizing the math required to interpret data; in formulating mathematically and in analysing it.

This difficulty, a consequence of the experienced education by these teachers, emerges when these students or even teachers become aware of MM. They learn about it in a single subject in an initial or continued teacher education course, with a limited number of hours if compared to the time experienced in 'traditional teaching'. As a result, obtaining effective results depends on the interest the participants involved have in following the modelling pathway; that advances gradually departing from practical and conscious activities about the importance of learning.

The current educational structure in various countries does not lead to the integration among math and the other disciplines of the course, neither the diverse areas of knowledge. As a consequence, *there are difficulties students and also teachers have in making use of modelling in the classroom practices, particularly in the initial and continued education, due to the time spent in this educational structure.* Paraphrasing Salles (2007), MM will only reach a status of legitimized knowledge if it resists to the tests imposed by the group that 'maintains' the traditional process in the teaching of math. The incorporation of MM for education can establish a dynamic relation between the 'traditional' and the 'innovations'. Allowing for a change in the conception of math teaching departs from reactive interactions among the teachers in the group. This kind of argument present in many these 42 papers suggests another style of thought.

Departing from the categories: *justification*, *process*, *possibilities* and *difficulties*, it may be claimed that the styles of thought displayed by these 42 paper, of the 32 'first authors' converge in the understanding that modelling can contribute not only to improve mathematical teaching and learning, but also to provoke reaction and interaction between the body of teachers and the body of students involved in the on-going and necessary production of knowledge. It is a mutual share of acquired experiences. These authors, participants of the ICTMAs, make explicit their knowledge gained in the interactions among the theories of MM for Education and their practices of modelling in classrooms. As Wenger (1998, p.45) points out, "this collective learning results in practices that reflect both the pursuit of our enterprises and the attendant social relations". These interactions conduct to a style of thought.

According to Fleck (1979), so that a style of thought can be constituted, it passes through the phases of instauration, extension and transformation that occur by means of interactions of distinct groups in the circulation of ideas intercollectively and intracollectively. This circulation of intracollective ideas occurs among experts and the circulation of intercollective ideas among non-experts. This process implies seeking for knowledge that brings increment to the existing data and yet, creates a collective a thought. It is worth highlighting that these styles of thought, that come since the teachers-researchers of the first phase (1965-1975) in the classification proposed by Blum, Niss and Galbraith (2007), are present in the community, independently of the country, the geographical distance, the educational system.

These styles may be considered the scientific knowledge of the ICTMA. This knowledge was consolidated by facts, theories and interpretations shared by the community of practice. Perceiving that the process involved in MM for education is even more relevant when dealing with issues that allow having a particular set of data that can be better studied using specific methods and as consequence, become charmed with the solution and the validity of this solution.

FINAL REMARKS

The papers analysed reveal that these researchers who circulate in the conferences organized by ICTMA constitute a collective of thought, indicating the concern and support to the official documents and to the issues related to teacher education. The collective of thought about modelling in math teacher education courses follows institutionalized standards. It means that the events in the modelling process in courses for teachers or future teachers enunciate necessary changes in the educational structure, based on values, goals and other inspiring stimuli that come from the different people within the system.

To Fleck (1979), the researcher's style of thought designates the formal aspects of his/her research that comprises all ways of expressing the units associated in the process and in the results. The style comes from shared practices and shared knowledge. When a certain style is shared by a group of people, a collective of thought is established. It may be claimed that in the ICTMA conferences we encounter groups that share the same style of thought that is composed by collectives of thought. This sharing occurs when the researcher identifies in his/her research object traces that are recognized and valued by the community, that are present in other studies about similar themes. There are hues of this style of thought that arrange facts and fit in the MM theory for education dominant in the discussions of mathematical education. These various elements coalesce around different conceptions of knowledge production, but when confronted, as pointed out by Salles (2007), constitute an extension of the established style of thought.

This fact indicates that the style of thought about MM for Education that circulates in the ICTMAs is established in the knowledge of different groups of researchers. And according to Fleck's theory (1979), the changes in this area will become noticeable as a collective process, in which the transformations will be construed by the community of thought, when the ideas circulate in the ICTMA conferences. The importance of analysing scientific productions throughout time in MM in the international scene is justified by the panorama that this type of research results, providing a map of the contributions, needs and challenges related to MM. According to Witter (1996), it is by means of this type of research that a basis of scientific data may be formed. This basis consolidates certain knowledge and thus, allows for scientific advancement.

Investigating how each style of thought a group adopts or how it is incorporated by the style of the community of thought, how styles are established and transmitted is the beginning of another study. Diverse presuppositions may be formulated. Some may direct the researcher's steps to the origins of M&A in the teaching practices of the school system. It is relatively simple to identify how the ideas and proposals are disseminated. These proposals and ideas, written or verbalized, carry styles of thought, express attitudes or ideals that deal with experiences, beliefs, values and casual sequences, whose order is diverse.

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THE USE OF THEORY IN TEACHERS' MODELLING PROJECTS – EXPERIENCES FROM AN IN-SERVICE COURSE

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We discuss how to support teachers' use of theory on modelling and on learning mathematical concepts in an in-service course on project work and modelling for upper secondary teachers. The course is centred on the teachers' experiments with modelling projects in their own classes. The paper is a case study on how to facilitate the interplay between theory and development of teaching practice. Our analyses focus on modelling as a means for learning mathematics, and on theories on the learning of mathematical concepts. The theories have potentials for improving the teachers' practice, but they need concretisation and re-contextualisation in the teachers' projects in order to be helpful for the teachers.

INTERPLAY BETWEEN THEORY AND PRACTICE IN MODELLING

The research question is: *How to integrate theories on modelling and on the learning of mathematics in an in-service course on modelling for teachers at upper secondary level so as to be helpful for the teachers' development of their own practice?*

During the last three decades, theories in the area of teaching and learning mathematical modelling have developed in close connection with the inclusion of modelling in secondary mathematics curricula, and through interplay with the development of practices of teaching in this area. These theories rest upon the definitions of key concepts related to modelling such as the notion of a mathematical model, the cyclic modelling process and modelling competency. The theories include justifications of mathematical modelling as part of mathematics curricula in general education and suggestions for design of teaching in different contexts. In more recent research, specific learning potentials and difficulties related to modelling and ways of enhancing students' learning of mathematical concepts through modelling have been researched (Zbiek and Conner, 2006), (Blomhøj and Kjeldsen 2006, 2010b). This development opens for a closer interplay with theories on mathematical learning in general. However, also in this area there is a scarcity of studies investigating concretely how theories can be of use for teachers in developing their practice.

Research in mathematics education tends to focus more on the development of theories than on how these theories can function in practice in processes of improving teaching and learning. Hereby, the process of doing research is separated from the process of applying theory in developing practices of mathematics teaching. It is a challenge to find ways to bring theory into play in the practice of teaching. Action research seems to be a fruitful approach to teachers' professional development in that respect (Krainer, 1999). Boaler (2008, p 103) found that the teachers' involvement in experimenting with their teaching seems to be an important factor in successful use of theory in practice. The development of our in-service course and our related research can be seen as a form of developmental research practice with exactly this aim (Blomhøj and Kjeldsen, 2006).

THE STRUCTURE, ORGANISATION AND PURPOSE OF THE COURSE

The course is designed and taught by the authors in collaboration. It is advertised on an open marked for professional development courses. 12 to 20 mathematics teachers participate in each course. The participating teachers come from different schools. The course runs from 5 to 8 months within a school year. In this period, the teachers are involved in experimenting with their teaching. They develop and implement their own modelling projects. The course begins with a seminar over three full days. During the seminar, the teachers work in groups of two or three developing a problem oriented modelling project of the group's own choice. After the seminar they implement their modelling project in their own classes, covering 10 lessons and two written homework assignments for the students. There is a one-day midterm seminar, and the course finishes with a two-day seminar. The teachers write a report that documents their experiments with their modelling projects. The reports are published and disseminated on an internet portal for school teachers in Denmark.

At the first seminar, the participants are introduced to mathematical modelling and problem oriented project work, as well as to theories on the learning of mathematic as presented in the next section. In the first phase of the development of their projects, the groups are asked to emphasize the following four issues: (1) Intentions for their own development. What is it in particular that they want to experiment with as a teacher in their project organised modelling course? (2) What are the main intentions for the students' learning in the course? (3) How to set the scene for the students' problem oriented project work? (4) How to evaluate the students' learning through observations and/or product evaluations? The groups' first proposals are presented and discussed in the first seminar, and then further developed afterwards. Two weeks after the first seminar, the preliminary descriptions and student materials for the modelling projects are distributed to all participants. Within a period of two months the groups finish the detailed planning and each teacher try out his or her modelling project in at least one class. In some cases, it is possible for two teachers working at the same school to observe parts of the course in the colleague's class. This has proven to be supportive for the teachers' reflections and for their further cooperation with improvements of the projects. We therefore encourage participation in pairs of teachers from the same school. The one-day midterm seminar is held after the period with experimental teaching, with the aim of supporting the teachers in reporting their projects and their related reflections in a form that could be helpful for colleagues wanting to do similar modelling projects. A first version of the reports is distributed to all participants a month later. These are discussed at the final two-day seminar after which the groups receive written feedback and suggestions for improvements before publication of the reports. This organisation allows detailed discussion of the teachers' projects and their relation to the introduced theories.

THE USE OF THEORIES IN THE COURSE

At the first seminar, the teachers are introduced to theories in three different domains: (1) Theory on problem oriented project work (Blomhøj and Kjeldsen, 2010a). Here the emphasis is placed on the importance of formulating a problem, which can guide the students' modelling process, and enable them to take control of the process supported by milestones and supervision. The way in which the scene is set for the project in the class, and the explicit demands for the students' project reports should be the main didactical instruments for guiding the students' project work. (2) Theory on mathematical modelling, e.g. justifications for modelling, modelling competency and the modelling cycle (Niss et al., 2007), (Blomhøj and Kjeldsen, 2010a). The emphasis is on illustrating and discussing the modelling process, reflections in relation to concrete examples, and on the teachers' ideas for modelling problems for their projects. (3) Theory on the learning of mathematical concepts. This element relates to modelling as a vehicle for supporting students' learning of mathematics (Blomhøj and Kjeldsen, 2010b). Here we present and discuss theoretical ideas related to: The important role of representations for the learning of mathematical concepts (Steinbring, 1987); the process-object duality in concept formation (Sfard, 1991); concept images (Vinner and Dreyfus, 1989); and the RME-idea concerning the change of role of a model from "a model of some particular object/situation" to "a model for the learning of a mathematical concept" described by Gravemeijer (1994). In parallel with diSessa and Cobb (2004), we argue that such theories have a lot to offer for improving teaching experiments.

However, the theories need to be concretised and re-contextualised in relation to the teachers' particular modelling projects, in order to be helpful for the teachers. That is the main reason for the design of our in-service course. In the course, the participating teachers are developing their own modelling projects, and through our interplay with the teachers in this process, it is possible to bring the theories into play in relation to the teachers' modelling projects. Until now we have developed three different forms of intermediate representations for presenting and discussing with the teachers, the potentials of the various theoretical ideas for the development of their practice of teaching modelling. These are: (1) Detailed and concrete descriptions of the modelling process behind the models in the teachers' projects. Such unfolded modelling processes enable the teachers to become aware of and discuss the potentials for supporting the development of the students' modelling competency in the planned activity and to help students in case of difficulties, without spoiling the essential modelling challenge; (2) Schemes spanning all the different representations of particular mathematical concepts and their interpretations in a given modelling context. Such schemes can help capture the potentials for supporting the students' mathematical learning and they can be a tool for analysing the students' modelling activities in relation to their mathematical learning potential. Figure 2 represents such a scheme for the project presented in the next section; (3) Construction of anticipated dialogues between the teacher and a group of students facing some particular modelling challenges or learning difficulties. Discussing such dialogues can help teachers prepare for supporting the students' learning in modelling activities without taking over the students' tasks. In the present paper we focus on the use of (2) as a tool for spanning the mathematical learning potentials in a modelling activity and as a tool for analysing the students' activities in a modelling project.

The methodological approach in our research and development of our course is similar to that of critical mathematics education (Skovsmose & Borba, 2004), where there is a distinction between three types of situations (teaching practices): The current situation (CS) is the practice of the teacher before the course. The imagined situation (IS) is the practice the teacher imagines to establish. The arranged situation (AS) is the practice that emerged in the experiment when the teacher implemented his/hers modelling project in class. Three processes combine the three situations: (1) Experimenting; (2) Analysing; and (3) Pedagogical imaging, see figure 1.

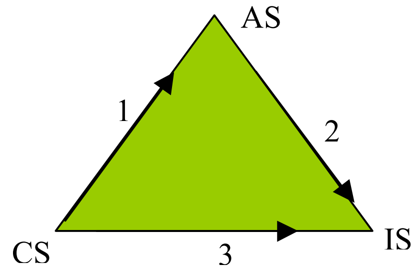


Figure 1: The methodological triangle in critical mathematics education.

In our course, we use theory on mathematical modelling and on the learning of mathematics to establish a shared idea about an imagined situation concerning a concrete modelling project suggested by the teachers (process 3). We help the teachers to use elements of theory as a basis for designing their projects (the arranged situation) (process 1) and for describing their aims for developing their practice of teaching in relation to the imagined situation (process 3). At the final seminar, each project is analysed with respect to the relation between the arranged situation and the teachers' ideas about the imagined situation (process 2), which they have described in their reports. Together with the teachers we reflect upon and develop new ideas about the imagined situation and how to further develop the projects for the next time. In relation to these processes, we focus on developing our interactions with the teachers in order to support their reflections on their projects.

A MODELLING PROJECT ON THE DECAY OF ALCOHOL AND THC

This problem oriented project work in mathematical modelling of the decay of alcohol and THC (tetrahydrocannabinol, the active drug in hash) was developed by four teachers from three different high schools. The project was implemented in three first year classes (age 15-16 years) in ordinary high schools in Denmark.

Before deciding for this theme for their project, the teachers discussed the ethical aspects of the theme. Most Danish young people age 15 and older drink alcohol at parties, and some drink too much! In the current situation hash is illegal to sell and possess but legal to smoke. In average, one or two students in a class of 30 students can be expected to have had some experiences with hash. The teachers found the theme relevant and motivating for the students. They sought it could make the students reflect critically upon their own alcohol habits and possibly prevent them from experimenting with hash. The theme was mathematical relevant since the decay of alcohol and THC is essentially linear and exponential, respectively.

The teachers formulated seven learning goals for the students in the project work:

1. provide the students with a positive experience on using their mathematical skills to answer interesting and relevant questions from their life world
2. support the students' conception of modelling and applications of mathematics
3. teach students to have a critical outlook on mathematical models
4. support the students' learning of linear and exponential functions
5. develop the students' understanding of the parameters in the two models
6. train the students to communicate mathematics
7. support the students' IT competences

These learning intentions were inspired by the theories introduced at the in-service course. They fall into three groups: (I) aspects of developing the students' modelling competency (1-3); II) aspects of developing the students' concept images and their mathematical understanding of linear and exponential functions (4-5); (III) aspects of developing the students' IT and communication skills (6-7).

The students worked in groups of three to four. They were given a set of four exercises and the following task of writing a kind of newspaper article:

> "Write an article for students of your own age about the decay of alcohol and THC in the human body. In the article you should also explain the mathematics you have used to complete the exercises. Your answers to the exercises and your graphs should be integrated into your article." (Our translation from the teachers' report)

In the first two exercises, the students were presented with a set of realistic, but not authentic data showing the decay with time of some amount of alcohol or THC respectively in a person. The students were asked to draw graphs using T-Inspire or Excel, to describe the graphs in their own words, to determine the time for the amount to decrease to the half two times consecutively (to experience a fundamental mathematical difference between the two cases), to determine the mathematical expression for the functions represented in the graphs, and finally to interpret, in their own words, the significance of the parameters of the functions in the two contexts. The students were encouraged to find data on the web for the decay of alcohol and THC in the human body to compare with the models they had developed from the given data. In exercise three, the students were given data for the amount of alcohol in some popular drinks, and asked to model the decay of the amount of alcohol they themselves had consumed at their last party. Finally, they were asked to compare the decay of hash and alcohol.

In the following, we focus the discussion on the second group of learning intentions. We analyse the students' reports and the teachers' related reflections with respect to these learning intentions. In the next section, we illustrate how the theories on the learning of mathematics can help span the potentials of the project for supporting the students' learning of the two types of functions. Our analyses rely on discussions that took place at the in-service course, the teachers' reports, their design of the project, the tasks given to the students and the articles written by six of the student groups, two groups from each of the three classes. We round off by discussing the relation between the potentials of the projects and what was fulfilled in the project.

Regarding the first three learning goals, the teachers' evaluations show that aspects of the students' modelling competency were invoked in all three classes. In the articles written by the students, they interpret their linear and exponential graphs for the decay of alcohol and THC, respectively with respect to the modelling context. They translate back and forth between the graphical representation and what it tells us about the amount of alcohol and THC in the body as times go by. Several groups of students reflected upon the validity of the models in the sense that they questioned whether the parameters in a model that was based on data for one particular person are valid for another person. The rate of decay of alcohol is mainly due to the liver and is only slightly depending on the body mass. A constant rate of 8 g/hour is a good estimate for a normal grown up person. For THC one finds a first order decomposition with a half-life period close to three days.

The students came to reflect upon the differences in the way in which the substances decrease in the two situations through the questions about how many hours it takes for the amount of alcohol and THC, respectively to decrease to half of the initial amount. They used the model with their estimation of their intake of alcohol at their last party. Many of the students were surprised to realize how many hours it actually takes, according to the model, before the alcohol in their body has vanished.

With respect to supporting the students' learning of linear and exponential functions and the significance of the parameters it is unclear, whether the students realized the fundamental properties of the two types of functions. As one teacher wrote:

> "The idea was that the students should realize that the half time was a constant in the exponential case and not in the linear case. Many students didn't realize that because they used the graphs [to determine when half of the amount had decayed and when half of the half of the amount had decayed] and they reached two different approximations [for the exponential function]."

The students were able to reach an expression for the linear decrease of alcohol and the exponential decay of THC. They were also aware that the parameters in the linear function measure the initial amount of alcohol and the amount of alcohol that disappears per hour – except that in some of the articles, the students were not accurate in distinguishing between the amount of alcohol and the concentration of alcohol in the body. However, as one of the teachers wrote:

> "Surprisingly many of the students had problems de-mathematizing the parameters a '$y=a\cdot x+b$' and a ' $y=b\cdot e^{ax}$ ', and interpreting the significance of these parameters for the decay of alcohol and THC, respectively. The main problem was the understanding of a indicating the [absolute] amount of decrease of alcohol per hour and a being a number determining the relative decrease of THC per hour as e^{a}. …So next time I will use different names."

The article, the students were asked to write, trained the students' competency in communicating in and with mathematics. In the articles, the students sometimes reasoned with mathematics and other times within the modelling context, e.g.:

> "In all these calculations we have seen that hash is in the body for a longer time than alcohol, and therefore it can damage one's ability to learn if one smokes hash regularly."

The students performed the calculations on the models constructed with the given sets of data, and for these it took much longer for the body to get rid of the THC than the alcohol. However, the model cannot say anything about learning disabilities!

The teachers' intentions for the students' learning were more or less fulfilled. The students used mathematics to model situations from their life world. Nearly all students were able to reflect upon and criticize the models to some extent. The modelling of alcohol and THC created teaching situations, in which the teachers became aware that the students had bigger problems than anticipated with understanding and interpreting the parameters in the linear function and the exponential function both mathematical and in the modelling contexts. In order to remedy this, the project can be further developed by designing and using model eliciting activities as explained by Ärlebäck, Doerr and O'Neil (see this proceeding).

THE USE OF THEORY IN DESIGNING AND ANALYSING THE PROJECT

In our in-service course the teachers were introduced to the model for concept formation in mathematics developed by Anna Sfard (1991) and the basic idea that the access to a mathematical concept goes through the meaning of its representations and their relations (Vinner and Dreyfus 1989), (Steinbring, 1987).

These elements of theory can be combined to form a tool for capturing the potentials for learning mathematics through modelling activities. We view this as a beginning development of a methodology for integrating theory from mathematics education into teachers' experimentation and development of their own teaching practice. In the scheme in figure 2, we illustrate this idea in relation to the concept of a linear function and the different forms of representations that can come into play in the modelling of the decay of alcohol. In each cell in the scheme there is a representation of the concrete model of the alcohol decay and of the general linear model in the same form of representation. In each form of representation both the process and the object aspect of the concept can be represented. All these representations can be interpreted and be given meaning in the modelling context. In the representations, the mathematical properties can be pinpointed and related to the properties of the general linear model or to the general exponential model. Hereby, the modelling context can mediate meaning also to the general mathematical model and to the mathematical concepts involved. We believe that the transition from a process to an object understanding of a mathematical concept can be supported in all forms of representations in the scheme. However, this needs to be further researched. The scheme can help span the learning potentials in modelling situations with regard to a particular concept, and hereby be of help for the teachers both in designing projects and in their dialogical interactions with their students.

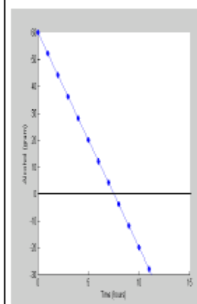
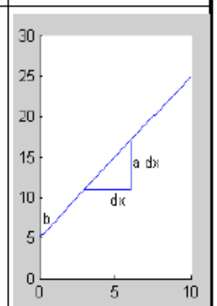
	Natural language	Numerical	Algebraic	Algorithmic (Excel)	Graphic
Process	8 gram of alcohol is removed by the liver per hour. 12 gram is added per drink (beer)	x 0 1 2 y 60 52 44 -8 -8	$f(x+1)=f(x)-8$ $f(0)=60$	B2=-8 A5=0 A6=A5+1…. B5=60 B6=B5+\$B\$2 ….	
	x times the slope plus the constant yields y. One extra unit of x course a change in y of the value of the slope	x 0 1 2 y b $a+b$ $2a+b$ $+a$ $+a$	$f(x+\Delta x)=$ $f(x)+a\Delta x$; $f(0)=b$	Can be generalised by change of parameter and initial value	
Object	After five drinks and x hours: $y=60-8x$ gram of alcohol is left in the body	x 0 1 2 3 y 60 52 44 36	$y=-8x+60$	B2=-8; B3=60; A5=0 A6=A5+1…. B5=\$B\$2*A5+\$B\$3 B6=\$B\$2*A6+\$B\$3 …….	
	A linear combination with constant sum.	A tabel of (x,y) with $y=ax+b$	$f(x)=ax+b$	The algorithm is general due to parameters.	

Figure 2. Shows process and object aspects of representations of the alcohol model.

The teachers were introduced to the learning potentials spanned in the scheme at the first seminar of the in-service course. However, the way the scene was set for the students' modelling work in the project did not systematically challenge the students to work with the different representations. The algebraic and graphic representations are dominant in the students' articles. Moreover, in these two forms of representations, the process aspects are not visible in the students' work.

In the final seminar of the in-service course, the teachers reflected themselves about the unfulfilled learning potentials in the project. Suggestions for how to guide the students to work with the different representations in the scheme the next time were discussed. One way could be to ask the students explain the model in all five representations from both the process and the object perspective in the articles. Also, the learning potentials in comparing the schemes for the linear function and the alcohol model with that of the exponential function and the THC model need to be addressed explicitly in the requirements for the students' articles. Another way could be to develop model eliciting activities directed towards the different representations (see Ärlebäck, Doerr and O'Neil; Doerr, Ärlebäck and O'Neil, this proceeding)

CLOSING REMARKS ON THE THEORY-PRACTICE RELATION

To support the students' learning of mathematics is an important objective for including mathematical modelling in secondary mathematics. However, in order to meet this objective we need to develop the practice of teaching modelling based on theories on the modelling and on the learning of mathematics. This can be done through theory based experimental work in the context of in-service courses or developmental projects. To facilitate the teachers' use of theory in such contexts it is important that we develop intermediate representations and modes of collaborations, which can mediate powerful theoretical ideas to teachers, so as to be helpful in their development of practice. The representation scheme presented in this paper (see figure 2) is an example hereof, and a beginning for developing a methodology for teachers' use of theory in experimenting with and developing their teaching practice.

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BARRIERS AND MOTIVATIONS OF PRIMARY TEACHERS FOR IMPLEMENTING MODELLING IN MATHEMATICS LESSONS

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Mathematical modelling is one central competency within the German education standards in mathematics for primary schools (from grade one to grade four). However, a lot of primary teachers do not know how to deal with modelling in the classroom, what can be often recognized during in-service teacher training or classroom observations. To get more insight into reasons for this reserve, a quantitative study was developed with the aim to investigate central barriers and also motivations of primary teachers for implementing modelling in mathematics lessons. The results show three essential barriers for primary teachers: 1. lack of material; 2. time pressure; 3. Assessment. In the paper we will go deeper into these and further results.

INTRODUCTION AND RESEARCH QUESTION

The national and international research area of mathematical modelling at the secondary and tertiary education levels is rather well established and a lot of modelling problems, materials and practical ideas for teachers are available. Although mathematical modelling in primary schools (grades 1 to 4) is discussed within the international debate as well, there are still rather few research studies. In Germany in particular, there are only very few studies focusing on this topic, and the development of modelling problems for primary classrooms did not really start until the national education standards for mathematics at the primary level (in which modelling is one of five competencies that pupils ought to achieve in grades 1 to 4) became compulsory in 2003. But there is still a big gap between the demand that modelling really starts in primary school and is an essential part of mathematics, on the one hand, and the experiences from everyday school practice, on the other hand, where many teachers do not like to deal with this topic since it seems to be "too complex and too hard for young kids". Adequate starting points for avoiding this mismatch are, in particular, modelling seminars with practical elements for future primary and secondary teachers at university. We have a lot of experiences with these seminars (see Borromeo Ferri & Blum, 2009) and we are convinced that these teachers will have another view on teaching modelling when they are in school. The biggest problem is to convince those teachers who have taught mathematics for a long time in the traditional way and have heard about modelling, but do not know what it really means and how it can be implemented in their daily mathematics lessons. This special situation with primary teachers in Germany has led us to investigate barriers and motivations for teaching mathematical modelling in primary school more deeply and more systematically.

In the international literature, there are no quantitative findings concerning these aspects for primary school teachers. So our central research question was:

What are the main reasons which hinder or motivate primary school teachers to implement modelling in their mathematics lessons?

THEORETICAL BACKGROUND

A lot of research studies have revealed positive effects of modelling problems in elementary school (Bonotto, 2004) with examples such as shopping in the supermarket or timetables for trains. Also Verschaffel (2002) emphasizes the possibility of modelling activities in primary school, in particular in the context of arithmetic operations (see also Usiskin, 2007). Especially the research of English (2002, 2006) concerning modelling activities of 10- and 11-year-olds are impressive. Young students created, for example, a shopping-guide in the context of a modelling problem (see Mousoulides & English, 2008). Starting with Modelling Eliciting Activities (MEA) already in kindergarten and subsequently in primary school is in Lesh's sense (see e.g. Lesh & Doerr, 2003) the basis for effective modelling in upper grades. A lot of Lesh's case studies show that primary kids are actually able to deal with modelling problems and to get remarkable results, in particular if the teachers themselves like modelling, have knowledge of how to communicate it in the classroom and have stimulated so-called "thought-revealing-activities". We also have used one of Lesh's examples (the "Big Foot" problem) successfully in grade 4. Summarizing existing studies about teaching and learning mathematical modelling, the crucial role of the teacher becomes evident. So modelling can only be learned effectively if there are teachers who have appropriate competencies in this field (Borromeo Ferri & Blum, 2009).

Besides the above-mentioned aspects, our main hypotheses are based on further results of empirical research. Within the LEMA project (**L**earning and **E**ducation in and through **M**odelling and **A**pplications, project director: Katja Maaß) questionnaires, mainly for secondary teachers, were developed concerning beliefs (Maaß & Gurlitt, 2009) and also concerning the question which obstacles and motivations the teachers have for integrating modelling in their mathematics lessons (Schmidt, 2010a). The main results of Schmidt's quantitative study with more than 50 secondary teachers in a pre- and post-test design have revealed three main barriers for teaching modelling: 1. The time needed for working on modelling problems; 2. Lack of materials (no access to suitable modelling problems); 3. Assessment (teachers do not know how to give marks for modelling activities). More generally, the following six categories of obstacles for the implementation of modelling in mathematics classrooms have been identified in the educational debate (compare, e.g., Blum, 1996, 2011; Pollak, 1979; Kaiser-Meßmer, 1986; Maaß, 2004; Burkhardt, 2006; Ikeda, 2007): 1) organisational obstacles (such as the time needed to deal with modelling problems in the classroom); 2) student-related obstacles (lessons become more demanding and less predictable); 3) teacher-related obstacles (non-mathematical competencies and broader beliefs are needed, lessons become more demanding and less predictable, assessment becomes more complex); 4) material-related obstacles (are there enough suitable examples?);

5) systemic obstacles (such as expectations of parents, scientific associations and other pressure groups, or regulations in examinations); 6) research-related obstacles (are there reliable empirical results as a basis for teaching modelling?)

METHODOLOGY AND DESIGN OF THE STUDY

Questionnaire development

The empirical study of Schmidt (2010a, 2010b) was particularly interesting for us because she developed, in the context of the LEMA project, a questionnaire on the basis of findings from those empirical studies in mathematical modelling which have reconstructed relevant obstacles and motivations for teaching modelling (see the end of the previous section). So the construction of her questionnaire can be described as an inductive and deductive procedure. The developed "inductive" items came from theory and the "deductive" items from interviews with experts. This questionnaire was the basis for our own questionnaire development, but we modified items and scales for the purpose of applying it with a sample of primary school teachers. From the original 56 LEMA items, we kept 37 und constructed 6 new ones, based on two extended piloting activities. A central theoretical background for our questionnaire was "The offer-and-use model" (Figure 1) according to Helmke (2006), because the categories of this model (teacher, education, context etc.) were the basis for the structure of our scales (see below).

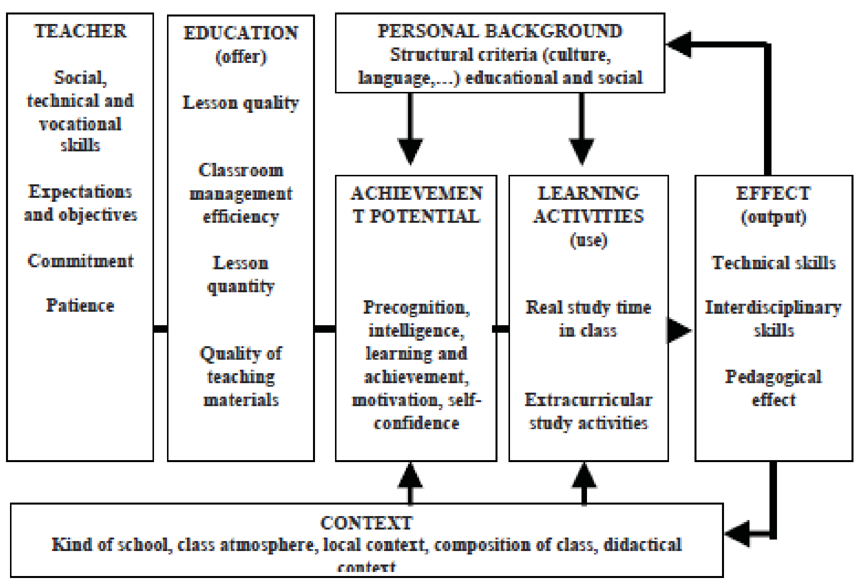

Figure 1: "The offer-and-use-model", Helmke (2006)

Altogether our questionnaire comprised 14 scales with 43 items and additionally one open item asking for personal comments and experiences with mathematical modelling. Besides demographical questions concerning age, years of teaching, experiences with modelling and subjects studied at university, the scales are labelled as follows:

"*context*" (sample item: The parents of the pupils in my class do not like if I teach modelling problems in mathematics lessons), "*differentiation*" (sample item: modelling problems support stronger and weaker pupils at once), "*time*" (in the sense of lesson time for teaching modelling), "*role of the teacher*", "*lesson planning*", "*motivation of the pupils*", "*material*", "*creativity*", "*independence*" (sample item: pupils are able work on their individual solution processes), "*excessive demand*", "*assessment*", "*long-term effects in mathematics lessons*", "*applying mathematics in real life*", "*long-term effects beyond mathematics lessons*".

The answer format corresponded to a 3-level Likert scale (Rost 1996) from "strongly agree" to "do not agree" (see Figure 2). The numbers of items per scale differ (from 2 to 6). Of course, only scales with a satisfactory reliability (see below) were used for further analyses. It is important to keep in mind that this questionnaire is focusing on teacher's personality and so gives only feedback about subjective ideas and attitudes concerning the topic of modelling. The questionnaire was discussed with several primary teachers before using it in the sample.

Hypotheses: On the basis of the theory and of the results of Schmidt's study within the LEMA-project, in combination with the research on modelling in primary school as well as experiences from teacher education, we assumed the following six scales as examples of *barriers* for modelling: "context", "time", "lesson planning", "material", "excessive demand" and "assessment". On the other hand, we supposed the following eight scales as *motivations* for teachers to include modelling: "differentiation", "role of the teacher", "motivation of pupils", "creativity", "self-dependence", "long-term effects in mathematics lessons", "applying mathematics in real life" and "long-term effects beyond mathematics lessons".

Design of the study and sample

The study was mainly *quantitatively* oriented, because on the basis of experiences in in-service teacher training we knew some of the barriers and motivations, but we still had no empirical knowledge about, for example, a comparison between teachers who studied the subject or not, and so we wanted to get more generalizable results. The data collection started in March 2012 and was completed in April 2012. Our sample comprised 71 primary teachers (female: 64, male: 7) from 8 of the 16 German states and was a convenience sample. Often two or three teachers from the same school got the questionnaire. We were glad about the relatively big sample for this study, having in mind how hard it is to get teachers, particularly teachers who did not study the subject, to spend time for working on such a questionnaire. The schools were in suburban regions respectively in smaller cities. The mean age was 44. The majority of the teachers (43 in total) have studied mathematics as a subject at university. The indicated frequency of integrating modelling was (in total): never: 20; seldom: 37; monthly:

9; weekly: 5. So we can say, as a first result, that for the majority of the teachers, modelling is not an essential part of mathematics lessons.

Data analysis

The underlying theory for the data analysis is the so-called Expectancy Theory or VIE-Theory (Vroom, 1964) which itself is based on the concept of Instrumentality (Peak, 1995) coming from motivation research. Vroom has built his theory on three variables which influence the motivation of a person (see Brandstätter 1999, p. 351ff): Expectancy, Instrumentality, and Valence. In our questionnaire, we left out Expectancy. *Instrumentality* is the belief that a person will receive a reward if the performance expectation is met. *Valence* is the value that the individual places personally on the rewards based on his/her needs, goals, values and sources of motivation. In order for the valence to be positive, the person must prefer attaining the outcome to not attaining it. Expectancy Theory of motivation can, for instance, help managers understand how individuals make decisions regarding various behavioural alternatives. The model below shows the direction of motivation when behaviour is energized: Motivation = Instrumentality **x** Valence.

The following example of our study shows the procedure:

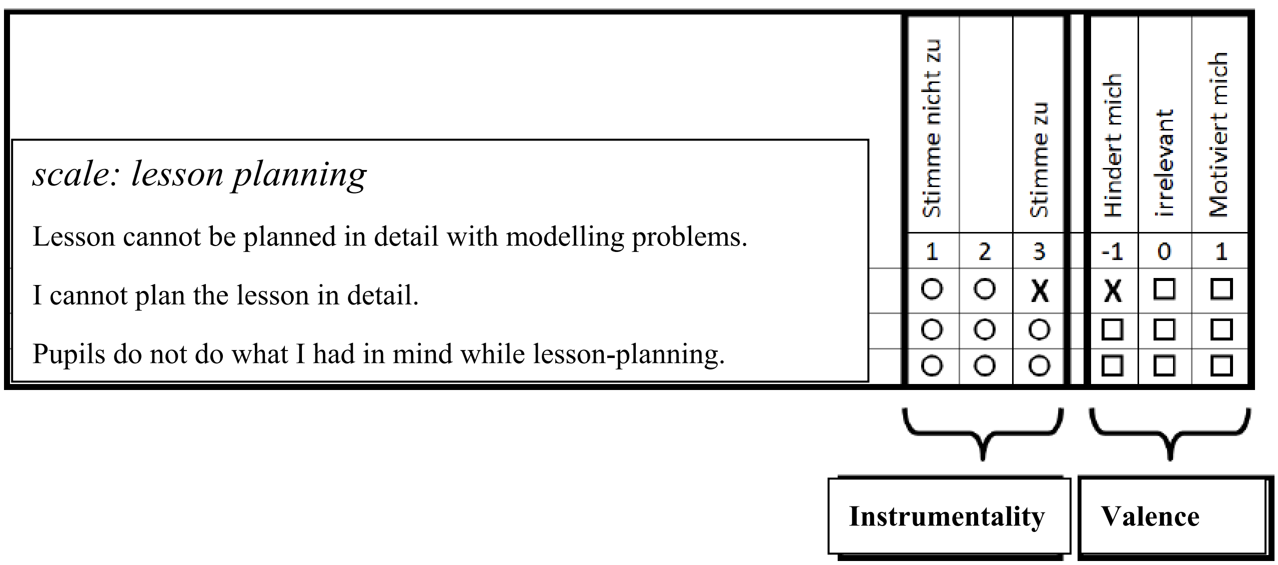

Figure 2: Example of data analysis (scale: lesson-planning)

The figure shows values from 1 to 3 in the first dimension and from -1 to 1 in the second. For the product of Instrumentality and Valence in this example, we get: 3 x (-1) = -3. Overall the motivation values may range from -3 to 3. It should be noted that “motivation” in the usual sense means positive values for motivation here.

In this way, all the data were coded and the software SPSS was used. The reliabilities of 12 of the 14 scales were satisfying, with Cronbach’s α in the range between .65 and .84. Only the two scales “context” and “motivation” had smaller reliabilities; hence these two scales will not be contained in the following diagrams.

EMPIRICAL RESULTS

Relevant aspects from teachers' perspective – an overview

Very briefly we will first show how strong the test persons agreed to the statements in these scales, so we look at *the left part of the rating scale (Instrumentality)* of the questionnaire. The higher the level of agreement, the more relevant is this aspect for the teachers. In the following boxplot, those scales are shown which can be interpreted as *barriers* (see our hypotheses):

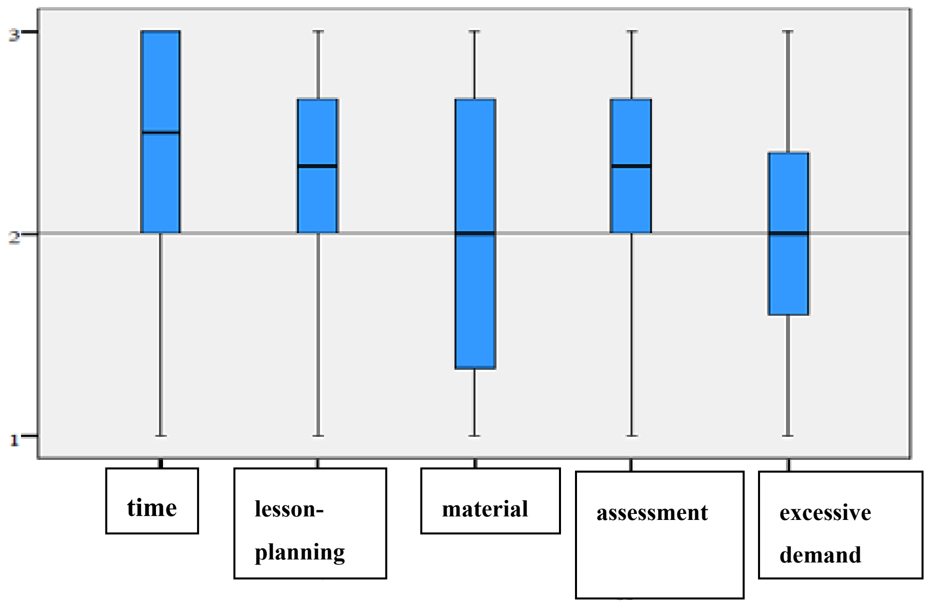

Figure 3: Relevant aspects from teachers' perspective

The values between 1 and 2 show a tendency towards rejection by the teachers, whereas the values between 2 and 3 show a tendency towards agreement concerning the relevance of these aspects. In particular the scales "*time*", "*lesson planning*" and "*assessment*" have high relevance for the teachers. Regarding "*material*" and "*excessive demand*" different attitudes of teachers become visible. For the scales that can be interpreted as *motivations* from the teachers' perspective, the scale "*independence*" got the highest agreement from nearly all teachers. But also the other aspects ("long-term effects in mathematics lessons", "applying mathematics in real life", "long-term effects beyond mathematics lessons", "creativity", "role of the teacher") show a surprisingly high level of acceptance. Summarizing the first rating scale in more detail, the highest agreement came from the category "*learning activities*": 87% of the teachers have the opinion that their pupils could be more creative mathematical thinkers when working with modelling problems, and 85% think that the level of self-dependence will be better. The highest value within the category "effects" was received by the aspect "relevance for everyday life" with 78%. The above-mentioned aspects only show the level of agreement. In the following it is interesting to see whether this agreement is more a barrier or more a motivation for integrating mathematical modelling.

Barriers and motivations: For investigating the expected barriers or motivations both dimensions in the questionnaire (Instrumentality and Valence) were multiplied and so linked in the sense of the Expectancy Theory. Regarding to our hypotheses we expected barriers concerning all five scales. Analyzing these results in more detail, for 50% of the teachers' "*time"* is seen as a barrier; only 28% are indifferent and for 22% it seems to be motivating. The aspect "*material"* is also a barrier for 42% of the teachers, 41% are indifferent and 17% are motivated. Concerning "*assessment*" the opinions of the teachers were different, and against our expectations, the aspects of "*excessive demand"* and "*lesson-planning"* were not seen as barriers.

Expected reasons of motivation: For seven scales of the questionnaire, answers were expected that tend towards teachers' (positive) motivation to integrate modelling. Especially the aspect of self-dependence was evaluated very positively: 91% of the teachers voted to be influenced positively, 6% were indifferent and only 3% saw a barrier for it. Concerning the "long-term effects beyond mathematics lessons" even 96% of the teachers are motivated, only 6% voted for indifference and for 1% it is a barrier. Only the scale "differentiation" did not show such high positive effects: 69% rated it as positively, 17% were indifferent and 14% regarded it as a barrier. Looking at the other scales, also "*creativity"* and "*applying mathematics in daily life"* had similar positive values.

Differences between teachers who studied mathematics as a subject or not: In Germany not all primary teachers have to study mathematics as a subject; it depends on the particular state. 28 of 71 teachers have not studied mathematics. The analyses show no big discrepancies concerning the aspects of "*time"* and "*lesson-planning"* and the aspect "*material"* seems for both groups of teachers to realize a substantial obstacle. However the aspect "*assessment"* illustrates considerable differences: teachers who did not study mathematics see here a barrier to teach modelling and for the other teachers "*assessment*" is a strong motivator.

Influence of experiences in teaching mathematical modelling concerning barriers and motivations: A small part of the sample (14 teachers) often uses modelling problems in class, 37 persons only seldom and 20 never. This was a good basis for analyzing differences in the attitudes of the teachers with respect to how often they do modelling activities. Already the aspect "*time"* made it clear that there are substantial differences: Experienced teachers do not see a barrier, but for inexperienced teachers time is a strong argument against modelling. Very similar results can be seen for the aspect "*material"*. Experienced teachers also show a strong motivation concerning the material. A reason is, of course, that they often use such problems and know where to get these. Regarding the expected reasons of motivation, the scales "*long-term effects in mathematics lessons"*, "*applying mathematics in real life"* and "*long-*

term effects beyond mathematics lessons" were a stronger motivator for the inexperienced teachers. We think that experienced teachers recognized the positive effects of these aspects for modelling, too, but for them these aspects were less relevant compared to others.

SUMMARY AND DISCUSSION

Although mathematical modelling is a central competency in the German education standards for mathematics and should thus be a compulsory part of mathematics lessons, a lot of primary teachers still do not have these aspects in their mind and a lot of them are afraid of integrating modelling in their classrooms. For this reason we wanted to investigate barriers and motivations of primary teachers more systematically. The three essential barriers are *material*, *time* and *assessment*. For teachers experienced with modelling, *time* was not such a strong barrier as it was for the inexperienced teachers. One can suppose that the problem of time is rather a prejudice concerning modelling problems and their possible complexity. Probably different teachers set different priorities, because some are willing to invest time for modelling and others do not like to use this kind of tasks. The aspect of *assessment* was a barrier for many teachers. For teachers who did not study mathematics it seemed to be a much stronger barrier than for those who studied the subject. This could be related to their professional education. While teachers who did not study mathematics are fine with tasks which have unique results (right or wrong solution), the specialized teachers are able to assess pupils who work on open problems and so produce multiple solutions. The aspect *material* could not be allocated to a group, but was rated as a barrier for the main part of the sample. Besides the barriers a lot of motivating reasons for modelling could be found, in particular *self-dependence* of the pupils, *creativity*, *long-term effects in mathematics lessons, applying mathematics in real life, long-term effects beyond mathematics lessons*. Also the changing *role of the teacher* was rated very positively.

The results of the study gave new insight into attitudes and opinions of primary teachers for implementing mathematical modelling in their lessons. All answers of the teachers made clear that they have recognized the benefit of modelling problems and modelling activities in general. All the barriers mentioned before can of course be eliminated or at least reduced by suitable professional development activities. With our results, we have now a better basis for meeting the needs of the teachers in such professional development activities.

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PROJECT TEACHING AND MATHEMATICAL MODELLING IN STEM[1] SUBJECTS: A DESIGN BASED RESEARCH STUDY

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In this paper we introduce our concept of a design based research study on the implementation of a new course STEM. The idea is to create a 3-years course for students of lower secondary level that integrates several STEM subjects, e.g. mathematics, computer science and biology. The subjects are not taught separately but are working on the same overall topic, seeing through the eyes and using the techniques of the respective discipline. After motivation and introduction of our concept we explain the real-world problem for the current round of the study and outline some first results and experiences.

Keywords: Mathematical Modelling, Design Based Research, STEM subjects, Project Teaching, Active Learning

INTRODUCTION

Beginning with the first Pisa study in 2000 (OECD, 2001) there is a worldwide discussion of an appropriate education in the so-called STEM subjects[1]. One cause was the alarming fact that in many countries the majority of secondary school students fail to reach proficiency in math and science (Kuenzi, 2008). Further studies showed that a lack of substantial subject matter knowledge of teachers is an important reason. Another trigger for the still growing interest in the discussion of education in STEM fields is the demand of industry for highly qualified young people with a degree in one of the STEM fields.

One significant reaction to the above mentioned debate was to strengthen the role of mathematical modelling in teacher training as well as in the curricula. In Germany, this can be seen from the fact that mathematical modelling is mandatory in most of the recently introduced master programs for academic training of mathematics teachers. Moreover, there is a strong increase in the number of publications on mathematical modelling of *real-world*, *realistic* or *authentic problems* over the past decade.

Definition 1: An *authentic problem* is a problem posed by a client, who wants to obtain a solution, which is applicable in the issues of the client. The problem is not filtered or reduced and has the full generality without any manipulations, i.e. it is posed as it is seen. A *real-world or realistic problem*, is an authentic problem, which involves ingredients, which can be accessed by the students in real life.

[1] STEM = Science, Technology, Engineering, Mathematics (in German: MINT)

RESEARCH QUESTIONS

Along working on a real-world problem we want to study the following research questions:

- How can project teaching in STEM subjects be improved?
- How does project learning and modelling in a 3-years real world project affect the sustainability of the techniques and knowledge used throughout working on the problem?
- How can tablet computers be used effectively throughout such a project?
- How do real-world problems affect the intrinsic motivation of the students to learn mathematics/computer science? Here we use standard instruments as QCM, see e.g. (Rheinberg, Vollmeyer & Burns, 2001)
- How do real-world problems affect Self-Efficiency of the students? Here we use standard instruments as in (Jerusalem & Schwarzer, 1999)

Furthermore we want to investigate the usage of tablet computers in STEM and the dependence on the above topics. Here, the application of tablets is twofold:

- **Tool:** Tablets are used for presentation and to create learning portfolios, as a calculator or measuring device (e.g. oscilloscope).
- **Blackbox:** Software is used to provide complex models and techniques which can not be dealt with at the current students' competence level (e.g. Fourier analysis, smartphone programming).

LEARNING BASED ON NEURO BIOLOGY

Very recently it was shown in Neuro Biology, that the chance for a sustainable learning is not only based on the subject itself but also on an emotional learning environment (no cognition without emotion), see e.g. (Beck, 2003), i.e. if the students have an emotional link to the learning group as well as to topic itself. This is based on the dogma of Neuro Science stating that every process of the brain is resulting from integrated processes of single neural cells, i.e. processes which are linked not only to thinking but also to feelings and acting have more sustainability just because they are more integrated and developed in the brain structure. To obtain these links it is important, that children make their own experiences in as many different forms as possible. In (Hüther, G., 2003) there is mentioned how education and learning processes can be blocked.

An optimal learning effect can be achieved, if one looks carefully at the questions a child is posing and to answer them completely and uniquely. (Singer, W. 1999, p.62).

Moreover learning has to be done with examples not by instruction and preaching (Spitzer, 2002). But it is important that the students have a structural impulse. Here we see parallels to the inductive teaching methods from active learning, see (Beck, 2003, p.5). This is the theoretical fundament of the project presented in this article.

IDEA OF THE JUNIOR ENGINEER ACADEMY "STEM"

Although there seems to be an agreement on the importance of the incorporation of modelling problems into their math lessons on a regular basis teachers face several problems: Often, there is a lack of time since they need to cover the whole curriculum for final examinations. Secondly, many real-world problems are quite complex and do not seem to be suitable for young students (or students at school at all) or as short time projects, respectively. Moreover, many problems cannot be worked on just from a mathematical perspective because they make strong demands on other STEM disciplines – and this is challenging for both students and teachers! Of course, problems can always be simplified, but if this essential step of the mathematical modelling process is taken over by the teacher the students get no training in a very crucial point of the whole process. Later, when working as a STEM professional, nobody will be there to do this first step for them and hence, they will eventually fail to solve their real-world problem.

In summary, it seems to be very difficult to establish a meaningful education in the STEM fields just be modifying the way of teaching for single subjects. We have some experiences from a long-term experiment introducing a significant number of small but realistic modelling projects in regular math lessons (Bracke & Geiger, 2011). But during the study we were facing many of the above-mentioned problems, and this is why we started to make up a new concept for STEM lessons that is currently under evaluation as part of a pilot project between the Felix-Klein-Centre for Mathematics[2] and the Heinrich-Heine-Gymnasium[3] in Kaiserslautern, Germany.

Our concept for STEM lessons

In our pilot we did not just want to develop a new concept for math or physics classes but the idea was to use another organizational structure. Therefore, we chose the Heinrich-Heine-Gymnasium, a high school with a branch for highly gifted students. They already had a compulsory elective lesson called "STEM" which can be chosen by students entering 7th grade. The former concept was to divide the lessons of this 3-years course into three parts: During the first year the students were taught in computer science, during the second year they had additional math lessons and the third year was dedicated to one natural science, i.e. biology, chemistry or physics (in alternating order). Throughout the whole 3-years course the students had three hours of classes per week. It is important to note that the contents of the whole

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[2] The Felix-Klein-Center for Mathematics in Kaiserslautern (http://felix-klein-zentrum.de) is a cooperation of the Fraunhofer ITWM and the Dep. of Mathematics, University of Kaiserslautern.

[3] http://www.hhg-kl.de

course was intended to have no intersection with the standard curriculum of 7th – 10th grades[4]. The main reason to choose this particular school was not the fact that they have a special branch for highly gifted students but the already existing organizational structure. In our opinion it is not a necessary condition to work with highly gifted students: In a similar project conducted by the TheoPrax Centre in Pfinztal, Germany, teachers work with ordinary school classes of grade 8–10 (TheoPrax, n.d.).

Together with the school's administration as well as with teachers of STEM subjects a new course "STEM" was set up: Of course, it is still a compulsory elective course and the time scope is three lessons per week throughout grade 7, 8 and 10. But there is a common topic for every 3-years course and every week there are lessons in mathematics, computer science and one natural science[5]. In the first round starting in 2010 – which is now in its third year – the topic was *Planning Sites of Wind Parks* and besides mathematics and computer science, the students get lessons in physics. The second round beginning in 2011 has the topic *Batteries, Accumulators and Fuel Cells: The Search for the Super-Storage* with chemistry as the natural science. The new topic for the third round is *Bioacoustics – Automatic Recognition of Bird Voices* and as the reader might guess, biology was chosen to be the natural science taught in that course.

Just reading those topics, the whole project sounds quite ambiguous for students of lower secondary level and therefore would be avoided by many teachers at first glance. Our intention was to work on real-world STEM problems that are interesting for the students. The course is not about only learning techniques but about learning concepts and solving problems. The teachers are seen not only as instructors but also as co-workers on the STEM topic. For every course we have a team consisting of regular teachers and "external teachers": The external teachers are graduate students (for computer science lessons in round 1 and 2) and mathematicians for math lessons in round 2 and math/computer science lessons in round 3, respectively[6]. Additionally, each subject has an external consultant, an expert from the University of Kaiserslautern or the Fraunhofer ITWM.

The concept encompasses the regular lessons (three hours of classes per week as mentioned before) and additionally, there are excursions and workshops throughout the whole 3-years course. Examples for excursions are visits of university labs (to conduct own experiments) or other institutes/companies. The regular workshops are

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[4] In the special branch of the school students skip 9th grade by concept, i.e. the lower secondary level consists of grades 7, 8 and 10.

[5] The natural science is chosen to correspond to the overall topic of the course.

[6] The second author teaches mathematics in the second round of the project and both authors do a team teaching of mathematics/computer science/physics in the third round.

starting from *Team Building* and *Time/Project Planning* up to *Creativity Techniques* and *Conflict Management*.

An important part of the pilot is financing: For the first round the Felix-Klein-Centre and the Heinrich-Heine-Gymnasium have been awarded with a so-called Junior Engineer Academy by Deutsche Telekom Foundation. This covers the external teachers, excursions, workshops and some materials/devices that are not in the regular budget of the school. Since the first year was very promising both partners immediately decided to extend the pilot to have at least three rounds, i.e. in total a 5-years period.

Example: Bioacoustics – Recognition of bird voices

Since the topics of the first two rounds in our project were related to physics and chemistry, respectively, the obvious idea for the current topic was to find a challenging project involving biology.

While for the other topics we started from a concrete problem (planning sites of wind parks, optimal storage of electrical energy), the starting point for the current round was the external expert[7], who is a specialist in ornithology and thus has no problem to distinguish between a whole variety of birds just by hearing them singing. For ordinary people, on the other hand, the identification of birds by their voices is a very hard task and most people can only recognize a few very popular species by their sounds. For similar tasks like plant identification where there exist tools – books and smartphone apps – that support amateurs in learning the identification process to a good level.

The recognition tools of birds are mostly books and even smartphone apps[8] that are based on visual features. But currently there is no application that relies on bird voices as the only (or at least main) feature for the identification: The project WeBIRD[9] uses a huge database and claims a high accuracy for the recognition process but the corresponding smartphone app has not yet been published. J. Hammarberg (Alexandria Institute (Aarhus, Denmark)) concludes in a blog report (Hammarberg, 2012) that there is some research on the automatic recognition of bird voices[10] but there seems to be no current approach that can be used for an application based on smartphone techniques.

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[7] Hans-Wolfgang Helb was assistant professor for zoology, ecology, behavioral biology and ornithology at the University of Kaiserslautern for 35 years.

[8] Audubon Birds, BirdsEye and iBird Pro (apps for iOS & Android based systems)

[9] WeBIRD: The Wisconsin Electronic Bird Identification Resource Database. Project leader is biologist Mark Berres, assistant professor at CALS, Madison, Wisconsin.

[10] C.f. (Jankovic and Köküer, 2011) or (Lopes, Lameiras Koerich, Nascimento Silla & Alves Kaestner, 2011)

Thus we chose the topic *Bioacoustics – Automatic recognition of bird voices* for the current round of our project. The goal for the students is to generate methods and implement software to identify birds not by their visual appearance but by their songs. This is an actual problem in ornithology, where the ultimate goal - a smartphone app that simplifies the identification process - would provide two important contributions for clients from ornithology:

- Support for amateur ornithologists and those who are interested in that field.
- Produce a tool for monitoring biodiversity in ecosystems by non-invasive methods.

From the mathematical point of view the techniques required for this project such as *Fourier analysis* and *Hilbert space theory* are far beyond standard school mathematics. Therefore many mathematics teachers at first glance would maybe refuse to carry out the project. The same problem arises in computer science, where students do not at all know techniques as *machine learning, neural networks* and *smartphone programming*.

We started the course with a presentation of the general, complex problem and the expert explained the ornithological background. We also highlighted the ultimate goal of a smartphone app having the features described above as a *nice to have*.

Since this real-world problem is a current research topic, i.e. a client is interested in a solution, we think that the students are more intrinsically motivated to work on this problem and to learn the things they need to know in order to do this (c.f. the section on teaching concepts and design based research). This intrinsic motivation will be one of the authors research topics within this design based research study.

In our opinion the question if such a complex topic can be given to lower secondary level students depends strongly **on the expectations the teachers have**. Moreover, gathering some understanding about the complexity and the exact points which make the problem that difficult would be a big success.

Some comments on introductory lessons and first experiences

In this subsection we describe our idea for and some results from the introductory phase of the course, i.e. the first seven weeks[11]. In the course we have 19 students, 12 girls and 7 boys, who chose STEM as their elective course for the next three years. Although they are highly gifted, this does not imply that everybody is very good in all of the three subjects – but as mentioned before we do not expect this heterogeneity to be a problem at all. Besides working on the problem itself one important goal of the first weeks was to obtain a deeper insight into the social and

[11] Remember that we have three hours of lessons per week (one each in mathematics, computer science and biology).

mathematical or computer science abilities of the class. Vice versa, the students should have a chance to acclimatise in the teaching style of active learning, which was not much used in the years before.

We do not present details on the biology part here since the authors are teaching mathematics/computer science and at least for the introductory phase there is no need for a close ties between the subject. Of course, there is an exchange of information between all teachers on a regular basis. All we want to mention concerning the first biology lessons is that the main goal was to gather some basic knowledge on the anatomy of birds and the way they are able to produce sounds. Later on, it will be very interesting to investigate the role of the singing of the birds and why they are able to sing various and sometimes rather complex songs: Is it a matter of genetic inheritance or of learning from their parents – or maybe a mixture of both?

Concerning the mathematics/computer science part we first have to note that despite having only three subjects in the course we knew that also a lot of physics would be involved right from the beginning.

In fact, in the first lessons the students investigated the physics question: *What is sound?* We therefore decided to replace computer science by physics for the first weeks. Our first lesson started with the four diagrams shown in Figure 1 and the question of their meaning. The diagram was made by the freeware program Audacity.

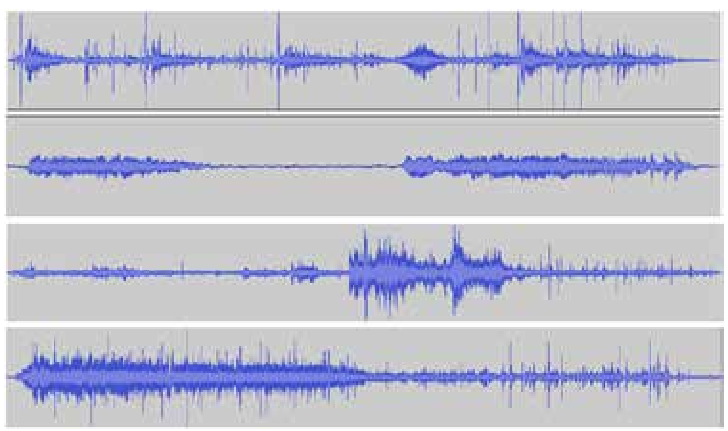

Figure 1: What's that? (Audio tracks visualized with *Audacity (Freeware)*)

The students had an intuitive interpretation and although there are no axis labels they all related these diagrams to sounds. They even gave some qualitative interpretations of the kind of sound they expected behind each graph. This is quite astonishing since of course by labelling and scaling it is possible to assign very different meanings. It would also be interesting if the quasi-symmetry with respect to a horizontal axis (which corresponds to a zero amplitude of the signal in this case) was an implicit factor for the interpretations of the students.

During the first lessons we did a lot of physical experiments to learn more about the nature of sound and the main teaching concept was *learning circles*. A lot of time[12] was spent on working in groups as well as on preparation and presentation of the results. For example, after the first series of experiments the students were asked to prepare a documentary video for each of the experiments. They first had to write their own scripts, then do the corresponding recordings and finally prepare an audio commentary that was mixed together with the video.

The next step was to obtain common definitions for the physical notions of a *tone*, a *complex tonal sound*, a *sound* and a *bang*. To assist the students during this discussion we introduced the oscilloscope as a measuring device[13] and performed some experiments to get more clarity about the characteristics of the different types of sound.

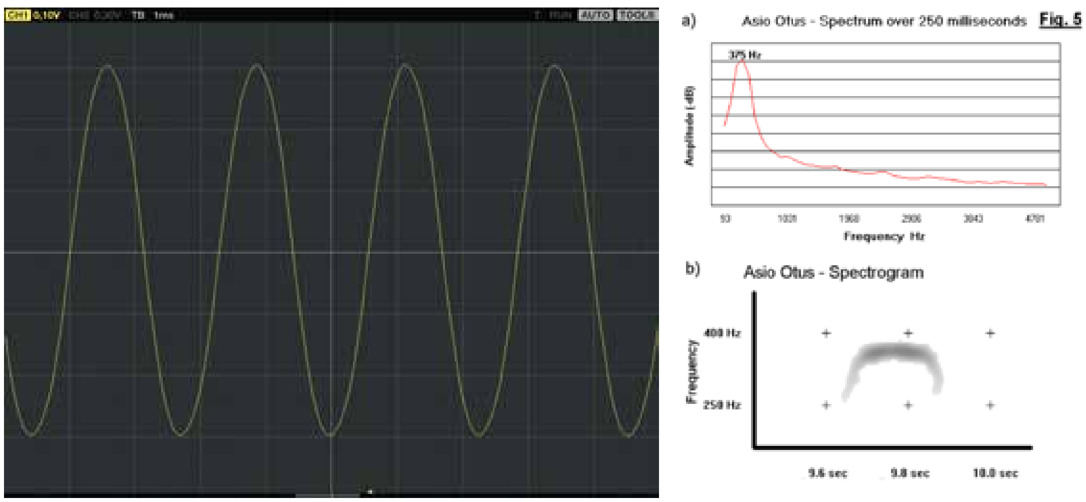

Figure 2: Sounds on the oscilloscope (left) (iPad app *Oscilloscope*) , Example for a sonogram, from http://www.birdsongs.it (right)

Since during their research the students found the terms of amplitude, frequency and wave length we used screenshots of the oscilloscope (where the iPad app *Oscilloscope* was used) (c.f. Figure 2 (left)) for a tone they have actually heard before and asked them for an analysis, guided by some questions like *Where are the zero-passings/minima/maxima of the graph?* They created value tables for each of the features and plotted the corresponding graphs. Of course, this can be done for different tones and the next goal we agreed about is to derive a mathematical characterization for what seems to be the basis for all the graphs we have seen so far.

[12] In a classical physics course the teacher would probably have spent at most half of the time to achieve the same goals.

[13] An iPad with the app *Oscilloscope* was used. The feature to connect the tablet to a beamer was extensively used.

In Mathematics the students learned besides the notion of a sine oscillation how to use proportionalities. This was indeed used to obtain the oscillation length. Furthermore the notion of function and graph was introduced. The students learned how to superpose graphs and obtain the graph of a sum of functions. In a next step this techniques are used to investigate the phenomenon of “anti-sonic”, i.e. the fading of a sine signal by superposing it with the same signal at a different phase.

After that, the computer science part starts, since we have to process and analyse data of sound recordings. One main goal for the first year of our course (concerning mathematics/computer science & physics) is the ability to produce and read so called *sonagrams* or *spectograms*, i.e. frequency-time-plots, that display more information than a simple waveform (amplitude-time-plot). An example can be seen in Figure 2 (right).

TEACHING CONCEPTS AND DESIGN BASED RESEARCH

One important point in our teaching concept is to look carefully at the questions the students are posing and trying to answer them completely and uniquely (cf. Singer, 1999). If necessary, we are going to add experiments or mini-lessons to achieve this goal. Educational methods taking this knowledge of learning into account are about 100 years old and go back to Pestalozzi, who already stated that an effective learning is learning with head, heart and hand. By Vygotsky’s theory (Vygotsky 1978), it is crucial to choose the lessons according to the individual zone of proximal development of the students in order to obtain a sustainable knowledge and to improve already existing skills individually. Of course it is not necessary that the teacher poses problems and questions, also the students their selves can ask for problems they are interested in. In our opinion this will even augment the motivation for learning techniques to solve the problems.

Furthermore in our opinion it is important, that at the end of the project we have an end-product. As in a typical order in an industrial company, there should be a product the students have to construct. This is the main goal, which can of course be manipulated during the period. Towards this, the students claim smaller problems to solve, according to their interest and the needs to the final product. With this setting we try to give the structural impulse mentioned above. The main idea of this project is to use the so-called Design-Based Research model, see e.g. (Collective, T. D. - B. R., 2003), to improve the techniques during the process. We plan first to practice the above-mentioned strategy to the group of students to adapt the methods according to their educational success.

NOTES

1. In German, the common analogue for STEM (Science, Technology, Engineering, Mathematics) is MINT which stands for Mathematik (Mathematics), Informatik (Computer Science), Naturwissenschaften (Natural Science), Technologie (Technology). In our opinion the definition of MINT is too restrictive. Thus we include other disciplines

like geo sciences, social sciences and economic sciences. Moreover, STEM explicitly incorporates the engineering disciplines in addition to technology whereas in MINT they are included in the term technology.

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THEMES FOR MATHEMATICAL MODELING THAT INTEREST DUTCH STUDENTS IN SECONDARY EDUCATION

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In this paper we describe empirical research into the question which possible themes for mathematical modeling interest Dutch high school students. We investigate possible differences between boys' and girls' interests. We also study differences between students in two different streams of the Dutch school system, called havo and vwo. Our study confirms in the mathematical domain the conclusion of international surveys about students' interests in the natural sciences that there are marked differences between boys and girls. However, our study did not find such differences between the havo and vwo streams, although Dutch literature suggested otherwise.

Keywords: Mathematical modeling, high school students, students' interests, empirical study

INTRODUCTION

Modeling is widely seen as an important part of mathematics education. To motivate high school students for mathematical modeling, the modeling context should be appealing to them. Indeed, interest plays a role in determining what people want to learn (Dohn, 2010). Interests that come from the individual itself last longer, whereas interests that are externally induced last for only a short period of time (Carmichael, Callingham, Hay, & Watson, 2010). However, interest is not just a mean. To quote Lavonen, Byman, Uitto, Juuti, and Meisalo (2008, p. 9): "Since Herbart (1841/1965a, 1841/1965b), modern pedagogy has emphasized the value of interest not only as a mean, but as an educational end in itself." Referring to Hidi, Renninger and Krapp (2004), Lavonen et al (2008) write on p. 9: "…interest-based motivation to learn has positive effects both on the studying processes and on the quantity and quality of learning outcomes."

This paper focuses on the question which topics for mathematical modeling high school students are interested in. Whether these topics satisfy other suitability criteria for modelling problems for these students is a different matter, which we did not investigate. However, if one wants to design modelling problems for high school students, then it is useful to know which themes do or do not interest these students. In this paper we restrict ourselves to this question.

The research was carried out as part of a Dutch project, called *Havo-Pro*, which aims to help teachers improve their *havo*-education. '*Havo*' is a stream in Dutch secondary education. It is sandwiched between a large pre-vocational stream called '*vmbo*', which takes about 60 percent of primary school pupils, and a small pre-university stream called '*vwo*', which takes about 20 percent of the pupils. It is meant for students with 'intermediate cognitive abilities'. From now on we will simply use the Dutch acronyms '*havo*' and '*vwo*' to refer to these streams. For more information about the Dutch school system I refer to eacea.ec.europa.eu/education/eurydice.

The third and fourth grades of *havo* are perceived by many educational professionals (teachers, school leaders, policy makers) as problematic (Vermaas & Van der Linden, 2007). Many students in those classes, usually 15 or 16 years old, seem to lack interest, and to under-achieve (Vermaas and van der Linden, 2007), resulting in stagnation. Lack of motivation is identified as the main problem. With our research into *havo* students' interests we hope to help alleviate this motivational problem.

Hamer (2010) addresses these issues for science and mathematics education. (In the Netherlands, it is customary to group the natural sciences and mathematics together under the heading '*bèta*'.) Whereas at *vwo*, more than 50% of the students choose the science, rather than the humanities track, at *havo* less than 30% of the students do so. *Havo* students tend to think that only especially talented students are qualified to choose the science track (Hamer, 2010). Furthermore, these students tend to underestimate their abilities when it comes to mathematics and science. This holds in particular for girls. Van Langen and Vierke (2008) report that only 39% of the boys and 19% of the girls choose the science stream, even though grades indicate that 65% of the boys and 54% of the girls are sufficiently talented. The lack of *havo* students entering technological tertiary education is thus not a consequence of absence of talented students. Schreiner & Sjøberg (2010) report that lack of interest in science among high school students is a widespread phenomenon in developed, wealthy countries, such as the Netherlands. This confirms the relevance of this phenomenon and shows the importance of investigating students' interests.

Additional support to study students' interests about possible modeling themes is given by Ryan & Deci (2000). According to this paper, the main factors contributing to intrinsic motivation are authenticity, competence and relatedness. Modeling tasks usually involve team work, which can contribute to relatedness. Modeling themes that align with students' (daily life) interests can contribute to authenticity and competence as perceived by the students.

As the title of her report indicates, Hamer (2010) lists 10 points of attention for mathematics and science teachers which may help to improve *havo* students' motivation, such as 'working in groups', 'activating students', 'connecting theory and practice', and 'accessibility'. From a constructivist point of view this is hardly surprising. Hamer (2010) claims that these issues are especially important for the *havo* stream. Mathematical modeling problems seem to have the potential to naturally incorporate these recommendations. Other points of attention were concerned with procedural clarity, guidance, and positive feedback. These points are important in the actual design and execution of modeling problems. However, given the motivational problems of *havo* students described above, to design mathematical modeling problems, one first has to address the crucial question which possible topics for mathematical modeling *havo* students are interested in.

Since almost all schools with a *havo* stream also have a *vwo* stream, we investigated both groups at the same time. There is a second reason for considering both *havo* and *vwo* students. As teacher trainers at Delft University of Technology (except for the first author), we wish to develop a module which we can use to train our pre-service teachers how to design modeling problems for high school students. These pre-service teachers will usually teach at both *havo* and *vwo*. Therefore we are interested in *vwo* students' interests, too. Furthermore, it is interesting to investigate which (if any) differences exist between *havo* students' and *vwo* students' preferences. Since Schreiner & Sjøberg (2010) found remarkable differences between boys' and girls' interests, we will also study these differences.

So we arrive at the following research questions for this paper:

1. Which themes for mathematical modeling are interesting to students in second, third and fourth year (students aged 13 – 16) of *havo* and *vwo* ?
2. What are the differences in interests between *havo* and *vwo* students?
3. What are the differences in interests between boys and girls?

THEORETICAL BACKGROUND

Researchers have recently performed an international study into high school students' interests in the natural sciences and technology. This research project is called ROSE, an acronym for 'Relevance of Science Education' (Schreiner & Sjøberg, 2010). To investigate students' interests, the international research team used a questionnaire. Schreiner and Sjøberg (2010) report that there are remarkable differences between countries, but even more striking differences between boys' and girls' interests. Lavonen, Byman, Uitto, Juuti, and Meisalo (2008) used the questionnaire to investigate interests of Finnish high school students. They also found remarkable differences between boys' and girls' interests.

Other well known large-scale studies such as TIMSS and PISA do not measure the affective dimensions of science education. Indeed, as Schreiner & Sjøberg (2010) write on p. 4: "It is a worrying observation that in many countries where the students are on top of the international TIMSS and PISA score tables, they tend to score very low on interest for science and attitudes to science. These negative attitudes may be long-lasting and in effect rather harmful to how people later in life relate to S&T [science and technology] as citizens."

Constructivism also teaches that educators should not ignore the affective dimensions. Furthermore, one should connect to pre-existing knowledge, and align with real world problems students are familiar with and that appeal to them. Hamer (2010) suggests that this is of key importance especially for havo.

Since research into interest is usually done using questionnaires, we decided to use a questionnaire, too. For mathematics education, however, we did not find a suitable questionnaire in the literature, so we had to develop one for ourselves.

METHOD

The validated ROSE questionnaire served as a starting point in the development of our own questionnaire. To generate potential themes of interest related to mathematics education, we looked at the Dutch mathematics curriculum, topics used in national modeling contests *Wiskunde A-lympiade* (Alympiade, 2011), and mathematical games. We also used the results of surveys into trends concerning young people (see Youngmindz, 2011). We used these different sources to minimize the risk of overlooking potential themes of interest. These trends were grouped into gaming, music, technology (gadgets, computers), environmental engagement, money, health, and sports. All our items were related to one of these groups, and we tried to give the different groups equal importance in our questionnaire. Finally, we tried to cover different branches of mathematics, deterministic and probabilistic, discrete and continuous, using computers or not. Exemplary questions can be found in table 3.

The questionnaire consisted of 70 questions, 51 of which dealt with the theme of this paper: students' interests. The first 50 items were of the ROSE type, i.e. questions of the form "Would you like to learn about…?" using a four point Likert scale with extremes marked "no" and "very much". The remaining item was an open question, asking for an interesting topic that was missing in the preceding questions. This question did not reveal new topics, except 'fashion' and unserious answers.

DATA COLLECTION AND ANALYSIS

The questionnaire was answered by 287 students, aged 12 to 18 (M = 14.46, SD = 1.16) from 7 Dutch secondary schools. Since we will be interested in differences between boys and girls, havo and vwo students, we list their numbers in table 1.

	girls	boys	total
havo	53	80	133
vwo	76	78	154
total	129	158	287

Table 1: questionnaire sample

Due to missing values, the data analysis below may use fewer students.

To verify the structure of the questionnaire, we first ran a principal component analysis (PCA) on the 50 items. The overall Kaiser-Meyer-Olkin measure (KMO) was 0.90 ('superb' according to Field, 2009), and all KMO values for individual items were >.79, so the sample size was adequate for factor analysis (Field, 2009). Bartlett's test of sphericity $\chi^2(1225) = 6376$, $p < .001$, showed that correlations between items were sufficiently large for PCA (Field, 2009 Using a scree plot (see Figure 1), we decided to keep 5 components, which together explained 48,2% of the variance. We discarded the 13 items with absolute loadings < .4 (Field, 2009). All Cronbach α's were >.78 and they were not improved by deleting more items, so statistical reliability of the 5 subscales was good.

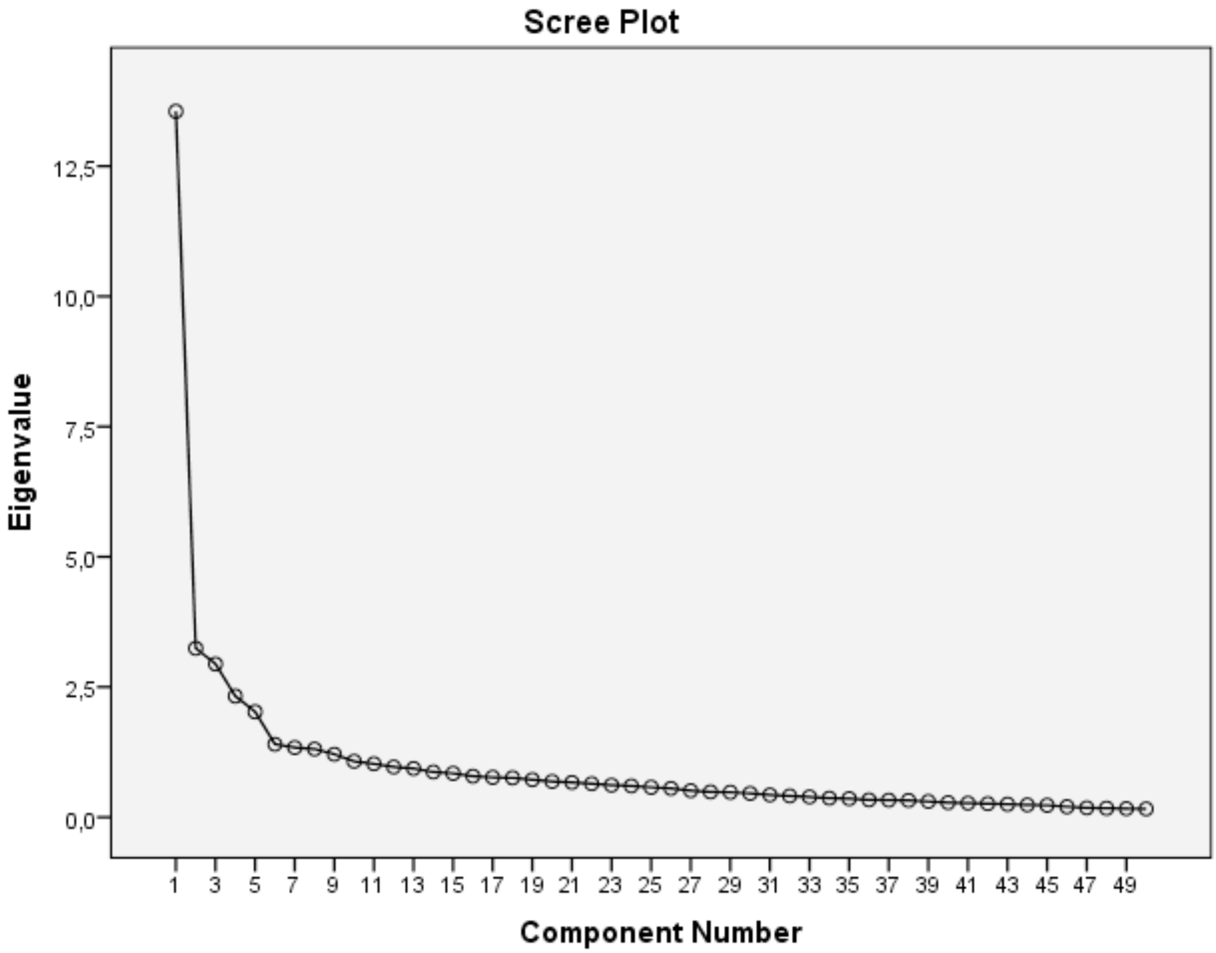

Figure 1: Scree plot of eigenvalues

Looking at the pattern matrix (oblique rotation), the five factors turned out to centre around the themes 'environment and climate', 'sports', 'finance and politics', 'electronic devices and computer games', and 'finance and consumership'. Recall that to set up the questionnaire we used the seven categories 'gaming', 'music', 'technology (gadgets, computers)', 'environmental engagement', 'money', 'health', and 'sports'. 'Health' and 'music' have disappeared, 'gaming' and 'technology' have merged into one factor, whereas 'money' has split into a 'politics' and a 'consumership' version.

The number of items in the pattern matrix for our five factors were 9, 5, 6, 9, and 8, respectively. Interpretation of the first factor was a little problematic, since 4 out of the 9 items did not fit our description 'environment and climate'. Similarly, the third factor 'finance and politics' contained 2 items dealing with voting systems. Finally, the fifth factor 'finance and consumership' contained 2 items about music, one about epidemics and one about lotteries. Fortunately, the dissident items in the third and fifth factor had small loadings, except for the epidemics item in factor 5.

We computed factor scores by averaging the remaining questions for each factor: 5 for factor 1, 5 for factor 2, 4 for factor 3, 9 for factor 4, and 4 for factor 5. In table 2 we give the mean factor scores. Mean scores which are higher than the average 2.5 of the Likert scale are set in boldface. We have performed *t*-tests to compare boys with girls, *havo* students with *vwo* students. For (2-sided) *p*-values < .10 we give the effect score $r = \sqrt{t^2 / (t^2 + \text{df})}$. Effect scores near .1 are small, effect scores near .3 are medium (Field, 2009).

	Mean *N*	*M*(girl) *N*	*M*(boy) *N*	*p* *r*	*M*(havo) *N*	*M*(vwo) *N*	*p* *r*
F1	1.87 281	1.77 128	1.95 153	.02 .14	1.85 129	1.89 152	.62
F2	**2.62** 285	2.44 129	**2.77** 156	.002 .18	**2.53** 132	**2.71** 153	.11
F3	2.26 280	2.01 126	2.47 154	< .001 .29	2.30 130	2.23 150	.42
F4	**2.69** 284	2.49 128	2**.85** 156	< .001 .23	2.60 132	**2.79** 152	.08 .11
F5	**2.60** 283	**2.72** 127	2**.50** 156	.02 .14	**2.54** 132	**2.65** 151	.23

Table 2: Factors and means

Table 2 shows clearly that differences between boys and girls are quite substantial, whereas differences between *havo* and *vwo* students are negligible. Not a single difference has 2-sided *p*-value < .05. There is only one difference between *havo* and *vwo* with $p < .10$, namely 'electronic devices and computer games'. The effect is small ($r = .11$), *havo* students being a little less interested than *vwo* students.

Differences between boys and girls were more pronounced. All factors have 2-sided *p*-values < .05, boys being more interested than girls, except for the fifth factor. However, since the sample was quite large, small differences can already be significant. Indeed, the first and fifth factor have small effect size ($r = .14$). The largest difference with medium effect size (r = .29) was found in the factor score belonging to 'finance and politics'. The remaining two factors 'sports' and 'electronic devices and computer games' are a little more popular among boys (effect sizes .18 and .23, respectively).

The most popular factor is 'electronic devices and computer games' with M = 2.69, especially among boys (M = 2.85) and *vwo* students (M = 2.79). Girls seem to favour the slightly mysterious factor 5, which we dubbed 'finance and consumership'.

Now we have a rather clear picture on the level of our five factors, we return to the individual items. Which items had the highest mean scores, corresponding to highest level of interest? In table 3 we give the top ten items for boys and girls. We indicate which factor (if any) the items belong to. Since we know that differences between *havo* and *vwo* are small, we do not separate between these two groups.

CONCLUSIONS

We found that the themes which interested our students most were 'electronic devices and computer games', 'finance and consumership', and 'sports'. Surprisingly, there were no significant (2-tailed $p < .05$) differences between students in the *havo* and students in the *vwo* stream. At first sight, this seems to be at odds with Hamer (2010), which deals with differences between these two streams. However, that paper deals with *pedagogical* differences between *havo* and *vwo*, not with subject matter. On the other hand, there were significant and sizeable differences between boys and girls. This confirms the findings of Schreiner and Sjøberg (2010), and Lavonen et al. (2008).

Teachers and other developers of teaching materials for secondary schools should be aware of these findings when selecting topics for mathematical modeling assignments. To attune to the needs of *havo* students, they should also take into account the list of ten points of attention by Hamer for *havo* education.

Although we did not investigate this in depth, we expect that the favourite themes mentioned above are suitable for modeling tasks. Indeed, many such modeling tasks have already been designed more or less successfully in the Netherlands and elsewhere. Furthermore, one can use our results to improve existing modeling tasks.

boys		girls	
item	mean	item	mean
Mathematics and the design of videogames (F4)	3.20	What is the probability to win the first prize in a lottery? (F5)	3.11
Computing probabilities in a casino	3.15	What is the cheapest cell phone contract? (F5)	2.94
What is the probability to win the first prize in a lottery? (F5)	3.13	How does an i-pod work? (F4)	2.91
Mathematics and making money (F3)	3.10	How much do peers spend on cell phones? (F5)	2.80
How does an i-pod work? (F4)	3.09	How do pin codes work? (F4)	2.77
What is the most popular game console? (F4)	3.04	How does an mp3 player work? (F4)	2.71
Can one predict world records in sports? (F2)	2.97	Computing probabilities in a casino	2.71
How to develop a game strategy (F4)	2.96	Can one predict world records in sports? (F2)	2.67
The optimal training scheme for a top sportsperson (F2)	2.85	Mathematics and making money (F3)	2.62
How does a computer work? (F4)	2.82	Mathematics, music and noise (F5)	2.60

Table 3: Top 10 items for boys and girls.

For example, in order to improve students’ motivation we modified a modelling task about traffic jams by adding a financial component about the costs involved.

Of course, the design of modeling problems involves much more than identifying topics of interest. For example, which activities are best suited to develop certain modeling competencies at a certain level? Interesting and important as these questions are, they were not part of the research described in this paper.

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STUDENTS' MODELLING OF LINEAR FUNCTIONS: HOW GEOGEBRA STIMULATES A GEOMETRICAL APPROACH

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The use of technology has crucial influences on mathematical modelling. We present a study with a class of 8th graders solving tasks involving linear models, where students regularly used GeoGebra. Our aim is to examine students' approaches to application and modelling situations in the "technology world" of the modelling activity. The results show that students skipped much of the algebra work, and rather chose a geometrical approach, harnessing the affordances of the tool.

INTRODUCTION

The impact of digital tools in mathematics teaching and learning is, to a great extent, concerned with the changes that the use of technology operates in the forms of understanding and approaching mathematical ideas and processes. In many ways technology challenges the traditional hierarchy and disconnection of mathematical topics and it also reshapes the nature and purpose of mathematical representations in doing mathematics. This has clear implications on a learning context that involves exploring and applying mathematical models given "the way technologies can provide multiple connections in mathematics, supporting a student's holistic development of mathematical understanding" (Pead, Ralph & Muller, 2007, p. 318).

Developing and exploring mathematical models with technological tools reveal new sides that go far beyond the idea of gaining more computational or graphical power in dealing with mathematical models. In fact, as some researchers have been suggesting, the well-known modelling cycle needs to be re-conceptualised to integrate a third world – the technology world (Greefrath, 2011; Greefrath, Siller & Weitendorf, 2011; Siller & Greefrath, 2010). Not only the modelling cycle can be augmented to include a third world where the computer model and the computer results are fundamental parts but, most importantly, the impact of digital tools occurs at all stages of the modelling cycle. The formulation of the mathematical model and the computer model are fused together and the same goes for the application of the mathematical model and the implementation of the computer model.

As Greefrath (2011) points out, one of the consequences of using digital tools is the "algebracising" of numerical data. In fact, different available software and computer packages can provide algebraic representations of real data inputs, offer a graphical representation of the generated algebra, and additionally allow establishing connections between them dynamically.

In the case of real problem situations involving linear change, the mathematical models involved are typically associated with the concept of linear functions, including its algebraic formulation, together with its tabular and graphical representation. However, when using GeoGebra students may easily get an algebraic expression of a linear function just by plotting two points of the graph and choosing the tool to create a line through two points; the equation of the line appears as an independent object. Therefore the process of creating and applying a mathematical model may change significantly. The act of building a model is fused together with the computer outputs resulting from entering a table of numerical data and then translating it into other mathematical representations (a graph or an equation).

Given such significant consequences of computer use in students' access to different interconnected representations of linear variation, it is important to investigate how students' approaches to linear models are influenced by the affordances of GeoGebra.

This study focuses on a class of 8^{th} graders developing mathematical and computer models with GeoGebra, and aims to investigate how students' ways of formulating and applying linear models are shaped by the technological tool.

THEORETICAL FRAMEWORK

The emphasis given to particular goals behind mathematical modelling in education has enabled researchers to distinguish among different perspectives (Kaiser & Sriraman, 2006). Accordingly, both a conceptual modelling and a contextual modelling perspective are important roots to situate the theoretical stance of this study. Realistic situations (some of them requiring experimental data collecting) are seen as providing the opportunity to elicit students' broad understanding of linear functions, like constant rate of change and parameters involved in linear graphs. Following that initial basis, contextual problems are seen as opportunities for conceptual development, connecting different aspects of linear models, like relating tables to graphs and to equations in finding answers to particular problem-based questions.

Different theoretical standpoints have been suggesting possible ways of looking at the effects of introducing digital tools in students' understanding of mathematical models. One way of addressing the interplay between modelling and technology focuses primarily on the medium that the modeller is using and stands on the idea of co-action, which acknowledges an interactive influence between the user and the technological tool (Moreno-Armella, Hegedus & Kaput, 2008).

> (…) we introduce the idea of co-action to mean, in the first place, that a user can guide and/or simultaneously be guided by a dynamic software environment (Moreno-Armella, Hegedus & Kaput, 2008, p. 102).
>
> The student and the medium re-act to each other and the iteration of this process is what we call co-action between the student and the medium (Moreno-Armella & Hegedus, 2009, p. 510).

From that point of view, the use of technological multi-representational tools frames students' representational choice (Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009) and this is reflected on their modeling approaches when solving modelling tasks.

Studies on the strategies developed by students in problem solving tasks have shown that they tend to avoid the algebraic treatment of the problem and prefer non-algebraic routes as those based on arithmetic reasoning, on trial and error, working backwards, etc., all summing up to a certain compulsion to calculate rather than to do algebra work (Stacey & MacGregor, 2000).

A similar movement of deferring algebraic approaches is reported by Yerushalmy (2000), where students made intensive use of technology in a function approach to school algebra, through modelling tasks. Based on linear break-even situated tasks proposed to pairs of students in three interviews, separated in time, the study shows how students' strategic approaches to the problems evolved: from graphical representations to predominantly numerical methods, to relations between quantities, to graphs of functions and finally to the algebraic expressions.

> The learning starts with graphical representations of variations, used later on to analyze patterns of numbers by watching the behaviour of the increments, moves on to analysis and construction of relations between quantities that are defined globally, to accurate graphs, and then to explicit expressions (Yerushalmi, 2000, p. 142).

As already stated the technology world brings in more opportunities for students to decide which representational modes they find the most efficient to formulate and apply mathematical models to real situations. Therefore students' approaches to mathematical modelling are likely to evolve within such versatile contexts and to match different levels of mathematization (from the real world to mathematics or within the mathematical world). In such evolving processes of representational decisions co-action between the student and the tool becomes an important concept in that the tool offers an answer (e.g. the algebraic expression) as a result of an alternative representational act from the user (e.g. plotting a graph).

Our study looks at the ways in which students approach linear models in a multi-representational environment, based on an intentionally designed set of tasks, and especially considering how translations are produced from one representation to another as a result of a co-action process.

RESEARCH METHOD

The teaching experiment supporting this study was developed in a class of 8th graders aged 12-14, the majority being 13 years-old, from a public school located in the metropolitan area of Lisbon. The class is characterised by an overall good level of mathematics achievement. The teaching experiment was designed in line with new Portuguese curricular orientations, according to which understanding in algebra topics is to be supported by modelling real situations. A sequence of 7 tasks was developed over a period of one month and took seven lessons with different durations: 3 lessons of 90 minutes and 4 lessons of 45 minutes. All lessons took place in a computer lab where students were organised in pairs, each pair having one computer to work on. In each task students were to formulate and apply a mathematical model within a problem situation by selecting information, interpreting the situation, getting ways of representing it mathematically and applying it to find answers to specific questions about the real context. Some of the tasks required that students engaged in real data collection, either from an experiment performed in the class or outside. All the tasks were conceived to activate the use of mathematics, namely the concepts of linear variation and linear function, in connection to contextualised questions. The tasks evolved from problem situations aimed at eliciting linear models' general properties to more focused problems where the context was explored to stimulate the use and application of linear models for obtaining particular solutions.

Students had acquired an early level of knowledge about GeoGebra previous to the teaching experiment; in general, they were familiar with the Graphics View, the Algebra View and the Spreadsheet View of the software but they were not fully aware of the fact that once they created geometric constructions on the Graphics View the equations were displayed in the Algebra View, as a result of the interactive nature of graphics and algebra in GeoGebra. So the teacher decided to allow students to discover for themselves any additional details of the program while performing the tasks. Students were not bound to use the computer but rather they were free to choose how to solve each task. Accordingly, in the same task, some students could work with GeoGebra, others with paper and pencil, and even others could use both.

For this study, different types of data were collected: the work produced by the students (written records, files created in GeoGebra with access to the construction protocol), the daily log of class observation, and information recorded on audio and video of two pairs of students.

A qualitative methodology and a case study design with strong descriptive and interpretative dimensions were adopted. From each of the pairs that were videotaped one student was elected to become the core of a case. Nevertheless, the other member of the pair and whole class discussions were sometimes integrated in the case.

In the following we present a segment of the case of Pedro, one of the students who belonged to a videotaped group, which in a way reveals exemplar instances of the kind of modelling approaches that took place in the class when GeoGebra was used as a multi-representational tool.

DATA DESCRIPTION AND ANALYSIS

A selection of four tasks is presented for the case of Pedro and his partner, Diogo. The selected tasks (1, 4, 6, and 7 of the sequence) are intended to illustrate the development of students' approaches to modelling over time.

Task A: Water filling

In the class students performed a simple experiment of filling beakers with water from a tap and recording the volume of water after successive equal time intervals. The water flow was changed and the experiment repeated two or three times.

1. In filling the beakers with water what are the variables involved and what is the relationship between them?
2. Find an equation to describe each experiment. Describe how the water flow affects the equation.
3. What is the volume of water after 15 seconds, in each case?
4. How long does it take to fill up a volume of 5 liters, in each case?

Pedro and Diogo entered the collected data organised by columns in the Spreadsheet View and plotted the several points (ordered pairs of time and volume) in the graphical area. Then they created several straight lines in the Graphical View until they found the one that they considered the best fit. The students then realised that the equation of each line was automatically shown in the Algebraic View of GeoGebra.

Pedro: Look, it also makes the equation!

To answer the question of finding the volume after 15 seconds, Pedro and his partner decided to enter the equation x=15 in the input bar and obtained the correspondent vertical line (equations are immediately displayed in the Algebra View and graphed in the Graphics View).

Teacher: What are you doing then?

Pedro: We are making the intersection between this and the other lines.

Teacher: Oh, very well thought out.

Pedro: And we have discovered the points of intersection (in the Algebraic View).

Their next action was to inspect the intersection points appearing in each of the linear graphs and relating the displayed y-coordinates of those intersection points to the volumes of water after 15 seconds, in each case. Following the same reasoning, the two students entered the equation y=5000 (volume in ml) and determined the intersection points between the horizontal line and the linear graphs. By reading the x-coordinates of the intersection points, Pedro answered: Experiment 1 – 435 sec; Experiment 2 – 141 sec; Experiment 3 – 115 sec. Thus they obtained the answer for the time (the x-coordinate) needed to fill the beaker with 5 litres, for each water flow.

Task B: The stack of shopping baskets

On a trip to the supermarket in their extra-school time, students conducted some data collection, measuring a stack of shopping baskets with a variable number of baskets. They measured the height a stack of one basket, two baskets, and so forth. As the number of baskets increased, students recorded the height of the stack on a table.

1. Create a graph that fits the data collected in your table.
2. Find a model to represent the height of a stack of shopping baskets depending on the number of baskets in the stack.
3. What is the height of a stack of 50 baskets? Explain.

In the class, using the information recorded in their tables, they started to create a graph in GeoGebra, showing no difficulties in performing this action. Pedro and his partner began by using the Spreadsheet View and entered the data organised in columns, then marked the points in the Graphics View and obtained a straight line. Instantly, in the Algebra View, they got the expression y=8x+30 (figure 1). At this stage they understood that the rate of growth was related to the height of the outer edge of a basket overlaying the previous one in the stack.

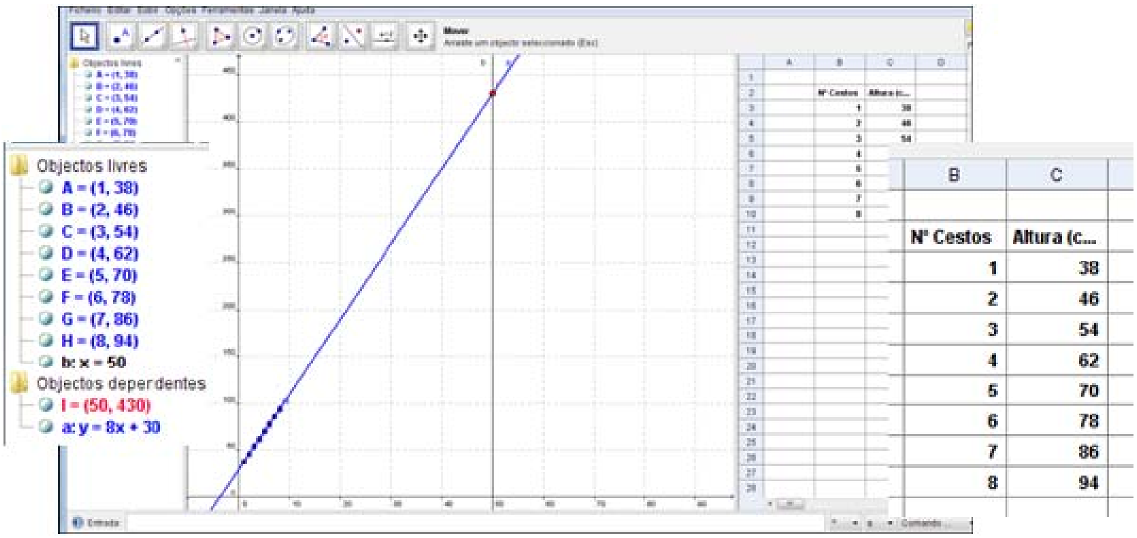


Figure 1: Computer model of the height of the stack (screenshot of students' file)

The question of finding the height of a stack of 50 baskets showed different approaches in the class. Two approaches were used among the several pairs of students. Some took the algebraic expression obtained in GeoGebra and used paper and pencil, assigning the value 50 to x to compute the y. Other pairs chose to work in GeoGebra, entered the equation of the vertical line and by intersecting lines obtained a point and its coordinates. This was the case of Pedro's group who arrived at point I(50,430), as shown in figure 1.

Pedro's explanation on the processes used was given in the following way:

Pedro: But we also found another way to solve the equation.

Teacher: How is that?

Pedro: Well, x was number of baskets, and as the number of baskets had to be 50, we made 8 times 50 plus 30. And we got 430.

Teacher: And besides, how did you do it the other way?

Pedro: It was like this: we draw the line x=50 and intersected it with the line already there and it gave the point where y was 430.

Task C: The cost of having your own car

The problem statement provided some data on the monthly cost of owning a car depending on the distance travelled. Students started by using the Spreadsheet View to enter the given data.

The cost of owning a car depends on the number of kilometres travelled per month. Based on the information published by "Time Magazine", the cost changes linearly with distance, and it is 336 € per month for a distance of 300 km and 510 € per month for a distance of 1500 km.

1. Find an equation that expresses cost versus distance.
2. Make an estimate of the monthly cost to a travelled distance of: 1000 km; 2000 km.
3. Find the maximum distance not to exceed a monthly cost of 600 euros. Explain.
4. Find the y-intercept in the graph of cost versus distance. How do you interpret it?

Then they plotted the two points from the table and draw the line through them in the Graphics View. Again GeoGebra allowed students to have the graph of the function and its algebraic expression simultaneously displayed (figure 2).

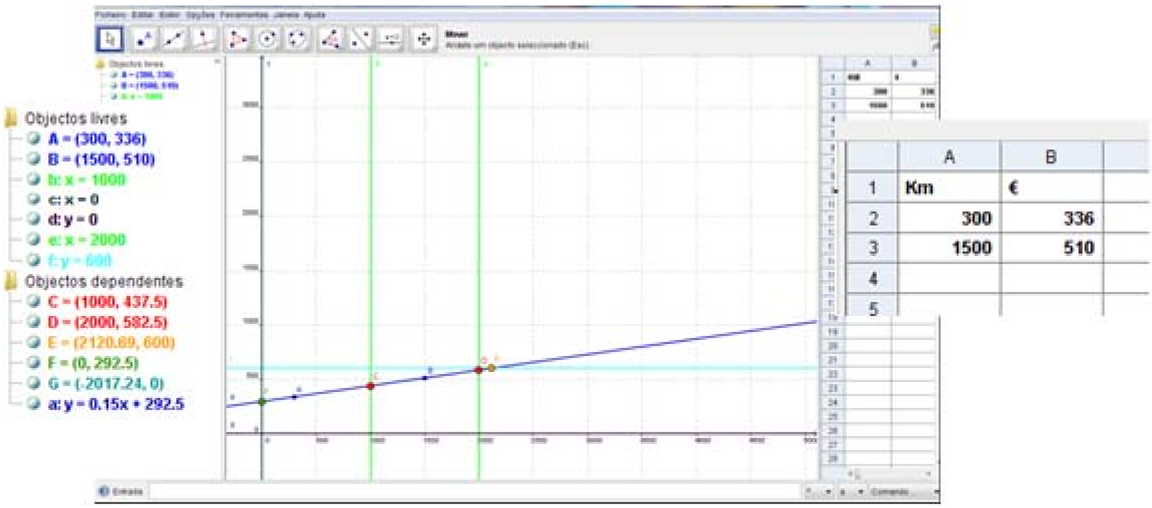


Figure 2: Model of the cost depending on the distance (screenshot of students' file)

To obtain an estimate of the monthly cost in the case of a travelled distance of 1000 km per month, students graphed the vertical line by entering the equation x=1000 and then determined the intersection point of the two straight lines. To calculate the cost for a distance of 2000 km, the procedure was similar to the previous one, leading to the coordinates of another point.

Students were also asked to determine the maximum distance not to exceed the monthly cost of 600 €. Soon they decided to plot the horizontal line of equation y=600 and, by looking for the intersection of the two lines, as happened in the previous cases, they answered to this question (figure 2).

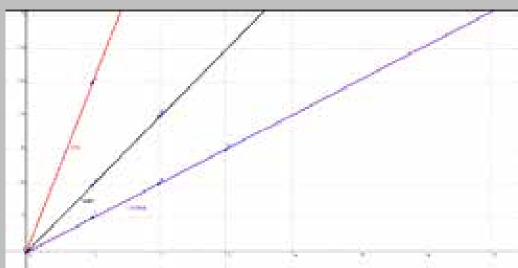
Finally they interpreted the meaning of the y-intercept of the graph in the context of the problem. They conclude that, even if the car was parked (0 km travelled), the owner had to pay expenses (292.5 €). The students were quite surprised at this result and gave it a contextual meaning as shown in the dialogue:

Diogo: It's the taxes, it's for the government!

Pedro: The meaning of this value? It means that I may not drive at all but I still have to pay at least 292.5 euros.

Task D: Speed in typing on a computer

The problem statement on the speed of typing a text on a computer included a graph with three linear functions plotted. To answer the questions, students would have to examine the graphs. They could just use paper and pencil or they could also use GeoGebra to solve the task.

> Ana, Beatriz and Carolina are learning to type a text on the computer. Their teacher tested their speeds and measured the time (in minutes) and the number of words written, obtaining the given graphs.
>
> 
>
> 1. What is the fastest student?
> 2. How many words can each one write per minute?
> 3. How long does it take each of them to write a text with 520 words?

Pedro chose to plot the given graphs in GeoGebra, thus transposing the situation to the computer where he continued to develop his work. Then he explained:

Pedro: So far I plotted the lines shown in the problem in GeoGebra and then it gave me the equations of the lines. Now I will graph the line y=520 and intersect it with the previous lines to know how many words can they… (pause) how many minutes each of them takes to type 520 words.

This way, the student obtained the intersection points and gave the following answer:

Pedro: Ana takes 20.8 minutes, Beatriz takes 52 minutes and Carolina takes 104 minutes.

The teacher questioned the student on the use of GeoGebra in getting the equations for each case and checked if Pedro would be able to do it without the software. Pedro was one of the fastest students in using the software to obtain graphs from tabular data and reading out the algebraic outputs; he was also very alert to the possibility of entering an equation and getting the correspondent graph. So, he moved forth and back from one representation to the other and it was important to see if he was just relying on the tool or if the tool was offering him the chance to see the formal mathematical model embedded in multiple interconnected representations.

Teacher: Now tell me one thing, you chose to use GeoGebra but could you have done it without GeoGebra?

Pedro: If I had not done it with GeoGebra... (pause) then I would have to find the expression.

Teacher: How? For example, in the case of Ana, which is the expression?

Pedro: Well it is a straight line passing through the origin, so the model is y=kx. And as 1 corresponds to 25, we get y=25x.

CONCLUSIONS

Along the sequence of tasks, the formulation of mathematical models developed, in most cases, with the use of the computational tool. Mostly, the modelling process started with some numerical information and students used the Spreadsheet View to convert this tabular information into a graphical representation. The results show that students used the graphical representation of the mathematical models as the main source for developing their analysis of the models, which allowed them, among other things to *avoid* the traditional way of solving equations. The action of the students (plotting a graph) was followed by a prompt from the tool (the equation). This in turn generated new actions from the students: entering formulas to get vertical or horizontal lines. The tool then provided points of intersections and their coordinates; the students looked at the values that were displayed to find solutions for the equations and thus applying the model to find their answers. The way in which the computer was used illustrates an iterative process of co-action between the students and the tool. Moreover there is also action and reaction between the computer model, the mathematics world and the real context in providing meaning for the variables and for the algebraic expressions.

The case of Pedro indicates that, in many situations, the students were able to get the solutions algebraically. Therefore it seems that working with algebraic expressions and solving linear equations was not a strong obstacle to most of the students. The option for the geometrical manipulation of linear models is echoing previous research that highlights students' preference for non-algebraic approaches (Stacey & MacGregor, 2000). In our study, this kind of preference can also be explained on grounds that go beyond potential difficulties in algebraic manipulation. The affordances of the computational tool were assimilated by the students and reflected on their use of geometrical objects (lines, intersections, points, coordinates) to come across alternative approaches for exploring the model.

The geometrical representation became their object of reference in the modelling process and moreover it became a means to obtain the algebraic equation in the Algebra View. This seems to be a good example of the "algebracising" mode of the software used (Greefrath, 2011). In fact, at the beginning students were not actually aware of this utility provided by the software but quickly started to appropriate it in their exploration of the models. Similarly they realised that by entering an equation in the input bar (x=k, or y=k), the geometrical object immediately came up in the Graphics View. Consequently, this automatic translation from geometry to algebra and vice-versa had an influence on students' modelling approaches, namely in their representational choice. Regarding the application of models, the strategies used were essentially geometrical, taking advantage of the possibility of plotting new lines and relating the objects in the Algebra and Graphics Views. This is the main reason why many of the equations involved in the problems were solved from a geometrical point of view, by intersecting lines and obtaining the coordinates of the intersections.

The data reveal how students were guided by and simultaneously guided the computational tool to explore and understand linear models, showing the co-action between the student and the medium (Moreno-Armella & Hegedus, 2009) in the "technology world" of the modeling activity (Siller & Greefrath, 2010).

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TEACHING PRACTICES AND MODELLING CHANGING PHENOMENA

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While much research has demonstrated the positive impact of mathematical modelling on student learning, considerably less research has focused on the teaching practices that are needed to support modelling approaches to student learning. In this study, we examined the characteristics of teaching in a classroom setting where the students engaged in a sequence of model development tasks designed to support their abilities to create and interpret models of changing physical phenomena. The results illustrate the demands that modelling tasks place on teachers, ways of responding to those demands, and suggest needed pedagogical shifts in teaching practices.

Over the past three decades, much research has focused on the potential of mathematical modelling to impact student learning throughout K-16 mathematics. But despite the evidence from research on the positive impact on student learning, progress has been slow in the widespread adoption of mathematical modelling as a classroom practice (Blum & Borromeo Ferri, 2009; Maass, 2011). While many factors influence changes in schooling, one crucial factor in any kind of change in classroom practices is the teacher (Godwin & Sutherland, 2004; Ruthven, Deaney & Hennessy, 2009). But the teaching practices associated with mathematical modelling have received somewhat limited attention from researchers (c.f., Blum & Borromeo Ferri, 2009; Doerr, 2006; Lingefjard & Meier, 2010; Maass, 2011; Wake, 2011; and others). As we have argued elsewhere (Doerr, 2007; Doerr & Lesh, 2011), the knowledge needed for teaching mathematics through modelling and applications appears to differ in some significant ways from traditional approaches to teaching mathematics. In this paper, we examine the nature of teaching practices that support student learning through mathematical modelling, and thereby contribute to an understanding of the knowledge needed for teaching mathematics through modelling. To that end, we investigated the characteristics of teaching in a pre-college classroom setting where the students engaged in a sequence of model development tasks to support their abilities to create and interpret models of changing physical phenomena.

THEORETICAL BACKGROUND

Over the last twenty years, researchers have documented the difficulties that students encounter in learning to create and interpret models of changing phenomena (Carlson et al., 2002; Michelsen, 2006; Thompson, 1994). To address these difficulties, we designed a model development sequence (Doerr & English, 2003; Lesh et al., 2003) to support the development of students' abilities to model changing physical phenomena. This modelling approach to student learning is what Kaiser and Sriraman (2006) identify as a "contextual modelling" perspective, emphasizing tasks that motivate students to develop the mathematics needed to make sense of meaningful situations. Many researchers have used model eliciting activities (MEAs) developed by Lesh and colleagues (Lesh et al., 2000; Lesh & Zawojewski, 2007) to investigate the development of students' conceptual systems (or models) in a wide range of settings and contexts. In this study, however, we moved beyond a single model eliciting activity to design a model development sequence whose goal was to produce a conceptual system (or model) that can be used to make sense of a collection of structurally similar physical world contexts.

A model development sequence begins with a model eliciting activity, in this case, designed to confront the student with the need for the construct of average rate of change. This construct is central to students' abilities to create and interpret models of changing phenomena. The MEA is followed by one or more model exploration activities and model application activities (c.f., Doerr & English, 2003; Lesh et al., 2003). Model exploration activities focus on the underlying structure of the elicited model and on the strengths of various representations and ways of using representations productively. Model application activities engage students in applying their model to new situations, which can result in further adaptations to their model, extending or deepening understandings of representations, and refining language for interpreting and describing the context. In this study, the model development sequence was intended to support the development of the students' generalized understanding of average rate of change and their abilities to create and interpret models of changing phenomena. Throughout model development sequences, students are engaged in multiple cycles of descriptions, interpretations, conjectures and explanations that are iteratively refined while interacting with other students and participating in teacher-led class discussions.

The diversity and complexity of the multiple cycles of the development of the students' models places substantial knowledge demands on the teacher, as teaching "becomes more open and less predictable" (Blum & Borromeo Ferri, 2009, p. 47). As Maass (2011) found in her study, responding to the openness of modelling tasks was especially challenging for teachers with a "static" disposition with its strong focus on final examinations and on teacher-centered pedagogies. The openness of modelling tasks, which is in part a consequence of the diversity of student thinking, leaves the teacher with the need to develop strategies to support the students in making progress with the task, but without directly showing the students how to resolve their difficulties (Lingefjard & Meier, 2010). However, as Lingefjard and Meier note, "it is obviously not enough to ask the teacher to avoid giving a solution to their problem" (p. 106). What the teacher needs is a range of strategies to draw on and, just as importantly, a set of rationales that will enable her to interpret the events of the classroom, select tasks to further the development of students' models, and engage students in the self-evaluation of their models (Doerr, 2007). Such a range of strategies and rationales would provide the basis for responding to students without doing the task for them, or as Blum and Borromeo Ferri (2009) characterize it, for maintaining the balance between providing sufficient guidance for the students while preserving student independence. Characterizing such strategies and elaborating their underlying rationales by the teacher is the focus of this study. To that end, our study was guided by the following question: what are the characteristics of the teaching practices that support students' abilities to create and interpret models of changing physical phenomena when engaged in a model application activity?

METHODOLOGY

This study used design-based research as an approach to studying teaching and learning in the classroom with the intent of contributing to theories of teaching while producing outcomes that are useful in naturalistic settings (Cobb et al., 2003). This design experiment began with the collaborative framing by the researchers and the teacher (third author) of a model development sequence that was intended to support students in developing their concept of average rate of change and in creating and interpreting models of changing physical phenomena. We first describe the model application activity that occurred near the end of the model development sequence, elaborating the intended learning goals for the students. We then describe the context in which the teaching occurred and the iterative cycles of analysis that occurred to understand the teacher's actions in the classroom and her interpretations of those actions.

In the model application activity examined in this paper, students were asked (1) to create a model of the intensity of light with respect to the distance from the light source, (2) to analyze the average rates of change of the intensity at varying distances from the light source and (3) to describe the change in the average rates of change as the distance from the light source increased. The students were given flashlights, meter sticks and light sensors to use with their graphing calculators to measure and collect data of how the light intensity varied with the distance from the light source. Light intensity changes with the distance from the light source at a non-constant rate and can be modelled by an inverse square function. In earlier model exploration tasks, the students had explored the patterns of change in linear and exponential functions. However, the patterns of change for an inverse square function are not approximated by either of these patterns, and hence students needed to draw on other known functions for this context and provide a rationale for choosing a function. Since all of the students had taken a prior course in physics, it is nearly certain that they had studied the inverse square law that applies in this situation. However, we did not assume that the students had investigated or understood or would recall that the reason for the inverse square law in this context is related to the geometry of the sphere. Thus, having students make sense of the relationship to the surface area of the sphere and ways of representing that relationship was part of this model application activity. In addition, students needed to apply their model of average rate of change in a context where the independent variable was not time, but was distance. Finally, students had to interpret negative average rates of change in terms of the light intensity and distance. Overall, this model application task required the students to apply and extend their concept of average rate of change to a new context of changing physical phenomena.

CONTEXT AND PARTICIPANTS

The sequence of model development tasks formed the basis for a six-week course for students who were preparing to enter their university studies. The teacher had four years of experience teaching secondary and college students; this was her third year teaching the summer course. There were 35 students in two sections of the course, all of whom had volunteered to participate in the study. Eleven of the students were female and 24 were male. All students had completed one year of physics and four years of study of high school mathematics; 21 students had studied calculus in high school and 14 had not studied any calculus. Pairs of students completed the model application activity over the course of three lessons. Throughout their work on the task, the teacher led several whole-class discussions that involved students in discussing their emerging models of light intensity and representations of how that intensity changes. The teacher also engaged in conversations with pairs of students as they created their model of light intensity and considered how the intensity changes with respect to distance.

DATA SOURCES AND ANALYSIS

The data sources included videotapes of all class sessions, written field notes and memos, class materials such as worksheets and a record of board work, the teacher's lesson plans and annotations made by the teacher during the lesson. Following each lesson, there was an audio taped debriefing session with the teacher, which captured the teacher's reflections on the lesson and any changes to the plans for subsequent lessons. The model application activity took place over three lessons; each lesson lasted one hour and 50 minutes. Our analytic approach was a collaborative examination of the teacher's actions in and interpretations of classroom events. The analysis of the data took place in two phases. Consistent with the iterative approach of design-based research, the first phase of analysis took place during the six weeks of teaching. In this phase, the research team met with the teacher and regularly engaged in discussion about the model development sequence, the progress of the class as a whole, and our observations about students' thinking about average rate of change and their use of mathematical representations for expressing their ideas. Analytic memos were written by members of the research team to document their emerging understandings of the teaching practices and observations about student learning.

In the second phase of the analysis, members of the research team viewed the videotapes and wrote a detailed script of each lesson, identifying the nature of the teacher's activity and the teaching dilemmas that occurred in each lesson. Following the principles of grounded theory (Strauss & Corbin, 1998), codes were developed to categorize the teaching practices. As we analyzed the practices, we sought confirming and disconfirming evidence in the teacher's lesson plans and annotations during the lesson, and with the teacher's perspective on the lesson from the de-briefing interviews. We present two of the results of our analysis of the teaching practices that supported the students as they created their models: (1) revealing and revising student ideas; and (2) developing and refining representations. We also discuss the difficulties encountered by the teacher in balancing the tension between guiding students and maintaining their independence.

RESULTS

Revealing and revising student ideas

Throughout the model application activity, the teacher engaged the students in revealing and revising their ideas about how the light intensity changed as the distance from the light source increased. This episode occurred at the beginning of the activity. The teacher asked the students about their intuitive ideas on the changes in light intensity, based on their everyday experiences with light. She posed the following question: "Imagine the tail lights of a car moving at a constant speed away from you. Is the light intensity (1) fading at a constant rate, (2) fading slowly at first and then quickly, (3) fading quickly at first and then slowly, and (4) unsure." The students responded to this question using a student response system (also known as voting systems or "clickers"), with the results shown in Table 1. The teacher routinely used the option of "unsure" to discourage students from guessing and to encourage students who see difficulties or ambiguities in a question to continue thinking, without being forced to choose a particular response.

Responses	*Number and Percent Response*
Fading at a constant rate	8 57%
Fading slowly then quickly	4 29%
Fading quickly then slowly	1 7%
Unsure	1 7%

Table 1: Student responses to the rate at which light intensity changes

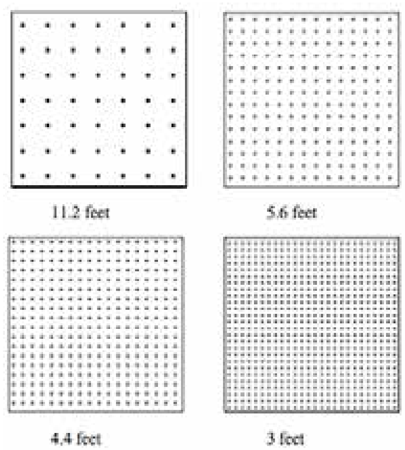
The teacher engaged the students in a discussion about their reasoning and found that students had several different perspectives on this context. One student argued: "no matter how big the light is, you can see it at different distances"; this argument suggests that the light intensity does not change with respect to distance. Several students offered an argument in support of fading at a constant rate by reasoning that "the speed of the car is constant." Others focused on the constant speed of light and reasoned that even though the constant speed of light is different from the constant speed of the car "it's like running in a train" where one can simply add the speeds. However, some students were sceptical about the relationship of the constancy of the speed of the car and of the speed of light to the intensity of the light. One student posed the question: "the car moves constantly, but how do you see the light?" and another asked: "we're talking about intensity. How does that relate to the speed of light?" This discussion was entirely an argument among the students, and revealed their reasoning about how the intensity of light changes at different distances from the light source.

At this juncture, the teacher stepped in, leaving their arguments unresolved, and gave them a task that would enable them to evaluate and potentially revise their ideas. The teacher signalled this as she said: "We are going to sort this out." She gave them data collection equipment that they could use to measure light intensity at different distances from a point source of light. The teacher deliberately did not discuss their ideas further because, as she later commented, she did "not want to give it all away." Rather, she intended for the students to engage in collecting and analyzing data that would enable them to answer this question for themselves. By collecting and graphing data, the students evaluated the alternatives and came to the resolution that the intensity of light decreased at a non-constant rate as the distance from the light source increased.

Developing and refining representations

In keeping with the methodology of design-based research, this model application activity was developed through several iterations. In previous versions, we found that students encountered difficulties in developing meaningful representations of the change in light intensity with respect to distance from the light source. Hence, as we began this modelling task with the students, we explicitly focused on students' images of light intensity. In the lesson following the data collection, the teacher posed a question, analogous to that in the previous episode, but intended to further develop students' representations of light intensity. The students were asked to interpret a "dot" representation of intensity at various distances from a flash bulb and to find the intensity at two unknown distances (see left side in Figure 1). The students initially had difficulty understanding and using this representation. The teacher then introduced the table representation shown on the right in Figure 1. The students recognized that an equation fitting this data would be useful, as one student commented that we "need an equation, but we don't know what it would be."

At this juncture, the teacher polled the students to find out which parent graph they thought would best correspond to the table of data, thus revealing (as in the previous episode) students' ideas about a possible symbolic representation. Most of the students focused on two of the answers, with 57% (n=8) choosing an exponential function, 29% (n=4) choosing $y = 1/\sqrt{x}$, and with 7% (n=1) each choosing y=1/x and y=1/x^2. As before, the teacher asked the students to resolve the question of finding an appropriate equation to fit the data. Using their graphing calculators and working with partners, the students rejected $y = 1/\sqrt{x}$ as a parent graph. Two pairs of students came up with two distinct functions: $y = 1400(1/x^2)$ and $y = 715(0.58)^x + 12$, both of which fit the given data reasonably well. However, this response from the students had not been anticipated by the teacher in her planning and left her uncertain, in the moment of teaching, as to how to proceed.



distance	*intensity*
3	169
4.4	81
5.6	49
11.2	12.25

Figure 1: A dot and table representation of light intensity

Unlike the previous episode, where the teacher knew that collecting and graphing data would enable the students to evaluate and revise their ideas about the changing intensity of light, it was less clear how to engage the students in a critique of these two functions, especially since both functions were a reasonably good fit of the data. The teacher juxtaposed the projection of the graph of each function and the data, and turned the question over to the students, asking "which [function] makes more sense?" Several students saw the exponential function as "more accurate" and one student argued that the graph of $y = 1400(1/x^2)$ would show up in the second quadrant and hence "wouldn't be accurate to the data." Still uncertain as to how to engage the students in a critique of these functions, the teacher re-polled the students as to which parent function would best model the data. This time, 86% (n=12) of the students chose an exponential function and 14% (n=2) chose y=1/x^2. Re-polling the students gave the teacher some additional time to think about how to proceed; during this time she quickly conferred with a member of the research team who suggested focusing the students' attention on the long term behaviour of both functions. The teacher linked the long term behaviour of the function to the students' intuitions that the intensity of light should get "closer and closer to zero as we get out further and further." This led them to reject the exponential decay function, which did not approach zero.

For the teacher, this somewhat partial resolution was critically important, since the inverse squared behaviour needed to be understood as a meaningful and explanatory representation of the change in light intensity with respect to distance, not simply as a "good fit." However, the issue we wish to raise is that knowing how to further the students' own thinking, in the moment of teaching, was neither obvious nor easy from the perspective of the teacher. In this episode, as the teacher ended the lesson, she focused the students' attention on the critical question of *why* an inverse squared representation was reasonable. She said that the "thing I want you to think about is 'why'? Why does this inverse square function make sense in this situation?" To answer this question, the students would need to further develop their ideas about representing how light intensity changes in terms of the distance from the light source.

In the next lesson, the teacher again focused the students' attention on making sense of how light intensity is changing with respect to distance. She began by asking the students about "why it [an inverse square function] would make sense?" and "How do you think about light coming out of a light source?" Several students responded with ideas about light going in "all directions equally," "travels evenly," and "in all directions." The teacher pursued these ideas and asked: "what image do you think of when you think of all directions equally?" One student offered an image of rays: "near the point source, they are really close. But then they go apart. … As they [the rays] get farther from the point source, they get farther from each other. … And that's why the intensity is less." Several other students offered an image of "spheres" moving out from the light source.

The discussion continued as the teacher built on these images, with student generated representations of enlarging spheres and re-visiting the dot-based representation of intensity; this discussion eventually led to the formula for the surface area of the sphere. The students had moved from the dots representation, to a table representation (both shown in Figure 1), to a symbolic representation, to images of rays and spheres, and to the formula for the surface area of a sphere. At this juncture, the teacher was again faced with deciding what to do next. Rather than guide the students through bringing these ideas together, the teacher turned these elements of representing their model of changing light intensity back to students, asking them to think "about all these ideas and put some of this together … One of the questions is why do you think light behaves this way [as an inverse square]?" She encouraged them to use the representations that had been discussed as "ways to reason about that" and thus develop and refine their representations of changing light intensity.

DISCUSSION AND CONCLUSIONS

This study began with the design of a model development sequence intended to develop students' abilities to create and interpret models of changing phenomena. The design of that sequence provided multiple opportunities for students to generate and revise ideas, to interpret and give meaning to various representations, and to reason about changing phenomena. When students are engaged in such modelling activities, teachers are likely to encounter substantial diversity in student thinking, including student approaches that can not be fully anticipated. This places substantial new demands on teachers to respond with strategies that support students in making progress with the modelling task, but without overly directing students in resolving their difficulties. The results of this study highlight two teaching practices in response to those demands. First, explicitly revealing the diversity of student intuitions about changing phenomena provided an opportunity for the teacher to engage students in revising their ideas. Second, supporting students in developing and refining their representations of changing phenomena enabled the teacher to make visible representations that could be useful in providing explanatory descriptions about the behaviour of changing phenomena. In engaging in these practices, the teacher encountered moments that caused tension in knowing how to productively proceed in the lesson without simply directing the students toward some known solution. In both cases, the teacher made an important pedagogical shift from a practice of teacher evaluation of student thinking to engaging students in the self-evaluation of their ideas. Such a practice is well aligned with preserving student independence (Blum & Borromeo Ferri, 2009).

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DEVELOPING A CRITERION FOR OPTIMAL IN MATHEMATICAL MODELLING

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***Abstract**: Optimisation of real-world phenomena with the help of mathematics constitutes one direction of mathematical modelling. Thus, actions carried out while working on optimisation tasks will be located in the modelling cycle. Aspects of the concept of optimal are discussed on a meta-level. In order to identify processes of optimisation, some transcripts of students working on an optimisation task are analysed. Successively changing the perspective on the problem to be solved, starting from an individual and broadening to a collective (or economic) perspective, seems to be a successful approach which leads to an extraction of a suitable criterion for "optimal" in certain cases.*

***Key words**: mathematical modelling, optimising, conception of optimal*

INTRODUCTION

The term "optimal" plays an essential role in everyday life, whether it is advertisement that praises a product's optimal features or practices in life that have to be improved or optimised. So, optimising is daily routine, although often on a subconscious level. Because one is confronted with the word, it is important to be able to reflect on its meaning. This reflection contains asking in what sense something is optimal and – maybe more crucial – for whom it is optimal. Furthermore, it is necessary to know different aspects of the concept of optimal to be capable of optimising professionally or scientifically. Finding optimal solutions for real world problems with the help of mathematics is part of mathematical modelling. When Maaß (2010) lists "objectives linked to the implementation of modelling", especially two items seem to be relevant in the context of a conception of optimal:

- "The students should be able to apply mathematics in their everyday life and their professional life.
- Mathematics is supposed to help students in understanding their world and in critically viewing mathematical information in the sense of active citizenship." (Maaß, 2010, p. 289)

The Duden (2007), a dictionary of German language, describes the word "optimal" as *best possible under given restrictions, with regard to a goal to be achieved* (Duden, 2007, p. 1238; *my translation*). These are also main aspects of the notion of optimal in mathematics. The challenge lies in capturing restrictions as well as the goal and in translating it into mathematics. Modelling tasks which encourage an examination of the word "optimal" should contain questions where neither the restriction nor the goal is obvious a priori. Restrictions, on the one hand, can often be determined by applying assumptions and collecting additional data. On the other hand, the goal to a high degree depends on subjective and intuitive assessment of the situation and on the perspective from which the situation is considered. Therefore, the goal has to be negotiated with participants. For some goals this might be more complicated than for others. For example, cost minimisation can be considered as a clear goal where a consensus is found soon (except from sources of costs). In contrast, striving for maximum fairness can be expressed in many different ways which requires reasonable compromises.

This paper argues that certain modelling tasks can foster the development of a conception of optimal. An analysis of students working on optimisation tasks suggests that there are ways of elaborating a criterion for "optimal" that seem to be more successful than others.

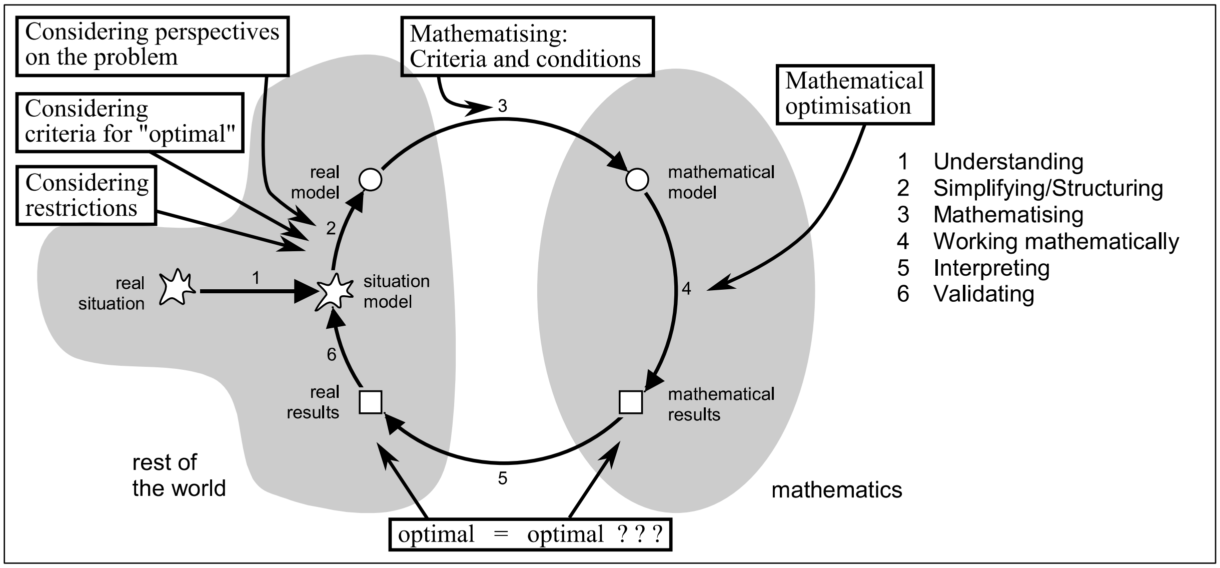
THEORETICAL BACKGROUND

Schreiber (1979) describes *optimality* as a fundamental idea in the Bruner sense. In this context optimality is seen as *property of forms, variables, numbers etc. satisfying a given condition best possibly* (Schreiber, 1979, p. 167; *my translation*). While this highlights the relevance of an attributing (static) meaning of optimisation for mathematics education, Schupp (1992) identifies *optimising* as fundamental and focuses on the procedural (dynamic) meaning. In formulating levels of optimising he emphasises the last level called "meta-optimisation", in which a reflection on optimising with students takes place. This includes besides others the reflection on solving processes, strategies as well as on intra- and extra-mathematical meaning of optimising (Schupp, 1992, p. 114). In the context of the reflection on the best solution method it is hinted that a decision concerning this matter depends on the situation, the criterion, the question what is to be understood as "better" (or best), and on the individual preferences (Schupp, 1992, p. 159). In this sense this perspective on optimisation includes uncertainty and a need for critical considerations of decisions and results which is characteristic for mathematical modelling in general.

Understanding optimising related to real-world phenomena as one direction of mathematical modelling, the actions of optimising can be located in the modelling cycle (see figure 1). Relevant conditions and restrictions have to be identified and structured. These conditions and restrictions can be expressed in a *real model*. But a real model requires even more: Finding a suitable perspective on the problem or weighing up different perspectives as well as choosing a criterion for optimal can be considered as *simplifying and structuring* and takes place in the *rest of the world*. Is there a certain perspective extracted and an objective criterion for optimal is expressed, a *real model* was created. Objective criteria are considered as those, where anybody else would decide on the fulfilment of the criteria in the same way. The translation of the criterion and the conditions into mathematics leads to a *mathematical model*. In the case of linear optimisation, for example, this could be expressed as an objective function and side conditions. Working mathematically – i.e.

mathematical optimisation – provides *mathematical results* which have to be interpreted to get *real results*. Validating concerns the correspondence between real results and the situation (Borromeo Ferri, 2006). This contains the question whether “optimal” in the mathematical sense can be accepted as “optimal” in the real-world sense. The answer to this question may lead to acceptance or refusal of the solution.

Figure 1: Main aspects of optimising as part of the Modelling Cycle by Blum and Leiß



(2006, p. 1626).

In a single optimisation task there are several ways to deal with different perspectives. While above it is described that one perspective has to be extracted, it is also possible to keep all perspectives in order to mathematise them separately. This allows a comparison of the perspectives concerning the results they lead to. Some perspectives may lead to the same result, so that an extraction of one perspective is not necessary.

Especially in steps 1 to 3 and in steps 5 to 6 of the modelling cycle meta-knowledge on optimisation is needed, in the first steps in order to realise the need of finding different perspectives and in the last steps in order to be able to critically question an “optimal” solution in a mathematical sense. Furthermore, results from validating may provoke a shift in perspective which leads to a revised real model.

METHOD

The analysis is based on data which was gained by Busse (2009). In the context of his PhD-thesis four pairs of 16-17 year old German students were videotaped while working on modelling-tasks outside lesson hours. Afterwards each student watched the video record that was showing him or her. In doing so the students were asked to verbalise thoughts concerning the real-world context which emerged while working on the tasks. Both the student and the researcher could interrupt the video record to initiate this verbalisation. This so called stimulated recall was recorded likewise. The transcripts concerning one of the modelling-tasks (see Fig. 2) are analysed here. Although the task does not mention optimisation, it constitutes an optimisation task as the question on the "best position" of the common house is implied.

In a little wood a home for aged people has been built. In the figure the seven residential buildings are marked by black dots. There are paths in the wood so that the aged people do not have to walk through the undergrowth. The paths are marked by bold lines. On the path between the two crossings (marked by a dotted line) a common house is planned to be placed. This common house is meant to serve for afternoon coffee and evening events. The question is where exactly on this path the common house is to be built.

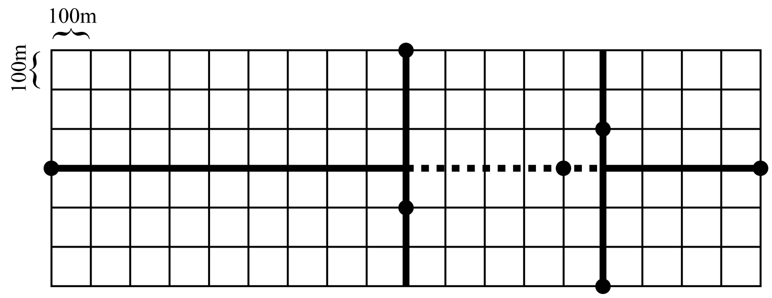


Figure 2: Modelling-task "Home for Aged People" (Busse 2011, p. 40).

The analysis includes three steps:

- Text segments concerning a *certain perspective on the task*, a *criterion for optimal*, a *restriction* or a *solution* are extracted. These aspects can be on an explicit or implicit level.
- The working process is traced by considering the aspects mentioned in the first item.
- An abstraction from the working process is made.

PRELIMINARY RESULTS

As a first approach towards a solution, three pairs of students state that the common house *can be placed anywhere*. The remaining pair expresses negative knowledge by considering that the common house *should not be placed at the right crossing*. Whereas in the first case no criteria for a "good" place for the common house seem to be available, there are hints for such criteria in the second case, although implicit and vague. For example, one criterion could be a low maximum range from the residential buildings to the common house. These different utterances do not permit judging on the student's work but they rather constitute an approach to the task which concludes that there is a lack of information.

In further examination of the problem the four pairs proceed in a very different way. One pair is first irritated by the lack of information and needs hints from the researcher. Then these students intuitively mention a criterion and a solution which they accept without or with an unconscious and intuitive validation in the sense described by Borromeo Ferri (2006). It takes them less than five minutes to finish the task so that the full potential of the task is not taped. The chosen criterion and the matching solution seem coincidental.

The question is what is necessary to accept the working process as suitable, in which way criteria can develop and how these criteria can influence a solution. This is described in the following by having a closer look on another pair's working process. Its members Heinrich and Ingo reflect on different perspectives and matching criteria intensively.

After reading the instruction Heinrich spontaneously states a criterion [1]:

51	H:	Oh. So that everybody has the lowest possible distance.

This focuses on the distance between the residential buildings and the common house and expresses a possible need of every single aged person. In this way the problem is regarded from an *individual perspective*. The fact that a shorter distance for one person could mean a longer distance for another person constitutes a conflict caused by this criterion which is not seen consciously at that time. Heinrich repeating this statement in variation (line 100) a goal comes to the fore soon:

139	H:	But I'm assuming that as little as possible uh- so, that all have [...] a distance that is fair.

So *fairness*, which in the stimulated recall turned out to be seen as equality, dominates the working process from now on. In the stimulated recall Heinrich verbalises his thoughts:

154	H:	[...] that if possible all people will have the same distance- or better, so that, that it is not unfair [...] that not most of the people have a short distance and one person has a very long distance.

This highlights the relevance of the residential house at the left end which will be furthest away from the common house. In order to reduce the longest distance between the residential buildings and the common house Heinrich votes for the common house to be at the left crossing (line 741). Referring to this position for the common house he concludes:

769	H:	Actually this is fairest. Because then the longest distance is nine hundred [pointing at the house at the left end].

This perspective can be called a *group-related perspective* because certain groups – here groups that are disadvantaged – are considered. It constitutes a shift from the individual perspective towards a more abstract one.

Whereas Heinrich is the more active in the promotion of this solution Ingo brings into account a new perspective. The new perspective could be described as a *collective or economic* one. He states:

915	I:	We can also do it so- uh that you uh- test it for all and then you take the sum of the distances to all houses [...] and where the sum is lowest.
927	I:	[...] Look. You sum-up all distances from every house. [...] Distances to the house in the middle. You sum it up. Then you have a sum. And

you make this for every possible position of the [common] house. That means six all in all.

944	I:	[...] Then you compare the sums and where the total distance is lowest-
948	H	There it is best. In fact that's true.
951	I:	But the problem is that this is not fair.
981	H:	No, that's not fair.

This point of view can be described as a *collective or economic perspective* because individual needs remain unconsidered. Instead, it is argued in a more unemotional and economic way. A time-consuming calculation that is carried out by the students results in the best position for the common house being at the same position as the house on the dotted line. A long discussion on whether the one or the other solution is the most suitable leads to the acceptance of the first one. So, the group-related perspective with the matching criterion is chosen as appropriate in this case. In the stimulated recall it is argued (line 955) as indicated in the transcript above that the second solution might be good in other contexts, but in the context of aged people where long distances are covered laboriously it is refused.

Reference back to aspects of the concept of optimal

In the case of Heinrich and Ingo the solving process includes main aspects of the concept of optimal. On the one hand, that means an intensive discussion on the aim of an optimal solution by considering different perspectives, here called individual, group-related and collective perspective. On the other hand, the conditions and restrictions were taken into account thoroughly.

Abstraction of the case: From individual to collective perspective – choosing the "right" one

A natural behaviour in determining which solution works best is to wonder which solution would work best for oneself, assuming oneself being one of the persons affected by the decision to be made. Concerning the *Home for Aged People* task one consequence of this perspective could be:

Everybody wants to have the shortest (or longest – if noise is expected) possible distance between his residential building and the common house.

A conflict occurs when *better* for one person means *worse* for another person. This can be the case in many situations when several people are involved. Therefore, a new perspective is necessary. From the perspective of involved people that are disadvantaged, a group-related criterion is generated which diminishes disadvantages. This criterion can be expressed as:

The longest distance should be lowest possible.

Changing to a more collective and economic perspective may lead to the following statement:

The sum of distances of all aged people should be lowest possible.

Successively the perspective changed from individual criteria via criteria of certain groups to a rather collective and economic criterion where individual needs remain unconsidered.

Solely applying the collective and economic criterion brings a wide difference in distances. One person would have to walk much longer than another. Under the condition of fairness – which is seen as equality – this criterion seems unsustainable for the students so that the group-related perspective is considered as the "right" one.

It should be mentioned, that not one perspective is of higher quality than another per se – it is worth considering the needs of every single person (individual perspective) – but considering many perspectives as a whole constitutes a great level of understanding of the situation and the concept of optimal.

CONCLUSION

Mathematical modelling is challenging for students. Managing the lack of information is one obstacle. Optimising provides further difficulties, for example, the choice for a suitable perspective and the matching criterion for optimal is not clear a priori. The transformation of the perspective from which the pair described looks at the problem suggests that this approach fosters successful optimising and therefore can be used as strategy in the rest-of-the-world-part of the modelling cycle. The transformation of the perspective can be characterised as changing successively from individual to collective needs with respect to existing restrictions.

Working on optimisation tasks like the one described here can foster student's understanding for the need to consider many perspectives on an optimisation problem. This plays a main role in terms of the conception of optimal. And this is what enables students to question models and "optima" declared by others critically.

However, there might be other determining factors and strategies that provide good results. In further studies it should be explored, whether the strategy presented here turns out to be sustainable in other optimising tasks as well and whether other determining factors and strategies can be identified. The impact of student's meta-knowledge about optimising is a question that has to be explored likewise.

Furthermore, sources for the shift of perspectives should be explored. That contains the question on what causes a change of perspective.

NOTES

1. The statements of Heinrich and Ingo are part of the transcripts produced by Busse (2009). They partly have been translated from German into English by the author. For the reason of better readability, pauses and accentuations are not displayed. Names have been changed.

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MATHEMATICAL MODELLING DISCUSSED BY MATHEMATICAL MODELLERS

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This paper examines and discusses how two mathematical modellers work and what aspects of mathematical modelling they emphasise. Based on interviews it was found that they operate differently in terms of work organisation and work tasks. They also emphasised different aspect of modelling, one stressing mathematical aspects and the other focusing on other aspects, like how models are being used in the society.

Introduction

The relevance of using mathematics in and for out-of-school activities, in particular in and for waged labour, is one main argument for teaching mathematics in education (Romberg, 1992). However, the synergy between mathematics used in different workplaces and mathematics taught and learned at school is not always straight forward, but one *major issue* described as an interface between mathematics and a workplace is mathematical modelling (Sträßer, Damlamian & Rodrigues, 2012). Mathematical modelling is described both in curricula and research literature as a link between education and workplace settings.

The Swedish curriculum for upper secondary school emphasizes, in the section of the aim of the subject mathematics, the use of mathematics in relation to workplace situations and to use investigating activities in an environment close to practice (Skolverket, 2011). One investigation activity is mathematical modelling, which is described as one of the seven teaching goals, i.e. to develop students' ability to "interpret a realistic situation and design a mathematical model and to use and validate a model's properties and limitations" (ibid, p. 91, my translation). The descriptions from the Swedish curriculum above indicate the use of realistic modelling activities in the mathematics classroom with a relation to workplaces, at least if the modelling problem is chosen adequately.

One example of educational research literature addressing the issue of modelling and workplace mathematics is the proceedings from the EIMI-study (Educational Interfaces between Mathematics and Industry) conference (Araújo, Fernandes, Azevedo, & Rodrigues, 2010). It includes several papers often related to engineering and modelling. Other examples of research literature that include modelling at a workplace are focusing on what Skovsmose (2006) would call *operators*, like bankers (Noss & Hoyles, 1996), telecom technicians (Triantafillou & Potari, 2010) and operators in a chemical plant (William & Wake, 2007). *Operators* are employees that make their working decisions based on apparatus, technology, with input and output of numerical values, in contrast to *constructors* who develop the technology, and *consumers* who evaluate models used for decisions based on information gathered from reading, watching, and/or listening to statements (Skovsmose, 2006). A common finding related to *operators*' use of mathematics is illustrated by Noss and Hoyles (1996), who investigated bankers' use and understanding of models and modelling. They found that the bankers mainly used computer aided tools with input and output values and the bankers did not consider the underlying mathematical structure of the models they used. The understanding of the mathematical models was preserved to the *rocket scientist*, as the bankers called the *constructor(s)* of the models. However, Noss and Hoyles (1996) did not interview or discuss the situation with the *rocket scientist* themselves, at least not in that paper, which could have been one way to create a communication link between the *constructor* and the *operators.* Overall there seem not to be much attention in research literature focusing on those that are *constructors* in the field and who call themselves mathematical modellers. How do they work? What aspects of modelling do they emphasise? What mathematics do they use? What are their views about modelling? What challenges do they meet in their work? How do they communicate with *operators/ consumers*?

The questions stated above are explored in an ongoing research project, where this pilot-study is a part. The research project will contribute to the ongoing research in mathematics education about modelling and to the understanding and conceptualization of workplace mathematics in that answers may develop new insights into pedagogy and curricula, links between school and workplace, and how mathematical meanings are created in and out of school contexts. This paper will focus on the first two questions above, with aim to examine and discuss how two modellers work and what aspects of mathematical modelling they express as central during their work.

WORKPLACE MATHEMATICS

The goals within the research area of workplace mathematics are several, such as to explore what and how mathematics is used in specific professions (Noss & Hoyles, 1996), to identify discrepancies and similarities between what mathematics is taught in school and what mathematics is needed in the workplace (Triantafillou & Potari, 2010), to analyse communication between *operators* and *consumers* (William & Wake, 2007), and to find strategies that will improve a general curricula that better prepare students for future work (Wake, 2012). What seems to be accepted by educational researchers is that workplace mathematics is not identical to school mathematics. Workplace mathematics is situated dependent and more complex, including specific technologies, social, political and cultural dimensions that are not found in any educational settings (e.g. Noss & Hoyles, 1996; Wedege, 2010). For example the linguistic conventions of representing mathematical models (formula, graph, table) (Triantafillou & Potari, 2010; William & Wake, 2007) are different in mathematics education and in some vocations. Even though the models in a workplace are specific, they offer a potential together with metaphors and gestures to facilitate communication of mathematics between *operators* to *consumers* (William & Wake, 2007). To allow communication about development and validation of mathematical models are also described as “principals for strategic curriculum design” that support workplace mathematics (Wake, 2012,p. 1686). Other principals given by Wake (2012) are: to take mathematics in practice into account; facilitate activities that pay attention to technology; and, to let students criticise mathematics used by others.

METHOLODGY

According to Wedege (2010) a researcher investigating mathematics at workplaces should consider two closely linked approaches, a *subjective approach* and a *general approach*. A *subjective approach* focuses on the workers' abilities and their (subjective) needs in their specific workplace, whereas a *general approach* focuses on (general) demands from the labour market and the society for “formal” (school) mathematical competencies needed in a workplace. A heuristic theoretical model by Salling Olesen (2008), addressing both these approaches, is suggested as a helpful research tool for investigating the dynamics of workplace learning and especially for workplace learning in mathematics (Wedege, 2010). Workplace learning is described “as the process in which individual workers learn by participating in work as a specific activity” (Salling Olesen, 2008, p. 115). An investigation of workplace learning at a specific occasion could be seen as a snapshot of what skills, emotions, knowledge and commitments the worker(s) have developed up to that point in time.

The model is described by Salling Olesen (2008) with the use of a triangle, where each corner is the centre of a small circle (see the figure on p. 119), to illustrate a relation between the three components *the societal work process* (division of labour, type of tasks and work organisation), *the knowledge available* (discipline, craft, methods and skills used in a workplace), and *the subjective working experiences* (individual/collective life history and their subjectivities like values, norms, emotions, etc that appear to be profession specific). Inside the triangle the words experiences, practices, identification and defensive responses are written to illustrate “that learning in the workplace occurs in a specific interplay of experiences and practices, identification and defensive responses” (p. 118). For example, a mathematics teacher may say “this modelling task is useful as a class activity (learned by experience), but it doesn't fit into our school made tests (learned through practice)” and “to calculate the half-life of Caesium we do in mathematics (learned through identification), but to set up models for radioactivity belongs to physics (learned through a defensive response)”. The suggested theoretical model is helpful for this paper, since “the model pays particular attention to the cultural nature of the knowledge and skills with which a worker approaches a work task, whether they come from a scientific discipline, a craft, or just as the established knowledge in the field” (p. 118) and “we can also see general subjects and skill such as literacy and mathematical modelling in this perspective” (p. 124). Modelling used in a vocation may be seen as the craft and the discipline will refer to mathematics. In addition the three components (*the societal work process, the knowledge available* and *the* *subjective working experiences*) may indicate the origin of the given reasons why the modellers emphasise some aspects of modelling more than others.

One appropriate method to capture the complexity of workplace mathematics is to use observations in a workplace together with interviews (Wedege, 2010). For this paper, I have used and developed semi-structured interview questions that pay attention to Salling Olesen's (2008) model and the research aim. The main source used in the construction of the interview questions is the set of critical questions developed by Jablonka (1996) for analysing mathematical models. According to Jablonka, the key aspect when someone is working with mathematical modelling is to judge the quality of the mathematical model. The interview questions are stated in the appendix together with a description of their purpose (to describe **how they [the modellers] work** and examine the modelling **aspects emphasised**) and their relation to Salling Olesen's (2008) three components (*societal, subjectivity*, and *knowledge*).

This is a pilot study and both the participants Adam and Ben (fictitious names) were previously known to me. The interviews were conducted and audio taped in June 2012 and lasted about 40 minutes (Adam) and 90 minutes (Ben) and later transcribed, summarized and analysed based on the categories **how they work** and **aspects emphasised** together with the three components of Salling Olesen (2008).

RESULTS AND ANALYSIS

Case Adam

Adam got his PhD in numerical analysis working with solutions to partial differential equations. He held a post doc position for a year and a half and after that he has worked with climate modelling and aerodynamic problems. Recently, part of his position is situated at a university working to develop new methods to solve differential equations. Mathematical modelling is very central to him, he said. He gets his working problems, with an aim to describe/simulate a reality, from meteorological institute and aircraft manufacturers. Briefly, the problems consist of a set of differential equations, developed by some physicist, which Adam solves by constructing computer programs. The programs consider initial values and constraints and they are used to simulate and compare to real data. The division of labour at his company is constructed after individuals' different abilities (i.e. numerical analyst, meteorologist, geophysicist and computer scientist). They work collaboratively, often in pairs, to understand what the best way is to solve something, but the collaboration is also about everyday problems like how something is going to be delivered. The main tool used for communication is mathematics, "you cannot formulate anything without it [mathematics]", said Adam. Artefacts used are whiteboards, computers or anything that can illustrate and/or simplify the problem to find a solution. He expressed that "mathematical modelling means to translate physics to mathematics" and gave an example about the movement of a pendulum that can be modelled by a differential equation. In addition he expressed that he is not involved in all steps of the modelling work: his competence consists of "translating mathematics to a computer model that will emulate the 'real' mathematical model". Adam mentioned that there are several difficulties while solving the equations like how to represent the move from the continuous model to a discrete model, to make simulations that are both accurate and stable and produce a result that someone can trust. The programming languages used are Fortran, C or C++, but he also uses other ICT tools like Mat-lab and statistical toolboxes in the modelling work. He expressed that most of the programming he has learned in his vocation. Doing climate simulations one needs to know input values, such as how the climate is now, where on the earth you are, how the vegetation is, how fast the earth turns. In order to minimize problematic data they use many different measure series made by satellites, which measure thousands of things. However, measurement errors and techniques are not his field of expertise, Adam said. You can verify and control a model, because you know some expected values, but it is difficult in practice with computer codes to actually get these values. This part he expressed as a very central part of his vocation and a bit frustrating, because the computer programs he writes do not always do what they supposed to do. The results the modelling team produce are predictions, therefore it is not possible to know which solution is correct. The validity of the results is based on historical data and climate trends. Nevertheless, a critical point brought up is that these models are just predictions that the *operators/consumers* need to consider, and there is one unit at the workplace dealing with communication between *constructors* and *operators/consumers*.

The mathematical modelling (the craft) in Adam's work situation seems to be originating from pure mathematics (discipline), in particular the solving of differential equations, and may be considered as intra-mathematical modelling. He has to reformulate the given task in the mathematical domain, select relevant data, translate the mathematical model to a computer model, solve the computer model and interpret and evaluate the result, and finally evaluate the validity of the computer model. Adam's interpretation of modelling work and what aspects he emphasised may be influenced from all three components from Salling Olesen's (2008) model. Maybe most important for his reasons to emphasise the mathematical aspects of modelling is the work task (*societal*) described as differential equations. Other reasons that he expressed were the work organisation (*societal*) with predefined division of labour (numerical analyst, meteorologist etc) and his experience of teaching mathematical modelling courses at the university focusing on differential equations (*societal*). There are things that he has learned through work, i.e. programming and other methods related to ICT, which he expressed as a central part of modelling (*knowledge*). More reasons for his emphasis on the mathematical aspects may be the way they communicate with the use of mathematics (*societal*) mediated by artefacts (whiteboards, computers etc) as a part of their work practice (*subjectivity*). In addition, according to Adam, his description of modelling was similar to those of his colleagues, which may have evolved through their practice (*subjectivity*).

Case Ben

Ben has a PhD in mathematics with a thesis on probability theory. His working experience, where he explicitly worked with mathematical modelling, is wide. He has experienced modelling from a variety of practices, such as a municipality, the military defence, consulting companies and he has worked at different universities. Some examples of work tasks are: constructing water conservation plans (constructing a reality), simulating the interplay between humans and their recourses (simulating/ creating reality), and to develop measurement instruments for identification and estimation problems (constructing tools). His modelling knowledge has not come from general education, he has learned through his vocation, especially programming (Fortran). He argued that one of the strengths of mathematics is that you have a notation that makes it possible to present research and findings in a compact way and to identify cause and effect. Ben described modelling by describing how he worked with modelling. The modelling tasks, he said, are given to him by supervisors or companies. When companies ask for help they have often thought through the problems and want to get help with the mathematical parts. However, he stressed, "as a mathematical modeller one must first make the complete problem clear to oneself, it is not enough with the last part /.../ this process to identifying the problem and formulate the problem is a very long and slow process". He continued to express that he does not necessarily always end up with the same problem as the one given to him. To identify what processes, what variables, and what quantities are needed is important, but most important is to know what type of data exist or can be developed. Also, the consumer's (the company's) purpose must be taken into consideration; otherwise it may be problems to put the paper product into action. Validating is also expressed as an essential part of the modelling work and described as difficult. Ben is a bit concerned that people often draw too far conclusions from their models, especially when models are built only on simulated data, because reality is something else and more complex, "the only positive one can get out of a simulation is if it doesn't work, than it won't work in real life either, but you cannot be sure of that either". However, if the problem is about economy or efficiency of something than it is possible to put the result back into practice and confirm whether it was a saving or not. A result is often one among other results (a maximum can be flat and several values may give almost identical results) and then the consumer has to consider the outcome. Ben expressed that the consumer often wants to have a yes or a no and he needs to explain that the world is not dichotomous. For communication it is useful if the consumers understand the mathematical model, but sometimes they do not understand the model, which may be problematic, especially if they like the result and can use the model. Other times the consumers do understand the model, but they are not interested to control the underlying reasons and assumptions, which may also cause problems. The "misuse" of mathematical models is frequent according to Ben and he gives an example, which he has read recently in a statistical journal and refers to the ad hoc and quasi methodology used in PISA. Ben expressed that he works individually quite often, but that there are regular meetings with the consumers to make reconciliations. A problem with these meetings is that the consumers have to build up a certain body of knowledge for the meetings to be constructive, which may be too much to ask for, he said. Ben also said, that he has become a bit skeptical towards mathematical models used in the society, “one doesn’t solve society problems with the use of mathematical models – they may be used in negotiations by one or the other part”. The best negotiator often wins and the best option is not always picked, but that is democracy. ”You cannot talk about any un-political neutral mathematical models”, he added.

In contrast to Adam’s description of modeling, with an emphasis mainly on the mathematical domain, Ben’s description is wider and including aspects related to non-mathematical issues. Ben stressed the following aspects: to identify and formulate the problem; to identify relevant processes, variables, quantities and existing or none existing data; and validating the model. He also expressed a concern how models are being developed and used in the society and in companies as well as emphasized that communication between *constructors, operators and consumers* about mathematical models is a factor for a healthy democracy. Much of his reasons can be analyzed from different components of Salling Olesen’s model (2008). The design of the working task (*societal*) seems to effect Ben’s expressions. His working tasks are quite general and he needs to clarify and formulate a problem for himself to be able to identify possible variables, and processes, which may be why he expressed these aspects as central. The division of labour (*societal*) plays a part. Ben often works individually (*subjectivity*), which means that he has learned through his practical experiences (working with these tasks), and through communication with consumers and employers, what aspects are valued as important in his community of practice. In line with Adam, Ben expressed that he had learned programming (*knowledge*) and that this was useful for his occupation and a part of the modelling work. The concern about how models are used in society and about the political commitment of modelling in society may stem from an individual (*subjectivity*) conviction based on experience from being a *constructor* (developing models used in the society), *operator* (used and tested colleagues’ models for society) and *consumer* (reading and listening to explanations of models, such as PISA).

DISSCUSSION AND IMPLICATIONS

Both Adam and Ben call themselves mathematical modellers but their descriptions of how they work with modelling at their respective workplace are quite different. Adam works in a modelling group where the members have different specific roles and Adam’s role is to solve well-defined problems (solve differential equations). The division of labour may be one *societal* reason why Adam mainly emphasises mathematical aspects of the modelling and he does not put too much attention to other aspects because it is not a part of his position. Ben on the other hand has a wide experience of working individually (*subjectivity*) with more open problems, which may be a reason for stressing aspects related to non-mathematical issues. An aspect he brought forward is his critical approach to how models are being developed and used in society where the best models according to the modeller is not always the model chosen in practice, since there are more stakeholders that come into play in society (i.e. politicians, companies, negotiators, etc) with their own purposes. Similar and other social aspects of modelling that one ought to consider in mathematics education are elaborated and reflected about by Jablonka (2010). However, there are also similarities identified, for example that both modellers have learned programming at the workplace which was seen as a typical *knowledge* for their vocation, both expressed the importance of qualitative data and validation, and both described communication about mathematical models between *constructor*, *operator* and *consumer* as an essential part of the mathematical modelling. This last similarity identified, i.e. that communication between different practitioners with use of mathematical models is important for mathematics education, is discussed by Wake (2012) and William & Wake (2007). As was discussed above, modelling can function as a link between school mathematics and workplace mathematics. Ben's expressions and the statements about modelling in the Swedish curriculum (see the introduction) highlight the importance of interpreting a realistic situation and to evaluate a model's properties and limitations in modelling work, which means that these aspects also should be emphasised in mathematics education. However, both modellers expressed that validation is difficult in their work, and according to Jablonka (2010) validation mostly is a missing part in classroom practice, because the result is almost never put back into action in out of school settings. Still and maybe more problematic for modelling to be the ideal interface between industry and mathematics education, is the difference in objectives. In industry mathematical modelling is “the gateway into the use of mathematics” (Sträßer et al., 2012, p. 7872) whereas in education modelling is a mathematical classroom activity either as an aim in itself (to develop modelling competencies) or as an aim to develop a broader mathematical ability (didactical tool to learn mathematics) (see e.g. Blum & Niss, 1991). To develop *a* modelling competence, will be difficult to pursue based on this study since the two modellers presented such different descriptions. However, only few students will end up as mathematical modellers and thus the other aim, to use modelling as didactical tool to develop a broader understanding of mathematics, might be more useful for students. Modelling as a didactical tool could be used in teaching about mathematics hidden in technology, which is one aspect of modelling emphasised both in research literature (e.g. Jablonka, 2010; Noss & Hoyles, 1996; Wake, 2012) and in this study as important. One example of activities is to critical analyse mathematical models developed from technology and are used in the society. Not just to develop students' mathematical understanding of technology and to gain knowledge about the importance of communication between *constructors* and *operators/consumers*, but also to develop a critical view of how mathematical models are used in the society, which is an important ability of a critical citizen in a democracy (Skovsmose, 2006).

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Appendix

Interview questions	***The main aim of the interview question is to find:***
1.What is your academic background?	Individual life history (*subjectivity*)
2.What are you working life experiences before you got here?	Individual life history (*subjectivity*)
3.What is your vocation and what role does mathematical modelling play in your vocation?	Individual life history (*subjectivity*) and the institution with its culture (*societal*)
4.What does mathematical modelling mean to you? Make a general description how you work with a modelling problem (from start to end).	Individual/ Institution's view of modelling, both **how they work** and **aspects emphasised** (*subjectivity, societal and knowledge*)
5.Have your view on modelling changed during the years? (If yes) How?	Individual life history, change in **aspects emphasized** and why (*subjectivity and societal*)
6.Who gives you the problems to work with? What are the aims with the problems you get?	Work tasks (*societal*), **aspects emphasized**.
7.How do you work with mathematical modelling in your vocation (by yourself, in groups) If it is group work how/what communication take place? What types of artefacts are used?	Work organisation and communication (*societal*) as well as methods used (knowledge), **how they work**.
8.What type of problems do you work with?	Work tasks (*societal*), **aspects emphasized.**
9.What kind of models do you develop (static/dynamic, deterministic/stochastic, discrete/continuous, analytic/simulations)?	What mathematics/ methods are used (*knowledge*), **aspects emphasised**
10.What are the connections between input and output?	Methods used (*knowledge*), **aspects emphasised**
11. How was the necessary measurement data obtained? Is there a way to control the quality and the origin of the data? Can you give example of values and quantity of the data?	Methods used (*knowledge*), **how they work and aspects emphasised**
12. What factors may have affected the investigated phenomena (measuring instrument or its use)?	Methods used (*knowledge*), **aspects emphasised**
13. Is it possible to control the result? What types of assumptions have been made according to the context? Who decide what assumptions are being important? What is the accuracy of the result?	Methods used (*knowledge*), individual/ institution's view on assumptions (*subjectivity and societal*), **aspects emphasised**
14. How does the solution contribute to understanding and action?	individual/ institution's view on the result (*subjectivity and societal*)**, aspects emphasised**
15. What is an acceptable solution? Who set the goal for the mathematical activity? Who defines the criteria? Are there other solutions?	Methods used (*knowledge*), individual/ institution's view on criteria used/defined (*subjectivity and societal*), **aspects emphasised** and **how they work**
16. Are there any risk to use the result? If so, how is that considered? Is ethical issues discussed?	Methods used (*knowledge*), individual/ institution's view on ethical issues (*subjectivity and societal*), **aspects emphasised** and **how they work**
17. Is mathematical modelling something that was a part of your education in school or something you learned in your vocation?	Individual work life experience (*subjectivity, societal and knowledge*)

INVESTIGATING STUDENTS' MODELING COMPETENCY THROUGH GRADE, GENDER, AND LOCATION

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Abstract

Main purpose of this study was determining the levels of competency of modelling in grade 9th and 10th Iranian students. In this regard we investigate the role of grade, gender and living location in the modelling competency and barrier level. In this study 779 students are participate (366 students from rural location and 413 students from urban places). Students asked to solve a real world problem. Data source was students worksheet and field notes. For analysing of the data, SPSS was used. Especially Mann-Whitney U test was employed. Finding of this study show there isn't a gender difference in mathematical modelling competency. There is no difference between rural and urban students performance, but grade 10 students have better performance than grade 9 students.

INTRODUCTION

Iranian educational system is centralised. Mathematics education goals are set at the national level and the Ministry of Education develops the syllabi and textbooks (Kiamanesh 2005). Mathematics textbooks distribute in all over the country in each school year. All students have same mathematics textbooks. Mathematics teachers asked for using these textbooks in their teaching as a main material. Upon TIMSS advanced mathematics teachers questionnaire, about 95 percent of mathematics teachers in grade 12 use official mathematics textbooks as a main material in their teaching (TIMSS official website).

Recently, a new reform of school mathematics curricula was started in Iran. In the third edition of national curriculum, there are some part about modelling and application. Mathematics textbooks in the grade 9, 10, and 11 started to change from 2008. These new versions of mathematics textbooks were based on the previous version and have similar chapters, although their order is changed. One of the goals of this new version of math textbook was modelling and application. So, writers of textbooks modified new textbooks upon real world application approach.

There is common myth (Rafiepour, Stacey and Gooya, 2012) about application of mathematics between scholars in Iran. If learners have good background in mathematics then they can apply their mathematical knowledge in solving real world problems. Although this myth rejected by several researchers in mathematical modelling community such as De Lange (2003) and Niss, Blum & Galbraith (2007). So we need to do a large scale study for showing that students haven't good results in solving real world problems.

The aim of this paper is studying Iranian students' mathematical modelling competency. Indeed below research questions leading our study:

- Is there any gender difference in mathematical modelling competency?
- What is the role of grade in mathematical modelling competency?
- Are there any differences between students performance on the rural and urban places?

LITERATURE REVIEW

Many countries consider modelling and application in their teaching and learning approaches in last two decade (Niss, Blum, Galbraith, 2007). In Iran educational system, and especially in new version of math curricula, noticed to modelling and application. Although researches upon content analysis of new Iranian math textbooks show that modelling problems are scarce in the textbooks but there are some standard application problems in the textbooks (Rafiepour, Stacey, and Gooya, 20012; Rafiepour, 2012).

When we call modelling approach, we mean a process that started with a problem situated in the extra-mathematical world (EMW) – perhaps a real world everyday problem or a problem from another discipline such as physics or biology. The modelling process continues with formulating the EMW problem in mathematical terms. This is called vertical mathematization by Freudenthal (1991). When this process is complete, the mathematical problem can be solved by the application of mathematical concepts and solution processes. Finally the mathematical solution must be interpreted to provide an answer to the EMW problem, and checked for its adequacy in answering the original question. A new cycle of formulation to improve the model may then begin. In the formulation stage, the problem solver faces a problem situated in a real context or science context, and then gradually trims away aspects of reality, recognizing underlying mathematical relations, organising according to mathematical concepts, and describing the stripped down problem in mathematical terms. In the interpretation stage, the problem solver considers the mathematical result(s), and uncovers their meaning in terms of the real context. In figure 1 a simple diagram of modelling cycle presented.

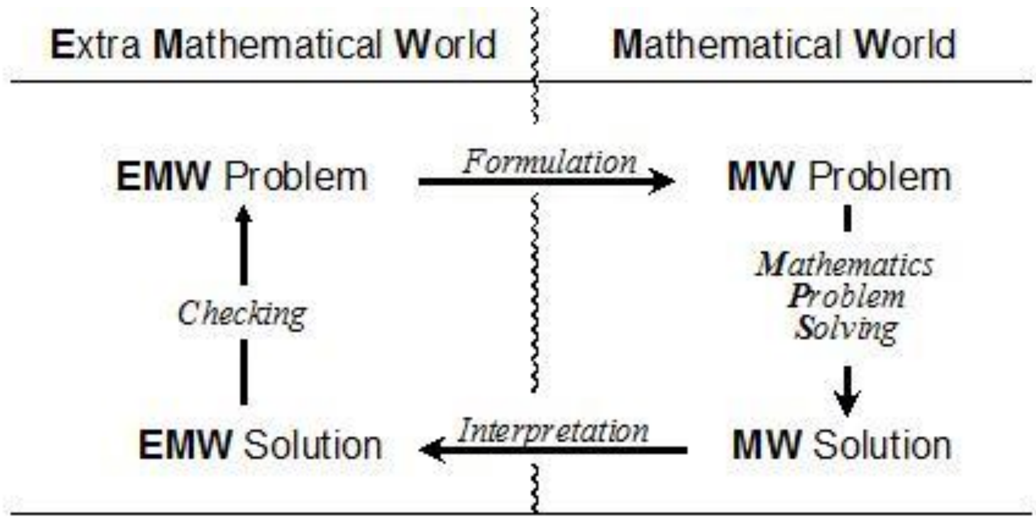

Figure 1: A model of the modelling cycle (adapted from OECD, 2006)

There are several definitions for modelling competency. For example Blum and Kaiser (1997, cited in Maab, 2006) define it by several sub-competencies. Other researchers such as Ikeda and Stephens (1998) address it as an assessment schema. Blomhøj and Jensen (2003) define modelling Competency as someone's insightful readiness to carry through all parts of a mathematical modelling process in a given situation.

However in this study we use Maab (2006, p. 117) definition: "Modelling competencies include skills and abilities to perform modelling processes appropriately and goal-oriented as well as the willingness to put these into action".

For assessing modelling competency, there are several theoretical frameworks. Such as Jensen (2007), that introduced a multidimensional competence-based assessment for Assessing Mathematical Modelling Competency. His model contains 3 dimensions: *Degree of coverage, Radius of action, and Technical level* (Blomhoj & Jensen, 2007). This model isn't as much as easy to use for assessing students' mathematical competency in large scale study. So we didn't use this model as our theoretical framework for mathematical modeling competency assessment.

In this study we use the framework that provide by Ludwig and Xu (2010, p. 80) with six different consecutively levels (Figure 2). Indeed, we need a framework to use it easily for lots of students' worksheet.

- Level 0: The student has not understood the situation and is not able to sketch or write anything concrete about the problem.
- Level 1: The student only understands the given real situation, but is not able to structure and simplify the situation or cannot find connections to any mathematical ideas.
- Level 2: After investigating the given real situation, the student finds a real model through structuring and simplifying, but does not know how to transfer this into a mathematical problem (the student creates a kind of word problem about the real situation).
- Level 3: The student is able to find not only a real model, but also translates it into a proper mathematical problem, but cannot work with it clearly in the mathematical world.
- Level 4: The student is able to pick up a mathematical problem from the real situation, work with this mathematical problem in the mathematical world, and have mathematical results.
- Level 5: The student is able to experience the mathematical modelling process and validate the solution of a mathematical problem in relation to the given situation.

Figure 2: six level for assessing mathematical modelling competency

METHOD

Participants

In this study we use two step cluster sampling. At first stage we identify all districts of Kerman (one of the south provinces in Iran). Then we determine urban and rural district via official document produced by educational Administration. Finally in each district we determine one school and collect data from selected classes in grade 9 and 10. In sum 779 students from 33 classrooms were participated in this study. 413 students were female and 366 students were male. 366 students were from rural places (17 classrooms) and 413 students were from urban places (16 classrooms). All of these students were in grade 9 and 10 (15 and 16 years old) in Kerman. Details of participants show in table 1. These students haven't any special teaching toward mathematical modelling before the test. They just familiar with some standard application through the new version of Iranian mathematics textbooks in grade 9 and 10.

	Gender	Grade 9	Grade 10
Rural	Boy	121	59
	Girl	120	66
Urban	Boy	120	67
	Girl	83	143

Table 1: participants of this study

Data Collection Instrument

For data collection, we use a real world problem that involve to pealing a pineapple. This problem get from Ludwig and Xu (2010, p. 80). We use only one problem for data collection because of time limitation. Pealing a pineapple problem choose for our study because our pilot study shows that this problem was interesting for most of students.

In the data collection session, we go to the classroom and show a picture about pealing the pineapple as below and ask students to explain that Why does the salesman peel the pineapple in this way? Is there any mathematical way for describing salesman manner?

Figure 3: method of salesman for pealing a pineapple

All of the Students activities were videotaped then transcribed. Students' worksheets and field notes of researchers were considered as a data set in this study.

Data Analysis

For analysis of the data two mathematics teachers code all students worksheet separately upon figure 1 schema. In this process, coders use film of the students' activities and filed notes. Degree of agreement between coders was reported more than 85%. Indeed each coder read every student (779) response and decides about level of competency for each student upon table 2. There are only a few disagreements (about 80 cases) between two coders to allocating level of competency for each student's response. For responding to the research questions we apply Mann-Whitney U test for two independent groups.

RESULTS[1]

The level of students modelling competency in rural and urban places in grade 9 and 10 reported in table 2 and 3.

Level of competency	Grade 9		Grade 10	
	N	%	N	%
Level 0	94	39	21	16.8
Level 1	89	36.9	37	29.6
Level 2	44	18.3	53	42.4
Level 3	12	5	7	5.6
Level 4	2	0.8	7	5.6
Level 5	0	0	0	0

Table 2: The level of students modelling competency in rural places

Level of competency	Grade 9		Grade 10	
	N	%	N	%
Level 0	66	32.5	53	25.2
Level 1	80	39.4	76	36.2
Level 2	56	27.6	67	31.9
Level 3	0	0	2	1
Level 4	0	0	3	1.4
Level 5	1	0.5	9	4.3

Table 3: The level of students modelling competency in urban places

CONCLUSION AND DISCUSSION

Gender difference is one of the components in equity issue. In the pilot study we confronted with different performance in the case of boys and girls in solving real world problem. So we concentrate on gender issue as a first research question. In the first research question we investigate the role of gender in the mathematical

1 -Output of SPSS omitted because of the space limit.

modelling competency. Upon output of SPSS, at the 0.05 significance level we couldn't find gender difference between students, thus our hypothesis about gender difference was rejected. Although we saw there are some sort of gender difference in solving real world problem, but for discovering this differences we need more in-depth qualitative research.

In the second research question we search for finding a role for grade of students in modelling competency. Indeed we would like to address the impact of grade in modelling competency. Upon output of SPSS, at the 0.05 significance level we find, students in grade 10 was better performance than students in grade 9. As we seen, students in grade 10 have better performance in real world problem solving than grade 9 students. This result is in the line with Ludwig and Xu (2010) results.

In the previous pilot study, we observed that students who living in rural places have better conception about real world problems, therefore we decide to examine this hypothesis. In the third research question we investigate the role of place of living (rural/urban) on mathematical modelling competency. There is no statistical significant difference between rural and urban places in mathematical modelling competency. Indeed rural and urban students have similar performance on solving real world mathematics problem.

Upon new reform in mathematics curricula and mathematics textbooks in grade 9, 10 and 11 in Iran, and considering mathematical modelling and application in new series of textbooks, it seems to be necessary to investigate Iranian students' performance in mathematical modelling competency. Current study examines Iranian students' performance in solving a real world problem and determines their level of mathematical modelling competency. This finding could be use as a base for doing further research in Iran and comparative study with other countries. Descriptive results show weak ability of Iranian students' modelling competency; although there are a few students that can achieve level 5 of mathematical modelling competency. These students are able to experience the mathematical modelling cycle completely and validate the solution of a mathematical problem in relation to the real world situation. This is a good point. Indeed we are in the start point of a long way; there are lot of things to do.

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SOLUTION AIDS FOR MODELLING PROBLEMS

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One option to help students to process modelling problems is the use of solution plans. Some of these solution plans will be introduced. As part of a qualitative study a solution plan in connection with a modelling problem was used in Grade 6. The students were observed and interviewed during it. The assessment strongly shows differing work processes but comparable written solutions from the students dependent on the solution plan.

MODELLING AS COMPETENCY

Mathematical modelling is one of the six mathematical competencies that are accounted for in the German educational standards for mathematics. Students should acquire the skills, based on diverse mathematical content, to translate between reality and mathematics in both directions. Due to the high significance of this competency for classes, solution aids for students when working on modelling problems in math class will be discussed in this article. That is why the competency of modelling will be briefly introduced in the following.

The core of mathematical modelling was already described by Pollak (1977) as interplay between mathematics and the "rest of the world" (see Fig. 1).

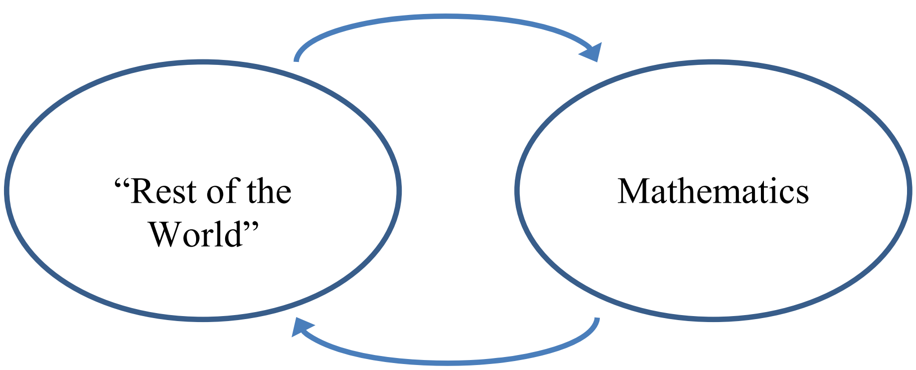

Fig. 1: Mathematics and the "rest of the world" (cf. Pollak 1977)

Modelling competency is described more precisely in Blum et al. (2007) as the ability to perform the respective required process steps while switching back and forth between reality and mathematics adequately in regard to the problem as well as to analyse given models or comparatively assess them. Modelling cycles (similar to that in Fig. 1) describe the different sub-processes of modelling activities in different detail and with different perspectives (Kaiser & Sriraman, 2006). A selection of these so-called partial competencies is listed in Table 1.

Partial competency	**Indicator**
Understanding	The student constructs her/his own mental model of a given problem situation and thereby understands the question.
Simplifying	The student separates essential and un-essential information of a real situation.
Using mathematics	The student translates appropriate simplified real situations in mathematical models (i.e. term, equation, figure, diagram, function).
Interpreting	The student relates the results obtained in the model to the real situation.
Validating	The student checks the results obtained in the model based on the real situation.

Table 1: Partial competencies of modelling (a selection)

The conscious division of modelling into partial processes is a possible way to reduce the complexity for those teaching and those learning and to set up suitable problems. Such a view of modelling especially makes it possible to train individual partial competencies in a targeted way and thus to construct extensive modelling competency. Also the view of the partial processes of modelling can be used to create solution plans for students and to thereby make solution aids for processing modelling problems available. Sjuts (2003) describes activities of planning, monitoring and checking, that are also initiated by solution plans, as *procedural meta-cognition*.

SOLUTION AIDS

Solution plans can aid the processing of modelling problems. These plans are used in the context of strategic aids and not in the sense of programmed learning. Blum (2006), for example, developed a solution plan for students as part of the DISUM Project that is based on a simplified model building cycle (see Fig. 2).

This solution plan contains four steps called *understanding the problem, creating a model, using mathematics* and *explaining results.* Every step is explained to the student with a question and some clarifying points.

Blum’s solution plan belongs to the so-called indirect general strategic aids because although he does refer to general specialised modelling methods he does not give any concrete assistance that is based on the tasks in steps 2 and 3 of this solution plan and the commonality of the strategic aid is abandoned in favour of content-oriented pointers. Because of the pointers to equations and the Pythagorean Theorem it is a matter of a content-oriented strategic aid. This solution plan can be prepared for students. Its usage can also be practiced with the help of example problems.

1	Understanding the problem	What is given, what is sought?	Read the text precisely Imagine the situation exactly Make a sketch
2	Creating models	Which mathematical relationships can I establish?	Fill in missing entries, if required, i.e. set up equations or plot triangle
3	Use mathematics	How can I solve the problem mathematically?	I.e. work out the equation or use Pythagorean Theorem, write down the mathematical result
4	Explain the result	What is my end result? Is it logical?	Round off the result and relate it to the problem – possibly back to 1, write down answer

Fig. 2: Solution plan for modelling problems (Blum, 2006)

In a study by Schukajlow et al. (2010) as part of the DISUM project, significant differences in student achievement in modelling was verified using this solution plan in regard to the Pythagorean Theorem area of content. The class with the solution plan proved to be the more effective form of teaching and learning. In addition, the students in the solution plan group were also more aware of using cognitive strategies, in other words the solution plan.

A shorter solution plan is used by Zöttl & Reiss (2010) in the content area of geometry. This is reduced to three phases, namely

- Understanding the task,
- Calculating,
- Explaining results.

As part of the KOMMA Project, completed solution examples were used in addition to the solution plan above from Zöttl & Reiss (2008) that consisted of a problem and the description of the solution steps. In the area of mathematical justification and verification positive effects could already be ascertained when such solution examples were used (Reiss & Renkl 2002).

An alternative solution plan can be found in Greefrath & Leuders (2013). In the set up of this solution plan the problem solving steps of Polya were taken more strongly into consideration than in the plans cited above. In his book *How to solve It* Polya developed a catalogue of heuristical questions that are supposed to help process problem solving tasks. Here the problem solving process is divided into the following sections (Polya, 1973): Understanding the problem, devising a plan, implementing the plan, review. Schoenfeld (1985) follows up on it and describes certain steps in more detail. At the end of the problem solving process he differentiates between verification and transition. The proposed solution plan for learners at the start of secondary school therefore contains five steps and can be used for modelling as well as for problem solving tasks:

- Understanding the problem: Formulate it in your own words.
- Choose the approach: Describe assumptions and plan the calculation method.
- Performing: Perform the calculation.
- Explain the result.
- Checking: The result, calculation and approach.

The solution plan is similar to problem solving activities in order to use it more frequently than a solution plan specifically for modelling problems. Because every modelling problem is also a problem, the strategies suitable for problem solving are also helpful for modelling.

Certain authors also use a simplified modelling cycle as solution aids for the students. As part of a qualitative study in Grades 7 and 8, Maass (2004) studied the modelling competence of students and by the end of the study could reconstruct proportionate meta-cognitive competencies in a large percentage of the students. A result she also describes is that the students sensed the knowledge of the modelling process and the depiction of the cycle as an orientation aid.

But disadvantages to such solution plans have also been named. Meyer and Voigt (2010) lead the way saying that a solution plan dependent on a modelling cycle with a structured formula for processing practical calculation problems from the 1960's and 1970's can be compared, where they were offered to students as alleged solution aids and turned out to be additional learning material.

Another option for a solution aid for processing modelling problems that does not address solely the modelling process but can also be used for general problem solving situations is asking and answering indicative questions or simpler questions (Greefrath & Leuders, 2013). Here the students learn to ask and answer questions about the modelling problems. Two goals can be achieved hereby. Firstly, it is easier to recognise which information the text in the problem really supplies – possibly even information necessary to solve the problem – and which must be procured in a different way. Secondly, the modelling problem to be processed is disassembled into partial steps that can initially be processed individually reducing cognitive load (Sweller, 1988) before the partial results are then put together to solve the problem. Later the students can ask themselves such questions and decide whether they can be answered with help from the text.

STUDY DESIGN

As part of a qualitative study up to now three pairs of students from 6th Grade of a secondary school were observed while working with the solution plan from Greefrath

& Leuders (2013) and subsequently questioned. A qualitative study was chosen to get information about the processes while working with the solution plan. Of course the generalizability of such a qualitative Study is low, but the goal was to get information about the processes in detail. Up to now there are no emprical results on students working with this specific solution plan.

The students' activities while working on the task and the subsequent interviews were filmed. Until now three such interviews were evaluated. The students worked on two problems one after the other. The first problem served to understand the solution plan and to put the given steps into the right order. The second problem consisted of using the solution plan. For this the above solution plan from Greefrath & Leuders (2013) was presented together with the following problem:

> Work out the following problem according to this solution plan: What amount of liquid do I drink every week?

The videos of the observations and interviews were completely transcribed and evaluated following Grounded Theory (Strauss & Corbin, 1990). Here the transcripts were worked through line-by-line and the individual lines of text openly coded. In this way categories were developed which were then used to evaluate the results. The coding was done by two independent persons in order to achieve highest possible interrater reliability. (Hilmer, 2012)

RESULTS

After viewing the openly coded lines of text the following categories were developed that were then used to code the interviews:

- Orientation on the plan and example
- Fulfilling the requirements of the plan
- Difficulties in implementing the plan and example
- Incorrect or incomplete conversion

The fulfilment of the requirements of the plan will be observed in more detail in the following. Also the individual five steps of the plan were individually coded as part of this category and used for the following analysis. These are the first empirical results for this special solution plan.

Interview 1:

This tandem is strongly oriented on the given solution plan but they especially considered the concrete example of the first problem. The students mixed up the phases *Choose approach* and *Perform* and the test persons oriented themselves alternately on the question, example and plan. Although the plan did encourage meta-cognitive processes, it did not supply optimisation of problem processing. Figure 3 depicts the phases of processing of this pair of students.

PHASE	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Understand the problem	x					x									
Choose approach			x		x			x		x			x		
Perform		x		x					x			x			
Explain result											x			x	
Check							x								x

Fig. 3: Solution phases in Interview 1

The mix-up of the second and third steps of the solution plan becomes clear in figure 3. Even though the observation shows that both students could not orient themselves strongly on the given plan, they still answered the question as to whether the plan had helped them:

> "I thought it was good that I could always look there. If I had to do it by heart I think it would be more difficult. This way I had a kind of comparison"

Interview 2:

This pair of students oriented themselves clearly on the given solution plan. Difficulties only came up partially in the description of the corresponding phases. A content error occurred in the last processing step. Here the necessary plausibility observation does not succeed. Figure 4 shows the phases passed by this pair of students and shows a definite difference to Interview 1:

PHASE	1	2	3	4	5	6	7	8
Understand the problem	x							
Choose approach		x		x				
Perform			x		x		x	
Explain result						x		
Check								x

Fig. 4: Solution phases in Interview 2

This pair of students shows the sequence that should be given by the solution plan much clearer. Interestingly, one of the two students pointed out in the subsequent interview that he sees a connection with the known solution aid for written problems: Question, calculation, answer. Altogether the use of such a solution plan was seen as being very positive by the students. The written solutions of the pair of students clarified the clear sequence and the maintained structure of the solution plan.

Understand the problem: How much do you drink in one week?

Choose approach: I drink ca. 2 litres a day. I would calculate 2 x 7.

Perform: 2 l x 7 = 14 l

Explain result: I drink ca. 14 l a week and ca. 2 l a day.

Check: That would be 9 x 1,5 litre bottles and one 0,5 litre bottle.

Interview 3:

This tandem oriented themselves on the phases of the solution plan. All the phases are listed except for the last phase. But they oriented themselves more on the example given in the first problem than on the plan itself. This led to problems in some places with the abstraction of the example. The phases *Choose approach* and *Perform* were processed separately and successfully. The following two phases however were mixed up and shortened. The overview shows the sequence of the phases:

PHASE	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Understand the problem	x					x			
Choose approach			x						
Perform		x		x			x		
Explain result					x			x	
Check									x

Fig. 5: Solution phases in Interview 3

This pair of students was very positive about the solution plan and finds the added explanations to be important:

Interviewer: Did the plan help you?

Student 1: I would say that if it wasn't there, the explanations, only understanding the problem, choosing the approach, I wouldn't have gotten it. Here it says to formulate in your own words, describe assumptions and plan the calculation method. That helped.

Interviewer: And if you had done the question without the plan? Would you have done it the same way?

Student 2: It would have been more difficult and would probably have taken a bit longer.

This group of students also noted down the written solutions according to the given plan structure. (Hilmer, 2012)

DISCUSSION

Of course this qualitative study with only three pairs of students is limited, but the three interviews assessed show that the student solutions were influenced by the given solution plan. Especially the written solutions receive a structure clearly adapted to the solution plan. Here it becomes clear that the choice of solution plan can have a great influence on the written solutions of the students.

However the actual solution path of the student pairs do not always follow according to the given plan. Even if the written, fixed solutions of the students all have the same structure the solution process of the tree pairs differs clearly. This reminds one of the individual modelling routes that were described by Borromeo Ferri (2007). The decisive question in judging the effectiveness of these plans as solution aids is whether the solution process plays a role in the modelling process or only influences the result. Apparently there are students that have greater difficulties dealing with such a plan and others who only need a short introduction to work with the plan.

Regardless of whether the solution plan sustainably influenced the solution process, the six students interviewed made positive comments about the solution plan and felt supported by it. This highlights in a certain way the results of Schukajlow et al. (2010). Also the use of finished solution examples, like in our study in the first problem, seems to appeal to some students.

In the near future this study will be continued with additional cases and the observation of diverse solution plans. A detailed study of the solution processes seems indispensable since the difference were visible in the solution processes; however the written solutions of the students only exhibited few differences.

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MERGING EDUCATIONAL AND APPLIED MATHEMATICS: THE EXAMPLE OF LOCATING BUS STOPS

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Abstract: *The reduction of complex mathematical models for solving real-world problems to a level applicable in secondary education is considered to be a very important goal to increase motivation and understanding. This is, for instance, witnessed by its inclusion in mathematics curricula of many countries and the OECD study PISA (OECD, 1999). The actual implementation of this goal of merging educational and applied mathematics (MEAM) is, however, still in its infancy and rather difficult. In this paper we present a case study in which we discuss our experiences with a two-day modelling project for German high-school students in the age group of 15-19 years. As real-world problem we introduce the problem of locating new bus-stops in public transportation systems. The mathematical optimization tools to tackle this problem reduce to questions solvable by basic geometry. The paper also includes an evaluation of this case study with regard to the motivation of students and our conclusion for the mathematics curricula of schools.*

Keywords: Applied mathematics, modelling, motivation, empirical research.

INTRODUCTION

The optimization problem of locating new bus stops in an urban area is just one of many mathematical real-world problems found in industry and society which are also suitable to be taught in schools. Research on how to merge educational and applied mathematics (MEAM) and how to insert it to the curricula of secondary education is still in the early stages. Integrating real-world problems to everyday school life is an aim we want to achieve to prepare and advance students for their academic studies and later employment, as well as motivating students of all capability levels for the subject mathematics (Blum et al., 2002). It is important that students learn and know the connection between mathematics and problems found in industry and everyday life as this can improve their understanding of mathematical issues as well as their motivation and enjoyment. The OECD study PISA (Programme for International Student Assessment) also emphasizes on bonding real-world and mathematical problems (OECD, 1999) which shows how important it is to integrate management mathematics to regular school lessons.

As applied mathematics and modelling real-world problems play a decisive role in industry and science the goal we pursue is to regularly integrate this to lessons in school, i.e. to merge educational and applied mathematics. The merging does not only have to take place by means of modelling problems, but can also mean to integrate these topics to school lessons in a more classical way. By emphasising specific curricular mathematical content, challenging applied mathematical topics can be used to repeat, combine or develop new and old mathematical subjects.

Since the 1990s handling modelling problems in regular school lessons or during extra-curricular projects of 2 to 5 days is a successful activity started by a group of industrial and management mathematicians at the University of Kaiserslautern (Bracke et al., 2011). Embedding real-world modelling into regular school lessons was discussed and evaluated by Bracke et al., combining the knowledge from selected topics of a whole school year using a modelling cycle as described in Figure 2 by Blum et al. (2005). This study came to the conclusion that real-world modelling tasks can be integrated into regular school lessons of a whole school year and that the attitude of students towards this method was very positive. Kaiser et al. (2006; 2010) studied authentic modelling problems with students during modelling weeks and describe problems which are mainly suitable for students of an upper secondary level in school. Their study also showed that most of the students taking place in the evaluation would appreciate if such authentic problems were included in school mathematics more often, since they liked the opportunity to apply mathematics in real life. Maaß et al. (2007; 2011) emphasise the insertion of modelling problems into regular school lessons on a lower secondary school level. Precise modelling tasks were developed and evaluated during the project STRATUM with the successful aim to improve modelling competencies of lower achieving students. Methods to teach authentic problems concerning combinatorial optimization were discussed by Lutz-Westphal (2006). The focus does not lie on modelling problems but the integration of combinatorial optimization problems to regular school lessons. A feedback questionary was given to participating students to evaluate what they liked or disliked about the topics. The feedback was on the average very positive. We can hence say that applied mathematical problems as well as modelling problems in general seem to enhance the joy and understanding for mathematical applications.

Management mathematical problems seem to be particularly suited for modelling activities, since these problems are often very close to the everyday-life experience of the students (Bunke et al., 2007; Kreußler et al., 2012). Operations research, including the area of mathematical optimization, is a field of study which implies a vast number of applications from industry and everyday life. Geometrical methods can often be used to solve optimization problems which provide an excellent opportunity to integrate management mathematics to lessons in schools while staying within the framework of given curricula at the same time. Several examples of this type have been developed in the research project MaMaEuSch: Management Mathematics in European Schools (Hamacher, 2001-05) or can be found in the book Hamacher et al. (2004).

In the subsequent sections we consider the optimization of bus stop locations and its integration to secondary mathematical education in schools. At first, we present the mathematical background and a solution strategy in order to highlight the advanced mathematics behind this topic. Afterwards, we discuss the difficulty of reducing the presented mathematics to a level suitable for schools and present a modelling project studying the motivation of students concerning management mathematical topics.

PLANNING BUS STOPS – A GUARANTEE OF ACCESSIBILITY

Due to the increasing growth of population in urban areas the optimal location of bus stops becomes a more and more important task to maximize acceptance and convenience of public transportation systems and to minimize pollution, noise and congestion. To achieve these goals the public transportation network of a city should be an efficient and worthwhile alternative to travelling by car. This includes better prices, small travelling times and short distances to the nearest bus stops. We should also take into account that construction costs and travel times both increase with each additional bus stop. Therefore, we need to find a compromise between short travel times and enough bus stops within reach for all customers. The distance a customer is willing to go to the nearest bus stop is called the *covering radius*. Research by Murray (2001) showed that approximately 92% of the already existing bus stops in Brisbane, Australia, were redundant, assuming a covering radius of 400m for bus stops. The reduction of bus stops could hence lead to an enormous gain in time and, as a result, to an increasing number of customers.

We now want to specify the exact problem statement. We are given an already existing public transportation network, i.e. the roads of a city, represented by a planar undirected graph $G = (V, E)$. The edges $e = (v_i, v_j) \in E$ represent possible bus routes between the nodes v_i and $v_j \in V$, whereas the set V consists of important breakpoints and junctions of the given transportation network. There is given a finite number of customer locations $P = \{ p_i \in \mathbb{R}^2, i = 1, \dots, m\}$ and a covering radius $r > 0$. The *Continuous Stop Location Problem (CSLP)* now wants to find a minimal number of bus stops that cover a given number of customers within the covering radius r while all points on graph G are possible locations for new bus stops (Schöbel et al., 2009). The set of all points on the edges of graph G can be defined as $T \coloneqq \cup_{e \in E} e = \{ x \in \mathbb{R}^2 : x \in e, e \in E \} \subseteq \mathbb{R}^2$. Given a covering radius $r > 0$, a customer location $p \in P$ is *covered* by a point $s \in T$, if $d^p(p, s) \leq r$. Furthermore, we define the *unit ball* B_p of a customer location $p \in P$ as all points $x \in \mathbb{R}^2$ whose distance to p is less than or equal to 1. $B_p^r \coloneqq p + rB_p = \{ x \in \mathbb{R}^2 : d^p(x, p) \leq r \}$ then defines all points $x \in \mathbb{R}^2$ whose distance from $p \in P$ is less than or equal to the covering radius r. To solve *CSLP* we use the ideas of Schöbel et al. (2009) who present a suitable algorithm for our problem. In the first step we need to check whether *CSLP* is solvable at all. This can easily be done, since *CSLP* is solvable as soon as each customer location $p \in P$ is covered by at least one point $s \in T$, i.e. if $B_p^r \cap T \neq \emptyset$, $\forall\, p \in P$. Since we are given a continuous set T of possible bus stop locations at the start, our goal is to reduce this to a finite dominating set $S_{cand} \subseteq T$ for which we know that it must contain at least one solution S^*. *CSLP* can then be stated as a well-known *Set Covering Problem* and hence be solved by any algorithm known to solve this, for example the greedy algorithm suggested by Chvatal (1979). The algorithm to solve *CSLP* can be summarized as follows. We are drawing circles $B_{p_i}^r \, \forall \, p_i \in P, i = 1, \dots, m$ with a given covering radius r around all customer locations. The intersection points of these circles with the given road network as well as given junctions and breakpoints form the candidate set S_{cand} of possible bus stops. Assuming the same construction costs for all bus stops we can set the costs $c_j = 1 \, \forall \, j$ without loss of generality. We then sort these candidates in decreasing order of the number of customer locations covered by them. Using the greedy method of Chvatal (1979) we start choosing bus stop locations beginning with the candidate that covers the maximum number of customers, continuing by always choosing the candidate of highest coverage left. As soon as all customer locations are covered the optimal candidate set S^* is found.

A BUS STOP MODELLING PROBLEM

Since the optimal planning of bus stops in a city forms an important up-to-date subject it is a good example to demonstrate students the application and usefulness of mathematics outside school. To find out whether working on exercises dealing with management mathematics can foster the interest and pleasure of students for the subject mathematics a number of schools in Germany, more specific the State of Rheinland-Pfalz, were visited by mathematical staff of the University of Kaiserslautern. Two-day modelling projects were given to groups of three to five students currently attending 10th to 13th form in school, i.e. students at the age of 15-19 years. A variety of different projects were presented at the start such that students were able to choose a project matching their interests. One of the projects was called "Where should bus stops be positioned?" where the Continuous Stop Location Problem discussed above was a possible modelling tool. This project was presented and dealt with in two of the participating schools. The students were free to choose a project of their interest. In each school one or two groups of students worked on the bus stop modelling problem which came to a total of 14 students.

The students were given an exercise sheet explaining the significance and importance of optimal bus stops with respect to time and money. The data included a specific road network with information about the position and number of customers at the start (see Figure 1) together with the task to find a minimal number of bus stops for the given road network such that as many customers as possible are able to reach a bus stop within a covering radius of 400m. The discussion of the students with regard to the specific radius is a very interesting example of the extended modelling cycle, but is beyond the scope of this paper. The students were then confronted to tackle the given task by running through a modelling cycle which is well-known in tackling real-world problems and has, for instance, been described in Blum & Leiß (2005) for educational mathematics (see Figure 2). The concept of a modelling cycle was explained to and discussed with all students at the beginning of the two-day modelling period in order to prepare them for the task lying ahead. While the students were working on the projects, teachers and university staff acted according to the "principle of minimal help" (Aebli, 1978): Students had the freedom to independently discuss and develop ideas; the modelling staff only interfered in small doses to avoid dead-end streets in the modelling process.

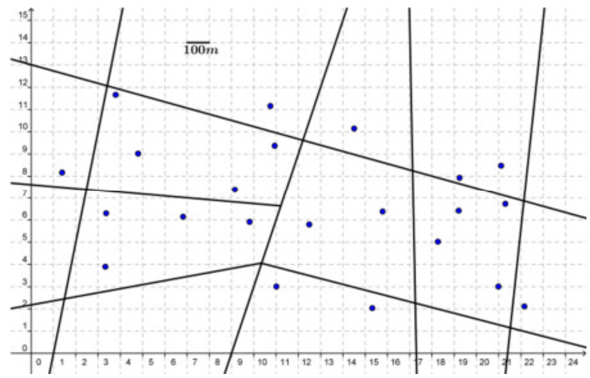

Figure 1: Road network given to students during modelling project (lines indicate streets, points the position of buildings, e.g. hospitals, schools, supermarkets, etc.)

One of the goals of the two-day modelling projects was to improve and strengthen the general mathematical competences (e.g. arguing mathematically, mathematical modelling, communicating) required by the curricula of the German school system (MBWJK, 2007). Additionally, management mathematical topics like locating bus stops were used to motivate students with projects concerning relevant real-world problems as well as improving their understanding of the application of mathematics outside school. Finally, as the previous example shows, these subjects automatically combine and repeat many individual basic contents of school mathematics.

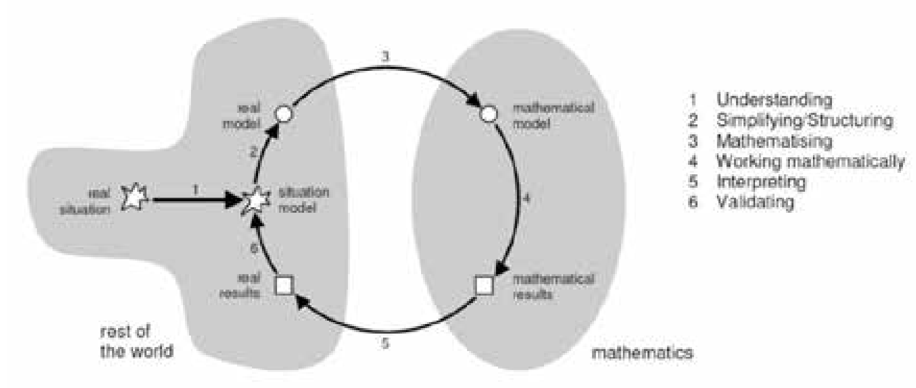

Figure 2: Modelling cycle according to Blum & Leiß (2005)

After having developed a strategy to tackle the problem, the students were asked to generalize their approach such that it could be applied to any other city. In addition, the students examined the city centre of their own city and were able to present optimal bus stops often matching the in reality existing stops, confirming the efficiency of their strategy just developed.

Student Results

The outcome of the two-day modelling project was, in general, very impressive. On their own account the students developed strategies very similar to the algorithm solving *CSLP* that we have presented above. This can, for example, be seen in the following student solutions (Figure 3). Since most of the students assumed an Euclidean distance measure, circles with the given radius of 400m were drawn around each of the marked customer locations. Afterwards, the students sought for those areas where as many circles as possible intersected and which were then marked red (see left picture of Figure 3). The points where those areas intersected with the transportation network were then defined to be possible places for new bus stops. Choosing points from the highest to a lower number of intersections the number of required bus stops was minimized, exactly as done in the greedy algorithm discussed earlier.

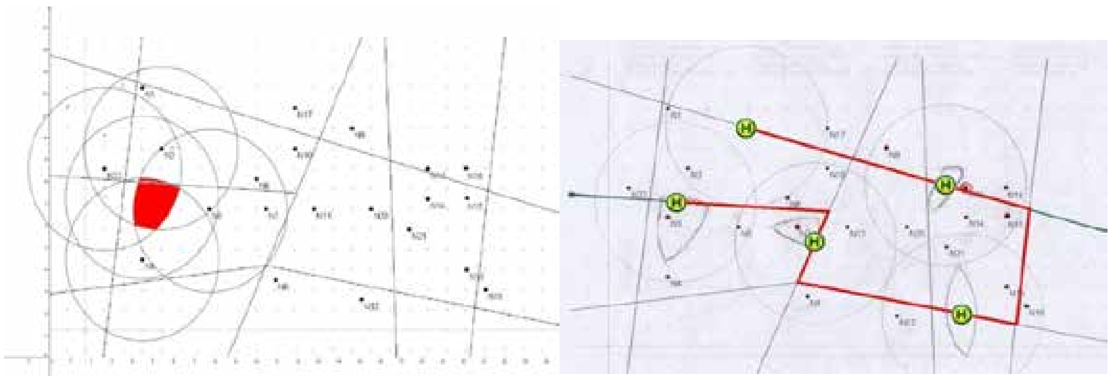

Figure 3: One solution by students of 11th form

Another group of students of 11th form improved this method even more by including the weights, i.e. the number of people at a customer location. The size of the radii drawn around each location was then adapted with respect to the importance of the place considered. The more customers likely to travel from a certain location, the more important it became and, hence, had a smaller radius drawn around its location. The maximum radius allowed remained to be 400m. Other students not only presented solutions for an optimal route but also thought about a most customer-friendly or economic solution, ensuring that a certain percentage of customers was being covered. Some students observing the city centre of their own town developed a grid of triangles placed over the map such that each pedestrian would be able to use a bus stop within reach of 400m. The corner points were then adjusted to the streets as well as taking important places of the city into account. As no distance measure was given beforehand, the students were able to develop alternative ideas to Euclidean distances integrating the nature of street networks into their models.

Evaluation of Survey

At the end of each modelling project the students were asked to fill in an evaluation sheet asking multiple-choice as well as free-answer questions. They were, for example, asked about their overall impression, how they liked the choice of their topic and if they could identify and understand the connection between mathematics and business as well as its usefulness outside school. This empirical survey was used as a preliminary stage for further research planned in this area. It gives a first impression of the usefulness and effect that the integration of management mathematical topics can have on students in school and justifies its embedding into everyday lessons. Altogether, the feedback was very positive. The multiple-choice questions could be answered on a scale from 1 to 6, where 1 means “very good/much”, decreasing to 6 standing for “very bad/little”. The following table lists some questions with their corresponding average and median student answer, demonstrating the results of the 14 students working on the project “Where should bus stops be positioned?”

Question	**Average**	**Median**
Overall impression	1,9	2
Choice of topic	2,3	2
Were you able to identify the connection between maths and business?	2,0	2
Did you learn what mathematics is useful for outside school?	1,9	2
Did you enjoy working on the project?	1,8	2

These answers show that the students enjoyed the MEAM approach and found the topics of management mathematics on the average “good”. This was also confirmed by the free-answers. For instance, the question “What did you like most?” was answered with statements like “Using classic school mathematics to solve real-world problems.”, “Getting to know applications of mathematics in practice.” or “The material to work on.” In addition to this, observing the students was also very meaningful. It was obvious that the students were working very enthusiastically and enjoyed the teamwork with their schoolmates. Often, students continued working on their projects in the evenings at home, presenting further ideas and solutions the following day. Even students who, at first, seemed a little uninterested or usually displayed a low competence level in mathematics were inspired by the topics and worked very hard and enthusiastic in the end. Since all the students who were tested were very motivated to work on topics concerning management mathematics we draw the conclusion that the inclusion of applied mathematics to the curricula of schools will increase the motivation and understanding of the subject mathematics in schools.

EMBEDDING INTO THE CURRICULA OF SCHOOLS

Optimizing bus stops in a city is a topic covering a lot of different subject-matters specified in the curricula of secondary schools. By didactically reducing and adapting the contents to the abilities of the students in question this topic could also be discussed with students in lower grades. Especially the required geometrical competences to solve the problem can already be achieved in 5th and 6th grade. The curriculum for 5th and 6th grade of the German school system in the State of Rheinland-Pfalz (MBWJK, 2007) lists general mathematical competences like “arguing mathematically”, “mathematical modelling” or “communicating”, as well as content-related competences that should be learned by students of this age. These topics include drawing circles and identifying center point, radius and diameter, getting to know the difference between a straight line and a line segment, using the Euclidean distance to describe the shortest distance between points and learning how to use dynamic geometry software. These subject-matters already form a good background to study the Continuous Stop Location Problem on a visual, geometrical basis. The curriculum for 7th and 8th grade introduces the absolute value, while the theoretical calculation of intersection points as well as solving systems of linear equations using matrices is not covered until 11th or even 13th grade (MBWJK, 1998). The difficulty in reducing the topic of optimizing bus stops to the competence level of a group of students lies with the different backgrounds and knowledge the students might have gained so far. It might be necessary to introduce the theory of graphs in one class while another group of students might have already studied and used these in other subjects, e.g. computer science. Management mathematical topics are, hence, also very suitable for interdisciplinary education and can show students the important connections of their subjects in school.

There is a variety of possibilities how to insert the topic of optimizing bus stops into everyday lessons at school. This could, of course, be done by means of a modelling problem as shown above, usually taking at least 3 to 4 lessons of 45min each. In doing so, the teacher should not be expecting a specific solution but remain open for all kind of solutions presented by the students. Most of the teachers may find this very hard as they are used to pose questions to the class where they know the exact answer beforehand. For this reason, continuing education training courses for interested teachers with the aim to learn how to organize and teach modelling projects in school are regularly offered at the University of Kaiserslautern and find great acceptance among teachers. On top of that, the teachers should of course be familiar with the mathematical content concerning the modelling task. As we have pointed out, modelling projects dealing with subjects of management mathematics are very attractive to all students tested and are a great possibility to not only combine different topics learned in school but to also improve the capability of working in a team, discussing mathematical matter as well as improving modelling competences, which are all part of the mathematical competences required to be learned according to the German curricula for secondary schools. Another method would be to include this topic into the standard lessons, either using it as a motivating example or to strengthen already learned tools. This could, for example, be done at the beginning of the topic "Linear Algebra & Analytical Geometry" in 11th – 13th grade (MBWJK, 1998) driving towards the exact calculations of geometric intersection points. Motivated by this interesting example and its usefulness, students could at first develop geometric ideas guided by their teacher, learning afterwards how to implement these ideas theoretically.

SUMMARY AND CONCLUSIONS

The optimal planning of bus stops in urban areas is a management mathematical topic suitable to be taught in schools. Dealing with up-to-date mathematical problems can lead to greater motivation and understanding of the subject mathematics and its application outside school. This was tested in a preliminary survey during two-day modelling projects with students of various schools in Germany and showed how enthusiastic and interested students of all competence levels can be when dealing with management mathematical topics. Integrating real-world problems to lessons in school can not only increase motivation but also prepare students for further studies and their working life after school. This can, for example, be done by introducing modelling problems. Their solution procedure combines different topics learned in school and also enhances mathematical competences like “mathematical modelling” and “communicating”. Management mathematical topics are also a good opportunity to be included into the curricula of secondary schools while being compatible with the guidelines given therein. Examples like the optimal planning of bus stops can be used to motivate at the beginning of a new topic as well as strengthen already learned facts. Hence, we conclude that management mathematics should be included in the lessons in schools more often as this increases motivation, understanding and pleasure for the subject mathematics.

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ABOUT MATHEMATICAL MODELING IN THE REALITY OF THE CYBERNETIC WORLD

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In this article, we explore the relation between mathematical modeling (MM) and the cybernetic world, understood as any environment produced with digital technologies. We seek to provide evidence that the MM practiced in the reality of the cybernetic world has singular characteristics that interfere in the process of constructing models. Toward this end, we discuss reality from a theoretical perspective, in order to understand the reality of the cybernetic world. This understanding is related to data collected while working with university-level mathematics students as they develop electronic games using the Scratch programming environment. We conclude that the reality of the cybernetic world is one dimension of reality with specific characteristics related to space, time, and plasticity, which opens up new possibilities for using MM in mathematics education.

INTRODUCTION

Research and teaching practices that link mathematical modeling (MM) and digital technologies, pointing to a synergy between the two, are increasingly common (Campbell 2010; Siller & Greefrath 2010; Greefrath 2011; Borba & Soares 2012). According to Kaiser (2011) and Dalla Vecchia & Maltempi (2012), despite the existence of different conceptions of MM, the issue of reality is always present, leading us to assume that there is an intrinsic relation between MM and reality. The nature of reality is a delicate question, however, which is further complicated in the context of digital technologies, generating descriptors such as: reality of the cybernetic world, augmented reality, hyper-reality, virtual reality, mundane reality, physical reality, etc. Thus, we question whether the reality to which MM refers encompasses the reality of the cybernetic world, and seek to identify the specific characteristics of MM in the context of this reality.

Seeking answers to these questions, we present a theoretical discussion regarding reality followed by results of a study conducted with university-level mathematics students, aiming to understand mathematical modeling as it is practiced in the cybernetic world.

REALITY AND REALITY IN THE CYBERNETIC WORLD

In this article, we assume the perspective of reality proposed by Bicudo (1999, p. 31, our translation), understood as:

> a dynamic, temporal, historical whole, perceived in the man-world encounter, not separate from the perceiver, who speaks of it and interprets it, building a web of

meanings in the intersubjectivity upon sharing experiences and communicating interpretations.

Thus, reality can be understood as lived reality that occurs in specific times and spaces, constituted in the natural field where all thoughts, actions, and perceptions of the different subjects who live in it originate. We ask, however: Can the cybernetic reality be understood as a dimension included in this reality?

Research by Bicudo & Rosa (2010) suggests that it can. However, considering the cybernetic world as reality requires conceiving of it from a perspective that differs from that of modern science, which speaks of physical, objective reality by referring to the place and time where measurable entities, which can be physically manipulated, are located or placed. As Bicudo and Rosa point out, from this perspective, the cybernetic world cannot be considered real, given that the "whereabouts" of this world presents itself in a different manner. In philosophy, however, we find that the real, and consequently, reality, can have distinct ways of showing themselves, involving, among other aspects, the actual and the virtual (Granger 1994).

The actual can be considered that situation or entity which shows itself to the observer who contemplates the here and now, i.e. that which appears in mundane reality. Actualization is the process by which something moves from a situation of potentiality to the actual state, in which potentiality signifies "characteristic of what is potent, of that which has the force to be, which carries within itself the potential to become" (Bicudo & Rosa 2010, p. 24, our translation). According to these authors, the opposite of actual is the non-actual, i.e. that which was not actualized and is in a state of potentiality, including the virtual, which "refers to the way, in general, that it could become actualized through actions that are together with the available materials and the techniques, in particular applications, explanations of the empirical, etc." (p.27, our translation). In other words, the essence of the virtual lies within itself, and is not necessarily linked to the actual, although it may contain situations that may become actualized. For this actualization to occur, the act and the material must occur at the same time, rendering it dependent on them, in such a way that the nature of the material conditions the manner in which the actualization takes place. In addition, according to Granger (1994), the reference is responsible for the relation between the actual fact and the virtual fact, and in the reality of the cybernetic world, this reference is based on the scientific and technological device being used, which supports the actualizations that occur in it (Bicudo & Rosa 2010).

Thus, the virtual is real but not actual, because it has not yet happened through acts and the material at hand. According to Granger (1994), the sciences (mathematics in particular) do not deal directly with facts presented to the observer in mundane reality, i.e., actual facts; rather, they deal with virtual facts. In this way, we could say that mathematics is virtual, but has the potential to branch out into actual facts. Physics and chemistry, for example, avail themselves of mathematics, but they use it as a tool in their theoretical constructions, which are not exact like mathematical theory, since no matter how theoretical they are, they operate with mathematical apparatuses (mathematical knowledge, formulas, ways of operating, etc.) and the empirical dimension in which our acts take place. According to Bicudo & Rosa (2010), computer science also operates with mathematics (which, as we said, is virtual) and, unlike physics and chemistry, deals with another dimension of available "material and form", namely: communication, information, and the creation of programs within a landscape pre-established by mathematical knowledge. And upon actualizing the virtual (the potentiality present in the mathematical apparatus) through form and the available material and effectuated acts (decisions), reality is being constituted which, therefore, also encompasses that which is addressed in the cybernetic world.

Thus, actualizations in the sphere of the cybernetic world emerge from the relation of the human being with the scientific apparatus, through commands, languages, and actions that occur in the encounter between human and computer as well as in the communication between humans and others subjects that is made possible by the system of technological reference. In this broad dimension of reality, which opens up to the experience of the subject, space and time are of a different nature, being more branched out and fluid, such that actions may not actualize in the same manner as they do in everyday lived reality without digital technologies. Thus, we see the reality of the cybernetic world as assuming a plasticity that enables the construction of environments in which the physical relations established in the mundane can either be experienced or totally ignored.

Understanding the cybernetic world as a dimension of reality, as a modality of lived reality, implies the emergence of a set of possibilities that can be investigated, in the philosophical as well as educational fields. Based on the above, it is our understanding that the essential aspects of the reality of the cybernetic world are associated with singular time, space, and plasticity, constituting a basis for investigation of the mathematical modeling process when the situations involved address this broad dimension of reality. Thus, in a broad sense we assume MM to be a dynamic and pedagogical process of building models grounded in mathematics ideas that refer to and aim to address problems of any dimension encompassed in reality.

MATHEMATICAL MODELING IN THE CONSTRUCTION OF ELECTRONIC GAMES

The cybernetic world is a singular space where the actions taken may not be actualized in the same manner as they are in everyday lived reality without digital technologies. Aiming to investigate MM in the reality of the cybernetic world guided by qualitative research methodology, we ministered a course entitled "Construction of Electronic Games". We believe that the construction of electronic games paves the way for the locus of the happenings that are related to the situations of the game to be the reality of the cybernetic world. This enables investigation of MM in which the "where" of the happenings is a space that differs from the classical physical notion, thus making possible advancements in understanding of the nature of MM practiced in the reality of this world.

Eight university-level mathematics students participated in the course, which took place in eight 4-hour periods from May to July, 2009. The set of information composed of their words and gestures and their interactions with each other and with the software and other media constituted a fundamental part of the data analyzed. Data was collected using written notes, cameras and mainly by means of the Camtasia software, which makes it possible to capture simultaneously the image of the computer screen and images and audio of the students as they interact with them. Parts of the video and audio recordings were transcribed, categorized and analyzed according to our theoretical referential.

The main software program used was Scratch, a free software, developed at the Massachusetts Institute of Technology (MIT), which was conceived based on many ideas from Logo. It is a visual programming language that allows users to construct, interactively, their own stories, animations, games, simulators, songs, and art. The commands are composed of blocks that are dragged to a specific area and connected, creating a program that can be executed. Some excerpts from a program made with Scratch are presented in Figure 1.

Within the perspective of MM adopted, we consider a program, or part of a program, made with Scratch to be a model, as we assume that it is the result of an association of situations from reality (actual facts) with concepts related to mathematics (virtual facts), by means of a reference. In this case, Scratch assumes the role of reference which apprehends the situation being investigated, enabling the manipulation of concepts and symbols according to logical formal rules (propositional calculus). The set of commands that can be actualized in the cybernetic world is limited, but they can be interwoven to configure more complex commands. This reference makes it possible to work with a set of pieces of knowledge associated with science and that branch out and become interwoven as users/authors interact with the situations they wish to represent using the Scratch language.

RESULTS AND ANALYSIS

In this section, we analyze data collected with one pair of students in interaction with the second author of this paper and his assistant, who ministered the course. These students decided to create a game in which a car, controlled by the player, would navigate around obstacles that appeared on a road. At first the students created a model (part of a program) responsible for the movement of the car that allowed its vertical movement over the entire window (area used to execute the program). However, the car´s movement extrapolated the region occupied by the road in the image, giving the impression that the object was floating. Laura, one of the students, then posed the question: *And how do we get it to stop floating?*

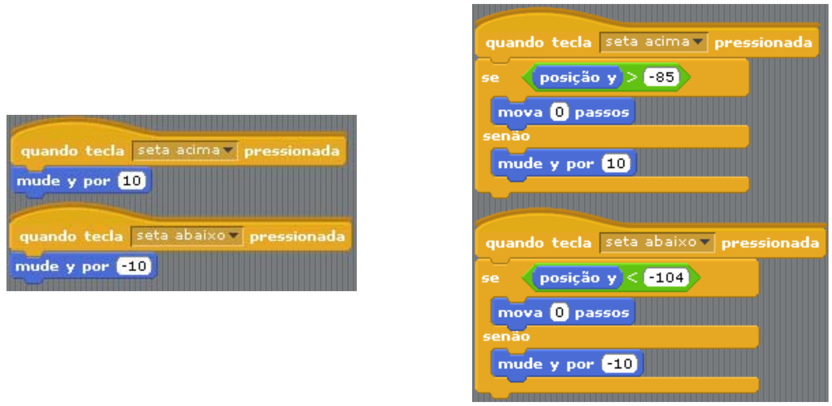
The effort to re-create a movement similar to an automobile in mundane reality was what guided the consequent steps to solve this problem, as can be observed in Laura´s description of her objective: *I want to... like, if it reaches this height here, this (x,y), it stops. It moves zero steps. Otherwise it will go there, up above. I don´t want it to go beyond this height; in this case, I can´t go above or below the road. I want it to go only on the road.* Thus, it can be observed that the student´s reference is clearly a fact that actualizes in the mundane world, which is the movement of the car in this dimension of reality.

On the other hand, the thing that generated all this questioning was not a specific situation that occurred with an automobile and experienced by the student in mundane reality, but rather actions that involved the manipulation of Scratch while creating a game, and which were actualized on the computer screen. The situation, in this case, only revealed itself to be problematic to the student within the reality of the cybernetic world, since it allowed the structure that was built to become actualized in a different way compared to everyday experience. Thus, there was a sort of tension created between what was imagined by the pair of students and what was actualized in cyberspace, creating a problematic field that led to various actions by the students.

With this, one can see how the cybernetic world opens the way for "realities that are possible, projected, and invented" (Bicudo & Rosa 2010, p. 20), and that this creative process can configure itself in the tension generated between aspects that refer to situations inspired by mundane reality and the potentiality provided by the plasticity of the cybernetic world which, through actualization, makes it possible for occurrences experienced in this space to extrapolate the physical situations of the everyday.

The model for the movement of a car, created by this pair of students using Scratch, ceases to be something imagined or thought and becomes a fact that is subject to manipulations of the language itself. This manipulation of the language involves a web of concepts, notably supported by mathematics, which allows the situation being investigated/modeled, when apprehended by language, to be altered or modified, enabling actualizations in the cybernetic world that are distinct from those that occur in mundane reality (a floating car, for example).

To achieve their objective, the pair of students made changes in the model associated with the movement of the car throughout the construction process. In the first model, presented in Figure 1a, there are in fact no limits to the movement of the car. In Figure 1b, however, the model includes conditioners that allow movement only within stipulated limits. Thus, although the player decides how to move the object, all the actions are previously established by the model. Unlike mundane reality, the fact actualized by the model in the reality of the cybernetic world is completely determined by the model itself, which is also characteristic of this dimension of reality in the face of the model constructed.



(a) **(b)**

Figure 1: Initial model (a) and final model (b) made using Scratch

In this sense, we understand Bicudo & Rosa (2010, p. 28) when they state that "[…] reality in cyberspace is virtual because it is based on the sciences, notably mathematics", i.e. it is a dimension of reality, but it lends itself to virtual adjectification because it is totally supported by the scientific apparatus. This common base enables a complete determination of the operations, which are actualized in the reality of the cybernetic world.

According to Bicudo & Rosa (2010), this determination is compensated by the multiplicity of ways in which these actualizations can take place. In the specific case of MM, which involves a process of construction, this multiplicity does not reveal itself only in the possibilities that the (ready-made) model offers, but also in the very construction of the model itself, allowing a multiplicity of possible paths which influence the final structure, depending on the way they are actualized in the reality of the cybernetic world, and the objective of those involved. This aspect can be observed at two different moments during the interaction between the Laura and the researcher:

Laura: And how do we get it to stop floating?

Researcher: You need to put some conditioners, right? It´s going to stay stopped there, right? [Referring to the horizontal movement].

Laura: It stops.

Researcher: Does it stop or... does it "move" zero steps?

Researcher: Or... stop moving. So you can put, change X to Y, you can put an "if". An "if-else"...

Laura: Put an "if-else", then, in the middle of this here [of the commands]?

Researcher: Right. "if-else" or "if", I´m not sure. You would have to test it there to see what works.

The researcher´s last suggestion indicates that one of the ways to find a solution is to test the different possible paths, indicating that the idea of multiplicity is present not only in the actualization of the game by the player, but also in the process of constructing the game itself. The following excerpt indicates a new multiplicity of paths to be followed:

Laura: When the *y* mouse is equal to, in this case, to this point *y* here [pointing again with the mouse to the position of the car and observing that it is located at y = -65]

Assistant: Equal or less. I´m not exactly sure what the objective is there.

Laura: OK, so I´m going to experiment.

In this excerpt, the research assistant expresses doubt regarding the path to be taken, indicating two paths ("equal or less"). These distinct possibilities are taken up by Laura, and her subsequent actions show that the use of the word "experiment" was associated with the use of all three symbols that the command could assume, thus presenting a multiplicity of paths.

In this way, we affirm that the MM activity presented involved a problem that resulted from the tension created between what was imagined by the students and its actualization in the cybernetic world, which revealed aspects that differ from the actualization of an automobile in mundane reality. This difference, in the analysis presented, occurred by means of the apprehension of the phenomenon by the reference, which is given by the Scratch language. Because it has the same structure as the cybernetic world, the constructions of the model and of space itself are interwoven, showing that, in the reality of the cybernetic world, the model (which is the apprehension of the phenomenon by the reference) completely determines the possibilities of the player´s action. In addition, the follow-up actions related to the problem of the car were associated with the change and adaption of the model to the situation desired by Laura, which consisted of keeping it within the limits of the road. This process involved experimentation with various different commands. When the car failed to behave as desired, changes were made. These changes in the model continued until the model was adapted to the desired situation.

This is consonant with some views of MM that see the effort to adapt the model to the situation as an important stage in the modeling process, as in research carried out by Bassanezi (2004), Biembengut & Hein (2007), Borromeu Ferri & Blum (2010), and Kaiser, Schwarz & Tiedemann (2010). According to these authors, the adaptation of the model, or of the solution the model provides for the situation to which it refers, is denoted by validation, and can be consolidated through comparison of the data obtained using the model to the empirical data. If the comparison is unsatisfactory, the model is re-formulated to find a better solution.

When another pair of students in the course was analyzed, however, we observed that they did not proceed to reorganize their model when faced with a situation that failed to behave as they wanted. Rather, they sought to re-organize the entire visual environment in which the model was referenced and would become actualized. These students decided to create a game in which the objective was to navigate on a map of a fictitious city by changing two variables, denoted “meters” and “degrees”. The map was initially created based on an image the students obtained from the Internet (Figure 2).

Figure 2: Map in the Scratch programming environment

When the MM addresses a problematic situation from the cybernetic world, it can be influenced by mundane reality, as discussed above. Nevertheless, it is still possible to see aspects related specifically to the context of the cybernetic world. The experience of this second pair of students is an example of this, in which the “where” characteristic of reality in cyberspace shows itself decisively in the way they develop their work, broadening the mundane possibilities. It is precisely based on this potentiality of creation and re-creation of spaces that we believe the MM process can manifest itself differently compared to cases where the reference derives from the everyday, since it is supported by a reality that can be constructed, imagined, thought and re-thought, showing an environment that is plastic and transforms itself.

Some of the problems faced by this pair of students involved the relation between the scenario (map) and the object that moved in it. The object was supposed to return to the stipulated starting point upon touching the edge of the streets, but this was not happening. The following excerpt shows the researcher attempting to help the students and illustrates one of the singular aspects of the cybernetic world, plasticity:

Researcher: So I think the best thing would be to put a stronger border here. [Referring to the edge of the streets in the image represented on the map]

Fernanda: Do it with Paint, maybe. [Referring to the software Paint, which can be used to edit images. This idea was later abandoned in favor of using the software Excel].

The researcher shows that one possible solution to the problem presented by the model created by the students would not involve changing it, but rather changing the background scenario that limited the action of the object. The researcher´s suggestion reveals a plastic view of the cybernetic world that was immediately corroborated by the student. This agreement led the students to change the scenario. Initially, they aimed to totally deconstruct the model, i.e., to build a completely new one. However, the use of the Excel software allowed them to re-build it based on the original map. It was at this moment that the student, Fernanda, again showed that she considered the space where the model was being constructed to be malleable, revealing an aspect that is very distant from a reference to the physical, mundane reality which is already actualized. This aspect can be observed mainly when the students say: *We can make it thin, like this, and afterwards we can augment it, OK?*

Our interpretation is that there was a concern regarding the borders and the thickness of the streets, perhaps influenced by the researcher when he suggested making the border "stronger". However, when the student says, "We can make it thin, like this", we believe the student was, at first, attributing importance to the outline of the streets in the construction of the map; but the rest of her comment, "and afterwards we can augment it", revealed that the malleability of the construction of the environment was already incorporated. In other words, the student was not initially concerned about all the details that constituted the map and could influence the commands used, since the environment constructed could be modified to fit the desired format.

This observation strengthens the idea of the plasticity of the reality of the cybernetic world and reveals a singular aspect when we perceive the problematic situations of this construction from the perspective of mathematical modeling. In fact, when the reference is mundane reality, the model is normally "refuted"; however, in the reality of the cybernetic world, it is possible to refute the "reference", and maintain the main idea. With this, one observes the existence of a potential space where it is possible to consider naturally situations in which a model was invented by the students. The plasticity of the cybernetic world, allows, at least potentially, the creation of a space in which imaginative aspects can be actualized, thus representing a differential for the practice of mathematical modeling itself.

NOTES

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PARENTAL ENGAGEMENT IN MODELING-BASED LEARNING IN MATHEMATICS: AN INVESTIGATION OF TEACHERS' AND PARENTS' BELIEFS

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As part of a larger project aimed at promoting mathematical modeling as an inquiry based approach to mathematics and science, this study aimed to: (a) describe teachers' and parents' beliefs about inquiry-based mathematical modeling and parental engagement, and (b) explore the impact of a modeling-based learning environment on enhancing parental engagement. The research involved three sixth-grade teachers and 32 parents from one elementary school. Teachers reported positive beliefs on parental engagement and parents' potential positive role in implementing innovative problem solving activities. Parents expressed strong beliefs on their engagement and welcomed the modeling approach. Implications for parental engagement in mathematics learning and further research are discussed.

Keywords: Parental engagement, teacher beliefs, modeling, interdisciplinary problems, model eliciting activities.

INTRODUCTION

This study argues for an inquiry-based approach in the teaching and learning (IBL) of mathematics, one that is based on a models and modeling perspective (Lesh & Doerr, 2003). A modeling based IBL approach can serve as an appropriate means for bridging complex real world problem solving with schools mathematics (English & Mousoulides, 2011). This bridging is necessary, as complexity gradually appears in all forms of the society and the education, and new forms of mathematical thinking are needed (English & Sriraman, 2010). With this into account, researchers and educational policy makers highlight the need to promote an IBL approach in mathematics that enhances students' abilities in designing experiments, predicting complex systems, manipulating variables, working in teams, and effectively communicating with others (Lesh & Zawojewski, 2007).

Integrating such an innovative approach in mathematics is not a straightforward process. It conflicts with various factors, including national curriculum requirements, the need for more time and resources, teachers' beliefs and practices, and parents' beliefs and attitudes towards such innovations. The significance of parents' role in mathematics education has been documented in a number of studies (see Epstein et al., 2009), and parental engagement has been documented as a positive influence on children's achievement, attitudes, and behaviour. However, to achieve appropriate parental engagement teachers have to find appropriate methods to involve parents, especially when it comes to use innovative methods, like mathematical modeling (Deslandes & Bertrand, 2005).

There is a lack of studies that focused their research agenda on the parents' role during the implementation of innovations, such as mathematical modeling. To this end, the present study focuses on designing a teaching experiment on mathematical modeling, one that also involves parents, in an attempt to create an environment for engaging parents and creating a relationship between teachers and parents. The study further explores teachers' and parents' beliefs as a means to further improve parental engagement and communication between parents and school.

THEORETICAL FRAMEWORK

The theoretical framework review focuses on two strands: (a) instructional interventions to promote mathematical inquiry through a modeling perspective, and (b) parental engagement in the mathematics classrooms with an emphasis on teachers' and parents' beliefs.

A Modeling Perspective in Inquiry Based Learning in Mathematics and Science

In successfully working with complex systems in elementary school, students need to develop new abilities for conceptualization, collaboration, and communication (Kaiser & Sriraman, 2006; English & Sriraman, 2010). In achieving these abilities, a number of researchers propose the use of an inquiry-based approach in the teaching of mathematics, one that builds on interdisciplinary problem-solving experiences that mirror the modeling principles (Haines, Galbraith, Blum, & Khan, 2007; Lesh & Doerr, 2003). Specifically, the present study proposes introducing Engineering Model-Eliciting Activities (EngMEAs) within the mathematics and science curriculum; realistic, client-driven problems based on the theoretical framework of models and modeling (English & Mousoulides, 2011).

Engineering Model Eliciting Activities (EngMEAs) have been in the focus of our work for the last few years (see Mousoulides, et al., 2008; English & Mousoulides, 2011). EngMEAs provides an enriched modeling approach by offering students opportunities to repeatedly express, test, and refine their current ways of thinking as they endeavour to create a structurally significant product for solving a complex problem. The development of the models necessary to solve the EngMEAs has been described by Lesh and Zawojewski (2007) in terms of four key, iterative activities: (a) Understanding the context of the problem / system to be modelled, (b) expressing / testing / revising a working model, (c) evaluating the model under conditions of its intended application, and (d) documenting the model throughout the development process. The cyclic process is repeated until the model meets the constraints specified by the problem.

Parental Engagement

Parental engagement has been documented as a positive influence on children's achievement in mathematics, regardless of cultural background, ethnicity, and socioeconomic status (Epstein et al., 2009). A significant body of research provides solid support for building parental engagement, by reporting a constant correlation between increase in parental engagement and increase in student achievement and students' success, socially and emotionally (Cutler, 2000; Epstein et al., 2009).

Active parental engagement, however, is quite difficult to be maintained. Therefore, programs of parental engagement should be carefully designed and implemented, taking into account all related variables and barriers (Musti-Rao & Cartledge, 2004). Musti-Rao and Cartledge (2004) argue that schools should be less concerned about parental engagement in schools, and more concerned about determining what role parents can play so that parents productively involve themselves in their children's education. They suggest inviting parents' experiences in science, technology, and engineering into discussion, and including parental engagement strategies in teacher professional development courses. They also propose a number of strategies for engaging parents, such as mathematics and science fairs, community involvement utilizing science and engineering experts, and the establishment of a clear communication between teachers and parents, in an attempt to bridge teachers' and parents' beliefs and expectations (Musti-Rao & Cartledge, 2004).

Epstein and Van Voorhis (2001) identify teacher beliefs as an important barrier to creating effective relationships between home and school. They report that teacher beliefs could impact the relationship between parents and teachers. Teachers enter the teaching profession with personal (positive and negative) beliefs about parental engagement that may not be addressed until they interact with students' families and that might result in not involving parents and other community members in the classroom. Teachers who believe that parental engagement could contribute to positive student learning outcomes are more likely to be involved in parental engagement practices than teachers with negative views.

Parents' beliefs on mathematics teaching and learning and the significance of their engagement might also impact parental engagement. Often, the beliefs and expectations between families and educators are not shared collectively, and in many cases parents might have negative beliefs that can lead to stereotypes regarding the relationship between them and teachers. In order for parents' beliefs to change into positive ones, parents should be open to invitations to be engaged in school mathematics, while more parental engagement training on how to work with parents and communities is needed for teachers (Cutler, 2000; Epstein, et al., 2009).

THE PRESENT STUDY

The Purpose of the Study

This study investigated teachers' and parents' beliefs on parental engagement during the implementation of a modeling activity, and examined the impact of Twitter®, a contemporary technological tool, as a means to facilitate: (a) parental engagement and (b) the interactions between the students, the teacher, and the parents.

Participants and Procedures

The research presented in this study was part of *PRIMAS*, a larger research design that includes: (a) inquiry-based mathematics and science instruction, (b) integration of engineering model-eliciting activities as a part of the mathematics and science instruction, and (c) examination of various forms of parental engagement, including workshop participation, participation in classroom activities, and communication with teachers. During their participation in *PRIMAS*, a longitudinal four-year project on Inquiry and Modeling Based Learning in Mathematics and Science, 62 elementary and secondary school teachers in Cyprus participated in workshops on inquiry- and modeling-based teaching and learning in mathematics and science. During workshops teachers were provided with appropriate student and teacher materials for integrating modeling in their day-to-day practices.

This study followed three teachers who implemented modeling activities in their classrooms. The teachers were from one public K-6 elementary school in the urban area of Nicosia, the capital of Cyprus. One teacher used to teach both mathematics and science in one class, while the other two teachers used to teach mathematics and science respectively in a second class in the same school. The two classes (22 and 19 eleven-year old students), their parents and the three teachers worked on the *Water Shortage* activity, during the second year of the *PRIMAS* project.

Prior implementing modeling activities in their classrooms, the three teachers attended three five-hour workshops on afternoon or Saturday sessions. An additional workshop on the use of Twitter® and on parental engagement good practices, also took place. In one of the first three workshops and in the additional workshop, one parent from each student family was also invited to participate. Twenty-eight out of the 41 parents accepted the invitation and participated in the two workshops, which were conducted by *PRIMAS* personnel.

The three workshops provided teachers with some insights on IBL, mathematical modeling, and on parental engagement in mathematics. An introduction to the *Water Shortage* activity took place in the last workshop. During the second part of the workshop teachers and parents were introduced to Twitter® and on the possibilities it could provide for the mathematics classroom, as an online technological tool which can break down the rigid classroom schedule barriers and allow teachers, students, and parents to collaborate. Teachers were actively engaged during the workshops, as they shared questions, suggestions, and examples from their own practices and beliefs.

The Implementation of the Water Shortage activity

The *Water Shortage* model eliciting activity entailed: (a) a warm-up task comprising a mathematically rich newspaper article, designed to familiarize the students with the context of the modeling activity, (b) “readiness” questions to be answered about the article, and (c) the problem to be solved, including tables of data (see Table 1). The activity asked students to assist the local authorities in finding the best possible country that could supply Cyprus with water.

The problem was implemented by the teachers and the author. Working in groups of three to four, the students spent five 40-minute sessions on the activity. During the first two sessions the students worked on the newspaper article and the readiness questions and familiarized themselves with Google Earth and spreadsheet software. During group discussions students identified the significance of the problem and submitted a relevant tweet (one Twitter® account was created for each group of students). Twelve parents commented on students' tweets, by emphasizing that water shortage was among the country's most important problems. Some parents further provided additional sources of information. During the second session students reviewed their parents' comments and suggestions that were provided through Twitter®.

Country	Water Supply per week (m^3)	Water Price (m^3)	Tanker Capacity (m^3)	Tanker Oil cost per 100 km	Port Facilities for Tankers
Egypt	3 000 000	€ 4.00	30 000	€ 20 000	Average
Greece	4 000 000	€ 2.00	50 000	€ 25 000	Very Good
Lebanon	2 000 000	€ 5.20	30 000	€ 20 000	Average
Syria	3 000 000	€ 5.00	30 000	€ 20 000	Good

Table 1: The provided data for the four countries

In the next two sessions students worked on solving the problem. They developed a number of appropriate models for solving the problem, and shared these models with their teachers and parents. During model development students were prompted by teachers to share their ideas with their parents. To facilitate model sharing, a public Wiki was created, in which students could easily upload their files. Student then shared the links to their models with their parents, using appropriate tweets. Twenty-six out of the 28 parents that participated in the implementation of the activity followed their children's tweets and provided feedback and suggestions to their models. In total, during model development parents and teachers sent more than eighty tweets. However, a significant number of these tweets just encouraged students to continue the good work, while only a few tweets actually provided constructive feedback and identified weaknesses in students' models. During the last session students wrote letters to local authorities (as required by the activity), explaining and documenting their models/solutions. Finally, a class discussion focused on the key mathematical ideas and relationships that students had generated.

Interviews

All three teachers (two females and one male) and six parents (three females and three males), randomly selected out of the 28 parents that were involved in the study, participated in individual semi-structured interviews. Three areas of interest were investigated: (a) participant's (teacher or parent) beliefs on mathematical modeling and the implementation of the EngMEA, (b) parental engagement, and (c) their experiences during the EngMEA implementation with regard to collaboration and communication. The interviews were conducted right after school or in the early evening. Each interview lasted between 45 to 60 minutes and all interviews were audio recorded and later transcribed. Data from the semi-structured interviews were summarized through sequential analysis. A grounded theory approach was adopted. Themes were identified and clustered through axial coding, which was conducted in AtlasTI software.

RESULTS

Results are based on the qualitative analysis of interviews. The results are presented in terms of the themes that arose from the sequential analysis of teachers' beliefs and practices and parents' beliefs, with regard to mathematical modeling and parental engagement.

Teachers' beliefs

Teachers' responses could be summarized in three broad categories: (a) student cognitive goals and teaching effectiveness, (b) student affective goals, and (c) constrains of using a modeling based approach. Teachers had positive beliefs on the impact of IBL modeling-based approaches in students' cognitive gains. "This open approach get the children to think critically, set their own questions to reach the solution, and become independent in solving quite complex problems", a teacher commented. A second teacher added: "I really enjoy teaching [in a setting like this]. This is the only way to teach higher order thinking skills and problem solving skills." Teachers were also emphatic in commenting on the affective goals of using a modeling approach. They claimed that such approaches could help students develop "a love for mathematics and science". A teacher reported: "Some students rarely participate in more traditional lessons. Now, they repeatedly said that they like those activities [Primas modeling activities] very much". Teachers were also concerned with the constrains that are related to mathematical modeling, and especially with time: "There is so little time for extra curricular activities … modeling activities require a lot of time and efforts; how can you complete, even a part of it [activity], in 45 minutes?" Another dimension of teachers' comments was the use of Twitter®: "At the beginning I was sceptical … but then I was impressed by parents' involvement and how it [involvement] benefited students' work … I will continue using it [Twitter®] in my lessons, when possible. Well, this is obvious from the number of my tweets!" Another teacher added: "I love it! It is great when students and parents are involved. I had the impression that the lesson was on going; 24 hours a day. This is teaching!"

Teachers explicitly underlined the importance of active parental engagement. One commented: "Parents have good ideas to discuss with their children … there was a fruitful interaction between parents and children at home … we had discussions [in class] almost everyday … based on the interactions with their parents at home". Another teacher added: "Workshops helped parents to be involved and to realise that everyone was valued and that their role was crucial to the success of the activity. But of course that was not easy; only half of parents participated, right? And I am not sure how many of them spent much time finding resources at home and provided accurate feedback to students' solutions". Another teacher shared the same belief, that for fruitful parental engagement careful planning and organization were needed.

Another theme that arose during teachers' interviews was the impact of parental engagement in students' growth and involvement. Teachers reported that the activity could not be the same without parental engagement, which was clearly positive and constructive. Teachers also proposed that: "This approach was a good example and we clearly need more good examples in engaging parents in mathematics, which, I believe, is beneficial for students' achievement". Another teacher pointed: "Is there a better way to engage parents both at school and home? I do not think so. And I am confident that such appropriate engagement will help children improving their grades and attitudes towards mathematics and science".

Parents' beliefs

Parents were enthusiast and welcomed the modeling activity. They found the activity interesting and challenging not only for their children, but also for them. A parent, who was actively involved throughout the activity, commented: "I have been always good in maths … but the activity opened new horizons for me … I frequently visited the Wiki and commented on students' tweets. It was great!" Another parent said: "Such activities could help our children to develop important skills, needed beyond school … I am very happy that we use such approaches in our school". Although interesting, the activity was also found to be difficult. A parent mentioned: "Well, I could not know for sure that the solution was correct, and that was somehow annoying. Perhaps more guidance could help".

All six parents explicitly commented on the partnership climate that was generated. "I had the feeling that we [parents and teachers] were equal partners", one parent commented. She said: "It was far better than sitting at the back [in the classroom] to watch a lesson. We were actively involved and we had constant communication with our children and the teacher. It was really good". Another parent added: "I was following teacher's comments and suggestions and I tried to build on these, by discussing the problem at home with my child … Yes, it helped our communication".

To improve parental engagement in schools, parents seemed to unanimously agree that good communication and active engagement was the key. Parents suggested that a variety of engagement strategies could assist all parents to be engaged. One parent noted: “Attending lessons is not bad, but it cannot be the only way of engagement. I enjoyed the two workshops very much, although I had to leave work early. We should have workshops more often”. Another parent mentioned a strategy currently employed in the school: “Last year children had to work on two projects. Those projects were not focused on maths, but required some maths and science. I would like to see more projects like these, in which I can work with my child at home”.

Although quite satisfied with the situation, parents explicitly mentioned that they expected from school and teachers to do more, in order to enhance their (parental) engagement. It was revealed that school’s climate had a significant impact on the overall effectiveness of parental engagement. From parents’ responses a number of factors were uncovered, showing what schools should do in order to encourage and enhance parental engagement. A parent mentioned that schools should promote parental engagement using various methods, and not only by expecting from parents to be engaged. She said: “Schools and teachers must actively seek and promote the parental engagement. Not all parents are engaged by default”. The importance to implement initiatives that engage students was also mentioned by two parents. One of them mentioned: “Such activities are one of the best ways to engage parents, because their children are also engaged. When children are excited and discuss their mathematics work at home, parents are more inclined to be engaged in mathematics”.

DISCUSSION

The purpose of this study was to examine teachers’ and parents’ beliefs on parental engagement in mathematics teaching and learning, with a focus on modeling as a problem based approach. The results supported the expectation that such an approach was likely to positively affect teachers-parents’ partnership and possibly student outcomes (Epstein et al., 2009). The environment generated, provided opportunities for parents and teachers to establish appropriate communication and collaboration venues, which resulted in improved students’ models (English & Mousoulides, 2011). The modeling activity implementation as a means to engage parents in school mathematics could be considered successful; teachers were increasingly refocusing their teaching to incorporate and respond to students’ ideas and needs, and parents were responding positively to their new roles as engaging partners in their children learning. With regard to Twitter®, results revealed that Twitter® assisted in generating a safe, shared knowledge space in which teachers and parents gained insights into students’ learning, helped students to develop better solutions, and provided more opportunities for reflection and discussion.

During interviews, teachers expressed positive beliefs towards modeling, although they identified challenges and demands in their knowledge and other institutional constrains (Epstein et al., 2009). Using a modeling perspective in mathematics teaching, teachers had opportunities to adopt a more interdisciplinary, real world based approach, in which students’ role was central and parents’ impact had the potential to be very positive. Clearly, teachers’ beliefs were important in determining parental engagement in their classroom. Positive teachers' beliefs and attitudes were needed to maintain the best possible parental engagement and to build mutual understandings and collaboration for the improvement of mathematics teaching (Cutler, 2000).

Interviews with parents revealed that they hold positive beliefs and attitudes towards innovations in the mathematics classroom, like a models and modeling perspective. Parents also reported significant positive beliefs towards their engagement in schools, indicated at the same time the necessity for the school and teachers to take actions. Parents identified that a clear and constant bidirectional communication venue was urgently needed and they stressed that the modeling environment could be a successful method to achieve this goal. Parents also commented that many schools might focus their partnership activities on forms of engagement, like classrooms observations and formal meetings with teachers, although these were not perceived to be the most effective form of engagement activities.

The findings from this study suggest a need for researchers to expand their definitions of parental engagement, beyond traditional ideas of school and classroom norms, to include a dimension related to active parental engagement and technology rich modeling environments. Despite its limitations, this study provides new insights into the importance of modeling related parental engagement practices in mathematics teaching and learning. It suggests that teachers and schools that have positive beliefs towards parental engagement and facilitate the use of inquiry- modeling-based approaches are more likely to have positive active parental engagement and probably better students' learning results. Unquestionably, students need high-quality instruction to improve mathematics learning. However, if schools, teachers, and parents work together in creating appropriate, collaborative environments, they are more likely to see higher students' learning outcomes.

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STUDENTS' DISCUSSIONS ON A WORKPLACE RELATED TASK

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This study is situated in a mathematics course in the vocational education programmes in Norway. The students worked in small groups to make a budget for a hair salon. The real world connection made the students puzzled by the big numbers they calculated, and offered the students opportunities to discuss and evaluate their answers. They did not trust their own calculations without checking with others and, unsolicited, the students started comparing answers and discussing with other groups. The students were in the first stage of an inquiry process, but rarely moved on to ask their own questions or question their own estimates.

Key words: vocational education, real world, inquiry, task design, discussion

BACKGROUND

In Norway most of the adolescents (over 90%) continue with education after the 10 years of compulsory schooling (Statistics Norway, 2012). About half of them attend vocational education programmes to get a trade certification. These vocational education programmes are mostly structured as two years of school based courses followed by two years apprenticeship. In the vocational education programmes less than half of the students finish on time (The Norwegian Directorate for Education and Training, 2011), and one of the reasons for this is that they fail in mathematics (Norwegian Ministry of Education and Research, 2010).

During the first year all vocational students attend a mathematics course consisting of 84 hours course work. It is a goal stated by the government that this mathematics course should be oriented towards their vocation (The Norwegian Directorate for Education and Training, 2010a), though the content of the curriculum is the same for all nine vocational education programmes. The course consists mostly of topics the students have met before, numbers and algebra, proportionality, percentages, and geometrical shapes. Some new topics include price indices and accounting. From observations of lessons and conversations with teachers my impression is that students have forgotten, are confused about or never have understood parts of the mathematics they met earlier. The mathematics teachers are usually unfamiliar with the mathematics of the students' future jobs.

In a collaborative project with mathematics teachers from different vocational schools in the county, we, didacticians [1] and teachers have started to build a community for improving the teaching and learning in the mathematics course in the vocational programmes (MTPL research group, 2011). This paper is about how a task connected to their future employment encouraged students in a vocational education programme to discuss and use inquiry in mathematics. The research is based on findings from the pilot study of my doctoral research. The research question discussed in this article is: How and in which situations do students show an inquiry approach on a work place related task?

THEORETICAL PERSPECTIVES AND RELATED RESEARCH

The project is based on the key ideas from the LCM/ICTML and TBM-LBM projects (Carlsen & Fuglestad, 2010; Jaworski et al., 2007) about creating learning communities of teachers and didacticians. This learning community is supported by workshops, discussions between didacticians and teachers and school visits. Inquiry is an important concept in the community (Jaworski, 2005), both as inquiry into mathematics teaching and learning, and as a goal for student activity. Wells (1999) describes inquiry as ''a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them (Wells, 1999, p. 121)''. The tasks are provided for the students to experience an inquiry approach to mathematical activities. This inquiry approach can be visible when the students investigate, experiment, ask their own questions and try to find their own answers.

Skovsmose (2001) writes about different milieus of learning, and make distinctions between the tradition of exercises and landscapes of investigations. In both educational paradigms there is the possibility for division into mathematics, semi-reality and real-life references. Inspired by this model and the inquiry goal, the teacher and I aimed to design a task that would be in the landscape of investigations and connected to real-life. The task should be possible to develop in different directions, encouraging the students' inquiry during the problem solving activity.

To make the students aware of the use of mathematics in their future workplace, the task was connected to the students' future vocation by use of artefacts like receipts and authentic rental fees. Williams and Wake (2007) note that teachers, students and workers find it difficult to identify mathematics in the workplace. Furthermore research has revealed students' difficulties with word problems based on real-world situations (Greer, Verschaffel, & Mukhopadhyay, 2007; Verschaffel, De Corte, & Lasure, 1994). The students often disregard the context and give answers that do not make sense in the real world, for instance that one needs 31 reminder 12 buses to transport soldiers (Silver, Shapiro, & Deutsch, 1993). This signalled there was a challenge to make the students aware of and to use the real world connection when they solved the task. The given task was intended to encourage the students to use realistic assumptions and give them the possibility to evaluate their answers.

METHODOLOGY AND RESEARCH SETTING

I first met the teacher in workshops at the university, and he volunteered his class for the pilot study. Information about the research project was given to all participants.

One student declined to participate in the research, and therefore worked apart from the class during the research period. My case is situated in a class of 15 female students aged 16 and 17 who study in the vocational education programme ‘Design and Crafts’. This programme aims at education for jobs like jewellers, dressmakers, hairdressers and florists. According to the teacher, the students’ performance ranges across all attainment levels and there are some low achievers in his class, which is typical for the vocational mathematics courses. There are episodes in the collected data where this is evident, for example one of the students sang the multiplication song to calculate nine times five after saying she is not good at the nine times table.

The class was about to start working on an economics unit, where the competence objective was that the students should be able to ‘’make budgets and do accounting using different tools’’ and ‘’calculate taxes’’ (The Norwegian Directorate for Education and Training, 2010b). Together the teacher and I prepared a vocational oriented task about a budget for a hair salon. The task was revised several times before it was used in the classroom, mostly to make the level appropriate and the text clear. Many of the students planned to become hairdressers, related vocations like skin care workers, or other occupations where they would work in a small business. In addition the students, regardless of their future vocation, had experiences being customers at a hair salon. The students spent about two hours (three lessons) working on the budget. The teacher suggested, and I agreed, that I should present the task since I had formulated it, and take the leading role in the classroom with the teacher as observer and assistant. The students worked in small groups of three or four. Both the teacher and I circulated in the classroom to help the students.

The first task was supposed to help the students reflect on their own experiences as a customer at a hair salon. They were asked to write down ‘’what is a usual price for a lady’s hair cut?’’, ‘’what is the highest price you could have paid to get your hair cut?’’ and ‘’what is the lowest price YOU could have cut someone’s hair for? (Remember that you do not get paid the whole price yourself)’’. After a class discussion about this the students continued with the second task. This task was to ‘’write down the most important expenses a hair salon has in a month’’ and ‘’write down the most important income a hair salon has in a month’’. Then they were given a sample budget on paper and information on rental fees (per square meter), sample prices on haircuts, electricity, phone prices and so on, and the third task asked them to ‘’use the information you are given. If some important expenses or income is lacking you need to discuss to get realistic numbers’’. They were also to calculate the loss or profit. This third task took most of the allotted time for the students.

To do these tasks the students needed to decide prices, how many and what types of haircuts they could manage in a day, which and how big expenses the salon had and so on. This is in line with the inquiry idea where the students should investigate and ask their own questions. From a mathematical point of view there are possibilities for the students to learn about budgeting, estimation, modelling how many customers they can expect per day, calculating sales tax and area to determine rental fees. Much of this information is connected to rate of change, like haircuts per day, area needed per worker and rent per month. These rates of change are also connected with each other so that if one has many hair colourings per day one need to have more employees or fewer regular haircuts. The students mostly needed 'elementary' mathematics like addition and multiplication to start on the budget, but they had to decide and estimate the important information. When the students finished a paper version of the budget, they started to use a spreadsheet. But they used it only to record their data, and did not apply formulas to be able to alter their input. It could have been possible to make a dynamical budget model on a spreadsheet for the students to encourage them to try out different scenarios. The task opened up the possibility for the students to collaboratively inquire into the mathematics of budgeting, and together decide what a good and realistic budget is. There was no definitive correct answer to the task, and the students were challenged to explain and defend their choices, both by the teacher and when the groups talked to each other.

In order to document the activities the lessons were video recorded with one camera. When the task was introduced the video camera had a view of the students and the projector screen. During the students' work with the task the camera was focused on one group with three students, and their discussions with the group next to them. The students were instructed to work as normal as possible and try not to be disturbed by the presence of the camera. After the lesson, the students' notes were collected, and I discussed the students' work with the teacher. This session was audio recorded. The recordings both from classroom activity and my conversations with the teacher were transcribed. I translated the Norwegian transcript into English aiming at the closest fit of words and meanings. No students were interviewed about the tasks.

From the transcripts it appeared that the size of the numbers often puzzled students. This made me curious, and I identified all episodes where the students discussed their answers. The episodes presented below are selected from this set of episodes and are representative of the work done by the students on this task. To observe if the students began to inquire into the mathematics I looked for engagement, if the students expressed puzzlement, what they were puzzled about and if something in the task or their work triggered their curiosity. In addition I noted how and with whom the students sought to discuss and evaluate their answers, and if they asked the teacher, the other students or did something else.

SOME FINDINGS DEDUCED FROM THE DATA

In this paper I focus on the episodes where the students discussed their answers. These episodes were analysed looking for how and in which situations the students showed an inquiry approach in their work. This analysis showed that:

1. The students were often surprised by and wondered about their results.
2. The students compared and validated their results by discussing with other groups or the teacher/didactician.
3. The students evaluated their results, and sometimes reconsidered their initial assumptions.

These observations are possible to link to the notion of inquiry. It can be observed that the students sought to understand their answers, started asking questions and discussed with each other. The following excerpts are from the work of small groups A and B, which were located next to each other. The groups worked separately, but did sometimes interact. In the transcriptions behaviour are written in italics.

The students were often surprised by their results

The students in both group A and B were often surprised and disbelieving of their results. This can be observed in the following episode at the beginning of the first lesson. Group A decided that their classroom was about the right size for their salon, and made an estimate of the size of the room to be about 50 square meters. The rental fee was given as 1200 NOK per square meter, about 160 euro.

Student A1: We'll say 50 then. 50 times 1200. *They take up cell phone calculators.*

Student A2: First to calculate it. Damn. *Pressed something wrong on the calculator.*

Student A1: 60 000.

Student A2: *Laughter.* Okay.

Student A1: Sick. Gosh.

Student A2: In a month?

Student A1: Yes. If we have about this [size of room].

Student A2: Wow. *All three students focusing on their notes and looking surprised.*

This was also evident in the other groups. In group B student B1 claimed loudly that one cannot pay 42 000 in rent to which student A1 replied ''we have 60 000 monthly'', while the other members of group A giggled. Student B1 showed her surprise by exclaiming ''yes, wow!'' This surprise can be connected to the real world setting of the task, and that the students had a sense of how much money this is. For the students 60 000 is about equal to three or four months' salary when they start their apprenticeship. This puzzlement made group A and B compare and discuss informally their answers with each other.

The students validated their results by discussion

To confirm their estimates and numbers the students approached both the teacher and didactician and each other. Often the students first asked the teacher/didactician, and then compared with the other groups. It therefore looked like validating with their peers was at least equally important as feedback from the teacher. Here group A and B were still wondering about the rent, and asked me about it. I explained that ''well, but it is actual real numbers''. The students giggled and exclaimed ''goddamn. Oh my god''. I then tried to point the students to that ''... You earn some [money] also. You do not earn 450 kroner one day in a month, in a way, on a lady's haircut. You earn some more'' to which student B1 contradicted with ''not 42 000 NOK''. But student A1 then said ''yes, yes, you do [earn] that. Oh my God.'' and giggled again.

The students did not appear to be satisfied with my answers, and were still hesitant about the big numbers. I did not provide a ''right'' solution, but tried to encourage the students to reflect on if there may be an explanation for their seemingly big numbers. They continued to work in separate groups after this exchange, but did not change their answers even though they appeared unsure. The students might also have thought that since I did not say that it was wrong, their solution had to be fine.

The lesson ran out of time, and we continued the next mathematics lesson, four days later. The students worked on deciding income from different items, like ladies' haircuts, product sales and hair colourings. After doing this the students added up to get their total income and group B again asked me about the validity of their results, and started discussing with group A.

Student B1: But we have only 4 haircutters, and we earn [income of salon] almost a million a month. Is it very unlikely to earn that much in a month? *Addresses Trude.*

Student A1: We have hundred and fourteen [114 000].

Student B1: Hundred and fourteen. What?!

Trude: Then you can evaluate why you have that much difference.

Student B1: Oh my god, we have a million a month.

Student A1: What?

Student B1: Yes. *Smiles and giggles.*

Student A1: Yes, but this [114 000] is a week.

Student B1: Oh, you have calculated a week. Well then.

I tried to ask the students to evaluate why they had different answers, but the comparison between weekly and monthly income appeared to end the progress of the discussion. The students were still interested in the numbers, and when a student from another group came by two minutes later they again discussed their income.

Student A1: Yes, we earn this much a week. Hundred and fourteen thousand.

Student C1: Week?

Student A1: *Giggles.* Yes.

Trude: Weekly?

Student A1: Yes, we calculated a week.

Student C1: We earn, about, eeh, around 75 000 a month.

Student A1: What? *Asks with a sharp voice.*

Student C1: A month. That is lots.

Student A1: That is little. They have a million. *Points to group B.*

The students then continued working without changing their answers. Here I observe that the students compared the answers with each other, and they did not agree about what is too much income. When the group with a million in income was drawn into the comparison everything else appeared small, even though there is a big difference between the other groups. The students' continuation of the work could indicate that the strangeness of the big numbers was reduced with the comparison with the other groups. There may also be a misunderstanding between the students about earnings, income and profit. Both the students and I used the words interchangeably, and this may have caused some of the puzzlement over the amount of money generated.

Evaluating the results and reconsidering assumptions

There were episodes where the discussion between students led to changes in their estimates for the budget. This can be seen by the following episode where group A started to make the budget in Excel after finishing the paper version. Their own notes did not show their estimates, so the group reversed the operations to figure out their earlier estimates. For instance they took 240 000 in earnings from hair colourings, and divided by the colouring price and got 240 persons. The students had assumed that they could colour four persons per employee every day, so four colourings times three employee times twenty days a month. When making the budget for a week they did not reflect on this, but now they started questioning their estimate.

Student A1: Then it is 240 we colour a month. We can't colour 240 persons a month!

Student A2: I believe we can colour about 100 persons, no, not even that.

Student A1: Not even that. Maybe 50, 70 something. Then this is wrong again. How many should we say that we can colour, 50 or 70?

They decided on 70, and continued working. There are also situations where the students compared their results with other groups, and identified the factors that made the differences between the budgets. When group B had finished their budget and calculated the profit they approached group A to compare the results.

Student B1: How much did you have in profit?

Student A1: Wait a minute. First we had three hundred [300 000], now we have two hundred and eighty four [284 000].

Student B1: *Shows her notes with the profit 122 160 kroner.*

Student A1: But you have more employees, we have a bigger salon [in square metres]. You pay 42 000 in a month, we pay 60 000 [in rent].

Student B1: We have four employees.

Student A1: We have three.

Student B1: Yes.

Student A1: Then it is a bit …

Student B1: We earn actually more [than you].

Here the students realised that one of the important variables in the budget is the number of employees and therefore the ability to generate high earnings. It seemed important to have the best salon, and they gave the impression of being satisfied with having the salon with the highest profit. These competitions between the groups are also evident in other episodes, and the students discussed what is included in the prices in their different salons, like free coffee, the work space and so on.

DISCUSSION

The students looked like they engaged in the task, but this could be an effect of the video recording, and the novelty of a different teacher/researcher. The students were interested in the camera, and there are instances of them singing and dancing in front of it. I will still argue that the students did not act for the camera while working. When they fooled around they did not do mathematics, until they remembered their task. When they started working again they gave the impression of being focused on the task and not the camera. I therefore believe that what they said during their mathematical activity is what they would normally express in the classroom.

The task of making a budget for a small business, like a hair salon, is something many of the students will encounter in their working life. This may have given the making of the budget some authenticity even though it was clearly a task in the school setting. It could be that the real world setting made the students willing to question their answers and discuss them with members of the other groups. This questioning may be an indication that the students were engaging in an inquiry process and started to seek understanding with the others. The puzzlement with the answers, and thereafter discussions seemed to contradict the research on word problems where the students mostly ignored the real world setting.

At the start of the lesson the students and I spent some time discussing different prices at hairdressers, and this gave me the impression that students associated this with their own experiences. This starting activity was helpful when the students made estimates and worked out the budget. The students may have been more aware of the connection to the real world, and therefore more critical of the realism of their answers. When they are making a budget in their work life they will need to estimate the customer base, and how much income the small business can generate. The task presented in this study could help them to understand how different expenses and incomes are connected to each other.

The task planned to be consistent with Skovsmose's landscapes of investigations with connections to real life references. My findings show that the students displayed a willingness to wonder. They were often surprised by their results, and could be on the way toward an inquiry stance in mathematics. They were seeking confirmations from the teacher and didactician and the other groups to validate their results, and while solving the task the students discussed and evaluated their answers with the others. It looked like they were more likely to trust their calculations when they realised that the other groups had gotten answers that had similarly big numbers. After working with the task for a while they did not seem so puzzled by the big numbers. The students reconsidered and revised their answers when they discussed if they really could colour the hair of 240 persons a month. I sometimes tried to encourage the students to ask questions and stimulate them to discuss what made the differences in their answers, and therefore what would make the most impact on the budget and profit. This was not an easy process to start, and even though the students discussed their answers it rarely went further than comparisons between the groups.

To summarise, the findings suggest that the students showed an inquiry approach in the work place related task when they got puzzling answers. The students tried to correct or validate their solution through asking questions to the didactician or the teacher, and by comparing or discussing with the other students. They sometimes revised their answers, but did not start asking their own questions. This questioning could have been further encouraged if one had more comparisons in the whole class, and challenged the students to defend their choices. This could have been done by comparing the different groups' budgets on a spreadsheet managed by the teacher, and thereby opening up possibilities for discussion and explanations.

To further develop the research and the real world connections the task could be expanded with the students investigating how hair salons make their budget, and what income and expenses they have. They could gather information on a typical daily income of a hair salon, and then revise their budgets. This real data might prompt more discussion and evaluations of their budgets, and possibly be the start of more inquiry processes for the students. To find out why the students did not change their answers it would be informative to interview some students. I will continue the research with other vocationally oriented tasks and other students.

NOTES

1. As in the LCM project I use the term didacticians for the researchers/educators from the university since both teachers and university researchers can engage in the research (Jaworski, 2005).

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TEACHER BEHAVIOUR IN MODELLING CLASSES

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The paper deals with the question of teacher behaviour in modelling classes and focuses on teacher interventions. Different aspects of interventions are considered, such as triggers of interventions and teachers' purposes as well as effects and consequences of interventions. The presented results indicate that interventions which are based on the demand of explaining the state of work and invasive interventions that are triggered by a lack of students' progress of work can foster students' independent solving processes.

Keywords: Teacher interventions, mathematical modelling, empirical research

INTRODUCTION

Mathematical modelling has been one of the main research areas at the Working Group of Didactics of Mathematics at the University of Hamburg already for a long time. During this time various aspects of mathematical modelling have been investigated. In recent years, the focus has been on fostering students modelling competencies most effectively. At the moment, two different PhD theses are carried out. One of them compares the effectiveness of a holistic versus an atomistic approach (for detailed information see Grünewald, 2012). The other one focuses on the effect of different kinds of teacher interventions (for detailed information see Stender, 2012).Within recent research the question of teacher behaviour in modelling classes and the effects of teacher interventions were analysed. In this paper we will introduce selected results of these current research studies based on two master theses.

THEORY

Mathematical Modelling in the classroom

As mathematical modelling is the topic of this working group, we do not want to outline mathematical modelling in general. But as mathematical modelling in the classroom deals with different aims (see Kaiser & Sriraman, 2006 for an overview), we want to stress those points which are important for our understanding of modelling, which is reflected by the modelling tasks chosen for the studies. Departing from the realistic or applied perspective on modelling described by Kaiser & Sriraman (2006) the modelling tasks dealt with need to be complex in order to foster the pragmatic aims required by the realistic perspective on modelling. This means that the modelling process starts with a question to answer and some background information. Then students therefore have to find out the information they need to work on the task. Sometimes this information can be calculated by given photos, sometimes the students have to read them up and sometimes they have to make assumptions on the basis of their own experience. The grade of complexity depends on the students' experience with modelling tasks as well as on the time they have to work on the problem. Another characteristic is that the task is not implemented in a special content, so the students are completely unsure which mathematical techniques are useful for solving the given problem. Furthermore, we try to choose authentic tasks, i.e. tasks which are relevant at least for a special group of people, at the best for the students themselves.

Scaffolding in mathematical modelling classes

Teaching is a complex process. In order to support students effectively, teachers not only have to have strong knowledge about different contents, but also about different types of teaching methods and adequate assistance. One theoretical approach that deals with tailored and temporary support that teachers can offer students is scaffolding. Because scaffolding has been studied extensively in the last couple of decades, slightly different approaches exist [1]. However, the central goal of all scaffolding approaches is to enable students to solve problems on their own. For this purpose, students are supported in a very practical way when they are not able to solve given problems or when they get stuck. The support takes place on both the cognitive level (required strategies and concepts) and the meta-cognitive level (instructing self-regulated learning). The main principle is a consequent orientation of the students' individual learning process, which Van de Pol et al. (2010) calls *contingency*. A condition, therefore, is the willingness and the competency of teachers to be responsible for the demands of thinking and understanding of students. So the teacher should have content and diagnostic knowledge. In particular, when students work on complex modelling tasks and can choose mathematical techniques on their own, the teacher must be able to decide in a short period of time, if the students approach is expedient or not. Depending on how self-regulated students are in their working process, the teacher reduces the support, which is called *fading* in the sense of Van den Pol et al. (2010), because the teachers are *transferring the responsibility* to their students.

Within the framework of scaffolding, Hammond and Gibbons (2005) developed a model, which differentiates between scaffolding on a macro- and on a micro-level: on the one hand, teachers can provide or foster different didactical settings (e.g. group work) and consider different students' characteristics (e.g. thinking styles, beliefs). This kind of assistance on a macro-level is called *designed-in-scaffolding*. This can be planned before attending class and has to be based on pedagogical content knowledge as well as on didactical knowledge about the modelling process. On the other hand, teachers can intervene at special times while students are solving mathematical problems. This kind of assistance focuses on a micro-level and Hammond and Gibbons call these interventions *interactional scaffolding*. Interactional scaffolding cannot be planned in detail. The single interventions have to be based on pedagogical content knowledge as well as on didactical knowledge about the modelling process. Whether these interventions are adequate or not depends on the circumstances in which they occur. Consequently, adequate intervention must be a consequence of a teachers' diagnosis of a students' difficulties while solving mathematical problems.

When talking about different kinds of interventions, one has to distinguish between the trigger, the level, and the effect of interventions. Concerning triggers, Leiß distinguishes between invasive and responsive interventions: every time teachers intervene on their own initiative, the intervention is called invasive. If students ask for help, the intervention is called responsive (Leiß, 2007: 105f.).

The most well-known distinction between different levels of interventions is the taxonomy of assistance according to Zech (2002). He differentiates motivational, feedback, strategic, content-oriented strategic and content-oriented assistance. The intensity of the intervention decreases gradually from motivational assistance to the content-oriented assistance. This classification has been used several times to describe possible assistance in modelling processes (see Leiß, 2007; Maaß, 2007). Based on this categorisation, Leiß created a descriptive analysis of adaptive teacher intervention in the modelling process. The analysed interventions were classified by their trigger, level, and intention (see Leiß, 2007). The main results of Leiß' study were, among others, that strategic interventions are included in the intervention-repertoire of the observed teachers only very marginally and that the teachers often choose indirect advice where students have to find only one step by themselves in order to get over their difficulty. Further studies like Link (2011) cannot confirm these results. This is significant because in these studies it was found that, in particular, strategic interventions lead to metacognitive activities of learners (see Link, 2011). The mentioned studies have one recommendation in common: Maria Montessories' principle "Help me to do it by myself".

Besides different triggers and levels of intervention and the knowledge of the modelling cycle, teachers should also be aware of the role of metacognition within mathematical modelling for a basis of possible interventions. In recent research on metacognition, a distinction between declarative metacognitive knowledge (i.e. learning strategies, person and task characteristics) and procedural metacognitive skills (i.e. controlling, monitoring and self-regulation) is made (Schneider & Artelt, 2010). Stillman, Galbraith, Brown & Edwards (2007) developed a theoretical framework for studying students' procedural metacognition. Galbraith & Stillman (2006) point out that reflections should be related to mathematical content and the processing decisions for fostering the students' modelling competencies. Only in this way can students become better modellers and not just solvers of separate problems.

To sum up, in order to foster students modelling competencies teachers should see themselves as *moderators or facilitators of knowledge* rather than as *disseminators of information* (see Herget & Torres-Skoumal, 2007).

PROJECT SETTING AND SAMPLE

The research results we want to present in the following were achieved within different frameworks. For this reason the conditions of data collection and, of course, the research questions were different in detail, although both projects were focused on teacher behaviour in modelling classes. In the following, we will briefly describe these different conditions as well as the samples and methods used to collect and analyse the data. An overview of the different project settings and samples of the two studies is given in table 1. Beutel and Krosanke (2012) as well as Meyer (2012) analysed teacher behaviour in modelling activities of grade 9 students, but Beutel and Krosanke collected data during modelling days, while Meyer collected data from two of six modelling activities in double lessons that were integrated in normal math classes. The modelling days [2] are carried out once a year for all grade 9 students in one secondary school (Gymnasium). Over three days they are asked to work on one complex modelling task, which they can choose between four different tasks. The students work in small groups and are supervised by two university students who were prepared for this within a master seminar at university. So one can say that these students are experts concerning the special tasks and novices concerning teaching in general. In 2012, 160 students attended the modelling days and were looked after by 32 university students, so each small group contained about 10 students. All groups that had chosen one special task ("traffic lights or roundabout") were videotaped. During the other modelling project (ERMO, Grünewald 2012) different forms of arrangements (holistic and atomistic approach) were tested against each other in order to evaluate, which is more effective in fostering students' modelling competencies [3]. For this purpose, around 20 classes of the 9th grade of six secondary and district schools were divided into two groups: group A tackled modelling problems according to the holistic approach and group B tackled modelling problems according to the atomistic approach. The intervention period started in February 2012 with a teacher training. During the intervention period the classes performed six modelling activities. Before the first and after the fifth modelling activity the students wrote a modelling test. In addition to these tests, they filled in a learning questionnaire at the end of each modelling activity and the teachers filled in short questionnaires about the run of the modelling activity. During the modelling activities the students in the holistic group dealt with complete modelling problems with an increasing complexity of tasks. The students of the atomistic group dealt with sub-processes of mathematical modelling separately. The modelling activities of both groups were designed with an autonomously-orientated learning environment – such as small group work, the principle of minimal help and a demand for reflection.

Table 1: Framework of the studies

	Beutel & Krosanke	**Meyer**
Students	One group of four students in grade 9, secondary school	Two classes in grade 9, divided into small groups, secondary and district schools
Teachers	University students, experts in modelling, novices in teaching	Experienced teachers, novices in modelling
Length of analysed solving process	2 days, each lasting 5 hours	2 lessons, each lasting 1.5 hours
Modelling tasks	One complex modelling task, students chose on their own	Two modelling tasks, all students working on the same task given by the teacher
Data collection	Videotaping of the whole solving process	Audiotaping of the solving processes, interviews with the teachers after each lesson
Data analysis	Qualitative content analysis	
Research aim	Analysis of teacher interventions and the effects and consequences on students behaviour	

Beutel and Krosanke chose one of the videotaped groups of students who worked on the task "traffic lights or roundabout" and analysed the interventions given by the university students. The reasons for the selection of this group were the quality of video and tone, the level of performance and – most importantly – the level of communication (in this particular group every aspect and every assumption were discussed, calculations were read aloud). This special group only consisted of four students who had very low experiences with modelling tasks and were looked after by one female and one male student. The videotaped students worked for ten hours on the question whether traffic lights or a roundabout is more effective for crossroads. Meyer watched the behaviour of two different teachers who were willing to be observed and interviewed from the holistic group during the first and the third modelling activity. Moreover, the two teachers were recorded, so Meyer was able to analyse the teachers' verbal interaction with the students in detail. After each lesson she did an interview with both teachers.

Beutel and Krosanke, as well as Meyer, analysed their data using qualitative content analysis by Mayring (2010). Beutel and Krosanke reconstructed learners' problems as well as the teachers' behaviour, both in relation to the different steps of the modelling process. This proceeding allowed a more sophisticated look at the potential and the effects of teacher interventions during the modelling process. In contrast to other studies, Beutel and Krosanke analysed the effects of interventions in the context of the complete modelling process by short-term and long-term considerations connected with the solution process of the students. In addition, a more differentiated view was sought on the success of the teacher interventions. So the main aim of the study was to describe the effects of intervention at various levels and to analyse them according to their appropriateness. Meyer encoded all interventions following Leiß (2007), with regard to the three main categories: trigger, level, and intention. The main aim was to investigate teachers' behaviour in modelling lessons, divided into introduction, group work and presentation phase. Furthermore, Meyer tried to relate the teachers' behaviour to the concepts of the role of a teacher as a *disseminator of information* respectively a *moderator or facilitator of knowledge* (see Herget & Torres-Skoumal, 2007).

RESULTS

Beutel and Krosanke, as well as Meyer, analysed teachers' behaviour in different kinds of modelling classes. In the following, we will outline the results of both studies referring to two aspects: different types of triggers of interventions and teachers' purpose of the intervention as well as effects and consequences of interventions.

Types of triggers and teachers' purpose of the intervention

The interventions identified by Beutel and Krosanke were mostly invasive. A detailed view reveals that these interventions were given after a period in which students were not working on the task. So Beutel and Krosanke conclude that teachers intervened not to guide students to solve the modelling task but obviously their aim was to help the students because they had recognized problems. The strategic intervention which occured most often was the request to present their state of work. As well as strategic interventions, feedback and content-related interventions occurred.

Meyer encoded the interventions of both teachers as mostly invasive, but while the interventions of teacher A mainly referred to organisational aspects of the modelling activities teacher B provided much more content-related help. The analysis of the observed math lessons shows clear differences in the teachers' intervention behaviour and suggests the typing of teacher A as *moderator or facilitator of knowledge* and teacher B as *disseminator of information*. Teacher A as *moderator or facilitator of knowledge* acted in a restrained manner during the different phases of the modelling classes. This, for example, can be seen in the fact that he did not immediately correct students' mistakes such as inadequate assumptions in the solving processes. Teacher B as *disseminator of information* intervened much stronger and more often during the modelling classes than teacher A. Teacher B, for example, corrected students' mistakes immediately and controlled the solving processes by setting the steps of solutions that he considered adequate. Instead of providing content-oriented assistance, teacher A's interventions were mostly motivational and strategic, for example the students were asked for the problem formulation and encouraged to make appropriate assumptions, as well as to use their own context knowledge.

Effects and consequences of interventions

Concerning the effects and consequences of interventions, Beutel and Krosanke reconstructed that the strategic intervention *presentation of state of work* had potential both for students and teachers. One consequence of this particular kind of intervention is the reflection and the structuring of present results and present action. As a result, Beutel and Krosanke could reconstruct students' ability to solve a partial problem after an intervention was given; the realization of the importance of obtained results and thus their incorporation into the solution process and the verbalization of previously remained intuitive insights. Another consequence, as mentioned before, was that Beutel and Krosanke were able to confirm this special kind of intervention as a diagnostic tool for teachers.

Looking at the effect of interventions, Beutel and Krosanke differentiate between short-term and long-term effects. However, a definition of effects was impossible because of the complexity of effects. But in trying to define effects, Beutel and Krosanke inevitably had to look at interventions which led to metacognitive processes. They were able to reconstruct effects on the declarative level as well as on the procedural level of metacognition. In most cases, strategic interventions are the trigger of such processes, but content-related interventions and feedback can also lead to metacognitive processes. Every time Beutel and Krosanke analysed feedback as a trigger of metacognitive processes, feedback was given in combination with a content-related intervention. They also point out, that metacognitive processes that were triggered by teacher interventions do not always lead to progress in the solution progress; sometimes the interventions did not influence the solving process at all.

As a result of the behaviour of teacher A as a *facilitator of knowledge*, Meyer observed the students as being encouraged to think and work for themselves. Due to the many teacher interventions of teacher B as *disseminator of information* during the solving processes the students were hardly able to work for themselves. Through numerous hints the students were directed to appropriate solutions.

While teacher B mainly focussed on the mathematics and the correctness of the real solutions, teacher A aimed at organizing and supporting the students' individual learning processes. Meyer assumed that the behaviour of the *moderator or facilitator of knowledge* can especially help students work independently on modelling tasks and promote the development of students' mathematical modelling competencies.

CONCLUSION

Both studies dealt with the question of teacher behaviour in modelling classes. In both studies the effects of teacher interventions were one analysed aspect. While the study by Beutel and Krosanke aimed at describing the effects of an intervention at various levels and analysing them according to their appropriateness, the study by Meyer focussed on describing the role of the two teachers as *moderator or facilitator of knowledge* respectively as *disseminator of information*.

Both studies describe the observed teacher interventions as mainly invasive, while different types of interventions could be reconstructed, for example motivational, strategic and content-oriented assistance. However, differences between the preferred types of intervention among the teachers could be identified. The study by Meyer shows exemplarily two different types of teacher behaviour in modelling classes, teacher A as moderator or *facilitator of knowledge* and teacher B as *disseminator of information*. While teacher A mainly used motivational and strategic assistance, teacher B provides more often content-oriented support. The study by Beutel and Krosanke describes the role of the teachers as moderator or facilitator of knowledge i.e. the teachers did not aim at guiding the students to solve the modelling task in a prescribed way. Concerning the effects and consequences of interventions, both studies reconstructed mainly strategic interventions.

The different results of these studies point out that invasive interventions are not to be rated as more appropriate than responsive interventions or the other way around: invasive interventions, which are of a organisational nature or which are carried out because teachers diagnosed a lack of progress or students' helplessness, seem to be valuable for the students' solving process.

A second point is that teacher B often expressed his own uncertainty in how to intervene in a strategic way. The university students who were acting as teachers in the other project were trained for this work over several weeks. According to this one can conclude that the usage of strategic interventions can be promoted by specific training activities within teacher education and that they are also accessible for young students.

A third point is that for all of the researchers it was difficult to find out which interventions were effective and have consequences. This is due to different aspects: the effectiveness of an intervention does not only depend on the intervention itself but also on the student to whom it is given. Thereby, one cannot generalize the effectiveness of special types of interventions. Another reason is the definition of effectiveness: can you classify an intervention as effective, if it enables the students to continue their work for only for a few minutes? What about long-term-effects? Meyer could classify the teachers she had observed as *moderator or facilitator of knowledge* (teacher A) and as *disseminator of information* (teacher B) and reconstruct that the behaviour of teacher A promotes students' independent solving processes much more than those of teacher B. If the results of this case study can be generalized, it would help us to answer the question of which interventions are adequate.

NOTES

1. An overview on different approaches of scaffolding is given by Van de Pol et al. (2010).
2. The modelling days are one important part of Peter Stenders PhD project (Stender, 2012) and ares based on the idea of modelling weeks, which are carried out since several years at different universities (Kaiser & Schwarz 2010).
3. A definition of modelling competencies is for example given by Maaß (2006).

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STUDYING THE TEACHING/LEARNING OF ALGORITHMS AT UPPER SECONDARY LEVEL: FIRST STEPS

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KEYWORDS

Algorithm, Algorithm Workspaces, Modelling competencies, Programming, Simulation.

SUMMARY OF ONGOING RESEARCH WORK

The new French curriculum for upper secondary mathematics introduces building algorithms as a task for every topic. The underlying assumption is that an algorithmic approach can help students access to mathematical notions. The design and construction of algorithms by students is located within a more general framework of modelling approaches in mathematics. The first steps in this work include: **1.** Initial questions **2.** A specification of the knowledge at stake **3.** The elaboration of a theoretical framework taking into account general frameworks in math education and in computer science education, as well as frameworks related to modelling **4.** The preparation of a didactical engineering **5.** A projected study of students' modelling competencies.

Initial questions

According to the new French upper secondary curriculum in mathematics, teaching algorithmic should help to give sense to a number of studied concepts. The questions are: **1.** How teaching can go beyond this objective so that algorithmic becomes a learning object? **2.** How working on algorithm can be considered as a modelling task?

The knowledge at stake

1. The conception and execution of algorithm. **2.** The concept of algorithm: as a "tool", as an "object". **3.** The "Algorithmic Workspaces". **4.** The modelling competencies.

Theoretical frameworks

Our theoretical framework is based on dialectic "tool-object" (Douady, 1984), a transposition of "Geometrical Workspaces (GW)" (Houdement and Kuzniak, 2003, 2006) to "Algorithmic Workspaces" (AW). Further, we will use the description of the levels of modelling competencies (Henning and Keune, 2007). In this approach, an algorithm will be seen as a model of a real world situation. This transposition GW to AW makes possible to start by thinking the interactions between objects, artifacts and paradigms when analysing tasks, and helps to analyse the respective positions of students and teachers, under influence of curricula. We propose to structure the AW into a network of components: **1.** sets of objects; **2.** artifacts "programmable"; **3.** a set of theoretical ideas that help to create and justify algorithms on the objects for execution by the artifacts. The paradigms algorithmic allow to interpret the contents of the components and to define their functions on three levels to algorithms:

Level I: an intuitive approach of algorithms of real life situations. At this level, for instance, the algorithm's effectiveness follows naturally from its description.

Level II: a "natural" axiomatic of algorithms. Effectiveness of the algorithm used, its efficiency and complexity are questioned. The algorithm can become an object.

Level III: formal treatment of algorithms (e.g. Turing machines).

At all three levels, just like in geometry, representations of algorithms involve specific registers (Duval, 2006) with conversions and treatments.

Preparation of a didactical engineering

The method is based on the idea of a didactical engineering (Artigue, 1992). In a framework of modelling approach in mathematics, a random process' simulation tasks is analysed. We consider a random phenomenon based on number of births within a family that stop either: – after the birth of the first son – after the family has four children. We expect from the students that they can implement and understand simulation with diverse instrumental choices and representations of randomness. It is implemented with three $11^{th}/12^{th}$ grade classes.

Projected study of students' modelling competencies when building algorithms.

In a modelling approach, the student is confronted with the following challenges:

- describe the various steps between the observation of reality and the construction of the mathematical model developed;
- move to the simulation, and then write, in natural language, one (or more) algorithm (s) the model obtained, and the simulation defined;
- encode the algorithm(s) obtained for use in machine language;
- do relevant interpretation of the results obtained.

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THE ROLE OF MODELING ON EFFECTS OF IRANIAN STUDENTS

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Abstract

In this study, twelve problem (4 modeling problems, 4 word problems and 4 pure math problems) used a pre-test and post-test for 92 grade 10 Iranian students to examining their effects(enjoyment, value, interest and self-efficacy) about different type of problems. There are three session teaching units between pre and post test as an intervention. Results of this study show that effects of students were improved.

Keywords: Modeling Activities- Effect- Enjoyment- Self efficacy.

In this study, 92 Iranian students (39 girls and 53 boys) in grade 10 participated. The main purpose of this study was investigating impact of modelling activities on effects (enjoyment, value, interest and self-efficacy) of students about different type of mathematical problems (Modelling problems, Word problems, and Pure Math problems). This study was in the line of German educators' research (Schukajlow, et al, 2011).

For data collection, we use questioner contain 14 problems- 4 problems were modelling, 4 of them were word problems and last 4 were pure mathematics problems. These problems come from research literature about modelling in the case of modelling problems, and from Iranian textbooks in the case of word and pure problems. This questioner was used in pre-test and post-test for assessing the effect of students. Between these two tests, there is an intervention. This intervention was containing 3 modelling activities (different with pre- & post- test) that teach to students in 3 ordinary classrooms session-about 75 minutes for each session.

In pre-test and post-test, students asked for self-reporting about 12 problems without solving these problems. This self-report tool comes from Schukajlow, et al (2011), and contains 4 statements for each 12 questions as below:

- I would enjoy after solving this problem....
- I think it is important to be able to solve this problem....

- It would be interesting to work on this problem...
- I am confident I can solve this problem...

Students asked to answer these statements with 5 point Likert scales contain strongly disagree; disagree; neither agree nor disagree; agree; strongly agree. These data used for SPSS. For data analysis, ANOVA, T-test and Pearson correlation was used. Other sources of data collection were classroom observation and semi-instructed interview with students.

Results of this study show that the effect of Iranian students about modelling problems was lower than other problems. While in Schukajlow, et al (2011) study, students have same effect about different type of problems. Iranian students have more enjoyment about word problems and pure math problems than modelling problems. They also have less self-efficacy about modelling problems than other two types of problems. These finding was different with Schukajlow, et al (2011) study. We think this different result is rooted in stronger background of German students in modelling activities in both research and practice.

Another interesting finding of our study was about Iranian students' opinion about modelling problems. They think these types of problems did not belong to mathematics. This finding is in the line with Rafiepour and Gooya (2010) research. Rafiepour and Gooya (2010) discuss about Iranian teachers' opinion about real world problems. In some cases, Iranian math teachers didn't believe to real world problem. Indeed, these Iranian mathematics teachers thought real world problems didn't belong to mathematics but belong to other field of knowledge like economy, biology and so on.

All students in their interview said they had enjoyed with modelling activities and they would like to have such activities in the official curriculum and textbooks.

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG7: MATHEMATICAL POTENTIAL, CREATIVITY AND TALENT

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This was the second meeting of the WG *Mathematical Potential, Creativity and Talent* that was established at CERME 7 in 2011. The goal of the WG is to draw the attention of the mathematics education community to the field of mathematical creativity, mathematical potential and mathematical giftedness. The WG was formed to encourage discussion among researchers in the field of mathematics education and research mathematicians, and to promote empirical research that will contribute to the development of our understanding in the field. Following the debate at WG7 at CERME-7, we carried on an international exchange of ideas related to research on the didactics of teaching highly able students as well as on the promotion of creativity in all students.

The papers accepted for presentation in the WG served as a starting point for the group discussion. Participants were not expected to give a formal presentation of the papers. At each session a keynote speaker was assigned. Keynote presentations were allotted 15 minutes followed by 5 minutes of questions from the audience. All other participants were given 5 minutes for presentation of the main ideas excerpted from their papers with another 5 minutes of questions from the audience. We varied the forms of the discussions (e.g., whole group discussion, small group discussion) on different topics assigned by the group leaders

Four central topics were assigned for discussion by the group participants: (1) mathematical reasoning, creativity and giftedness, (2) mathematical problems for the gifted and mathematical problems for the development of creativity, (3) teachers and the teaching of mathematically gifted students, along with teaching all students with and for creativity, and (4) research methodologies in research on creativity and giftedness.

MATHEMATICAL REASONING, CREATIVITY AND GIFTEDNESS

The following questions led the discussion on the topic of mathematical reasoning, creativity and giftedness: How can gifted children and twice-exceptional children be helped to develop their potential and capabilities? How does mathematical creativity express itself at the preschool level and how is it observable? What kinds of reasoning (and to what degree of creativity) do talented students use when solving a non-routine problem? What contextual variables, personal characteristics and math-specific abilities combine to allow high potential in math?

Marianne Nolte told about the city of Hamburg's inclusion policy and how it has led to joint school education of disabled and non-disabled children. The goal of the Center for Gifted Students at the University of Hamburg is to develop the potentials of all children to the fullest possible extent. The center pays special attention to gifted and twice-exceptional children who are at particular risk of talent loss. In this work the problem of non-recognition is demonstrated based on four case studies on special talents combined with developmental impairments. Melanie Münz argued that in research on mathematical creativity early childhood is seldom taken into count. She illustrated instances of two children, four and five years of age, coping with a mathematical task while mathematically creative ideas emerged spontaneously from the interactions between the children. Elisabeth Merlot shared her observation that some successful participants of mathematical Olympiads are under-achievers in school mathematics and argues that more attention should be devoted to high ability students in school.

MATHEMATICAL PROBLEMS FOR THE GIFTED

At this session Alexander Karp shared with the participants his analysis of the problem sets found in textbooks for the mathematically gifted. Oleg Ivanov and Konstantin Stolbov presented their conception of typologies of mathematical problems, and Romualdas Kasuba discussed the importance of learning mathematics with pleasure. Karp suggested that in order to advance our understanding of the role of mathematical problems in realisation of high mathematical potential, the sets of problems for the mathematically gifted were compared with the sets in the ordinary textbooks. He argued that this analysis leads to the conclusion that gifted students are assumed by the textbooks' authors to learn some algorithms more easily and better and to apply new ideas to other domains more easily and quickly than ordinary students. This analysis leads to raising two concerns: First, practitioners and theoreticians seem to arrive at the same ideas independently, while collaboration between them could be especially productive. Second, the role of these more challenging problems in the development of the gift is not quite clear; specifically, whether it would be beneficial for “ordinary children” to be taught using problems similar to those given to the “gifted”.

The participants stressed that there are many approaches to and classifications of problems. Ivanov and his colleagues suggested a two-dimensional model for classification of problems. Dimension 1 corresponds to the schemes of mathematical reasoning (i.e., algebraic, analytic, combinatory-algorithmic, or what is called by the authors 'syllogistic'); while Dimension 2 describes the kinds of activities and approaches typically used by students for solving a problem. Kinds of activities may include codifying, interpreting, restructuring, transforming and transferring, etc., etc. This classification can be of help in demonstrating the similarity of activities and approaches across the subject domain boundaries and in this way demonstrate the unity of the mathematics. This approach can help educators in developing new problems and activities, and in identifying the need to develop additional tasks in different domains.

Discussion of the paper by Kasuba led to the suggestion that in order to make problem solving pleasurable many conditions need to be fulfilled. A problem can and even should be challenging while still being “solvable” and age- and development-appropriate. A problem should be attractive, meaning, for example, that it should have an interesting formulation and representation. A problem should have an effective solution, first of all meaning that by “doing very little” we “obtain a lot”.

Participants agreed that it is not an isolated problem, but rather the interconnected set of problems that helps to develop mathematical creativity. It was stated that, except for very artificial and overloaded tasks, almost any problem can be made useful in the process of education if it is included in a meaningful environment. This mathematical environment should demonstrate to students the openness of the study and the reasons and the rationale for the study. It was emphasized that problem solving sessions are particularly useful if and when pedagogical values are combined with mathematical values. The concept of “problem solving space” was coined and used to describe the entire problem solving process in the class, which includes the supportive and organizational role of the teachers and developing an especially conducive atmosphere of creativity.

TEACHING AND TEACHERS OF MATHEMATICALLY GIFTED STUDENTS ALONG WITH TEACHING ALL STUDENTS WITH AND FOR CREATIVITY

Florence Mihaela Singer, Jarmila Novotná, and Els De Geest presented didactical ideas directed towards the development of mathematical creativity in all students. They focused their attention on teachers' role in teaching with and for creativity. Mihaela Singer presented a case-study of a professional who teaches in a mixed ability class, and analyzed mismatches between the teacher’s self-efficacy beliefs and his specialized content knowledge. Jarmila Navotna focused on the effects of didactical heterogeneisation on students’ creativity, namely students’ ability to come up with new solutions to problems that are novel to them. Els De Geest discussed pedagogical approaches of experiential learning and the use of pedagogical constructs in an undergraduate distance-learning mathematics education course and whether these can lead to creativity regarded as ‘possibility thinking’.

Participants were involved in an experiment in which they were asked to imagine that they are school principals and have to interview a new teacher for 'teaching for creativity'. A set of qualities that should be examined were formulated, such as excellent and broad mathematical subject knowledge and competence in observing and interpreting occurrences in the classroom including learning processes, children's original ideas and unexpected initiatives. The list of teachers' qualities also included willingness to change practice, interest in further professional development, creativity in interpreting the curriculum, variability in teaching and risk-taking.

With respect to teacher preparation for teaching the gifted along with development of teachers' beliefs, mastery in pedagogy, robust and deep content knowledge at the university level and school level, the participants stressed the importance of knowledge of extracurricular mathematics and of elementary mathematics from an advanced standpoint. These teacher education programs have to be enhanced by intensive problem solving, observation of practice and mentoring by a personal adviser who is an expert in the education of the mathematically gifted.

RESEARCH METHODOLOGIES IN RESEARCH ON CREATIVITY AND GIFTEDNESS

Discussion of methodologies in research on mathematical creativity and giftedness was based on multidimensional examination of mathematical giftedness performed by Roza Leikin in collaboration with her colleagues from the University of Haifa, Israel (Ilana Waisman, Miriam Lev, Nurit Paz-Baruch, Shelley Shaul and Mark Leikin). Since mathematical giftedness is an extremely complex and ill-defined construct, this study searches for deep insights into its nature and attempts to define it. It connects two fields of educational psychology – mathematics education and gifted education. The connection between theories of mathematics education and theories of giftedness is reflected in the sampling procedure and research tools specifically designed for our study. The study has a multidimensional character as it examines and compares the mental activity of participants with different levels of mathematical exceptionality in three dimensions: cognition, neuro-cognition and mathematical creativity. The study suggests that mathematical giftedness can be defined as a combination of general giftedness and excellence in mathematics. This combination leads to the manifestation of cognitive, neuro-cognitive and creative properties at an upper edge of a continuum (revealed on a number of tasks). At the same time, several tasks led to the observation that mathematical giftedness can be characterized by qualitatively different phenomena in students.

SUMMARY

The Working Group entitled *Mathematical Potential, Creativity and Talent* encourages discourse between research mathematicians, mathematics educators and educational researchers. This discourse is reflected in papers included in the proceedings of the WG.

Based on the discussion that took place in the WG, we argue that further development of clear definitions of concepts related to creativity and giftedness should be encouraged. Characteristics of mathematically gifted students should be identified through careful systematic research and through the design of identification tools. Mathematics education researchers should pay more attention to mathematical creativity and its advancement among both students and teachers. Teachers' characteristics should be better described and teacher education programs directed towards developing creativity and advancing mathematically talented students should be carefully designed.

MEMORY AND SPEED OF PROCESSING IN GENERALLY GIFTED AND EXCELLING IN MATHEMATICS STUDENTS

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The present study examined the memory and speed of processing abilities associated with general giftedness (G) and excellence in mathematics (EM). The research involved four groups of 16-18 years old participants varying in levels of G and EM. A total of 160 participants were tested on a battery of three memory tests and five speed of processing tests. Working-memory was found to be related both to G and EM factors. The results reveal that G factor is related to high-level short term memory and that E factor is associated with high-level visual-spatial memory. Gifted students who excel in mathematics (G-EM group) outperformed the other three groups in all speed of processing tasks. The results can contribute to the theoretical knowledge regarding the similarities and differences in memory and speed of processing abilities in G and EM groups.

Key words: general giftedness, excellence in mathematics, memory abilities, speed of processing

INTRODUCTION

Despite increasing interest in gifted students and their education, little empirical data is available concerning the cognitive skills of gifted individuals who excel in mathematics. To date, much of the research on the relationship between different cognitive abilities and mathematical competencies has been conducted on low achieving mathematics students (e.g., Swanson & Jerman, 2006). Few studies have examined highly mathematically gifted adolescents (e.g Dark & Benbow, 1991; Swanson, 2006). Previous studies on mathematical ability and memory were undertaken mostly on children in elementary school (e.g., Hoard, Geary, Byrd-Craven & Nugent, 2008 ; Bull & Johnson, 1997; Smedt et al., 2009) while very few examined high school students (e.g Dark & Benbow, 1991). The contribution of speed of processing to mathematical excellence was also examined in several studies (Fry & Hale, 1996; Hoard et al., 2008;) but no study has yet examined the associations between general giftedness (G) and excellence in mathematics (EM), with regard to memory and speed of processing abilities.

BACKGROUND

General Giftedness, Memory and Speed of Processing

Earlier studies identified a connection between measures of intelligence and Working Memory (WM) or Short Term Memory (STM). For example, Ackerman, Beier and Boyle (2005) conducted a meta-analysis of the literature from 1972 to 2002 that examined the relationship between WM or immediate memory (i.e., STM) and intelligence. The study revealed a correlation between measures of STM and general ability and observed a link between simple span memory and intelligence. Similar results were obtained by Carroll (1993) who demonstrated an average correlation between immediate memory factors (i.e. tasks requiring storage and retrieval of information) and general ability.

In addition, a number of research studies have focused on the hypothesis that speed of processing is part of one's intelligence. The faster the speed of processing the higher the IQ score (Deary, 1993; Deary, 2000; Finkle & Pederson, 2000).Vernon (1983) investigated the relationship between several measures of speed of cognitive information-processing and WISC (Wechsler Adult Intelligence Scale) and Raven Advanced Progression Matrices scores in tests on intelligence. The results suggested that the speed factor accounts for 65.5% of the variance in intelligence scores. The conclusion was that individual differences in intelligence can be attributed, to a moderate extent, to variance in the speed or efficiency with which these operations are performed.

A considerable amount of recent evidence suggests that elemental speed of processing abilities (e.g., encoding speed, efficiency of short-term memory storage and processing, and simple and choice reaction time) may be related to intellectual giftedness (Dark & Benbow, 1991; Kranzler, Whang & Jensen, 1994; Vernon, 1983). As to the relationship between giftedness and memory ability, some studies have demonstrated that gifted children display a higher rate of memory capacity compared to their non-gifted peers (Gaultney, Bjorklund, & Goldstein, 1996; Harnishfeger & Bjorklund, 1990). For example, Calero, García-Martín, Jiménez, Kazén & Araque (2007) compared WM capacity between children with high and average IQ aged 6 to 11 years and found that high IQ children had significantly higher scores on WM than their average-intelligence coevals.

Excelling in Mathematics, Memory and Speed of Processing

Memory abilities are thought to be critical to many aspects of mathematical learning (e.g., Meyer et al., 2009). In particular, WM storage is regarded as being essential for solving complex (multi-step) arithmetical problems (Hoard et al., 2008). Children who excel in early mathematics learning tend to have a high WM capacity (Hoard et al., 2008; Meyer et al., 2009; Passolunghi, Mammarella & Altoe, 2007). Dark and Benbow (1991) showed that individual differences in WM span were associated with intellectual giftedness in mathematics. Testing intellectually gifted 13 and 14 year-olds, they found that mathematics achievement was related to enhanced performance on a WM span task with digits and other tasks involving the detection of stimuli, whereas verbal precocity was related to enhanced performance on tasks involving verbal stimuli. Additionally, Hoard et al. (2008) found that intellectually gifted individuals had an advantage in visual-spatial memory (VSM).

Speed of processing seems to be important as a predictor of variations in arithmetic performance (Durand, Holme Larkin & Snowling, 2005; Fry & Hale, 1996; Hoard et al., 2008). Bull and Johnston (1997) found that processing speed, as measured by visual matching, crossing-out tasks, and perceptual motor speed, was a unique predictor of arithmetic skills, independent of reading ability, for 7 year-old children. Taub, Floyd, Keith & McGrew (2008) showed that processing speed was significantly related to quantitative knowledge for children of ages 9 to 13.

Some studies examined the contribution that speed of processing and short-term memory make to arithmetic calculation (Berg, 2008; Johnson, Im-Bolter & Pascual-Leone, 2003; Geary & Brown, 1991). Case et al. (1982) found a linear relationship between speed of processing and storage capacity of the working memory. In a study of 6 to 11 year olds, they reported that faster counting speed predicted higher counting spans. The explanation for these results is that the faster a child processes relevant information, the more information the child can retain over a short period of time.

Accordingly, this study examined the memory and speed of processing abilities associated with factors of general giftedness and excelling in mathematics. The aims of the present study were as follows:

1. To examine which memory abilities are associated with G and EM factors

2. To examine the differences in speed of processing among students, according to G and EM factors.

METHOD

Participants

We report herein our findings on 160 10th -12th grade students (16-18 years old) right-handed male and female students who were recruited for the study (see Table 1). The participants were subdivided in four experimental groups, determining the research population by a combination of G and EM factors:

G-EM group: students who are identified as generally gifted and excelling in mathematics;

G-NEM group: students who are identified as generally gifted but do not excel in mathematics;

NG-EM group: students excelling in mathematics who are not identified as generally gifted;

NG-NEM group: students who are identified as being neither generally gifted nor excelling in mathematics.

Table 1: Description of study groups

	Generally Gifted (G) Raven ≥ 27	Non-Gifted (NG) Raven ≤26	Total
Excelling in mathematics (EM) SAT-M ≥26 or HL in mathematics with math score ≥ 90	*n*=34	*n*=44	*n*=78
Non-excelling in mathematics (NEM) SAT-M ≤22 and RL in mathematics with math score ≥90 or HL in mathematics with math score ≤80.	*n*=36	*n*=46	*n*=82
Total	*n*=70	*n*=90	*n*=160

Tasks and Materials

Memory abilities tests

1. Short-Term Memory - Digit Span test (WISC III; Wechsler, 1997)

The test involves two parts: during the first phase the participant is asked to listen to random digits read out loud and then to repeat the digits in the same order. If the participant recalls the digits correctly, another trial is administered. Successive trials are administered using random digits increased by one digit and so forth until the participant fails two attempts. The maximum possible span is nine digits.

During the second phase the participant is asked to listen to a random set of digits read out loud and then to repeat the digits backwards. If the participant recalls the digits correctly, another trial is administered. Successive trials are administered using random digits increased by one digit and so forth until the participant fails two attempts. The maximum possible span is eight digits. The measure of both test parts constitutes a standard score based on Israel's norm scale scores (from Hebrew version of WISC III).

2. Working Memory for Digits and Letters test (WISC III; Wechsler, 1997)

The participant is asked to listen to a mixed series of letters and digits and then to rearrange them by first repeating the digits in the correct order, and then the letters in the correct order. If the participant recalls the digits and letters correctly, another trial is administered. Successive trials are administered using random letters and digits increased by one letter or digit and so forth until the participant fails two attempts. The maximum possible span is eight letters and digits. The measure of the test constitutes a standard score based on Israel's norm scale scores (from Hebrew version of WISC III).

3. Visio-Spatial Working Memory test (Corsi, 1972)

This block-recall task consists of ten blocks arranged randomly on a wooden board. The test involves two parts: during the first part the researcher points at a sequence of blocks at a rate of one per second. After the researcher completes the sequence, the participant is asked to replicate the sequence. If the participant recalls the sequence of blocks correctly, another trial is administered. Successive trials are administered adding one more block each time and so forth until the participant fails two successive attempts. The maximum possible span is ten blocks.

During the second part, the researcher points at a sequence of blocks at a rate of one per second. After the researcher completes the sequence, the participant is asked to replicate the sequence backwards. If the participant recalls the sequence of blocks correctly, another trial is administered. Successive trials are administered adding one more block each time and so forth until the participant fails two successive attempts. The maximum possible span is ten blocks. The measure of both test parts constitutes a standard score according to the accepted Israeli scale (from Hebrew version of Visio-Spatial Working Memory test).

<u>Speed of processing tests:</u>

1. Visual- matching test (Woodcock-Johnson Tests of Cognitive Ability, 2001)

The test consists of rows that include one target symbol and 19 additional symbols. The participant has to circle all the symbols that are identical to the target symbol. The time limit for the assignment is 120 seconds.

2. Cross-out of numbers test (Woodcock-Johnson Tests of Cognitive Ability, 2001)

The test consists of 60 rows, each with six numbers. The participant has to circle two identical numbers in each row. The time limit for the assignment is 120 seconds.

3. Digit-symbol test (WISC III, 1997)

The test consists of a code table displaying pairs of digits and symbols, and rows of double boxes with a digit on the top box and nothing on the bottom box. The participant has to use the code table to determine the symbol associated with each digit (the test consists 133 digits), and to write as many symbols as possible in the empty boxes below each digit. The time limit for the assignment is 120 seconds.

4. Symbol-search (WISC III, 1997)

The test consists of rows marked by one target symbol and five additional symbols. The participant has to decide if the target symbols appear in the row of symbols and to mark YES or NO accordingly. The test consists of 60 items and the participant has to mark as many items as possible within 120 seconds.

5. Simple arithmetic exercises (Openhaim-Bitton, 2003)

The participant has to solve as many simple arithmetic exercises as possible within two minutes. The accuracy and number of correct answers will be examined.

The dependent variables in each test were: accuracy (in %) of correct answers.

Data Analysis

To investigate the questions addressed in this study, multivariate analysis of variance tests (MANOVA) were used to compare the scores of participants in each test. The between-subject factors were: G and EM factors and the within-subject factors were the scores on all memory and speed of processing tests.

RESULTS

Memory tests

MANOVA revealed two significant effects. First, a main effect of the G factor ($F(3,150)= 4.32$, $p<.05$) was obtained. In this case, univariate ANOVA tests showed that the source of differences between the groups was the STM test ($F(1,152)= 12.20$, $p<.01$) on which G students achieved higher ($M=12.6$, $SD=2.7$) STM scores as compared to NG students ($M=10.7$, $SD=2.4$). Results of performance on the memory tasks (means and standard deviations) are presented in Table 2.

Table 2: Performance of participants on three memory tests: standard scores for correct answers in each memory test

Memory tasks		EM	NEM
		M (*SD*)	*M (SD)*
STM	G	12.8 (2.7)	12.4 (2.8)
	NG	10.9 (2.3)	10.5 (2.5)
WM	G	10.6 (2.2)	10.8 (2.7)
	NG	10.8(2.7)	9.7 (2.2)
VSM	G	12.5 (2.1)	11.8 (1.5)
	NG	12.0 (2.5)	11.3(2.2)

Note: G=Gifted; NG=Non-Gifted; EM=Excelling; NEM= Non-Excelling STM=Short-Term Memory; VSM= Visual-Spatial Memory; WM= Working Memory

Additionally, a main effect of the EM factor was found to be marginally significant on the VSM score ($F(1,152)=3.65$, $p<.06$). In this case, EM students scored higher ($M=12.2$, $SD=2.3$) than NEM students ($M=11.5$, $SD=1.9$).

An interaction of G × EM factors was found with regard to the WM total score ($F(3,152)=3.22$, $p<.05$). The WM scores of EM students were similar in G ($M=10.6$, $SD=2.2$) and NG students ($M=10.8$, $SD=2.7$). However, the WM scores of NE students were significantly higher for G ($M=11.7$, $SD=2.4$) than for NG students ($M= 9.7$, $SD=2.2$).

Speed of Processing tests

MANOVA revealed an overall significant main effect for the G factor ($F(5,149)=2.50$, $p<.05$) and the EM factor ($F(5,149) = 4.35$, $p<.01$). In this case, univariate ANOVA tests showed the source of differences between the groups was the Cross-out of numbers and Simple arithmetic exercises tests. In this case, G students outperformed NG students and EM students were more accurate than NEM students in both tests (see also Table 3).

Additionally, univariate ANOVA tests showed an interaction between G×EM factors with regard to the Symbol-search test ($F(1,153)=4.11$, $p<.05$) and the Digit-symbol tests ($F(1,153)=5.88$, $p<.05$). The accuracy of NG students in both tests was similar in EM and NEM students. However, the accuracy of G students in both tests was significantly higher for EM than for NEM students (see also Table 3).

Table 3: Performance of participants (in %) on five speed of processing tests

SIP tasks		EM	NEM
		M (SD)	*M (SD)*
Visual matching	G	68.3 (12.3)	61.1 (14.8)
	NG	61.1 (11.2)	62.4 (11.6)
Cross-out of numbers	G	75.2 (6.2)	71.1 (5.6)
	NG	70.2 (8.4)	68.4 (10.0)
Digit-symbol	G	67.0 (11.4)	61.7 (10.0)
	NG	63.7 (7.6)	65.1 (9.3)
Symbol-search	G	76.0 (12.4)	69.7 (12.0)
	NG	70.9 (9.1)	71.9 (11.2)
Simple arithmetic exercises	G	97.0 (5.0)	89.8 (10.8)
	NG	92.3 (9.8)	83.6 (14.6)

CONCLUSIONS

The aim of the present study was to examine the memory and speed of processing abilities associated with general giftedness (G) and excelling in mathematics (EM). Results showed that there are different memory abilities associated with G and EM factors.

The findings revealed that G is related to high STM ability and the EM factor was found to be linked with high VSM. In addition, the results demonstrated that the WM scores of students excelling in mathematics were comparable in G and NG individuals. However, the WM scores of NEM students were significantly higher for G students as compared to NG students.

The results regarding speed of processing tasks show that the gifted students who excel in mathematics (G-EM) outperformed the other three participant groups in these tasks. The findings, however, discerned discrepancies between tests: in some tests a difference in performance was associated with both G and EM factors while in other tests, only the G factor was associated with the differences between the groups.

The findings obtained in the study partly support previous observations and suggest that memory and speed of processing abilities seem to be important factors in explaining mathematical giftedness. The findings add to the theoretical knowledge pertaining to the cognitive processing of G-EM and G-NEM students and can enlighten educators and instructional designers, thus enabling them to better plan effective educational programs tailored to the unique educational needs of gifted students excelling in mathematics. With this in mind, educational programs for G-EM students should address the observation that these students possess high abilities in visual-spatial memory and in information processing and should therefore implement the use of visual aids in teaching mathematics in gifted classes.

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STUDENTS' PICTURE OF AND COMPARATIVE ATTITUDE TOWARDS MATHEMATICS IN DIFFERENT SETTINGS OF FOSTERING

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The fostering of promising students in mathematics can be done in different ways. By the results of a survey in different settings we try to give advices for a suitable way of selecting and fostering potentially mathematically gifted students. Therefore we look at the students' picture of and comparative attitude towards mathematics.

Keywords: mathematical giftedness, fostering, selection, beliefs, attitudes.

INTRODUCTION

There are different ways of selecting promising students for reasons of fostering (in mathematics) consequently leading to different settings:

One way is to offer additional courses or materials for volunteers, which means that the *interest* of the students is the main motivational force. The general importance of this aspect was already identified by Kruteskii (1976), for example, who lists a "general synthetic component", a "mathematical cast of mind", i.e. the necessity of an aptitude and interest for mathematics in order to be successful within this subject. He states "It is expressed in a selectively positive attitude toward mathematics, the presence of deep and valid interests in the appropriate area, a striving and a need to study it, and an ardent enthusiasm for it" (Kruteskii, 1976, p. 345). The mathematician Kurt Devlin appears to have the same opinion when he declares that "whatever it is that causes the interest, it is that interest in mathematics that constitutes the main difference between those who can do mathematics and those who claim to find it impossible" (Devlin, 2000, p. 275).

Another way to identify promising students is to rely on teachers' choices, so it is not only the students' interest driving them to the courses but also a *qualitative* external selection process. Linke & Steinhöfel (1986, 1987), amongst others, discussed this process.

And a third way would be a *quantitative* procedure of testing the students (see Nolte, 2012, Kontoyianni et al., 2011, Hagborg & Wachman, 1992, Bittker, 1991, Löser, 1985, amongst many others) or simply choosing those with the best marks in mathematics (see, to some degree, BSHA in the next section, for example).

In practice, selection processes often are designed as multilevel mixtures of these three ways (see Wagner & Zimmermann, 1986, for example). From a psychological perspective, these selection methods represent different aspects of the causality between a giftedness potential on the one side and performance or assessment, respectively, on the other side. We already discussed these aspects in Brandl (2011) and Brandl & Barthel (2012) and gave some initial results concerning theoretical insights and corresponding practical consequences there.

In this paper we want to look at different settings of fostering again in order to compare the students' perceptions of and comparative attitudes towards mathematics. This was already done for just one setting in Brandl & Barthel (2012), where we concluded that there are strong hints that mathematically gifted students should be fostered separately from their (old) classmates. The reason was/is just their absolutely different attitude towards mathematics and therefore the absolutely different learning atmosphere they deserve or create, respectively, for and/or by doing mathematics.

THE DIFFERENT SETTINGS

We did our empirical survey in three different settings. The first is a German boarding school for high attaining students, where successful candidates are chosen by very stringent assessments, representing the third selection method sketched in the introduction ("*quantitative* selection"); the second setting, the fostering courses at our university, represent a mixture of mostly the first ("*interest*") and a little of the second ("*qualitative* selection") selection method. As a contrast to these two settings we added a "normal" class from a regular higher secondary school.

Boarding school for high attaining students (BSHA)

In order to be selected for this German boarding school for high attaining students, applicants have to fulfill several requirements: in the main subjects – German, Mathematics, a foreign language and natural sciences – students need to achieve at least the mark "good"; an average mark "good" in the last two school reports; a general IQ score of approximately 130 points in the intelligence structure test I-S-T 2000 R (Liepmann et al., 2010) and a successful participation in a two-day assessment center concerning social skills. The sample group consists of 8 classes with 113 students from 11^{th} and 12^{th} grade (which represent almost all students within these grades). Their age ranges from 14 to 18 with an average age of 16; the number of boys and girls is balanced. On a scale from 0 (fail) to 15 (excellent) the median of their marks in the subject mathematics in the last school report is 12.

Fostering courses at university (FCU)

Offering of special fostering courses in mathematics at schools or at universities is quite common and has a long tradition. However, there are different kinds of selection processes for these courses, including identification by a teacher, special tests for mathematical giftedness, students' interest (self-selection) or a combination of those; see Nolte, M. (2012) amongst many others, for example. At the University of Passau, these kinds of courses have been offered for several years. The participants are split into at least two groups consisting of 13-to-14-year-olds and 15-

to-16-year-olds. There are no tests for IQ or mathematical giftedness; mostly the students are attending the courses because of their own interest in the subject. Unfortunately, the numbers of participants (and so the samples) are rather small: 9 "younger" and 6 "older" ones. Typically, the participants of the "higher" course are former participants of the "lower" course.

Regular higher secondary school (RHSS)

In order to compare the results from the data collected at the BSHA and the FCU with a "normal" school, identical questionnaires were given[1] to 23 students of a class in a 11th grade at a regular higher secondary school ("Gymnasium"). The average age is the same as at the BSHA and the sample consists of 9 female and 14 male participants. As might be expected, their average mark in mathematics is much lower than that of the BSHA sample.

METHODOLOGICAL REMARK

One of the methodological issues of this study is that the number of participating students in the survey is different for the three settings (BSHA, FCU and RHSS). So, the conclusions drawn from the evaluation of the three groups cannot be interpreted in a general way based on representative quantitative empirical data. However, as *all* students from 11th and 12th grade of BSHA and *all* students from the courses of FCU were chosen, the results may still be representative for the different settings in a *qualitative* way; and as *all* students from one whole class from the 11th grade at RHSS were chosen, the RHSS sample can serve as a representative *qualitative* counterpart. The more or less qualitative results of this survey can so be seen as a starting point or a pre-study, respectively, for a deeper and representative quantitative research.

PICTURE OF MATHEMATICS

The questionnaire given to the students was identical[2] for all three settings. One of the foci of this questionnaire is students' notion of mathematics; answers could be given on a bipolar 5-point Likert-type scale. The categories on the horizontal axis in the following figures are:

1: Calculations, rote learning, algorithms

2: Problem solving, proofs, riddles

3: Creative process of building a theory

4: Tinkering[3] with beautiful things

5: Competence to get a safe and well-paid job

6: Modelling, language of nature

7: Symbols, numbers, patterns, formulas, abstract structures

8: Texts, theorems, laws

9: Something for freaks

Boarding school for high attaining students (BSHA)

For the BSHA we got the following profile:

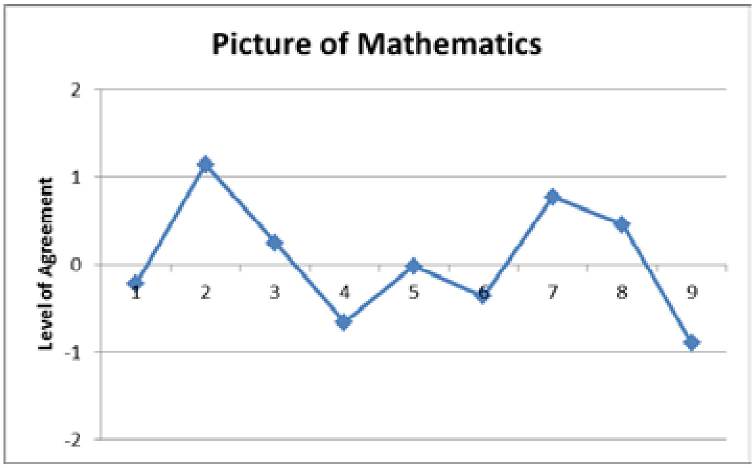

Figure 1: Picture of Mathematics (of 11th and 12th grade altogether) at the BSHA.

The diagram shows an ambivalent picture of mathematics: though the students of the BSHA do not believe that mathematics is something for freaks (9) but is rather problem solving and proofs (2), patterns (7) and theorems (8) instead of rote learning (1), however, they rather don’t see it as a tinkering around with beautiful things (4) or the language of nature (6). This might be caused by the fact that the lessons at the BSHA are bound to the general curriculum necessary for the central-posed A-level exams.

Fostering courses at university (FCU)

Participants in the FCU answered like this:

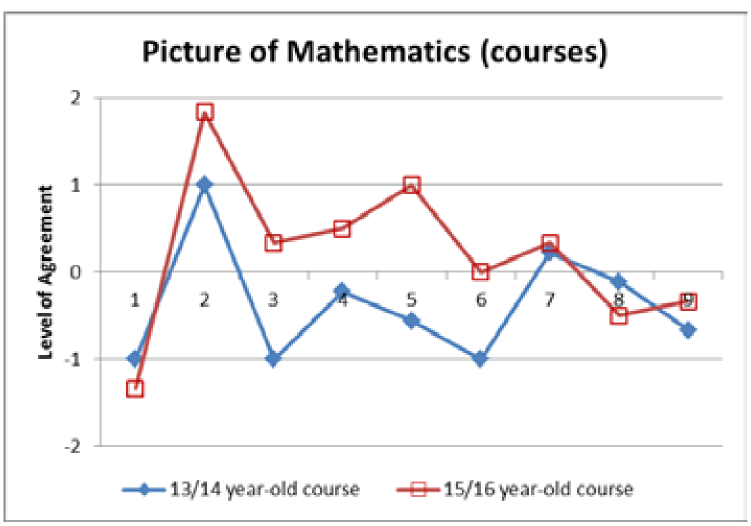

Figure 2: Picture of Mathematics of the 13-to-14-year-old and 15-to-16-year-old ones at the FCU.

Whereas the curve for the younger students is quite similar (despite being a little bit more negative) to the one at the BSHA, the older students (fitting better to the group at the BSHA and the RHSS) show a generally more positive picture except for the numbers 1 (rote learning) and 8 (theorems). The main difference from the BSHA diagram can be seen in the far more positive values concerning problem solving (2) and tinkering around with beautiful things (4). We attribute these differences to the freely chosen mathematical contents of the lessons and the way of dealing with mathematics within these courses.

Regular higher secondary school (RHSS)

The 11th grade at the RHSS delivered the following picture:

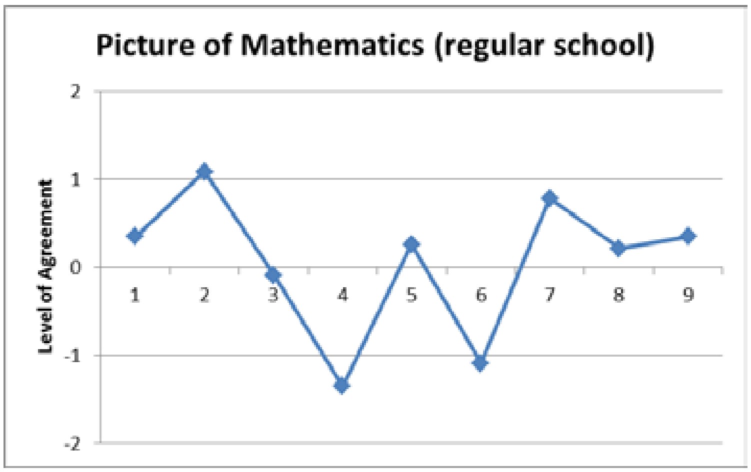

Figure 3: Picture of Mathematics of an 11th grade at the RHSS[4].

First, the line in figure 3 is completely different from the line of the older ones in the FCU, illustrating the diverging perspectives of mathematics of interested students and those who are not. Second, although being (surprisingly) similar in most points to the diagram in figure 1 for the BSHA (items 2 to 8), there is a tendency to see mathematics as some kind of rote learning process (1) and something for freaks (9).

COMPARATIVE ATTITUDE TOWARDS MATHEMATICS

The data for the different attitudes towards mathematics came from answers on a 7-point Likert-scale. The students had to rate their own attitude towards mathematics as well as estimate the one of their classmates at the current school (or the university courses, respectively), their classmates at their – for BSHA: old – school (dispensed for RHSS) and society as a whole. The categories on the horizontal axis in the following figures are:

1: ugly, terrible	------	aesthetical, beautiful
2: hard	------	easy
3: boring	------	interesting, exciting
4: incomprehensible	------	logical

Boarding school for high attaining students (BSHA)

The results are shown in figure 4:

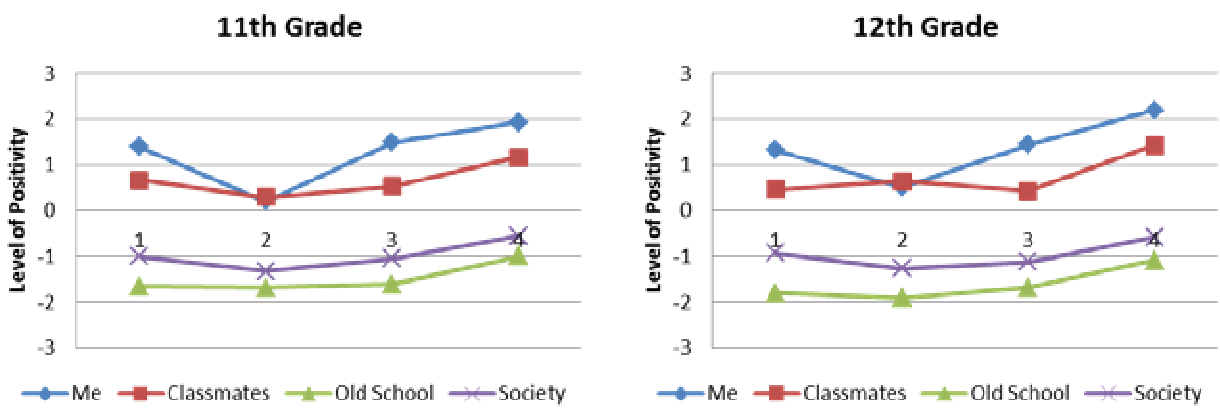

Figure 4: Attitude towards Mathematics, separately for grades 11 and 12, at the BSHA[5].

All students rated themselves more positively than the rest (classmates, old school, society). This might suggest a sort of “I’m sure that the others don’t find math as interesting and nice as me”-thinking. Furthermore, there is a remarkable gap between the estimated attitudes of the current classmates compared to the old ones. Of course, all members of the sample were selected because of their very good marks in mathematics. Nevertheless, the difference of the two lines shown in figure 4 indicates that they must have felt like strangers in their old classes. Additionally, the attitude towards mathematics in society is always rated quite low, but higher than that in their old class (see also Brandl & Barthel, 2012).

Fostering courses at university (FCU)

The attitudes of participants of the FCU are shown below:

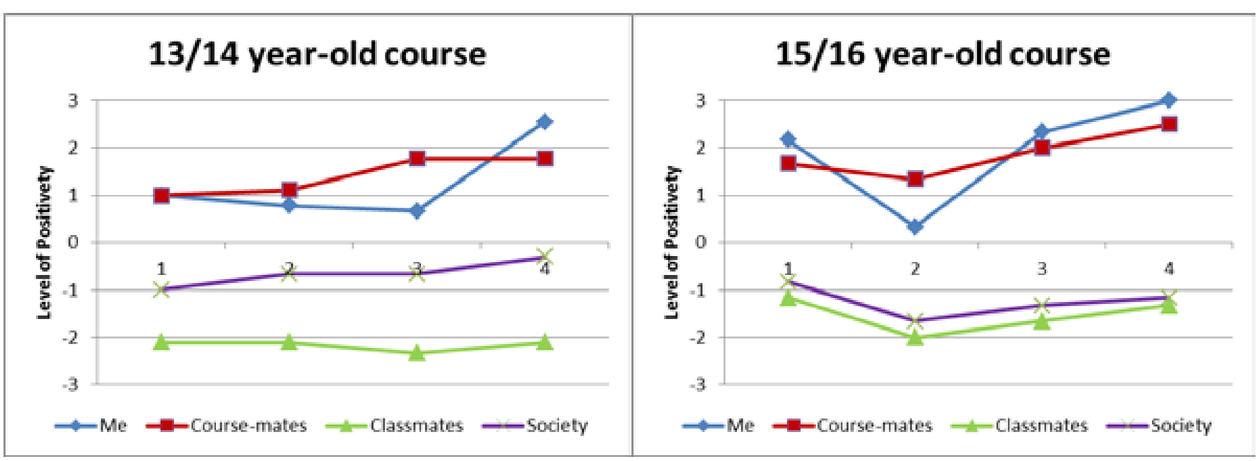

Figure 5: Attitude towards Mathematics, separately for the 13-to-14-year-old and 15-to-16-year-old ones, at the FCA.

First of all, just like for the BSHA students, the students themselves and their course-mates show a positive attitude towards mathematics in contrast to the estimated negative one of their "normal" classmates and society. The gap between the current course-mates and the "normal" classmates is quite huge compared to the analogical gap (between "classmates" and "old school") in figure 4. Furthermore, the students' rating of society's attitude as more positive than that of their "normal" classmates is similar to that of BSHA students.

Regular higher secondary school (RHSS)

In contrast, at the RHSS the following rating was given:

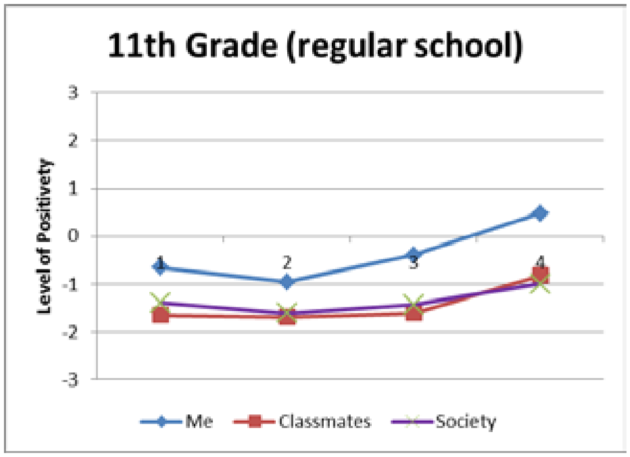

Figure 6: Attitude towards Mathematics at the RHSS[6].

Clearly, there is only one slightly positive rating for the individual logical sight ("Me", 4). All of the other ratings are, on average, negative, which is a marked difference from BSHA and FCU students. Of course, this results from the fact that now not only the highest attaining or interested ones are asked. Surprisingly, the levels for the estimated attitudes of classmates and society are at almost the same height as it is for BSHA and FCU. So this seems to be very stable across students from different attainment levels. Another stable characteristic, one that held across attainment levels, was the gap between "Me" and all the others, although in this instance the classmate group are current classmates instead of former classmates. One distinctive feature in the RHSS survey responses is the nearly identical curves for classmates and society; these curves were separated to a greater degree in the BSHA and FCU survey responses.

HYPOTHESES / QUALITATIVE CONCLUSIONS

Based on these observations in the diagrams from figures 1 to 6, we draw the following qualitative conclusions that may serve as hypotheses for a larger and deeper quantitative study:

- The picture / perspective of mathematics of somebody who is interested in mathematics differs essentially from one averaged over an "ordinary" class; it seems[7] also to be more positive than that one averaged over a "high-attaining" class.

- In general, the beauty of mathematics and the possibility of thinking of mathematics as just a tinkering around are only stated by students who were "chosen" for special courses because of their interest in mathematics. So, on the one hand, this stays in line with the findings in Kruteskii (1976) and Brandl & Barthel (2012) that the strong motivational force *interest* and the ability for *aesthetical sensation with mathematics* are correlated in some way. On the other hand, it seems as if an institution depending on an official curriculum (instead of a free choice of contents) is not able to establish any kind of playful aesthetical enjoyment by doing mathematics in class for a (often large) group of highly different interested and motivated students.

- There is a gap between the (more positive) attitude towards mathematics of *students themselves* and the estimated attitudes of their *current classmates* at BSHA and RHSS. We interpret this as a sort of "I'm sure that the others don't find math as interesting and nice as me"-thinking in all math classes. This psychological fact may be used by the teacher, as the awareness of the students' individual understanding of their potential in and their attitude towards mathematics can help to establish a more comfortable atmosphere in order to foster the students according to their individual needs.

- The levels for the estimated attitudes of their old (BSHA) / "normal" (FCU) / current (RHSS) classmates and society in all three settings always are at almost the same (negative) height; so this seems to be quite stable. This could reflect the "negative image" of mathematics in society, which perhaps is projected onto or adapted by the classmates. Furthermore, this may be a signal that the "mathematical climate" in ordinary math classes at regular (secondary) schools may be not supportive for establishing successful learning processes in mathematics. This is underlined by the fact that in the case of BSHA and FCU the estimated attitude of the old / "normal" classmates is rated even lower than that one of society.

- In the context of actual fostering by separation of the particular students there is also a significant gap between the (more positive) attitude towards mathematics of *students themselves* and the estimated attitudes of their *old* (BSHA) and "*normal*" (FCU) classmates, respectively. It is biggest when looking at the interested ones (FCU).

  Probably the environment of an ordinary math class (like at RHSS) is not suitable and supportive for promoting mathematics as something beautiful, challenging and joy-bringing. Students from BSHA / FCU may have felt / feel some kind of alienated in their old / "normal" classes. Based on these preliminary results, separating them from their old / "normal" classmates seems to be a promising way to give them an appropriate surrounding for performing and learning mathematics.

However, in individual problem-centered interviews with all the (eight) mathematics teachers at BSHA almost all teachers confessed that the main problem when it comes to the students' performance (especially in mathematics) seems not to be their more or less existing giftedness potential, but the psychological hindrance of a *narcissistic wound* / *shock* that comes from being confronted with just best-of-students in class and the eventual loss of this status for oneself. So, apart from aspects more or less related to (mathematical) beliefs and attitudes discussed in the previous paragraphs, pedagogical and psychological issues strongly connected with performance and assessment (And not the giftedness potential in the first place!) have to be considered seriously and may even be seen as decisive if a separation/selection is considered.

Hence, additional non-performance orientated courses (as realized in FCU, for example) seem to be suitable settings.

We hope that these preliminary and qualitative results can serve as guiding hypotheses in order to be confirmed in a deeper quantitative research.

NOTES

1. This survey was done in the context of the exam thesis of Christian Barthel, which was supervised by the author of this paper. So the results described within this paper concerning the RHSS are based to some part on Barthel, C. (2011).

2. The questions related to the estimation of the comparative attitudes of the students themselves in relation to other groups were modified slightly to fit the context of BSHA, FCU and RHSS (example: 'classmates' for BSHA was changed to 'course-mates' for FCU).

3. This is motivated from the description/definition of mathematical giftedness in Ruelle (2007).

4. Figure 3 is related to figure 20 in Barthel (2011, p. 53).

5. Figure 4 is partly taken from figure 2 in Brandl & Barthel (2012) and appeared first in Brandl (2011c).

6. Figure 6 is based on figure 26 in Barthel (2011, p. 64).

7. Probably the sample size of FCU is too small to come to a stronger conclusion.

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POSSIBILITY THINKING WITH UNDERGRADUATE DISTANCE LEARNING MATHEMATICS EDUCATION STUDENTS: HOW IT IS EXPERIENCED

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This paper reports on the Creative Thinking in Mathematics Education Enquiry (CTMEE) at The Open University. The study investigates whether the pedagogical approaches of experiential learning and the use of pedagogical constructs in an undergraduate distance learning mathematics education course can lead to creativity seen as 'possibility thinking' (PT) (Grainger, Craft and Burnard, 2007). Data consist of 23 quantitative and qualitative responses from students to an on-line questionnaire. It is the analysis of the latter that are discussed in this paper. Findings suggest that such pedagogical approaches can indeed contribute to developing possibility thinking. This paper offers a descriptive categorization of how the features of the PT framework are manifested with undergraduate distance learning mathematics education students.

Keywords: possibility thinking, creativity, experiential learning, e-learning

INTRODUCTION AND BACKGROUND

Possibility thinking (PT) is, in essence, thinking about possibilities. PT has been described as being the process of thinking that leads to creative development (Aristeidou, 2011). It is thus seen as at the core of and driving force for creative thinking (Craft, 2000, 2001; Craft and Jeffrey, 2003; Grainger et al., 2007; Craft, Cremin, Burnard, Dragovic and Chappell, 2012; Craft, McConnon and Paige-Smith, 2012). PT is about 'everyday creativity', also referred to as 'little *c* creativity' or 'what if' thinking. It is about trying out different possibilities, identifying problems and solving these (Craft, 2002); it is about'thinking in novel and valuable ways about the world using imagination as close to the notion of creativity' (Craft, 2000, p.9).

Grainger et al. (2007) developed a framework for identifying and analysing possibility thinking of teaching and learning. The framework was originally developed within the context of primary school children. The features of this framework for identifying an analysing PT involve:

PT feature	What this could involve…
Play	As a result of time for immersion, ideas incubate and questions emerge through playful encounters

	[with others or the tasks/concepts]. Extending of boundaries. Engagement, interest and motivation.
Immersion	Opportunities and time for extended periods to immerse in particular activities which are frequently revisited.
Question posing	'What if' scenarios. Questioning, generating ideas. Questions of students treated with deep respect and interest. Making predictions, compensation, improvisation, testing.
Self determination	Ownership. To exercise agency (one is the agent in the change/activity) and autonomy (self chosen action; self directed acted; self initiated activities). Expected to take risks.
Risk taking	Challenges with no clear-cut solutions. Developing courage to take risks. Contributions are valued.
Being imaginative	'As if' thinking. Being imaginative and imagining. Consider what might be, alternative world frames. Can position oneself differently and postulate reasons for this.
Making connections	Connections between ideas and activities and between one's own and others' lives.

Table 1: Features of PT and what these could involve. Amalgamated and adapted from Grainger et al. (2007).

The Centre for Mathematics Education in the Open University (OU) offers distance learning courses in mathematics education at undergraduate and postgraduate level. The pedagogical approach taken in all these courses is one of experiential learning (Kolb, 1984; Dewey, 1938) where students are asked to undertake mathematical tasks for themselves, reflect on this experience, try the tasks out with learners, and reflect on both experiences. The courses also use *constructs* which act as labels for experiences, such as *do-talk-record*, *generalising and specialising, conjecturing & convincing, imagining & expressing* (Pólya, 1962; Mason, Burton and Stacey, 1982). These pedagogical underpinnings of experiential learning and constructs are made explicit in the course materials and are referred to in research publications (Mason & De Geest, 2010).

The CTMEE project intended to address the following research aims:

1. To find out whether, and if so, how, the features of the PT framework resonate with adult learners in a distance learning setting?

2. To find out whether experiential learning and the use of constructs in the teaching of mathematics contribute to developing possibility thinking
3. To find out whether there are any there any other pedagogical approaches within the courses that nurture possibility thinking

This paper reports on the findings to the first research aim. Findings to the second and third research aim have been reported in De Geest (2012).

RESEARCH DESIGN

Data was collected via an on-line questionnaire of students on the course 'Developing Algebraic Thinking' (course code ME625; Mason, Graham and Johnston-Wilder, 2005). This is a third level undergraduate distance learning course that can count towards Graduate Diploma in Mathematics Education and/or a BSc(Honours) Mathematics and its Learning . The course is open to everyone, though it is intended particularly for students working or aspiring to work in mathematics education. It integrates development of the core ideas of algebra with relevant pedagogical constructs and principles and aims to extend awareness of how people learn and use algebra. Examples of mathematical tasks in the course are:

> It was reported that during 1992 in a certain county, although black students were only 10% of the school population, 40% of school exclusions were black. This means, it said in the report, that black students are 6 times as likely to be excluded as white. Justify, comment and construct a general method for making similar calculations from other similar data.
>
> (Task 9.3.4, Mason et al. 2005, p189)

And

> Write down two numbers
>
> Write down two numbers which total 36
>
> Write down two numbers with a difference of 8
>
> Write down two numbers with a total of 36 and a difference of 8
>
> (Quickie 10.1, Mason et al. 2005, page 199)

The questionnaire was developed based on the model of possibility thinking of Grainger et al. (2007). It consisted of statements which respondents were asked how these fitted with their experiences first within the context of experiential learning (part 1), then in the context of the use of constructs (part 2), with response options 'happens often/sometimes/a few times/not at all'. For example, to find out whether the students had experienced the feature of the PT framework of risk taking as a result of experiential learning in the course, the statement read:

> Trying out tasks for myself, reflecting and trying the tasks with others in ME625…
>
> prompt me to take risks in my thinking and my practice

This was followed by response options of ‘happens often/sometimes/a few times/not at all’. To obtain exemplification of their experiences, respondents were invited to describe a particular incident of how/when this happened. It is the analysis of these qualitative responses that are discussed in this paper.

All 120 students registered on the course were invited to take part in the study shortly after the due date of the final assessment but before they received their results. Twenty-seven responses were received, of which twenty-three were useable (four did not go past the consent part of the questionnaire). All had completed the questions relating to experiential learning (part 1), and 17 completed both sections. Twelve respondents provided 39 exemplifications of their experiences. Responses varied in length from short statements to more elaborate responses. Several responses mentioned specific tasks, or specific constructs.

The qualitative responses were analysed using a grounded theory approach such as constant comparison and were informed by the PT framework (Grainger et al., 2007). The analysis questions used were ‘what are the students experiencing in terms of the PT features?’ and ‘what triggers/influences this experience?’.

RESULTS AND ANALYSIS

The analysis provided exemplification of how features of PT, as described earlier in this paper, were manifested with the students:

Play/playfulness

Playfulness seems to take the form of exploring and experimenting. Such activities are no longer considered a waste of time. Assessment tasks where mentioned as leading to extensive exploring. Important also was mentioned that having the toolkit on how to explore from using constructs. Knowing how to get unstuck seems important so students can work through being stuck when exploring and experimenting as illustrated by the responses:

> “I now keep a notebook with me so I can explore 'What if...' whenever this may occur to me.”
>
> “Manipulation and drawing is no longer a waste of time.”
>
> “On tasks for TMAs [assignments for assessment] I often ended up exploring so much that I had far too much information to answer the questions!”
>
> “The constructs have made it possible for me to explore the inner aspects of tasks. I was able to work through being stuck most of the time by using the course constructs. “
>
> “Previously I would have stopped as soon as the going gets tuff and would therefore have missed out on the inner aspects.”

Immersion

There were no explicit responses indicating the students had experienced opportunites and time for extended periods to immerse in particular activities which are frequently revisited. Perhaps it might be implied from the course design and the students responses about playfulness which seems to suggest students were engaging with tasks for some time.

Question posing

Students reported asking ‘what if’ questions by being required to extend and adapt tasks. They had learned ways to extend such tasks by using constructs such as *Dimensions of Possible Variation* (DoPV). They had become aware of these DoPV from being presented with tasks which had many-right-answers and/or many-right-ways to solve the tasks as illustrated by the responses:

> “Being asked to extend tasks prompted this [posing questions] at first but then it started to become more natural.”

> “If I understand an idea then I can still further my understanding by asking question like 'what if...'. “

Self determination (autonomy and agency)

Self determination, directed action and self chosen action in the forms of autonomy and agency seems to be very present with the students. Many responses mentioned experiencing ownership of ideas, of learning and of feeling valued. Experiential learning and reflection stimulated students to develop their own conjectures about theories of learning and of mathematics learning based on new and old experiences. This is aided by the tasks with their focus on many-right-answers and many-right-ways to solve the problem: they seem to validate and respect own ways of thinking and at same time offer new ideas and approaches. The requirement for adapting and extending task, the stimulus to create own examples and been given a toolbox of constructs and frameworks to do so, means students are in charge of their own learning. They report feeling interested, focussed and motivated as a result and as illustrated by the responses:

> “Being told something is 'correct' or 'incorrect' is not always helpful. In fact, for me, it is usually rarely helpful. More helpful and important is the reasoning, which make something valid or invalid. Experiential learning has the advantage that it does not force a view. On the contrary, one discovers through experience the dimensions of variation of a problem and its associated range of permissible change. This means that there is an element of real ownership associated with ideas.”

> “Lots of ideas which I was able to construct my own tasks from.”

> “That I can decide what is the best way for me and that by doing this I can encourage my learners to find the best way for them.”

"I could see that the tasks I encountered and took on were very much up to me; how did I extend them? What resources worked for me?"

"If you create your own examples or expand one that is given you are taking charge of learning and that makes you more interested and focused."

Risk taking

Students report feeling comfortable with taking risks, indeed one response read 'no fear of failure'. Taking risks meant a change in teaching practice by moving from using textbooks to investigative mathematics, using an approach of experiential learning in the classroom. Taking risks also happens within the students' own learning of mathematics by moving out of their comfort zone and challenging themselves by doing mathematics tasks which they consider beyond 'their level'. Important seems to be the ability to know how to get unstuck which can be achieved by using course constructs and frameworks. It was also mentioned that the impact the tasks had on the student's own motivation, encouraged to take more risks in classroom. It is not clear what that motivation is. The course and task design's emphasis on many-right-answers/many-right-ways contributes to the students feeling that their thinking and ideas are valid and valued. This could give a sense of feeling safe. The requirements of the assessment tasks push the student out of their comfort zone as illustrated by the responses::

"I have been encouraged and required to think and work in new ways, without fear of failure."

"Seeing the impact of the tasks on my own motivation encouraged me to "take more risks" in the classroom."

"When I was stuck I used different constructs to get unstuck."

"Using techniques from constructs and frameworks to get you out of stuck situations."

"Having constructs at hand I feel capable of dealing with problematic situations."

Being imaginative

Students reported feeling imaginative as having new and different experiences of and approaches to learning mathematics. The 'low entry high ceiling tasks, the many-right-answers/many-right-ways design and the requirement for developing own examples and adapting tasks all seem to stimulate being imaginative. Experiential learning and reflection makes students become aware of their own learning and on working on developing a (new) structure of thinking. Making connections between mathematical ideas as a result of the task design leads to solving problems in more than one way as illustrated by the responses:

"Experiencing the connection between different concepts allowed me to solve a problem in various ways and also to prompt learners to seek alternative ways of solving a problem."

"The experiential approach to learning actually allows for all sorts of approaches to be tried. This is because the approach requires you to start with what you perceive to be important and, through a collection of smaller tasks, assess the appropriateness of your choice. From the experience of the process and the actual results obtained you can make further, more informed, choices."

Making connections

Students made connections with other areas of mathematics, other mathematical approaches, and with becoming aware of own learning. Working on imagery seems to offer a way to connect abstract ideas. The emphasis in the course and task design of that there is not one right way or right answer, but many, seems lead students to being able to empathize with differing ways of learning of other people. Reflection lets students become aware of their own learning, of their assumptions about learning and the role of their own past experiences and make connections between these. These are then challenged as a result of having to do the tasks themselves, and having to adapt and then try out the tasks with learners. Several respondents reported a change in their teaching as a result, and in particular and change in their questioning. Another aspect of development concerns being able to express and talk about learning and pedagogies by using constructs and frameworks as vocabulary as illustrated by the responses:

"I was looking at it with fresh eyes. I could see how other might approach it. I could develop the question, adapt for different learners."

"By learning mathematics in new ways I have become more aware of what works for my learners."

"Throughout the book our preconceived ideas are challenged."

"Returning regularly to the construct showed me how the way I learn relied on what I previous knew and the way I had previously tackled similar problems. Also showed me why less experienced learners may struggle to make these connects."

DISCUSSION AND CONCLUSION

This is a small case study specific to one course in mathematics education of the Open University and is thus limited in scope and range. It involves self-reporting and this may not be entirely reliable. It is also likely that respondents willing to spend about 20 minutes answering a questionnaire might be biased to be positive about their experiences as a student on the course and would have a positive story to tell. It is also not clear what the implications are of the timing of the research, that is the students were invited to take part in the study shortly after the due date of the final assessment but before they received the results.

The intentions of the study as reported in this paper were to find out whether the features of PT (Grainger et al., 2007) which was developed in a classroom setting with primary aged school children would resonate with the experiences of adult learners, in a distance learning setting, in the context of mathematics education learning. It seems that these students experienced the features of PT, although there was not sufficient data to get insights into what the feature of immersion would entail in this setting.

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TYPOLOGIES OF MATHEMATICAL PROBLEMS: FROM CLASSROOM EXPERIENCE TO PEDAGOGICAL CONCEPTIONS

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In this paper we propose a new classification of mathematical problems. Usually one speaks about low-level and high-level problems, exercises and authentic mathematical tasks. We aim to classify problems in terms of the activities and schemes of solutions needed to solve them. Moreover, we intend to introduce a two-dimensional model of classification. Finally, we suggest an approach aimed at developing an inductively-associative form of thinking.

Keywords: classification of problems, schemes of solutions, kinds of mathematical activities, associative thinking

INTRODUCTION

Students' different styles, approaches and problem solving strategies have been a matter of interest of many mathematics educators during the last decades (Schoenfeld, 1980 and 1992, Selden & Selden, 2005, etc). By learning problem-solving, "pupils should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations" (NCTM, 2000). One of the main problems of teaching mathematics is selecting the tasks which best 'promote and develop mathematical talent' (Chamberlin, 2010). V. G. Dorofeev (1983) suggested using tasks in, as he put it "rich problem neighbourhood". This means that for a given problem other problems exist which are connected to it by mathematical content, schemes of reasoning, and/or a circle of mathematical notions used in the solutions. The idea of a recent study by R. Leikin (Leikin, 2007) was to teach students to find diverse solutions of a given problem, which form what the author calls the 'solution space of a mathematical problem'. These papers include similar ideas since in both works pupils are accustomed to looking at the interrelations between problems, methods, notions, etc. A. P. Karp (1997) has suggested discriminating different types of combinations of problems, such as "prompting" (when the first question prompts the answer to the second one) or "control" (when one question helps to verify the answer to the previous one).

In this paper, pedagogical characteristics of separate problems are explored and discussed regardless of the problems' level of difficulty, though, we are mainly interested in pupils with advanced academic capabilities. When talking about teaching problem-solving, it is not enough to distinguish between *low-level* and *high-level* tasks. Chamberlin says that "HOT (*high-order thinking* – author's comment) tasks are those in which the problem solver needs to engage in cognition to successfully solve the problem" (Chamberlin, 2010, p.66). But what if a pupil is able to solve one HOT task but is unable to solve another? Can the cognition processes in these cases differ from each other?

THE STARTING POINT: ANALYZING PRACTICE

The highest priority of any educational process must be mathematical (intellectual) development. When studying mathematics, pupils solve many problems all the time. However, problem solving means more than just a straightforward execution of standard procedures. "Advanced ways of thinking, such as analyzing, conjecturing, defining, proving, generalizing … can and should be developed from elementary school onward" (Selden & Selden, 2005). Based on our intuition and experience, we have selected the following problems by means of which it may be possible to, so to speak, "enter the child's mind".

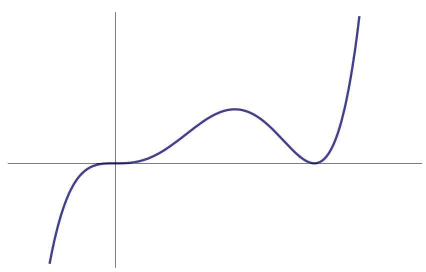
1. Find the range of the expression a^2-2a+b for $a\in[-2,3]$ and $b\in[-2, 1]$.

2. The numerator of a given fraction is increased by 1 and the denominator is increased by 2. Compare the fraction obtained with the given fraction.

3. Check whether or not 100903027 is a prime number.

4. Suppose x^{199} and x^{213} are both rational numbers. Is it true, then, that the number x is also rational?

5. Give a formula for a function whose graph looks like the curve in the following figure.



6. Does there exist a line tangent to the parabola $y=x^2-x+5$ and parallel to the line $y = 2011x$?

7. Find the largest value of the fraction $\frac{n^2}{2^n}$ where n is a positive integer.

Here are the solutions of these problems (given by the authors). Pupils' solutions (and mistakes) will be discussed later on.

Problem 1. The key point of the solution consists in rewriting the given expression in the form of $(a+1)^2-b-1$. Now, since $-1 \le a+1 \le 4$, the interval [0, 16] is the range of the expression $(a+1)^2$. Since $-2 \le b \le 1$, the interval [−2, 1] is the range of the expression $-b-1$. Consequently, since "the letters a and b vary independently", the interval [−2, 17] is the range of the given expression.

Problem 2. First of all, let us formulate the problem in the algebraic form. Let k be the numerator of the given fraction and let n be its denominator. After increasing the numerator by 1 and increasing the denominator by 2 we obtain the fraction $\frac{k+1}{n+2}$. In order to understand which fraction is greater, consider their difference. Thus we obtain $\frac{k}{n}-\frac{k+1}{n+2}=\frac{2k-n}{n(n+2)}$. Consequently, the initial number is greater if and only if the inequality $2k>n$ holds. The obtained inequality is equivalent to the inequality $\frac{k}{n}>\frac{1}{2}$. Thus, if the given number is greater than 0.5, we obtain a number which is less than the given one. And if the given number is less than 0.5, then the obtained number is greater than the given number. Finally, if $\frac{k}{n}=\frac{1}{2}$, then $n=2k$, therefore $\frac{k+1}{n+2}=\frac{k+1}{2k+2}=\frac{1}{2}$, which means that the obtained number equals the initial one.

Problem 3. Certainly, it is possible "to guess" the desired factorization. However, a natural way to answer the question consists of using the notion of decimal representation of integers. For example, the number 103 is "one hundred and three", or "one hundred plus three". Since the given number can be rewritten in the form of

$$100900000+3027 = 1009 \cdot 100000 +3 \cdot 1009 = 1009 \cdot (100000 +3) = 1009 \cdot 100003,$$

it is a composite number.

Problem 4. Certainly, the rationality of some power of a given number does not imply the rationality of this number. It is well-known that the number x such that $x^2=2$ isn't rational. On the other hand, if the numbers x^2 and x^3 are both rational then their quotient $x=\frac{x^3}{x^2}$ is rational, too. This observation is the key to solving this problem. Since the numbers x^{213} and x^{199} are rational, their quotient $\frac{x^{213}}{x^{199}}=x^{213-199}=x^{14}$ is rational as well. Similarly, the quotient of the number x^{199} and the fourteenth power of the number x^{14} is rational, so the number $x^{199-14\cdot 14}=x^3$ is rational. Moreover, the number $\frac{x^{14}}{x^{12}}=x^2$ is rational; consequently, the given number $x=\frac{x^3}{x^2}$ is also rational.

Problem 5. Certainly, there exists a rather routine method of "graphing functions". However, here we have the converse task. Let us try to find a polynomial $p(x)$ whose graph looks like the given curve. First of all, one has to describe properties of a function defined by its graph. Obviously, number $x=0$ must be its root with multiplicity of three or more and the number $x=2$ must be a root with multiplicity of two or more. Let's consider functions $y=x^3$ and $y=(x-2)^2$. Their product is a function with the described properties of its roots. Therefore, let $p(x)=x^3(x-2)^2$. We have to check that the behaviour of the function is similar to a function with a given graph and, in particular, that the function $p(x)$ has three intervals of monotonicity. For that purpose let us calculate its derivative,

$$p'(x)=3x^2(x-2)^2+2x^3(x-2)=x^2(x-2)(5x-6).$$

Therefore the function $p(x)$ increases on the intervals $(-\infty, 6/5]$ and $[2, +\infty)$ and decreases on the interval $[6/5, 2]$.

Problem 6. It is well-known that two lines in a plane are parallel if their slopes are equal. The slope of the line tangent to parabola $y = x^2 - x + 5$ at point (x_0, y_0) equals $2x_0 - 1$. The slope line $y = 2011x$ equals 2011. Thus, these lines are parallel to each other if and only if $2x_0 - 1 = 2011$, consequently, if $x_0 = 1006$. Therefore, the answer is positive; there exists a line tangent to the given parabola which is parallel to the given line.

Perhaps, it would be more natural to use the condition of tangency of a line to a parabola in terms of multiplicity of roots of an equation. Indeed, line $y = 2011x + b$ is tangent to graph $y = x^2 - x + 5$ if and only if the equation $x^2 - x + 5 = 2011x + b$ has a unique solution, that is, if the discriminant of the quadratic equation $x^2 - 2012x - (b + 5)$ equals zero. Obviously, the equation $2012^2 + 4b + 20 = 0$ has a solution; thus, the desired tangent line exists.

Problem 7. For an "expert" mathematician, it is obvious that the given sequence decreases starting with a number represented by *k*. In order to find this number, one has to solve the inequality $\frac{n^2}{2^n} > \frac{(n+1)^2}{2^{n+1}}$. By multiplying both sides by the common denominator we obtain the inequality $2n^2 > n^2 + 2n + 1$, $n^2 > 2n + 1$, and $(n-1)^2 > 2$, which is valid for all $n \geq 3$. Consequently, the fraction attains its largest value for $n = 3$. Thus, the answer is 9/8.

For a novice learner, it would be reasonable to use the following "heuristic strategy" (Schoenfeld, 1980, p.794): "Given a problem with an integer parameter *n*, calculate special cases for small *n* and look for a pattern." The following table contains the result of the calculation of the initial terms of the given sequence $x_n = \frac{n^2}{2^n}$.

n	1	2	3	4	5	6
x_n	0.5	1	1.125	1	$\frac{25}{32} \approx 0.781$	$\frac{9}{16} \approx 0.563$

Now the behaviour of the given sequence becomes completely clear. One only has to prove that this sequence decreases starting with its third term.

These problems appeared to be extremely hard for pupils from ordinary classes. The usual situation in a classroom is the following one: pupils look at the statements of the problems, sit still, and do not know what to do. In their attempts to give solutions to some problems they wrote that "Since the denominator of the fraction increased by a bigger number than its numerator, the value of the fraction decreased" (Problem 2) or "Since the power of an irrational number may be a rational number, the rationality of x^{199} and x^{213} does not imply the number *x* is rational" (Problem 4) without giving any justification for their assertions. Unfortunately, the majority of Russian pupils are still used to solving problems only by means of prescribed rules. Usually, when asked to solve a problem, they are only expected to recognize one suitable method. As a result of such teaching and learning, they get stumped if, for example, more than one method is needed to solve a problem (as is the case in the solution of Problem 6). The results found among pupils from classes with an advanced mathematical education were different, though even they were not completely successful. Certainly, while the facts that should be proved in Problems 1-7 appear to be rather unexpected, their solutions do not require “complicated mathematics”, no one needs to know or use “sophisticated techniques” to solve them, as all necessary notions and methods are familiar to all the pupils. So, what makes them so difficult?

Let us examine the solutions presented above. The notion of the range of a function (expression) is used in the solution of Problem 1, whereas the solutions of Problems 2 and 3 are purely algebraic. In the first case we suggest speaking about *the analytical scheme of reasoning*, in the second one – about *the algebraic scheme of reasoning*. Certainly, since in Problem 5 one deals with properties of functions, the scheme of reasoning in the solution of this problem is analytical. Despite the fact that the solution of Problem 7 is based on solving a rather routine inequality, the scheme of reasoning is analytical since the main idea consists of studying *the behaviour* of the given sequence. The solution of Problem 4 is different since its key point is the construction of an algorithm. The corresponding scheme could be called *a combinatory-algorithmic* scheme of reasoning.

Now, let us examine the starting points of all these solutions. In Problem 7, the calculation of the terms of the given sequence for small n made clear the behaviour of this sequence. In other words, we carried out *a mathematical experiment* which resulted in *setting a hypothesis*.

The solution of Problem 1 consists of three steps. First of all, we have *to codify* the statement of the problem in order to write it in a symbolic form, namely, the fraction $\frac{k}{n}$ is given and the fraction $\frac{k+1}{n+2}$ is obtained. In this situation, G. Pólya (1973) used the word *translation*; he wrote that “to set up equations means to express in mathematical symbols a condition that is stated in words; it is translation from ordinary language into the language of mathematical formulas”. We prefer to use the word “codifying” to describe a more general situation. For example, one may codify a subset of a finite set by means of a sequence of 0s and 1s.

In the second step we rewrote the inequality $a > b$ in the form of $a - b > 0$ and transformed the difference of fractions. Finally, we have *to interpret* correctly the obtained condition on the numbers k and n. Indeed, by rewriting the inequality $2k > n$ in the form of $\frac{k}{n} > \frac{1}{2}$, we obtain the condition on the value of the given fraction.

We propose to distinguish problems by the kinds of activities used to solve them. We have already identified such kinds as: *a mathematical experiment, setting a hypothesis, codifying,* and *interpreting*. Certainly, there also exists such a kind as *using a standard method*. For example, two standard methods were used in the first solution of Problem 5.

It is noteworthy that the mathematical experiment that can be carried out in the solution of Problem 1 shows that the obtained fraction may be more or less than the given one. Indeed, $\frac{1}{3} < \frac{2}{5}$, though $\frac{3}{2} > 1 = \frac{4}{4}$. Although we are not able to set the correct hypothesis after this experiment, it can protect us from reaching the wrong conclusion. However, the mathematical experiment in Problem 3 is sure to fail since the smallest divisor (larger than 1) of the given number equals 1009, which is the 169th prime number.

The activity employed in searching for the solution of Problems 1 and 3 may be called *restructuring*. One is engaged in the same kind of activity when performing substitutions in order to reduce a transcendental equation to an algebraic one, or, more generally, when representing some given function as composite of other simpler functions.

In the second solution of Problem 6, a problem from calculus was transformed to a problem concerning multiple roots of polynomials. The same kind of transformation occurred in the solution of Problem 5. Thus, ideas and methods from some themes of mathematics were used to solve problems initially stated in terms of another theme. Such kinds of activities may be called *transforming and transferring*.

To conclude this section, let us consider one more problem (Koichu, 2010, pp.258-259).

Problem 8. N players take part in an individual tennis competition. What is the overall number of games, if a player who loses a game leaves the competition?

B. Koichu observed that in solving this problem "the most frequently used start was considering the problem for small values of N". However, later on he says that this approach "led the solvers away from the immediate solution". When one sees that the number of games for $N = 2,3,4,$ and 5 players equals, correspondingly, 1,2,3 and 4, he or she will, obviously, generalize a hypothesis in which the number of games equals $N - 1$. Thus, the mathematical experiment leads to a correct hypothesis. To prove it, one has only to observe that any game excludes one player; therefore, after $N - 1$ games there will be only one player left – the winner of the competition. Thus, to solve this problem it is reasonable to use such kinds of activities as *carrying out a mathematical experiment* and *setting a hypothesis*. On the other hand, the proof of the hypothesis is purely logical. Certainly, any solution is an example of logical reasoning; we suggest calling such schemes of reasoning *syllogistic* (in particular, a proof built upon *reductio ad absurdum* is syllogistic). Therefore, B. Koichu's observations merely shows that his students are accustomed to reasoning algebraically and not in a syllogistic way.

A TWO-DIMENSIONAL MODEL

We introduce a new typology of problems (in a certain sense a two-dimensional typology), some of whose components were known previously (Vedernikova & Ivanov, 2002). The main idea consists of using the classification “kind of activity—scheme of reasoning” to assess the process of mathematical education. In the previous section, we introduced such schemes of reasoning, as: *analytical, algebraic, combinatory-algorithmic,* and *syllogistic* and such activities, as: *using a standard method, a mathematical experiment, codifying, interpreting, restructuring, setting a hypothesis, transforming and transferring*.

If we want to develop different aspects of students’ mathematical thinking such as ability to formalize mathematical material, to generalize it, and others (Krutetskii, 1976; see, also, *Standards for Mathematical Practice* in *Common Core Standards*, 2013), given problems have to be diverse in terms of the kinds of activities used for their solution and in terms of their schemes of reasoning. In a certain sense we looked for a transition from ‘the tacit expertise’ to ‘the grounded science’ (Ruthven, 1993). The problem is that it is impossible to describe these characteristics in terms of the typologies of mathematical problems that are known in Russia and other countries (see, for example, Sarantsev & Miganova, 2001, and Chamberlin, 2010).

In the following table we present our analysis of the solutions of the discussed problems. When solving highly difficult mathematical problems employment of several complex reasoning schemes are required. Thus it seems important to devolve problems that can be solved by implementation a particular reasoning scheme.

#	***Kind of activity***	***Scheme of reasoning***
1	restructuring	analytical
2	codifying; interpreting	algebraic
3	restructuring	algebraic
4	experiment	combinatory-algorithmic
5	transforming and transferring	analytical
6_1	using a standard method	analytical
6_2	transforming and transferring	algebraic
7	experiment; setting a hypotheses	analytical
8	experiment	syllogistic

As you can see, there are no problems with coinciding pedagogical characteristics; all of these problems are distinct. Therefore, this set in particular may be used successfully by a teacher for assessment of pupils’ mathematical development.

The development of mathematical thinking should be one of the main goals of teaching mathematics in schools, especially, at an advanced level. We gave problems

1-7 to pupils aged 15-16 who started to learn mathematics at the advanced level only in the autumn of 2012. Here is the description of one of their works.

One pupil gave an excellent solution for Problem 3; he restructured the given number and factorized the obtained numerical expression. He codified Problem 2 correctly; however, he made mistakes in operations with fractions. He made two common mistakes in the solution of Problem 1, namely, he concluded that if $a \in [-2,3]$, then $a^2 \in [4,9]$, and, consequently, the interval $[-2,13]$ is the range of the expression $a^2 - 2a$. He gave the correct solution for Problem 6 (the algebraic one). Through this process, a teacher is better able to understand strengths and weaknesses of the pupil's mathematical knowledge and development.

PAIRING PROBLEMS

In the last part of our paper, we develop the approach outlined in the section "Instead of a Conclusion" of the book by Ivanov, 2009. The main (and well-known) idea is that "What you have been obliged to discover for yourself leaves a path in your mind which you can use again when the need arises." George Christoph Lichtenberg (1742-1799).

The first example deals with problems which may be called *pre-tasks*. In a situation when your pupils are unable to solve the suggested problem, it is worthwhile giving them another one. To be more precise, you suggest that your pupils handle another problem with the same kind of activity and scheme of reasoning used in its solution (according to the table above).

For example, suppose you gave Problem 3. It is likely that they cannot solve it. You may suggest going on to the next problem by saying something like "Never mind, try to solve another problem".

Problem 9. Check whether or not for any positive integer *n* the number $n^5 + n^3 + n^2 + 1$ is a composite.

The point is that when factorizing a polynomial one is restructuring an expression. In our case, since $n^5 + n^3 + n^2 + 1 = n^3(n^2 + 1) + n^2 + 1 = (n^2 + 1)(n^3 + 1)$, this number is a composite for any positive integer *n*.

It is likely that the majority of your pupils will be able to do this exercise successfully, since most pupils are accustomed to factorizing polynomials. Your next sentence should be: "Surely you have guessed how to solve the previous problem…" The point is that the same kind of activity and the same scheme of reasoning are used in the solutions of Problems 3 and 9. Thus, you have given your pupils "a hint", but only "an implicit hint".

There are also problems which may be called *post-tasks*. Consider the following problem.

Problem 10. Examine the behaviour of the function $f(x) = \frac{x+a}{2x+b}$ (here both *a* and *b* are positive numbers) on the interval $[0,+\infty)$.

The solution is rather standard. For example, one can find the derivative, $f'(x) = \frac{2x+b-2(x+a)}{(2x+b)^2} = \frac{b-2a}{(2x+b)^2}$. Thus, if $b > 2a$, then $f'(x) > 0$, which implies that this function increases on the given interval; similarly, if $b < 2a$, then $f'(x) < 0$ and the function decreases. It is worth noting that the condition $b > 2a$ is equivalent to the condition: $f(0) = \frac{a}{b} < \frac{1}{2}$.

Now, ask pupils the following question: *"Can you see some relationship between this problem and the problem within the given set?"* The point is that Problem 2 is a straightforward corollary of Problem 10. And the number 0.5 in the answers of these problems appears because of the fact that the line $y = 0.5$ is the asymptote of a hyperbola which is the graph of the given linear-fractional function.

Do you agree that it is useful to pair problems?

CONCLUSION

A famous Soviet psychologist S. L. Rubinstein said that: "Man's development does not coincide with the content of his knowledge and skills and is not determined by the coherence of operations inherited to man, but by the culture of his internal intellectual processes." We cannot judge the situation in other countries, but in Russia teaching mathematics even at the advanced levels often means simply broadening the scope of familiar mathematical notions, methods and algorithms. However, as mentioned at the conference "Teaching Mathematics in Mathematics and Science High Schools" (Saint-Petersburg, 2012), we cannot attain a deep understanding of mathematics merely by giving our students a lot of problems to solve, most of which are exercises in applying known methods and algorithms. As a result, the majority of our pupils are only able to apply standard methods and become helpless even in a simple situation in which they are required to use two techniques in order to solve a problem. We hope that the results of our research will be useful for teachers, providing them with tools which can help them to analyse and enrich their pedagogical strategies as well as help them in constructing concrete lessons.

To conclude, the proposed ideas may also be instructive in pre-service and in-service mathematics teacher education (see, Ivanov & Il'ina, 2001, and Ivanov, 2009).

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MATHEMATICAL PROBLEMS FOR THE GIFTED: THE STRUCTURE OF PROBLEM SETS

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This paper discusses the structures of problem sets from textbooks for the mathematically gifted and suggests ways to analyze these structures. It is argued that seemingly purely technical details, such as the presence and quantity of simple problems or the grouping of problems, reflect important aspects in the understanding of gifted education, and more broadly, of the education of all students. An analysis of existing textbooks along the lines proposed here, it is argued, is useful for working teachers and textbook authors. But it may also interest a wider circle of mathematics educators, making it possible to pose a series of questions that may lead to new research.

Keywords: textbooks, gifted students, problems, problem sets, schools with an advanced course in mathematics.

INTRODUCTION

This article is part of a large study of Russian mathematics textbooks in which I am currently engaged. The analysis of textbooks and their history constitutes a complex methodological problem (Schubring, 1987). It is notable, however, that researchers usually focus on the theoretical material presented in a textbook, as it were, while devoting less attention to problems. In part, this is explained by the fact that problems were quite slow to appear in textbooks in general. On the other hand, even today, despite the fact that the importance of problem solving has become universally recognized, problems in textbooks are often seen as mere exercises, whose aim is to develop various skills, and which are consequently more or less typical and traditional, and therefore of little interest for discussion.

Here, we will briefly discuss the distinctive features of the structure of problem sets from textbooks for the mathematically gifted (or from special problem books for these students). For this category of students, problems are especially important (although they are, of course, important for all other students as well). The discussion will focus on details that may be regarded as technical, but which are founded, as we will argue, on important considerations about the philosophy and organization of education. Note that the present study is not yet completed and represents only a relatively small part of the research being conducted, or more precisely, offers only certain examples of this research.

ON THE PRINCIPLES AND GOALS OF THE STUDY

The principles (methodology) for analyzing problem sets are insufficiently developed. The analysis of problem sets is sometimes confined to simplistic quantitative parameters (how many problems are given per topic) and in the best cases to indications of different types of assignments -- multiple choice, short answer, or essay question. Although no complete theoretical generalization of methodological approaches is offered in this paper, examples of deeper analysis will be demonstrated.

The research question which this study attempts to answer is: what are the differences between sets of problems from textbooks for the mathematically gifted and sets of problems from "ordinary" textbooks. Important characteristics of the problem sets which need to be analyzed to address this question include (but are not limited to) the following:

- *Quantity.* As mentioned above, this characteristic is frequently analyzed. It is important, however, to go beyond simply identifying the numbers of problems in different sections, and to connect these parameters with others. The quantity of problems in a textbook has changed over time. At a certain point, historically, textbooks contained no problems at all for students to solve on their own. Gradually, the number of such problems grew, and separate problem books began to appear as supplements to textbooks. The increase of the quantity of problems was, of course, not necessarily the same across all topics, and specific distinctions between different cases deserve to be analyzed. The same number of problems in two sections can be achieved in very different ways, say, by providing identical or similar tasks or by posing original questions every time; one more direction of study is consequently the deeper exploration of quantity in each case. Another interesting theme for analysis is the relation between the main, theoretical part of a textbook, as it were, and the number of problems in the textbook, as various problems were moved over into the theoretical part and vice versa -- various theoretical concerns were shifted to the problem section, and so on.
- *Difficulty of problems.* This very notion requires a definition. Sometimes a distinction is drawn between difficulty as a psychological characteristic, which consequently can be determined only through empirical investigation, by offering students a problem to solve, and difficulty as a particular characteristic of the text, for example, the number of operations involved in the most widespread algorithms for solving a problem. Certainly both types of difficulty present interesting topics for study; in the work described here, however, we will focus on the second of them.
- *Arrangement in groups.* Identifying different groups of problems within each section, we can explore how different the groups are across the entire book. By

looking at the whole set of problems in a textbook or in some section of it as one mega-assignment, we can study the relations between different groups of problems given in the textbook. The theoretical principles of the organization of the groups can be identified (even if the textbooks' authors did not explicitly express these principles), and so on.

- *Mathematical subject matter.* It is natural to expect that problems' subject matter will correspond to the syllabus and this is, of course, typically the case. However, more precise quantitative analysis usually reveals that certain topics occupy special positions—the problems corresponding to them are incomparably more numerous and sometimes they are present, in one way or another, in virtually every section.
- *Structure of problems.* This question is highly significant. The structure of a problem is in fact a complex phenomenon, in particular, because the givens of a problem may be stated in highly dissimilar ways. Studying a problem's structure, for example, includes studying the relations between different possible forms of the representation of the mathematical material used in the problem.
- *Interconnectedness of problems.* This parameter also effectively pertains to structure, but the structure in question here is that of the set of problems as a whole (or the mega-assignment). Researchers have often attempted to represent the structure of an assignment in the form of a graph (Stolyar, 1974). Likewise, one may attempt to depict the whole set of problems belonging to a section in a similar fashion, representing the problems themselves as vertices linked by edges with other problems to which they are linked in one or another manner. The resulting picture is often informative. However, the fact is that reality is even more complicated and substantive. Not only is the resulting graph a directed one (the order of the problems is important), but the connections between the problems can vary greatly: a problem may serve as a hint for the problem that follows it, or it can represent the translation of another problem into a different mathematical language, and in both cases the two problems will be connected, but connected in different ways.

In addition to these characteristics, a few other are of interest, such as *non-mathematical subject matter,* which includes, for example, historical and political information incorporated in the problems; or *pedagogical characteristics,* such as are manifested in special forms of delivering content to students or even in the special selection of data in the problems with a view to making them more accessible for students. As has been said already, this paper presents only a small part of the research, and no full analysis of even the characteristics mentioned above will be undertaken below.

A BIT OF HISTORY

Let us state in somewhat greater detail which books will be analyzed and under what conditions these books are used. At the end of the 1950s, so called specialized mathematics schools appeared in the USSR (Karp, 2011; Vogeli, 1968; 1997). Sometimes these schools are called schools for the mathematically gifted, and this is justifiable, since many of their students have belonged to this category; however, it seems more correct to refer to them more neutrally as schools with an advanced course in mathematics.

These schools built their own curricula and traditions of teachings with the participation of leading Russian mathematicians, including Kolmogorov, Gelfand, Smirnov, Lavrentyev, and many others. Students were admitted to these schools on a competitive basis, often being selected from a whole large city or region. From the 1960s on, textbooks for teaching in these schools began to appear in one or another format.

Later, the number of such schools grew substantially. Special textbooks and problem book for such schools began to be published, sometimes in quite large numbers (for example, Alexandrov, Werner, Ryzhik, 2006a, b; Vilenkin, Ivashev-Musatov, Shvartsburd, 1995a, b; Galitsky, Goldman, Zvavitch, 1997; Karp, 2006). While at the beginning of their existence schools with an advanced course in mathematics included only the two upper grades, later a four-year course of advanced study, encompassing grades 8 to 11, became widespread.

It must be said that the curricula of such schools, whether provided by the Ministry of Education or formulated in the schools themselves, usually include many advanced sections that are either not studied in ordinary schools at all or studied there in incomparably less depth and detail. Below, we will subject those sections to analysis which at least in name correspond to those studied in ordinary schools. This will make it easier for us to draw comparisons and contrasts. Note that aside from textbooks and standard problem books, there also exists a vast supplementary literature for mathematics schools, which includes, for example, Olympiad problem books. These will not be analyzed below.

PROBLEM BOOKS FOR GRADES 8-9

Let us look in more detail at one of most widely used problem books in schools with an advanced course in mathematics: the problem book for grades 8-9 by Galitsky et al. (1997). This text began to be published in the mid 1990s and has gone through many editions since. Reflecting the curriculum that existed at the time of its publication, the problem book contains the following chapters:

> Review; Divisibility of integers; Square roots; Quadratic equations; Inequalities; Powers with integer exponents; Functions; Equations and systems of equations; Word problems;

Powers with rational exponents; Sequences and progressions; Trigonometric expressions and their transformations.

Practically all of these topics (except perhaps the divisibility of integers) were covered in one way or another at certain points in ordinary schools as well. We will see many more distinctive features, however, if we compare separate groups of problems. Consider, for example, the small and auxiliary section "Incomplete quadratic equations."

This section contains 16 assignments, each of which except the last four includes several—from three to six—problems. The assignments united under one number are not necessarily identical. For example, the first of them (5.1.) contains incomplete quadratic equations of the form $ax^2 = b$ with coefficients of different types (fractional, negative, etc.) as well as equations that are reducible to them, including equations that may be reduced to incomplete quadratic equations by introducing a new variable (for example, $4-9(2-5x)^2 = 0$). The remaining assignments are devoted to the following questions:

Solving incomplete equations of the form $ax^2 + bx = 0$ and equations reducible to such equations; Incomplete equations with letter coefficients; Rational equations reducible to incomplete quadratic equations; Incomplete equations with absolute value notation; Problems on investigating equations with parameters; Word problems.

For comparison, consider an analogous section from the textbook by Dorofeev et al. (1999), written somewhat later than the problem book just cited and for ordinary schools. The problems in this textbook are divided into two groups, A and B, the second of which is addressed to stronger students. Section A contains 14 assignments, half of which contain six problems each (this time, the problems are, if not completely analogous to one another, then very similar). These problems are devoted to solving various types of incomplete equations and equations that are reducible to them. We should note the inclusion of problems—nonstandard for such a section—in which students are asked to solve incomplete cubic equations on the model of incomplete quadratic equations (for example, $x^3 - x = 0$). The remaining problems in this section are word problems.

Section B contains 12 assignments, which include equations that are reducible to incomplete quadratic equations, including those that may be reduced to incomplete quadratic equations by introducing a new variable, solving equations with letter coefficients, investigating equations with letter coefficients, and once again, word problems.

It may be noted at once that even if the topics in the two problem book are the same, a problem from Galitsky et al. (1997) is usually more difficult, if only in the sense that its solution requires more steps. Compare, for example, problem 490 (b) from Dorofeev et al. (1999) and 5.3. (d) from Galitsky et al. (1997). In the former, students are asked to solve the equation $ax^2 - x = 0$; in the latter, the equation $a(x^2 - 6x + 9) + 4 = 0$. Another factor, however, is probably even more important.

Even in comparing the problem book by Galitsky et al. (1997) with the text by Dorofeev et al. (1999), which is also written in such a way as to allow students who are gifted and interested in mathematics to use it, we can see that the differences between them consist not only in the technical difficulty of the problems, but in the different principles underlying their organization. The following three distinctive characteristics may be identified in books for mathematically advanced students:

- Fewer problems in which students are asked to directly apply and practice employing an algorithm that they have studied;
- Greater attention to developing some idea or transferring it onto other objects;
- More connections with other, previously studied sections.

BOOKS FOR GRADES 10-11

Textbooks for grades 10-11 with an advanced course in mathematics, as has already been said, contain many extra sections by comparison with ordinary textbooks. (For example, Vilenkin et al., 1995a, 1995b contains sections on limits and continuity, complex numbers and operations with complex numbers, elementary combinatorics and elementary theory of probability, while the textbook by Alimov et al., 1996 does not cover any of these topics.) Let us again compare sections that are present in both types of textbooks. For example, solving exponential equations and inequalities.

This section in the textbook by Vilenkin et al., 1995b is very short. Students are given only 14 equations and inequalities (arranged in two groups). The textbook by Alimov et al., 1996 offers 107 problems arranged in 26 groups. A comparison between these numbers, of course, tells us little if we do not also take into account the number of problems in each textbook overall (the textbook for ordinary schools also has more problems in general).

It is noteworthy, however, that in Alimov et al., 1996 the problems are arranged in such a way as to be next to problems that are very similar to them conceptually (within the same group). Sometimes, there is a certain development within a group. For example, group № 15 begins with the equation $3 \cdot 9^x = 81$ and ends with the equation $6^{3x-1} = 6^{1-2x}$. In both cases, naturally, the solution is based on the use of the fact that if $a^b = a^c$ (where a is a positive number not equal to one), then $b = c$. Nonetheless, the assignments are not identical.

They differ, however, far less than the assignments in the corresponding group in Vilenkin et al., 1995b. Here, the first equation is $4^{x-1} + 4^x = 320$, but already the third equation is $2 \cdot 3^{x+1} - 5 \cdot 9^{x-2} = 81$, in other words, an equation of a completely different type. Note also that Vilenkin et al., 1995b contains no completely elementary equations or inequalities such as $a^x = a^b$ or $a^x > a^b$ at all, while Alimov et al., 1996 contains quite a number of them.

On the other hand, Vilenkin et al., 1995b does contain assignments that connect the topic being studied with topics studied earlier. These include, for example, the following, in our view rather artificial, equation and inequality: $5^{2+4+...+2x} = 0.04^{-45}$ and $\left|2^x - 2\right| - \left|2^x - 1\right| \geq \left|2^x + 1\right| - 5$, in order to solve which students must know how to find the sum of a geometric progression and to solve modulus inequalities. In other words, the distinctive characteristics that we noted earlier are present here as well.

DISCUSSION

Above, we deliberately discussed two typical algorithmic (technical, computational) sections. Neither solving quadratic equations nor solving exponential equations and inequalities can be considered topics that throw into some special relief the kind of conceptual understanding and depth that we would like to regard as typical manifestations of mathematical giftedness.

Of course, it is not difficult to cite examples from other topics, or problems characteristic of another way of organizing instruction. Probably the most vivid example of such a special way of organizing instruction is the system of instruction used in several Moscow schools that makes use of "leaflets" or "sheets" (Davidovich, Pushkar', Chekanov, 2008). The problems used in this system of instruction are selected in such a way as to allow students themselves to arrive at the recognition of the fundamental theoretical principles of the course being studied, deriving and proving the corresponding propositions on their own.

In the West, a similar system has been called the "Texas method" or the "Moore method," since it is often associated with R.L.Moore, who made use of it (Parker, 2005). The fact that this system may be used only with highly gifted students, and even then only under special conditions, is obvious (for example, in Moscow's school № 57, at a class being conducted according to the system just described, in addition to the teacher-supervisor, there are four or five teaching assistants to whom students present their solutions, who clarify certain details, etc.). The discussion above, however, concerned no special sections and no special conditions.

The three distinctive characteristics of the structuring of problems in the educational literature for the gifted that we have identified need to be theoretically interpreted and viewed in the light of other theoretical insights into the gifted. The distinctive characteristics of the thinking of the mathematically gifted were already identified by Krutetskii (1976), who took into account the opinions of practicing teachers. It may be said that practicing authors of textbooks rely on the following notions about the thinking of the gifted (which are reflected in the distinctive characteristics of the structures of problem sets discussed above):

- Gifted students grasp rules and algorithms more quickly and retain them better;
- Gifted students more easily make connections with other sections of mathematics that they have studied and are studying, more freely transferring what they have learned to other fields.

Here, we might ask to what degree these considerations (which largely parallel those formulated by Krutetskii) are borne out not only by practical experience, but also by organized studies. Let us note that the practical question concerning the number of assignments-exercises that students should be given for the purpose of practicing the application of some rule deserves serious investigation in general. It is not difficult to point to situations in which students are given literally hundreds of identical assignments so that they might memorize rules sufficiently thoroughly. On the other hand, one may also cite textbooks that offer only two-three identical problems. Of course, sometimes this is done because the authors in principle do not approve of memorization and consequently do not set themselves the task of facilitating it. But sometimes this is not the case—the authors think that certain things do have to be memorized thoroughly, but believe that even two same-type, one-step problems will suffice, while the rest will be learned in the course of working on more difficult assignments. Whether this is the case, or whether it pertains only to mathematically gifted students, or even whether this is not always true even for these students, or whether the outcome depends not on giftedness but on some other facts, are questions that, in our opinion, have not been sufficiently investigated.

In another paper (Karp, 2007), we discussed the notion that possession of great knowledge is evidence of talent. As can be seen, the authors of textbooks assume that gifted schoolchildren remember more than ordinary ones, and boldly weave into problems materials covered earlier. On the other hand, one could say that authors do this because they make it their goal to get schoolchildren to remember more than is demanded of them under ordinary conditions (and as experience shows, they achieve this goal). It would be interesting to collect data (experimental or drawn from teaching experience) about the success—or conversely, failure—of such an approach with "ordinary" students. What happens if they are treated in the same way as those who are recognized as gifted?

In the assignments analyzed above, individual ideas are developed only to a modest extent (which cannot be compared, for example, with the way in which ideas are developed in the aforementioned "sheets"), although this still goes beyond what is found in "ordinary" textbooks. More generally, shifts from one group of assignments to another are usually based on rather simple considerations—usually, they are due simply to the fact that another topic is being covered, that is, another type of assignment and another algorithm for solving such assignments. In another paper (Karp, 2002), we pointed out that the structure of problem material may be quite complex and discussed the importance of studying the "morphology of the problem set." It is noteworthy that, despite the virtually universal recognition accorded to the importance of creating opportunities for independent discovery by the students, in practice such opportunities, which are opened up by the nonstandard structuring of problem sets, are not very numerous in textbooks. This points to interesting directions for improvements in working both with gifted and with interested students, and, indeed, with all other students as well.

CONCLUSION

The practice and theory of working with mathematically gifted students are, we would argue, not infrequently divorced from one another (which happens often in mathematics education in general). Working teachers often do not know about the work of psychologists, although it would be useful for them, while theoreticians in their turn not always consider it necessary to become familiar with the work of working teachers. Meanwhile, comparisons and contrasts between different approaches are evidently fruitful.

The composition of textbooks is undoubtedly based on definite theoretical principles, even if these are not always explicitly formulated, and perhaps not even fully recognized by the authors themselves. Analyzing existing experience and identifying such principles, we would argue, is useful to working teachers and theoreticians alike.

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LEARNING WITH PLEASURE: TO BE OR NOT TO BE?

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Creative education often addresses the question what are students able to understand immediately and what can they not. Importantly, to answer this question, teachers need not only consider students' knowledge and abilities, but their feelings and motivation as well. Therefore, the issue of pleasure in the process of learning deserves to be studied. It is hardly possible (at least at this stage) to present a rigorous quantitative study of relevant facts, but still, this paper aims to present some thoughts in identifying some principles for working with students .The paper offers a summary of decades of teaching practice in action.

Keywords: Learning with pleasure, interest, motivation, problem solving, proof.

THE LEARNING WITH PLEASURE: AWAKENING INTEREST

The goal of this paper is to analyze the practice of writing problems and organizing working sessions on students' problem solving activities. Challenging problems were a focus of a special ICMI study (Barbeau and Taylor, 2009). This work to some extent continues along the same lines, concentrating on one specific theme – making problem solving enjoyable. The definition of pleasure and enjoyment in general and in mathematics education specifically has been the subject of many studies which hardly can be reviewed here even superficially (let's just remind that the famous book of Rademacher and Toeplitz (1957) was called in English translation *The Enjoyment of Mathematics*). The author would limit himself to saying that the pleasure which we are feeling in process of dealing with mathematics is an intellectual pleasure often associated with the accomplishment of an important task (it can be compared with the pleasure that one feels after finishing building a house).

To understand the issue in question better, we need to formulate some general pedagogical assumptions about students and their learning in general. Based on our observations we assume a child can achieve astonishingly much when properly instructed and motivated.

CASE STUDY OF AN *OPTIMISTIC NUMBER*

Let us start with the analysis of one problem. Let us name a natural number *optimistic* if it has more than one digit and if these digits, taken from the right to the left, are strictly decreasing. So in this case 12 is the first *optimistic* number and, obviously and consequently, the number 123 456 879 is the last optimist. We ask to multiply the optimistic number by 9 and then count the sum of its digits. For instance, taking that first optimistic number 12 and multiplying it by 9 we'll get 108 with 9 as the sum of digits 1 + 0 + 8.

So what else could we get? For another example take the largest optimistic number 123 456 789. Multiplying by 9 we'll get 123 456 789 · 9 = 1 111 111 101 – again with 9 as the sum of its digits

$$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 0 + 1.$$

What more could we get? Or do you already believe that only 9 as the sum of its digits is the uniquely possible? What to do and how to start?

AWAKENING THE NATURAL CURIOSITY

It is very unlikely that the student is not impressed by these "strange" results. How can it be? Is it a miracle? Typically, students consider more numbers and try to check the statement. Whatever their next step will be, students will be engaged in thinking. Their natural curiosity is awaked.

Why is it here and not in many problems offered in school? Probably, the reason is that the problem starts with a very clear and observable result. Understanding the problem is the first stage among Polya's four stages (Polya, 1973), and is really important. The problem can and even should be challenging but we need to prepare students by starting with something what student can see right there in the classroom. The first principle is "start with something known and understandable". The second principle is also here: "try to say something unusual and unexpected". The cognitive conflict between clearness and unexpectedness awakes the curiosity.

CASE STUDY OF *DISTRIBUTING COINS*

The problem to be discussed below was reported to the author by the famous Latvian professor, problem composer and educator Agnis Andžans. The author of the problem and the author of this paper presented the problem many times to different audiences, starting from students of Grade 3 and 4 and finishing with students of Vilnius University. The process of the solution is the same.

Firstly, the lecturer asks how to distribute 12 identical coins between two purses putting in one purse twice as many coins as in the other one. The clear answer – 4 and 8 – in all audiences is given immediately, without the slightest delay.

And now the lecturer proposes to take 10 coins instead of 12 and achieve exactly the same result – that in one purse there would be twice as many coins as in the other one. It needs to be added that to divide coins into parts is not permitted.

The audience is puzzled. Those more educated even write the equation $x+2x=10$ which can not be solved in integers (and dividing coins is not permitted). Those less experienced simply say "No, it is not possible!" The lecturer is supposed to behave in such way as to insist that a solution is still possible. Sometimes somebody finds the solution. If this is not case then the lecturer announces that he will now make two “moves” and some people will immediately, unavoidably present the answer. The two “moves” are simple: the first movement imitates some small circle (purse) and another – another circle (purse), large one.

Immediately, some listeners come with the solution: one purse with 5 coins must go into another purse which holds already the remaining 5 coins. The audience is happy. The pleasure of problem solving is achieved.

WHY WAS THE SOLUTION PLEASURABLE?

The task was selected according to the principles formulated above -- the situation is clear as well as the cognitive conflict. The next step comes with the solution -- it is enjoyable. Why? First, it is understandable -- no person in the room can say that he or she do not see how it works even if he or she did not solve the task. Second, it is unexpected. The person solving the problem had some limitations, some stereotype in the mind. This limitation was removed -- the feeling of intellectual freedom makes the problem solver happy.

CONSIDERING MORE CHALLENGING PROBLEMS: CASE STUDY OF THREE BROTHERS

The previous problem is just a puzzle and although some mathematics is involved it hardly relies on any sophisticated knowledge. Let's consider a situation that is a bit more challenging.

Let's take any person and assign to this person any number – positive, negative, or zero. The most natural way to model this process with a real world situation is considering a financial capacity of the person. Indeed, a person can have a positive banking account, have nothing, or, unfortunately, even be in debts, that is have a negative account. In line with it we define the financial capacity of any group of persons (maybe containing a single person only) as the sum of numbers assigned to each person in that group. A financial capacity of any group of persons (again, maybe containing a single person only) is said to be *remarkable* if, and only if, that sum in question is positive. And now the problem comes.

Consider any group of three brothers with the *remarkable* financial capacity. Prove that there is a pair of brothers with the *remarkable* common financial capacity.

The formulation of the problem is as required simple and understandable. The solution is also clear -- let's just assume that the statement is not correct. In other words, let's assume that

(A) The total financial capacity of all three brothers is *remarkable*.
(B) The total financial capacity of any pair of all three brothers is not *remarkable.*
Now some mathematical language should come in the play. Let x, y and z be correspondingly the numbers assigned to these three brothers or these are their single

financial capacities. Then the first condition that their total financial capacity is *remarkable* means that the sum of all x, y and z is positive. But, according to the second condition all three pair-wise sums $x + y$, $y + z$ and $z + x$ are not positive, because, as told, all total financial capacities of any possible pair are not *remarkable*. Adding these three non-positive sums we'll get the double sum of all three numbers x, y and z, or

$$2(x + y + z) \leq 0.$$

But then $x + y + z \leq 0$ what is clear contradiction to the given fact that the total financial capacity of all three brothers is *remarkable* – or to the condition 1.

EXTENDING THE PROBLEM

Are you happy? Well, it is a nice problem, but it is not a surprise. Neither the result, nor the method was unexpected. The point, however, is that once we familiarized ourselves with the situation we can go further. The next question comes if we "slightly" change the word -- instead of asking about *one* pair ask about *all* pairs.

A problem with three brothers now sounds like that: Given that the total financial capacity of three brothers is remarkable, is it necessarily true that the total financial capacity of all pairs of these brothers is remarkable?

The solution is even easier that the previous one but of very different nature. Let's simply consider an example; let the first brother have 6, the second 5 and the third brother -10. Obviously, the sum of all three numbers is positive, while two sums of two numbers are negative. The answer to the problem is no, it is not necessarily true.

In solving this problem we solved one more about having two pairs of brothers with the total remarkable capacity. Even having them is not necessarily true.

The case of three brothers is explored but now the time for the four brothers comes. Now the list of questions might be longer and contain the following questions in addition to the question about one pair which is answered affirmatively:

(A) Are there necessarily at least two pairs with the common *remarkable* financial capacity?

(B) Are there necessarily at least three pairs of brothers with the common *remarkable* financial capacity?

Then you can pose a general question about n brothers with the total *remarkable* financial capacity (this problem was offered in one Olympiad in Belarus).

THE PLEASURE OF ACTING AS A MATHEMATICIAN

The author of this paper discussed the problem in many audiences. Obviously, the financial model for this problem helps solvers to understand the problem better and feel more comfortable with it. It would be wrong, however, to attribute the interest which the audience has to the matter of money only. Clearly not. The reason for the pleasure is again purely intellectual: participants enjoy the process of asking questions and extending the problems. Typically, this work is done by the research mathematicians, while students are assigned to the lower job of answering somebody's else questions (the analysis of problem posing from the social point of view is offered by Brown and Walter, 1990, and by Watson and Mason, 2005). Here they can act as a mathematician and, again, this new freedom is enjoyable.

A FEW "MINOR" DETAILS: CASE STUDY OF *DOUBLY THRICE-SMART NUMBERS*

The problem is supposed to be attractive. In writing attractive problems a teacher should remember that children likes games, they like tales, they like unusual situations and words. It is advisable to combine the mathematical side with the play.

For example, it is well known that for any large integer it is possible to write it as the sum of a few summands. It can be done in several ways. If the number gets smaller the degree of freedom "gets smaller". Applying some special and unusual terminology, we may try to make the situation to appear more attractive.

Let us call the number *doubly thrice-smart* if it is possible to write it as the sum of three addends in two ways with no of those six attracted addends being used twice.

Clearly 111 or 21 is *doubly thrice-smart* because of
$111 = 100 + 10 + 1 = 104 + 4 + 3$, and $21 = 1 + 9 + 10 = 2 + 4 + 15$.
On the other hand, it is clear that 6 is not *doubly thrice smart* because of
$6 = 1 + 2 + 3$ is clearly the only possibility to write 6 as the sum of three different addends. So the question – where is the boundary between these integers that are *doubly thrice smart* and which are not, might be interesting for the arithmetical investigation.

The solution of the problem, although technically not difficult, contains some substantial ideas. It may be helpful to apply Polya's strategy of working backwards and think what can be the numbers involved in the sums for our number (let's call it x). Obviously, six smallest positive integers are

$$1, 2, 3, 4, 5, 6.$$

Assume that this *doubly thrice smart* x is obtained using these numbers in any order and in any way twice. That implies that 2x equals the sum of these numbers which is

$$1 + 2 + 3 + 4 + 5 + 6 = 21.$$

Two conclusions are following. First, we cannot operate with these numbers only -- the number 21 cannot equal 2x. Second, numbers smaller than 11 are not *doubly thrice smart* -- even operating with the smallest possible numbers we obtain something greater. It means that the smallest possible candidate to be *doubly thrice smart* is a number 11. And 11 is indeed such a number because 11 can be written either as

$$11=2 + 4 + 5 \qquad \text{or as} \qquad 11 = 1 + 3 + 7$$

so 11 is indeed the smallest such integer and the rest follows.

OBSERVATIONS FROM THE CLASSROOM

This problem according to the experience of the author, is understandable for children of the age of 10. Importantly, they even are already able to understand the concept of proof.

By saying this, the author doesn't mean that the abstract reasoning is necessarily developed by all or even by many third-graders. When teaching this problem in Grade 3, the author faced the situation that the conclusion that no number smaller than 11 can be *doubly thrice smart* was taken as the one established by authority of the teacher. Still, students realized that the statement had to be somehow supported and validated (the authority of the teacher was one of the possibilities, particularly when the practical experience also demonstrated to the children that there are no required representations for the numbers smaller than 11). In Grade 4 at the same school it was possible to achieve much more: evidently, a good proportion of pupils present were able to understand even the abstract part or the proof.

In any case even those who missed the abstract part enjoyed the task. The atmosphere of the game rather than the dull and routine school activity was helpful, and the funny wording was useful in establishing this atmosphere. Another important feature was that the problem permitted actions on the different levels of cognition. While some students were concerned with more abstract tasks, other could enjoy the more practical and concrete search for numbers. The opportunity to work differently seems to be an important characteristic for making problem solving a pleasurable activity.

WHAT IS AN EFFECTIVE HINT? THE CASE STUDY OF THE ST. PETERSBURG PROBLEM

The problem sounds as a task for the low grade. Nine numbers are given:

15, 16, 17, 18, 19, 20, 21, 22 and 23.

We need to put them into 9 entries of a 3 x 3 table in such a way that the sum of any pair of neighboring integers is always different (neighboring cells are those sharing the side).

Let us note that placing them in the order is not suitable. Indeed in that case 16 shares the common side with 17, as well as 15 with 18 do. Because both pairs have the same sum 33 that way of placing them is not the required one.

15	16	17
18	19	20
21	22	23

Again, the teachers can find themselves in the situation similar to the one with the purses described above. The audience does not see the way to solve the problem. It would be desirable to give a hint, but still, not providing the entire solution. One way to do this may be to suggest thinking about how to make the sides more like diagonals so that neighbors will not be neighbors anymore. The hint although very small and more emotional than practical can suggest the right direction of thoughts. Indeed, imagine that the table given above is rotated around its center. We will obtain the following table in which no "wrong" pairs are visible.

15	17	20
16	19	22
18	21	23

Indeed, all 12 of neighboring pairs prove to provide different sums:

$15 + 17 = 32$, $17 + 20 = 37$, $15 + 16 = 31$, $17 + 19 = 36$, $20 + 22 = 42$, $16 + 19 = 35$, $19 + 22 = 41$, $16 + 18 = 34$, $19 + 21 = 40$, $22 + 23 = 45$, $18 + 21 = 39$, $21 + 23 = 44$.

The effect of achieving much by undertaking practically nothing is quite memorable. This little hint proves to be very efficient - students learn to consider their "vague" feelings and make them more accurate.

THE SUPPORT AND ENCOURAGEMENT OF THE TEACHER MEANS A LOT!

This paper deals mainly with the mathematical side, identifying criteria for a good problem and for a good solution. The pedagogical dimension is, however, at least equally important. The pedagogy has already been mentioned when talking about the wording, but the behavior of the teacher is of major value too. The paper was started with the problem about the optimistic number. Now, near the end of the paper, we should praise the optimistic teacher: a teacher who believes in the students' abilities and interests and shares these interests with them. This disposition of the teachers is felt in the classroom and influence the audience.

CONCLUSION

The author spent many years teaching how to solve problems and has written several books in three languages trying to report some of his impressions (Kasuba, 2008a, 2008b, 2009, 2012). This experience represented in the discussion above leads to posing two questions.

The first one deals with regular school mathematics: the tasks discussed above are usually considered to be part of so called extracurricular mathematics, or mathematics for the circle. It is important to consider what can be done with regular school mathematics about the equation of the straight line and the law of sines to be pleasurable. The author strongly believes that the principles for the teacher are the same as stated above:

- start with something known and understandable;
- try to say something unusual and unexpected;
- provide the feeling of intellectual freedom;
- give the chance to act as a mathematician;
- make use of minor details being age appropriate;
- be optimistic and provide a supporting behavior.

It is another story that, surprisingly, the experience of pleasurable learning in the regular classroom is not very often discussed. The author believes that such experiences should be collected and analyzed. Also, the samples of activities and problems which proved to be helpful in this work should be made broadly available. The question here is not just a technical issue -- making the learning pleasurable in the context of a regular classroom includes some important reorganizations permitting specifically the happiness of intellectual freedom.

It is important to emphasize here the difference between "pleasurable" activity and easy activity as well as non-mathematical activity. Unfortunately, too often in order to keep students engaged and make them have fun mathematics is substituted with some irrelevant stories or pictures or videos. The job of the mathematics educator is to make mathematics education pleasurable, not to provide pleasure instead of mathematics education. Open and critical discussion of the existing controversies is badly needed.

The second question deals with the methodology of research in mathematics education. This paper represents the notes of the practitioner attempting to generalize his observations. Given the development of methodology and research instruments in mathematics education (Schoenfeld, 2007), it would be desirable to find some ways to complete more rigorous studies on the issue of learning with pleasure. Clearly, we have some instruments which permit measurement of the satisfaction of the group. The author, however, is afraid that so far these instruments cannot distinguish the highest levels of intellectual pleasure (just as it is difficult to distinguish by means of questionnaires the difference between good and entertaining performance and major events in the history of theater).

The author believes that we need to make use of qualitative studies but rigorously organized. Specifically, we need to collect episodes of learning mathematics with pleasure at different levels, starting probably with those who later became involved in mathematics, but not limited to them. What was pleasurable? How was the pleasure achieved? What was the role of the teacher?

Answering all these questions seems to be both of practical and theoretical importance. The author would like to think that this paper contributes to raising these questions.

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THE CONNECTION BETWEEN MATHEMATICAL CREATIVITY AND HIGH ABILITY IN MATHEMATICS

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This paper presents part of the findings from a large-scale study that explores the relationship between mathematical creativity, mathematical expertise and general giftedness, a relationship which is not obvious. We distinguish between relative and absolute creativity in order to evaluate mathematical creativity in school children. This paper demonstrates that general giftedness and excellence in mathematics has a major effect on secondary students' creativity associated with production of multiple solutions to mathematical problems. However, these effects are task-dependent. Thus, we conclude that different types of Multiple Solution Tasks (MST) can be used for different research purposes.

Key words: Mathematical Creativity, Multiple Solution Tasks (MST), General Giftedness, High Level of Mathematical Instruction.

RATIONALE

This study employs multiple solution tasks to explore students' creativity in mathematics. It continues a series of studies that were directed towards the design and validation of the research tool used herein. (Leikin and Lev 2007, Leikin 2009, Leikin & Lev In press-2013 , Levav-Waynberg and Leikin 2012, Guberman and Leikin 2012). These earlier studies also examined the relationship between mathematical creativity and the level of the participants' mathematical ability. All the studies led to several hypotheses which we are currently examining in this large-scale study. We hypothesized that (1) between-group differences are task dependent and (2) in the originality-fluency-flexibility triad, fluency and flexibility are of a dynamic nature, whereas originality is of the "gift" type.

BACKGROUND

Creativity

There is no single, authoritative perspective or definition of creativity (Mann 2006). Our study follows Torrance's (1974) definition of creativity with four components: *Fluency* refers to the continuity of ideas; *Flexibility* is associated with changing ideas; *Novelty* is characterized by a unique way of thinking; *Elaboration* refers to the ability to generalize ideas. Of these four components, novelty, or originality, is widely acknowledged because creativity is viewed as a process having to do with the generation of original ideas.

While drawing a connection between high abilities and creativity, researchers express a diversity of views. Some claim that creativity is a specific type of giftedness (e.g., Sternberg 2005), while others feel that creativity is an essential component of giftedness (Renzulli 1978), and still other researchers suggest that they are two independent characteristics of human beings (Milgram and Hong 2009). Thus, analysis of the relationship between creativity and giftedness, with a specific focus on the various fields of mathematics, is important for better understanding of the nature of both mathematical giftedness and mathematical creativity.

Mathematical creativity

One of the complexities related to the relationship between mathematical giftedness and mathematical creativity is rooted in the contrast between viewing mathematical creativity as a property of the professional mathematician's mind (Ervynck 1991) and the opinion that mathematical creativity can and must be developed in all students (Sheffield 2009). Naturally, creativity in school mathematics differs from that of professional mathematicians. Mathematical creativity in school students is evaluated with reference to their previous experiences and to the performance of other students who have a similar educational history. Leikin (2009) suggested that viewing personal creativity as a characteristic that can be developed in schoolchildren requires a distinction between *relative* and *absolute* creativity. *Absolute creativity* is associated with discoveries at a global level. Our work deals with *relative creativity*, which refers to mathematical creativity exhibited by school students when evaluated in relation to their previous experiences and to the performance of other students who have similar educational histories. The current study accepts the relative perspective on creativity while evaluating originality of students' solutions.

THE MODEL FOR EVALUATION OF MATHEMATICAL CREATIVITY

Multiple solution tasks

A multiple solution task (MST) is an assignment in which a student is explicitly required to solve a mathematical problem in different ways. Solutions to the same problem are considered to be different if they are based on: (a) different representations of some mathematical concepts involved in the task, (b) different properties (definitions or theorems) of mathematical objects within a particular field, (c) different properties of a mathematical object in different fields.

Table 1 demonstrates an example of a multiple solution task (Jam problem) and depicts 10 different solutions to the problem.

Table 1: Jam problem-Multiple solution task

Group of solutions	Description
	Jam Problem Mali produces strawberry jam for several food shops. She uses big jars to deliver the jam to the shops. One time she distributed 80 liters of jam equally among the jars. She decided to save 4 jars and to distribute jam from these jars equally among the other jars. She realized that she had added exactly 1/4 of the previous amount to each of the jars. How many jars did she prepare at the start? 
A	System of equations -1 $\begin{cases} xy = 80 \\ 1.25x(y-4) = 80 \end{cases}$ System of equations -2 $\frac{x}{y} \Rightarrow \frac{x}{y-4} = \frac{5x}{4y}$
B	Equation -1 $\frac{4}{x} = \frac{1}{4}$ Equation -2 $\frac{4}{x-4} = \frac{1}{4}$ Equation -3 $1.25x = x + 4$
D	Equation in 2 variables $4x = \frac{1}{4}x(y-4)$
E	Diagram
C	Insight: Fractions/ Percents — 1/4 of initial amount is 1/5 of the new amount. 4 jars are 1/5 of all jars, thus there were 20 jars at the start.
F	Insight Solution — 4 jars equals one quarter of the remaining amount of jam → 20 jars in total
G	Insight Solution — Jam from each of the 4 jars was distributed among 4 jars – overall all the jam from 4 jars went into 16 jars. Thus there are 20 jars in total

The scoring scheme

The evaluation model was first introduced by Leikin (2009) and then employed in the research of Levav-Waynberg and Leikin (2012) and Guberman and Leikin (2012).

Table 2: Scoring scheme for evaluation of creativity (based on Leikin 2009)

	Fluency	Flexibility	Originality	Creativity
Scores per solution	1	$Flx_1 = 10$ - for the first solution $Flx_i = 10$ - solutions from a different group of strategies $Flx_i = 1$ - similar strategy but a different representation $Flx_i = 0.1$ - the same strategy, the same representation	$Or_i = 10$ $P < 15\%$ or for insight/ unconventional solution $Or_i = 1$ $15\% \le P < 40\%$ or for model-based/ partly unconventional solution $Or_i = 0.1$ $P \ge 40\%$ or for algorithm-based/ conventional solution	$Cr_i = Flx_i \times Or_i$
Total score	$Flu=n$	$Flx = \sum_{i=1}^{n} Flx_i$	$Or = \sum_{i=1}^{n} Or_i$	$Cr = \sum_{i=1}^{n} Flx_i \times Or_i$

n is the total number of appropriate solutions

$P = (m_j / n) \cdot 100\%$ where m_j *is the number of students who used strategy* j

Fluency (Flu) refers to the pace at which solving proceeds and the switches taking place between different solutions.

To evaluate *flexibility* (Flx), we established groups of solutions for the MSTs. Flexibility embedded in a problem is evaluated according to expert solution space. Two solutions belong to separate groups if they employ solution strategies based on different representations, properties or branches of mathematics.

Originality is evaluated by comparing individual solution spaces with the collective solution space of the reference group.

In the decimal basis we used in scoring, the total score indicates the originality and flexibility of the solutions in the individual solution space of a participant. For example, if the total flexibility score for a solution space is 21.3, we know that it includes 2 solutions that belong to different solution groups (based on different solution strategies), 1 solution that uses a solution strategy similar to a former solution but differs in some essential characteristics, and 3 solutions that repeat previous ones.

The *creativity* (Cr) of a particular solution is the product of the solution's originality and flexibility: $Cr_i = Flx_i \times Or_i$. The use of the product of flexibility and originality scores enables evaluation of the most creative solutions, with the highest score ($Cr_k = 100$) given for a flexible and original solution. This also addresses the fact that previously performed solutions cannot be considered to be creative. The total creativity score on an MST is the sum of the creativity scores for each solution in the individual solution space of a problem: $Cr = \sum_{i=1}^{n} Flx_i \times Or_i$.

The model for evaluation of creativity applied with a particular set of MSTs constitutes the research instrument in this study.

THE STUDY

Research goals

There are two main interrelated goals in this study:

(1) To examine relationships between mathematical creativity, general giftedness, and mathematical excellence.

(2) To explore the power of different types of MSTs for the identification of between-group differences related to mathematical creativity as reflected in multiple solutions produced by the students.

The test

The problems included in the test differed with respect to:

(1) *Mathematics topic* to which the problem belongs in the school curriculum

(2) *Complexity*

(3) *Conventionality* of the problem and conventionality of the solutions, requiring insight in order to produce the solutions (following Ervynk 1991).

The test consisted of five problems (Leikin & Lev, accepted). We focus here on 2 of these problems (see Table 3) to discuss task dependency of the effects of G and EM factors on mathematical creativity (see the Population section below).

Correctness of the solution for a problem was evaluated according to the complete solution produced by the student to the problem. For a complete solution a student received 25 points. Creativity components were evaluated according to the scoring scheme (Table 2).

Table 3: Two problems on the test

Topic	Problem
Word problems 	**Jam problem:** Mali produces strawberry jam for several food shops. She uses big jars to deliver the jam to the shops. One time she distributed 80 litters of jam equally among the jars. She decided to save 4 jars and to distribute jam from these jars equally among the other jars. She realized that she had added exactly 1/4 of the previous amount to each of the jars. How many jars did she prepare at the start?
System of equations	$\begin{cases} 3x+4y=14 \\ 4x+3y=14 \end{cases}$

Population

A sample of 191 students was chosen out of a population of 1200 10^{th} -11^{th} grade students (16-17 years old). The sampling procedure was aimed at investigating the effect of G and EM factors (see Table 4).

G factor: Students for G groups were mainly chosen from classes of gifted students (IQ>130). Additionally, the entire research population was examined using Raven's Advanced Progressive Matrix Test (RPMT) (Raven, Raven & Court, 2000) (see Table 4 for the sampling criteria).

EM factor: All 1200 students studied mathematics at high and regular levels (HL, RL). The level of instruction is determined by students' mathematical achievements in earlier grades. Instruction at HL differs from that at RL in terms of the depth of the material learned and the complexity of the mathematical problem-solving involved. Additionally, excellence in mathematics is examined using the SAT-M (Scholastic Assessment Test in Mathematics, adopted from Koichu, 2003). (See Table 4 for the sampling criteria).

After completion of this stage, 191 of the initial 1200 students were subdivided into four experimental groups, determining the research population according to varying combinations of the EM and G factors, as presented in Table 4.

The fifth group of students – S-MG (super gifted in mathematics) students included G-EM students who were members of mathematical Olympiad teams or who study university mathematics while attending high school. These students received recommendations from research mathematicians familiar with their achievements.

Table 4: Target population

	Gifted (G) IQ>130 Raven > 27/30	**Non-Gifted (NG)** Raven < 26/30	**Super Gifted (S-MG)**	**Total**
Excelling in math (EM) SAT-M >26 or HL in mathematics with math score > 92	G-EM N=38	NG-EM N=51	S-MG N=7	96
Non-excelling in math (NEM) SAT-M <22 and RL in mathematics with math score > 90 or HL in mathematics with math score < 80.	G-NEM N= 38	NG-NEM N=57		95
Total	76	108	7	191

Data analysis

A multivariate analysis of variance tests (MANOVAs) were used to compare the scores on each component of creativity that participants received in each problem.

Between-subjects differences were examined for each one of the problems and each one of the creativity components for G factor, EM factor and interactions between G and EM factors.

Within-subjects differences were examined for their performance on the different tasks.

FINDINGS

As mentioned earlier, we present herein our findings related to the two problems. These two problems are regular curriculum-related problems which have unconventional (insight-based) solutions.

Table 5 presents the percentage of students with different levels of fluency (the number of overall solutions produced by a student) and flexibility (the number of different solutions produced by a student). We learn from this data that students in all the groups were more successful, fluent and flexible in solving the system of equations than in solving the word problem. This may be due to the fact that in contrast to the system of equations, word problems similar to the Jam problem are seldom used for mathematics lessons in 10^{th}-11^{th} grades. From Table 5 we learn that, although there is a connection between fluency and flexibility in students' problem solving performance, these two parameters measure different mental abilities. Production of multiple solutions does not mean production of different multiple solutions. Clearly students from the G-EM group (including S-MG students) differed significantly in their flexibility when solving both problems. Statistical analysis – comparisons of column means – (which we do not present here due to space limitations) supports this observation: participants from the G-EM group differ from participants of all other groups in the fluency and flexibility of their problem solving performance.

Table 5: Fluency and Flexibility

%		Jam problem						System of equations						
No. of solutions (Flu) / No. of groups of solutions (Flx)		0	1	2	3	4	6	0	1	2	3	4	5	8
G-EM (N=38)	Flu	5.3	15.8	23.7	55.2			0	0	7.9	71.1	21		
	Flx	5.3	29	52.5	13.2			0	5.3	36.8	50	7.9		
G-NEM (N=38)	Flu	28.9	31.6	13.2	23.7	2.6		5.3	0	5.3	68.4	15.8	2.6	2.6
	Flx	28.9	44.7	26.4				5.3	5.3	47.3	39.5			
NG-EM (N=51)	Flu	27.5	23.5	21.6	27.4			2	2	9.8	78.4	7.8		
	Flx	27.5	47	23.5	2			2	9.9	47	47.1			
NG-NEM (N=57)	Flu	50.9	24.6	17.5	7			1.8	14	21	54.4	7	1.8	
	Flx	50.9	33.3	15.3				1.8	28	43.9	26.3			
S-MG (N=7)	Flu			14.3	85.7						42.9	57.1		
	Flx		42.8	14.3	42.9					28.6	71.4			

Table 6, which presents Means and SD that we obtained for all the examined criteria on both problems, provides additional support for the observation of the specific qualities of mathematical reasoning in G-EM students.

MANOVAs demonstrate effects of EM and G factors on all the examined criteria (Table 7). Significant main effects of G and EM factors were found for Cor, Flu and Flx criteria on Jam problem, while only the G factor had a main effect on Or and Cr criteria. In the system of equations, significant main effects of G and EM factors were found for Flx, Or and Cr. Only the G factor had a main effect on Flx criteria. For the system of equations we found an interaction between EM and G factors with respect to students' flexibility related to solving the system of equations: G factor strengthens the effect of EM factor; that is, excelling in mathematics students who are gifted in mathematics are significantly more flexible than their non-gifted counterparts, whereas no significant differences appear in the flexibility of EM and NEM students among NG students.

Table 6: Means and SD

		G-EM		G-NEM		NG-EM		NG-NEM	
	N	38		38		51		57	
		Mean	SD	Mean	SD	Mean	SD	Mean	SD
	Cor	23.68	5.66	16.58	11.46	18.14	11.27	11.23	12
	Flu	2.29	.93	1.39	1.22	1.49	1.17	0.81	.97
	Flx	12.92	5.46	7.65	5.2	8.11	5.54	5.09	5.23
	Or	2.86	4.43	1.44	3.41	0.93	2.71	0.6	2.24
	Cr	27.58	44.52	9.1	27.17	8.58	27.1	4.16	18.5
$\begin{cases}3x+4y=14\\4x+3y=14\end{cases}$	Cor	25	0	23.68	5.66	24.51	3.5	23.42	5.01
	Flu	3.13	.53	3.13	1.21	2.88	.65	2.56	0.95
	Flx	16.39	6.96	12.51	4.76	11.51	2.57	11.01	2.04
	Or	2.42	3.98	1.17	2.71	0.32	.19	0.26	.09
	Cr	22.72	40.14	9.78	27.23	1.5	1.97	1.1	.02

Table 7: Effects of G and EM factors

Between-subjects effects		G-factor	EM-factor	G×EM
		F(1,180)	F(1,180)	F(1,180)
	**Cor**	**11.642****	**19.253*****	.004
	Flu	**18.496*****	**23.939*****	.430
	Flx	**21.105*****	**26.703*****	1.971
	Or	**8.548****	3.364	1.326
	Cr	**7.347****	6.716	2.532
	F(5,176)	**5.115*****	**6.981*****	1.565
	Wilk's Λ	.873	.834	.957
$\begin{cases}3x+4y=14\\4x+3y=14\end{cases}$	**Cor**	.355	3.617	.032
	Flu	**9.897****	1.518	1.518
	Flx	**25.555*****	**12.066****	**7.167****
	Or	**21.049*****	**4.072***	3.284
	Cr	**20.528*****	**4.091***	3.612
	F(5,176)	**8.894*****	**3.091***	**3.146***
	Wilk's Λ	.833	.935	.934

*p<.05 **p<.01 ***p<.001

SUMMARY

This study examines relationships between mathematical creativity, general giftedness, and mathematical excellence. It also explores the power of different types of MSTs for the identification of between-group differences related to mathematical creativity as reflected in multiple solutions produced by the students. Five groups of 10th grade to 11th grade students who varied in their level of general giftedness and in the level of mathematical instruction participated in this study.

The study demonstrates that G students have higher scores on all the examined criteria. Both EM and G factors have a main effect on students' fluency and flexibility associated with MSTs. The G factor has a main effect on originality. G and EM factors interact on the flexibility criterion: the EM factor has a more significant effect on flexibility and originality criteria among G students.

The effects of EM and G factors are task-dependent and are related to the level of insight embedded in the task and the familiarity of the conventional solution. Task dependency of the findings, as well as different effects that were discovered in the study, demonstrate that EM and G factors are interrelated, but different in nature.

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MATHEMATICAL CREATIVE SOLUTION PROCESSES OF CHILDREN WITH DIFFERENT ATTACHMENT PATTERNS

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In research about mathematical creativity seldom the early childhood is taken into count. The paper investigates the mathematically creative solutions of Kindergartners (in Germany children between 3-6 years attend Kindergarten). Starting point is the longitudinal study MaKreKi (mathematical creativity of children) in which theories of mathematics education and psychoanalysis are amalgamated for the investigation of mathematical creativity. In this paper two episodes of two children between 4 and 5 years old are presented, while they cope with a mathematical task. It focuses on the mathematical creative ideas emerging in the interaction between the involved children and the accompanying person and how the children deal with the mathematical instructions.

Keywords: mathematical creativity, attachment theory, early childhood mathematics

INTRODUCTION

Attempts to define mathematical creativity seem to lead to more than 100 contemporary definitions of creativity (Mann 2006), but mathematical creativity in early years is rarely examined. Thus the central research question is, how does mathematical creativity express itself at the age of preschool and how is it observable?

Following Urban (2003) a theory of creativity has to consider the “4P-E Structure” of creative thinking and acting, which embodies the interactive structure of the factors: problem, person, process, product and environment. The existence, the range and recognisability of possible problems to be solved creatively are determined by meta- environmental factors like evolutionary and social-historical developments, macro-environment like economic, material, cultural and political conditions and micro-environmental factors such as socioeconomic conditions of the family (Urban 2003). From a socio-constructivist point of view, the individual ability of mathematical creativity develops in the course of many interactions with other members of the culture. The paper focuses on the mathematical process while children are working on mathematical tasks in mathematical situations of play and exploration. Therefore it highlights on the negotiations of meanings (Brandt & Krummheuer 2001) between the children and the accompanying person during the interactive process while coping with mathematical task and in a specific cultural system. Beside this situational, micro-sociological access the paper also refers to psychoanalytically- based attachment theory, in which a function of the culture is understood as an aspect of the relationship between mother and child. Bowlby’s theory of attachment suggests that children come into world biologically pre-programmed to form attachments with others, because this helps them to survive (1969). The neonate develops special relationships with her parents. In the first years of life the child develops an ‘inner working model’ through child-parents-interactions (Bowlby 1969). This ‘inner working model’ contains the early individual bonding experiences as well as the expectations, which a child has towards human relationships, derived from these experiences. They conduce to interpret the behavior of the caregiver and to predict his or her behavior in certain situations. So the attachment between mother and child has a great impact on the social- emotional and cognitive development of the child. After the first year of life this ‘inner working model’ becomes more and more stable and turns into a so called attachment pattern.

Children are confronted with mathematical tasks and contents from different domains of mathematics as they appear in their everyday life. In the MaKreKi project these contents are presented in form of mathematical situations of play and exploration (Vogel 2013) for the children regarding their assumed mathematical competencies. Within in these situations a competent adult guides the children. He has some knowledge about the mathematical contents, which are intended and a minimal set of instructions like questions or allegations. The competent adult also has some hints of possible reactions and expressions of the children, so that he is somehow prepared for possible mathematical tasks and solutions emerging in the context of the mathematical situation of paly and exploration. All these information for the accompanying person are documented in ‘didactical design patterns’ (ibd.). On the situational level the presentation of these mathematical situations of play and exploration initiate processes of negotiations of meanings, which necessarily do not be in in accord with the described mathematical domain nor the activities that are expected in the design pattern.

Following Urbans idea of the “4P-E Structure” this article examines macro-environmental factors like cultural conditions e.g. the intended mathematical domains/contents and the expected mathematical tasks and solutions presented in the mathematical situations of play and exploration as well as the interactive negotiations of these themes (micro-social factors). Beside this the paper also involves psychodynamic aspects of early childhood development (micro-environmental factors). In the research design of the MaKreKi project children are dealing with mathematical problems and tasks guided by a competent adult. This adult can be seen as a representative for the parents, because it might be reasonable to assume that children shows similar behaviour in this situation like they would with their parents, because of their stable ‘inner working model’ and their attachment pattern. The attachment system is now widely studied in the life cycle, but little is said about creativity as a concomitant of this system (Brink 2000). Therefore the second research question is, which correlation exists between the attachment pattern of children and their mathematical creativity?

THEORETICAL PERSPECTIVES

The following section introduces theoretical perspectives of mathematical creativity in early childhood and offers psychoanalytical considerations about attachment theory and the connection to creativity.

Mathematical Creativity

Mathematicians and researchers in mathematics education as well as psychologists have examined mathematical creativity under their various scientific viewpoints (Hadarmard 1954, Sriraman 2004). A clarification of concepts of creativity is difficult and additionally complicated by its relationship to the concepts of intelligence, giftedness and problem solving. According to Urban and Sriraman, who offer a definition of mathematical creativity, which keep in mind the "4P-E Structure" of creative thinking, creativity is understood as the ability of creating a surprising, meaningful product, which extends the body of knowledge and as the process that results in unusual/novel/insightful solutions to a given problem (Urban 2004; Sriraman 2011). Sriraman (2011) notices the difference between creativity at the professional and school levels and suggests that the ability to produce original work can be regarded as mathematical creativity at professional level whereas the process that results in unusual solutions can be seen as mathematical creativity at school level. For mathematical creativity at preschool level exists no definition.

Because of the relative lack of current research in mathematical creativity in early childhood, this article deals with the following four aspects of mathematical creativity (Sriraman 2004) regarding the presented definition of mathematical creativity at school level:

• *Choice*: Poincaré (1948) described as a fundamental aspect of mathematical creativity the ability to choose from the huge number of possible combinations of mathematical propositions a minimal collection that leads to the proof. With regard to the age group of interest under this *choice* aspect of mathematical creativity the production of (unusual) relations between mathematical examination and experiences and the playful contact with mathematical methods is understood.

• *Non-algorithmic decision-making*: According to Ervynck (1991), mathematical creativity articulates itself not when routine and/or standard procedures are applied but when a unique and new way of solving a problem emerges. With regard to the age group one is able to shift the accentuation and speak of the "divergence from the canonical" (Bruner 1990, p.19).

• *Adaptiveness*: Sternberg & Lubart (2000) characterize creativity, as the ability to present an unexpected and original result that is also adaptive.

• *De-emphasizing detail*s: In his study, in which he investigates the ideas and thoughts to mathematical creativity from famous mathematicians, Liljedahl (2008) has discovered that detail does not play any role during the incubation phase of creativity. Many of the participants mentioned how difficult it is to learn mathematics by attending to the details, and how much easier it is if the details are de-emphasized.

Attachment theory

Attachment theory originates from Bowlby (1969) and postulates the central role of attachment behaviour for individual development. Bowlby perceives the attachment system as the central source of motivation. In his approach the antagonism between attachment and exploration has a highly relevant explanatory power. Both systems cannot be simultaneously activated. If a child feels secure, it can activate his exploration system and explore his surroundings. If it perceives a danger, the attachment system is activated. The child interrupts its exploratory behaviour and seeks safety by its parent. Four attachment pattern are described (Ainsworth et al. 1978): *Insecure-avoidant*: The 'insecure-avoidant' child (A) experiences that its mother feels best when it shows no intense affects itself and behaves towards her in a controlled, distanced manner with a minimum of affect. *Secure*: The securely attached child (B) has, thanks to its sensitive mother, a chance to build up a secure relationship to her in which the whole spectrum of human feelings in the sense of communication with another, that can be perceived, experienced and expressed. *Insecure-ambivalent*: The ambivalently attached child (C) has spent its first year with a mother, who sometimes reacts appropriately, and is at other times rejecting and overprotective, i.e. on the whole, inconsistent and for this reason she reacts in a way that is unpredictable for the child. *Insecure-disorganized*: The disorganized/ disoriented attached child (D) could not build up a stable inner working model, as its mother (or father) suffered under the consequences of an acute trauma (for example, the dramatic loss of an important person). They were psychically so absorbed by this loss that they could hardly take up a coherent relationship with their infant.

Relating this approach to the topic of mathematical creativity of young children, the results of empirical attachment research point to the fact that the shaping of domain-specific (mathematical) creativity can not only be localized in the potentially stimulating mathematical contents in the child's milieu but also in the type of attachment of the child to its parents. Grossmann describes the link between the attachement pattern of the child and the 'successful cooperation' (in german: gelingende Gemeinsamkeit) in a child-parent play situation more detailed (Grossmann 1984). The 'successful cooperation' of this play situations correlates with the delicacy feeling of the mother and a more delicacy feeling leads very often to a secure attachment pattern of the child (ibd.). Mothers of secure attached children seem to be more reserved, gentle and they show more efforts in handing over the lead to their children in play situations instead of mothers of insecure attached children. Mothers of insecure attached children are often strict and controlling and they have more instructional ratio in play situations than mothers of secure attached children (ibd.). Significant differences between children with a secure attachment pattern and children with an insecure attachment pattern in play situations are also described in the study of Grossmann. Secure attached children are more often initiators of the common play and they seem to be rather extroverted instead of insecure attached children, who wait for instructions.

METHODOLOGY

Regarding the theoretical considerations and the attempt to identify mathematically creative moments in mathematical interactions of preschool children, in the following there is conducted an analysis of interaction, which based on interactional theory of learning (Brandt & Krummheuer 2001). It focuses on the reconstruction of meaning and the structure of interactions. Therefore it is proper to describe and analyze topics with regards to contents and the negotiation of meaning in the course of interactional processes. The negotiation of meaning takes place in interactions between the involved people. These processes will be analyzed by an ethno methodological based analysis, in which is stated that the partners co-constitute the rationality of their action in the interaction in an everyday situation, while the partners try constantly to indicate the rationality of their actions and to produce a relevant consensus together. This is necessary for the origin of own conviction as well as for the production of conviction with the other participating persons. This aspect of interaction is described with the term ‘accounting practice’ (Lehmann 1988, p. 169). To analyze these ‘accounting practice’ of children in mathematical situations, the reconstruction and anlaysis of argumentation of Toulmin (1969) have proved to be successful. Four central categories of an argumentation are "data", “conclusion", “warrant" and "backing". The general idea of an argumentation consists of tracing the statement to be proven back to undoubted statements (data).

For the diagnosis of the attachment pattern the MaKreKi project apply the Manchester Child Attachment Story Task, so-called MCAST (Green et al. 2000). This is a story telling test that has good reliability and validity.

FIRST INSIGHTS

The following section presents extracts of interpretations of two children Nina and René of the MaKreKi project while they are participating in the mathematical situation of play and exploration called ‘Ladybug’. Both children are examined paired with one of her/his closer friends and at least one adult person, who acts as nursery teacher.

The ‘Ladybug-situation’

In this situation the children can differentiate between similar objects, which differ according to their size and color. The objects are pictures of ladybugs, which differ in size (small and large), in color (red, green, yellow), and in the spots on ladybugs in three ways (shape, amount of spots, size). The design pattern suggests the following mathematical activities to the children through material and designated instructions and impulses: Counting and determination of quantity; Arrangement and comparing of sets e.g. in respect of the number of elements on the back of the ladybugs; Identifying mathematical structures. The 'Ladybug-situation' consists of two parts. In the first part the children are dealing with little ladybug cards. Typical instructions of the accompanying person are: 'Look what I brought.'; ‚Put together all ladybugs which belong together'; ‚Can you find further groups or families of ladybugs?'; ‚Why do these ladybugs belong together?' In the second part the children are dealing with big ladybug-cards, which have small and large spots on their backs. Usually the accompanying person offers a triplet of big ladybug-cards and asks: Which one does not belong?

Case study one: René's solution process in the 'Ladybug-situation'

René is a four years and 9 months old boy who lives with his parents and his older sister in a small city. His father works fulltime in a computer firm and his mother remains at home. He shows insecure-avoidant attachment pattern (A). The attachment pattern is measured by the MCAST (Green 2000).

Beside René there are two persons involved: Lisa, a four years old girl from René's preschool and a member of our research team, who conducted the conversation with the two children. The following interpreted episode refers to the end phase of a collective processing of the task. René, Lisa and the member of the research team invented a familial system of description: The small ladybugs represent kid-bugs and the big one mom-bugs, dad-bugs, or parents-bugs. All little ladybugs lie around the carpet.

The guiding adult has put a triplet of 3 red ladybug cards in the centre of the carpet:

Figure 1: Triplet of ladybug-cards

René has mentioned that the ladybug with 19 little triangles and Marie has mentioned that the bug with seven big triangles does not belong to the group:

René comes up with the solution that *both* bugs with many and small triangles do not belong. His justification has two aspects:

- Comparing the figures of the small and the big ladybug-cards, he concludes, that the bugs of the small cards should also only possess small figures on their tops.
- The two cards with the many and small triangles cannot exist in the system of the cards at all.

If one understands the figures of the ladybugs to be people's hands, René's argument is that parents do not have hands of the size of kids, this is impossible. They cannot be parents and children "at the same time", as he says. With respect to the three aspects of mathematical creativity mentioned one can conclude: René's solution is based on a surprising choice of a familial system of description for the comparison of the ladybugs. Hereby he does create a somehow non canonical combination of size and family-members. Furthermore on the level of speech, he expresses this unusual choice by a linguistic adaption of the size of ladybugs by using a familial metaphor. He says that the big ladybugs would be "already big". The wording of "big" can appear in the size-system of description and in a familial system of description. The guiding adult seems to have difficulty in comprehending René's approach. Possibly she shares Marie's solution. So she asks René for an explanation two times ('why' and 'what do you mean by saying this'). With respect to the interactional setting it is René, who takes the part as the competent partner and explains his position to his counterpart. In this situation he presents a very deep argumentation as the toulmin scheme shown:

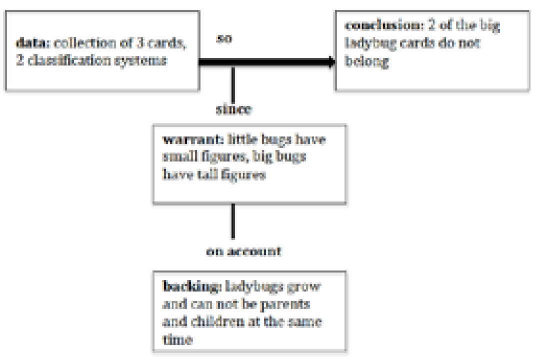

Figure 2: Toulmin scheme of René's argumentation

In René's argumentation one can see that he connects the first part (finding family members, making groups of little ladybugs because of their relationships regarding their spots (amount, shape) or their colors) with the second part (separate big ladybugs, which do not belong together) of the mathematical situation. So he transforms two operations into one under disregarding the detail that only one ladybug does not belong to the triplet. In the end the guiding adult forces an agreement and asks if it is all right for René to take Marie's solution.

Case study two: Nina's solution process in the 'Ladybug-situation'

Nina is a five years and five month old girl who lives with her mother in a German major city. Her parents are divorced. Nina shows secure attachment pattern (B).

Beside Nina there are two other persons involved: Samira a five years old girl from Nina's preschool and an accompanying person from the research project. At first the children and the adult person have dealt with the little ladybugs. They have discovered various families of ladybugs where the color and the number of spots determine to which family a ladybug posse. At the end of this phase Nina mentioned that all ladybugs of the same family are grown in the same stomach. After that the little ladybug cards are moved to the edge of the table. The presented scene begins with the second part of the ladybug situation: The guiding adult has put a triplet of big yellow ladybug cards on the table and asked: Which one does not belong?

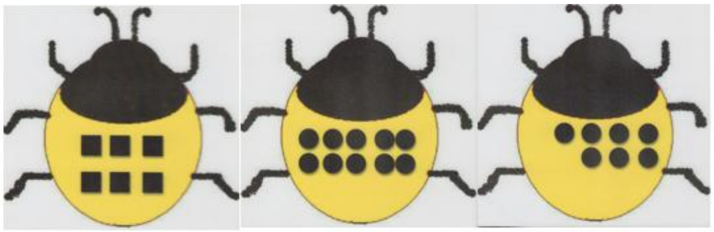

Figure 3: Triplet of ladybug-cards

Nina comes up with the solution that two big ladybug cards are wrong and only one ladybug card is right. The right one has ten circles on his back, which correspond to the number of little yellow ladybugs. She creates a non-canonical solution. Her surprising choice includes aspects of relationships similar to a mathematical function: The number of elements on the back of the ladybug and the quantity of babies, which appertain to the bug. Each spot represent one of his babies. She extends her functional relationship by determining the color as a feature of the functional relationship between the ladybugs: The big yellow bug can only have little yellow ladybug kids. By reconstructing Nina's argumentation it is obvious that she connects the first part of the mathematical situation (finding families/groups of ladybugs) with the second part (which one does not belong?):

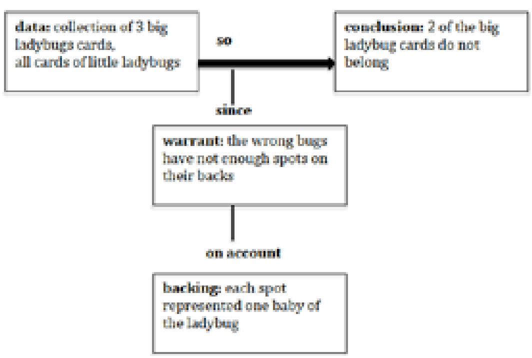

Figure 4: Toulmin scheme of Nina's argumentation

Nina disregards the detail, that only one ladybug does not belong to the triplet. On the level of speech Nina is able to formulate her non-canonical solution with appropriate expressions, so the identification of the little yellow ladybugs as babies of the big yellow ladybug can be seen as a linguistic achievement to describe the functional relationship between the little and the big ladybugs in an adaptive way. Samira shares Nina's solution, but the guiding adult seems to be surprised. Similar to René, Nina has to give an explanation of her one card solution, what is expected by the guiding adult.

Summary and Prospect

Two cases of the MaKreKi study were discussed and it has been shown that the approaches mentioned in the theoretical perspectives are useful to describe creative mathematical processes of young children at preschool age. The cases illustrate that children who can be seen as mathematical creative are able to change the perspective on a mathematical task, although a clear instruction from the guiding adult focuses another perspective. Nina as well as René offered an one card solution instead of the expected two card solution, because of the connection they have discovered between the two parts of the ladybug situation. As seen in the two episodes the final definition of the problem situation is a matter of the negotiation of meaning in the concrete situation of interaction. Instructions may have a strong impact on children's interpretation of mathematical tasks and so only mathematical creative children are able and have the confidence to see more possibilities and perspectives than the canonical solution, which is forced by the comments of the guiding adult. Both children are initiators (Grossmann 1984) of their non-canonical solutions. In case of Nina as a child with a secure attachment pattern this observation is in accord with Grossmann's results. Following Grossmann children with an insecure-avoidant attachment pattern like René are often less autonomous in play situation. In the mathematical situations of play and exploration this is not the case for René. He often is the initiator, too. So the behaviour of children in play situations may be linked to the context of the situation as well as to their attachment pattern.

Regarding the construction of children's mathematical thinking it is important to understand and honour their non-canonical solutions, which might be the first steps of the development of mathematical creativity in early childhood. Additional analysis of young children's non-canonical solutions can help to describe, understand and identify the mathematical potential of young children. Till now the connection between mathematical creativity and the attachment pattern in early childhood is not satisfactorily investigated. Therefore a conceptual framework has to develop which examines the cultural and the situational impact as well as the influence of the attachment pattern on the development of mathematical creativity in early childhood and connects with creative mathematical abilities of young children.

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TWICE EXEPTIONAL CHILDREN - MATHEMATICALLY GIFTED CHILDREN IN PRIMARY SCHOOLS WITH SPECIAL NEEDS

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Inclusion in Hamburg leads to joint school education of disabled and non-disabled children. The goal is full development of the potentials of all children. Gifted and twice-exceptional children in particular are at risk that their potential is not recognized. Based on four case studies in the article, the problem of non-recognition is demonstrated by both, special talents and developmental impairments.

INTRODUCTION

Children differ in many ways. They have different interests, show a different attitude to work and come from different familiar backgrounds which is why they speak different native languages and have different genetic potentials. Compensatory education, heterogeneity, diversity and inclusion are keywords which arose in different times of the German pedagogical discussions. All these terms have in common that they were intended to develop the awareness of these differences. In all times these key-terms were, and they are still today, the foundation for constructing concepts I order to give all children an adequate access to education. The "CONVENTION on the RIGHTS of PERSONS with DISABILITIES of the UN (2006)" lays a legal basis for people with functional limitations and handicaps. The goals of this convention, which in the meantime is signed by many states, are the following:

> "(a) The full development of human potential and sense of dignity and self-worth, and the strengthening of respect for human rights, fundamental freedoms and human diversity;
>
> (b) The development by persons with disabilities of their personality, talents and creativity, as well as their mental and physical abilities, to their fullest potential." (Bundestag 21. Dezember 2008, p. 18)

At the moment, in Germany approaches for the implementation of the idea of inclusion are discussed. The realization of inclusion in Hamburg for example means that all children with or without disabilities[1] are taught together. Based on special conditions, this is already implemented in various schools under the heading "integration". This concept demands a lot of effort and care. Many teachers do not know whether they have sufficient practical knowledge for being able to teach children with such diverse conditions appropriately.

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[1] Excluded are blind and deaf children.

Particular attention is paid to children with disabilities. But we should not neglect gifted children with special needs. In particular, a high intelligence enables these children often to compensate limitations and peculiarities. However, this is often associated with the fact that a special talent is not easily recognized.

Since school year 1999/2000, at the University of Hamburg we are fostering mathematically talented children of the third and fourth grade within the framework of our project called PriMa[2]. After a talent search process we foster about 50 children of grade three until the end of grade four at the University. This project is regularly attended by children with specific performance deficits, diseases or impaired developmental processes.

How many children are affected?

Although we ask the parents to provide us with additional information in case of such impairments, this information is only given voluntarily. Therefore, until now we do not have accurate data about the whole sample of children participating in our fostering project. This year there are five out of 50 children with special needs, of whom we know that for sure. Furthermore, the results of a questionnaire which was answered by 115 parents of our project show that more than 14% of our children are affected by partial performance disturbances or impairments.

		Frequency	Percentage	Valid percentage	Accumulated percentage
Valid	No	**97**	**84,3**	85,1	85,1
	Yes	**17**	**14,8**	14,9	100,0
	Total	114	99,1	100,0	
Missing	System	1	,9		
Total		115	100,0		

Table 1: Partial functional-performance disturbance

What do these children need?

Method

To learn more about the needs of the children and to acquire more practical knowledge, case studies have been proven to be an appropriate approach. In this article four children are presented. All children were tested as mathematically gifted. For our considerations several results were taken into account:

[2] PriMa is a cooperation project of the *Hamburger Behörde für Schule und Berufsbildung* (u.a. *BbB*), and the *William-Stern* Society (Hamburg), the University of Hamburg. (for further information you are invited to visit the website http://blogs.epb.uni-hamburg.de/nolte/)

a) Individual observations conducted during the fostering lessons.

b) Records of the children written while solving mathematical problems.

c) Discussions with the parents gave us information about the learning biography of their children.

d) The school report of the second class gives insight in the observation of the teachers and their assessment of the children's performance.

e) The results of the children collected during the talent search process. These include half standardized observations during trial lessons, the results of a mathematics test developed by us, the results of intelligence testing and results of testing verbal competences.

For measuring the IQ we used CFT 20 R in combination with a vocabulary test and a number sequence test. As an academic achievement test in mathematics we used DeMat 2+. The stumbling words test is based on finding useless words. The reading test proves the degree of understanding of a short story. For observing the performance in the trial lessons (MTR) we developed special lists which contain cognitive components of problem solving and patterns of action which are important in solving mathematical problems (Nolte 2006; 2012a; 2012b). We constructed the maths test based on complex problems which enabled us also to show cognitive components of problem solving and patterns of action.

Justin, 8 years old

Justin participated in the project during our third year. Because we were still in the process of developing the talent search process, we do not have as many information about him as with the other children. Additionally Justin did not participate in the intelligence test.

Sometime after starting with our fostering program, Justin lost his motivation. He was no longer in the mood to come to the university and did not feel well. The tutors reported that he had difficulties to understand our problems and that he did not start working on his own. Justin did not like to explain his solving process in writing. But this is also the fact with many other children. After talking with his mother we supposed that perhaps his reading capabilities were not as good as it is normal for a gifted child. Therefore we conducted further diagnostics and tested his reading skills. By that we have found out that his reading skills were lower than the average rank.

His school report tells that Justus is a high-performing student in all subjects. His capabilities in the acquisition of writing and reading are just as positive as those shown in mathematics. He is described as an inquisitive, thoughtful and socially well-integrated boy. Some hints about problems are given by the advice to speak clearly, and to write stories more detailed. He is learning Italian in school. Here it is pointed out that it takes practice to learn new words.

Why is Justin unhappy and unmotivated?

Our tasks for mathematically gifted children are presented with little redundancy. The language is demanding due to this fact and because we pay a special attention to mathematical correctness. Although the problems are presented orally and with examples, to work on them requires an understanding of the texts of the worksheets. These kinds of phrasings differ from those normally used in school. The mixture between everyday language and special mathematical language is called "Bildungssprache" (‚academic discourse' or ‚academic language': Gogolin 2009; Gogolin and Lange 2011). Understanding our phrasings is even more difficult due to the complexity of the mathematical content, the low redundancy and the high level of mathematical correctness. Therefore, reading and understanding the text in our problems is significantly harder for Justin than for other children.

In school Justin is able to participate in reading contests, but it is difficult for him to understand our tasks. He gets the impression to be not intelligent enough to understand the mathematical contents. Therefore, he feels unsafe and withdraws. His tutors get the impression that he is no longer interested in solving mathematical problems. His mother notices that he does not feel well. She thinks his performance in reading is not as high as expected. Since his performances in school are in an above average range, his teacher thinks, his mother is too ambitious.

Reading Test	Trial lessons (MTR)	Math-Test
ca. PR 35	One of the best	34th of 135

Table 2: Justin Test Results

The test clearly shows discrepancies between his mathematical capabilities and his reading capabilities. Justin felt this discrepancy between his superior mathematical skills and his less than average performance in reading, but could not cope with them. We talked with him about his weakness and developed with him mechanisms to compensate them. The tutors were informed and the mother took care for additional support concerning his weaknesses in reading.

Justin is a child with a dissociated performance profile, which results from a partial impairment of performance.

Justin needs:

- Understanding of others for his behavior
- Understanding his situation and developing methods to compensate his weakness
- Strategic support in the current situation to compensate his weakness
- Long-term professional support in the field of reading acquisition

Lars, 9 years old

Lars is a shy and cautious boy. At the beginning of our lessons Lars was motivated, but usually more and more he lost his interest. First interaction and communication with others was difficult, perhaps also due to the fact, that his workplace and his way of problem solving were chaotic. Before two meetings he changed his place and went to another group of children. First the children got into contact and showed only a weak interest in working. Obviously Lars accepted and enjoyed the situation. During the last session he worked excellent. He seemed very motivated and very attentive even in plenary discussion.

His school report tells that Lars is a high-performing student in all subjects. Also his capabilities in the acquisition of writing and reading are nearly as positive as those shown in mathematics. Only his spelling is described as good, not as very good. Hints to show more efforts are given for keeping his working place in order. He is described as well integrated and supporting, but not always controlling anger.

IQ	Vocabu-lary	Number sequences	Academic achieve-ment test	Stumbling words test	Reading test	Trial lessons (MTR)	Maths-Test
151	118	136	PR 97	PR 58	41 of 50 points	Place 95 of 370	Place 76 of 226

Table 3: Lars' Test results

Also with Lars discrepancies between his high intelligence and his verbal capabilities can be observed. Comparing the IQ with his verbal capabilities there was a difference of about more than two standard deviations. His performance in a mathematical school test was very high (PR 97). Nevertheless the observer during the trial lessons did not propose him for participation in the project.

His writing is not appropriate for a third grade child, especially of his intelligence.

Table 4: Lars' writing

How can these discrepancies between his school reports and his performance in our tests be explained?

Lars goes to school with a very high proportion of children with a migration background. In these situation his weak writing and reading skills are not that much noticeable because many children have very poor performance in the acquisition of reading and writing.

But Lars was unhappy and shy, and we got the impression that he needs further support. The parents told us, that Lars shows weaknesses in the auditory perception. This partial impairment of performance has an important impact on his capabilities of understanding verbal / auditory information. It also influences the development of skills in reading and writing. That might be the reason for misunderstandings in communication processes and furthermore means more effort and stress in participating in school (Nolte 2000; 2004). We confirmed the parents in our opinion, that their son has very high potentials but needs additional support concerning his impairments. Obviously Lars can take care for himself and we hope that he will continue working with motivation.

Lars *needs*

- In the current situation, people who insure that communication and interactions shall be clearly
- Support to develop metacognitive competences for organizing his work
- Long term intervention for the development of his auditory perception and his written language skills.

Karen, 9 years old

Karen is a quiet and reserved student. She works very carefully and is keen to make any mistakes. After getting used to the situation at the university she is actively involved in the plenary discussions. She writes only sophisticated solutions, which show excellent considerations. She seems not to feel being under pressure. Her meticulous work can be regarded as the reason for working sometimes very slowly. Also Karen's reading capabilities did not match with her mathematical performance. In the stumbling words test she reached a percentile rank of 58. Due to her slowness she could not finish the whole test. The results of other tests did not match with her remarkable performance during the trial lessons and the maths test. Here results show a discrepancy of nearly two standard deviations between the total IQ and the result of the vocabulary test. The good results in the reading tests can be explained by the possibility to get the necessary information from the context.

IQ	Vocabu-lary	Number sequences	Trial lessons (MTR)	Math-Test	Academic achieve-ment test; Mathe-matics	Stumbling words test	Reading test
120	91	112	Place 12 of 370	Place 36 of 226	PR 97	PR 58	46 of 50 points

Table 5: Karens' test results

Analysing her worksheets, it is striking that she works very carefully. She expresses herself well and seems to have a remarkable memory.

With Karen we see a discrepancy between the test results concerning IQ, verbal capabilities, and the mathematical achievements in the teaching process. A spelling mistake gave us the crucial evidence. This was an error typical for children with impairment of phonological processing. In the meetings with parents, we learned that Karen needed several surgeries between one and seven years old because of problems with her ears. This affects the auditory development and thus about phonological consciousness. Her very good results in our fostering program show her ability to compensate her weaknesses by very carefully and reflected approaches. But this is the reason for very slow work which does not allow time limitations. So we suppose that her test results do not show her real capabilities.

Karen's needs

- more time than other children
- no more pressure

Dissociated performance profile

All three children can be regarded as an example for children with specific performance deficits. Their test profile show significant variations. In all cases, these discrepancies are not identified in school. All children need additional support to develop their potential.

Leon

At the beginning of our fostering program the parents informed us that Leon is autistic.

> "In addition, gifted individuals and those with autism are also similar in the way they sometimes have a compulsive preoccupation with words, ideas, numbers and foods; perfectionist personalities; a rigid fascination with an interest; a need for precision; intellectual rigidity; a lack of social skills; the need to monopolize conversations and situations; the ability to concretely visualize models and systems; an intense need for

stimulation; difficulties in conforming to the thinking of others; and a tendency toward introversion" (Cash 1999, p. 23).

IQ	Vocabu-lary	Number sequences	Trial lessons (MTR)	Math-Test	Academic achievement test in Mathe-matics	Stumbling words test	Reading test
151	118	136	Place 9 of 370	Place 25 von 226	PR 97	PR 58	42 of 50 points

Table 7: Leons' Test results

Leon feels completely comfortable working with us at the university. He is fascinated by doing mathematics. His achievements are impressive. Most of his considerations he does on mind using an almost unlimited number space. He prefers to work alone and can be distracted by anything when it comes to mathematics. Talking to others he explains very precisely his ideas at a very high level of language. But due to his need for precision the participation in discussions with others is difficult. Also plenary phases are a challenge for him, especially if other contributions are not as exact and completely as his considerations. It's hard for him to endure that others are not as fast as him and also not as capable as him in understanding mathematical content deeply.

For the tutors, working with him it is a pleasure and a challenge at the same time. He takes everything what is said literally.[3] Therefore, the tutors check their communication with him constantly. Furthermore, his brilliance in mathematics requires emotional support for other children. His rigidity in conversations is a challenge to cope with and they are confronted with the fact that their contributions often are at a lower level of mathematical insight. Due to this experience some of the other children got the impression of being less able as they are.

Dissociated development

With Leon the cognitive skills and the social-emotional skills differ strikingly. He gets therapeutic support and continuously develops his skills to behave in a group.

Discussion

At the university we are used to children who are exceptional in different ways. We are well aware of the requirements arising from the fact that some children do experience the first time, not to be the best. Rather, they must learn to work in a

[3] One day, when his mother picked him up after the lesson she said: "Today we have to rush home!" That for him was the signal quickly to go home. He took the elevator, he found the subway station, the correct line, get off to the right place, the right bus. We got a call that he was at home at the very moment when the police wanted to start searching for him.

group with similar interested and talented children. It is very important to us to support the children in developing their social and emotional skills as well as their mathematical competencies.

The cases lead to the question whether high abilities allow hiding developmental disorders and impairments during regular classroom lessons, and also the other way round whether developmental disorders and impairments can hide high cognitive abilities. Apart from Leon where a clear diagnosis was present, our observations were not always consistent with school reports.

We assume that the children come voluntarily to us because they are interested in mathematics. So it is an important signal for us, when they lose motivation. Also discrepancies in test results can be faced as hints for problems concerning development. Both Justin and Lars showed by their awareness that something is wrong by behaving not appropriate. This was different with Karen. She behaves very well, works very hard, but she stresses herself. Perhaps there are different ways to compensate such irritations with boys and girls. Because we have experience in the context of giftedness and developmental disorders, we are cautious in judging behaviour and performance.

> “To reach their full potential, twice-exceptional students need a balanced educational program that nurtures their gifts and talents while providing intervention for their disabilities.” (Schultz 2012, p. 120) and later “Many school programs seem designed to either remediate weaknesses or develop gifts and talents but are unable to address both simultaneously (Schultz 2012, p. 120).

Intervention as well as the development of challenging learning environments is based on the information of teachers about normal and abnormal development. Interpretations of behaviour or performance are as broad and as good as the knowledge of the observer. For working with children with developmental disabilities the knowledge of the nature of the problem is essential. Knowledge about developmental disorders as possible causes of unclear or even contradictory observations in the classroom is a prerequisite for the development of appropriate methods. Here we need the balance between searching for impairments and overlooking them. Not every problematic child is ill or disturbed. The first step in analysing a children’s problem should begin with looking for rational reason for the shown behaviour. A lower motivation to work on mathematical problems can also be caused by the situation in the classroom or that a child would rather play football. With the next step, however, questions should be asked concerning the learning preconditions of the child in connection with a particular task. After this the responsibility of the teachers lies in the construction of appropriate methods to match as well as possible the strengths and the weaknesses of the child. To meet the high demand of working with impaired and / or twice exceptional children additional support for teacher is necessary.

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TEACHING HIGHLY ABLE STUDENTS IN A COMMON CLASS: CHALLENGES AND LIMITS OF A CASE-STUDY

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The case-study presented in this paper is built around a middle school teacher's beliefs. The paper focuses on identifying links between self-efficacy beliefs, content knowledge, classroom practices and mathematical beliefs of a professional who teaches in a mixed-ability class. The data come from a semi-structured interview, group discussions, and individual work on a problem-posing task. The results revealed a mismatch between the teacher's self-efficacy beliefs and his specialized content knowledge. The teacher was found to posses most of the expert features except theory and schema change – a capacity to change mental frames due to an inquiry process. These findings might explain both his success in training high ability students and failure to address adequate tasks for orchestrating students' understanding.

Keywords: Self-efficacy beliefs; Specialized content knowledge; Mental frames; Mixed ability class; Expert thinking

INTRODUCTION

Instructional practices are strongly influenced by teachers' beliefs. On the one hand, we consider beliefs about mathematics, its teaching and learning and, on the other hand, affects and beliefs regarding personal efficacy. The study of teachers' beliefs continues to represent a strong line of research, as many surveys and publications can witness about (for example, Maasz & Schloeglmann, 2006). It has been shown that teachers who regard mathematics as a static body of knowledge, with procedures and rules to apply, also consider the skillful application of these as goal of mathematics learning, and will focus much less on aspects of understanding the origin of the procedures or that of the concepts (Thompson, 1992). This "traditional view" of mathematics will manifest in classroom practices as the direct transmission of the rules, followed by solving problems as illustrations to the rules and practicing them on a new, but structurally identical problem (Stigler & Hiebert, 1997). Such findings of strong interdependence between beliefs and classroom practices were confirmed by more recent studies, too (for example, Stipek, Givvin, Salmon, & MacGyvers, 2001).

Teacher self-efficacy belief has been defined as teacher's belief in his/her abilities to organize and execute courses of action to bring about desired results (Tschannen-

Moran, Woolfolk-Hoy, & Hoy, 1998) and has been related to student motivation, achievement, student's self-efficacy beliefs and, also, to teacher's classroom practices (for a review, see Tschannen-Moran and Woolfolk-Hoy, 2001). Teachers with high self-efficacy beliefs might think of themselves as highly effective teachers – a thought that, in turn, influences their need to reflect and adjust their classroom practices. A way to inspect the concordance of such belief with reality is to focus on teachers' specialized content knowledge (SCK) as part of the mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008). The construct of mathematical knowledge for teaching refers to the knowledge teachers need in their practice. It has several components: common content knowledge, specialized content knowledge, knowledge of content and teaching, knowledge of content and students, and knowledge of curriculum. SCK represents a body of mathematical knowledge beyond the one taught to students and different from mathematical knowledge used by research mathematicians or applied in professions where mathematics is a tool. It is mathematical knowledge that is specifically needed in teaching. It is needed in carrying out tasks such as: "recognizing what is involved in a particular representation, finding an example to make a mathematical point, modifying tasks to be either easier or harder" (Ball, Thames, & Phelps, 2008).

The interplay between beliefs construct, the mathematical knowledge for teaching and the classroom practices, in all its complexity, has not been studied, as far as we know. However, a huge body of research indicates that the interactions are not one-way, simple causal relationships, but rather they are characterized by continuous feedbacks, often pulling in opposite directions.

In our article we look at the case of a middle-school teacher, habituated in training high-ability students in regular settings. Our interest is on identifying links between his self-efficacy beliefs, his SCK, his classroom practices and mathematical beliefs, both on concordant and discordant cords, and to look at the ways in which his practices might impact the students' learning and understanding of mathematics.

METHOD

Design and procedure

The paper presents a case study. The data were collected through participant observation, semi-structured interview, and documentation (as defined by Yin, 1994). We describe the circumstances of data collection in the next paragraphs.

The authors of the paper organized a 3-day interactive seminar on the topic of problem posing. In all, 43 mathematics teachers of various levels of education, from primary to upper-secondary, participated in that seminar. The seminar proceeded as follows:

In an initial session, teachers were shown several examples of problems and a discussion was initiated about the formulation of a problem. Then, the trainers undertook a qualitative analysis and discussed techniques for problem modification and/or posing.

During the next phase of the seminar, the participants had the task to pose, individually or in groups of 2-3 people, multiple-choice problems. For this assignment, a special requirement referred to defining the distracters: they should reflect, as much as possible, typical mistakes – including erroneous strategies, misconceptions, computing errors, etc. Each posed problem had to include an indication/hint for its solving and a brief comment addressed to the student regarding the choice of each of the distracters.

On the last day of the seminar, every participant presented a problem, posed by the person or selected from those posed by his/her group. These problems were analyzed and critically evaluated during interactive discussions between all the participants at the seminar. Several criteria were taken into consideration: the level of difficulty, the quality of formulation, solution hints, etc.

The present study idea came from the experiences within this seminar. One of the participants (referred hereafter as Mr. T.) got remarked for his perspective on mathematical knowledge and teaching strategies. We concentrated on his case: we analyzed his interventions during the group discussions, we invited him to an interview, and we documented on his personal history as a teacher.

Intervention during the seminar

We start by listing below the problem that Mr. T. presented for the seminar discussions, along with his given solution. This problem was selected by Mr. T. as being the most relevant for the posed problem activity of him and his team. Some other of his interventions will be presented within the discussion section.

Problem 1. *The number of integer solutions for the equation* $\left|x - |3x-2|\right| = 8$ *is:*
A) 1; *B)* 2; *C)* 3; *D)* 4; *E)* 0.

We include below the idea of the solution of this problem as Mr. T. presented it:

Equation $\left|x - |3x-2|\right| = 8$ *is equivalent with* $x - |3x-2| = \pm 8$.

$$x - |3x-2| = +8 \Rightarrow |3x-2| = x-8 \geq 0 \begin{cases} 3x-2 = x-8 \Rightarrow x = -3 \in \boldsymbol{Z}, but -3 < 8 \\ 3x-2 = -x+8 \Rightarrow x = \frac{5}{2} \notin \boldsymbol{Z} \end{cases}$$

$$x - |3x-2| = -8 \Rightarrow |3x-2| = x+8 \geq 0 \begin{cases} 3x-2 = x+8 \Rightarrow x = 5 \in \boldsymbol{Z}, and\ 5 \geq -8 \\ 3x-2 = -x-8 \Rightarrow x = \frac{-3}{2} \notin \boldsymbol{Z} \end{cases}$$

In conclusion: x = 5.

Interview

The starting point in the interview was the following problem:

> Problem 2. *In the sequence 1, 4, 7, 10, 13, ... the numbers are counted by three's, meanwhile in the sequence 4, 9, 14, 19, ... the numbers are counted by fives. Which are the first four common terms in the two sequences*?

The interviewers firstly addressed questions related to teaching, such as: ”For what grade would you consider this problem appropriate? In what chapter/content would you include it? What goal could it serve? What helping/bootstrapping questions would you address to your students?” Later, Mr. T. was asked to modify Problem 2 and, then, to explain the criterion used for modification.

The interview was coded using codes referring to beliefs and knowledge approach: a. What is a “good class”; b. Who is a “good student” (how should he/she behave, what should he/she do); c. What does it mean ”solving a math problem”; d. What alternate ways of solving might be; e. How should a solution be communicated and noted down; f. PP techniques; g. Criteria for judging the posed problem; h. Explicitly state the purpose of problem modifications actions. We consider the codes a-e as relevant to the teacher’s mathematical beliefs, meanwhile the codes f-h as being relevant to his SCK.

Documentation

Given our interest in his personal history as a teacher, we collected information from a diversity of sources: mass-media; autobiographic file, and discussions with a school district responsible who supervises Mr. T.

Mr. T. is a middle school teacher, from a small town in Romania. He has been teaching for 33 years. Lately, one of his former students (now college student) had exceptional results at national and international contests, being a medalist at the International Mathematical Olympiad (IMO) and at the Balkan Mathematical Olympiad (BMO). The student asserted, in interviews given to mass-media that his middle school teacher had an important role in his development as mathematician.

Mr. T. considers himself a successful teacher in preparing students for mathematical competitions, as he revealed it in the interview. He said about his student: „I’m helping him to *understand* mathematics. He is a really good student.”

RESULTS AND DISCUSSION

We focused on the following research questions: Is there any relationship between high personal efficacy belief and the specialized content knowledge (SCK)? In what ways mathematical beliefs and personal efficacy beliefs relate to classroom practices? To answer these questions, we analyze the data gathered from Mr. T’s interventions during the seminar, the interview, and other sources of documentation.

Beliefs on teaching and learning

During the interview, we asked Mr. T. about the level of difficulty he would associate to problem 2 if using it in class. Mr. T. insisted that this problem has an above average level of difficulty, proper for classes with highly able students and earliest possible to be tackled it is in grade 5. He justified this assertion as follows:

> I consider it is an above average difficulty problem, because it asks the student to write a general expression of a term…because, like this, step by step, you have given 4 elements, but I could give 444 [*common elements*].

For Mr. T., a problem needs to hide a challenge, and in this case, the problem is seen as consistent only if we take into account its potential for generalization. By having this perspective, he thinks from the very beginning to "take away" the intuitive support. Mr. T. gave an algebraic solution to this problem instantly. Indeed, in the presence of an algebraic solution, the numerical exploration of the problem might become uninteresting. The question is: does exploration have value for children's learning? As expressed in the interview, Mr. T. thinks that such exploratory initiative should be reprised in students. As such, at our question about the possibility that some students might first observe the regularity in the succession of common terms and then extrapolate that in order to obtain the 444 common terms, Mr. T. made a refusal gesture and said:

> We don't tell stories at mathematics. The student must have the capacity for synthesis, to write just what is strictly needed, nothing more. Along with the required arguments!

Mr. T. perceives the above problem as difficult because the only solution method accepted by him is the formal, algebraic solution. Consequently, for his classroom practice, essential is to transmit the mechanisms of formal solutions, which, it is, of course, difficult for 5^{th} graders. By his attitude of refusing simple explorations, Mr. T. gives the impression that he focuses only on the high performing students, ignoring the others. Moreover, this exclusion seems conceptually automatized in him: he was unable to come up spontaneously with a simpler problem of the same type; his proposals, made at the insistence of the interviewers, were still complicated.

Returning to problem 1, posed by Mr. T., we assume that his goal was to force a case-analysis on the student's behalf in order to identify possible variants and exclude the impossible ones. It seems that Mr. T. carefully defined the numbers in the problem, so that the solution phases allow highlighting possible errors.

The solution algorithm given by him is, of course, correct, but the learning of the algorithm is not necessarily leading to the understanding of the reasons for the unique solution. From a didactical point of view, this solving might not be a proper one. From mathematical perspective, the exercise can be identified as one of finding the solution of a polynomial equation. In such problems, establishing the domain of the involved functions is essential and conditions the solution of the equations.

We observe that Mr. T. evaluates the problem based on an algorithm that is correct from mathematical point of view, but looses the profound justification for which certain conditions appear. This justification is linked, in fact, to the understanding and explicit use of the concept of function.

During the seminar discussion, it was suggested to solve this problem through a graphical representation, which would better reflect the involved mathematical phenomena (see Figure 1). Moreover, if the representation is done on a grid, the integer solutions could be visualized. Such visualization would help to avoid errors that could occur during a formal solution.

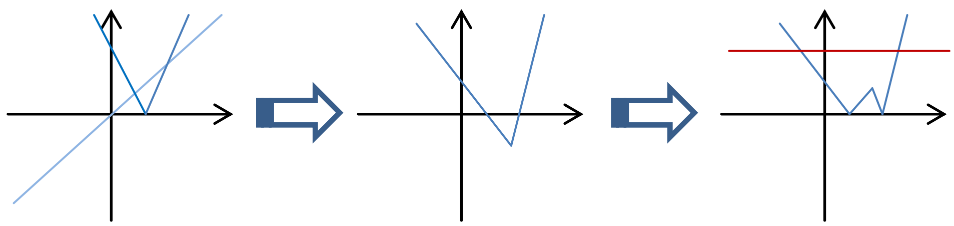

Fig. 1: A graphical solution for the Mr. T's posed equation: Compare x to $|3x\text{-}2|$, draw the graph of $x \to x - |3x - 2|$, then visualize the equation $|x - |3x - 2|| = 8$.

Mr. T. demonstrated no interest for the graphical solving: his answer was that the other approach (algebraic, presented by him) is more interesting. We hypothesize that his reaction was determined by the fact that a graphical solution would eliminate the traps hidden in the "algebraic" solution.

Specialized content knowledge

We now analyze the solutions proposed by Mr. T. to problems 1 and 2 in more detail, in order to reveal his underlying knowledge and understanding.

For the first problem, the solution consists of an algorithm of repeated steps in which the critical elements are: checking the non-negativity conditions, and checking if the partial results belong to the set requested in the problem. We compared this solution with the ones given to the same problem by five mathematicians, teaching at the university level. We noticed that the ideas mathematicians used for solving problem 1 were the same as Mr. T. used, eventually containing some not-so important shortcuts. We can assert that in solving problem 1, Mr. T. behaves as an expert mathematician.

In solving problem 2, Mr. T. wrote down the generic terms of the two arithmetical progressions, and then he arrived to a Diophantine equation and identified the set of solutions. It is clear that Mr. T. knows the general solution mechanism of Diophantine equations and he can particularize that one for the current problem. Therefore, when he was asked to formulate a more difficult problem, he modified the question by asking for 10 common terms of the given sequences that, besides being common terms, are also divisible by 13. He asserted that the solution strategy will not change, because we will still arrive to a Diophantine equation. Again, his behavior exhibited a level of mathematics expertise. According to Glaser (1997), an expert possesses: structured, principled knowledge (the expert rapidly accesses the underlying meaningful patterns) and principles stored in coherent chunks of information and proceduralized and goal-oriented knowledge (recalling a principle or a rule and efficient implementation of it). In addition, an expert makes use of an effective problem representation, which allows a qualitative assessment of the nature of the problem and the development of a mental model that reduces and organizes the problem space. Relevant is also automaticity to reduce attention demands. Another dimension of expert behavior refers to self-regulatory skills (Glaser, 1988, 1997). Experts develop metacognitive skills that control their performance in particular areas of knowledge, they learn to monitor their problem solving by predicting the difficulty of problems, allocating time appropriately, noting their errors or failures to comprehend, and checking questionable solutions.

The solution strategies of Mr. T. for the presented problems bring evidence for the above features identified for an expert by Glaser. Mr. T. spontaneously framed the problems into more general classes. This assertion is reinforced by one of his seminar interventions that we present bellow.

During the initial session of the seminar, the following problem, given at the admission exam to the Faculty of Mathematics, was analyzed:

> Let $G = \{(a, b) \mid a, b \in \mathbf{R}, a \neq 0\}$, and the algebraic operation defined by $(a, b) * (c, d) = (ac, ad + b)$. Show that G is a group that is non-abelian.

Some of the students enrolled to this exam proved that the operation * is not commutative, without verifying that (G,*) was indeed a group, and consequently have lost most of the score for this subject. The majority of the participants at the seminar agreed that the solver's omission was generated by the use of the word "that", which, in this case, filters the correct decoding of the problem. Several modified formulations were proposed, considered as more accessible/ recognizable by students, like: "G is a non-abelian group"; or in two steps: a) "G is a group; b) G is not an abelian group". Mr. T. had a different opinion. He sustained that the wording is mathematically correct and that any discussion about reformulation is a waste of time, since a "serious" student, in the context of such problem, will automatically prove both.

Mr. T. has the conviction that a student who created routines of solving problems of that category (verification of the properties of an operation) will prove first that the structure exists and, then, will prove certain properties of it. Such a student will be not deviated by the preposition used in the problem text: he will correctly solve even those problems with fuzzy formulations, because he knows that type of problem. In other words, he expects students to behave also as experts without considering the process for becoming so – a quality indispensable for classroom teaching.

Creativity training versus competition training

Mr. T. is, in certain interpretation, an efficient and successful teacher, if we focus on some of his students' performance in math competitions. We saw, in the above discussion, that Mr. T. focuses his classroom practices to promote a formal understanding of mathematical concepts, refusing the idea of explorations /investigations. The refusal of a gradual approach is expressed by a classroom practice strongly related to his views on mathematics and knowledge of mathematics. His inability to simplify a problem and use it as starting point for learning illustrates that his SCK is at imbalance with his self-efficacy beliefs.

From his conceptual thought it seems to be missing what Glaser calls *theory and schema change* (Glaser, 1988) – that is: a dimension of expert thinking that reflects a process of interrogation, confrontation, conflict, and discovery. Mr. T. is reluctant to explorations in problem solving but also in changing his personal approaches of various issues. Therefore, we ask, what explains then his success in the creativity and/ or competition training? In order to answer, we analyze the manner in which Mr. T. changes a given problem. For example, at the request to modify problem 2, he proposed the following:

> *I could keep one of the sequences, and for the second we consider one with a bigger ratio. I'm thinking to still ask for the first 4 common terms or the first 10, to extend them more and to force them to…for example, 2, 22, 42, 62 …*[with a ratio of 20].

Mr. T. keeps one sequence unchanged. With regard to the problem question, for him, requesting the first 4 common terms, or the first 10 is the same thing because, due to the low number of terms, the students could still proceed by trial and error. The choice of the second sequence with a higher ratio has the purpose to farther the terms from the origin of the axis such that the numbers with which the students should work are bigger. Such situation would force the student to take an algebraic approach and (eventually) to accept the necessity of an algorithm in solving the problem. Moreover, Mr. T. chooses the ratio as 20 (multiple of 5). Why 10 (number of terms) and 20 (ratio)? We suppose that he is governed, subconsciously, by a certain mathematical aesthetic, a sense of beauty that rises from the fundamental nature of the base 10 that leads to "beautiful", "elegant", "round" numbers.

In a previous study (Voica & Singer, 2012), we discussed the relationship between problem posing and mathematical creativity in terms of cognitive flexibility. We concluded that, in problem posing, high performing students express a functional type strategy – they vary one single element of the starting problem, to control the quality of the newly obtained statements. Mr. T. proceeds similarly, because he keeps unmodified a part of the data and varies a single parameter – the ratio of the second sequence. It is possible that this "mono-dimensional" cognitive flexibility of the teacher, similar to the one of high performing students, allows him to do an efficient training of creative students. This behavior properly fits a way of thinking that focuses on efficiency in problem solving, an aspect requested within mathematics Olympiads.

CONCLUSIONS

Mr. T. has strong positive self-efficacy beliefs. He proves to have a mathematical content knowledge similar to an expert mathematician; however his SCK mismatches his perception of efficacy. As solver and trainer for International Olympiad style competitions, Mr. T. is interested in efficiency. From this point of view, Mr. T.'s objective is the acquisition of solving algorithms for classes of problems. More complex these are more are the chances that the student turns into a highly performing one. However, from a teaching perspective, a practice focused on algorithms leads to a failure on profound understanding of mathematical ideas and concepts. The algorithms targeted by Mr. T. are of high level of abstraction and complexity. He seems to have "cut off" SCK and kept only those mathematically "delicate areas" that he teaches in a systematic manner to highly performing students. These students are trained, for example, in avoiding "traps" such as the ones concerning checking initial constraints over the data and results.

However, a fundamental problem might be his "traditional" view of mathematics: mathematics is a set of formalized knowledge where performance is essential. This view explains his belief that knowing a complex algorithm means knowing mathematics – there is nothing more to look into there. Such beliefs lead to specific classroom practices, where the children should be the ones to "keep the rhythm" (as some performing students do). To be a good student means, for him, to work a lot after you have been taught the algorithm. However, such an approach can have a negative impact on students in a mixed ability class and, maybe, even on high performing children.

Problematic issues derive from the fact that he has no way of knowing if the students really "understood" (because the question does not address this target) and if the students really develop their own creativity (because the lack of opportunities for exploration could lead to inhibition and misunderstandings). He perceives learning as a product, not as a process: Mr. T. does not accept that a child can be coached towards the solution with helping hints that deepen understanding – it is a one-chance approach for him: "either you get it from the first moment, or you don't". Such view can have a counteractive impact on students' self-efficacy perceptions, as the feedback they receive is a negative one. Besides the existence, in general, of a traditional view of mathematics, the source of such beliefs can be linked to the vision the general public (including parents and students) share about what it means to be good at mathematics: for most, the only criteria is the performance in competitions.

In the same time, high-ability students can have additional sources of information or interest in individual learning: in general they receive training from multiple sources. They can overcome certain deficiencies in understanding by being exposed to diverse training practices. Even the former student of Mr. T. admitted that, besides his middle-school teacher, he had also other trainers, including parents.

Contrary to the case of the students with high potential and interest in mathematical performance, the other students are usually ignored by such teacher. Breaking the vicious circle asks for much more than highlighting ways to increase teacher's SCK; there is a need to change teachers' mentalities and beliefs. Beyond Olympic preparation, all the students, gifted or less gifted, should be prepared for a successful professional life and optimum social inclusion; in today's dynamic world, a training on problems typologies and complex algorithms is far from being enough.

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MATHEMATICAL CREATIVITY AND HIGHLY ABLE STUDENTS: WHAT CAN TEACHERS DO?

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The text focuses on the effects of didactical heterogeneisation on students' creativity, namely on students' ability to propose such solutions to problems that are novel for them. The key questions for mathematics education are: Should the teacher direct his/her teaching towards good mastery of algorithms or towards development of students' creativity? Should all students or only the highly able ones be given the opportunity to work creatively?

Keywords: Didactical heterogeneisation, didactical differentiation, students' creativity, teachers' didactical culture.

INTRODUCTION

Formulation of the question addressed by this text is relatively simple: If learning of mathematics is based on coming up with new solutions to new problems (from the student's perspective) and not only on mere reproduction of algorithms, then mathematical creativity is undoubtedly in the center of mathematics education; some students feel freer in allowing themselves to come up with innovative solutions. Who are these students and should they (often termed "intelligent", "highly able" or "gifted") be regarded as an additional problem in the teacher's work (sadly this perception of the situation is not rare)? One must not forget that it may not be possible for the teacher to work with differences in their students' abilities as is recommended by didactics. This is the thesis that we will defend here.

When Alfred Binet (1857-1911), the French psychologist and inventor of the first usable intelligence test (known at that time as Binet test and today as IQ test) (1905) was asked for a definition of intelligence, he answered: "Intelligence, it is what my test measures."; if he had been asked for possible grounds of a talent in mathematics, he would have probably answered with the same irony: "Intelligence!". Such an answer would have left teachers skeptical as to the way of dealing with differences in creativity (i.e. the ability to create "something new") of their students when teaching. More precise definitions, presented by psychologists – e.g. Julian de Ajuriaguerra, who introduced the term "over-gifted" – will all be enigmatic to the question of the origin of giftedness; they will stay faithful to its evangelic origin (Matthew, XXV, 14). We are neither psychologists nor neuropsychiatrists; as didacticians we will approach the problem of highly able students independently of any ideology (e.g. of the type "for" or "against"); indeed, even if a didactician is not able to state whether specific educational policy of differentiation is profitable for one or another category of students, his/her work can still contribute to clarification of the intentions and probable outcomes brought about by this organization.

ORGANIZATION OF THE EXPERIMENT

This research uses a larger set of observations. The aim of this set of observations was to explore the impact of teachers' didactical variability (i.e. their ability to organize situations – and therefore knowledge – for teaching arithmetic) on students' mathematical culture (Novotná, Sarrazy, 2011).

The following paragraphs describe the conditions of these observations.

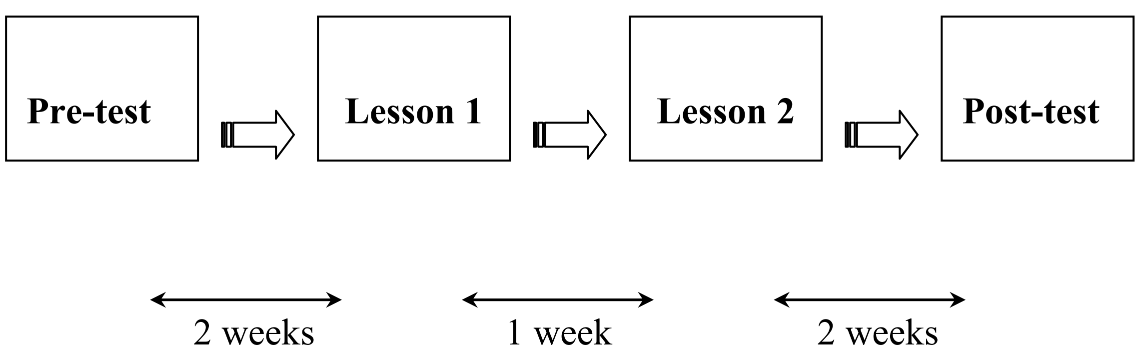
The studied population

The representative sample participating in the experiment consisted of 112 French pupils aged 9-10 from 7 primary school classes. Their school level was evaluated using a standardised test enabling us to position the pupil's level in relation to the French school population. The pupils were then divided into three groups corresponding to the criteria developed by the authors of the test: "Good": the mark (x) in TAS is in the interval <8; 10>; "Average": $x \in$ <5,5; 8); "Weak": $x \in$ <0; 5,5). There is a strong correlation with the teachers' evaluation (χ^2; $p < .001$).

The conditions of the observation

The conditions of the teachers' teaching were meant to be as close to their usual work as possible; the conditions were discussed in two perspectives: a) the topic of the lesson; b) the time of the observation. The teachers had to perform two one-hour lessons; there also was a pre-test and a post-test.

The course of the experiment was standard; it proceeded according to the following scheme:



The topic of the lesson

The topic taught was meant to be appropriate to the pupils' level and simultaneously was meant to be of novel nature in order to avoid deviations possibly caused by what the teachers had already taught, in other words in order to limit the effects of didactical memory of the class (Brousseau, 1997).

The topic selected for the two lessons corresponds to the fourth additive structure of Vergnaud (1983) because it allows integration of the two above described methodological restrictions. This structure works only with positive or negative transformations ("gain" or "lose") without any indication of the initial numerical status. The following is an example of such problem:

> Lou plays two rounds of marbles. She plays the first round and then the second. In the second round, she loses 4 marbles. After the two rounds she wins 6 marbles. What happened in the first round?

The pre-test and post-test consisted of 22 problems of this type (selected from 24 types of problems for this type of structure); each problem contained at most two numbers smaller than 10. The level of difficulty of these problems depends on the position of the unknown (possibilities: 1st transformation, 2nd transformation or compound transformation) and on the transformations that can be either of the same sign or of the opposite sign. For example, the above given problem *Lou* is very difficult for 9-10-year old pupils, while the below given problem *Dominika* is much easier, although a non-negligible proportion of the pupils produces an incorrect answer anyway ("*She has* 2 marbles altogether" instead of "*She won* 2 marbles altogether.")

> Dominika plays two rounds of marbles. She plays the first round and then the second one. In the first round, she wins 6 marbles. In the second round, she loses 4 marbles. What happened in the whole game?

To avoid any influence on the teachers' organization and structuring of the lessons, the teachers had no access to evaluation before the post-test – this condition was negotiated as part of the research contract.

This paper does not give us enough space for description of the observed lessons (and it is not necessary for our purposes)[1]: for our purposes we focus on the consequences of pupils' learning as a function of the initial level (shown in the pre-test). Thus we want to find the possibilities the teachers have (whatever their teaching style is) for differentiation of teaching with respect to their pupils; cognitive abilities.

ANALYSES AND RESULTS

Figure 1 represents the distribution of success in the pre-test:

[1] For a more detailed description of teaching activities focusing on this type of knowledge see (Chopin, 2011).

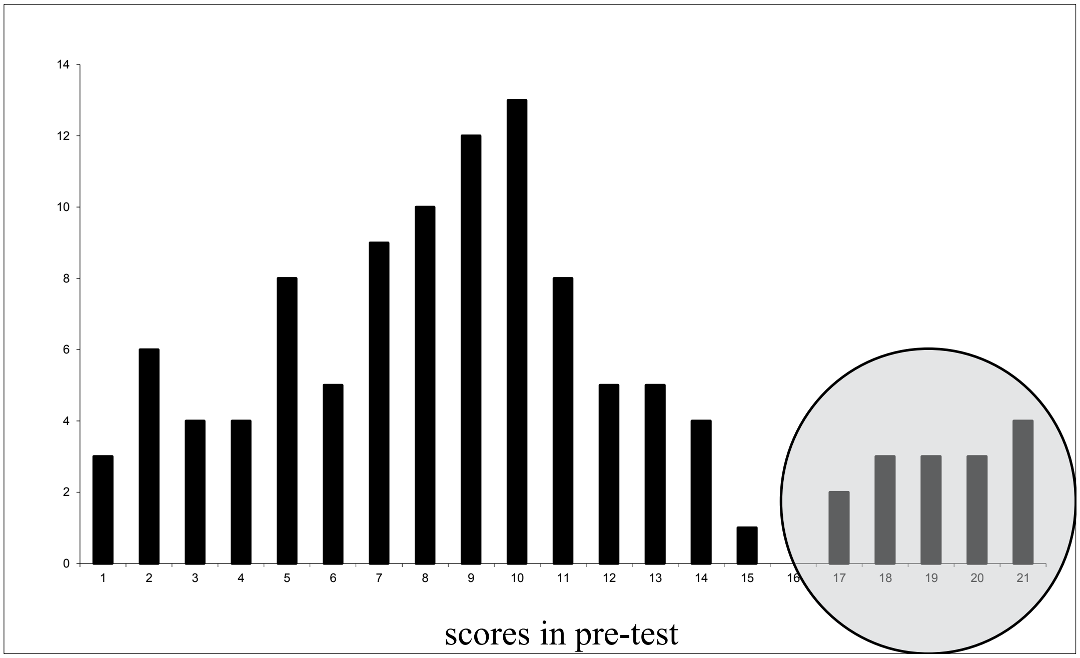


Figure 1: Success in the pre-test

In the following analyses 15 pupils who were successful in the pre-test at least in 17 problems out of the given 21 are considered as highly able.

Who are they?

They represent 13.4% (n = 15) of the participating pupils. There were more boys (73.3%) than girls (26.7%) – out of 51% boys and 49% girls in the whole set of pupils (χ^2= 2.64; ns; p = .11). The majority of them come from higher social classes (54%), 46% to middle classes. All parents of the pupils involved are secondary school graduates and 2/3 of them have a university degree. A questionnaire for the families confirmed that the parents' practices in the process of their children's upbringing were flexible (their children could negotiate the rules) and "curiosity" and "critical approach" were the dominant values in their educational approaches. To put it briefly, these are all students who succumb to rules which, whatever their degree of precision, must nonetheless be applied differently according to the context and situations.

In the context of the class the psychosocial status of these pupils is high in case of 87% of them and their majority are well aware of it (they neither underestimate nor overestimate their abilities). Unlike the majority of pupils, they always ask the teacher as soon as they do not understand something in mathematics lessons. They do not interact more than other pupils (t = 0.11; p = .91) but their interactive profile is significantly different from the others (χ^2 = 10.74; p = .005): they do not ask to speak, they simply speak.

Theoretical model of the study

Didactical treatment of heterogeneities of competences by the teachers

It is a generally accepted fact that all teaching and learning, at least in the school environment, tries to develop knowledge of maximum possible number of pupils in the limited amount of time. This development surfaces as a decrease in the number of pupils' errors; in other words, as a reduction of heterogeneity of pupils' decisions on how to proceed, answers etc. that are acceptable for the teacher. But the teacher has no tools for *direct* handling of heterogeneities of pupils' differences in their giftedness, the differences in their attitudes to mathematics, of the time and attention they are ready to devote to it etc. What he/she does in his/her teaching in the beginning is that he/she complies with this original heterogeneity, trying to optimise it. In fact, no matter what the level of the class is (weak, advanced or very advanced), a too ambitious lesson would be too difficult for a considerable proportion of the pupils and a too simple lesson would also be unacceptable (loss of time). Individual pupils are often more than happy and comfortable in their positions of "good pupils", "weak pupils" etc. For several reasons, these categories of classification must be considered as functioning of didactical systems as such, independently on the initial abilities of individuals (Brousseau, 1997).

Let us now explore the following two questions:

1. Did the teaching enable learning for the largest possible number of pupils?
2. Is this learning equally distributed with respect to the pupils on different levels?

Results

1. Not all the pupils benefitted equally from the teaching: pupils of the above average and average levels profited most (about 58% of pupils) – see Table 1:

	School level			
	Highly able	***Good***	***Average***	*Weak*
Number	15	**17**	**48**	32
m	0,7	**4.45**	**6.55**	2.32

Table 1: Means of success in the post-test

2. Those good and average pupils who were the worst in the pre-test made the biggest progress in the post-test (and vice versa). This shows a strong correlation between the level of success in the pre-test and in the post-test. This does not hold for weak pupils.

	School level			
	Highly able	***Good***	***Average***	*Weak*
ρ	-.57	- 0,78	- 0,73	-0,34
p.	.02	2.10^{-5}	.0001	.15

Table 2: Correlation of success pre-test/improvement (Spearman's rho)

Conclusion: Teaching is efficient only when we start from the threshold of the initial abilities; this can be called according to A. Marchive (1997) and referring to Vygotsky a "zone of proximal teaching", in which the teacher can teach with reasonable outcomes.

The presented results allow us to acknowledge the following two statements. No matter what the level of the considered class (*F*(pre-test) = 2.60; p. = .02) is:

- if the difficulty is lower, more pupils advance and heterogeneity decreases (r = -.87; s.; p. <.001);
- if the difficulty is high, the progress pupils make results in an increase of heterogeneity (R = +.83; S.; P. <.001).

DISCUSSION

Would the principle of didactical differentiation (using different things according to the students' initial level) allow us to support the conditions of creativity for all students, i.e. to support their learning?

As indicated in the previous text, regardless of the initial level of the class, any progress in knowledge in the class leads to an increase in heterogeneity. The lower the progress is, the more the heterogeneity is reduced. This implies that grouping pupils according to their level of giftedness does not bring optimisation of their learning, whatever their initial abilities are (there are researches that support this hypothesis – e.g. Duru-Bellat, 1996 ; Mingat, Duru-Bellat, 1997). The fact is that when teaching, the teacher must inevitably differentiate among pupils. We must be aware of the fact that the fall of very good pupils to weak positions if they attend large, above average classes will result in loss of courage, decrease in self-confidence etc. This is the price we must be ready to pay for an increase of the average level of the class. The following scheme (Fig. 2) illustrates this phenomenon: If a class A is divided into two groups, one group consisting of good and rather good students and the other of weak or rather weak ones, then the first group's level will be higher than the level of the second group; but it is interesting that the teacher will be forced to adjust his/her level of teaching according to the level of the group and will therefore create a new heterogeneity in the group. From the perspective of heterogeneity, we can see the same phenomenon in the two groups although the level of the first is higher than of the second.

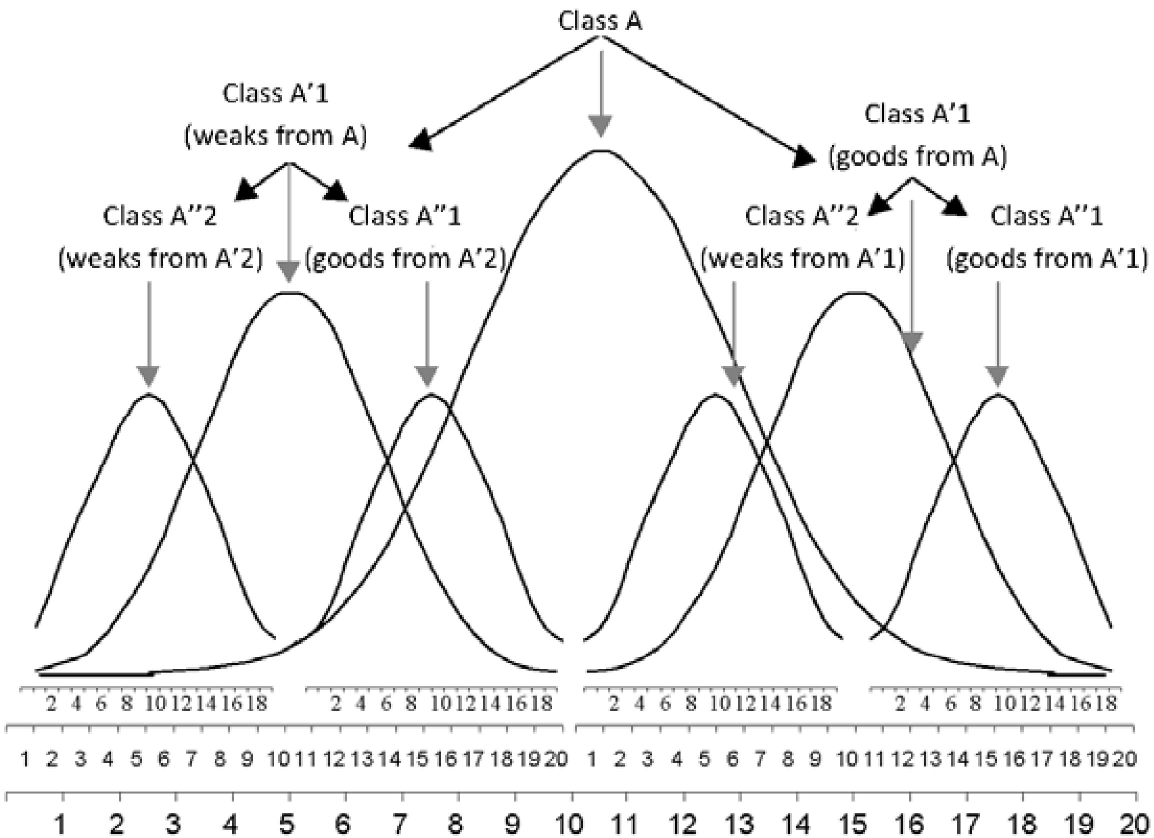


Figure 2: Division into classes according to students' giftedness

The decision to teach in such a way as to improve the performance of the best pupils at the expense of weaker pupils is political. It is not the task of mathematics educators to judge this choice. Their role is to help clarify this situation. As Mc Dermott & Varenne (1995, 343) claim the place that is reserved for pupils from cultural minorities says much more about how our institutions work, about the values they bring than about the expected cognitive priorities. These ideas are "clearly adapted to the functioning and institutions that, across a formal educational system, serve to political and economic goals". This is certainly not the question of glorification of egalitarianism or of radical elitism. We only warn of the dangers of the situation when modern democracies produce categories of individuals who are not able to communicate outside the boundaries of their own cultural community. Our results suggest that research motivated by legitimate concerns about effectiveness and equality is of great potential, as long as it pays sufficient attention to the conditions of organization of such teaching/learning situations in which everybody can adjust the knowledge to make it useful for his/her life. We are fully convinced that there exists a happy medium between acceptation of *indifference to differences* (P. Bourdieu) and its total refusal. It is the task of didactics to show this happy medium or at least to help to clarify it.

The set of these results evokes the fundamental question of orientation of education: Should it orient towards good mastery of algorithms or allow students to be creative in application of these algorithms in new situations (for this is the core of creation: it does not lie in rediscovery of some algorithm but in how the student really applies it in new situations)? It seems that both these orientations must be present simultaneously, which causes a paradoxical relationship: The more the teacher makes his/her teaching algorithmic, the more he/she limits the opportunities for creation for his/her students; the less he/she makes it, the less the students' knowledge is important and the less the students have the opportunity to create something new (as shown e.g. in (Sarrazy & Novotná, submitted to ZDM for 2013) the best students allow themselves to create new relations).

The Theory of didactical situation is born from the theorization and the scientific study of conditions enabling to overcome this paradox; although its recognition in the scientific community is high[i], its dissemination and its use in teacher training remain strongly limited, as it is shown by Marchive (2008). Should it be regretted? Certainly, as teacher training shows to be an important lever allowing teachers to leave this pointless debate. For the teacher, it is fundamental to trust students' creativity, but this pedagogical belief often leaves them helpless when they are to prepare conditions for: pedagogical intention itself is powerless face to face students' incomprehension.

We believe that it is desirable to increase teachers' didactical culture; in fact, if a didactician contributes to clarification of the conditions under which a student may be given the chance to create new knowledge (this contribution does not depend on the student but on the mathematical culture itself), the teacher's responsibility remains to manage the socio-affective conditions that allow the student to get involved in the adventure, which nobody can experience instead of them: the adventure of grasping the whole world in one day on their own, gaining the profit from it. How can one imagine that students would be able to produce something new, had they never had any opportunity to experience it? This is our noble mission: to organize the conditions for such mathematical creativity; the fact that some of them succeed in this adventure better than others is not, as we tried to show, the teachers' responsibility as long as he/she creates the conditions allowing the possibility of this adventure for all students.

ACKNOWLEDGEMENT

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[i] Its founder, Guy Brousseau is the first laureate of the Felix Klein medal for long-life research in the field of mathematics education, awarded by the International Commission on Mathematical Instruction.

BRAIN POTENTIALS DURING SOLVING AREA-RELATED PROBLEMS: EFFECTS OF GIFTEDNESS AND EXCELLENCE IN MATHEMATICS

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This event-related potentials (ERP) study investigates the impact of general giftedness (G) and excellence in mathematics (EM) on behavioural performance and cortical brain activity while solving geometric area-related tasks. We report on findings of comparative data analysis based on 74 right-handed male students. Effects of the G and EM factors emerged at the behavioural and the neuro-cognitive level. We found that giftedness is expressed in more efficient brain functioning. EM does not guarantee success in solving the problems and exerts a different effect on the cortical brain activity in G as compared to NG subjects. When performing area-related tasks, G can compensate for the lack of EM.

Key words: Geometric reasoning, Area, Event related potentials, General Giftedness, Excellence in Mathematics, Neural efficiency

INTRODUCTION

A considerable body of research has been conducted towards understanding the neural foundation of mathematical cognition (e.g., Santens et al., 2010). In addition, extensive neuroscience research has focused on human intelligence including individual differences in general intelligence (e.g., Neubauer & Fink, 2009; Deary et al., 2010) and on mathematical giftedness (O'Boyle, 2008). However, these studies have not gone beyond arithmetic, logic and spatial mental ability. That is why we have chosen to focus our attention on electrical brain activity (by means of Event Related Potentials – ERPs) associated with solving advanced mathematical tasks.

Additionally, to the best of our knowledge, differences between giftedness and expertise in mathematics have not been addressed in brain research to date. This observation resulted in the integration of 4 research population groups, divided according to varying combinations of general giftedness and mathematical expertise.

BACKGROUND

Studying areas of figures in school

Studying geometry in high school involves analysing geometric structures, characteristics and relationships (NCTM, 2000). Mental images of geometrical figures represent mental constructs possessing simultaneously conceptual and figural properties (Fischbein, 1993). Geometrical reasoning is usually associated with visual and logical components which are mutually related. The area of figures overlaps with content areas of geometry and is considered to be a significant topic of school mathematics (NCTM, 2000). Mathematics researchers and educators have suggested that knowledge of the properties of the basic shapes and congruence, geometric motions and area measurement concepts are all closely interrelated (Clements et al., 1997). Moreover, the integration of geometric knowledge and area measurement can be important for the conceptual understanding of area measurement (Huang & Witz, 2011). Problems involving comparison between the areas of two figures can be solved without using area formulae, thus enabling researchers to study student's purely conceptual understanding (Huang & Witz, 2011).

Mathematical abilities, cognitive skills and brain research

Literature review demonstrates quite consistent findings that connect different mental operations associated with mathematics and the location of brain activation. For example, research shows that attention control processes and general task difficulty (Delazer et al., 2003) are associated with the prefrontal cortex, while mental rotation (Heil, M., 2002) and visuo-spatial strategies in mathematics (Sohn, et al., 2004) are associated with the parietal cortex. The brains of the mathematically gifted show enhanced development and activation of the right hemisphere (Prescott et al., 2010) as well as enhanced brain connectivity and an ability to activate task-appropriate regions in both brain hemispheres in a well-orchestrated and coordinated manner (O'Boyle, 2008). There is strong evidence for special development of prefrontal and posterior parietal regions of the brain and their enhanced intra-hemispheric connectivity (e.g. Jung & Haier, 2007).

The goal of the current study was to investigate the impact of general intelligence, as well as of excellence in mathematics, on performance in short area-related geometric problems using electrophysiological measures.

MATERIALS AND METHODS

Participants

Seventy-four right-handed male high school students from northern Israel (16-17 years old) participated in this study. The students were sampled as shown in Table 1.

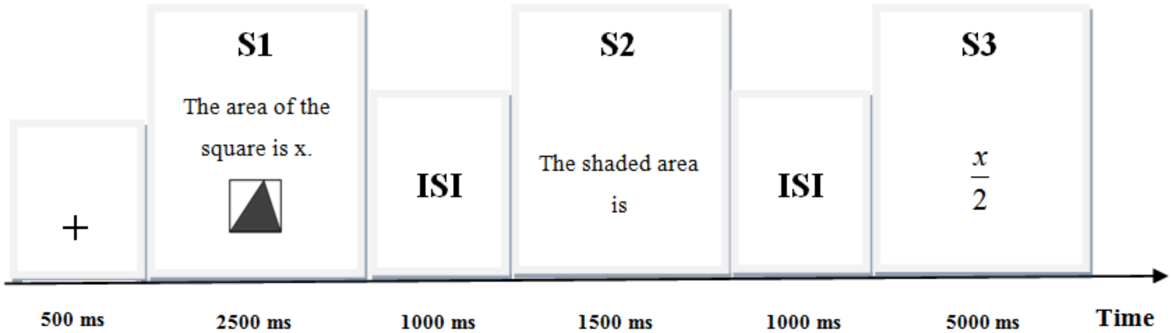
Table 1: Research population

	Gifted (G) IQ>135, Raven >28 of 30	**Non-Gifted (NG)** 100<IQ<130, Raven < 26 of 30	Total
Excelling in mathematics (EM) SAT-M >26 of 35 or Math score > 92 in high level mathematics	20	17	37
Non-Excelling in mathematics (NEM) SAT-M <21 of 30 or Normal level of mathematics instruction	20	17	37
Total	40	34	74

Stimuli and Procedure

A computerized geometry test was designed using E-Prime software (Schneider, Eschman, & Zuccolotto, 2002). The test included 60 tasks. Each task on each test was presented in three windows with different stimuli (S1 – Introducing a situation stage, S2 – Question presentation stage, and S3 – Answer verification stage) that appeared consecutively. Figure 1 presents the sequence of events and an example of a task. Time periods were determined by a pilot study with 15 participants.

Participants received a drawing of a geometric object. Part of this drawing was shaded. The participants had to determine what area of the drawing was shaded or what the area of the geometric object was in reference to the shaded part.



S1 – Introducing a situation; S2 – Question presentation; S3 – Answer verification;
+ – Fixation cross; ISI – Inter Stimulus Interval

Figure 1: The sequence of events and a task example

Behavioural data analysis

MANOVA was applied to reaction time for correct responses (RTc) and accuracy (number of correct responses - Acc) of the performance in order to examine effects of between-subject factors (G factor and EM factor).

ERP Recording and Analysis

Scalp EEG data were continuously recorded using a 64-channel BioSemi ActiveTwo system (BioSemi, Amsterdam, ND). [We do not present the technical details of data recording and analysis due to the space constraints of this paper]. ERPs (Event Related Potentials) are electrophysiological measures reflecting changes in the electrical activity of the central nervous system related to external stimuli or cognitive processes occurring in the brain. The ERP waveforms were time-locked to the onset of S1, to the onset of S2 and to the onset of S3. The grand average waveforms (average of students waveforms) were calculated for each stage.

Early components (P1 and P2) and late potentials, as well as the electrodes and time frames for statistical analysis, were chosen based on a preliminary examination of grand average waveforms and ERP topographical maps (electrical voltage distribution) (Table 2).

Table 2: Electrodes chosen for statistical analysis

	Time frame (ms)	Selected electrodes
P1 component	S1, S3: 100 – 180; S2: 100 – 200	P3, Pz, P4, PO3, POz, PO4, O1, Oz, O2
P2 component (for S1 only)	180 – 250	AF3, AFz, AF4, F3, Fz, F4, FC3, FCz, FC4
Late potentials	250 – 500, 500 – 700 and 700 – 900	AF3, AFz, AF4, F3, Fz, F4, FC3, FCz, FC4, P3, Pz, P4, PO3, POz, PO4, O1, Oz, O2

We performed the following statistical analysis related to early components and late potentials.

(1) In order to examine early differences in amplitude and latency of early components associated with perceptual processing of each stage, we conducted repeated measures MANOVA for each task's stage using Laterality (3 levels: P, PO, O) as a within-subject factor, G factor (2 levels: G, NG) and E factor (2 levels: EM, NEM) as between-subject factors. **(2)** In order to examine the mean overall activity we performed repeated measures MANOVA on RMS (i.e., the square root of the mean of the squared potentials from each common referenced electrode) using Time (3 levels: 250-500, 500-700 and 700-900 ms) as a within-subject factor, G factor (2 levels: G, NG) and E factor (2 levels: EM, NEM) as between-subject factors. **(3)** The mean amplitudes were averaged over six electrode sites (ELS) (PR - right posterior (P4, PO4, O2), PM - middle posterior (Pz, POz, Oz), PL - left posterior (P3, PO3, O1), AR - right anterior (AF4, F4, FC4), AM - middle anterior (AFz, Fz, FCz) and AL - left anterior (AF3, F3, FC3)). In order to examine the differences in electrical activity in the aforementioned electrode sites, the repeated measures MANOVA was performed on the ERP mean amplitude considering the ELS (6 levels: 6 electrode sites – PL, PM, PR, AL, AM and AR) as within-subject factor, the G factor (2 levels: G, NG) and EM factor (2 levels: EM, NEM) as between-subject factors. This was done for each stage of task problem solving. *P-values* were corrected for deviation from sphericity according to the Greenhouse Geisser method.

RESULTS

We report here only the main effects and significant interactions.

Behavioral data

Table 3 demonstrates reaction times for correct responses and accuracy (mean and SD) of the performance on geometric tasks found for the four groups of participants.

Table 3: RT for correct responses (RTc) and Accuracy for different groups (Mean, SD)

	Acc (%)			RTc (ms)		
	G	**NG**	**Total**	**G**	**NG**	**Total**
EM	81.9 (7.1)	75.4 (9.0)	78.9(8.6)	1223.0(314.9)	1344.5(419.4)	1278.8 (366.4)
NEM	81.3 (6.8)	73.3 (8.6)	77.7(8.5)	1453.4 (441.2)	1218.4 (382.0)	1348.9 (426.9)
Total	81.7 (6.9)	74.4 (8.7)	78.2 (8.7)	1328.2 (395.9)	1283.4 (400.5)	1309.8 (391.5)

Acc – Accuracy; RTc – Reaction time for correct responses

The MANOVA showed a main effect for the G factor (F (2, 68) = 9.137, p < .001). The follow-up ANOVA analysis showed a main effect of G factor on accuracy (F (1, 69) = 15.285, p < .001) and marginally significant interaction of G × EM factors involving RTc (F (1, 69) = 3.741, p =.057). Figure 2 demonstrates these effects and interactions.

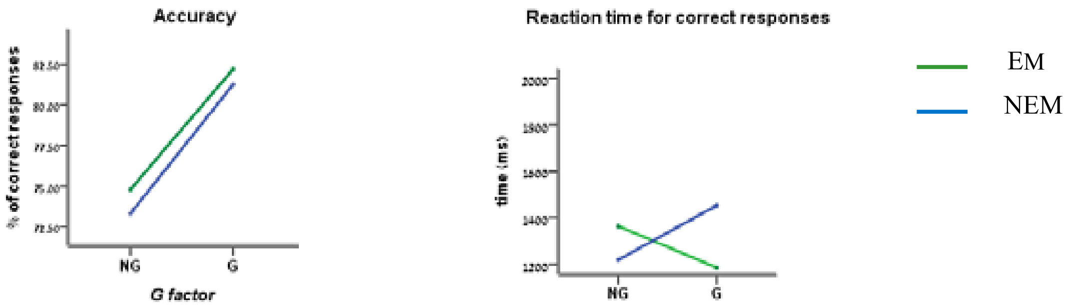


Figure 2: Accuracy and RT for correct responses in the four experimental groups

G participants were significantly more accurate than their NG counterparts. The accuracy of G-NEM individuals was similar to the accuracy of G-EM and much higher than those of NG-EM. However, the RTc of G-NEM was the longest of the four groups of participants.

Electrophysiological scalp data

The grand average waveforms for four experimental groups for each tasks' stage are displayed in Figure 3.

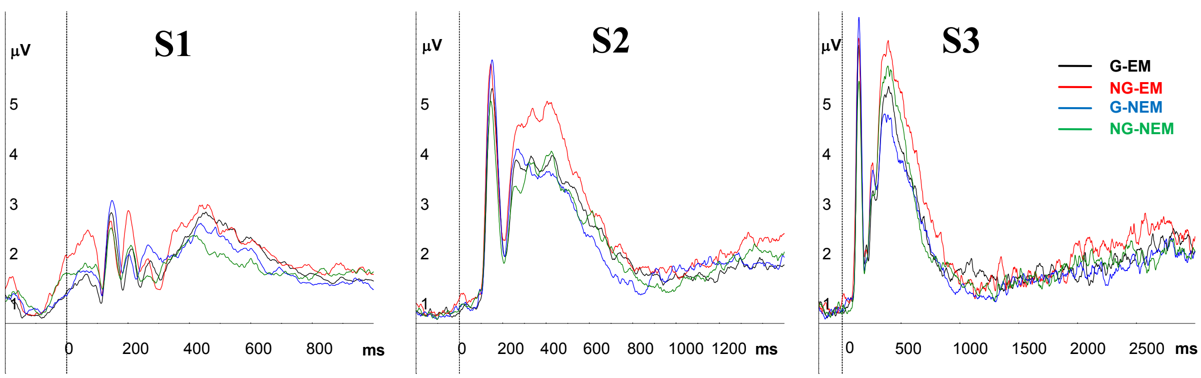


Figure 3: The grand average waveforms at each stage

From the grand average waveforms observed in Figure 3, one can see that posterior P1 is elicited in each of the three task's stages: S1 – Presentation of a situation, S2 – Question presentation and S3 – Answer verification. Anterior P2 is elicited only at Stage 1.

The significant effects on latency and amplitude of P1 and P2 are shown in Table 4.

Table 4: Main effects and significant interactions of latency, amplitude of P1 and P2

Stage	Peak	Latency		Amplitude	
S1	P1	N.S.		Laterality × EM at P electrodes	F (2, 140) = 4.464*
	P2	Laterality × G at AF electrodes	F (2, 140) = 3.464*	Laterality × G at F electrodes	F (1.730,121.088) = 4.918*
		G factor × EM at AF electrodes	F (1,70) = 6.712*	Laterality × EM at F electrodes	F (1.730, 121.088) = 3.750*
				at FC electrodes	F (1.811, 126.757) = 3.232*
		at F electrodes	F (1,70) = 5.285*	Laterality × G × EM at AF electrodes	F (1.917, 134.170) = 3.266*
S2	P1	Laterality × EM at O electrodes	F (1.611,112.782) = 3.985*	N.S.	
		G factor at O electrodes	F (1,70) = 6.613*		
S3	P1	N.S.		G × EM at O electrodes	F (1,70) = 4.852*

* $p < .05$ (p-values with Bonferroni correction)

The aforementioned results suggest that different participants groups perform early processing of an object in different ways. These differences were expressed in latency and amplitudes of P1 and P2 as well as in laterality of these peaks. Space constraints of the paper do not allow us to explain all of these findings in detail. Following the P1 and P2 components, we analysed late potential components. Table 5 represents main effects and significant interactions followed by pair-wise comparisons.

Table 5: Main effects, significant interactions and pairwise comparisons for each stage

Stage	RMS – root mean square		ELS – at electrode sites		
S1	***EM factor***	5.373*	***G × EM***	250-500ms	10.029**
	500-700 ms	4.838*		500-700ms	8.621**
	700-900 ms	6.422*			
	for NG: RMS(EM) >> RMS(NEM)	8.162**	for NG: Amp(EM) >> Amp(NEM)	250-500ms	10.805**
				500-700ms	9.096**
	250-500 ms	4.409*	for NG at PM: Amp(EM) >> Amp(NEM)	250-500ms	4.911*
	500-700 ms	7.481**		500-700ms	8.886**
	700-900 ms	7.037**			
S2	***EM factor***	5.298*	*N.S.*		
	for NG: RMS(E) >> RMS(NE)	5.132*			
	250-500 ms	5.093*			
S3	*N.S.*		***G factor***	500-700ms	4.911*

(1) $^{*}p < .05$; $^{**}p < .01$ (p-values with Bonferroni correction); (2) *d.f.* (1, 70); (3) Amp – Mean Amplitude; (4) RMS – Root Mean Square

We found a main effect of the EM factor at S1 and S2 stages for RMS, which is the measure of overall mean electrical activity. RMS of EM was larger than that of NEM participants. Further investigation of this effect showed that a significant difference between EM and NEM exists only for NG participants (for S1 stage see Figure 4). Interestingly, EM and NEM displayed similar RMS activity amongst G participants.

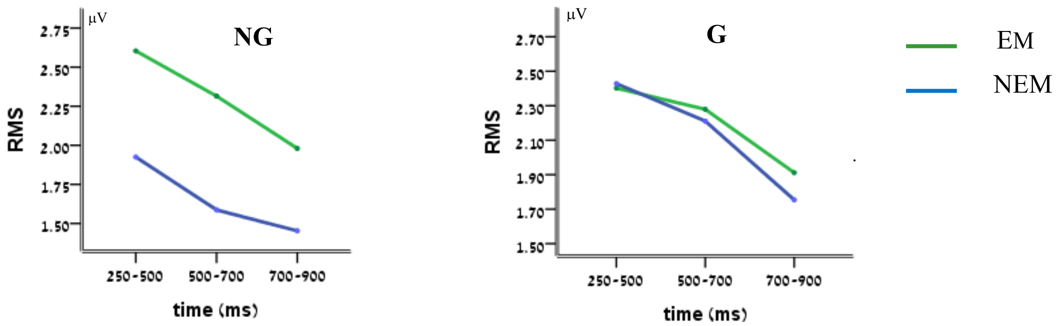


Figure 4: RMS activity at S1 stage for G and NG participants

To investigate more precise topographical distribution of the electric potential, we carried out an examination of the mean amplitude in the electrode sites defined above (AL, AM, AR, PL, PM and PR).

The statistical analysis found significant G × EM interaction at S1 stage (250-500 and 500-700 ms time frames). At these sites the mean amplitude of NG-EM participants was significantly higher than their NG-NEM counterparts. However, the mean amplitude in G-NEM students was only slightly different from that of their G-EM counterparts.

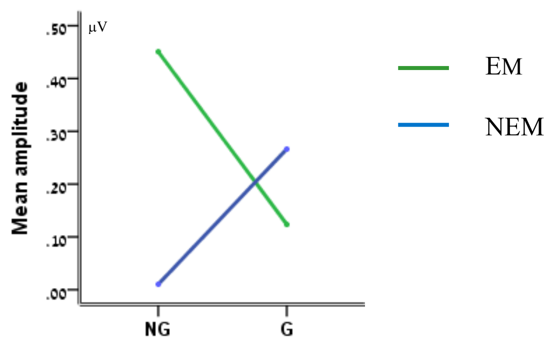


Figure 5: Mean amplitude at S1 stage (500-700ms) for the four groups

The most prominent difference between NG-EM and NG-NEM was at middle posterior (PM) electrode site (Figure 6).

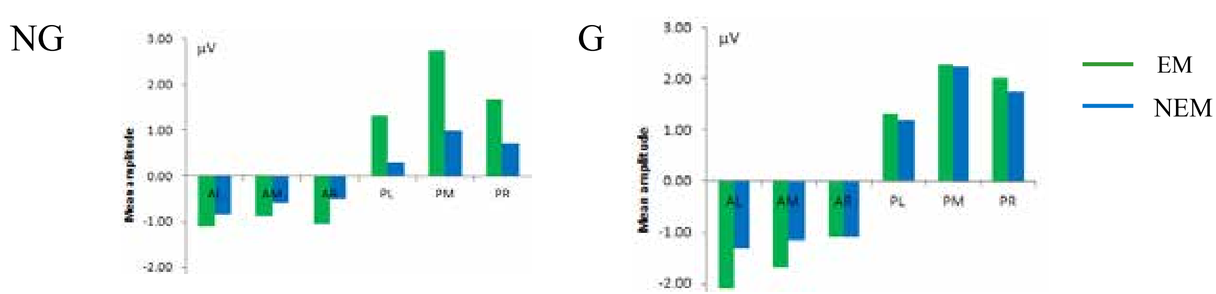


Figure 6: Mean amplitude at S1 stage (500-700ms) for G and NG students at six electrode sites

The statistical analysis revealed a main effect of the G factor at the S3 stage – answer verification. At 500-700 ms, G participants had a lower mean amplitude in the predefined electrode sites compared to their NG counterparts (see Figure 7). Pairwise comparisons revealed marginally significant differences between G and NG participants only amongst EM participants (F (1, 70) = 3.676, p = .059).

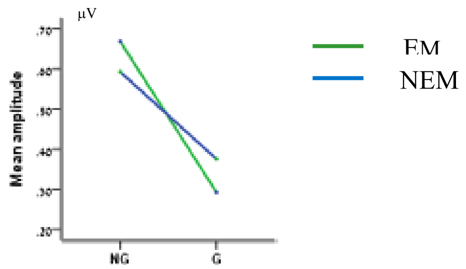


Figure 7: Mean amplitude of students in the four groups at S3 stage (500-700ms).

Figure 8 displays the scalp topography of participants in four groups at 500-700 ms as it is manifested in the answer verification stage (S3).

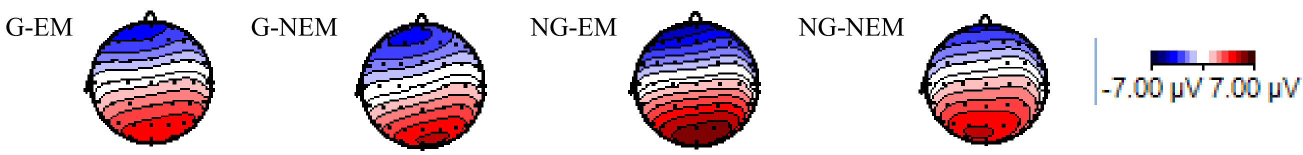


Figure 8: Scalp topography of participants in four experimental groups (500-700 ms).

From Figure 7 and Figure 8 we observe that G-EM participants have the lowest mean amplitude, whereas the NG-EM have the highest mean amplitude at the rear parts of the scalp. This suggests that G-EM devoted the least effort in solving area-related tasks. On the other hand, NG-EM struggled to solve this kind of tasks.

CONCLUSIONS

The present study investigates the differences in brain activity in gifted versus non-gifted and excelling versus non-excelling male adolescents when performing area-related tasks involving transition from a mathematical object to its property.

Behavioural data of the study demonstrated that the accuracy of gifted students was significantly higher than those of non-gifted. The accuracy of G-NEM individuals was similar to the accuracy of G-EM but much higher than that of NG-EM. Therefore, we can conclude that excellence does not affect accuracy in solving geometry area problems amongst G individuals. On the contrary, the RTc of G-NEM was the highest amongst the four groups and significantly differed from the RTc of G-EM. However, G-NEM participants were slightly slower but more accurate than NG-EM. Therefore, from the behavioural data it can be concluded that giftedness exerts a strong impact on performance in NEM participants. Thus, giftedness can compensate for the lack of excellence in mathematics in area-related problems.

Furthermore, electrophysiological data revealed significant differences between participant groups as it was manifested in latency and amplitude of early components (P1 and P2) and their lateralization. Thus, the analysis of early components suggests that early processing of the mathematical object amongst the participant groups was different.

The analysis of late potentials revealed that participants excelling in mathematics had higher overall mean activity (RMS) than their non-excelling counterparts. However, further pairwise comparisons detected that this difference was significant only amongst NG participants. The RMS measure was similar between EM and NEM amongst G participants. Similar findings were obtained from analysis of electrical activity at six predefined electrode sites during the S1 stage. At these sites the mean amplitude of NG-EM participants was higher than their NG-NEM counterparts. On the other hand, the mean amplitude of G-NEM participants was slightly different from that of their G-EM counterparts. The significant difference in mean amplitudes between EM and NEM participants was found only for NG students. This difference was most prominent at the middle posterior (PM) electrode site. Therefore, we can conclude that G-NEM and G-EM participants process the stage of introducing a situation (geometric figure with shaded area) and the stage of question presentation in the same way. This leads us to the conclusion that in area-related problems excellence loses its influence amongst the gifted participants.

The electrophysiological results demonstrated that in time period 500-700 ms at answer verification stage (i.e., S3) gifted students have lower overall mean amplitudes for correct responses at six predefined electrode sites. That is, they seem to exhibit more efficient brain activation during this task (e.g., Neubauer & Fink, 2009). The lowest electrical activity at the S3 stage was amongst G-EM participants, whereas the NG-EM had the highest electrical activity. This indicates that that NG-EM invested a great deal of cognitive resources in order to verify the given answer.

The present study provides evidence that different levels of giftedness as well as different levels of EM are reflected in the amount of cortical activation and in the behavioral measures. The electrophysiological data can provide a level of measurement and analysis that is difficult to approach by behavioral means. This may be relevant for conducting educational interventions for individuals with different abilities skills and for evaluating the effects of these interventions.

AKNOWLEDGEMENT

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MATHEMATICALLY ABLE BUT UNDERACHIEVING IN SCHOOL MATHEMATICS

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Teacher observation has shown that some pupils achieve very high on the Kangaroo Competition test (KC) but very low on the Swedish National test in Mathematics (SNM). This study will investigate the number of pupils who have high achievement scores on the KC (top 10%) but low achievement scores on the SNM (bottom 50%). Individual results on the SNM given in grade 6 (age 12) will be compared to results on the KC given in grade 7; concerning approximately 700 individuals. Results will give an example of the quantity of mathematically able pupils who underachieve in School Mathematics in Sweden. Data interpretation will connect this study to international research concerning mathematical abilities and mathematical achievement among mathematically able pupils.

BACKGROUND

Research stresses that early identification of giftedness, stimulation and support are important to prevent underachievement (Seeley, 2004). Sweden therefore has to support and stimulate those pupils who are mathematically able in early ages to prevent underachievement. TIMMS show less high achieving students in Sweden (Skolverket, 2009).

The Swedish curricula claim that teaching in mathematics should give pupils opportunity to develop mathematical abilities mentioned in the curricula (Skolverket, 2011). The purpose of the SNM is to measure abilities according to the Swedish curricula connected to different topics in mathematics; it aims to give the teachers support in equal and fair assessment and grading (Skolverket, 2012).

Mathematical abilities discussed in international research can often be connected to Krutetskii (Krutetskii, 1976). Krutetskiis definition can be used to analyse the problems in the KC (Pettersson, 2011). The purpose with the Kangaroo Competition test is to awake curiosity and inclination to learn mathematics (Nationellt centrum för matematikutbildning, 2013).

However, are those two ways of measuring mathematical abilities compatible? What if a pupil achieves high on the KC but low on the SNM, that pupil may attain low grades in mathematics and may not continue to study advanced mathematics.

So far in Sweden no research has investigated additional methods to find mathematically able pupils. Therefore this study will investigate if mathematically able pupils who underachieve in the Swedish school system can be made visible through the KC. These pupils could with accurate support and stimulation continue to study mathematics, technology or science at university advanced level; people who are needed for society in the future.

AIM AND METHOD

The aim of the study is to investigate the number of pupils who are high achievers on one mathematical test, in this case KC, but are not high achievers on the SNM. The SNM has a large impact on pupils' grades in mathematics, which in turn will guide pupils in their future choice of education. Thus there is a risk that the Swedish school system loses mathematically able pupils who with accurate support and stimulation could have had a successful education within mathematics, technology or science. The results will imply a discussion about differences in achievement in the two tests. The discussion will be framed by the how mathematical abilities are defined and measured in different contexts (the Swedish curricula vs. Krutetskii).

Individual results of the SNM given in grade 6 (age 12) will be compared to results of KC given in grade 7. The study covers a whole municipality (approximately 700 individuals). Pupils´ results will be compared relatively to each other, which will give ranked data. Data will be analysed with non-parametric tests using SPSS. Data interpretation will connect this study to international research concerning mathematical abilities and mathematical achievement for mathematically able pupils.

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CAN WE JUST ADD LIKE THAT?

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The aim of this qualitative study is to analyse the mathematical reasoning of a group of teacher education students, and ask if their mathematical reasoning can be categorised as creative mathematical reasoning. The students were not selected for mathematical giftedness or excellence. After an introduction, the research question is formulated and the methodology indicated. Finally, an excerpt from the transcripts of the study is given with a discussion and initial findings indicated.

Keywords The Four-C Model · Imitative reasoning · Creative mathematically founded reasoning · Dialogical approach.

INTRODUCTION

The Four-C Model of Creativity proposed by Beghetto and Kaufman (2009) recognizes the discoveries created by students when learning something as personal novelty, and thus to be creativity at a personal level. Lithner (2008) uses the terms imitative reasoning (IR) and creative mathematically founded reasoning (CMR). The basic idea here is that rote learning reasoning is imitative reasoning (IR), while the opposite type of reasoning (CMR) is creative and mathematically founded, meaning that the reasoning has personal novelty, is plausible and anchored in the mathematics of the given problem.

Research question:

How can the mathematical reasoning of teacher education students, not selected for mathematical giftedness or excellence, be described as creative mathematically founded reasoning?

METHODOLOGY

A group of teacher education students were given the following sequence:

$$0, 4, 10, 18, 28, 40\ldots$$

They were asked to find an explicit expression for the n'th term a_n of the sequence. I recorded the group working on this problem on video and prepared transcripts based on the recording. The transcripts will be analysed using the theoretical framework of Lithner (2008). To analyse the collaborative work in small groups I will follow the dialogical approach found in Bjuland (2004). Initial findings indicate that some of the reasoning of teacher education students is not imitative and hence creative mathematically founded reasoning, where the creativity is on the personal level.

THE STUDY

To solve the problem the students had to find the sum: $2 + 3 + \dots + n$. The students were familiar with the triangular numbers, but found that something was missing as indicated in the following transcripts.

1 Student 3: We are missing 1,

2 Student 2: we are missing 1, yes if we add,

3 Student 1: add 1 to each side,

4 Student 2: we have to add 2 … 2 … 2 times 1 … to both sides, because we have the number two …. Yes, if we try that, add 2 times 1, then you get an plus 2 times 1 equals 2, and then we get 1 plus 2 plus 3 plus 4 plus … plus n.

5 Student 1: yes don't we?

6 Student 2: yes,

7 Student 3: can we just add like that?

The students add 2×1 to both sides of the equation $a_n = 2(2 + 3 + \dots + n)$ which gives the equation $a_n + 2\times 1 = 2(1 + 2 + 3 + \dots + n)$. Since they are familiar with the triangular numbers, the problem is solved. Student 1 suggests in line 3 that they should add 1 to each side of the equation. In line 4, student 2 suggests that they should try to add 2×1 to both sides of the equation. This means that they make their own choices and try out their own ideas, which indicates that their reasoning is not imitative. Student 3 in line 7 asks if one can just add like that. One way to interpret this question might be that student 3 does not understand equations and the fact that one can add to both sides of an equation, but for student 3, the problem is rather that 1 is missing as indicated in line 1. Another way to interpret the question is that the idea used by the students to solve the problem, has personal novelty to student 3. The transcripts indicate that the mathematical reasoning of the students is not imitative and hence creative mathematical reasoning (CMR).

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PROPOSING A THEORETICAL FRAMEWORK FOR STUDYING MATHEMATICAL EXCELLENCE

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The excellence phenomenon has been fascinating mankind since ever. In particular, excellence in mathematics is an intriguing thing – why some are capable of a math performance at excellent level, and some aren't? What do this very restrict group do differently from everyone else that sets them apart? What contextual variables, personal characteristics and math specific abilities need to be present to allow the potential to do math at top level? These are some of the questions that can arise in the researcher's mind when trying to unravel this phenomenon. But the very different factors involved and the connections among them may constitute a difficulty. This poster is proposing a theoretical framework that attempts to give consistency to this array of concepts and the links between them.

OUR RESEARCH

We are attempting to clarify the components of mathematics excellence in university students, and to characterize their math reasoning. Relating to math reasoning, in particular, our interest lies in perceiving their creativity. To attempt to shed light on those issues we will invite some excellent mathematics university students to participate in an empirical study.

Besides those more practical aims we have another goal: to propose a theoretical model for studying mathematical excellence. To do so more effectively we intend to work in two phases. Firstly, we've examined the existing literature on the field and we developed a model (the one that can be seen here) based on previous research; Then, after conducting our empirical study – which isn't done yet – and both, understand the influence of the different external variables that affect the participants performance, and studying their inner qualities, we intend to go back to the initial model and refine it in the way that better fits the observed components and procedures. This later version of the model will be proposed to the research community to be used in future mathematical excellence research.

EXISTING MODELS

Among the existing theoretical models intended to the comprehension and development of talent and superior performance, there are two main perspectives. Some models emphasize the natural abilities while associating some personal traits and contextual factors, while others privilege the importance of long deliberate practice and self-monitoring.

OUR FRAMEWORK

Due to the diversity of the constituents of excellence and the different views on it, we think it's worthy to build a framework that takes in consideration the different beliefs and domains relevant to this phenomenon, and consequently allows a holistic approach on math excellence understanding. Therefore we suggest an umbrella framework, as seen in figure 1.

By observing the diagram, the central role of motivation is evident: as part of the needed general personality traits (other valuable traits are, e. g., intellectual curiosity, positive attitude towards math,…), as a couple together with talent that can lead to knowledge and also providing the necessary drive to engage in deliberate practice. We see the conjunction of abilities specific to mathematics (e.g., ability to generalize and to approach a problem in different ways, mathematical memory and intuition,…) and general personality traits as a potential to talent. There are considered personal catalysts in the form of self-regulation mechanisms (e.g., strategy choice, self-evaluation, adaptations,…) and social catalysts as a response to the combination of talent and motivation.

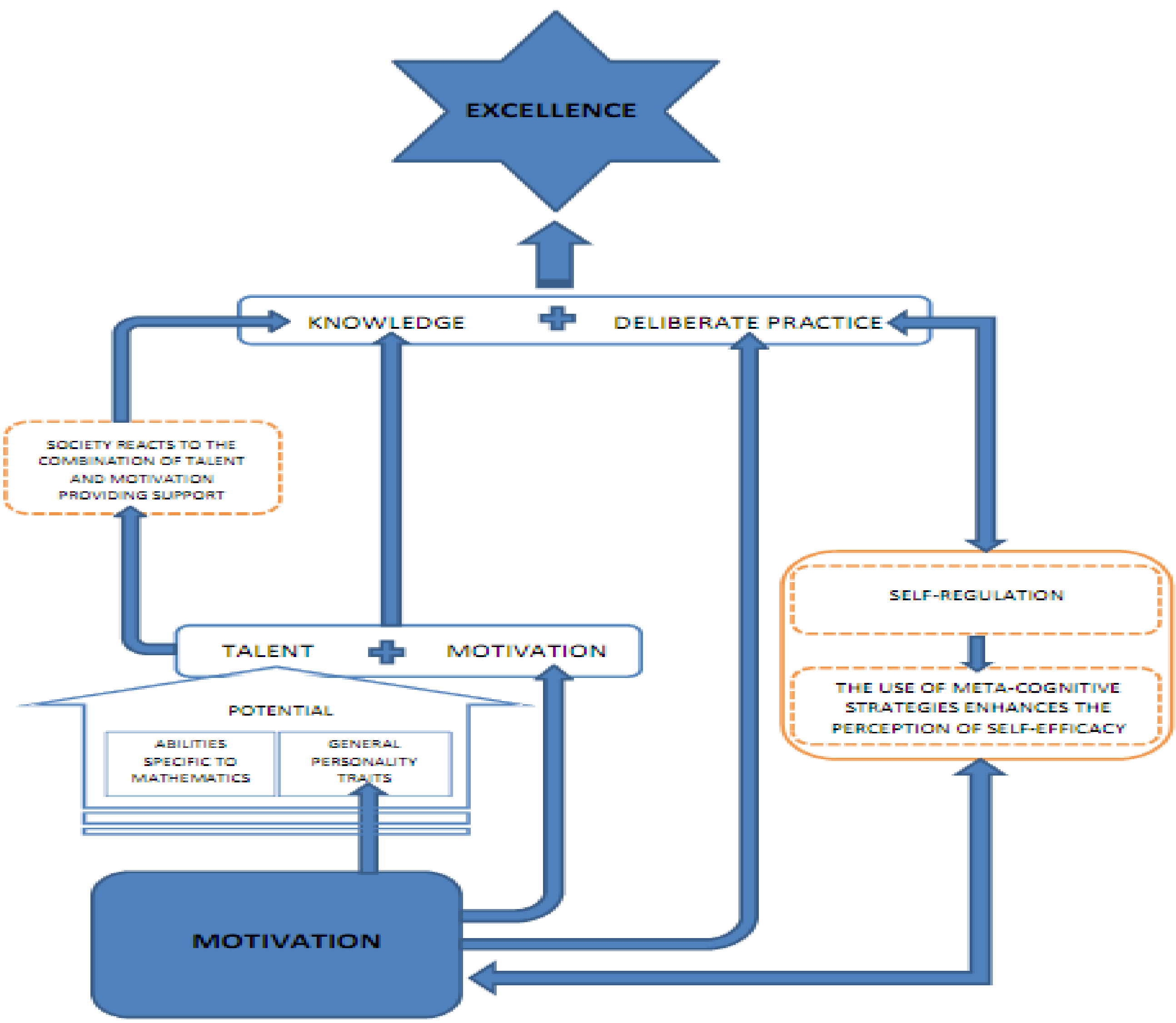

Figure 1: Purposed model of the concepts involved in math excellence

CREATIVITY AND CRITICAL THINKING IN SOLVING NON-ROUTINE PROBLEMS AMONG TALENTED STUDENTS

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The aim of this pilot study is to examine the extent of creativity and critical thinking in solving non-routine problems at a reasonable level, among students in the Mathematically Talented Youth Program. The study was conducted to answer following research question: What was the degree of creativity and critical thinking skills in problem solving among students who attended the Mathematically Talented Youth Program?

BACKGROUND

In general, creativity is defined as "the cognitive skill of proposing a solution to a problem or making something useful or novel from ordinary" (Hwang, Chen, Dung, & Yang, 2007). Mathematical creativity in school mathematics is usually connected with problem solving or problem posing (Silver, 1997).

Modern literature has provided a range of definitions of varying specificity in an attempt to describe and characterize what "Critical Thinking" is (Innabi & Sheikh, 2007). To cite just a few examples, McPeck (1981) defines it as the "skills and dispositions" necessary to "appropriately use reflective skepticism, " while Paul (1999) offers a more extensive description as "thinking that enables judgment, is based on criteria, corrects itself, and is context-sensitive."

The purpose of the current study is to analyze connections between creative and critical thinking. I present here shortly the study method and initial results.

METHOD

The questionnaire was handed out to three classes of eight-grade students in their second year in the Mathematically Talented Youth Program in central Israel. There was no time limit for the students to solve the problem. The students were encouraged to write their explanations on how they came up with the solution and to provide justifications of their thinking. All their written responses were first analyzed qualitatively to identify the types of reasoning used to solve the problem. The number of correct responses for each type of reasoning was calculated.

RESULT

The majority of the subjects (40.4%) were found to be at the most basic level of creativity and critical thinking. Students who developed a method from a given situation (17.5%) were found to be at the second level of creativity and critical thinking. Students who knew how to relate to the given figures in non-pre-defined manner and used their proofs in their solutions (15.8%) of the subjects were found to be at the third level of creativity. A high degree (64.1%) of clarity was found in these subjects' solutions: the arguments that they provided tended to be simple and straightforward. During the analysis some answers (26.3%) were categorized as "other" because they did not fit to any of the other categories.

Skills	**Skill of Creative thinking**	**Skills of Critical thinking**
Level 1: ***Comparison***	Create new or original criteria comparison	Analyzing the characteristics of the objects of comparison
Level 2: ***Raising points of view***	Developing the ability to step into the shoes of others and see things from a new angle	Justification and evaluation of points of view
Level 3: ***Argument***	Raising a variety of reasons that can support the claim	Taking a detailed personal position

Table 1: Categories of Creativity & Critical Thinking

DISCUSSION

It comes to no surprise that creative thinking is more inherent among students who exhibit mathematical accuracy and fluency, especially in the context of working with non-routine and novel mathematical tasks, requiring them to pose original and meaningful solutions (Binder, 1996). Students relied heavily on prerequisite knowledge and although presented their solutions with high clarity, they failed to express simplicity, structure, and cleverness in their solutions. Although the research literature on talented students support the notion that talented students are more creative, no specific high level of creativity and critically was found in this segment of the population.

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG8: AFFECT AND MATHEMATICAL THINKING

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Keywords: *Beliefs; motivation; attitudes; security; identity; goals; uncertainty orientation; problem solving*

INTRODUCTION

In the field of mathematics education, there has been an increasing interest in affect within the last fifteen years (Pantziara, Di Martino, Wæge, & Schloeglmann, 2011). At Cerme 8, the 25 participants of the Working Group 8 contributed to an inspiring and motivating discussion: 22 research papers and 3 poster presentations were initially submitted to the group and 17 papers were finally presented and discussed during the conference. In the proceedings, 17 papers and one poster presentation are included, each of which investigated some specific aspect of the complex world of affect.

We scheduled five of the seven conference sessions based on the themes of the presented papers: 1) Teachers' beliefs and self efficacy beliefs, 2) Teachers' beliefs and attitudes, 3) Teachers' and students' goals, security and identity, 4) Motivation, and 5) Students' emotions, beliefs and attitudes, and problem solving. Each participant's paper was presented and discussed within the whole group. Further discussion was encouraged by discussants who took up the role of a responder to the single paper in order to stimulate interaction. The five sessions were organized in such a way that enables the discussion to go beyond the scope of an individual paper to refine the conceptual frameworks of affect and to focus on its dynamic nature.

In the last two sessions, further discussion was stimulated in small sub-groups in order to explicate the relation between new concepts in the field and more traditional ones, to discuss the role of methodology in the investigation of affective issues, and to trace new directions of the research on affect. In particular, the discussion was inspired by the following questions: (a) How are the new affective constructs developed and what is their relationship to the more traditional constructs in the field? (b) How do the instruments and the context influence students' and teachers' affect? How can we reduce this influence? (c) Which directions of research on mathematics-related affect have not yet been explored sufficiently?

The structure of the affective domain

Much of the discussion in this affect group has focused on conceptual frameworks and terminology. The discussion was grounded in the graphic representation of the conceptual field of affect first introduced by Op't Eynde in Cerme 5 (Hannula, Op 't

Eynde, Schloeglmann & Wedege, 2007). The figure identifies three main conceptual categories (cognition, motivation and affect) and their partial overlap while it positions different affective constructs in relation to each other and in relation to the three main conceptual categories. This figure also identifies the socio-constructivist perspective on learning characterized by its focus on the situatedness of learning and problem solving and by the close interaction between (meta)cognitive, motivational and affective factors in students' learning.

The discussion was also initiated by Hannula's (2011) framework which focuses on the need to deepen our knowledge of the structure and dynamics in the affective domain. The framework can be seen as a meta-theoretical foundation for research on mathematics-related affect since it helps to identify similarities and differences between studies in the field, and it is probably useful for relating a variety of theories to each other. The most important notions in this framework are: (a) a distinction between trait and state-aspects of affect; (b) perceiving emotions, cognition and motivation in a synergistic relationship and (c) the identification of biological, psychological and social levels of affect.

Teachers' beliefs about mathematics and its teaching and learning were investigated extensively in the group (Arslan & Isiksal, Fauskanger & Mosvold, Pantziara & Philippou, Mosvold et al., Erens & Eichler). Revealing the long lasting discussion about the definitions and characteristics of beliefs, authors provided some of the more commonly used terms related to the terminology of beliefs like beliefs systems, conceptions, considerations, personal convictions, identity, knowledge and values. They also discussed categorizations of mathematics teachers' beliefs about the nature of mathematics, mathematics teaching and mathematics learning. Specifically, Fauskanger and Mosvold investigated the ways in which focused discussions based on Mathematical Knowledge for Teaching (MKT) items can be used to tap into teachers' beliefs about aspects of MKT. Pantziara, Karamanou and Philippou's study was concerned with teachers' beliefs and knowledge related to the Cyprus mathematics curriculum reform, while Erens and Eichler explored teachers' beliefs concerning the calculus domain to understand their teaching practices. Mosvold, Fauskanger, Bjuland and Jakobsen used two approaches of content analysis in order to learn more about pre-service teachers' beliefs from transcripts of focus-group discussions prior to field practice.

In relation to beliefs, teachers' self-efficacy beliefs were also introduced in the group by Arslan and Isiksal as well as Sarac and Aslan-Tutak. These authors used Bandura's (1997) framework to define self-efficacy beliefs as individuals' judgments regarding the capability of a specific behaviour. Sarac and Aslan-Tutak shared with the group the process of investigating teachers' trigonometry teaching efficacy and of categorizing them in terms of their efficacy levels. Arslan and Isiksal revealed the effects of an elective origami course on preservice teachers' beliefs and perceived self-efficacy beliefs in using origami in mathematics education.

Fuentes Rivera and Gómez-Chacón explored teachers' attitudes towards mathematics as a first approach to researching affective factors in the teaching and learning process within the Telesecundaria educative subsystem. They define attitudes as evaluative predispositions (negative or positive) that determine the personal intentions and influences on the behavior.

Other concepts related to affect were introduced by participants (Liljedahl, Charalambous & Rowland, Sánchez Aguilar et al., Gebremichael). Liljedahl provided a taxonomy of five goals that teachers own when they come to professional development programs and discussed with the group how teachers move from one goal category to another while being engaged in these programs. Charalambous and Rowland introduced a new concept to the group, the feeling of security, which mathematicians can draw from mathematics. They perceived security in terms of Riemann's (1970) four types of fear which correspond to four types of personal needs that are organized into two opposing pairs: (a) fear of *assimilation* versus fear of *isolation* and (b) fear of *change* versus fear of *stagnation*. Identity was discussed in the group as it serves to explain what makes a person feel like an able mathematics student and get involved and engaged in mathematical activities. The term was used by Sánchez Aguilar et al. in their study of factors that motivated Mexican female students to choose mathematics as a career. Ethiopian students' perceptions of the relevance of mathematics to their learning goals was described and analyzed by Gebremichael using the Engeström model (Cole & Engeström, 1993). The model depicts the activity system, which involves the interaction between the subject (students) and the object (motives, goals, learning school subjects including mathematics and material resources) mediated by tools and artifacts. As discussed, this interaction is also mediated by rules, the division of labor, and the community.

Based on the work done since Cerme 5, Wæge and Pantziara presented an overview of five motivational families of social cognitive constructs, identifying similarities and differences between self determination theory and achievement goal frameworks. Reviewing the research on the relationship between teachers' practice in the mathematics classroom and students' motivation, they revealed overlapping issues concerning teachers' practices that promote students' motivation in mathematics. Enhancing students' motivation was further discussed by Edwards and Deacon who explored the influence of classroom grouping on students' motivation. Specifically, they revealed the influence of grouping based on close friendships, and friendships by association on students' motivations to engage with mathematics. Di Martino and Zan focused on students' negative emotions towards mathematics, in particular by tracing students' fear of mathematics. Analysing students' narratives, by giving voice to the students about their relationship with mathematics, they revealed the origin of fear of mathematics. Also in this direction, Ader and Erktin investigated student's coping with negative emotions when faced with difficulties in mathematics.

Students' mathematical confidence was explored by Jagals and Van der Walt. Specifically, the authors investigated sources of low and high forms of students' mathematics confidence revealed through social, psychological and intellectual influences. In the same vein, Carreira, Ferreira and Amado described the help seeking and enjoyment patterns reported by students participating in an inclusive mathematics competition. They concluded that help seeking and enjoyment seem to be problem-dependent and that students seek help mainly from teachers and family. Students' preferences for solving tasks with multiple solutions and modelling were further explored by Schukajlow and Krug. In their study, the authors introduced uncertainty orientation (Sorrentino & Roney, 1999) which describes a person's typical way of dealing with complexity, uncertainty and abundant information. The authors assessed students' preferences for solving tasks with multiple solutions and uncertainty orientation before and after a five-lesson teaching unit promoting modelling competency.

Methodologies for investigating affect

Various methodological approaches, both quantitative and qualitative, were applied to investigating the complex and multifaceted nature of affect. Some papers developed questionnaires for studying teachers' beliefs, self-efficacy beliefs and attitudes (Arslan & Isiksal, Sarac & Aslan-Tutak; Rivera & Gómez-Chacón). The development of such instruments is basically related to also defining the concepts and their relationships. Other papers (Pantziara et al.) adopted questionnaires to apply them in a different socio-cultural context, revealing differences in the different settings. Questionnaires were also applied to gather students' uncertainty orientation, their preference for tasks with multiple solutions (Schukajlow & Krug) and their enjoyment while they are involved in challenging problems (Carreira et al.)

Other studies used qualitative methods to approach their goals. Focus group interviews were used (Fauskanger & Mosvold and Mosvold et al.) to investigate teachers' epistemic beliefs assuming that these give more realistic accounts of what people think. Semi-structured interviews were used in the study by Erens and Eichler and the study by Charalambous and Rowland while structured interviews were used in the study by Jagals and Van der Walt. Di Martino and Zan used a narrative approach referring to students' essays about their experiences related to mathematics. This approach was introduced as an interpretative approach capturing students' affect towards mathematics by giving voice to the students.

The use of a multi-method approach was adopted by several papers as a fruitful way to explore the affective domain. A combination of questionnaires and interviews was used by several presenters (Sarac & Aslan-Tutac, Aguilar et al., Gebremichael, Edwards & Deacon). Liljedahl gathered data from field notes, interviews and survey data to investigate teachers' goals. An intervention was applied by Schukajlow and Krug and questionnaires were used to measure the results of the intervention on students' affective constructs.

In the discussion, we focused on how the instruments and the particular context in which the study is conducted influence students' and teachers' affect and in what ways we can reduce this influence. Comparative studies were addressed and the issues of language and culture were further elaborated.

Discussion and further considerations

In our discussion, the development and clarification of the concepts and instruments in the domain remained an important focus. How do the studies presented in the group contribute to the theoretical frameworks and the refinement of existing affective constructs? In this discussion, the input of a combination of qualitative and quantitative methods was stressed. In particular, the methodological approach proposed by Mosvold and Fauskanger seemed promising to give deeper insight into teachers' beliefs. Erens and Eichler's methodology traced teachers' central and peripheral beliefs related to the teaching of calculus. Pantziara et al. applied a methodology for exploring dimensions related to teachers' inquiry and traditional beliefs while in Sarac and Aslan-Tutak's study an alternative methodology was employed as a more efficient mean to investigating teacher's efficacy beliefs. In the discussion of beliefs and efficacy beliefs, the robustness of these constructs was also considered. Are teachers' beliefs and efficacy beliefs stable across different mathematical domains (e.g. geometry, calculus), different social contexts (e.g. national and school level), and different populations (e.g. pre-service teachers, new teachers, in-service teachers)?

In an attempt to produce unity in the language and constructs of the affective domain, the second focus in the group concerned the relation between new constructs (e.g. security and uncertainty orientation) and more traditional ones (beliefs, attitudes, emotions, identity). What is the relation between mathematicians' security (Charalambous & Rowland), their mathematical identity, and their beliefs? How are students' uncertainty orientation (Schukajlow & Krug), beliefs and motivation related? Is uncertainty orientation part of identity?

Another focus of the group was to identify and interpret different affective constructs in relation to individual and social levels. In his study, Gebremichael adopted a social perspective in order to investigate students' perceptions of the relevance of mathematics to their learning goals whilst Di Martino and Zan relied more on psychological frameworks and applied a more in-depth analysis in identifying students' origins of fear of mathematics. Other studies (e.g Sánchez Aguilar et al.; Jagals & van Der Walt) attributed the origins of female students' attraction to mathematics-related careers and to students' confidence on both individual and social levels. In the same vein, a number of studies in the group highlighted the contribution of both individual and social characteristics to students' affective constructs like motivation (e.g. Edwards and Deacon; Carreira et al.; Schukajlow & Krug).

A focal point in the discussion was also the relation between different constructs in the affective domain and their connection to other areas in the realm of mathematics education. The complex nature of these relations was also discussed. In Liljedahl's study, teachers' goals may express what they believe and, conversely, beliefs presuppose and incorporate goals. In the study by Di Martino and Zan, the deep bidirectional interplay between emotions and cognition is stressed. The relation between students' motivation and achievement in mathematical areas like problem solving and modelling was incorporated in other studies presented in the group (e.g. Carreira et al., and Schukajlow & Krug).

Ultimately, the concluding remarks of the group included the necessity to deepen our knowledge of structure and dynamics in the affective domain by investigating further the theoretical framework proposed by Hannula (2011). Comparative studies on affect were also initiated by group participants. In addition, the comparison of studies investigating constructs related to different theoretical perspectives in a domain, for example motivation, was also suggested. Lastly strong longitudinal studies on affect were also welcomed.

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RESEARCH PAPERS OF THE GROUP

Arslan Okan & Işiksal Mine. The effect of the origami course on preservice teachers' beliefs and perceived self-efficacy beliefs towards using origami in mathematics education.

Carreira Susana, Tomás Ferreira Rosa, & Amado Nélia. Young students solving challenging mathematical problems in an inclusive competition: enjoyment vis-à-vis help-seeking.

Charalampous Eleni & Rowland Tim. Mathematics security and the individual.

Di Martino Pietro & Zan Rosetta. Where does fear of maths come from? Beyond the purely emotional.

Edwards Julie-Ann & Deacon Debra. Classroom groupings for mathematics learning: the impact of friendships on motivation.

Erens Ralf & Eichler Andreas. Reconstructing teachers´ beliefs on calculus.

Fauskanger Janne & Mosvold Reidar. Analyzing focused discussions based on MKT items to learn about teachers' beliefs.

Fuentes Rivera Maricela & Gómez-Chacón Inés. Attitudes towards mathematics of teachers in service of Telesecundaria: an exploratory study.

Gebremichael Tamiru Andualem. Ethiopian preparatory students' perceptions of the relevance of mathematics to learning goals.

Jagals Divan & van der Walt Marthie. Mathematics confidence: reflections on problem-solving experiences.

Liljedahl Peter. What teachers want out of professional learning opportunities: a taxonomy.

Mosvold Reidar, Fauskanger Janne, Bjuland Raymond & Jakobsen Arne. Using content analysis to investigate student teachers' beliefs about pupils.

Pantziara Marilena, Karamanou Marianna & Philippou George. Teachers' beliefs and knowledge related to the Cyprus Mathematics Curriculum reform.

Sánchez Aguilar Mario, Romo Vázquez Avenilde, Rosas Mendoza Alejandro, Molina Zavaleta Juan Gabriel, & Castañeda Alonso Apolo. Factors motivating the choice of mathematics as a career among Mexican female students.

Sarac Ayse & Aslan-Tutak Fatma. Investigating teachers' trigonometry teaching efficacy.

Schukajlow Stanislaw & Krug André. Uncertainty orientation, preference for solving task with multiple solutions and modelling.

Wæge Kjersti & Pantziara Marilena. Students' motivation and teachers' practices in the mathematics classroom.

POSTER PRESENTATION

Ader Engin & Erktin Emine. Handling negative emotions in learning mathematics.

THE EFFECT OF THE ORIGAMI COURSE ON PRESERVICE TEACHERS' BELIEFS AND PERCEIVED SELF-EFFICACY BELIEFS TOWARDS USING ORIGAMI IN MATHEMATICS EDUCATION

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Origami began to capture more attention in mathematics education literature because of its mathematically beneficial uses. This characteristic of origami affected the teacher education programs and some universities began to offer elective origami courses for preservice elementary mathematics teachers. In the current study, the effects of the elective origami course on preservice teachers' beliefs and perceived self-efficacy beliefs in using origami in mathematics education are investigated. Paired sample t-test results revealed that, there is not a significant change in the beliefs of preservice teachers but there is a statistically significant increase in preservice teachers' perceived self-efficacy beliefs with large effect size.

Key words: *origami in mathematics, beliefs, perceived self-efficacy beliefs*

Origami, which enables to fold various models from paper, is known as the Japanese art of paper folding (Franco, 1999). Although origami originated as a hobby more than 1200 years ago (Tuğrul & Kavici, 2002), in recent years, it has become a commonly used instruction tool for mathematics lessons (Boakes, 2009).

Origami could be used to promote the geometry knowledge of students since in the folding process, geometric principles are used and moreover, it is possible to fold two and three dimensional geometric models from paper (Cipoletti & Wilson, 2004; Georgeson, 2011). Although the most known application of origami in mathematics education is geometry teaching, origami could be used for the fields of algebra (Georgeson, 2011; Higginson & Colgan, 2001); calculus (Wares, 2011) and the list could be extended. In addition to the possible usage fields of origami in mathematics education, origami also helps to gain some skills for mathematics education. For instance, paper folding exercises would lead to an improvement on students' spatial visualization skills which are accepted as an important skill in mathematics education (Boakes, 2009). Furthermore, it helps to gain mathematical problem solving ability and using mathematical terms in the folding process would improve the mathematical language of students (Cipoletti & Wilson, 2004; Robichaux & Rodrigue, 2003).

Possible benefits of origami for mathematics education have affected some countries' mathematics education programs and Turkey is one of these countries. The Ministry of National Education (MoNE, 2009) defines origami as an instruction method which has various mathematical benefits such as improving mathematical problem solving skills, geometry knowledge and spatial visualization skills. In accordance with this view, origami activities to be used in mathematics education are given a place in the national mathematics education programs. However, treatment effects of origami were investigated in the national context with little research. In one of these studies, Kavici (2005) investigated the effect of origami exercises on preschool children's mathematical abilities and visual perception and it was found that children who had origami exercises for 11 weeks had significantly higher scores on mathematical ability and visual perception tests when compared with the children in the control group. In another study, Çakmak (2009) investigated the effect of origami on 4^{th}, 5^{th} and 6^{th} grade students' spatial ability. According to the results, she concluded that origami exercises significantly improved students' spatial visualization skills. Similar to the findings of Kavici (2005) and Çakmak (2009), in the study of Akan-Sağsöz (2008) it was found that 6^{th} grade students who had origami based mathematics instruction performed significantly higher on the test about fractions than the control group students who had not such instruction. Although research studies related with origami are limited in number, these studies found results in favor of using origami in mathematics education.

Consistent with the place of origami in the national curriculum of Turkey and the results obtained in the origami related research, some universities began to offer elective origami courses for preservice elementary mathematics teachers. In these courses, preservice teachers learn how to use origami effectively in mathematics lessons and experience how to overcome difficulties which may occur during paper folding activity. Although origami takes part in mathematics education programs and teacher education programs, it has not been studied much in terms of affective issues in the accessible literature. However, studies regarding the affective issues have an important place in mathematics education (Hannula, 2011). Among a wide range of affective issues, beliefs are attributed as an important component of mathematics education (Philipp, 2007). In spite of the consensus on the importance of studying beliefs, there is not a single belief definition in the literature (Pajares, 1992; Philipp, 2007). In the current study, the belief definition of Richardson (1996) was used as basis: "Psychologically held understandings, premises, or propositions about the world that are thought to be true" (p.103). Therefore, in its simple definition, beliefs towards using origami in mathematics education refer to individual considerations regarding the use of origami in mathematics education that are thought to be true. Although it is difficult to claim that there is a linear, unidirectional relationship between beliefs and behavior, it is generally accepted that beliefs shape predispositions towards behavior (Philipp, 2007). Therefore, investigating preservice teachers' beliefs could give a valuable view on their teaching decisions and thus, have a great importance (Pajares, 1992; Timmerman, 2004). In accordance with these views, it is believed that investigating beliefs of preservice teachers towards using origami in mathematics education and the effect of elective origami courses on their beliefs would lead to gain some insights regarding their future origami related teaching decisions.

In addition to beliefs, research on specific types of beliefs such as self-efficacy beliefs, provides important educational benefits (Pajares, 1992). Perceived self-efficacy beliefs refer to individual judgments regarding the capability of a specific behavior (Bandura, 1997). According to Bandura (1997), there are four basic sources of perceived self-efficacy beliefs: mastery experiences, vicarious experiences, verbal persuasion, physiological and affective states. Mastery experiences are personal interpretations on one's own experiences whereas vicarious experiences are interpretations on others' observed behaviors (Joet, Usher & Bressoux, 2011). Furthermore, perceived self-efficacy beliefs are affected from verbal persuasion which refers to verbal feedbacks that one gets from the people in their close environment and also affected from physiological and affective states which refer to one's physical and emotional situation at that moment (Bandura, 1997). In these four sources, mastery experiences are interpreted as the most influential source. In the literature, perceived self-efficacy beliefs' influence on future behavior and perseverance on doing that behavior are widely mentioned (e.g., Bandura, 1997; Brand & Wilkins, 2007; Joet et al.,2011). Therefore, investigating preservice teachers' perceived self-efficacy beliefs towards using origami in mathematics education could give some clues on their origami related possible teaching decisions. Furthermore, examining the change on perceived self-efficacy beliefs after participating in an origami course would enable the interpretation of the effect of these courses.

In brief, origami is seen as an effective way of teaching mathematics and when the issue is mathematics education, affective factors are interpreted as important components of mathematics education. Therefore, the current study aims to investigate the effect of the elective origami course on preservice teachers' beliefs and perceived self-efficacy beliefs towards using origami in mathematics education. For that purpose, the following research questions are investigated with the current study:

- Is there a statistically significant effect of the elective origami course on preservice teachers' beliefs in terms of the benefits and limitations of using origami in mathematics education?
- Is there a statistically significant effect of the elective origami course on preservice teachers' perceived self-efficacy beliefs regarding the use of origami in mathematics education?

METHOD

Research Design and Sample of the Study

The current study aims to examine the effect of elective origami course on preservice elementary mathematics teachers' beliefs and perceived self-efficacy beliefs towards using origami in mathematics education. In accordance with this purpose, one group pre-posttest experimental research was selected as the research design for this study.

In Turkey, there are five universities which offer elective origami courses for preservice elementary mathematics teachers and participants of this study were selected from one of these universities. In that course, preservice teachers are firstly trained to fold origami models and learn origami diagrams. Subsequently, they learn how to choose appropriate origami model and relate folding steps to mathematical concepts. Furthermore, they learn how to change the language of origami into the language of mathematics. Subsequently, teacher candidates prepare origami activities to teach different mathematical concepts and carry out these activities in that course in order to gain experience in origami based mathematics activities. There were 36 possible teacher candidates who participated in that origami course but data for the current study was collected from 33 preservice teachers since some were absent during the data collection process. Most of the participants were in the third grade of their university education and there were just three participants from 1^{st}, 2^{nd} and 4^{th} grades. Furthermore, in accordance with the general profile of education faculties in Turkey, most of the participants were female and there were just 5 male preservice teachers in that course.

Data Collection Instruments and Process

In the data collection process, Origami in Mathematics Education Belief Scale (OMEBS) and Origami in Mathematics Education Self Efficacy Scale (OMESS) which were developed by Arslan (2012) were used as the data collection instruments. OMEBS is composed of two dimensions called benefits of origami in mathematics education with 19 items and limitations of using origami in mathematics education with 7 items. All the items in OMEBS are in 6 point Likert format ranging from strongly disagree to strongly agree. Sample items from OMEBS are given in Table 1.

Table 1: Sample Items from OMEBS

Item	Dimension
Origami is beneficial to make some abstract mathematical concepts more concrete	Benefits of origami in mathematics education
Origami makes easy to teach geometrical concepts	Benefits of origami in mathematics education

Table 1: Sample Items from OMEBS (continued)

Origami cannot be used for mathematics lessons in crowded classes	Limitations of using origami in mathematics education
Using mathematical terms during origami activity helps to improve mathematical language of students	Benefits of origami in mathematics education
It takes long time to use origami activities in mathematics lessons	Limitations of using origami in mathematics education

OMESS is composed of one dimension named as perceived self-efficacy in using origami in mathematics education. There are eight items in 9 point Likert format ranging from insufficient to very sufficient and sample items from OMESS are given in Table 2.

Table 2: Sample Items from OMESS

How well do you feel…
to plan a mathematics lesson in which origami activities will be used?
to use mathematical language during origami activities?
to find solutions to the problems of students while relating origami activity to mathematics topics?

For both of the scales, detailed literature review and expert opinions during the item development process; moreover, high factor loadings in the exploratory factor analysis and good fit indices such as RMSEA, GFI, CFI in the confirmatory factor analysis were presented as the evidences for the validity of the scales (Arslan, 2012). In addition to the validity evidences, Cronbach alpha values were investigated in order to check the internal consistency of the data obtained in pretest and posttest. As can be seen in Table 3, for both of the scales, Cronbach alpha values were calculated above .80 in pretest and posttest. These values could be interpreted as good internal consistency reliability for the data collection instruments with this sample (Pallant, 2007).

Table 3: Reliability Analysis for Data Collection Instruments

	OMEBS		OMESS	
	Pretest	Posttest	Pretest	Posttest
Cronbach Alpha	0.862	0.945	0.958	0.937

OMEBS and OMESS were administered simultaneously to 33 preservice teachers as a pretest before the beginning of the elective origami course in the spring term of 2011-2012 academic year. After 14 weeks, scales were administered again as a posttest. Data obtained from these scales were analyzed with paired sample t-test in order to investigate whether there is a significant change in beliefs and perceived self-efficacy beliefs of teacher candidates.

RESULTS

Before conducting paired sample t-test, preliminary analyses were performed and one extreme point was removed from the data file. Furthermore, these analyses indicated that, there is no violation of the assumptions of paired sample t-test for the current data set. Therefore, it was decided that, the data file for the current study is appropriate to perform paired sample t-test.

Beliefs of Preservice Teachers in Using Origami in Mathematics Education

Preservice teachers' beliefs regarding the use of origami in mathematics education were investigated with the administration of OMEBS. As indicated in the methodology part, OMEBS has two dimensions which are benefits of origami in mathematics education and limitations of using origami in mathematics education. Paired sample t-test was performed for both dimensions of OMEBS. According to the results given in Table 4, there was an increase in teacher candidates' beliefs regarding the benefits of using origami in mathematics education from pretest (M=4.80, SD=0.50) to posttest (M=5.02, SD=0.81). However, this difference was not statistically significant, $t(31)=1.601$, $p=0.119$.

Table 4: Paired Sample T-test Results for the First Dimension of OMEBS

	Paired Differences			t	df	Sig. (2 tailed)
	Mean	Std. Deviation	Std. Error Mean			
postbenefit-prebenefit	,221	,780	,138	1,601	31	,119

Contradictory to the beliefs regarding the benefits of origami, preservice teachers beliefs' regarding the limitations of using origami in mathematics education decreased from pretest (M=3.48, SD=0.85) to posttest (M=3.34, SD=0.81). However, paired sample t-test results, which are given in Table 5, indicated that this difference was not statistically significant, $t(31)=-0.727$, $p=0.473$.

Table 5: Paired Sample T-test Results for the Second Dimension of OMEBS

	Paired Differences			t	df	Sig. (2 tailed)
	Mean	Std. Deviation	Std. Error Mean			
postlimitation-prelimitation	-,138	1,076	,190	-,727	31	,473

To sum up, according to the paired sample t-test results, it is possible to conclude that there is not a statistically significant effect of the elective origami course on preservice teachers' beliefs regarding the benefits and limitations of using origami in mathematics education.

The Impact of the Origami Course on Perceived Self-Efficacy Beliefs towards Using Origami in Mathematics Education

Data obtained from the administration of OMESS in pretest and posttest was analyzed with paired sample t-test. Analysis results, which are given in Table 6, revealed that, there is a statistically significant increase in preservice teachers' perceived self-efficacy beliefs in using origami in mathematics education from pretest (M=3.02, SD=1.76) to posttest (M=6.59, SD=1.56), $t(31)$=8.682, p=0.000. Therefore, it is possible to conclude that preservice teachers' perceived self-efficacy beliefs regarding the use of origami in mathematics education significantly increased after participating in the elective origami course. Furthermore, the eta squared statistic was calculated as 0.71 and it could be interpreted as a large effect size (Green & Salkind, 1997).

Table 6: Paired Sample T-test Results for the Data Obtained from OMESS

	Paired Differences			t	df	Sig. (2 tailed)
	Mean	Std. Deviation	Std. Error Mean			
postefficacy-preefficacy	3,572	2,327	,411	8,682	31	,000

DISCUSSION AND CONCLUSION

In the current study, it was seen that, preservice teachers held similar beliefs towards using origami in mathematics education before and after the course. They strongly believe that origami has various mathematical benefits such as making abstract mathematical concepts more concrete, improving mathematical language, spatial ability and proof ability. In addition to the mathematical benefits, they strongly believe that origami has instructional benefits to make mathematics lessons more effective. Preservice teachers' beliefs of this kind are also consistent with the studies in the literature (e.g., Boakes, 2009; Cipoletti & Wilson, 2004; Higginson & Colgan, 2001). Apart from the mathematical and instructional benefits of origami, preservice teachers do not believe that origami has various limitations to be used in mathematics lessons. They do not believe that, it is difficult to use origami activities in crowded classes and to plan origami based mathematics lessons.

Preservice teachers' beliefs regarding the benefits and limitations of using origami in mathematics education remained relatively stable after the elective origami course and no statistically significant effect of the course on participants' origami related beliefs was found. The reason might derive from the fact that, preservice teachers held already positive beliefs towards using origami in mathematics education before the course and kept their beliefs after the course. Another reason for this result might be derived from the nature of ‘belief’ since beliefs are generally resistant to change (Philipp, 2007). Therefore, one term course might not have a significant effect on their origami related beliefs.

When the issue is the effect of the origami course on preservice teachers’ origami related beliefs, it should be noted that they did not have a chance to use origami in real teaching environments during the course. In the literature, it is generally accepted that beliefs are influenced from past experiences (Kagan, 1992), but at the same time, beliefs are also shaped by current experiences (Philipp, 2007). Therefore, apart from elective origami course, if teacher candidates had a chance to use origami activities for mathematics lessons in elementary schools, their beliefs regarding the use of origami in mathematics education might be changed. Based on origami related teaching experience, they might have different beliefs regarding the benefits or limitations of using origami in mathematics lessons. Therefore, in addition to the elective course, enabling preservice teachers to use origami in real teaching environments might be beneficial to interpret their beliefs regarding using origami in mathematics education.

Contradictory to the beliefs of preservice teachers, there was a statistically significant increase in their perceived self-efficacy belief levels in using origami in mathematics education. Furthermore, eta squared indicated that there was a large effect size which is interpreted as so important in social sciences (Pallant, 2007).

In the literature it is mentioned that origami based mathematics lessons would be very beneficial if teacher relates origami folding steps to mathematical concepts effectively (Georgeson, 2011). Therefore, teachers should be knowledgeable on how to plan and organize origami activities for mathematics lessons (Cipoletti & Wilson, 2004). Before participating in the origami course, preservice teachers felt slightly efficient to use origami in mathematics education. However, after the course, they saw themselves quite efficient to plan and carry out origami activities for mathematics lessons. Therefore, it is possible to conclude that preservice teachers were not familiar with the own nature of origami based lessons before the course and felt slightly competent to use origami in mathematics education. However, posttest results revealed that origami course was rich enough to enhance preservice teachers’ knowledge and efficacy regarding the use of origami in mathematics education. Although elective course improved their efficacy level, more mastery experiences are suggested by

Bandura (1997) in order to improve perceived self-efficacy belief levels. Therefore, as stated above, enabling preservice teacher to use origami activities for mathematics lessons in elementary schools during the elective origami course might improve their efficacy level and similarly, using origami in other courses such as teaching practice, method courses might lead to an improvement on efficacy beliefs.

To sum up, current study aimed to contribute to the literature by examining the effects of elective origami course on preservice teachers' beliefs and perceived self-efficacy beliefs towards using origami in mathematics education. However, when the issue is affective factors, longitudinal studies might be beneficial to gain further insight (Hannula, 2011). Furthermore, studies with more participants and research studies comparing different countries would be beneficial to fill the gap in the related literature.

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YOUNG STUDENTS SOLVING CHALLENGING MATHEMATICAL PROBLEMS IN AN INCLUSIVE COMPETITION: ENJOYMENT VIS-À-VIS HELP-SEEKING[1]

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In this paper, we intend to describe the help seeking and enjoyment patterns reported by the participants in an inclusive mathematics competition. Results suggest that they seek help from a variety of sources, mainly family and teachers, and that they enjoy much the problems in the competition; in addition, help seeking and enjoyment seem to be problem-dependent. Some questions for future research are raised.

Key words: mathematics competitions, problem solving, help seeking, enjoyment.

CONTEXT AND RESEARCH QUESTIONS

Inclusive mathematical competitions as motivational environments

Around the world, the number of regional, national and international mathematical competitions has visibly increased, taking many different forms, contents and durations, and targeting considerably wide groups of students in terms of age and mathematical ability levels. Some competitions target highly talented students, as the International Mathematics Olympiad. Others, like the Mathematical Kangaroo contest, have an inclusive nature, welcoming students with various degrees of aptitude for mathematical problem solving and endorsing feelings of fun, joy, pride and mathematics loving.

Experience indicates that the challenging and competitive nature of enrichment projects is associated with students' positive affect towards mathematics and the development of problem solving skills. In fact, empirical research has proved that students' participation in mathematics competitions has an influence on their motivation for learning mathematics, especially at younger ages. Furthermore, both very good students and those who show some difficulties in school mathematics benefit from participating in such activities beyond the classroom (Freiman & Vézina, 2006).

Based on the review of several studies, Freiman and Applebaum (2011) claim that the advantages of mathematical competitions are strongly connected to affective and emotional factors such as: student satisfaction, self-efficacy, and cooperative skills as well as infused love and interest for mathematics. The context of mathematical competitions is therefore reaching much further than the identification of mathematically gifted students. Inclusive competitions are becoming important sites where mathematics is presented as challenging, exciting, accessible to average students, socially and emotionally engaging, and close to the daily aspects of students' lives.

Online competitions: the setting and the questions addressed

SUB12 is a regional online mathematical problem solving competition supported by the University of Algarve. It has been running annually since 2005, and it is intended for 5th and 6th graders (ages 10-12). It is a web-based competition with two distinct phases: the *Qualifying* and the *Final*. The *Qualifying* phase develops entirely at distance through the website and consists of a set of ten problems, each posted every two weeks. Participants are asked to solve each problem, at home, at school, etc., and to submit their answers by e-mail or using the online form available at the webpage. They can use several resources, approach the problems in different ways, and resort to different representations. Participants have total freedom in presenting their solutions (handwritten and scanned, using the computer to create images or diagrams…). The explanation of the solution process is a fundamental requirement for a complete answer. Regardless of the level of mathematical sophistication, all the correct and complete answers are equally valued.

Feedback is always provided to the participants by a team of senior mathematics teachers. Feedback is formative and encouraging, offering suggestions when needed to help overcoming obstacles, or recognizing successful answers. Students are allowed to submit revised solutions as many times as needed within the respective deadline. Help seeking is approved and even encouraged during the *Qualifying* phase given the inclusive nature of the competition.

Usually 10% to 15% of the total initial participants reach the *Final* phase as they have to submit correct and complete answers to, at least, eight of the ten qualifying problems. The *Final* is held at the university campus. Family members and also teachers accompany the participants but the latter compete individually and on-site.

Acknowledging the role of affective variables within inclusive mathematical competitions, we intend to get a picture of the SUB12 participants' enjoyment and help-seeking behaviour in relation to the problems proposed. In particular, we consider the following questions: (1) What is the significance of the help provided by the several partners that participants in SUB12 can resort to during the competition? (What is the quantitative dimension of this support? What is the most salient source of help and to what extent is the presence of external aid constant or variable over time?); (2) How do participants in SUB12 express themselves in relation to their higher or lower enjoyment with the various problems posed? Is it possible to detect situations of greater preference or dislike towards the problems released? and (3) What trends can be identified combining these dimensions?

THEORETICAL PERSPECTIVES

Challenging mathematical problems

In any learning context, performance must be understood in light of a strong bond between affect and cognition and recent theorizations are not reducible to the identification of causal links between affect and cognition. Understanding affect is now moving beyond the usual way of seeing it as the other side of cognition; instead, affect is viewed as a part of thinking: “Affect influences thinking, just as thinking influences affect. The two interact” (Walshaw & Brown, 2012, p. 186). In fact,

> Affect is as central to understanding the character of educational experiences as are motivation and cognition. Furthermore, affective, motivational and cognitive processes, while they can be separated conceptually and empirically, are interdependent in the ongoing experience of students. (Ainley, 2006, p. 391)

The very notion of challenge reflects how affect must be integrated with the cognitive aspects involved in it. By definition, a challenge presupposes an element of difficulty, raises the need to overcome an obstacle. Barbeau (2009) has thoroughly elaborated on the idea of challenges in mathematics education. His view stresses that mathematical challenges deliberately incite its recipient to attempt a resolution. A good challenge is one for which the individual has the necessary mathematical apparatus but is required to deal with it in innovative ways, thus implying the feeling of being intellectually alive and able to share some of the thrill of devising new approaches as mathematicians usually do. Such mathematical challenges are usually seen by the students as different from the regular problem solving activities in school and, even when perceived as difficult to grapple, the challenges boost feelings of enjoyment (Jones & Simons, 1999, 2000).

The various problems proposed in SUB12 briefly and succinctly describe a context-framed situation, casting a well-defined question, but they are expected to be seen by participants as challenges, which amounts to believing that students feel inwardly compelled to solving them. Therefore we propose a delicate difference between the idea of mathematical problem and the concept of challenging mathematical problem. A *mathematical problem*, usually conceived as a situation from which the initial and the final states are known but the process to move from the first to the last is not immediately available through mathematical techniques and reasoning, has its grounds on the cognitive components of the problem solving activity. In turn, a *challenging mathematical problem* includes a strong affective appeal by involving curiosity, imagination, inventiveness and creativity, therefore resulting in an interesting and enjoyable problem not necessarily easy do deal with or to solve (Freiman, Kadijevich, Kuntz, Pozdnyakov & Stedøy, 2009).

The idea of moderate challenge

Research has stressed the need for balance in the degree of challenging questions (or problems) posed to students and the idea of *moderate challenge* has come to the fore (Turner & Meyer, 2004). The propensity to perform a task seems to decrease in two situations: when one’s expectations about the probability of success are very high (too easy task) or very low (too difficult task). The preference for challenges rests on the situations in which the expected success is around 77%.

The environments in which help seeking is regarded as natural are consistent with those that value moderate challenges. Such challenges seem to be ideal for persuading students to try and encouraging them to explain alternative strategies, evaluate approaches and appreciate multiple possible solutions. Thus, the use of moderate challenges has much to gain when associated with features that are typical of contexts which value challenge. One of such features is viewing help seeking as legitimate and another is pressing for explanations and accountability for thinking. These two aspects are clearly present in SUB12 – not only is help seeking explicitly encouraged as reporting the solution process is required – and they are precisely the two essential categories that may describe challenge supporting practices, according to Turner and Meyer (2004). Interestingly it is reported that students who are given moderate challenges tend to reveal lower avoidance of help seeking.

On the occurrence of students' help seeking and help avoidance

When a participant seeks help, can we assume that the problem was actually seen by the participant as a challenge? If not, why? We may suspect that the degree of difficulty of the problem was probably too high, leading to the need of seeking help. Moreover, the very act of asking for help can compromise the challenging character of the task in the eyes of the participants since the sense of achievement, namely if it is equated with performance demonstration, may not be as full – the credit for having answered well goes not just to the participant but is shared with others.

Help seeking has been researched from the perspective of a behavioural type of self-regulation and has received increasing attention for its role in the learning process. Zusho and Barnett (2011) highlight some of the developments on the conceptualization of help seeking and help avoidance, namely stressing the social connotations of help seeking in tune with the costs involved: being perceived as needy and admitting failure or incapacity to accomplish a task. There is evidence that self-regulated and confident learners are more likely to display adaptive profiles of help seeking, which means looking for instrumental help: where the reasons to find help are the wish to learn and to understand the material, as opposed to a shortcut to get a task completed. In addition, empirical evidence suggests that help avoidance is a consequence of performance expectancy as help seeking is seen as a threat to self-worth and self-efficacy. In fact, low achievers tend to perceive greater threat in help seeking and therefore report higher levels of help avoidance; reversely, students with higher perceptions of cognitive competence show lower levels of help avoidance.

Furthermore, help seeking is adjustable to contextual factors, and patterns of help seeking are consonant with a caring, supportive and exploratory learning environment. This is also related to students' perception of moderate challenge where conditions of support and accountability for understanding are nourished and where a preference for challenging activities goes together with engagement and enjoyment. In such environments, students' preference for solving problems on their own may rise and help seeking becomes closer to seeking clues rather than answers (Zusho & Barnett, 2011).

On the relationship between challenges and affect

The meaning of moderate difficulty is not universal because different people perceive the same task differently. Even the same person can experience different levels of challenge in a certain task depending on having freely decided to engage with it or on having been required to perform it (Schweinle, Meyer & Turner, 2006). Although acknowledging the relativeness of moderate challenge, there are indicators supporting the claim that targeting moderate challenge is a favorable condition to develop positive affect. But other conditions should surround moderate challenge: a social environment that supports enjoyment, self-confidence and value in mathematics – a number of characteristics that SUB12 fulfills, all of which are grounded on the kind of feedback provided, which is a key element, amongst others, that contribute to the inclusive nature of the competition. Inclusiveness is part of the global aim of fostering enjoyment (by easing frustration, giving positive reinforcement, encouraging persistence, valuing cooperation more that competition – as reflected in the opportunity for children to participate in small groups).

> Optimal levels of challenge, coupled with affective and motivational support, can provide contexts most supportive of students' feelings of enjoyment, efficacy, and value in mathematics (Schweinle, Meyer & Turner, 2006, p. 289)

There seems to be a highly interactive relationship between positive affect, challenge, and value attributed to mathematics (and more precisely to mathematical tasks). In particular, we can refer to two kinds of threats to students' perceived ability, both possibly combined: the difficulty of the task and the need to seek for help. The two threats, especially when combined, can generate negative affect, which is observed in our study in terms of participants' level of enjoyment regarding different problems proposed in the course of SUB12.

METHODOLOGY

In this paper, we address two dimensions of 5th graders' participation in SUB12 concerning affect and emotion: help seeking and enjoyment in solving the challenging mathematical problems presented throughout the *Qualifying* phase. Data were collected through the participants' answers to a mini-questionnaire consisting of two multiple-choice questions included in the online form available on the webpage to submit the answer to each of the problems posted. The two items were straightforward and the answers were given by choosing a single option: 1) I solved the problem with the help of: a) Teacher; b) Family; c) Friends; d) SUB12; e) Nobody; and 2) I enjoyed the problem: a) A lot; b) So-so; c) Little.

Answering the questionnaire was mandatory when participants chose to use the online form to send their problem solution. However, the right not to respond was assured – alternatively to using the online form, participants could choose to send their answers directly to the SUB12's address using their personal e-mail. All participants in the competition are required to have an e-mail account.

The number of respondents to the questionnaire is slightly below half the number of participants enrolled and basically corresponds to those who used the online form to submit their answers. It is important to notice that as the competition unfolds the number of participants decreases (by attrition) and so does the number of respondents to the questionnaire (there were 469 replies at the beginning and 151 at the end, as shown in Tables 1 and 2).

Our analysis of the data is guided by the research questions and mainly intends to look for patterns that may help to understand the significance of help seeking and the level of enjoyment reported in each problem, and also possible associations between those. Since the mini-questionnaire was only launched on problem #2, the data consist of answers referring to problems #2 to #10. Our approach is mainly descriptive, based on the number of answers and percentages regarding each option per problem, by looking at such values across the series of problems.

DATA ANALYSIS

A first indication on how participants report on help seeking is quantitatively expressed for each of the nine problems in Table 1. Similarly, Table 2 displays the distribution of answers to the item on enjoyment per problem.

Prob. 2 - Help					
Nobody	Friends	Teacher	Family	Sub12	TOTAL
190	42	126	102	9	469
40,5%	9,0%	26,9%	21,7%	1,9%	100,0%
Prob. 3 - Help					
Nobody	Friends	Teacher	Family	Sub12	TOTAL
235	48	85	90	5	463
50,8%	10,4%	18,4%	19,4%	1,1%	100,0%
Prob. 4 - Help					
Nobody	Friends	Teacher	Family	Sub12	TOTAL
150	31	98	75	6	360
41,7%	8,6%	27,2%	20,8%	1,7%	100,0%
Prob. 5 - Help					
Nobody	Friends	Teacher	Family	Sub12	TOTAL
75	27	57	91	3	253
29,6%	10,7%	22,5%	36,0%	1,2%	100,0%
Prob. 6 - Help					
Nobody	Friends	Teacher	Family	Sub12	TOTAL
99	19	47	57	4	226
43,8%	8,4%	20,8%	25,2%	1,8%	100,0%
Prob. 7 - Help					
Nobody	Friends	Teacher	Family	Sub12	TOTAL
120	11	52	48	5	236
50,8%	4,7%	22,0%	20,3%	2,1%	100,0%
Prob. 8 - Help					
Nobody	Friends	Teacher	Family	Sub12	TOTAL
95	23	59	54	1	232
40,9%	9,9%	25,4%	23,3%	0,4%	100,0%
Prob. 9 - Help					
Nobody	Friends	Teacher	Family	Sub12	TOTAL
44	20	66	65	0	195
22,6%	10,3%	33,8%	33,3%	0,0%	100,0%
Prob. 10 - Help					
Nobody	Friends	Teacher	Family	Sub12	TOTAL
52	16	38	43	2	151
34,4%	10,6%	25,2%	28,5%	1,3%	100,0%

Table 1: Help seeking in each problem

Prob. 2 - Enjoyment			
A lot	So-so	Little	TOTAL
295	155	19	469
62,9%	33,0%	4,1%	100,00%
Prob. 3 - Enjoyment			
A lot	So-so	Little	TOTAL
340	115	8	463
73,4%	24,8%	1,7%	100,00%
Prob. 4 - Enjoyment			
A lot	So-so	Little	TOTAL
245	97	18	360
68,1%	26,9%	5,0%	100,00%
Prob. 5 - Enjoyment			
A lot	So-so	Little	TOTAL
136	109	8	253
53,8%	43,1%	3,2%	100,00%
Prob. 6 - Enjoyment			
A lot	So-so	Little	TOTAL
147	66	13	226
65,0%	29,2%	5,8%	100,00%
Prob. 7 - Enjoyment			
A lot	So-so	Little	TOTAL
169	55	12	236
71,6%	23,3%	5,1%	100,00%
Prob. 8 - Enjoyment			
A lot	So-so	Little	TOTAL
140	81	11	232
60,3%	34,9%	4,7%	100,00%
Prob. 9 - Enjoyment			
A lot	So-so	Little	TOTAL
114	58	23	195
58,5%	29,7%	11,8%	100,00%
Prob. 10 - Enjoyment			
A lot	So-so	Little	TOTAL
99	51	1	151
65,6%	33,8%	0,7%	100,00%

Table 2: Enjoyment in each problem

In what concerns the distribution of reported help seeking, the data show that it has a visible expression in all problems. Participants in SUB12 widely vary in their abilities to solve mathematical problems and this is an inherent feature of the inclusive nature of the competition. In general, a large percentage of these participants openly declared to have received help (help seeking was only slightly below 50% in problems #3 and #7). It is also important to recall that help seeking is actually encouraged and stimulated by the organization and explicitly stated at the webpage. Thus, fifth graders (the ones entering the competition for the first time) showed to be willing to seek for help in their problem solving activity during the *Qualifying* phase.

The two major sources of help, family members and teachers, reveal similar percentages, although there are pronounced differences in some of the problems (problems that are more related to daily life may become good opportunities for family members to help their children). It may well be that the same teacher is a source of help to a large number of participants – there are some teachers who reported in interviews that they gave support to their students throughout the competition, and so the help of teachers may cover a lot of participants. The third source of help is the participants' friends, although the amount of inputs from friends is clearly smaller. Finally, as students report it, the SUB12 (i.e. the organization of the competition to whom the participants contact by e-mail) is a residual source of help. However, all participants who sent an initial wrong or incomplete answer to each problem received feedback from the organization in order to reformulate their answer, and eventually this resulted in a correct solution. There is a discrepancy between the percentage of participants who claimed to have received help from SUB12 and the number of cases that actually succeeded after receiving feedback from the organization. A question then comes up: is it true that participants only recognize to have received help when they explicitly asked for it? The participants may not perceive the feedback constantly provided by SUB12 as actual help since such feedback is offered without being requested; it arises as a reaction (and is possibly seen as corrective stance rather than a means to help improving the work already done) to the answer sent by the participants.

The case of Luís (L) is illustrative of the referred situation (Figure 1). Luís sent a first answer to problem #5 which was not completely correct and stated that he had help from family and enjoyed more or less the problem. The SUB12 offered him feedback and some hints. He then reformulated his answer and resent it, claiming that nobody had helped him and maintaining the level of enjoyment. SUB12 answered again indicating a faulty answer and giving more feedback. Finally, Luís got the solution right and resent it, confirming having had no help and a more or less enjoyment.

There are few cases in which participants explicitly address the SUB12 asking for help to start solving a problem. In such rare cases, the fact is that participants declare that they had help from SUB12, recognizing to have sought such help.

From:	L
Sent:	Saturday, 17 March 2012 16:45:20
To:	Sub12

Q1: Family
Q2: So-so

Answer: (…)

From:	SUB12 Campeonato
Sent:	Sunday, 18 March 2012 11:09:00
To:	L

It is almost correct. You know that each side of the closed table is 60 cm long. You know that each leaf is 30 cm wide. Thus the open table is 60 +30 +30, which is 120 cm, long. But notice that the cloth falls 10 cm on each side, when the table is open. So you have to calculate the length of the cloth, increasing these 10 cm on each side. And the width of the cloth, what is it?
We wait for your correct answer.

From:	L
Sent:	Sunday, 25 March 2012 19:40:52
To:	Sub12

Q1: Nobody
Q2: So-so

Answer: (…)

From:	SUB12 Campeonato
Sent:	Sunday, 25 March 2012 20:12:39
To:	L

Hi, L!
We received your new answer to problem 5. You have to think better on your resolution
It's not completely correct yet. Note that the table cloth fell 40 cm on each side with the table closed. And the table is square, so how will the cloth be?...
We wait for your correct answer.

From:	L
Sent:	Tuesday, 27 March 2012 18:25:19
To:	Sub12

Q1: Nobody
Q2: So-so

Answer: (…)

Figure 1: Sequence of e-mails between Luís and SUB12

In what concerns enjoyment, the overall manifested feeling shows a general positive emotion of participants in facing challenging mathematical problems. However it can be noticed that in problems #5 and #9 the number of answers stating "much enjoyment" is lower. At the same time, for these problems there is an increase in the number of answers indicating "so-so" and "little" enjoyment. These two challenges are, in a sense, the deviants within the category of the enjoyment felt and are also those in which participants report having sought more help (more than 70%).

Given the volume of help sought in problems #5 and #9, we can infer that these problems raised more difficulties than others to the participants. This complexity seems to be associated with less enjoyment in solving these problems. This greater difficulty for participants also seems to indicate that the challenge was higher and therefore such over-challenging problems led to less positive emotions, namely, lowering the feeling of enjoyment. The greater search for help in these two problems may also have contributed to a lower enjoyment. There may have been participants asking for help on these two problems who did not feel this need in others, which may have contributed to a sense of weakness and thus to lower enjoyment. The data may also suggest that enjoyment may be problem-dependent, which resonates with the way students perceive the value and interest of different tasks. On the other hand these problems relate to mathematical topics in which students typically have difficulties (geometry and fractions), which may also contribute to explain their help seeking behaviour. So, help seeking also appears to be problem-dependent.

DISCUSSION AND CONCLUSIONS

As research suggests help seeking is an important matter in any learning context, being even more relevant within an inclusive mathematical competition. Our data reveal that participants significantly seek for help. Help provided by different sources positively contributes to the success in the competition and students' sense of accomplishment; in addition it positively influences the quantity and diversity of students enrolled. Participants seek help mainly from two sources: teachers and family. This is a sign of a great family involvement alongside with a presence of the competition in the school environment. Further research should follow to better understand how students perceive help seeking according to the different available sources, in particular SUB12. This source of help is especially intriguing since it seems to be recognized as such only when participants specifically ask for it.

In general, participants do enjoy the challenging problems proposed in the competition, and we believe that most of the problems can be considered of moderate challenge. This seems to resonate with prior studies which indicate that the challenging and competitive nature of enrichment projects such as SUB12 is associated with a positive affect towards mathematics and the developing problem solving skills (Kenderov, Rejali, Bussi et al., 2009). The problems where the level of enjoyment lowers are precisely those for which participants sought most help. Furthermore enjoyment decrease and help seeking increase are located in two specific problems: enjoyment and help seeking are problem-dependent. We argue that the degree of challenge of such problems is higher in that they involve mathematical ideas that are typically difficult in school curriculum. Another question can be raised: is the decrease of enjoyment related to a higher degree of difficulty or is it associated with the need to ask for help?

NOTES

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MATHEMATICS SECURITY AND THE INDIVIDUAL

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This paper complements a report (Charalampous and Rowland, 2012) regarding the feeling of security which mathematicians can draw from mathematics. The focus of this report is the potential of mathematics to empower the individual, offering possibilities both for distinguishing and transcending one's individuality. The research involved 19 *adult mathematicians working in universities and schools in Greece. Semi-structured interviews revealed that the participants distinguished mathematics from other sciences, and valued both the intellectual abilities which it cultivates and its social contributions.*

Key words: security, fear, mathematics, individual

INTRODUCTION

It is generally acknowledged that affect plays an important role in mathematics education. The literature has focused largely on negative responses to mathematics and on constructs such as mathematics anxiety, fear of failure, and mathematics avoidance (Zan, Brown, Evans & Hannula, 2006). However, mathematics may also engender positive experiences. For example, Russell asserts that:

> The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry (Russell, 1919, p. 60).

The topic of positive experiences emerging from engagement with mathematics is under-investigated. This paper continues the discussion on the potential of mathematics to contribute to one's sense of security, as introduced by Charalampous and Rowland (2012). There we explored the concept of security, as it emerges from the relationship of mathematicians to mathematics. We conceptualised security as the absence of existential fears and investigated two pairs of opposing fears: *assimilation* versus *isolation*, and *change* versus *stagnation*. In the previous paper we presented the findings concerning the latter pair, whilst here we focus on the former.

In the next section we draw out conceptions of mathematics that could be appealing to certain individuals in terms of security, and elaborate on our conceptualisation of security for the purposes of this study. We then proceed with an account of our findings from interviews with a sample of mathematics professionals.

THEORETICAL FRAMEWORK

In this section, we frame our investigation in terms of the nature of mathematics itself, and of security as a psychological phenomenon.

The nature of mathematics

A wide range of perspectives about the nature of mathematics has evolved since early Greek civilisation up to the present time (Davis and Hersh, 1980, Friend, 2007). Plato introduced the idea that mathematical objects pre-exist, awaiting human discovery. Early modern philosophy (Descartes, Spinoza, Kant) lent support to this conviction privileging reason over the sense-experience (Hutchins, 1952). Faith in the power of reason culminated in the quest for secure foundations for mathematics, i.e. a system comprising a few axioms and deductive rules from which all mathematics could be deduced. This dream had to be abandoned when Gödel proved that any such system complex enough to include arithmetic is necessarily incomplete. The collapse of the formalist project led to the re-conceptualisation of mathematical practice as a human and unavoidably fallible enterprise (Hume, 1978; Lakatos, 1976). This view of mathematics as negotiable and consensual is emphasised in social constructivism as a philosophy of mathematics (Ernest, 1998). Admittedly, modern epistemologies-ontologies of mathematics seem to place the mathematician on shifting sand, but nevertheless give them agency in an unfolding mathematical story.

Security

Psychology has recognised the need of security in various ways (e.g. Maslow, 1970). In this research, the concept of security is framed in two steps: (i) by reference to dictionary definitions of security as 'freedom from fear or anxiety' (e.g. www.merriam-webster.com); (ii) a typology of fear due to Riemann (1970). Riemann proposed four types of fear which correspond to four types of personal need and are organised into two opposing pairs. The first pair concerns the individual, the second their environment over time. Humans must achieve a balance between their need (a) to exist both as individuals and as part of an entity that transcends their restricted individuality and (b) to lead a life marked both by stability and innovation. The associated fears are (a) fear of *assimilation* [our translation] versus fear of *isolation* and (b) fear of *change* versus fear of *stagnation*. The effort to achieve balance also suggests that people try to avoid the extremes of each pair, and is consistent with defining security as absence of fear. More specifically about the fears which are handled in the paper: fear of assimilation is connected with losing one's individuality, the need to be distinguished from others and define one's self against others; on the other hand, fear of isolation is connected to avoiding loneliness either physically by belonging to a group of people or metaphysically by participating in an ideal 'essence' elevated beyond the human level (e.g. the universe or God).

The following quote illustrates the relevance of security to mathematics education and the potential application of Riemann's framework in order to interpret this feeling. Here Mendick (2005, p.175) comments on the pleasure that Phil, a young mathematics student, draws from his engagement with mathematics.

> Phil finds a security in mathematics that enables him to construct himself as intellectually mature and as distant from his working-class, minority-ethnic self.

Phil's response can readily be construed as a response to fear of assimilation; through his engagement and success in mathematics, he positions himself as distinct and distinctive, in terms of his intellectual capacity and his distance from his origins.

METHODS

We explored the concept of security with an opportunity sample of 19 Greek adult mathematicians. We chose adults because, on the whole, they could be expected to be more articulate about their relationships with mathematics. However, their stories reveal that this relationship began when they where young. So we expect that the results would be pertinent to young persons, as indicated by the earlier reference to Mendick (2005).

Nine of the participants were in university positions and ten were teaching in secondary schools. Among the first group there were five applied mathematicians (Faidra, Paraskevi, Periklis, Themis, and Vasilis) and four theoretical mathematicians (Alvertos, Dimitris, Kleitos, and Sofoklis). Three of the teachers were teaching in lower secondary schools (Marios, Sokratis, and Stamatia), and the rest were teaching in a technical upper secondary school ((Nestoras, Avgoustis, Aris, Eleftheria, Fanis, Loukas, and Thodoris). All the participants had at least an undergraduate degree in mathematics, which is necessary for teaching mathematics beyond the primary level in Greece. Most of them had substantial professional experience (15 years or more); Themis, Vasilis, Sofoklis, Periklis, Faidra and Eleftheria had been in post between 2 and 8 years. The names used in the paper are pseudonyms.

One semi-structured interview was conducted by the first author with each participant. Four issues were considered: the participant's personal history regarding mathematics (e.g. Describe the history of your relationship to mathematics. Is there any memorable incident?); their views about mathematics (e.g. What do you think distinguishes mathematics from the other sciences?); the relevance of mathematics in everyday life (e.g. Would you say that mathematics has helped you deal with your everyday life?); and their feelings about mathematics (e.g. What do you enjoy (or not enjoy) in your involvement with mathematics?). Views about mathematics were included since the history of philosophy of mathematics provides ontological and epistemological links to security, e.g. the mathematical certainty of Platonism. While discussing the participant's relationship with mathematics in a general way, unconscious feelings surfaced, which might have been difficult to access directly (Rubin and Rubin, 1995).

The utterances were coded as relevant to one of Riemann's four types of fear (see above). Several cases of multi-coding arose. Multi-coding *across* pairs is indicative of contribution of the dimension of time (second pair) to one's sense of individuality (first pair); Multi-coding *within* pairs suggests the dialectical relationship between their components, which is built as the individual strives to reconcile the two opposites. For example, secure sense of group membership can liberate a person to explore and express their individuality. As the data were revisited again and again, a constant comparison method lead to the emergence of broad themes and related sub-themes, associated with each type of fear. The themes and sub-themes associated to *fear of isolation* are listed in Table 1, by way of illustration. Note that themes 1-4 emphasise aspects of mathematics and mathematical activity that have the potential to offer protection against isolation, while themes 5-7 suggest *interconnections* with the other three types of fear and themes 8-10 acknowledge *limitations* in safeguarding against fear of isolation.

Themes	**Sub-themes**
Omnipresence	mode of thought; tool of all sciences; historical continuity
Social contribution	applications; social empowerment
Communication	conciseness and precision; impartiality and pluralism; real life
Teaching	initiation; communication with children; accessibility
Assimilation	initiation and success; talent and omnipresence; tool for/foundation of all sciences
Change	historical continuity and reliability; precision and one reality; omnipresence and connectedness
Stagnation	isolation and diversity; creation and social contribution
Limitations to isolation	teaching; communication
Limitations to change	omnipresence
Limitations to stagnation	historical continuity

Table 1: Fear of isolation: Themes and sub-themes

FINDINGS

As mentioned earlier, this paper reports on our findings relating to the pair *assimilation-isolation*. Our report is restricted to those themes which are directly related to these types of fear. The participant’s views are organised under the corresponding themes starting with fear of *assimilation* (relevant codes: abilities, self-confirmation, mathematical self, superiority to other sciences) and continuing with fear of *isolation* (relevant codes: omnipresence, social contributions, communication, teaching).

Fear of assimilation

First, we report participants' views which indicate how mathematics gave them a sense of being different and distinct individuals. Mathematics enhanced the participants' feelings of uniqueness and self-awareness, and protected them against fear of *assimilation*.

Abilities

The participants perceived the mathematical mode of thought as superior to any other kind. For example:

> The mind of the humanities stops; the mathematical mind is unsatisfied if the answer remains concealed, it goes on a discovering spree (Stamatia).

This mode implied the possession of above-average intellectual abilities which had been further cultivated by mathematics and were applicable to everyday life.

> . . . the ability to think abstractly requires an above-average capacity (Themis)

> . . . mathematics makes you think more; you think more quickly, more easily; you handle some problems in a more correct way (Vasilis)

> . . . the way the [mathematician's] mind functions, is rarely restricted to mathematics (Sokratis).

These views suggest that mathematicians enjoy admirable and respectable abilities which can boost one's self-esteem.

> I have observed that people who in my opinion have considerable potential, have a good relationship with mathematics (Themis).

Self-confirmation

Successful display of the aforementioned intellectual capabilities bestows a feeling of self-confirmation intensified by the perception that mathematics is difficult. For example:

> . . . [solving a problem] is as if you put test yourself to the test and [the 'examiners'] say to you: 'You passed' (Avgoustis)

> Sometimes you may think that you've found an original solution, and even though many others may have found it before you, you secretly feel proud; for a moment, you believe that you are the only one that has found [it] (Marios)

> [Mathematics] wasn't like history. It presented a certain degree of difficulty; and probably, I felt more successful when I did well in mathematics, while I didn't think it was as important when I did well in history (Faidra).

Some participants asserted that mathematics had increased their self-confidence; mathematics was regarded as a shield against mistakes.

> [Mathematics] has made me calm, confident in expressing my opinion without the need to necessarily persuade others (Aris)
>
> You feel confident about the quality of your capabilities . . . I believe that [mathematics] makes you set more and more difficult goals . . . Mathematics protects you against falling prey to lies . . . it saves you from traps (Stamatia)
>
> Mathematicians have confidence in their opinions; it stems from the fact that [in mathematics] you can check the actions you take, see if they are verified (Avgoustis).

Solving problems, either in mathematics or in everyday life, provides a limitless source of self-confirmatory opportunities. Repetitive positive experiences of that kind lead to confidence in one's ability to manage problematic situations.

Mathematical self

Some participants integrated mathematics with their identity. On the one hand, they traced their involvement with mathematics to personal traits and on the other hand, they recognised that this involvement had led to further cultivation of these traits. For example:

> One reason, why I like mathematics, is its consistency and its rationality; if they suit to one's character, then one is attracted to mathematics (Nestoras)
>
> Mathematical thought is characterised by strictness, precision, methodicalness and these affect your character as a whole and consequently your everyday life (Thodoris)
>
> I like [mathematics] because it is in my nature to be rational and I like to do things systematically (Marios)
>
> It is in my DNA (Eleftheria).

Most of them had entered the mathematical adventure in an early age.

> I knew how to count before I went to school; I also knew about fractions. At preschool age, my father asked me how I could share two apples between three children. I couldn't think of a way, and I answered embarrassedly: "I don't know". Then my father asked me if I knew how to share one apple between three children and I answered: "yes, each child will take 1/3". "Then why can't you share two?" asked my father and I realised what the answer was" (Alvertos).

Mathematics was a means to self-fulfilment and a valuable companion on the journey of self-awareness.

> It is a work of art; you don't need anything more, you feel content (Dimitris)
>
> Look at the person who manages to discover something after considerable effort. He exclaims 'Eureka' . . . he feels happy; he feels complete because he has used his intellect (Stamatia)
>
> Mathematics empowers people; it is a great tool, which can help humans reflect on their actions . . . it also offers stimuli for introspection (Nestoras)

> Mathematics teaches you how to think rationally, how to order assumptions and draw the best possible inferences. Therefore, up to an extent, it bestows power; [it bestows] the ability of self-evaluation and self-criticism (Fanis)

> Doing mathematics is an intellectual activity which leads you to maturation (Nestoras).

These views imply that mathematical skills play a crucial role in the process of shaping a secure identity. Consequently, they are regarded as an indispensable part of this identity.

Superiority to other sciences

Finally, mathematics was believed to occupy a distinguished position in the constellation of sciences: it is independent, it is at the cutting edge of research and it is the foundation of all sciences.

> Mathematics has an admirable trait; it is self-contained in the sense that it is not required to know another science in order to understand it (Themis)

> Research in mathematics continues; [it] precedes [research in] other sciences and it causes admiration (Fanis)

> . . . [mathematics] is the basis of all sciences; it is behind anything you can imagine. Even philologists say that syntax is the mathematics of language" (Marios).

These views indicate ways in which mathematicians experience and benefit from the importance and high status of mathematics. All in all, doing mathematics was connected to their sense of worth and being valued.

Fear of isolation

Here we report the participants' views which indicate how mathematics enabled them to feel a part of a greater whole. Mathematics enabled the participants to bridge the gap between themselves as individuals and the external world, protecting them against fear of *isolation*.

Omnipresence

The participants believed that mathematics transcends the human being, both statically and dynamically, thus allowing mathematicians to be part of something greater. First, mathematics is all around us, either as the language which discloses the secrets of the universe, or as a mode of thought to tackle problems.

> . . . but no matter if it is music, or a building, or the forces we realise around us, or anything, everything is linked to mathematics (Aris)

> Mathematics functions without being seen . . . It is the language of nature; there is not a different world of mathematics (Fanis)

> . . . the mathematical mode of thought is involved in all sectors of life (Thodoris).

Second, mathematics is an adventure of the human intellect throughout history.

> . . . when you learn a theorem, it is as if you communicate with the person that proved it (Avgoustis)
>
> When I teach . . . I try to follow the route that the human intellect has followed (Dimitris)
>
> . . . if you approach [mathematics] enough you can see its associations with other sectors of life and you can understand that [it] is part of the civilisation (Nestoras).

Social contribution

The participants concluded that since mathematics is everywhere mathematicians are useful. They claimed that mathematical progress could be translated into social progress. Mathematical knowledge was perceived as power, not only at the individual but also at the social level.

> . . . the various technological instruments which are produced, for example, an x-ray computed scanner. What lies behind its function? Differential equations do (Nestoras)
>
> Even football involves mathematics . . . when I see the ball moving in a trajectory tangent to the ground I know that the opponent must fall in order to intercept it (Aris)
>
> . . . through mathematics we try to interpret what is going on around us and express it in a language that we understand (Sokratis)
>
> . . . if you know a field well, you can promote scientific knowledge [in this field], contribute something, and this can definitely be translated as power (Periklis).

Communication

> I'm attracted to mathematics by its concise, symbolic language . . . All around the world, mathematical language has the same meaning for mathematicians (Fanis).

Some participants claimed that use of the precise mathematical language cultivated communicational skills.

> Mathematics influences the way I think and the way I organize myself; it has made me able to express myself precisely, examine many perspectives simultaneously; in that sense it has influenced my life since communication is facilitated and misunderstandings are avoided (Themis)
>
> I came to love mathematics when . . . I realised that mathematical training had helped me in comprehending social situations (Stamatia)
>
> Two persons who understand mathematics communicate with one another much better (Avgoustis).

Mathematics was also regarded as an advocate of impartiality and pluralism.

> Mathematics perceives its nature as international. Discrimination related to colour, height or hair is absent from mathematics; it plays no role in the process of problem solving. If one is taught by mathematics, then one learns to be free of prejudice (Sokratis)
>
> Multiple dimensions . . . help you see through someone else's eyes . . . [someone else's opinion] can be considered as another dimension (Sokratis).

The participants expressed the opinion that mathematics can bring people together, facilitate communication and solve misunderstandings, creating a better world to live in.

> [Modelling all sciences in mathematics] is desirable . . . afterwards we will be able to code life, to communicate better and avoid failures and mistakes. Most human problems are the result of ignorance. When I am ignorant I make mistakes and the more ignorant I am the more ready I am to deify something (Loukas).

Teaching

Teaching is not restricted to mathematics. It is connected to fear of isolation mainly through the opportunities it offers to communicate with young people and to initiate them into a community of experts. Mathematics was believed to offer the additional advantage of being accessible and tangible.

> . . . [e]ven though many claim otherwise, mathematics is tangible; . . . if you demonstrate to a logical human being that mathematics is based on logic, then mathematics becomes tangible (Aris)

> . . . there is an infinite field of questions even for the simplest rules, there are many applications and many levels of difficulty (Themis).

After all

> . . . [d]ifficult things don't exist; we name difficult what we don't know, and the less we know it the more we try to make it look impressive (Paraskevi).

CONCLUSION

In this investigation we regarded feelings of security cultivated through engagement with mathematics. Security was conceptualised as relief from fear by means of a framework based on Riemann's (1970) suggestion of four types of fear organised in two opposing pairs. In this paper we reported on the pair *assimilation-isolation*. The findings provide evidence that mathematics can be used as a means to achieve balance between these opposing fears. As many times before in the history of the philosophy of mathematics, the participants endorsed the belief that mathematics has special attributes which set it apart from other sciences. Mathematics is at the same time the leader and the foundation of all sciences; it is the language through which we translate, comprehend and 'conquer' the world (Guillen, 1995). Moreover, the attributes of mathematics empower the individual, who develops a repertoire of intellectual capabilities such as deriving logical inferences, finding applicable solutions or communicating with precision and conciseness. Mathematics was the tool which allowed the participants to shape a strong and confident identity (offering relief from fear of assimilation) and to connect to the world (mitigating against fear of isolation).

These findings suggest possibilities for further research: both in the direction of other cultures, investigating similarities and differences, and in the direction of younger participants, including mathematics students, to capture the development of feelings of security through the development of one's relationship with mathematics.

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WHERE DOES FEAR OF MATHS COME FROM?

BEYOND THE PURELY EMOTIONAL

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Abstract: *Fear of mathematics is a widespread emotion that has many negative consequences in students: it is a possible factor of local failure (since it prevents the best use of one's competence and knowledge) but also a possible factor of global failure (since it might lead to students giving up any engagement with mathematics). Investigating the cognitive origin of this emotion is fundamental to prevent it and to overcome its negative consequences. In this study we try to understand this origin giving voice to the students, analysing students' narratives about their relationship with mathematics.*

Keywords: *fear of mathematics, cognitive origin of emotions, affect*

INTRODUCTION AND THEORETICAL FRAMEWORK

In the last two decades, the specific field of research on affect in mathematics education has grown recognizing four specific constructs: beliefs, emotions, attitudes (McLeod, 1992) and values (De Bellis & Goldin, 1999).

Scholars have underlined the role of the emotional component in mathematics learning processes (Zan, Brown, Evans & Hannula, 2006) and in particular in problem solving processes (McLeod & Adams, 1989). The deep interplay between emotions and cognition is bidirectional: emotions affect cognitive processing in several ways, for example biasing attention and memory and activating action tendencies (Hannula, 2002); conversely the emotional experience is the result of a combination of cognitive analyses and physiological responses (Mandler, 1984). According to Mandler, it is not the experience itself that causes emotion, but rather the interpretation (influenced by individual's beliefs) that one gives to the experience. Understanding these interpretations is therefore central to answering the question of *where emotions come from*, and then to preventing and overcoming negative emotions towards mathematics (as hatred, anxiety and fear). These emotions may become very stable and have very negative influence on mathematical school performance, but more in general cause math-phobia also in adult age (Tobias, 1978; Buxton, 1981). Schlöglmann (2002) tells about stories of oppressive nightmares that are recalled even in adulthood.

In a longitudinal research study aimed to characterize students' attitude towards mathematics through the analysis of students' narratives (Di Martino & Zan, 2010) we suggested a model for attitude characterized by three dimensions: emotional disposition toward mathematics, vision of mathematics and perceived competence in mathematics. In this study we found recurrent patterns in the stories characterized by failures: a particularly interesting case - because it may end up in a generalized rejection of mathematics - is the recurrent pattern with the emotional dimension involving *fear*.

Ortony Clore and Collins (1988), proposing a categorization of the basic emotions according to their cognitive source, classify fear as a valenced reaction to events, focusing on consequences for self, and “prospect-based”. With the latter term the scholars mean emotions that are characterized as reactions to the prospect of an event, or to the confirmation (or disconfirmation) of the prospect of an event. In particular, fear arises when the individual is displeased about the prospect of an undesirable event. The variables affecting the intensity of fear are the degree to which the event is undesirable and the likelihood of the event. Then the cognitive component (belief) intervenes in the perception of the event, in the evaluation of the consequences for self (hence in the eliciting of the emotion itself), and in the evaluation of the likelihood of the event (hence in the intensity).

In this contribution we try to understand *where fear of math comes from*, analyzing how students narrate this emotion in their autobiographical essays.

METHODOLOGY

Scholars have used many different approaches to studying affect and in particular emotions: in our theoretical researches about affective constructs and their influence in the process of teaching and learning mathematics, we needed an instrument consistent with an interpretative approach, capable of capturing students’ emotions, beliefs and attitudes towards mathematics, giving voice to the students through the possibility of talking about the aspects they considered relevant for their own experience with mathematics (Di Martino & Zan, 2010).

We recognized in the narrative approach the key to reach our methodological goals. The primary focus of a *narrative approach* is people's expressions of their experiences of life. These expressions emerge in particular from autobiographical narratives, where the narrator tends to paste fragments, introducing some causal links, not in a logical perspective but rather in a social, ethical and psychological one (Bruner, 1990). These links become a bridge between the writer’s beliefs and emotions, highlighting the psychologically central role they play for the narrator.

We claim that in order to grasp the cognitive source of emotion, this pasting process is more important than an objective report of one’s experience:

> There is as yet no known objective measures that can conclusively establish that a person is experiencing some particular emotion (…) In practice, however, this does not normally constitute a problem because we are willing to treat people’s reports of their emotions as valid (…) we evaluate them as being appropriate or inappropriate, or justifiable or unjustifiable, not as being true or false. (Ortony, Clore, Collins 1988, p. 9)

Within a National Project in Italy, we asked schools across Italy to participate in a study about the students' attitude towards mathematics. In particular, we asked school directors to help us in collecting students' autobiographical essay "Me and mathematics: my relationship with maths up to now".

Essays were anonymous, assigned and collected in the class not by the class mathematics teacher. The study involved a large sample of students from different school levels: 874 from primary school (grade 1-5), 368 from middle school (grade 6-8), 420 from high school (grade 9- 13). The three different typologies of Italian high school (Technical Institute, Vocational School and Lyceum) were all represented.

In the field of affect the need for social and anthropological approaches, i.e. for studying affect in its natural contexts, is particularly stressed, motivating the use of non-traditional methods, such as narratives (see for example Kaasila, Hannula, Laine & Pehkonen, 2006).

As regards the analysis, we referred to Lieblich, Tuval-Mashiach and Zilber (1998), who identify two main independent dimensions in the process of reading and analysing a narrative: holistic versus categorical, and content versus form. The combination of these two dimensions produces four modes of reading a narrative: holistic – content; holistic – form; categorical – content ('content analysis'); categorical – form.

Our analysis of students' narratives about fear used two of these modes of analysis in two subsequent steps: categorical - content mode to select the stories characterized by *fear of math*; holistic – content mode to analyze in the selected essays the view of math associated with fear, and the *causal* links introduced by pupils (for example through utterances such as 'I am afraid *because*...').

RESULTS AND DISCUSSION

In the collected essays students often report (almost one student out of six) fear of mathematics, using several tokens that refer to fear (the most recurrent are anxiety, terror, nightmare, distress, worry, fright).

Our data shows that fear associated with mathematics is nearly always fear of *not being able*. We might therefore claim that the undesirable event that triggers the emotion is failure in mathematics:

> "When Miss Sandra explains something new, I get goose bumps, because I'm afraid I won't be able to do the exercises" 4P.106[1] - grade 4

[1] In the excerpts, the first number refers to the class level, the letter refers to the school level (Primary / Middle / High), the last number indicates the progressive numbering of the essay within the category.

> “Problems are the thing I like less because I’m afraid I won’t be able to solve them” 5P.135 - grade 5

If the undesirability of an event (in this case failure in mathematics) seems to be related to an individual’s goals, the probability that this event takes place is based upon two elements: the subject identification of the factors needed to be successful in mathematics, which we call *factors of success*, and the evaluation of one’s competence with reference to those factors. If failure in mathematics is viewed as a highly undesirable event, this subjective probability is, according to Ortony et al. (1988), what determines both the rising and the intensity of fear. In this framework, the individual’s beliefs about the stability/modifiability of factors of success in mathematics and about the indispensability of such factors (i.e. his/her theories of success, Nicholls, Cobb, Wood, Yackel & Patashnick 1990) come to be extremely relevant. As a matter of fact, a low perceived competence with respect to a certain factor of success considered unmodifiable and indispensable may induce a sort of *mathematical fatalism*. This status marks the renunciation to put all the available energies into the attempt to improve the situation (useless effort) and therefore may lead to the rejection of mathematical engagement. Moreover, in these cases, students feel virtually certain that the undesirable event (failure in math) will take place and they will then learn to associate fear to mathematics automatically and independent of the kind of activity they are given, the level of difficulty and any other consideration.

For example, amongst the factors of success that are most frequently associated with mathematics, in the essays we collected, there is intelligence:

> “In my opinion, to be good at mathematics one needs to be intelligent.” 5P.120 - grade 5

In general, in the essays that describe unease in the relationship with mathematics and that talk about intelligence as a factor of success, a judgment on one’s intelligence is given that appears definitive and unmodifiable:

> “I often think I am not that intelligent and I believe that when they tell me I am, and blame the fact I don’t study, they make me feel even more stupid, thick and useless” 5H.2 - grade 13

In other words students seem to have what Dweck and Bempechat (1983) call an 'entity' theory of intelligence, i.e. a vision of intelligence as a rather stable global feature in opposition to a vision of intelligence as a continuously expanding repertoire of abilities and knowledge (‘incremental’ theory).

Recognizing intelligence as a necessary condition to be successful in math has extremely negative consequences on those who fail, because it implicitly carries the message that failing is a proof of scarce intelligence:

> "I must say that sometimes I was rather good at some kind of reasoning but then as soon as I got things wrong I convinced myself I was dull. I wouldn't like you to misunderstand the word I've written, I used to think I was obtuse, close-minded, unable to understand and reason. Maybe I had made up in my mind a little image of mathematics, a subject I hate." 2H.118 - grade 10

The idea that failure in mathematics is a proof of scarce intelligence probably bears on the high undesirability of failure in mathematics with respect to failure in others school subjects, eliciting - as we can grasp by the above excerpt - a strong negative emotional charge towards mathematics. Not surprisingly, this negative emotional charge may lead students to hate mathematics and to choose to avoid engaging with it, rather than repeatedly failing, thus getting a confirmation of being '*scarcely intelligent*'.

From the analysis of the students' narratives another recurrent aspect emerges: in many cases students seem to simply identify mathematics success with school achievement. Therefore the recognition of success or failure is completely delegated to another person (the teacher) that becomes the object of fear, since he/she is an agent of an undesirable event for the student:

> "Last year I did not get on well with the teacher because I was rather frightened by him" 2H.51 - grade 10

Sometimes it seems not clear for the narrator himself if fear is related to the teacher, to mathematics or to both:

> "My relationship with mathematics is not great, but this does not mean that I don't study it, it's that when I hear the word, even when I see the teacher, I'm terribly scared, and I forget everything. Maybe, my only fear is the teacher and not mathematics, I never understood why" 3M.38 - grade 8

If the student identifies success with school achievement, his/her beliefs about the behaviour rewarded by the teacher come to be determinant in the recognition of factors for success or failure.

What emerges from the analysis of the essays is that success in mathematics is often identified with being quick at solving mathematical tasks (mainly at early school levels) as well as with not making mistakes. Therefore, there is a sort of identification failure – "slowness and mistakes", which brings about two worrying phenomena.

On the one hand, being slow, sometimes with the additional perception of being slower than others, becomes a proof of inability, with heavy negative consequences on the emotional side:

> "While I am tackling a problem and I don't manage to solve it, after some or many minutes I hear the voices of my classmates shouting they have finished. The queue

> to get the problem corrected by the teacher becomes longer and longer and this makes me even more anxious” 5P.110 - grade 5

On the other hand, fear of mathematics becomes fear (or even terror) of making mistakes since the very beginning:

> “During tests I’m so afraid to make mistakes that I put lucky charms on the desk” 3P.46 - grade 3

> “I don’t like mathematics: it scares me a little bit and makes me anxious because I’m always terrified of making mistakes” 5P.77 - grade 5

Sometimes time and mistakes are explicitly linked to each other. On the one hand needing time to do things in mathematics becomes frustrating and may lead students to hurry up and force *distraction errors*:

> “What made me really sad was the fact I was able to solve most of the exercises in the tests, but because I was slow I run out of time and couldn’t finish the test. So sometimes I speeded up and then I made mistakes in the calculations” 5H.14 - grade 13

On the other hand, fear of making mistakes may lead students to slow down and does not allow the students to finish what they are doing:

> “One thing I really hate is mathematics assessment. I actually HATE it; I can’t stand it because I’m afraid to make mistakes and hence I do everything very slowly and then time finishes before I can complete my work” 5P.170 - grade 5

Such vision of failure in mathematics not only has negative consequences, but it is far from the real experience of anyone who has studied mathematics at higher level, which shows that one needs to take all the necessary time and that in new and non repetitive problems errors are possible and sometimes functional for the process.

In a sense, we might therefore define *epistemologically incorrect* this characterization of failure in mathematics linked to a way of considering errors as something to absolutely avoid, which is so widespread and transmitted in the classroom. But it is most of all a losing vision, since it may not support the effort which is necessary in problem solving, coming to generate the widespread phenomenon of math-anxiety:

> “Many times, when there are problems in the classroom, I get anxious, I’m afraid I can make mistakes, also when I am at home I’m afraid to get it wrong, because I don’t like to make mistakes” 5P.63 - grade 5

Moreover, fear of making mistakes has negative consequences from the point of view of the non-optimal exploitation of the cognitive resources (metacognitive aspects), and pupils themselves are aware of that, describing it as the main cause of their difficulties in mathematics:

> "My problem is not being unable to carry them out, but rather fear to make mistakes, as a matter of fact up to now in the oral tests I am always afraid of making mistakes, of giving the wrong answer, even though I know things" 2H.52 - grade 10

> "I just give up when, even in front of a simple division, only because I'm uncertain or I'm afraid to make mistakes" 3M.61 - grade 8

The influence of fear of making mistakes on the optimal management of one's own resources, including cognitive ones, is felt by students themselves, to the point that some of them talk about their own fear of being scared:

> "When I do some mathematics tests, I am often, actually very often, scared and this causes problems in carrying them out, because I'm afraid that fear will make me make many mistakes, I get stuck because of that" 5P.97 - grade 5

Sometimes this fear of making mistakes works as an inhibitor (one prefers to stay silent rather than answering and risking to make a mistake):

> "I remember that when she asked me questions I was terrified not to know anything, and if she asked me something I was afraid to get it wrong, and I didn't answer the questions" 2H.120 - grade 10

> "In the classroom, when we were doing exercises, I was always afraid that the teacher could ask me something I would not have been able to answer and, for fear of making a mistake, I used to staying silent" 5H.29 - grade 13

Fear of math caused by fear of making mistakes may have another negative consequence: it can elicit fear of the new, of anything that may cause difficulties, going beyond the mere repetitive exercise.

> "I remember that once, in grade 3, when we started to learn divisions, I was sort of scared of learning, scared of going ahead with the program" 5P.211 - grade 5

This fear of learning and of going ahead is obviously a big obstacle for learning itself (it is particularly dramatic that a very young pupil has this fear) and, moreover, mortifies the possibilities of mathematics education promoting the development of a taste for discovery and for investigating difficult problems.

> "When we deal with something new I always fear it is difficult" 5P.125 - grade 5

In the context of school mathematics, easy/difficult and like/dislike are often identified in a perverse way.

The weight of fear of mathematics, its oppressing role and therefore the importance of fighting against this feeling with purposeful teaching interventions, comes up in the stories that narrate how this emotion can be overcome:

> "During the past few years my mind changed, and fear of mathematics changed with it. In the first year of primary school, I thought mathematics was a monster,

> with difficult numbers, which went around in my head all at the same time, with no rest. In the following years, the monster of mathematics became smaller and smaller and I got bigger and bigger. That tiny little monster later disappeared; and from that moment onwards I started to love mathematics. I was no longer annoyed by all those numbers, on the contrary, it made me happy, I was content, anxious to see the next problem to be tackled. I was there, ready with paper and pencil, ready to rack my brains. I found something in mathematics I had never found before, due to fear which increased and increased and never stopped, until my strength managed to have control over it and push away from my thoughts." 5P.219 - grade 5

In the stories talking about the overcoming of one's fear of mathematics, the teacher is often the decisive figure to determine the change:

> "I remember that every time I had a mathematics test at school I felt a strong pain in my stomach and in my head, which made me feel not so ready, so that when the day of the test arrived, my uncertainty led me to make mistakes. Fortunately this is not happening nowadays, and I cannot explain this change, but with this teacher I feel more confident and free from my fears." 2H.119 - grade 10

CONCLUSION

The analysis we carried out shows that fear of mathematics, that is often correlated to fear of failure, has a strong cognitive origin and at the same time, influences the possibility of managing one's own cognitive resources in the best possible way. Fear of math is a possible factor of local failure (since it does not allow for the best use of one's competence and knowledge) but also a possible factor of global failure (since it might lead to give up engagement with mathematics).

From the stories of reconciliation with mathematics emerges that getting over fear of mathematics is the primary and essential condition to be able to start a new and positive relationship with the discipline.

All these remarks charge the teacher with responsibilities, since he/she can convey but also modify a particular vision of mathematics: for example, the weight students attach to mistakes is influenced by the teacher's management and evaluation of mistakes.

In our study, a strong interplay between the rise and the intensity of fear of mathematics (viewed as a prospect based emotion) and the student's beliefs about success in mathematics (theories of success) and about self, emerges. Within a student's theories of success, the interaction among his beliefs about those we called 'factors of success', about his perceived competence related to these factors, and about their modifiability, appears particularly significant in order to interpret the evolution of fear toward fatalism. Consequently, in order to get over the students' fear of mathematics we have to work on both the vision of the discipline and one's own perceived competence, in particular sharing the modifiability of the *factors of success* in mathematics.

This action needs that the teachers are prepared to eventually modify their own beliefs towards mathematics and its teaching in the first place.

Moreover, another point of interest is the relationship between teachers' emotions towards math and their students' emotions towards math. This appears to be particularly important in the case of primary teachers that are not specialists in math, and often have had a troubled story with mathematics. Recent studies (Di Martino & Sabena, 2011) show that many pre-service primary teachers have fear of math and are terrorized by the idea of having to teach it.

These results confirms the need to go beyond the purely cognitive also in the research about teacher development, trying to provide teachers the theoretical instruments to interpret, analyze and counteract the students' fear of math, but also to overcome their own eventual fear of math.

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CLASSROOM GROUPINGS FOR MATHEMATICS LEARNING: THE IMPACT OF FRIENDSHIPS ON MOTIVATION

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The importance of small groups as a pedagogical tool in mathematics classrooms is widely researched and acknowledged to be a successful means of engaging students in mathematics learning. This small scale study examines the influence of close friendships, and friendships by association, in mathematics classrooms of 14-15 year olds on students' motivations to engage with mathematics. We use evidence from questionnaires and individual interviews to describe the motivational factors identified in two classes of students. Findings confirm the multi-faceted nature of motivation in interpreting classroom relationships and the differences in working relationships between groups of close friends and those of friends by association.

INTRODUCTION

The rationale for using talk in small groups as a pedagogic strategy in classrooms is well established. In a review of small group talk, Good, Mulryan and McCaslin (1992) describe "clear and compelling evidence that small group work can facilitate student achievement as well as more favourable attitudes towards peers and subject matter" (p. 167). While much research has been undertaken on the composition of small groups in classrooms in relation to gender, levels of mathematical attainment and age (see, for example, Bennett & Dunne 1991), there is little research on group composition by definition of levels of friendship. According to Slavin (1989), for effective collaborative learning, there must be *group goals* and *individual accountability*, but little is known about how social relationships impact on these factors. A review of group processes in the classroom by Webb and Palincsar (1996) identifies two individual actions associated with increased learning; 1) giving elaborated explanations to other group members, and 2) applying explanations (either received or self-generated) to solve problems or perform tasks (p. 854). Yet there is little research on how these group processes are affected by working with friends.

Friendship groupings in mathematics classrooms are a rarely researched phenomenon, yet they are often used as a pedagogical tool in mathematics classrooms, either briefly for a few minutes discussion or over longer periods of time. This study aimed to explore how the use of pedagogical strategies, such as regular and sustained group work, influenced the perceptions students have about doing mathematics. Since Hamm and Faircloth (2005) propose that motivation is subject-specific, students aged 14 to 15, working towards final mathematics examinations, are therefore in a key phase of schooling for investigation, given that mathematics represents one of only a few subjects students are required to study in England until the end of compulsory education.

Goos, Galbraith and Renshaw (2002) assert that, since the research undertaken on small group arrangements within mathematics have largely focussed on outcomes, there is a need for research examining how students think and learn as they interact with peers in small groups, emphasising the need for an exploration of the processes at work. Whilst much research exists concerning both theories of motivation and the influence of peers on learning, there is less evidence regarding how these areas interact, specifically within friendship groups in mathematics classrooms.

THEORIES OF MOTIVATION

Lord (2005) argues that several elements form learner motivation. He claims motivation is unique to each individual, can be expressed in a variety of ways, and is aimed towards an end point or a goal. How a student engages with learning is recognised by Brown as reflecting knowledge of themselves as a learner and the learning process, describing this as "metacognitive knowledge" (1988, p. 312). This reflects Lord's definition of motivation, that learning and motivation are both individual at the initial level. For learning to be truly effective, Brown argues that participants need to have reasons for learning and ownership of knowledge.

Ryan and Deci (2000) are adamant that, too frequently, motivation is defined as a single entity, claiming that individuals not only experience different kinds of motivation but how much motivation is experienced depends on situations and individuals. As Hannula (2002) has recognised, only outcomes of motivation are seen in students' behaviour(s). Therefore, labelling motivation as a single aspect underestimates the varying emotions students experience when completing different mathematical tasks. In analysing data in our study, we utilise Ryan and Deci's model of a continuum of motivation.

The most basic distinction between the types of motivation individuals experience, as identified by Ryan and Deci (2000, p. 55) is the contrast between intrinsic motivation and extrinsic motivation.

> "intrinsic motivation refers to doing something because it is inherently interesting or enjoyable and extrinsic motivation refers to doing something because it leads to a separable outcome".

Ryan and Deci present a continuum of motivation beginning with amotivation, the lack of motivation, through four stages of extrinsic motivation to intrinsic motivation. The distinctions between the different stages of extrinsic motivation represent the degree of control students experience, moving from amotivation, where individuals feel that control is impersonal, to intrinsic where control is internal and self-directed. The significance of the different levels of extrinsic motivation demonstrates that, whilst students may value an activity for its own sake, the enjoyment of the task, in and of itself, may not be the goal being worked towards. A student may value a task for its own sake but this is because it allows them to progress, perhaps, to higher education, requiring a particular examination grade. This reflects the notion of tasks leading to external rewards, hence having elements of extrinsic motivation, although the individual is controlling the effort personally.

Grouws and Lembke (1996) recognise that true motivation is intrinsic, coming from within individuals. They also acknowledge the significance of the classroom culture in establishing intrinsic motivation. This brings into focus questions concerning whether students generate motivation for mathematics from within themselves or through their classroom experiences.

Forman and McPhail (1993) take a sociocultural approach to their study of collaborative groups in classrooms. From this perspective, rather than locating the source of individual motivation and understanding within or between individuals, they locate it in sociocultural practices in which children have the opportunity "to observe and participate in essential economic, religious, legal, political, instructional, or recreational activities" (p. 218). Through guided participation, "children internalize or appropriate their affective, social, and intellectual significance" (p. 218). Findings by Skaalvik and Valås (2010) contradict Forman and McPhail in that these former authors argue that both self-concept and motivation develop with age and are closely connected to personal achievement. From very different viewpoints, these studies raise questions about the effect that friendships are likely to have on motivation.

In measuring attitudes students ascribe to different things, Hannula (2002) suggests that it is often difficult to connect what is being seen outwardly with the elements that are not seen. This reflects Pintrich and Schunk's (1996) arguments that motivation is a process and that only the product is what is seen. Hannula also observes that only when emotions are sufficiently powerful are they outwardly visible. Ollerton (2003) claims that mathematics, unlike other subjects, seems to cause students significant anxiety. Investigating the motivational beliefs students express concerning the ways in which they work in mathematics is important where emotions such as frustration, boredom and anxiety can be observed in outward behaviours. A study of friendship groupings is likely to explore both emotional and motivational relationships. One of the intentions of this study was to identify processes at work leading to the outcomes seen by teachers.

MOTIVATION AND FRIENDSHIP

Research by Berndt (1992, 1999) examines the influence that friendships during adolescence have on adjustment in and engagement with schooling. His 1992 study explored the impact adolescent friendships have on affective relationships with school. He identifies two elements that influence adolescents' friendships: the characteristics of the individual friends and the quality of the friendship. Therefore, how each individual is placed within their friendship groups will alter the influence they have. Berndt's (1999) study recognises that areas such as achievement and motivation may not be common features of conversations amongst adolescents, as participants in Ryan's (2000) study also indicated. Berndt also acknowledges the role of trust within friendships and how its presence allows friends to share experiences, emotions and rely on one another. It is possible, therefore, that relationships built on trust may be evident within mathematics classrooms between peers who are not friends.

If students are able to engage in relationships where trust exists, whether these are close friendships or friendships by association, the benefits may be greater than students working in isolation. Furthermore, if mathematics classes are organised around attainment levels, as is usually found in English classrooms, it is likely that each class will generate its own set of friendship relationships, dependent on the particular students within each class. Evidence that the more consistent these classes are over time, the stronger these structures become (Edwards, 2003) means that students in classes established over several years are likely to be aware of those they can trust within this context. However, if classes are arranged using attainment levels, the existence of friendships within these groups may not be utilised in the most beneficial way. Such a claim is highlighted by participants in Nardi and Steward's (2003) study in their descriptions of feelings of isolation and the sense that mathematics is not presented as a subject that allows opportunities to work with friends.

More recently, a large-scale quantitative study by Nelson and deBacker (2008), examined 253 middle school students' assessments of peer classroom climate, beliefs relating to a best friend's influence on achievement, achievement goals and self-efficacy. They found that positive outcomes relating to achievement were reflected by those students reporting a perception of being valued and respected by peers. As in Berndt's (1992) study, these authors also found that the quality of friendship and the relationship of best friends with academic achievement correlated directly with students' motivation for learning.

SETTING FOR THE STUDY

This study was undertaken in a large comprehensive secondary school in southern England with approximately 1700 students, aged 11 to 16. The school is deemed, by national inspection processes, to be successful, with mathematics examination results, at age 16, significantly above the national average. Within mathematics, students are taught in classes arranged by attainment level, determined by testing shortly after students transfer to the school at age 11. Each year group comprises 11 or 12 mathematics classes, with the mathematics classes studied here, at age 14 to 15, being in the middle of these attainment levels. There were a total of 62 students in these two classes. This age group, in this particular school, represented a gender imbalance of approximately 65% males to 35% females. This imbalance was similarly reflected in the two classes studied.

DATA COLLECTION

A questionnaire was designed to gather students' opinions about elements of their motivation, the influence of their peers which included the reciprocal nature of these relationships, and the use of peer groups as a tool within the classroom. The questionnaire items were allocated to a 'strand', such as *individual motivation*, *influence of environment/classroom culture*, and *influence of peers on motivation/knowledge* or *influence of peers on construction/accessing help*. For example, a statement from the *influence of environment/classroom culture* strand was "My motivation in mathematics can change based on what is happening around me". The questionnaire comprised of fifteen statements in which students were asked to express their level of agreement using a 5-point Likert-style scale. The lowest value (1) indicated "strongly disagree". The middle value (3) was given the description "uncertain" rather than "neutral" to allow students to state that they were unsure about an opinion on the specific statement rather than that they did not hold an opinion. The highest value (5) indicated "strongly agree". At the time the questionnaire was completed, students were offered the opportunity to participate in the interviews that formed the second part of this study.

Students were also asked to classify their relationship with others in the group in which they worked as '*All friends*' or '*Some friends*'. The former of these categories was described, for students, as peers who were friends outside of lessons, with the latter category as friends because of association with them within the mathematics class. The interview schedule was a semi-structured format, based around five broad questions, where the questions and plan for the interview were the same with each participant, but the ordering of questions and the use of specific follow-up questions was sufficiently flexible to probe particular issues identified in individual responses. The questions were structured to identify general elements of students' work in mathematics, their motivation(s), their interpretations of the relationships with their peers, the overlapping elements of motivation and working with peers, including the reciprocal nature of being a collaborative peer to others. Participation in the interviews was voluntary for participants who undertook the questionnaires. Nine participants, four females and five males, volunteered for the interviews.

OUTCOMES – ROLE OF FRIENDSHIPS

Findings from the questionnaire were analysed using the range of scale indicators from the Likert scale, since the sample size of 62 does not warrant the use of percentages. Here, we present those outcomes which focus solely on the friendship categories, self-identified by students as *All friends* or *Some friends* on the questionnaire.

Category	Questionnaire item	All friends		Some friends	
		Range	Scale	Range	Scale
Influence of peers on motivation	5: Working with my peers in mathematics lessons improves my motivation	2	3 – 5	4	1 – 5
Influence of peers on motivation	6: When I don't work with peers in mathematics, it makes no difference to my motivation	1	2 – 3	3	1 – 4
Influence of peers on motivation	7: Working in a group at a table in mathematics lessons improves my motivation	2	2 – 4	2	2 – 4
Influence of peers on motivation	10: Seeing my peers succeed in mathematics motivates me to pursue similar success	3	2 – 5	3	2 – 5
Influence of peers/knowledge construction	8: I understand things better in mathematics when I can discuss new concepts with my peers	1	4 – 5	3	2 – 5
Influence of peers/knowledge construction	9: I make better progress in mathematics when I work in groups	1	3 – 4	2	2 – 4
Influence of peers/accessing help	11: Being able to gain help from my peers in groups motivates me more than gaining help from the teacher	2	3 – 5	3	2 – 5
Peer contexts/ Reciprocal help	12: I am able to positively influence the motivation of my peers	1	3 – 4	4	1 – 5
Peer contexts/ Reciprocal help	13: My actions in mathematics lessons do not influence the motivation of my peers	1	3 – 4	3	1 – 4
Peer contexts/ Reciprocal help	15: My influence on my peers' motivation is no greater when working in groups than when working individually	2	2 – 4	2	2 – 4
Individual motivation	1: My motivation in mathematics comes from within myself	2	2 – 4	2	3 – 5
Individual motivation	14: Working in peer groups does not motivate me any more than working on my own	2	1 – 3	1	2 – 3
Influence of environment/ classroom culture	2: My motivation in mathematics comes from things around me	3	2 – 5	2	2 – 4
Influence of environment/ classroom culture	3: My motivation in mathematics comes from a range of different things	3	2 – 5	1	3 – 4
Influence of environment/ classroom culture	4: My motivation in mathematics can change based on things happening around me	2	3 – 5	2	3 – 5

Table 1: Comparison of ranges between responses for *All friends* and *Some friends*

This comparison of ranges of responses indicates that the *All friends* subgroup has greater consistency in their agreement on different statement categories and in responses to items with the same statement categories. This is shown through both influence of peers/knowledge construction and peer contexts/reciprocal help having a range of 1 for two statements each. The consistent responses to the statements in these categories highlight the learning enhancements that can occur in groups where participants are working with peers who are close friends. This evidence appears to support findings of both Berndt (1999) and Nelson and deBacker (2008) that trustful relationships amongst friends improve motivation for learning. The more varied ranges for the corresponding statements in the *Some friends* subgroup indicates that, whilst students value the contribution their peers are able to make, these are not as strongly held views as in the *All friends* subgroup.

It might be expected that, in groups where only some participants are friends, students place a greater emphasis on individual motivation. Evidence of this is seen where the statements reflecting individual motivation reflect greater agreement amongst the *Some friends* category. This is paralleled by the range of 1 for questionnaire item 14, where a low value might be expected; however, *Some friends* did not express disagreement as strongly as *All friends* about the difference not working with peers makes to their motivation. An inference of this finding might be that when students are working in peer groups, where only some members are their friends, they are required to generate more motivation from within themselves, rather than this coming from the closer friendships that they have elsewhere.

Using the category distinctions of *All friends* and *Some friends*, in groups where participants considered all members to be their friends, the responses are more consistent. A greater degree of individual motivation is expressed in peer groups where participants considered only some group members to be their friends. Evidence from the students' interviews indicates that their motivation in mathematics is also changeable, not necessarily linking to elements of the classroom culture, but rather to external factors, such as time of the day or week, or prior experiences in lessons in other subjects: "And I probably work harder on the Mondays than Thursdays [last mathematics lesson of the week] … cos on, like, a Monday I'm, like, all ready for work cos it's Monday morning … and then on Thursdays, it's a bit more laid back cos it's Thursday". The choice of working peers was also considered important to individuals: "I chose people that I knew, but at the same time I knew I could work well with them … through things like sports teams and things outside of school".

The findings from the interviews also indicate that students working in mathematics at age 14 to 15 view their peers as a valued resource for learning: "If they're your friends and, umm, maybe you can talk to each other which, like communication … you can help each other, which may help my motivation"; "I think it motivates me more if I ask my peers … they're easier to talk to sometimes … and they understand you more … as opposed to teachers who sometimes don't …". Students emphasised the contribution the classroom environment makes and how it influences changes within their motivation: "If you're not having a good day then it [motivation] doesn't, you don't really feel very motivated, just want to get the lesson over with … and if you're having a fine day, you don't mind learning and being motivated". They expressed a strong level of agreement over the benefits of being able to discuss new concepts with their peers: "If you don't want to ask the teacher, then asking them [friends] works as well". The interview responses also highlight how the physical arrangement of groups allows for greater flexibility and ease of accessing help from peers: "If you're in a, like, square table, then you can talk to anyone around the table … I think it motivates me more if I ask my peers …". In general, the students indicated a preference for gaining help from peers, rather than the teacher, due to understanding peers better and not appearing foolish through publicly requesting help. Such need for public acceptance amongst adolescents is identified by Warrington and Younger (2011).

DISCUSSION

Students in this study indicate that their motivation for mathematics is affected by several factors, supporting Lord's (2005) research. Evidence from questionnaires and interviews suggest that students place considerable emphasis on the classroom culture as a source of motivation for mathematics. However, whilst students recognise the influence the environment can have on their motivation, the personal element to motivation, described by Lord is also acknowledged.

In terms of Ryan and Deci's (2000) study, in which they argue that individuals not only experience different kinds of motivation but the extent of this depends on situations and individuals, all students in our study displayed some form of intrinsic motivation for mathematics. However, questionnaire responses also indicate that this is more likely to be a form of extrinsic motivation on Ryan and Deci's continuum of motivation. It is also evident in student interview responses that motivation for mathematics is not a single feature, supporting Valås and Søvik's (1993) research. These interview responses broadly reflect variations of extrinsic motivation moderated by an internal locus of control, as described by Ryan and Deci (2000).

These 14-15 year old students express an awareness of a relationship between their motivation for mathematics and their relationships with their peers, though the nature of this relationship is not explicitly identified in the interviews. What is highly evident, despite the small scale of this study, is that relationships with peers have a significant influence on these 14-15 year old students' motivation to work in mathematics.

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RECONSTRUCTING TEACHERS´ BELIEFS ON CALCULUS

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The teachers' instructional planning, their classroom practices that impact on their students' knowledge and beliefs could be understood as individual beliefs systems dependent from their actual teaching and learning experience. This individual experience might be contradictory when we regard different teachers or one teacher concerning different mathematical disciplines. For this reason, this report focuses on teachers´ beliefs with different levels of experience about their teaching of a specific mathematical domain, i.e. calculus that is a central part of the (German) curriculum at upper secondary level. After a brief outline of the theoretical framework and methodology of this research project, results of the qualitative reconstruction of different types of teachers´ beliefs on calculus will be explained.

INTRODUCTION

The importance of gaining knowledge towards mathematics teachers' thinking or beliefs has been emphasised by many researchers in mathematics education in various settings and projects for some reasons: on the one hand teachers' beliefs about mathematics and the teaching and learning of mathematics have a high impact on their instructional practice (Philipp, 2007; Eichler, 2011), on the other hand teachers' instructional practice, which is considerably determined by teachers' beliefs about their professional world (Calderhead, 1996), has a high impact on students' learning and beliefs concerning mathematics (Hiebert & Grouws, 2007). Moreover, the possibilities of changing the teachers' thinking about mathematics education depend on the teachers' beliefs towards teaching and learning mathematics (Franke, Kazemi & Battey, 2007). On these grounds there has been much effort in investigating mathematical beliefs of teachers all over the world in the recent two decades (Philipp, 2007). In this huge body of research, most of the study approaches concern teachers' beliefs on *mathematics* and the learning and teaching of *mathematics*. It is rarely considered though that – similar to the classification of mathematical subjects into fields such as algebra or probability theory – teachers´ beliefs on different mathematical domains such as geometry, stochastics or calculus may vary and may be associated with specific beliefs (Franke et al., 2007).

Research on (intended) curricula of experienced calculus teachers is rare (e.g. Tietze, 2000). Recent works aim primarily at investigating calculus lessons with the use of technology (e.g. Kendal et al., 2005) or look at calculus curricula of undergraduate courses at university. Although there is plenty of research on algebra teaching (Kieran, 2007), deeper investigations which focus on the development of teachers´ intended curricula independent of technology aspects are scarce.

As there are few investigations about calculus, which is a central part of the German secondary curriculum, domain-specific beliefs of secondary teachers referring to the teaching and learning of calculus are the main focus in this paper. Our specific interest primarily concerns the structure of beliefs that characterise calculus teachers' instructional planning (teachers' intended curricula) and, thus, impact on the teachers' classroom practice (teachers' enacted curricula), and the students' learning (Eichler, 2011). Reconstructing this structure possibly facilitates to partially identify which constituents are more central than others (Green, 1971). Further, regarding the system on calculus teachers' beliefs, we search for relations between different clusters of calculus teachers' beliefs that we call views (Grigutsch et al., 1998). We refer our findings to existing results for teachers of other mathematical disciplines. As part of a larger research programme it can be asked whether the degree of professional experience might have an impact on characteristics of the teachers' belief systems. Before we address the mentioned questions for this paper, an outline is given about the theoretical framework of the research programme, and a brief description of those parts of the method being relevant for this paper.

THEORETICAL FRAMEWORK

Research on teaching and curriculum has brought forward that a significant difference exists between the curriculum as represented in specifications by national or regional governments, sometimes accompanied by instructional materials, and the curriculum as it is actually enacted in the classroom by teachers and students. The various meanings of curriculum have been conceptualized by the work of Stein, Remillard and Smith (2007), who provide a curriculum model including four phases of a transformation process to describe a mathematical teacher's planning, the teacher's classroom practice and his students learning (see fig. 1).

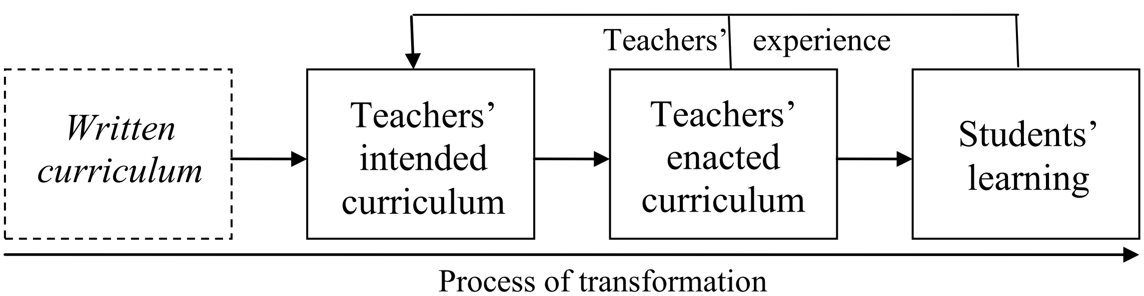

Figure 1: Four phases of the curriculum according to Stein et al. (2007)

The *written curriculum* involves both instructional content and teaching goals, often prescribed by national governments. The way the teachers interpret a written curriculum concerning content and goals is called the *intended curriculum*. The classroom practice involving interactions of a teacher with his or her students, and the instructional content "create something different than what could exist […] in the teacher's mind" (Stein et al., 2007, p. 321). This transformation of the intended curriculum is called the *enacted curriculum*. The individual students learning outcomes that are not necessarily intended by their teachers is called *students' learning*. In this article, the discussion is restricted to teachers' intended curricula.

A teacher's intended curriculum represented by content and goals can be understood as specific form of beliefs, if beliefs are defined as an individual's personal conviction concerning a specific subject, which shapes an individual's way of both receiving information about a subject and acting in a specific situation (Pajares, 2007). Regarding this definition, content and goals portray a teacher's conviction about an appropriate way of teaching mathematics. Thus a teacher's intended curriculum can be understood as a belief system that is characterised by a quasi-logical system of beliefs with different grades of importance or centrality (Green, 1971, Philipp, 2007). Another theoretical feature of a belief system involves that belief systems consist of several clusters of beliefs that may be contradictory. We hypothesise that a teacher's intended curriculum concerning the teaching of mathematics consists of clusters representing different mathematical domains that need not coincide with each other if the teaching goals are regarded, but are mostly consistent, if a specific mathematical domain, e.g. calculus, is regarded (Girnat & Eichler, 2011). In this paper, we restrict discussion of results on teachers' intended curricula concerning the teaching and learning of calculus.

In the following, belief systems of secondary teachers are characterised towards the teaching and learning of calculus as views representing the main instructional goals of the teachers (Philipp, 2007). There are four main categories which can be characterized by different features regarding the perception of mathematics in general (Grigutsch, Raatz & Törner, 1998) and calculus in particular:

- A formalist view stresses that mathematics/calculus is characterized by a strongly logical and formal approach. Accuracy and precision are most important and a major focus is put on the deductive nature of mathematics or calculus.
- A process-oriented view is represented by statements about mathematics being experienced as a heuristic and creative activity that allows solving problems using different and individual ways.
- An instrumentalist view places emphasis on the "tool box"-aspect which means that mathematics is seen as a collection of calculation rules and procedures to be memorized and applied according to the given situation.
- An application oriented view accentuates the utility of mathematics for the real world and the attempts to include real-world problems into mathematics classrooms.

Usually teachers' belief systems might consist of a mixture of all the four views outlined above. However the weight of each aspect varies from teacher to teacher. Although we assume that a teacher's belief system consists of parts of different views, our research shows patterns of views when we investigate teachers' beliefs referring to a specific mathematical domain: e.g., the research of Eichler (2011) yields that stochastics teachers primarily refer to application oriented teaching goals when they analyse both their instructional planning and their classroom practice. By contrast, teachers mostly disregard a direct connection of school mathematics and real world problems, if geometry is considered (Girnat & Eichler, 2011). Accepting the hypothesis, like Franke et al. (2007), that teachers' beliefs referring to the teaching and learning of mathematics differ, if they have geometry, stochastics or calculus in mind, it is worthwhile to investigate teachers' beliefs concerning the calculus domain to understand their teaching practice.

METHOD

Concerning the teaching and learning of calculus, we analyse teachers in different stages of their professional development including 10 pre-service teachers, 10 teachers in a practical phase after university studies (teacher-training college) that lasts 18 months and 10 teachers with a professional experience of at least five years. All the teachers are employed or would be employed at upper secondary schools (Gymnasium) in Germany.

In the first part of our research that we address in this paper, data were collected by semi-structured interviews comprising clusters of questions concerning several subjects: contents and goals of instruction, teaching and learning calculus, institutional boundaries and the nature of mathematics and calculus at school level in particular. Within these obligatory clusters, the teachers gave distinction to the interviews by describing their own typical examples but also had to react to specific prompts. These prompts include, for example, evaluating statements about adequate teaching, students' statements about calculus or different tasks from calculus text books. In addition, the teachers were asked to respond to questionnaires concerning mathematical beliefs (Grigutsch et al. 1998) and their teaching orientation (Staub & Stern, 2002).

The interviews were transcribed verbatim. Each transcript has a length of 30 to 40 pages. The first step of the analysis was to split the transcripts into episodes and label them in terms of the question clusters outlined above. Further, the episodes were analyzed by coding. Coding guidelines were adapted in compliance with qualitative data analysis (Kuckartz, 2012). First the relevant coding units were determined and then supplemented by inductive codings. This approach enables both the consideration of empirically-based individual content items (inductive aspects) and the significance and reconstruction of contents and goals (deductive aspects) that have been shown to be relevant in previous work on individual curricula e.g. of stochastics teachers (Eichler, 2011). Finally, the characteristics of each teacher were systematically arranged and summarized with respect to the projects′ research questions. Furthermore case summaries were compiled to highlight essential characteristics of each individual teacher.

RESULTS

In order to categorize and illustrate teachers′ beliefs concerning the planning and teaching of calculus by means of qualitative analysis, the deductive aspects of four different views (see above) were chosen. This method has been efficient in previous studies of intended curricula for other mathematical domains with respect to patterns of beliefs as well as to describe these views in depth. This involves the subjective teachers' definition of a specific view that represents the teachers' overarching teaching objectives. We illustrate a coherent view, in this case a formalist view concerning the subjective definition of Mr. C.

Mr. C.: „In general, exactness is crucial for me. That means to fit a necessary formalism as I know from my university studies. This also means that it must be possible to recognise a logical rigor. Sometimes I do more in that sense than the textbook actually demands."

Taking this teacher as a paradigmatic example, he did not mention aspects like to apply mathematics in real world problems or to learn problem solving, which means to emphasise the process of developing mathematical concepts. By contrast, for Mr. C., the main goal of calculus teaching seems to be emphasising the stringent and logical construction of a mathematical domain.

The identification of specific teachers´ views is always established in various parts of a single interview and we report only teachers´ views that are in some sense coherent throughout the whole interview. We illustrate this concerning this exemplified teacher. When Mr. C was asked to regard the expectations and needs of his students, he agrees consistently with a formalist view. For instance, Mr. C was asked to rate the statements of students shown in figure 2.

Which of the following comments on calculus would your students most often make?
Why? Which of these are the most important in your view?

I like calculus, because there is a connection to real life problems.

I like calculus, because hard nuts must be cracked and difficult problems can be solved.

I like calculus, because many exercises can be solved by similar procedures/patterns.

I like calculus, because the logic is clear and it follows strict mathematical rules.

Figure 2: Statements of students concerning calculus

As expected, Mr. C attaches particular importance to the last statement that represents a formalist view. Mr. C further explains his goals concerning his students' beliefs towards calculus:

Interviewer: How should your students characterise calculus?

Mr. C: Precise mathematics. Thus, on the one side that it is possible to understand how one develops mathematical ideas and how it is possible to build up a theory on the foundation of few basic ideas.

Summarising the beliefs of Mr. C concerning the teaching and learning of calculus, there exist several unambiguous examples of evidence for Mr. C´s formalist view. The high degree of coherence in different parts of the interview leads to the hypothesis that this formalist view is dominant and thus central in the belief system on calculus.

Regarding teachers´ beliefs on calculus, our study has so far revealed one particular aspect as some teachers stressed their need for formal and deductive nature of calculus which is characterised by accuracy, precision and a strongly formal and logical approach. However, in contrast to Mr. C, most of the teachers show a mixture of different views.

In particular, if teachers' hold beliefs that represent different views, we analyse relationships among the different views. We describe this analysis exemplarily by regarding Mr. A and Mr. B starting at the subjective definitions of their possibly central beliefs. In contrast to a (central) formalist view, these two teachers firstly delivered an insight on their views on applications:

Mr A.: „I quite agree with the emphasis on applications in the given example. That is certainly a way to motivate them (students), but nevertheless one should not reduce genuine calculus or the teaching of calculus to that topic. "

Mr. B.: „Examples for applications are quite suitable here, and with applications I always associate modelling of real data, […] increasingly introducing relevant applications into lessons may, for the students, succeed in a deeper insight into the concepts and ideas of calculus."

Both teachers have nearly completed their teacher training and have taught the first class of upper secondary level independently i.e. without being accompanied by a senior teacher. Mr A. supports the integration of applications as a principle of learning calculus at school for reasons of (student) motivation. In contrast Mr. B reckons that introducing real-world problems into calculus is a part of his system of aims and goals concerning his calculus teaching. The difference between the instructional goals of motivation on the one side, and solving real problems on the other, is stated by Förster (2011) concerning teachers who teach modelling, but, however, has not been reported in domain-specific research about teachers´ intended curricula so far. The aforementioned aspect seems to be relevant independent of the professional status of the teachers interviewed. Both views on applications can be found in all three groups of our sample.

In addition, the explicitly expressed goal of Mr. B. of integrating modelling tasks into his lessons is connected with another statement about dealing with the "formal logic" as a characteristic property particularly in calculus.

Mr B.: "…because I think that the formal derivation of integrals by limits is of no avail for secondary level students. It´s just too complex for most of them."

However, a general conclusion that applications are implicitly of primary importance than formality and logic cannot be drawn as the following quotation of Mr A. demonstrates:

Mr. A: "Calculus is more than just dealing with application-oriented tasks. Then, for example, one would not regard the precision and exactness of calculus and use applications as a means to an end."

Our hypothesis on basis of the present data is the following: If teachers hold a consistent formalist view on calculus, they do not mention any applications. The reversal conclusion, however, is not possible. Teachers who primarily favour applications in their calculus courses, e.g. Mr. A and Mr. B (see above), cannot be described as non-formalist. This example already demonstrates the abundance and need to differentiate the views of teachers on calculus.

As another aspect of relationships among different views or rather cluster of beliefs we describe in the following contradictory belief clusters that we call conflicts (of instructional goals). We reckon that the teachers' system of beliefs may have a quasi-logical structure and could involve contradictory clusters of beliefs with the paradigmatic example of Mr. E. Throughout the whole interview he speaks about the central role of logic in calculus lessons offering his perspective that exactness and logical rigour are necessary ingredients of secondary level calculus courses. Again, the degree of coherence of favouring formalist elements could provide an indication for a core belief. Yet, as he describes representative classroom situations, his subjective experience surfaces a conflict between his belief system about calculus and pedagogical processes in his calculus course.

Mr. E.: „ In my view it is quite important that there are formal definitions of concepts because you need them for proofs later on and it´s the tiny details that are particularly important…

In my class I clearly notice that students come to their limits concerning the degree of abstraction. [...] Remembering my own calculus course at school I can´t remember any bad experience with these formal aspects. So far I haven´t seen such a mismatch between teacher and students in maths."

Mr E. can be identified favouring a formalist view but probably will not enact his formalist view on calculus in the classroom in a predominant way because there is a conflict with the real situation he encounters in the classroom i.e. the students´ ability to understand the formal way of developing calculus ideas. Therefore this situation can be characterized as a conflict of objectives between his view on calculus and his teacher authority and responsibility. In particular when teachers are asked to reflect on representative examples of their actual teaching processes of specific elements of their calculus courses, the interview transcripts provide a deep and concrete insight of teachers´ subjective notions of their intended and enacted curricula and sometimes yield conflicts of a teacher's system of instructional goals. We hypothesise that a conflict of teaching objectives gives evidence for a central belief since peripheral beliefs might be superimposed if they show a conflict with central beliefs.

In addition to predominant beliefs that teachers show in different parts of the interview in a coherent way, the teachers also provide insights into some peripheral goals. These peripheral goals are neither coherent nor do they produce any conflicts between the intended and enacted curriculum. For example, the teachers indicate in some interviews the peripheral goal that calculus and the teaching of calculus is a collection of rules and procedures (i.e. toolbox) although their beliefs cannot be categorized globally as an instrumentalist view.

> Mr. F.: The main goal of every student is to perform well in his final exams – therefore calculation rules and procedures have to be thoroughly practised in class. Especially the calculus part of final exam tasks are alike in some respect, so practising is a substantial guideline for my course.

Often these views are motivated by normative aspects such as final exam tasks, yet seem to have a considerable impact on their actual teaching of calculus.

It is apparent though that for all teachers in our sample the preparation of the final exam (so called "Abitur") does indeed play more than a subordinate role in their calculus beliefs as an inductive feature. The driving force of the written curriculum and a focus on student achievement scores has a particular influence on the realisation of learning processes and, thus, the enacted curriculum.

DISCUSSION

Certainly the scope of this paper could not present an exhaustive discussion on teachers´ beliefs on calculus. We discussed two views with the teachers' subjective definitions, the formalist view and the application view. Concerning all teachers one of these views seem to be central in the calculus teachers' intended curricula since other beliefs, in particular the instrumentalist view with respect to exams, have become apparent as peripheral goals. We further discussed two indications of central goals, i.e. the coherence of beliefs concerning different aspects of the teachers' teaching and the existence of conflicts of teaching objectives.

The results of this study of teachers´ (central) beliefs on calculus as one important mathematical domain at upper secondary level suggest that the assumption (occasionally stated, e.g. Tietze, 2000) of a biased instrumentalist orientation of calculus teachers is not valid.

The establishment of a causal relationship between different clusters of teachers´ beliefs on calculus is an effort to reconstruct the network structure. At the current state of our analysis, it is striking that a formalist view could be a key factor to understand the calculus teachers' intended curricula: Those teachers who favour a formalist view as a central goal neglect any application-oriented view. We hypothesise that the individual characteristic of the formalist view considerably forms the calculus teachers' intended curricula.

Having outlined that teachers´ beliefs on calculus and its teaching are inherently different from the domain-specific beliefs of stochastics or geometry teachers, comparisons to other domains have to be drawn carefully, since we did not investigate the teachers' beliefs concerning to all mentioned mathematical disciplines. From a pragmatic perspective this would not have been possible as a single interview on calculus took about two hours. On the basis of present evidence we assume that differences in belief systems in the aforementioned domains surface in every secondary teacher. However, regarding the larger research programme, it has been conclusively shown that there exist differences depending on the mathematical domain being looked at.

At this point, the analysis of the present data cannot yet give a reliable answer whether e.g. pre-service teachers show a higher preference of collaborative activities in their pedagogical practice than experienced teachers who are assumed to favour a more instructivist view of learning. The development of teachers´ belief systems depending on their degree of professional experience as well as some verification whether expressed beliefs are guiding factors in actual classroom practice will be considerately analysed after a further collection of data.

With regard to the previous argument this qualitative study will turn towards the question by means of subsequent interviews in what way pre-service teachers´ beliefs are subject to change in the course of their professional development and whether there will appear a cohesion or fraction of their intended or enacted curricula.

Furthermore an attempt will be made to quantify relevant aspects of our sample with respect to the results of the qualitative approach, which will be evaluated by relevant quantitative data interpretation.

However, up to the present stage, our studies already underline the importance of a differentiated investigation of teachers' beliefs referring to specific mathematical domains like calculus. The focus on a deep and valid understanding of intended and enacted curricula of calculus teachers might further facilitate to understand teachers' classroom practice and to identify approaches to change both teachers' intended curricula, and teachers' enacted curricula.

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ANALYZING FOCUSED DISCUSSIONS BASED ON MKT ITEMS TO LEARN ABOUT TEACHERS' BELIEFS

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The mathematical knowledge for teaching (MKT) measures have become widely used among researchers both within and outside the U.S. The measures as well as the underlying framework have, however, also been subject to criticism. One aspect of the criticism relates to the MKT framework failing to include teachers' beliefs. This paper has a methodological focus and discusses in which ways discussions based on MKT items can be used to tap into teachers' beliefs about aspects of MKT. Two example studies will be used to indicate how analyses of teachers' focused discussions can provide researchers with important information about teachers' epistemic beliefs related to aspects of MKT.

Keywords: MKT, epistemic beliefs, focus group discussions

INTRODUCTION

Familiarity with the participating teachers' mathematical knowledge for teaching (MKT) is beneficial for teacher educators when planning and implementing professional development (e.g., Ball, Thames, & Phelps, 2008). Since beliefs influence teachers' interpretation, application and implementation of pedagogical approaches (Philipp, 2007)—as well as their approach to professional development and eventual gains from attendance (Fives & Buehl, 2010)—teacher educators will also benefit from being familiar with the participating teachers' beliefs. Teachers' beliefs about knowledge (including teaching knowledge)—often referred to as epistemological beliefs (e.g., Fives & Buehl, 2010)—are particularly important. According to Buehl and Fives (2009) distinct beliefs about different aspects of teaching knowledge exists, such as the source of teaching knowledge, the stability of teaching knowledge and the structure of teaching knowledge. In the present paper we focus on practicing teachers' beliefs about a fourth aspect: the content of teaching knowledge (as in Fives & Buehl, 2008)—in particular methodologies that can be used to gain insight into teachers' beliefs about teaching knowledge.

We study the adaptation and use of the MKT measures (Fauskanger, Jakobsen, Mosvold, & Bjuland, 2012). These measures have been used among researchers both within and outside the U.S. The measures as well as the underlying framework have been used and referred to by many, but they have also been subject to criticism. One aspect of the criticism is related to how the MKT framework fails to acknowledge the importance of teachers' beliefs (Petrou & Goulding, 2011; Schoenfeld, 2011). The importance of including beliefs in studies of teachers' knowledge has been emphasized, and some even argue for the equivalence of beliefs and knowledge (e.g., Beswick, 2011). Beswick (ibid.)

suggests that beliefs about mathematical content and pedagogy should be included in the MKT framework. Schoenfeld (2011) supports the idea of including teachers' beliefs, and he argues that this would increase the validity of studies on teachers' knowledge.

Despite the amount research on teachers' beliefs (e.g., Philipp, 2007), relatively few studies have focused on teachers' beliefs about teaching knowledge in general—teachers' epistemological beliefs (e.g., Fives & Buehl, 2010); even fewer studies focus on teachers' beliefs about the knowledge they need to teach mathematics in particular—their epistemic beliefs (e.g., Fives & Buehl, 2008). The MKT framework does not acknowledge the importance of teachers' beliefs (Petrou & Goulding, 2011), and we intend to investigate if and how teachers' reasoning—expressed in writing as well as in focus group discussions—may elicit their beliefs about MKT.

We invited teachers to participate in our studies, where we aimed at exploring how discussions based on MKT items can be used to study teachers' beliefs about aspects of MKT (See figure 2 for an example item). For the purpose of this paper, we will focus on methodological issues related to these FGIs in order to answer the following research question:

> In which ways can focused discussions based on MKT items be used to tap into teachers' epistemic beliefs?

We use two of our own studies as starting point for making this methodological discussion. Seven focus group interviews (FGIs) were conducted as part of the first example study, and the 15 participating teachers in these FGIs were selected from a larger sample of teachers. They had all given their responses to a set of 61 MKT items in a testing situation prior their participation in the FGIs. In the second study, six FGIs were conducted with 26 teachers based on their responses and written reflections related to ten MKT items.

CONCEPTUAL FRAMEWORK

Ball and colleagues (2008, p. 395) define MKT as "the mathematical knowledge needed to carry out the work of teaching mathematics", and this represents a further development of Shulman's (1986) theories of teacher knowledge. In his seminal paper, Shulman (ibid.) distinguished between 'Subject Matter Knowledge' (SMK) and 'Pedagogical Content Knowledge' (PCK). The MKT model builds directly upon this initial distinction (Figure 1).

In the MKT "egg", SMK is divided into three sub domains. 'Common Content Knowledge' (CCK) describes the mathematical knowledge that is common outside as well as inside the teaching profession. 'Specialized content knowledge' (SCK), on the other hand, represents a mathematical knowledge that is unique to the work of teaching. The third sub domain of SMK, 'horizon content knowledge', is mathematical knowledge not directly deployed in instruction.

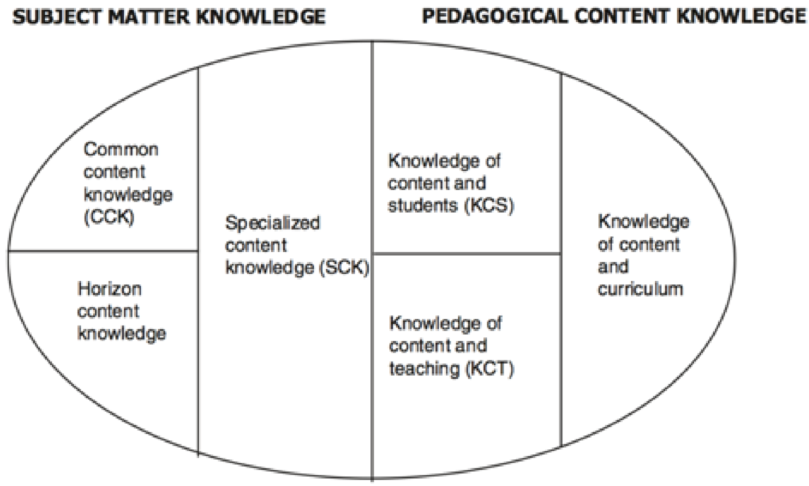

Figure 1. Domains of MKT (Ball et al., 2008, p. 403).

The right side of the oval contains three different categories of knowledge related to Shulman's PCK. The first category—'knowledge of content and students' (KCS)—is focused on students' mathematical thinking, knowledge and learning of mathematics. 'Knowledge of content and teaching' (KCT), which is the second category, refers to the knowledge used by teachers when designing mathematics lessons. The final category, 'knowledge of content and curriculum', includes (but is not exclusively related to) knowledge of grade levels where particular topics are typically taught, assessments and educational goals.

The MKT framework does not include beliefs. This has been criticized, and Beswick (2011) argued that beliefs about mathematical content and pedagogy should be included in the MKT framework. Philipp (2007), on the other hand, suggested that beliefs are closely related to knowledge, but he argued that a distinction should still be made between the terms. In this paper, we follow Philipp's suggestion and distinguish between knowledge and beliefs. We focus on the beliefs that teachers have about MKT, and we consider this to be an aspect of teachers' epistemological beliefs (e.g., Fives and Buehl, 2008). Teachers' epistemological beliefs are considered important by several researchers and several competing models that describe the nature of epistemological beliefs have been proposed, but general epistemological beliefs still seem to refer to "individuals' belief about the nature of knowledge and the processes of knowing" (Hofer & Pintrich, 1997, p. 112). In their attempt to clarify the research in this area, these researchers proposed that epistemological theories are composed of "certainty of knowledge, simplicity of knowledge, source of knowledge, and justification for knowing" (ibid., p. 133).

In order to avoid confusion, we use 'epistemological beliefs' with reference to teachers' general beliefs about knowledge and knowing in this paper. 'Epistemic beliefs' refer to teachers' domain-specific beliefs about knowledge and knowing—in particular teachers' beliefs about aspects of MKT.

Most of the research on MKT has been related to the measurement of MKT by the use of multiple-choice items, and the MKT items themselves are strongly connected with—and can even be seen as manifestations of—the MKT construct. When attempting to investigate teachers' beliefs about MKT, it might therefore be a good idea to use MKT items as a starting point for focused discussions. In the following, two studies will be presented as examples of ways in which focused discussions based on MKT items can be used to tap into teachers' epistemic beliefs.

Bryman (2004, p. 348) suggests that FGIs may give "more realistic accounts of what people think, because they [interviewees] are forced to think about and possibly revise their views". The use of activity-oriented questions in FGIs is highly recommended as productive supplements to oral questions and the importance of asking the participants to do something before the FGI is emphasized (Colucci, 2007). Inspired by other researchers who have used similar items in this way, we included MKT items that were well known for the teachers to focus the discussions. Below is an example from the public released MKT items that focuses on definitions. In the first example study the teachers discussed definitions, and in one of the items the main issue was whether or not 1 is defined as a prime number. The MKT item in Figure 2 is not the exact same, but it also has a focus on the definition of prime numbers.

2. Ms. Chambreaux's students are working on the following problem:

   Is 371 a prime number?

As she walks around the room looking at their papers, she sees many different ways to solve this problem. Which solution method is correct? (Mark ONE answer.)

a) Check to see whether 371 is divisible by 2, 3, 4, 5, 6, 7, 8, or 9.

b) Break 371 into 3 and 71; they are both prime, so 371 must also be prime.

c) Check to see whether 371 is divisible by any prime number less than 20.

d) Break 371 into 37 and 1; they are both prime, so 371 must also be prime.

Figure 2. Item 2 from the set of released items (Ball & Hill, 2008, p. 4).

In the second example study, an item focusing on place value and decomposing numbers was discussed. Since we are discussing two example studies in this paper, we have chosen to make separate presentations of the methodological issues and results for each study below.

STUDY 1—METHODS AND RESULTS

In the first study, 15 teachers participated in seven semi-structured FGIs. These teachers were selected from a convenience sample of 142 teachers. All the participants had a special interest in mathematics and mathematics teacher education and worked individually with a set of MKT items before they participated in the interviews. A complete form (Elementary form A, MSP_A04) with items from the LMT project[2] was used. This form had been translated and adapted for use among Norwegian teachers (Fauskanger et al., 2012), and it contained 30 item stems and 61 items in total. The form consisted of the following three sets of MKT items: number concepts and operations (27 items), geometry (19 items), and patterns, functions and algebra (15 items).

After the teachers had taken the test, they were given a short break before groups of two or three teachers' were invited to discuss the items. The initial aim with these discussions was to investigate whether or not our adaptation of the MKT measures was successful by bringing in the voices of the test-takers. In our previous analyses of these interviews, we learned that the practicing teachers also discussed different aspects of the knowledge they found relevant and irrelevant for their work as teachers—including aspects related to mathematical definitions (Fauskanger, 2012). This inspired us to analyze the FGIs with a focus on beliefs about MKT definitions. The results from this study was presented at the 2012 AERA conference in Canada (Mosvold & Fauskanger, 2012), and the main focus then was on what teachers' reflections on MKT items reveal about their epistemic beliefs concerning mathematical knowledge for teaching definitions.

The transcripts from these interviews were analyzed in two steps. First, directed—often called theory driven—content analysis (Hsieh & Shannon, 2005) was applied to the data. We began by identifying all that was discussed related to MKT-items focusing on definitions, and all that was said related to definitions when discussing other items as well. Both authors first searched the transcripts for occurrences of the words 'define', 'definition' and derived terms. When reading the transcripts, we discovered that words like 'concept' and 'formula' were used more or less as synonyms of 'definition'. We therefore searched the transcripts for these terms as well. In our separate analyses, we ended up with an almost perfect overlap of excerpts from the transcripts. Second, these excerpts were subject to further qualitative analysis to uncover subcategories related to what was said about definitions. For a subcategory to be established, the aspect in focus had to be discussed by the teachers in at least 2 separate interviews. Two researchers carried out independent content analysis of the data to ensure reliability.

Mathematics teachers need to know something about mathematical definitions, and the focus on definitions was present as one of the mathematical tasks of teaching in Ball and colleagues' (2008) presentation of the MKT framework.

When "choosing and developing useable definitions", teachers need to know the actual definitions.

A directed content analysis approach followed by qualitative analyses of data from these FGIs revealed different epistemic beliefs from the teachers concerning the relevance of MKT definitions as well as the task of teaching mathematical definitions. In our analysis, two categories emerged. The first category was: *Knowledge of definitions is an important part of teachers' MKT?* This category refers to teachers' common content knowledge and included the following two subcategories: *Definitions are important* and *Remembering definitions is not important*. Below is an excerpt from the transcripts that illustrates the first subcategory:

Interviewer: You suggest, in a way, more of the kind of tasks that focus on definitions, and less of the kind of tasks that focus on calculations, then?

Betty: Yes, I think that is correct.

Benjamin: Definitions are incredibly important as a pre-requisite, because if you don't have clear definitions and know a little about it, then you will easily be out of track.

Betty: And, what was said after the TIMSS study, what I have heard anyway, is that we score low on concepts. So, I believe it is more important to be clear about this than to be able to calculate correctly.

The teachers in our study seemed to agree that knowledge of definitions is important. Quite a few teachers suggested, however, that teachers do not have to remember the actual definitions—only know about them. Some teachers even claimed that knowledge of definitions—although it is arguably part of teachers' knowledge—is not crucial.

The second main category that emerged from our analysis was: *Choosing and developing useable definitions*—a category referring to teachers' SMK as well as PCK. It contained the following subcategories: *Adjusting to different groups of students*—highlighting teachers' knowledge of content and students—and *Inclusive definitions are confusing*—referring to teachers' common content knowledge. In the FGIs, some teachers suggested that certain definitions are less suitable for the lower grades—which is an example of the first subcategory—as represented by the following excerpt from the transcripts:

Karen: I think they [the MKT measures] should have been differentiated... As an example if one can have a rectangle that is not a parallelogram and that stuff [definitions of quadrangles]. (...). But we do not have [teach] it [definitions of different quadrangles] for the younger ones [students] we teach.

Ken: No, exactly.

This can be seen as supporting the claim that "choosing and developing useable definitions" is a relevant mathematical task of teaching, however the definitions chosen should—based on teachers' knowledge of content and students—be adjusted to different groups of students. This example study indicates that such a methodological approach—i.e. inviting teachers to respond to MKT items in a testing situation followed by focused discussions—might be a fruitful way to tap into teachers' epistemic beliefs related to MKT definitions. However, for future studies it is important to discuss if the teachers being interviewed after having conducted a test might bias the results.

STUDY 2—METHODS AND RESULTS

The second study was presented at the Norwegian conference "FoU i praksis" (Fauskanger & Mosvold, in press). In that study, our focus was on teachers' beliefs about MKT place value. This second study was conducted on the basis of what we had learned from the first study—indicating that FGIs based on teachers' responses to MKT items are suitable for eliciting teachers' views on which aspect of the MKT they consider relevant and/or irrelevant. In the second study, 26 teachers were asked to give written responses to ten MKT items from the "number concepts and operations" scale at home. Follow-up questions were added to the items, and they were asked to reflect upon which items best captured MKT that was important for them. Based on these written reflections interviews were conducted.

The interviews were recorded and transcribed, and the transcripts were analyzed in three steps. First, a double content analysis was made in order to reduce data. The first author read through all the transcripts and identified all that was said related to the item in focus. The second author conducted an independent content analysis in which the transcripts were analyzed by counting words in the text with the purpose of understanding the contextual use of words related to place value. Second, utterances were chosen as coding units, and the context unit was defined to be two utterances before and after each coding unit in which a key word was found. This was followed by a comparison of what was included in the first sample and not in the second, and the union of the two samples was used in the further analyses. In order to analyze teachers' beliefs about MKT, we decided to use the aspects of MKT as presented in Figure 1 as codes in this second part of our content analysis, and the sample was then coded by both researchers. In a third and final step, all the coded parts were analyzed in relation to aspects of MKT place value as highlighted in the research literature.

Our analyses indicate that the teachers as a group emphasize all aspects of MKT (Figure 1) when describing what is important for them as teachers. They agree that place value is an important base for students' future learning of mathematics, indicating that the teachers highlight several aspects of their PCK. When trying to make the teachers identify why it is important, one of the teachers responded:

Interviewer: What makes the place value system so important?

Doris: It is the foundation for counting and arithmetic. It is the basis for everything. The numbers wouldn't even have the names they have if it wasn't for the place value system.

Doris draws on her horizon content knowledge as well as knowledge of content and curriculum. Our findings also indicate that the teachers' utterances—in the FGIs—in some cases contrast important research findings. As an example, the teachers indicate column value to be more important than quantity value, whereas research emphasize that both are important related to multi-digit calculation (e.g., Thompson, 2003). A second example is that the teachers in this example study emphasize a decomposition of multi-digit numbers following the positions, whereas research emphasizes non-standard decomposition as an important base for multi-digit calculation (e.g., Jones et al., 1996). This is interesting since e.g. the standard Norwegian algorithm(s) for multi-digit subtraction includes non-standard decomposition, and it might indicate that this aspect is important to discuss in future professional development. Teachers' beliefs about MKT place value might thus provide a relevant starting point for professional development initiatives.

As in the first example study, this second study also indicates that using MKT items in FGIs might be a fruitful way to tap into teachers' epistemic beliefs about important aspects of MKT—such as their MKT place value. A three step content analysis of data from these FGIs as described above revealed different epistemic beliefs concerning the MKT place value. In conclusion, this example study indicates that inviting teachers to respond to MKT items and reflect upon the items in writing, followed by FGIs including the items might be a fruitful way to tap into teachers' epistemic beliefs.

CONCLUSIONS

Preliminary analyses of teachers' focused discussions related to MKT items in these two example studies indicate that such analyses can provide researchers with important information about teachers' epistemic beliefs related to aspects of MKT—in particular about the knowledge needed to teach place value (example study 2) and their epistemic beliefs about mathematical definitions (example study 1).

Different methodological approaches can be used when studying beliefs and the two example studies presented in this paper exemplify how focused discussions based on MKT items can be used to tap into teachers' beliefs about aspects of MKT. We have learned that analyzing these FGIs using different forms of content analysis can illuminate aspects of teachers' epistemic beliefs about MKT important for future professional development. Knowing, for instance, that teachers believe decomposition following the positions to be most important, professional development can be developed that allows us to discuss teachers' epistemic beliefs about decompositions.

So far, there is a scarcity of research related to epistemic beliefs about MKT (Fives & Buehl, 2010), and more research is needed. Our preliminary results indicate that different approaches to content analyses of teachers' reflections in FGIs based on MKT items can be particularly useful in this connection. Despite the obvious limitations of these two example studies, they support the argument that epistemic beliefs about MKT are important. Further studies of teachers' epistemic beliefs about aspects of MKT are called for, and we believe that such studies might have a potential to inform the further development of the MKT construct as such.

NOTES

1. Our research project has been supported by OLF, The Norwegian Oil Industry Association.

2. See http://sitemaker.umich.edu/lmt/home.

3. See http://ukm-stavanger.info for excerpts from the interview guide.

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ATTITUDES TOWARDS MATHEMATICS OF TEACHERS IN SERVICE OF TELESECUNDARIA: AN EXPLORATORY STUDY

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The results of a research carried out with in-service teachers of federal "telesecundarias" in the area of the city of Xalapa Veracruz, Mexico, will be shown in this article. The research explored the teacher's attitudes towards mathematics as a first approach on the research of affective factors in the teaching-learning process within this educative subsystem. The overall results showed no negative attitudes coming from the teachers but rather neutral with positive charge and slightly positive. However, in the subscales analysis the presence of a negative disposition in items related to anxiety and a low self confidence in some teacher's mathematical skills can be seen.

Keywords: Teachers in-service, telesecundaria, attitudes,

INTRODUCTION

The recognition of the importance of affective factors (emotions, beliefs, attitudes and values) in the quality of learning (Goldin, 2002; Gómez-Chacón, 2000; Hannula, Evans, Philippou, & Zan, 2004) and assessment processes of learning of mathematics has led to include these aspects in educational reforms in many countries in recent years (Martínez Padrón, 2008). Such is the case of Mexico with the secondary education reform in 2006 which has mainly focused on the competence-based learning, giving attitudes a leading role as well as conceptual and procedural contents. In recent years there has been an increasing emphasis on the development of better researching methods in attitudes (Leder & Forgasz, 2006). A particular point of interest was the measurement of some dimensions of self-concept (Bandura, 1997; Malmivuori, 2001), conceptual clarification (Di Martino & Zan, 2003; Ruffell, Mason, & Allen, 1998) and development in college stages (Galbraith & Haines, 2000; Gómez-Chacón & Haines, 2008; Liston & Odonoghue, 2008). As it comes on training future primary and secondary teachers some descriptive studies on attitudes were found. These focused on different aspects: mathematics as an object of study, its role in the society and science, its application in real contexts and the teaching-learning process of mathematics and the proposal of some measuring instruments on conceptions, beliefs and attitudes towards mathematics (Beswick & Callingham, 2011; Camacho, Hernández, & Socas, 1995; Hernández, Palarea, & Socas, 2001; Rico & Gil, 2003).

There is not a consensus in the definition of attitude among researchers in mathematics education (Di Martino & Zan, 2003, 2011; McLeod & Adams, 1989). For this job, we adopt Gómez-Chacón (2000) definition, who accurates it as:

> Evaluative predisposition (negative or positive) that determines the personal intentions and influences on the behavior. Therefore, it consists of three components: one cognitive that manifests on the mentioned attitude underlying beliefs, another one affective, that manifests on the work or matter acceptance or rejection feelings and one intentional or trending to a certain behavior. (p. 23).

And thus a multidimensional perspective was adopted for this paper.

This paper is focused on the teacher's attitudes towards mathematics. There are many factors that make this research important in order to clearly understand what is happening in this educational subsystem; first of all, the lack of enough research in this matter in the Mexican context (Juárez, 2010). Second, the contrasting results of previous researches. For instance, some authors have pointed out that teachers' beliefs and emotions towards mathematics influence the achievement of their students, as well as their beliefs and attitudes towards this discipline (Caballero, Blanco, & Guerrero, 2008; Ernest, 1989; Pezzia & Di Martino, 2011). It is indicated that the development of positive attitudes towards mathematics depends on the teaching style. Thus, negative attitudes may be appear when the teacher teaches instrumentally and the student attempts to learn relationally (Amato, 2004). On the other hand, other researches stated that affective variables such as conceptions, beliefs and attitudes towards mathematics play a determining role in the development of teaching practice (Frade & Gómez-Chacón, 2009; Hodgen & Askew, 2006; Philippou & Christou, 1998, 2003).

It is important to highlight the teaching context in this subsystem. In Mexico secondary schooling is provided in four modes; general, technical, for workers and "telesecundaria". This last one was founded in 1968 to provide coverage to rural areas where technical or general high schools are not built due to its high operation costs. Unlike the other secondary schools, "telesecundaria" teachers are in charge of all the subjects, likewise primary teachers. People with mayors in the "Normal Superior" with specialties such as English, Natural Sciences, Social Sciences, Math, Arts, etc.; and university courses in Physics, Biology, English and Mathematics are hired. Also professionals with other university mayors may be accepted, as long as they take complementary studies of Normal Superior or any Masters Degree in Education. To back up the teachers work a 15-minutes-long class is satellitaly broadcasted. This class is repeated daily during a learning sequence, giving the teacher the freedom to use it whenever it is more convenient. The methodology of this subsystem also includes the use of a book for each subject, one for the student and another one for the teachers. These books are distributed Nationwide, just as it happens in primary schools as well.

As the teaching context in primary schools and “telesecundarias” is quite similar, as stated before, some findings that researchers have reported in other countries on training primary school teachers will be used. It was found that many people at university stages have weak knowledge and negative attitudes, being some of them even afraid about mathematics (Beswick & Callingham, 2011; Frade & Gómez-Chacón, 2009; Philippou & Christou, 1998). While others have revealed moderated to positive attitudes among teachers in training and currently working (Caballero et al., 2008; Eudave, 1994). Additionally, a low performance in mathematics can be clearly seen in this subsystem as it has been shown by the international results in PISA 2006 and nationally in the results of ENLACE for its initials in Spanish (National Assessment of Academic Achievement in School centers). Therefore the need of researching on the telesecundaria teachers’ attitudes towards mathematics in order to set the ground for future researches that may be carried out. The research question we have raised is then the following: What are the attitudes towards mathematics of Telesecundaria teachers who do not have a training profile in math?

Teachers who have a mayor in mathematics were not taken into account in this research since it was assumed their attitude towards mathematic should be positive.

METHODS

Subjects

35 teachers attached to the Federal Telesecundarias of Xalapa, Veracruz, Mexico. 70 teachers with many different specialties compose the complete staff. As indicated before teachers with specialty in math according to the aim of the study, and also those teachers who due to their administrative functions (directors, supervisor, etc.) are not teaching classes were not included.

Instrument

The Likert scale, ATMAT by its initials in English of Ludlow y Bell (1996) was carried out. It was designed to measure attitudes and experiences related to mathematics and its teaching. This scale consists on 29 items, with options for three levels of agreement and three of disagreement. It was piloted with 240 undergraduate and graduate students in a mathematics teacher training institution in New England. Cronbach’s alpha coefficient as well as a factorial analysis were carried out in order to validate this scale. Cronbach’s alpha coefficient was 0.96

As the scale was applied in a Spanish-speaking context it was translated into Spanish and in order to avoid understanding problems from which it was piloted. This lead to separate item 1 in two, so that the final scale was composed of 30 items. Each one of them had five options ranged from “strongly disagree” to “strongly agree”, including the neutral or of indecision option, since we consider that in this way we would obtain more descriptive results. Cronbach´s alpha coefficient was verified for this version of the scale, obtaining a 0.754 value, which continues being a good reliability index, even if diminished with division of item 1.

The scale was divided into three subscales according to the three components of the multidimensional model of attitudes: Cognitive Subscale (C), Affective Subscale (A), Behavioral Subscale (B). The Affective Subscale is formed by items that refer on one side to the liking/disliking towards mathematics and on the other side to the anxiety caused by mathematics itself and the resolution of mathematical problems and also by the possibility of their teaching in different educational levels. The Cognitive Subscale is formed by items that refer to the beliefs about mathematics, mathematical problems, and self-efficiency and competitiveness beliefs. Finally the Behavioral Subscale is formed by items that show the tendency to act in concordance with their perspective of mathematics in learning and problem solving. This classification was validated by the criterion of two experts in research on attitudes in Mathematics Education. The drafting of the items in the subscales were of two types: positive (because they reflect a positive attitude) and negative (because they reflect a negative attitude). In the following table we present an item from each subscale according to the type of drafting as an example.

NO. ÍTEM	SUB SCALE	DRAFTING TYPE	STATEMENT
1	C	Positive	Mathematics is very interesting to me.
16	C	Negative	I have forgotten many of the mathematical concepts wich I have learned.
2	A	Positive	I enjoy math courses.
4	A	Negative	I feel a sense of insecurity when doing math.
17	B	Positive	I learn mathematics by understanding the underlying logical principles, not by memorizing rules.
12	B	Negative	I do not attempt to work a problem without referring to the textbook or class notes.

Table 1: Example of items in each subscale

After evaluating each teacher intervals were established to classify their attitudes as it is shown in the table 2. So if a teacher, for instance, obtained a score of 68, he/she would be in the interval of 60-74 and his/her attitude would be considered neutral with negative charge, symbolized as “I—”

Intervals	CLASSIFICATION OF ATTITUDES	SYMBOLOGY
30-44	Strongly negative	NN
45-59	Moderately negative	N
60-74	Neutral with negative charge	I--
75-89	Neutral with slight negative charge.	I-
90-104	Neutral with slight positive charge.	I+
105-120	Neutral with positive charge	I++
121-135	Moderately positiva	P
136-150	Strongly positive	PP

Table 2: Intervals, attitudes and symbols classification

Application

The schools of participating teachers were visited with their authorities' consentment. Teachers taking part in this research gathered in a classroom where they were given the necessary indications. They were invited to answer honestly as their names would be kept private. Afterwards it was proceeded to deliver the printed scale, giving them plenty of time to answer.

RESULTS

The results are presented in two sections, the first of them called global results. The total scores of the participants were addressed, which allowed us to characterize their attitudes. In the second section, results for subscales, a more detailed analysis according to the three components of attitude was done.

Global Results

According to the classification of attitudes raised, 23 teachers exhibit neutral attitudes, 4 of them are neutral with slight positive charge and 19 neutral with positive charge, the 12 remaining ones present positive attitudes, 11 moderately positive 1 strongly positive. It can finally be seen that they do not present negative attitudes.

To sum up it can be said that most teachers have neutral with positive charge and moderately positive attitudes towards mathematics (30 teachers). This is somehow consistent with the results obtained by Caballero et al., (2008), who found that Primary School teachers in training in the Extremadura University had not negative nor rejection attitudes. Eudave (1994), also found neutral with slight positive charge attitudes within high school in-service teachers. On the other hand, this is contrasting with results showing negative or even frightening attitudes among Primary School prospective teachers, especially at early training stages (Beswick & Callingham, 2011; Frade &

Gómez-Chacón, 2009; Philippou & Christou, 1998). We have compared our results with the ones obtained on studies over primary professors, because of the context similarity as we anticipated on the introduction.

Subscales results

Valuable information on the affective, cognitive and behavioral areas has been gathered after this research.

Affective subscale. Unlike what was found by Philippou y Constantinos (1998), who stated that "an alarmingly high proportion of students brought very negative attitudes to Teacher Education" (p. 196), most of the teachers show from moderately positive to positive attitudes on the different items that form this scale, specially on those about liking of mathematics, where no negative attitudes are shown. However, despite this positive tendency, we find high anxiety levels on 20% of the participants on items referred to "it makes me nervous to even think about having to do a math problem". Also on items referring the possibility of teaching mathematics on the next educational level, we find negative attitudes on between the 16.7 and the 33.3% of the participants.

Cognitive subscale. Most teachers showed favorable conceptions about mathematics, math problems and classes as well as self-confidence regarding their capacity of facing mathematical problems and their background on the subject. However, a 16.7% of the teachers of this study show a very low self-efficiency referring problem solving, for example the "I can draw upon a wide variety of mathematical techniques to solve a particular problem" kind of items. Similar results are to be found on items related with professional training, where from a 13% to a 20% of the teachers show a very low sense of competence. This is coherent with the results obtained by Caballero et al., (2008).

Behavioral subscale. In this topic there are neutral with positive charge attitudes. It can be stated that most teachers prefer learning through the real understanding of the logical intrinsic principles rather than memorizing rules although 26.7% manifests indifference. Many of them would be willing to work with problems which go beyond the difficulty level prepared for class while some other were undecided or reluctant to do so.

DISCUSSION AND CONCLUSION

We can observe on the global results that on one side, no negative attitudes are shown and on the other side the teachers attitudes are to be found mostly between neutral with a slight tendency to positive, to moderately positive. Watching the graphic carefully, it is to be noticed that the highlight occurs on the neutral attitudes (23/30) instead of the positive (7/30) ones. So the obvious question that comes out is, why does the highlight occur on the neutral attitudes? We can find the explanation on the results by subscales analysis, where some dimensions like the value and liking of mathematics are highly scored, while others, despite of being positively scored, negative and even very negative attitudes appear on percentages that go from 16.7% to 33% of the participant teachers. These issues can be grouped in two topics: The first one refers to the mathematical problems along the three subscales and the anxiety caused by facing them and a low self-efficiency perception at solving them, that seems to be caused by the feeling of not being secure of their previous knowledge and developed skills. Just as it was pointed out by Philippou & Constantinos (1998), who stated that the lack of confidence to solve problems is directly associated with negative attitudes. The second one refers to the possibility of teaching mathematics in different educational levels, particularly in the immediate superior level, where the 33% show a high insecurity level.

On the other hand and returning once again to the relationships among attitudes (cognitive, affective, behavioral) dimensions, we recognize some external factors that can influence each teacher's internal perspective. We consider that the attitude is part of this internal perspective. The external factors to consider are: teacher characteristics and teaching characteristics, in particular the method or problem solving (Wilkins & Brand, 2004; Wilkins & Ma, 2003). Let's consider each one of these factors (cognitive, affective and behavioral) more closely to obtain a better understanding of the teacher's perspective. Complementary to these scales, the information obtained from informal interviews, highlights the external factors that affect their perspective and their attitudes towards mathematics. The first external factor that emerged from the obtained information belonged to characteristics of the teachers when they were students. We see these characteristics as some of the most important, since teachers often have the power to affect other factors. These teachers described memories of nice teachers, funny teachers and devoted teachers. They also underlined personal attention that some students received from their teachers and the effect this had on such students' attitudes. So their "views" are related with the influence a teacher's behavior had on their attitude towards the class.

The second external factor of teaching characteristics is clearly related to the teachers' characteristics and their capacity to solve problems. The participants were encouraged to reflect on their mathematical experience and talk about their attempts to solve problems. Their knowledge of mathematics were insufficient to solve all the given problems, causing them anxiety when dealing with students and work at different levels. Significantly fewer teachers claim they have to memorize ideas. A medium-high percentage of teachers consider a dynamic and social perspective of mathematics. It is not a subject about concepts and procedures they have to memorize and they indicate the importance of working a problem-solving approach ("I learn mathematics by understanding the underlying logical principles rules, not by memorizing rules"). We note the apparent discrepancy in their responses when asked about their beliefs regarding themselves (self-efficiency). They indicate that their tendency is not to be able to solve problems. Whereas when they responded to beliefs about mathematics itself, their responses are more suited to social desirability.

All these findings and information supporting our statement that this group needs specific course in problem-solving, in order to improve the cognitive dimension of attitudes from mathematics as a body of procedures to be learned, to mathematics as a process of thinking~~.~~ These courses shall also reinforce their knowledge and mathematical skills, so that they manage to perceive themselves as more capable to solve problems and as consequence, their anxiety levels will diminish. This way, we may be able to reverse and prevent poor attitudes towards mathematics in “Telesecundaria” Teachers. Besides, since the secondary reform in Mexico on the mathematics subject is focused on problems solving, teachers have to be prepared, both, in knowledge and attitude, in order to the reform to be successful, as regards to improve the mathematics teaching and learning.

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ETHIOPIAN PREPARATORY STUDENTS' PERCEPTIONS OF THE RELEVANCE OF MATHEMATICS TO LEARNING GOALS

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This study investigates Ethiopian students' perceptions of the relevance of mathematics to their learning goals. Interview, classroom observation and survey were used to examine students' perceptions in the cultural, economic, historical and social context they are situated. The results indicate that participants in this study hold perception of the relevance of mathematics which is characterized by unknown future, trust, identity, empowerment to make informal decision and exchange value, which are held to various levels of strengths, with the characterizations trust, unknown future, and exchange value being strongly held. They are motivational factors to students' engaging in mathematics, and emotion is intertwined with them.

Key words: emotion, Ethiopia, identity, motivation, perception of relevance

INTRODUCTION

In Ethiopia the transformation of schooling from indigenous to modern one occurred on a contradictory situation: the religious and cultural background of the country on the one hand and the need for modernizing the country on the other (Wagaw, 1979). The contradiction seems to have contributed to the society's perception of schooling as a means of getting job rather than a means to solve societal problems. The students find themselves in this historical context of schooling. Literature suggests that students' perceptions about the relevance of tasks to their goals are motivational factor to their engaging in it (Wigfield, Hoa & Klauda, 2008). Students' goals are motivational factor for engaging in mathematics (Goodchild, 2001). Students' motivation to engage in mathematical tasks affects their performance, and their perceptions about accessibility of the goals affect their motivation (Hannula, 2006). Students regulate their emotions in order to achieve their goals (ibid). According to Wigfield, et al (2008) the importance of a task is associated with the individual's identity. There are diverse ways to motivate students to engage in mathematics. The rationale for their learning of mathematics could be one way (Wæge, 2007). This study is carried out in a school in the capital of Ethiopia. A preliminary result from a pilot study which exposed the characterizations of students' perceptions of the relevance of mathematics was reported in Gebremichael, Goodchild & Nygaard (2011). A questionnaire was designed and survey was undertaken based on this result. The purpose of this paper is to report on characterizations of students' perceptions of the relevance of mathematics with respect to their learning goals. The research questions I addressed are 'what are the characterizations of Ethiopian students' perceptions of the relevance of mathematics with respect to their learning goals; how strongly are these perceptions held, and do these perceptions have the same distribution across different categories of students. This paper is structured in such a way that the theoretical perspective is followed by methodology. Then, the qualitative and quantitative data analyses are presented, followed by the conclusion.

THEORETICAL PERSPECTIVES

In this study the theoretical perspective adopted is sociocultural theory, particularly, cultural historical activity theory (CHAT), which is discussed in more detail in Gebremichael et al. (2011). The Engeström model (Cole & Engeström, 1993) is used as an analytic tool to describe and analyze the students' perceptions of relevance. The model depicts the activity system, which involves the interaction between the subject (students) and the object (motives, goals, learning school subjects and the material resources) mediated by tools and artifacts. This interaction is also mediated by the rules, division of labor, and community (ibid). In their participation in the activity of schooling, students have motives. They belong to a community of natural or social science streams. The members of the school community are governed by the school rules. The school rules enforce that they should take school subjects including mathematics. They should cover certain topics over the year based on a mandatory textbook, which is a mediating artifact for their learning and the resulting perception.

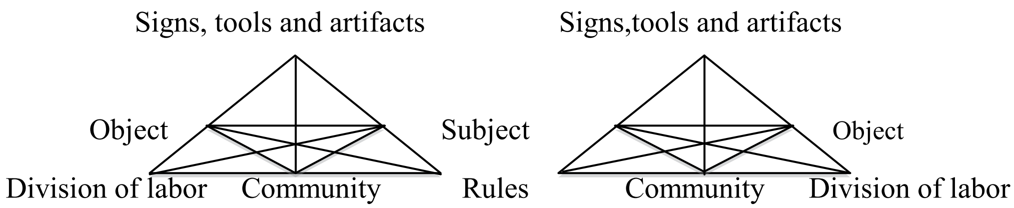

Figure 1: Interacting activity systems adopted from Roth & Lee (2007).

The students participate in a network of activity systems (Roth & Lee, 2007), and see figure 1. In their out-of-school lives students participate in the activity of family life where members of the family as a community share a common motive of survival of the family and a goal of enhancing the personal development of the student. According to the division of labor in the local activity system, students have responsibilities of mentoring their younger siblings as they are mentored by their elders. The rule enforces following and obedience to the elder. For many students, their parents are unlikely to know what they are learning, as the parents may not have the appropriate education. The preparatory school system is new which was only introduced in 2003 and replaced the freshman program of universities. The students in this study are situated in a historical context of schooling and learning mathematics. They have been learning mathematics before they were enrolled in preparatory school and their elders might have learnt preparatory mathematics. The school and the local activity systems interact; the model is dynamic and the components might embrace different entities that might change (Roth & Lee, 2007). Tension might arise within and between the components (Ibid). Emotion, motivation and identity are integral to each other and are distributed all over the activity system (Roth, 2007). It is in such a framework that the students' perceptions develop.

Following Vygotsky (1978) perception is understood in this study as a formation of meaning and making sense of encountered mathematical experiences.

METHODOLOGICAL ISSUES

This study employs a mixed approach, in which interviews and questionnaire are used to collect data in a school where I taught more than a decade ago. The students of this school are both sex. They are divided into two grade levels and two streams. The teacher further divided them into levels of achievement. These student categories were used for sampling both in the qualitative and quantitative data collection. They are of age 16 - 18, with a possibility of few exceptions. I undertook a pilot study using group interview, supported by classroom observation. A total of 24 students were selected for interviews by homeroom teachers (teachers who have a firsthand responsibility for a class of students). The department head selected 4 homeroom teachers who also teach mathematics, and they selected 6 students from their respective classes. 3 students of the same sex who are high, medium and low achievers were interviewed together. Analysis of the data resulted in the characterizations of students' perceptions. Using results of the pilot study, questionnaire was developed. The characterizations emerged were used as Likert scale items, and 335 students completed it. The data analysis is discussed next.

QUALITATIVE DATA PRESENTATION AND ANALYSIS

The five themes pertaining to students learning goals and the mediational process that regulated the development of these perceptions are presented here.

Mathematics is relevant because it is useful in an unknown future

The students are preparing for university studies and they have goals associated with it. Students form perception of the relevance of mathematics related to this.

Andualem: Is math useful for your future? How about the math you are learning now?

Habtu: I want to study astronomy and my brother told me that in addition to mathematics, physics is the base.

Abebe: I don't know the detail about astronomy and how much mathematical capacity it requires. Since mathematics is important in our everyday activities, it would be the same at that level. I think it would be important.

Meseret: [Mathematics] is a mother tongue. … In economics there is slope. We learnt it in 7th or 8th. We didn't know then that it has this use.

Makida: In books we don't see where to apply [it]. [It] has relation with other subjects and we apply it on them… at tertiary level.

As Habtu is participating in the activity of family life, he is mentored by his elder brother towards enhancing his personal development. The local community mediates his perception. Meseret considers her prior experience and projects it to her future. They perceive that preparatory mathematics is relevant to their future goals. The other subjects, school rules and artifacts mediate their perceptions. The school curriculum and the textbook do not give a rationale for learning the topics with respect to their future; their perceptions are motivational factor for engaging (cf. Goodchild, 2001; Wæge, 2007). It is discussed in “identity” that students set goals based on their relationship with mathematics, and in “exchange value” that mathematics determines their access to what they intend to attain in the future.

Mathematics is relevant because I trust the curriculum

There is a sense of trust by the students in the curriculum and/or the teacher.

Andualem: Why are you learning mathematics? Is it useful?

Asad: It is mentioned in the objective of the textbook. … I do not exactly remember specific examples. … I think it is because we should learn it.

Meada: In the objective it says ‘at the end of this lesson you will be able to understand’. … [Limit] is used in derivative.

Ruth: Our teacher usually tells us.

Azenegash: [The teacher] is our eye… If it were not relevant we wouldn’t have been taught. I think it is useful.

Asad perceives that they learn mathematics because they should learn it. In his participation in the activity of schooling to realize his motive of joining the university, he undertakes actions such as engaging in mathematical tasks and attending mathematics lessons. These actions are directed towards the goal of learning mathematics. He forms certain perception about the relevance of mathematics, which in this case is characterized by trust for the curriculum. When he is engaged in mathematical tasks the textbook is an artifact he uses. The textbook mediates his perceptions about the relevance of mathematics. Following Asad, another member of the same group, Meada explains what the objective informs and he gives a specific example. My review of the chapter on limits in the textbook confirms Meada’s statement. In one of the bullet points that lists the objectives it mentions that the concept of limit gives a basis for differential calculus. This provides a sense of rationale for their learning of the concept of limit if they know what differential calculus is for. The textbook tells what the student should expect to learn by the end of a chapter. This might prepare the students to what they should learn and what they should pay attention to. However, the textbook does not give a clearly articulated rationale for their learning of the concept of limit, particularly with respect to the students’ future goal. The perception emerged as a development from the tension between the object and the rules. The students have a high assumption of the teacher and the tension is mediated by the teacher. This is seen in Ruth’s and Azenegash’s narratives. The teacher and her/his practice mediate their perception which is characterized by trust. The rules, the division of labor in the school, where their role is to listen to the teacher, mediate their perceptions.

Mathematics is relevant because it gives an identity

Students form their identity in relation to mathematics and their future goal.

Andualem: Is math useful for your future? How about the math you are learning now?

Debesh: I want to study Banking and Insurance because it has mathematics …. I like mathematics … it is not difficult for me.

Essayas: I want to study law because my brother told me that it doesn't involve mathematics … economics, but [it] has mathematics; so I don't like.

Andualem: Do you find problems or examples which indicate that math has application?

Ruth: … Most social science students do not like mathematics. Only few students work hard. Thus our teacher always advises us.

Debesh perceives himself as someone who can do mathematics well. Essayas, on the other hand, perceives himself as someone who does not want to deal with mathematics. They are identified by their teacher as high and low achievers, respectively. In order to realize their motive of becoming university student, they attend classes and study school subjects towards achieving the goal of learning. There is emotion intertwined with their object and their decisions (cf. Roth, 2007). Their perceptions are mediated by the local community, and their emotion towards mathematics. Their emotions and identities are motivational factors in engaging in mathematics, and towards making a decision about their future study. Ruth is a high achieving and in other discussions as well she refers to the whole of students, and the social science when locating herself in the mathematics classroom. She is situated in the community of social science students who generally do not like mathematics and are taught by a mathematics teacher whose background is natural science. In the division of labor the teacher has the role of providing advice about working hard while the students have to listen to it. The teacher could tell them to work hard on mathematics and that it is important for their educational career. But, the student could not know how the mathematics they are learning would help them. The tension that Ruth experiences within her community seems to lead to a development of perception that is characterized by identity. Emotion is involved in here as well because the advice was initiated because of the social emotion exhibited among students (cf. Roth, 2007).

Mathematics is relevant because it empowers one to make informal decisions

Some students make judgments about what mathematics is for. Some perceive that mathematics and the other subjects are there for them to expose their talents.

Andualem: Why do you need to learn the mathematics you are learning now?

Erikihun: I want to study language or philosophy. … I am doing well in language. … Math and most of the subjects we are learning now might not be related to what we learn in the future. But, they help us to identify/know our interest and direct us to the future. We used to learn music; it is not important but if you have the interest then you will know. Some of us may end up in a field that doesn't involve math at all but others may need it.

In his history of participating in the activity of schooling he has been undertaking actions towards the goal of learning. In particular, he was attending to school subjects including mathematics, before he was enrolled in preparatory school. He is still attending to some of these subjects. He perceives that the other subjects are competing with mathematics for students' choice. Since he made other choices, he perceives that he doesn't need mathematics for his future, but he has to learn it because others in his group need it. The school rule enforces the topics that the student should cover in a year, and it has to enable him to understand why he is learning the subjects. His perception is mediated by other subjects, the school rules and community. It is motivational factor, and negative one (cf. Wigfield, et al, 2008). His emotion determines his future direction and is intertwined with his perception (cf. Roth, 2007). This characterization of perception is a development from tension within the object: the learning of mathematics and future goal are in tension.

Andualem: Do the mathematics textbooks reflect that math is useful? Do you see examples that show applications?

Netsanet: Before we use formulas, there are items which we do simply by observation, by looking at it attentively. That helps you to think and analyze; it broadens your mental capacity.

Netsanet perceives that mathematics is there to broaden her mental capacity. Her perception is mediated by the school curriculum, artifacts, the formula, and the prior meaning established in her about formula and other mathematical artifacts. Her perception of relevance about the mathematics she is learning observed to be exciting for her and seems to motivate her to work on mathematics.

Mathematics is relevant because it has exchange value

These students are supposed to score a qualifying grade to be admitted to the university, and Ethiopia is a poor country in which success in education and securing a job relates to sustaining the life of the individual as well as parents.

Andualem: Tell me your history in schooling and your experiences in mathematics?

Beza: We used to hear that 10th is the turning point for life. … [Studying] any social science would be ok [to be a hostess]. … Mathematics is compulsory.

She plans to become a hostess, and she perceives that the school that offers the training requires social science background of university. She perceives that success in mathematics is the gate keeper to joining the university and to achieving the goal of securing a job – becoming a hostess. This perception of relevance she attaches to mathematics motivates her engaging in mathematics.

Andualem: What do you plan to study? Is mathematics useful for your plan of study?

Ruth: Earlier I wanted to study law but it is 5 years. [I] study economics … then I can help my parents. … If I do not have the basis in mathematics I can't do it.

Ahadu: I want to become a [medical] doctor. … Whether one becomes a medical doctor or something else, learning mathematics is part of the process.

Ruth learns mathematics to sell it at the marketplace of learning economics so as to get job at the end, which enables her to sustain her family and study law which she really likes to study. The division of labor in her community – her responsibility to support her parents – mediates her perception of relevance. Her perception of relevance is a motivational factor for being engaged in mathematics. Ahadu perceives that one cannot make her/his way to the future without dealing with mathematics. His perception is mediated by the rules.

Andualem: What do you plan to study? Do you see application of mathematics?

Yirdaw: I want to be a private accountant. … Mathematics books from abroad are better at applications than domestic ones. … I prefer the [latter] for success in exams. But, for my interest I prefer the [former]".

Andualem: Do you find learning mathematics useful? Are the concepts you learnt this year of interest to you?

Alewi: I am not interested in it but it is required … I liked polynomial at the beginning. … when I scored poor at the first test, I turned my back to it again".

We see the tension between the mediating artifacts that are available for Yirdaw and the school rules that he has to succeed in examinations which are pertinent to the text book. The tension results in the development of his perception. Emotion that is associated with the artifacts is exhibited in Yirdaw's interest in using some of the artifacts than the others. Emotion also mediates students' perceptions. Though Alewi did not like it, because of the exchange value mathematics has, she put effort to succeed. Cognizant of the fact that her effort is in vain she dropped it as less relevant (cf. Hannula, 2006). Emotion is integral to and affects her perception of relevance. Her motivation is also seen to decline as a result.

PRESENTATION OF QUANTITATIVE DATA ANALYSIS

The quantitative data analysis and results are presented here. A large proportion of the students' parents do not have education that enables them help the students in their education. More than 63% of the mothers do not have any training above secondary level, while only 15% of them have training above secondary level. Only less than 6% of the mothers have university degrees. Their fathers' do not seem to be far different from their mothers'. More than 52% of the fathers do not have any training above secondary level, while 25% of them have training above secondary level. Only less than 16% of the fathers have university degrees. About 22% of the students mention neither their fathers' nor their mothers' levels of education. On the other hand, the students hold the five characterizations of perception to various degrees of strengths. The proportion of students who held perceptions characterized by trust is 71%; unknown future, 68%; exchange value, 63%; identity, 43% and decision, 33%. The first three are strongly held. Though the perceptions characterized by identity and decision are not strongly held, a significant proportion of students hold these perceptions. Identity especially is significant, when compared to the proportion of students who do not hold this perception, which is 31%. The item on future use of mathematics is about something which the students did not experience yet. A follow up question was asked, about their sources of information. For 64 % of the students their teacher is their source of information. Their mothers, fathers, brothers, sisters, or relatives are very rare sources.

Three of the five characterizations of students' perceptions did not show any significant difference based on the categories of gender, grade level, stream and levels of achievements. The distribution of items such as "preparatory mathematics is useful to get access to my future plan" is the same across all categories. This was tested using bar graphs as well as the Mann - Whitney U Test (and Kruskal-Wallis Test in the case of three categories) (Field, 2009). But, the students' responses for the item 'preparatory mathematics is useful in what I plan to learn in the future" had different distributions across the two streams. The results obtained from SPSS using the Mann-Whitney U test suggests that the null hypothesis that this characterization is the same across categories of stream should be rejected because its significance value is less than the significance level, 0.05. Similarly, the item "Preparatory mathematics is important because it helps me make decisions" does not have the same distribution across grade levels. 11th grade students held more strongly. There seem to be other variables such as economic benefit than mathematics which influence their decisions, because of students' economic backgrounds and the absence of appropriate information. Moreover, the 12th grade students need to make decisions about their future studies sooner, as they have to choose the fields of study before they are enrolled into the university. The distributions of these characterizations are the same across the other categories. The mandatory use of a single textbook together with the teacher's guide at each grade level, might make students' prior experiences with respect to schooling and mathematics classroom more or less the same. The natural science and social science students use the same mathematics textbook, which is prepared by experts and taught by teachers who have natural science backgrounds. Although there are subjects and few chapters in mathematics which are different for the two streams, the students' experiences in mathematics is more or less uniform. On the other hand, there are distinct differences in students' fields of study when they join the university based on their streams. This could be a possible reason for the difference in distribution of perception across the two streams about the relevance of mathematics to their future plan of study.

The situation of female and male students in schools is changing overtime. Over the past four decades there have been affirmative actions towards increasing the number of females in learning institutions and workplaces to compensate for the historical situation that disadvantaged the female. The number of female and male students in the preparatory school is balanced. The gender gap is becoming narrow, through years, in universities as well as workplaces. The number of female mathematics teachers is now increasing. A year before the data was collected the school principal was a women. During the data collection one of the two deputy school principals was a woman. These situations might have a part in creating a balanced sociocultural context for both sexes. The levels of achievements also had uniform distributions of perceptions, probably because of the high competition to join the preparatory school and very high success rate to join the university. These students were screened by a national examination and are considered as elites of their age group. The number of public and private higher learning institutions and intake rate is increasing. This being an important success indicator for the government, it highly publicizes it on the limited number of media the country has. This data was collected about four months ahead of the national and regional elections in the country, and these were major campaign instruments for the ruling party. The students get such information both in and outside the school. They are hopeful of joining the university.

CONCLUSION

For these students mathematics has exchange value, which transcends beyond the success in joining the university. The students perceive that mathematics is useful in their intended future field of study. The absence of clear idea about their future studies and what it requires in relation to mathematics on the one hand and the lack of information about why they are learning mathematics on the other, seems to lead the students to make their own speculations or accept that it is useful based on trust. Their trust for the curriculum seems to be affected by the lack of information as well as their regards for the teacher. Besides, for most of the students the sources of information about the future use of mathematics are their teachers. This has implications for the curriculum and/or textbook preparation as well as the teachers' role in designing the school curriculum of mathematics classrooms in Ethiopia. Though direct application problems to the students' future plan of study may be difficult to provide, it is important to inspire them by providing the rationales for their learning of the concepts (cf. Wæge, 2007). The necessary familiarization to their future fields of study with respect to mathematics needs to be done at the preparatory level. The mathematics teachers can be instrumental in enhancing the students' motivations by boosting their perceptions. Students' emotion, Identity and motivation are intertwined to each other. The former two influence one another and affect the later (cf. Roth, 2007). Popularizing mathematics, particularly, with respect to social science students needs attention. Understanding how students motivate themselves in order to engage themselves in the learning of mathematics is important. The characterizations of perceptions of relevance of mathematics have motivational effect on students engaging in mathematics. Future studies need to investigate the strengths of these characterizations as motivational factor for students.

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MATHEMATICS CONFIDENCE: REFLECTIONS ON PROBLEM-SOLVING EXPERIENCES

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The purpose of this paper is to present the findings of a qualitative inquiry into Grade 8 and 9 learners' mathematics confidence. Learners reflect on experiences with mathematics that have influenced their levels of confidence in the social, psychological and intellectual domains. A convenient purposive sample of four participants, individually interviewed with reflective questioning, provides insight into their reflective accounts on experiences with mathematics problem solving. Through content analysis, the researchers' findings confirm the dimensionality of mathematics confidence and present sources of participants' mathematics confidence.

Keywords*: Mathematics, mathematics confidence, reflection, metacognition, anxiety*

INTRODUCTION

Maree et al. (1997, p.95) classifies problems in study orientation as cognitive, external, internal and intrapsychological causes influencing achievement in mathematics. Sherman and Wither (2003, p.138) documented a case where a psychological factor, mathematics anxiety, caused an impairment of Grade 8 learners' mathematics achievement. There is a scarcity on literature focusing on metacognitive aspects and its link with affect in mathematics learning and teaching (Efklides, 2006, p.6). However, aspects of affect, such as values, beliefs and motivation do not primarily form part of this paper, although these aspects are considered in the larger study. Instead the focus is on mathematics confidence and the metacognitive reflections learners have on problem-solving experiences.

The findings of Bormotova (2010) confirm this link between metacognitive reflection and mathematics confidence. The current paper explores the nature of four Grade 8 and 9 learners' mathematics confidence by focusing on their reflections on experiences with mathematics problem solving.

In mathematics, an affect such as anxiety causes personal distrust of intuition and a consequent lack of effort, which are seen as learners' greatest barrier to achievement (Aschraf & Kirk, 2002, p.2; Thijsse, 2002, p.18). John (2009) proposes a model for metacognitive reflective practice and explains this as looking into one's own thoughts and feelings as a guide and aid to problem solving. According to John (2009), reflective practice consists of two phases. The first, where the learner recalls problem solving experiences, identifying the goals and achievements as reflected upon. Through such reflection, learners begin to understand their use of approaches towards mathematics problem solving. In the second phase they describe accompanying feelings and emotions that surrounds their memories and selected strategies. A deeper form of reflective account is established once the learner recognises how he or she feels, what they did and why. These emotions are metacognitively collective and could lead the learner along either of two different paths (Bormotova, 2010, p.29). The first gives rise to faulty beliefs that cause apprehensive thoughts about mathematics and threaten performance, thus interfering with the learner's thinking, memory processing and reasoning. Along the other road, confident learners may experience symptoms that are opposite to mathematics anxiety (Strawderman, 2010; Sheffield & Hunt, 2007, p.2). These include feelings of relief, fun, enjoyment and support, which aid with the solving of mathematical problems in a wide variety of contexts. Metacognitive reflection could ease mathematics anxiety, foster confidence and promote achievement in mathematics.

Some conscious and unconscious social, cognitive and personal evaluation practices are explored. The sources for low and high forms of mathematics confidence are scrutinised and contextualised.

CONCEPTUAL FRAMEWORK

Experiences with mathematics problem solving are either successes or failures and are accompanied by feelings of high or low confidence. Three integrated components feature in Strawderman's model for mathematics anxiety. In her study she adapted the model to include the opposite affect – mathematics confidence – in the social, intellectual and psychological domains (Strawderman, 2010, p.1). A natural overlap occurs between the boundaries where these domains coincide and our current study reflects upon these overlaps as portrayed in the conceptual framework. The discussion that follows provides insight into the theoretical influences in each domain. Metacognitive reflection, in this study, was seen as summative cognitive processes and involves thinking, examining, differentiating, detaching and serves as a self-explorative action (Kaune, 2006, p.350) and regards social, psychological and intellectual aspects of experiences with problem solving.

Social influences

Bergh and Theron (2009, p.86) explain the social domain in terms of Bronfenbrenner's model for ecological systems. The environment includes four levels where human influences develop. The first is a macro system, consisting of persons and organisations with the most frequent contact with the individual. A second is the meso system, consisting of interactions with groups such as schools. The exo system involves aspects outside immediate contacts, such as hospitals, social groups, chat rooms and clubs. The fourth system involves the micro system. This includes habits, socio-economic and political influences. The social domain represents external factors outside of the individual's control, which are contributed by persons such as family, peers and teachers (Strawderman, 2010, p.2).

Psychological influences

Reinforcement, from the psychological behaviourist view, has a positive or negative motivational effect on individuals and the outcomes of their actions (Bergh & Theron, 2009, p.156). Rewarding the correct action, approach or behaviour and punishing the incorrect has completely diverse consequences. Positive motivation involves setting goals and helping problem solvers to achieve those goals, which leads to the successful handling of more tasks that are complex. However, negative motivation can cause undesirable effects such as hostile behaviour and avoidance of certain tasks. It may also suggest alternative ways of doing the same wrong thing differently and instigate fear, which reduces the willingness to continue with a task. Hattie (2009), however, has found low correlations between motivation and achievement. The psychological region linked to internal processes involves the individual's behaviour and varies between becoming involved in and avoiding mathematics problem solving situations. The psychological domain extends further, relating affective factors – emotional history, familiar experiences and stimulus reactions associated with the individual's feelings of confidence, anxiety or discomfort, and pleasurable experiences. This includes skills and knowledge of the problem-solving procedures and strategies selected. Personal performance measured in this domain associates with the region of personal achievement and the perception thereof.

Intellectual influences

The intellectual domain entails cognitive influences. Reflecting on success or failure, the individual evaluates the appropriateness, acquirement and use of mathematics skills and concepts learnt, and consider aspects of their higher-order reasoning and reflective questioning. Guilford's structural model of intellect (Bergh & Theron, 2009, p.149) reviews the intellectual domain as a combination of various aspects involving visual, auditory, symbolic, semantic and behavioural constructs. The operations involved with these aspects appear mainly to be cognitive (memory, divergent production, convergent production) and metacognitive (evaluation and monitoring) in nature. The product of these operations is found to influence group settings, relationships, social systems, transformation of knowledge and it has implications for problem solving.

The three domains have individual characteristics as well as a natural overflow between their components; due to the cause and effect that one component has on another. In the social domain, family members, friends (peers) and society influence the values, beliefs and views the learner might develop. The result is a motivated or demotivated individual with positive or negative emotions. The engaging or avoiding of mathematics tasks and the intellectual components involved, either allows the individual to succeed or fail in his/her use of strategies or approaches.

Combining the components of social, psychological and intellectual domains, lead to an affective product in mathematics problem solving, namely mathematics confidence. This overflow and connection of the three domains can be observed in the theoretical framework illustrated below.

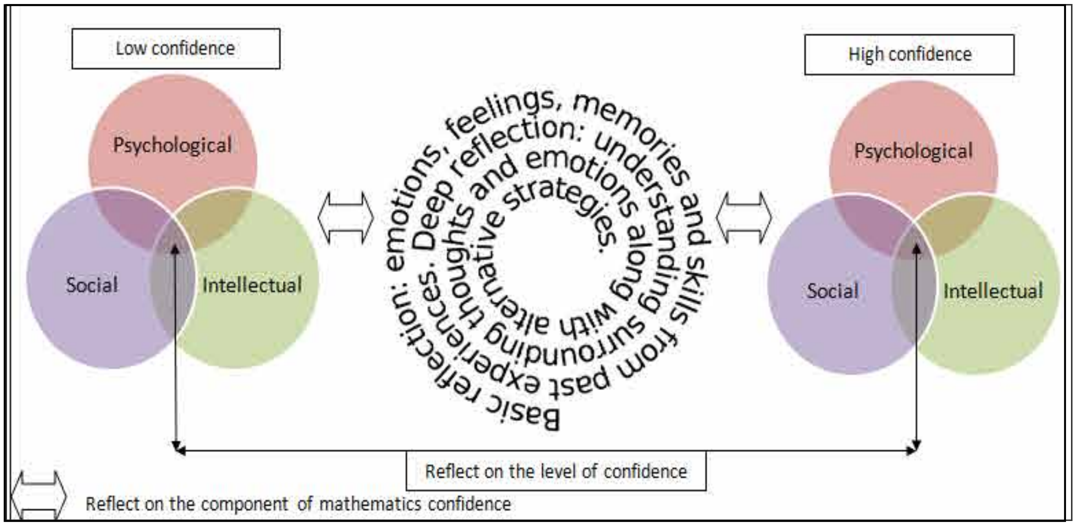

Figure 1 Conceptual framework for reflection on mathematics confidence domains
Source: Adapted from Strawderman (2010) and John (2009)

The primary research question this study seeks to answer is: what does learners' mathematics confidence entail when reflecting on mathematics problem solving experiences? To answer this question, the following methodology was employed.

METHOD

Three invited schools from the North West province in South Africa took part in the current study. This paper focuses on one aspect of this study, namely learners' confidence emanating from experiences with mathematics problem solving. As part of this qualitative, interpretive research design, data was gathered by interviewing four participants from the invited schools. According to their teachers, the purposively selected participants could express themselves verbally and would not be shy to share information about themselves. As arranged with appropriate authorities, the interviews were conducted in one of the classrooms at the participants' respective schools. The participants included three girls and one boy. Some biographical information of the participants is summarised in Table 1.

Table 1 Biographical information of participants

Participant	Age	Grade	Average achievement in 2010
Learner A	15	9	70%
Learner B	15	8	80%
Learner C	14	8	76%
Learner D	13	9	96%

Insight into their experiences with problem solving allowed the researchers to explore the sources and constructs of mathematics confidence. Interview questions, taken from the work of Wilson and Clarke (2004), Bormotova (2010) and Swanson (2006) included questions such as: Are you good at solving mathematical problems? How do you feel when solving problems in the classroom? Do your parents help or advice

you in problem solving? The questions promoted participants' thinking about experiences with problem solving and determined their perceptions about whether the sources contributed to their psychological state before, during and after such as experiences. The verbatim transcribed responses, analysed and coded, supported the themes identified by Strawderman (2010).
A-priori categories were determined in the theoretical framework and included the themes of the three domains: social, psychological and intellectual. The study involved an interpretive natural approach (Bormotova, 2010, p.73) towards an understanding of the research topic. Through content analysis, responses were categorised according to low and high traits in the three dimensions of mathematics confidence.

The results that are discussed in the next section point out that mathematics confidence has at least some roots in mathematics tuition during the primary school years – the effect of such tuition therefore has implications for senior phase mathematics problem solving.

RESULTS

Through reflection on past and more recent occurrences, participants 'made connections' with their feelings regarding mathematics problem solving. They identified the commencement of their confidence in the problem solving and developed an understanding and awareness of its foundations and variations. When asked when they first noticed a change in their mathematics confidence, all participants reflected on experiences in Grades 5 and 6. Their reflections on mathematics confidence could be traced back to incidents that had occurred between the ages of ten and fourteen years. Reflective responses included low and high confidence regarding social, psychological and intellectual aspects of mathematics problem solving.

Reflections on social aspects

Social interaction related to the context within and around mathematics problem solving. Participants compared their self-reflective knowledge with that of their peers. The social structure of the classroom, the arrangement of desks and teaching practices influenced the confidence of learners as social aspects.

The contributions of peers and family members were of a contrasting nature. According to the participants, parents who motivate and support their children and assist with homework do not always help to improve the latter's maths understanding.

Parental influences seemed to be minimal as one participant appeared to believe:

> Learner C: I didn't enjoy the work, even if my parents explained it to me.

Comparing themselves with peers, participants fused beliefs and views about their own capabilities. One participant stated a personal disliking of this quality:

> Learner B: I don't like to compete against someone.

This self-reflective knowledge promotes low and high confidence in problem solving. It is noteworthy that one participant in this study completely disagreed with the statements made by his peers and confidently said:

Learner D: I was like one with the group. Most kids would say that I don't like it and it's boring. Other people are still doing it, but not me. They don't have the passion, so they think, why should I do it? This year everybody tells us, our marks will drop because its high school…

Teachers' role in the social domain is evident:

Learner A: Most of them taught me something that wasn't maths…my math teacher was a real down-to-earth person that I could relate to.

Reflections on psychological aspects

Psychological traits of the learners' behaviour included a variation of feelings and emotions. Key aspects of the psychological state of participants' mathematics confidence included emotions and preferences with regard to likes and dislikes. Reflections on experiences integrated positive and negative emotions, feelings and associated thoughts, and an understanding of those sentiments. Participants disliked word problems the most. When asked what they liked and disliked about mathematics problem solving, three of the four participants indicated that their past and current teachers' teaching approaches and methodology contributed to their like or dislike of mathematics. Some of their reflections on the instruction methods included the following:

Learner A: Some say what you must do, they just give you the facts. Others make it practical.

Learner B: I liked the teacher but not the way he did it.

Participants seemed to understand, from experience, the characteristics of mathematics anxiety. It appears that their view on mathematics confidence revolves around their self-belief, pride and being scared of mathematics or failing the subject. One participant mentioned a high-confidence trait when comparing problem solving tasks in mathematics with tasks such as tests in other subjects:

Learner D: I just don't feel stressed like in other subjects. Maths is just there, it comes natural. I don't feel stressed. I feel excited to see what questions to expect. Anxious to see what's happening in the test [task].

Participants mentioned various teaching factors that they had experienced and to which they attributed their achievement in and understanding of mathematics during problem solving. One participant claimed:

Learner A: I couldn't understand, she [teacher] jumped around using different methods

Reflections on intellectual aspects

When asked what learners thought would make problem solving difficult or easy, all four participants responded with some regard to metacognitive components. Participants reasoned why they did not avoid mathematics:

Learner A: Even if you get it wrong, you still try to get it right. I just keep trying over and over again.

Learner B: I think it will boost my marks and I will become a better person.

A metacognitive trademark source of confidence was noted:

Learner A: When I know I can do it in another way, it's easy. When I know there's another way to solve the problem, I know I understand it.

Reviewing the steps used in problem solving contributed to the learners' selection and employing of strategies:

Learner C: I go through my work, wondering what I did wrong.

Avoiding mathematics was also an issue, as Learner C remarked:

Learner C: My body just gets into sleep mode. When the bell rings, I'm all energetic again.

It appears from the statement of at least one participant that dislike (psychological) and understanding (intellectual) of mathematics problem solving is related:

Learner D: Word problems are just something I don't like. I don't like it, because I don't understand it.

It is noteworthy that Learner D, in particular, seems to experience both low and high levels of confidence depending on the psychological and intellectual sources.

DISCUSSION

The findings suggest that it is possible that metacognitive reflection may regulate mathematics confidence in the social, psychological and intellectual domains. According to their reflections, participants recalled not only low confidence connected to unsuccessful experiences with problem solving, but also moments when their confidence was high and performance successful. The following discussion focuses on Strawderman's (2010) three domains of mathematics confidence.

Social aspects of mathematics confidence

Learners' attitude towards the subject can either be positive or negative (Maree et al.,1997) with strong correlations to achievement throughout the senior and further education phase (Grades 7-12) (Hannula, Maijala & Pekhonen, 2004, p.18). A number of other researchers (Ernest, 2002; Bormotova, 2010; Strawderman, 2010) confirm social aspects as a psychological factor empowering mathematical problem solving. It therefore seems that the academic environment must be of such a nature that it allows learners to learn how to learn.

Learners face and have to cope with the social context at school. According to McFarland (2011, p.3), learners in the senior phase experience changes associated with puberty, and have to make social and psychological adjustments in their everyday lives. Stankov, Lee, Luo and Hogan (2012) argue that social attributes include values and attitudes originating from society, and in particular from peers, parents and teachers. McFarland (2011, p.4) explains that adolescents focus on their peers and are greatly concerned about social acceptance. The findings of the current study seem to agree. The effect that peers might have on an individual's mathematics confidence involves a comparing of attitudes towards mathematics problem solving in particular.

Psychological aspects of mathematics confidence

Experiences involving Grade 5 and 6 learners indicate that, for these participants, mathematics confidence seems to start to change at around age ten or eleven, having its foundations in primary school years. Learners' dislike of teachers' approaches (Thijsse, 2002) towards problem solving and their preference for participating in group activities attest to the constructive role that reflection on mathematics confidence plays in respect of achievement in problem solving. Bormotova (2010, p.32) found that positive feelings and emotions enhance the learning process. These feelings keep the learner focused on the task and inspire new learning. Malmovuori (2006) found a connection, in terms of positive experiences, between regulating skills and strategies, and participants' enjoyment of mathematics problem solving.

Intellectual aspects of mathematics confidence

The knowledge and skills of problem solving strategies, as well as content knowledge, is vital in order to perform in mathematics. Participants reflected on their successes and failures with problem solving, and identified the level of difficulty or ease of using approaches learned and taught. Varying between complex and diverse components (Legg & Locker, 2009, p.471), learners reflected on the approaches used during problem solving and the experiences that contributed to their selection and use of strategies and knowledge. Typical traits (Learner C) displayed by learners with low mathematics confidence included avoidance of problem solving or metacognitive components such as evaluating and monitoring. Reflecting on metacognitive and mathematics skills occurred on a subconscious level. This regulative skill (Garrett, Mazzocco & Baker, 2006) differs between individuals. Strategies selected are acquired from knowledge of experiences in similar contexts, as suggested by Pantziara and Philippou (2011).

RECOMMENDATIONS

Several aspects of mathematics confidence and reflection require further investigation. These aspects include possible affective and metacognitive connections to mathematics problem solving. The following might serve as potential considerations for both researchers and teachers involved in mathematics education.

For researchers

The assessment of reflection on affect in mathematics – by using explorative and convergent mixed methods – may confirm and increase literature on the traits of the origins of mathematics confidence. It is suggested that the development of assessment methods to measure mathematics confidence might include social, psychological and intellectual factors. According to Bormotova (2010), journal-writing experiences as self-reflective reports may serve as a basis for reflection and could provide a narrative of the participants' mathematics confidence. Cultural and socio-economic differences could also be explored to determine the possible relationships between environmental backgrounds and their relation to mathematics confidence and reflection.

For teachers

Learners who experience low mathematics confidence or anxiety may build confidence in the presence of a supporting teacher (Thijsse, 2002, p.25). Symptomatic descriptions and identification of participants' mathematics confidence, especially during primary school years (Hannula, Maijala & Pekhonen, 2004), could alter the foundational challenges that learners experience with mathematics and possibly improve their attitude towards the subject. The use of standardised tests can assist with the measuring of learners' confidence in mathematics, identify and diagnose causes and resources to reduce the level of anxiety, and increase confidence. This, in turn, could guide the planning and implementation of relevant interventions. Curriculum development could perhaps cater for the social, psychological and intellectual needs of learners and aim to nurture ambition for mathematics, mathematics problem solving and math teaching.

CONCLUSION

Some of the most noticeable social sources include teachers' approaches to the teaching of problem solving strategies and learners' self-comparison with peers. Psychological aspects include feelings of stress when the learner does not understand the problem or what approach to use. A feeling of excitement is expected when maths comes *naturally*. As learner B mentioned, the social and psychological domains are connected with liking the teacher, but disliking the approach. Intellectual sources for learners' confidence appear to include persistence and an understanding and use of multiple strategies.

The results of this study indicate that confidence in mathematics has certain implications for the individual, and that the social, psychological and intellectual success of the community as a whole has its roots in diverse experiences with mathematics.

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WHAT TEACHERS WANT OUT OF PROFESSIONAL LEARNING OPPORTUNITIES: A TAXONOMY

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Teachers do not come to professional learning opportunities as blank slates. Instead, they come to these settings with a complex collection of professional learning goals. The research presented here takes a closer look at these goals across five different professional learning settings. The results indicate there exists a taxonomy of five categories of goals that teachers may approach professional learning with.

keywords: professional development, teacher goals, taxonomy

INTRODUCTION

Current research on mathematics teachers and the professional development of mathematics teachers can be sorted into three main categories[1]: content, method, and effectiveness. The first of these categories, *content*, is meant to capture all research pertaining to teachers' knowledge and beliefs including teachers' mathematical content knowledge, both as a discipline (Ball, 2002; Davis & Simmt, 2006) and as a practice (Hill, Ball, & Schilling, 2008). Recently, this research has been dominated by a focus on the mathematical knowledge teachers need for teaching (Davis & Simmt, 2006; Hill, Rowan, & Ball, 2005) and how this knowledge is developed within preservice and inservice teachers. Also included in this category is research on teachers' beliefs about mathematics and the teaching and learning of mathematics and the ways in which these beliefs change within the preservice and inservice setting (Liljedahl, 2010a). Some of the conclusions from this research speaks to the observed discontinuities between teachers' knowledge/beliefs and their practice (Skott, 2001; Wilson & Cooney, 2002) and, as a result, calls into question the robustness and authenticity of these knowledge/beliefs (Lerman & Zehetmeir, 2008).

The second category, *method*, is meant to capture the research that focuses on a specific professional development model such as action research (Jasper & Taube, 2004), lesson study (Stigler & Hiebert, 1999), communities of practice (Wenger, 1998), or more generally, collegial discourse about teaching (Lord, 1994). This research is "replete with the use of the term inquiry" (Kazemi, 2008, pg. 213) and speaks very strongly of inquiry as one of the central contributors to teachers' professional growth. Also prominent in this research is the centrality of collaboration and collegiality in the professional development of teachers and has even led some researchers to conclude that reform is built by relationships (Middleton, Sawada, Judson, Bloom, & Turley, 2002).

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[1] These categories, although presented separately, are not entirely distinct from each other.

> More accurately, reform emerges from relationships. No matter from which discipline your partners hail, no matter what financial or human resources are available, no matter what idiosyncratic barriers your project might face, it is the establishment of a structure of distributed competence, mutual respect, common activities (including deliverables), and personal commitment that puts the process of reform in the hands of the reformers and allows for the identification of transportable elements that can be brokered across partners, sites, and conditions. (ibid., p. 429).

Finally, work classified under *effectiveness* is meant to capture research that looks at changes in teachers practice as a result of their participation in some form of a professional development program. Ever present in such research, explicitly or implicitly, is the question of the robustness of any such changes (Lerman & Zehetmeir, 2008).

As powerful and effective as this aforementioned research is, however, it can no longer ignore the growing disquiet that somehow the perspective is all wrong. In fact, it is from this very research that this disquiet emerges. For example, the questions of robustness (Lerman & Zehetmeir, 2008) come from a realization that professional growth is a long term endeavour (Sztajn, 2003) and participation in preservice and inservice programs is brief in comparison. At the same time there is a growing realization that what is actually offered within these programs is often based on facilitators (or administrators or policy makers) perceptions of what teachers need as opposed to actual knowledge of what teachers really want (Ball, 2002). But not much is known about what teachers really want as they approach professional learning opportunities. The research presented here provides some answers in this regard.

METHODOLOGY

As articulated in Liljedahl (2010b), working in a professional development setting I find it difficult to be both a researcher and a facilitator of learning at the same time. As such, I generally adopt a stance of noticing. This stance allows me to focus on the priorities of facilitating learning while at the same time allowing myself to be attuned to various phenomena that occur within the setting. It was through this methodology that I began to notice that there was a distinct difference between the groups of teachers that came willingly to the professional development opportunities that I was leading and the teachers that were required, often by their administrators, to attend. This was an obvious observation. Nonetheless, it was as a result of this observation that, I began to attend more specifically to other differences. In doing so I began to notice, subtly at first, that the teachers who came willingly came with an a priori set of goals. With this less obvious observation I changed my methods from noticing to more directive research methods. I began to gather data from five different professional learning contexts over a period of two years.

Master's Programs

Teachers in this context are practicing secondary school mathematics teachers who were doing their Master's Degree in Secondary Mathematics Teaching. This is a two year program culminating in either a comprehensive examination or a thesis depending on the desires of the teacher and the nature of the degree that they are seeking. From this group I collected interview data and field notes during two different courses I taught in the program.

Induction Group

This group began as an initiative to support early career teachers (elementary and secondary) as they make the transition from pre-service teachers to in-service teachers. In truth, however, it also attracted more established teachers making it a vertically integrated community of practicing teachers of mathematics. Although this group now meets every second month for the duration of the study we met monthly. From this group I collected interview data, field notes, as well as two years of survey data.

Hillside Middle School

Hillside (pseudonym) is the site of a longitudinal study. For the last five years I have meet with a team of three to six middle school teachers every second Wednesday for an hour prior to the start of the school day. This group began as an administration led focus on assessment of numeracy skills, but after the first year took on a self-directed tone. The teachers in this group lead the focus of the sessions and look to me to provide resources, advice, and anecdotal accounts of how I have seen things work in the many other classrooms I spend time in. For the two year period that constitutes the study presented here I collected field notes and interview data.

District Learning Teams

Very much like the professional learning setting at Hillside, district based learning teams are self-directed. Teachers meet over the course of a year to discuss their classroom based inquiries into teaching. This inquiry runs throughout an entire school year, but the teams themselves only meet four to six times a year. The data for this study comes from three such teams that I facilitated in two different school districts. One of these teams ran during the first year of the study, the other two teams ran in the second year of the study. Like at Hillside my primary role is to provide resources, advice, and insights into their plans and reported classroom outcomes. The data from these teams consisted of field notes, interviews, and survey results.

Workshops

During the two years that I collected data for this study I was also asked to do a number of one-shot workshops. These were workshops designed around a variety of different topics either decided by myself or the people asking me to deliver the workshop. They ranged in time from 1.5 hours to 6 hours with no follow-up sessions. Data, consisting of field notes, comes from six such workshops. Data from two additional workshops consists of field notes and survey results.

Data

Field notes in the aforementioned settings consisted primarily of records of conversations I had with individual teachers during breaks as well as before and after the scheduled sessions. I used these times to probe more specifically about the origins of questions asked, motivations for attending, querying about what they are getting out of the session, and if there is something else they need or want. This sound very formal and intentional, but in reality, this was all part of natural interactions. In all, I collected notes on over 70 such conversations.

More directed than these natural conversations were the interviews. These were much more formal in nature and provided an opportunity for me to probe further about the conversations we had previously had or the things I had observed during our sessions together. Each interview lasted between 30 and 60 minutes. In all, 36 interviews were conducted over the course of the two years, resulting in 26 hours of audio recordings. These recording were listened to as soon as possible after the interviews and relevant aspects of the recording were flagged for transcription.

The survey used with the Induction Group, The District Learning Teams, and two of the Workshops consisted of an online survey instrument that was sent to the teachers prior to professional learning session. The survey contained five questions, the last two of these were of obvious relevance to the study.

4. *What do you hope to get out of our next session together? You can ask for understanding of mathematical concepts, teaching strategies, resources, lesson ideas, ideas about classroom management, networking opportunities, specific lesson plans, etc. In essence, you can ask for anything that will help you in your teaching or future teaching. List as many as you want but please be specific.*
5. *Please list something from a past session that you found particularly useful.*

The field notes, interview transcripts, and survey data were coded and analysed using the principles of analytic induction (Patton, 2002). "[A]nalytic induction, in contrast to grounded theory, begins with an analyst's deduced propositions or theory-derived hypotheses and is a procedure for verifying theories and propositions based on qualitative data" (Taylor and Bogdan, 1984, p. 127 cited in Patton, 2002, p. 454). In this case, the a priori proposition was that teachers come to professional learning settings with their own goals in mind and that these goals are accessible through the methods described above. With a focus on teachers' goals the data was coded using a constant comparative method. One of the things that emerged out of this analysis was a taxonomy of five types of goals that teachers come to professional learning settings with. To these I add a sixth theme which also emerged out of the analysis. Although not a goal per se this sixth theme deals with the resistance with which some teachers engaged in some of the professional developing opportunities.

RESULTS

In what follows I present each of these categories in turn, beginning with resistance and following it up with each of the five categories of goals.

Resistance

In the course of the two years of the study I collected data on a number of teachers who were flatly opposed to being part of the professional development setting I was working in. All of this data consisted of observation and conversations and came solely from the workshops and learning team settings. To a person, these teachers were participating in these settings at the request of an administrator or a department head. Left up to them, these teachers would choose to not attend.

First, these resistant teachers were present and they did participate in the sessions. They engaged in the activities, they asked questions, and they collaborated with others in the room. But this participation was guided by their reluctance at being there. As such, their contribution to the group was often negative, pessimistic, defensive, or challenging in nature. They would say things like "that will never work" and "I already do that". This is not to say that these teachers were the only ones to utter these types of statement, but rather that they only uttered these types of statements. Their questions to me were always challenging in nature with greater demands for evidence, justification, and pragmatism. These challenges were welcomed as they often provided others with an opportunity to engage in the content more critically. The call for pragmatism, in particular, was not unique to resistant teachers, but the goals for making that call were clearly different. When they challenged ideas based on their infeasibility the goal seemed to be to detract from the value of what was being offered; to invalidate it. When non-resistant teachers made the same call it seemed to be motivated by a goal to try to navigate the space between the ideal and the feasible; to find a way to make it happen.

The second reason I include this theme is that these teachers did not always remain resistant. There were several cases in my data where teachers who initially approached the setting with resistance softened their stance over time. In the workshop settings this was marked by a shift in the types and ways in which they asked questions, the ways in which they engaged in activities and interacted with their peers, and in the parting comments and conversations I recorded. In the learning team settings this was marked by the fact that between meetings, these initially resistant teachers, reported back at subsequent sessions about efforts made, and results seen, in their own classrooms.

The third reason for including this theme here is because I want to differentiate between the resistance a teacher may have to an idea in a professional learning setting and the a priori resistance a teacher may approach that setting with. In the former case I am talking about a healthy form of scepticism that, as mentioned, allows teachers to negotiate the space between the ideal and the real, between the theoretical and the practical. The later, however, is a stance that can prevent the uptake of good ideas and helpful suggestions. It can act as a barrier to learning and professional growth.

In all, out of the 70 conversations that I made notes on, 10 were with teachers who were, at least originally, resistant to being in the setting. Of these, four changed their stance over the course of the setting. However, my field notes record observations of many more such a priori resistant teachers as well as observed changes in some of them.

Do Not Disturb

This category of goals characterizes those instances where a teacher engages in professional learning because they want to improve their practice, but is reluctant to adopt anything that will require too much change. Ideally, what they want are small self-contained strategies, lessons, activities, or resources that they can either use as a replacement of something they already cleanly insert into their teaching without affecting other aspects of their practice. Such goals were rarely stated outright. Instead, they manifest themselves as overly specific statements of what it is they seek.

> "I was hoping to learn a new way to introduce integers".

> "I want something to do for the first 10 minutes of class."

> "A different way to do review."

All of these are indicative of situations where the teacher is looking to improve one thing about their teaching. The "don't let it affect anything else around it" is implicit in the specificity of the statement. In conversations or in interviews, however, this can sometimes come out more explicitly.

> "I'm happy with the rest of my fractions unit. It's just division of fractions that messes me up. I was hoping that you could show me a better way to explain it."

Delving deeper it became clear that in many of the instances where concern over disturbance and tight control over impact was important there was an underlying anxiety, most often around the deconstructing what they have worked hard to build up.

> "I've been teaching for seven years now, and I'm really happy with the way things are going. After the last curriculum revision and with us getting a new textbook I have worked really hard to organize all of my lessons and worksheets in math. I don't want to mess with that. So, please don't tell me anything that is going to mess me up. I really just want to know if there is a lesson that I can do using computers that will be fun and that I can just sort of insert into my area unit."

Less often this anxiety is around what they have worked hard to understand.

> "When I started teaching I was fine with math. But when I was given a grade seven class this year I sort of panicked about math. Especially the unit on integers. I had never understood it when I was in school and it took me a long time to teach it to myself. So, I don't really want to learn anything new that will rock the boat for me."

In other instances there didn't seem to an underlying anxiety, but just a pragmatic disposition that small change is good "less is more".

Willing to Reorganize

A slight nuance on the previous theme is when teachers want very specific improvements and they are willing to significantly reorganize their teaching and resources to accommodate the necessary changes. Although specific in nature, these goals do not come with limitations. They are stated with an implicit openness to the consequences that the desired improvements may necessitate.

> "So, yeah. I'm looking for an improved way to have my students learn how to do problem solving. Right now I do it as a unit in February, but it's not working. I've heard that other teachers do it throughout the whole year and I'm hoping to get some ideas around that."

Further probing of this teacher, as well as the others who made similar statements, revealed that they are not hampered by anxiety around invalidating existing resources or undoing things learned.

Willing to Rethink

Unlike the previous two categories, the goals that fit into this are much broader in scope and often welcome a complete rethinking of significant portions of a teaching practice.

> "I'm pretty open to anything. I mean with respect to differentiated learning."

From the interviews it became clear that for this teacher, as well as for those who expressed similar goals, there exists something in their practice that they want to bolster. In many cases these teachers want collections of resources that they could then organize and integrate into their teaching.

> "Anything to do with numeracy is good for me."

> "I'm looking for new ideas about assessment for differentiated learners."

In some cases, however, these teachers are branching out into new territories and are looking for a comprehensive package of what to do.

> "I'm hoping to introduce the use of rubrics into my teaching and want to get the rubrics I should use as well as instruction how to do it."

Either way, these teachers have a rough idea of what it is they want and are willing to rethink their teaching in order to accommodate new ideas. They do not have the anxieties of disrupting already held knowledge or resources that the teachers in the first category did and their goals are broader in scope than the second.

Out With the Old

The goals in the previous category were characterized by a willingness to rethink significant aspects of teaching practice. In the Out With the Old category, the goals are characterized by a rejection of a significant aspects of teaching practice. Teachers with these goals come to professional learning settings unhappy with something in their practice. This unhappiness goes well beyond the awareness that something needs to be improved that was seen in the previous three categories. For these teachers there is nothing to be integrated, there isn't a replacement of some aspect of their teaching to be made. They have already rejected the current paradigm and are now looking for something to fill the void that is left behind.

> "My kids can't think for themselves in problem solving. I don't know what I'm doing wrong, but it doesn't matter. I just need to start over with a new plan."

> "I can't stand the way group work has been working in my classroom. Or not working is a better description. I have given up with what I've been doing and am looking to learn something completely different."

This is not to say that these goals are coupled with blind acceptance of anything that fits the bill. The teachers who express these goals are often hypercritical of new ideas, usually as a result of their dissatisfaction with something that they have previously changed in their practice.

> "I spent a whole year trying to teach and assess each of the processes [communication, connections, mental mathematics and estimation, problem solving, reasoning, technology, and visualization] that are in the curriculum. In the end my students are no better at estimating or communicating, for example, than they were at the beginning of the year. My approach didn't work. I need a new way to think about this."

This is not to say that they are closed minded, but rather that they exert a greater demand on me, as the facilitator, to bridge the theoretical with the pragmatic.

Inquiry

The final category consists of those goals which align with the ideas and ideals of inquiry (Kazemi, 2008). As such, these goals consist, most often, of a general desire to acquire new knowledge and ideas about teaching. The teachers who express these goals are open to any new ideas and often come to professional learning settings without an agenda.

> "I'm not really looking for anything in particular. But, I'm eager to hear about some new ideas on assessment."

This is not to say that these goals are flighty and unrefined. The teachers whose goals fall into this category are often methodical in their change, pausing to ask exactly "what is it I am doing" and "if it's working". And if it is working they question "what is it that is telling me it is working". They want evidence of success, but they want it to come from their own classroom.

CONCLUSIONS

Much can be taken from the results presented above. The most obvious is that teachers who willingly come to professional learning settings do so with one of five possible goals. Further analysis reveals that these goals have some pseudo-hierarchal properties and that they reveal a discordance with the long held belief that single workshops are an ineffective means of creating professional growth (Ball, 2002). Finally, the results indicate that, in essence, teachers treat professional learning settings, and the people that facilitate such settings, as resources for the furtherment of their own professional learning goals.

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USING CONTENT ANALYSIS TO INVESTIGATE STUDENT TEACHERS' BELIEFS ABOUT PUPILS

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Student teachers' beliefs about teaching are arguably important to their learning and development as teachers. One aspect of these beliefs is that of a focus on the learners. In the study presented in this paper, we use two approaches to content analysis in order to learn more about student teachers' beliefs from transcripts of focus-group discussions. The analyses reveal that the student teachers focus less on the mathematical content and more on characteristics of pupils in their discussions. These characteristics were mainly related to the pupils' behaviour, their cultural background or their certain needs.

Keywords: student teachers' beliefs, learner orientation, content analysis

INTRODUCTION

Teacher education is important in many respects, among others to equip all teachers for effective teaching in the 21st century (OECD, 2011). Throughout their development—from student teachers to experienced teachers—they are faced with different challenges. Richardson and Placier (2001) conclude that novice teachers focus more on surviving in the classroom than on pupils' learning, and additional research is needed in order to learn more about how teacher education can contribute to a shift in focus in this respect.

There are several interesting issues to focus on in a study of student teachers' development—e.g. their beliefs about teaching (Fives & Buehl, 2010). Many consider student teachers' beliefs important, and there are different reasons for this. One reason is that student teachers bring with them strong beliefs about teaching into their teacher education, and these beliefs are important in relation to what the student teachers learn (Richardson, 2003). One conclusion from the Teacher Education and Development Study – Mathematics (TEDS-M) is that: "significant change is unlikely to occur unless teacher-preparation programmes explicitly address beliefs" (Tatto et al., 2012, p. 172).

Previous research has identified learner orientation as one of the main attributes of a high-quality learning environment (Bransford, Brown, & Cocking, 2000). The demands on teachers are increasing, and cultural awareness and regard for individual differences and needs are crucial aspects of a teacher's skills (ibid.). It is important to develop a focus on learners' attitudes, skills and understanding, and not least to use this actively when designing and implementing teaching.

Beliefs have been studied in mathematics education research for decades, and there has been a development of theories, focus and methods in relation to this area of research (see e.g., Philipp, 2007; Fives & Buehl, 2012). There are still, however, unanswered questions regarding beliefs and other aspects of the affective domain (see e.g. Hannula, 2011). In the following, we take a brief look at some of the main issues regarding concepts and categories that have been discussed in beliefs research.

In his handbook chapter, Philipp (2007) provides an overview of some of the more commonly used terms related to beliefs: affect (including emotions, attitudes and beliefs), beliefs systems, conceptions, identity, knowledge and values. All of these concepts have been used with various meanings by different researchers, and, according to Fives and Buehl (2012, p. 471), "the lack of cohesion and clear definitions has limited the explanatory and predictive potential of teachers' beliefs". A clarification of terminology is therefore important for determining research focus. Beswick (2012) provides a categorization of mathematics teachers' beliefs into beliefs about the nature of mathematics, mathematics teaching and mathematics learning (Table 1).

Beliefs about the nature of mathematics (Ernest, 1989)	Beliefs about mathematics teaching (Van Zoest, Jones, & Thornton, 1994)	Beliefs about mathematics learning (Ernest, 1989)
Instrumentalist	Content focused with an emphasis on performance	Skill, mastery, passive reception of knowledge
Platonist	Content focused with an emphasis on understanding	Active construction of understanding
Problem solving	Learner focused	Autonomous exploration of own interests

Table 1: Categories of teachers' beliefs (Beswick, 2012, p. 130)

The choice of methodology when studying beliefs is strongly connected with the researchers' view about the nature of these beliefs. For the purpose of this study, we consider beliefs as being held by individuals (Philipp, 2007), and all the aspects from the table above are relevant to consider in studies of teachers' beliefs. We analyse data from two semi-structured focus-group interviews with student teachers in order to answer the following research question:

> What can be learned about student teachers' beliefs about pupils from content analysis of their focused discussions prior to field practice?

The data material that we analyse and discuss in our attempts to answer this question is part of a larger, cross-disciplinary project entitled Teachers as Students (TasS).

METHODS

The TasS project—which is funded by the Research Council of Norway (project number: 212276/H20)—has a main focus on the field practice, which is part of the teacher education programme. It includes two data collection periods and two groups of participants: a control group (referred to in the project as the "business as usual condition") and an intervention group (referred to as the "lesson study approach condition"). Data collection in the control group was carried out in the spring of 2012, whereas the data collection for the intervention group is scheduled for the spring of 2013. For the purpose of this paper, we focus on the control group only.

Norwegian student teachers have 20 weeks of field practice during their four-year teacher education programme. For this research project, students in their second year participated. These students were just about to start their fourth three weeks long field practice. Two groups of student teachers from each of the subject areas: mathematics, science, English as a foreign language, and physical education were selected for participation in the control group condition. The intention was to have four student teachers in each group, but some of the groups ended up with only three student teachers in them. A focus-group interview (FGI) was carried out before (pre-FGI) and after (post-FGI) the field practice period in each of the eight groups. A main purpose with the FGIs was to investigate the student teachers' reflections about the mathematical content (or one of the other content areas involved in the study). Data collection also included video observations from student teachers' planning lessons with their practice teacher (pre-tutoring sessions), from carrying out lessons (two lessons in each group) and from evaluating lessons (post-tutoring sessions). In this paper, we analyse transcripts from the pre-FGIs conducted with the two groups of student teachers in mathematics (3 + 4 student teachers in these two groups).

Instrument

All pre-FGIs were carried out in a similar manner, and they were scheduled to last for approximately an hour and a half. Each interview consisted of the same four parts. The first part (5-10 minutes) contained introductory questions related to why they chose to enter into teacher education, why they decided to study mathematics, and what they anticipated as being the most interesting and/or challenging parts of being a teacher.

The second part of the FGIs (10-15 minutes) had a focus on the field practice they were about to start, and what reflections they had about their preparation for this period. The student teachers were asked to reflect on issues related to pupils, their own background knowledge and skills in the subject area and practical information given from the university.

Next followed a part of the interview where they were presented with the following case from a classroom situation (this part lasted for about half an hour):

> **Lise is teaching a class of sixth graders in mathematics. One issue she is very concerned about is getting feedback on ways that she can plan and teach which will enable her pupils' learning.**
>
> *Lise: The pupils in my class are fairly different when it comes to their attitudes to school work, in general, and mathematics, in particular. Some of the pupils are very motivated and hard-working whilst others don't seem to care at all. This is reflected in their educational achievement in mathematics, the effort they put into their school work and their engagement in the learning environment. How can I lift all these pupils to the next level? Some of the pupils are very interested in mathematics and do more homework than they are asked to and in addition they receive a lot of support from home. Other, less motivated pupils don't seem to work at all; neither at school nor at home. And moreover they don't seem to have learned anything from previous classes. In addition I have two pupils with reading difficulties and two minority language pupils who have only lived in Norway for a short time.*

Figure 1: Case from the interview guide.

They were then asked to discuss the challenges of the situation presented in the case, and how these challenges were related to the different aspects of teacher knowledge.

In the fourth and final part of the interviews (lasting approximately 30 minutes), the student teachers were asked questions related to the subject area that they were focusing on in their field practice (e.g. mathematics). In this part, they were asked questions about how they would plan a lesson—with a focus on fractions—in the class that was presented in the case from the previous part of the interview. The questions in this part were concerned with important aspects in the planning process, difficulties they believed students would have, ways of organising teaching in such a case, special concerns that they would have to be aware of as teachers, reasonable goals for such a lesson, and how they would assess their students in relation to the set goals.

Data analysis

The transcripts from the FGIs have been analysed by combining two approaches to content analysis. First, the transcripts were analysed by counting words in the text with the purpose of understanding the contextual use of words. This approach is often part of what has been referred to as summative content analysis (Hsieh & Shannon, 2005). Our summative analysis went beyond word count, however, and we tried to discover underlying meanings of the words used by the student teachers. The counting was used to identify patterns in the data and to initiate the process of developing contextualised codes. The word count was also a starting point for identifying the context associated with the word to try to discover the range of meanings the word had in the interviews.

The summative analysis thus provided insight into how particular words were used. This summative approach is, however, "limited by the inattention to the broader meaning in the data" (Hsieh & Shannon, 2005, p. 1285), and it was followed by a second phase of data analysis. In this phase, we used conventional content analysis to dig deeper into the data. According to Hsieh and Shannon (2005), this analysis is used in studies which aim at describing a phenomenon in order to understand it better—in this case "it" refers to the student teachers' statements. In conventional content analysis "researchers immerse themselves into the data to allow new insights to emerge" (Hsieh & Shannon, 2005, p. 1279) by reading the data word by word. In our study, we combined the use of these two common approaches to content analysis. The summative content analysis thus served as a way of reducing data, and it was then followed by the conventional content analysis where categories were developed as part of the analysis.

RESULTS AND ANALYSIS

In our attempt to study what can be learned about student teachers' beliefs from content analysis of their focused discussions prior to field practice, our combination of summative and conventional content analysis seems promising. In this part we will present results from the summative as well as the conventional content analysis, and we will discuss the results in light of previous research in this area.

The summative content analysis

After an initial reading of the interview transcripts, we realised that the student teachers' reflections had much less focus on the mathematical content than we first anticipated. This impression was strengthened by our initial attempts to analyse the data. As part of our summative content analysis of the transcripts, we generated a concordance in order to learn more about the words that were used in the discussions and the frequencies of those words. We started out with a focus on words related to mathematics.

Frequency	**Word/concept**
88	mathematics
88	fraction
30	task
26	addition
21	calculate
18	denominator/common denominator

Table 1. The most frequently used words related to mathematics

From this initial summative content analysis, it appeared as if the student teachers were discussing fractions quite a lot, and we started out by focusing our analysis on this. When we moved from a quantitative to a more qualitative analysis of the content, however, it became evident that these words were mainly mentioned briefly with little or no reflection. In one of the interviews, for instance, the interviewer asked about their preparation for the field practice. One of the student teachers followed up by saying that they were well prepared. When the interviewer asked about the mathematical topic they were focusing on, the student teacher replied: “It is algebra”. The reflections did not go deeper than just stating this.

We then decided to make a more open analysis of the content in order to learn more about what the student teachers were actually discussing in the FGIs. When generating a concordance that was sorted after occurrences of words rather than an alphabetically sorted list of the words that occurred, we observed that the most frequently occurring words were general words.

Frequency	Word (in Norwegian)	In English
1702	det	it
963	og	and
784	er	am/is/are
737	de	they/them/those
674	så	so

Table 2. The most frequently occurring words in the transcripts

Most of these words (e.g. like “and”, “is” and “it”) were used in different ways, and it was hard to discover any patterns. When making a more qualitative summative analysis of the most frequently used words, however, we realised that the word “they” was interesting. Throughout the transcripts, the word “they” was used almost exclusively with reference to pupils. It is not surprising that the students used this word to describe the pupils, and this result in itself is probably influenced by the case used. It was, however, through the results of the content analysis that we became aware of this, and we decided to follow up with further analysis to learn more about the student teachers’ beliefs about the pupils—as revealed by their use of the word “they”.

The conventional content analysis

From the summative content analysis, we followed up with conventional content analysis. We started out with analysis of the keyword (“they”) in its immediate context. In the first step of this analysis, the context was defined as five words before and after the keyword. During the analysis of these context units, the first attempts to categorise the data were made. Then we followed up by increasing the context to entire utterances. The transcripts were read word by word, and categories were developed inductively (Hsieh & Shannon, 2005).

From this analysis, the following categories emerged in relation to how the student teachers' referred to the pupils (as "they"):

- General reference to pupils
- Characteristics of pupils':
    - behaviour (B)
    - cultural background (CB)
    - certain needs (CN)

Below is an utterance from one of the student teachers in the two interviews. It has been chosen for illustration since it contains uses of the keyword "they" that relate to all the three subcategories of "characteristics of pupils". It also contains an example of other uses of the keyword (O).

ST: It is of course possible to provide assistance with homework. It is fair enough that all pupils have a right to receive such assistance. You can, I mean, if you have more than two minority-language [pupils], you have a right to get a teacher assistant. [You] can help them (CB) in this way. By doing that you can split your class. You get those (B) who are diligent, that they (B) might be seated in a group of their own and work with things they (B) are able to [do on their own], so that they (B) are always ahead of the difficult tasks while you still help those (CN) who are struggling. That you have those (O) two assistants who walk around and help, and... Yes. It depends on how much learning difficulties they (CN) have, those (CN) two other pupils, but this is where you have the opportunity to get in some extra help.

As we can see from the first few sentences in this excerpt, the student teacher is talking about the challenges of having pupils with a minority language in the classroom. (S)he points to a regulation concerning the right to receive help with homework (for the minority pupils), and another regulation concerning the right to have a teacher assistant in the classroom if there are more than two pupils with a minority language background in the class. When a teacher assistant is present to take care of the minority-language pupils, the teacher can more easily split the class into groups. This student teacher is obviously in favour of separating the pupils in groups according to level. It is interesting to observe that the best pupils are referred to as diligent rather than smart, clever or high achieving. From this statement, it is possible to suggest that this particular student teacher holds a belief where high achievement in mathematics is related to effort.

The weaker pupils, on the other hand, are referred to as "struggling", and the student teacher follows up by talking about them as pupils with a certain level of learning difficulties. So, whereas the best pupils are characterised by their effort, the lower achieving pupils seem to be characterised as pupils with learning difficulties, pupils who have certain needs, and pupils who are entitled to receive extra (external) support.

CONCLUDING DISCUSSION

In their distinction between three important aspects in research on beliefs about teaching, Van Zoest, Jones and Thornton (1994) mentioned learner focused as a third aspect. We have seen in the results from our analyses that the student teachers, when reflecting about teaching, had a strong focus on the pupils. It seems, however, that the student teachers—when talking about the pupils in our FGIs—did not focus much on mathematics, teaching or the pupils as learners; they focused more on characteristics of pupils. In this way, the student teachers' beliefs—although they were arguably about teaching—somehow did not fit inside of Beswick's (2012) table (see Table 1). The student teachers did focus on the learners, but they seemed to focus more on the pupils as individuals with different characteristics.

Our aim with applying content analysis to the transcripts of the FGIs was to learn more about student teachers' beliefs. An underlying assumption then, is that there is a connection between their beliefs and the words they use. After having made some early analyses, it appeared that reflections about the mathematical content were not dominating in the student teachers' discussions. Mathematical concepts were used, but they were mainly referred to rather briefly. When applying summative content analysis to the data material, we found that the FGIs contained more reflections about pupils than on mathematical content. The word "they"—which was mostly used with reference to pupils—were among the most frequently used words in the FGIs (table 2). This is not surprising as the discussions were obviously influenced by the case that was presented to the student teachers in the interviews. After having made even further analyses of the data material, using conventional content analysis, we learned more about the different aspects of these reflections about pupils.

When analysing their characterisations of the pupils, we could distinguish between reflections concerning the pupils' behaviour, their cultural background or certain needs that the pupils might have. The student teachers, in their reflections, seemed to believe that the responsibility for pupils' learning is to be placed outside the teacher. They related the possibilities to help the pupils to other factors than their own competence. One example of this can be found in the excerpt from the transcripts that we have discussed above. In this excerpt, the student teacher suggests that the pupils would learn if there were enough teacher assistants. Their focus here can be interpreted as being on surviving in the classroom rather than on pupils' learning (Richardson & Placier, 2001). This is, of course, only one example, but we found the same tendency in our analysis of the entire data material. In both interviews, there seemed to be a belief that the pupils represented different kinds of challenges to the teacher, and the student teachers might suggest that a solution to these challenges could be found outside and not inside themselves. Skott (2001) argued that teachers' beliefs about mathematics teaching are often obscured by the more general priorities related to organising the classroom, and this might be part of the explanation for the seeming lack of focus on mathematics and mathematics teaching in our FGIs.

Much can be learned about student teachers' beliefs by analysing their reflections in focused discussions. We decided to use a combination of summative and conventional content analysis in our analyses of data from FGIs with student teachers prior to field practice, but there are other kinds of analyses to make—each with different advantages as well as disadvantages or limitations. One of the advantages of using content analysis in our study was that it helped us discover what the student teachers mainly focused on in their discussions, and we could then use this in further analyses to learn more about their beliefs. The combination of approaches to content analysis also made it possible for us to discover reflections that were made by using a common word like "they". In order to learn more about student teachers' beliefs, it is also relevant to use the results of these analyses—as well as other analyses—to make a new evaluation of the interview guide. When looking back at the case—which was presented in the FGIs—it can be noticed that it was more focused on students than on mathematics. The case most certainly had an effect on the participants' reflections in the FGIs.

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TEACHERS' BELIEFS AND KNOWLEDGE RELATED TO THE CYPRUS MATHEMATICS CURRICULUM REFORM

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This paper presents some results of a larger study that investigates teachers' knowledge, beliefs, practices and enjoyment/confidence related to a mathematics curriculum reform. Data were collected from 100 in-service primary teachers through a questionnaire referring to teachers' background information, their beliefs, their enjoyment/confidence and their knowledge of the new mathematics curriculum. Findings revealed the existence of three factors concerning teachers' enjoyment/confidence, their traditional beliefs, and their inquiry-oriented beliefs. A description for teachers' knowledge of the new curriculum is presented. Correlations existed between their beliefs and their enjoyment/confidence.

Keywords: Teachers, knowledge, beliefs, curriculum reform.

INTRODUCTION

Cyprus launched a curriculum reform in 2008 and started implementing it since September 2011. As far as the new mathematics curriculum is concerned, the target was to move from traditional teaching of mathematics to more progressive approaches. It is, therefore, pertinent investigating the factors that can affect the successful implementation of the mathematics reform.

In mathematics education, a considerable body of research (Ernest, 1989; Stipek, 2001; Tompson, 1992; Wilkins, 2008) underpin the importance of teachers' knowledge, beliefs and attitudes in their effectiveness and their choice of instructional practices, and as a consequence their disposition towards the implementation of innovation. In this respect, we developed a study to investigate teachers' knowledge, beliefs, practices and enjoyment/confidence related to the new mathematics curriculum. In this paper we focus on teachers' beliefs, their enjoyment/confidence about mathematics and its teaching and their knowledge of the new mathematics curriculum.

BACKGROUND AND AIMS

Teachers' knowledge

Prior research refers to several factors that influence teachers' instructional practices and, therefore, the implementation of a new mathematics reform (Charalambous & Philippou, 2010; Handal & Herrington, 2003; Manouchehri & Goodman, 2000). We based our research on the model proposed by Ernest (1989) on teacher knowledge, beliefs and attitudes, as it represents an attempt to understand psychological factors underpinning the impact of curriculum innovation on mathematics teachers. A key difference in Ernest's model and other related models like Ball's, Thames' and Phelps' (2008) is the inclusion of beliefs and attitudes in the model.

Ernest (1989) suggests that teachers' knowledge of mathematics constitutes of several components: (a) pure subject matter knowledge that the teachers' needs in order to teach mathematics; (b) knowledge of teaching mathematics which involves pedagogical knowledge of the subject, and curriculum knowledge. Pedagogical knowledge of mathematic refers to teacher's knowledge of approaches to school mathematics like awareness of different ways of presenting mathematics and knowledge of students' methods, conceptions and errors. Curriculum knowledge refers to the knowledge of the curricular materials which mathematics instruction is carried out and assessed; (c) knowledge of other subject matter, which provides a knowledge of mathematics uses and applications (d) knowledge of organization for the teaching of mathematics, which refers to the knowledge of organising the mathematics instruction in individual work, co-operative groups, and to the management of practical activities; (e) knowledge of the students and school and (f) knowledge of education. In this paper we present results related to teachers' knowledge of teaching mathematics and particularly their curriculum knowledge.

As Ernest (1989) states, teachers' beliefs of mathematics and mathematics teaching were also found to influence teachers' instructional practices. More specifically, two teachers may have similar knowledge of mathematics but may teach using different methods and procedures due to their different beliefs of mathematics and mathematics teaching. The importance of beliefs and their impact on teaching has also been emphasized by several authors (Hannula, Evans, Philippou, & Zan, 2004; Stipek et al., 2001; Tompson, 1992; Wilkins, 2008).

Teachers' beliefs about mathematics, teaching and learning

There is still a discussion about the definition and characteristics of beliefs. According to Ernest (1989) "beliefs" consist of the teacher's system of beliefs, conceptions, values and ideology. Beliefs develop over time on the basis of related experiences, while the affective dimension of them influences the role and the meaning of each belief in the belief system (Wilkins, 2008). Teacher's beliefs consist of views about the discipline of mathematics, and about the teaching and learning of mathematics. Teachers' beliefs about the discipline of mathematics constitute the "rudiments of a philosophy of mathematics" (Thompson, 1992, p. 132), while their beliefs about the teaching and learning of mathematics may include the role of the teacher and the students in the teaching and learning situation, classroom activities, instructional approaches, mathematical procedures and the acceptable outcomes of instruction (Handal & Herrington, 2003; Thompson, 1992).

Many teachers may have more traditional beliefs about the discipline of mathematics and its teaching. These beliefs include that mathematics is a static body of knowledge which involve a set of rules and procedures that are applied to produce one right answer. Then knowing mathematics means to be skilful in performing procedures without necessarily understanding what they represent (Stipek et al., 2001; Thompson, 1992). In this traditional standpoint the teacher is in full control of the mathematics learning, providing students with a step by step instruction and allocating problems for practicing the procedure (Stipek et al., 2001; Tompson, 1992). The inquiry based beliefs include a dynamic view of mathematics, a problem centred view, a continually changing field of human creation and invention, open to revision (Stipek et al., 2001; Thompson, 1992). Inquiry-based beliefs refer to students' engagement in activities to construct their own knowledge, to reason, to be creative, to discover the knowledge and to communicate their ideas. The teacher shares the control with students playing a facilitator role, encouraging students to fulfil their own learning aims and construct meaning by themselves.

Teachers with more traditional beliefs were found (Stipek et al., 2001) to be more close to the entity theory of ability, supporting the view that ability is stable and immutable, leaving no room for development and minimizing the importance of effort. On the contrary, teachers with inquiry-based beliefs may be associated with an incremental view of ability in which the ability is amenable to change and increase with learning and effort (Cury, Elliot, Fonseca & Moller, 2006). As far as students' motivation is concerned, several studies (Patrick, Anderman, Ryan, Edelin, & Midgley, 2001; Stipek et al., 2001) showed that teachers with more traditional beliefs value the importance of extrinsic motivation, while teachers' with more inquiry-based beliefs value the importance of intrinsic motivation.

Apart from beliefs, Ernest (1989) includes in the model teachers' attitudes towards mathematics and its teaching, such as liking, enjoyment and interest in mathematics, and also teachers' confidence in their own ability in mathematics and its teaching. In Ernest's model, the interrelationship between knowledge, beliefs and attitudes is not explicitly addressed (Wilkins, 2008). However, studies (Karp 1991 in Wilkins, 2008; Stipek et al., 2001) found that teachers with more traditional beliefs are less confident and enjoy mathematics less than teachers who are related to more inquiry-based beliefs.

A considerable amount of studies has led researchers to consider teachers' beliefs as an important mediator in curriculum implementation (Charalambous & Philippou, 2010; Handal & Herrington, 2003; Tompson, 1992). It is assumed that in order for teachers to implement a curriculum reform their beliefs must be somehow aligned to the basic philosophical beliefs underlying the reform (Handal & Herrington, 2003).

Cyprus reform and the new role for the teacher

In September 2011, the new mathematics curriculum started to be implemented in grade one (first year of primary school) and grade seven (first year of secondary school) and it is expected to be in full operation by June 2017. It follows current trends in education, presenting mathematics as a dynamic tool for thought (Cyprus Ministry of Education and Culture, 2010).

The new mathematics curriculum has been designed according to four principles: (a) students should be involved in mathematical investigations, which enhance their curiosity and interest, related to already existing knowledge, based on real life situations and interdisciplinary questions, (b) emphasis should be paid on problem-solving, (c) ICT as an integral part of mathematics education and (d) students' experiences will be enriched through pedagogically rich examples, that arise from the active engagement with meaningful mathematical problems and concepts (Cyprus Ministry of Education and Culture, 2010). These principals are to a great extent congruent with inquiry-based mathematics instruction (NCTM, 2000; Wilkins, 2008).

For the effective implementation of the new curriculum, the role of the teacher in the mathematics classroom shifts from the traditional instruction in which the teacher transmits to students pieces of mathematical information to a more demanding role. While students engage in investigations, the teacher is expected to create "a community of inquiry" in which the students comfortably exchange ideas and justify their views (Manouchehri & Goodman, 2000; Wilkins, 2008).

As far as knowledge is concerned, the new mathematics curriculum expects teachers to have the necessary subject matter knowledge and knowledge of teaching mathematics (Ernest, 1989), to understand the core mathematical ideas of the curriculum material, to recognize the relationships among concepts, to be able to reason mathematically and use multiple representations of new mathematical concepts. The teacher is expected to build on students' thinking around different investigations and to connect their thinking to specific mathematical concepts. As other studies revealed (Manouchehri & Goodman, 2000; Stipek et al., 2001) teachers need confidence in their ability to make sense of mathematics and students' solutions and strategies, while they must encourage and reward students' efforts to solve mathematical problems.

Several studies have stressed that many reforms fail due to teachers' lack of mathematics knowledge and knowledge of teaching mathematics or because their beliefs are not congruent with the beliefs supporting the reform (Manouchehri & Goodman, 2000; Stipek et al., 2001). It is therefore evident that teachers' knowledge and beliefs have a determinant role in the success or failure of the new Cyprus mathematics curriculum. In this respect the aim of this study was:

- To investigate teachers' beliefs about: (a) the nature of mathematics (procedures vs. thinking), (b) mathematics learning (correct answers vs. understanding), (c) control of the classroom (teacher's control vs. students' autonomy), (d) the nature of mathematical ability (fixed vs. developing), (e) motivation (extrinsic vs. intrinsic).
- To investigate teachers' confidence and enjoyment of mathematics and mathematics teaching.
- To examine teachers' knowledge of teaching mathematics and particularly the degree of teachers' level of awareness of the new mathematics curriculum.

METHOD

Data were collected through a questionnaire from 100 in-service teachers from primary schools in rural and urban areas in Cyprus. Part of these subjects had earlier participated in professional development programs focusing on the mathematics reform. The questionnaire administered comprised of five parts: (A) teachers' background information, (B) teachers' beliefs, enjoyment/confidence, (C) teachers' knowledge of teaching and particularly their awareness of the new mathematics curriculum, (D) teachers' instructional practices, and (E) teachers' mathematical knowledge. Parts D and E of the questionnaire are not presented in this study since they are beyond the aims of this paper.

The first part of the questionnaire sought demographic data, including the subjects' educational background, such as the number of maths courses they took during their undergraduate studies and the number of seminars they attended during their careers related to mathematics education as well as the seminars they have attended specifically related to the new mathematics reform. Moreover, their experience in teaching mathematics and the class that they were teaching mathematics during the specific year were reported.

The second part comprised of items selected from the questionnaire by Stipek et al. (2001) that were related to teachers' beliefs and enjoyment/confidence about mathematics and teaching. Specifically there were 30 items measuring teachers' agreement on a 5 point Likert scale (1- strongly disagree, to 5-strongly agree). Each item was related to one of the two ends of bi-polar scale of each of six dimensions of mathematics: (a) the nature of mathematics (procedures vs. thinking), for instance: "The best way to understand math is to do lots of problems", "In every lesson teachers need to discuss how people use the math being taught to solve real-life problems"; (b) mathematics learning (correct answers vs. understanding), for instance: "Students who produce correct answers have a good understanding of mathematical concepts", "Children's reasoning in their mathematical problem solving is more important to assess than whether they solve problems correctly"; (c) control of the classroom (teacher's control vs. students' autonomy), for instance: "It is important for teachers, not students, to direct the flow of a lesson", "Good teachers give students choices in their math tasks"; (d) the nature of mathematical ability (entity vs. incremental), for instance: "Mathematical ability is something that remains relatively fixed throughout a person's life", "Improvement should be a major consideration when grading students"; (e) motivation (extrinsic vs. intrinsic), for instance: "Giving rewards is a good strategy for getting students to complete math assignments", "If children aren't working, it is probably because the task is not very interesting"; (f) teachers' self confidence and enjoyment of mathematic, for instance: "I feel confident that I understand the math material I teach".

The third part of the questionnaire assessed teachers' knowledge of teaching and particularly their awareness of the new mathematics curriculum. The eight items measured teachers' knowledge on, disagree or agree. Two spice items are:

"According to the new mathematics curriculum students should not come across many representations of a mathematical concept to avoid being confused" and "In the new mathematics curriculum an attainment target can be found in more than one class".

RESULTS

The analyses of the data revealed that teachers had various experience of teaching and educational background (table 1). The great majority of the teachers had more that 10 years of experience, while 66% had postgraduate degrees. At the beginning of the implementation of the mathematics reform, 24% of the teachers reported that they had participated in professional seminars focusing on the mathematics reform. In Cyprus dissemination of information related to the new reform might take place in schools, since informed teachers are expected to inform their colleagues.

Experience	Education		
	Undergraduate studies	Postgraduate	Phd
1-5 years	1	11	0
6-10 years	6	12	0
11-15 years	12	14	1
15-20 years	6	20	1
20 -more	6	9	0
(N=99) 1 missing	**31**	**66**	**2**

Table 1. Teaching experience and mathematical background

Regarding the first and second questions of the study, principal component analysis (PCA) was conducted on the 30 items with orthogonal rotation (varimax). The Kaiser-Meyer-Olkin measure verified the sampling adequacy for the analysis, KMO = .72 ("good" according to Field, 2009), x^2 (276) = 795.42, p<.001, indicated that correlations between items were sufficiently large for PCA. After various analyses 6 items were deleted due to their loadings on various factors or to their low loadings. We ended in three factors explaining 46.41% of the variance. Table 2 shows the factor loadings after rotation.

	Rotated factor loadings		
Item	**Confidence / Enjoyment**	**Traditional**	**Inquiry**
I think of myself as being good in mathematics.	.833		
I don't enjoy doing mathematics (reverse).	.822		
Math is my favourite subject to teach.	.801		
I enjoy encountering situations in my everyday life that require me to use math to solve problems.	.752		
I'm not competent enough in math to teach it beyond the elementary grades (reverse).	.647		
When my answer to a math problem doesn't match someone else's, I usually assume that my answer is wrong (reverse).	.631		

When I teach math I often find it difficult to interpret students' wrong answers (reverse).	.617		
The best way to understand math is to do lots of problems.		.807	
Students who aren't getting the right answers need to practice on more problems.		.711	
Students who finish their math work quickly understand the material better than students who take longer.		.681	
It is important for teachers, not students, to direct the flow of a lesson.		.653	
It is important for teachers to maintain complete control over math lessons.		.647	
Students who produce correct answers have a good understanding of the mathematics concepts.		.561	
The more students are concerned about grades and performance the more they learn.		.460	
Students who really understand math will have a solution quickly.		.451	
Giving rewards is a good strategy for getting students to complete math assignments.		.450	
Students will work hard on interesting and challenging math tasks, whether or not their work is graded.			.805
Improvement should be a major consideration when grading students.			.689
In every lesson teachers need to discuss how people use the math being taught to solve real-life problems.			.628
Students' reasoning in their mathematical problem solving is more important to assess than whether they solve problems correctly.			.602
The more students enjoy working on math tasks the more they learn.			.592
Effort should be a major consideration when grading students.			.584
There is usually one way to solve a math problem.			-.481
If students aren't working, it is probably because the task is not very interesting.			.429
Eigenvalues	5.50	3.19	2.40
% of variance	23.13	13.30	9.98
a	.85	.78	.63

Table 2. Summary of exploratory factor analysis results (N=100)

The items that cluster on the same factor suggest that factor 1 represents teachers' enjoyment/confidence, factor 2 represents traditional beliefs of mathematics and its teaching, and factor 3 represents inquiry-oriented beliefs of mathematics and its teaching. As it was assumed, the traditional view of mathematics comprised of dimensions for which high scores were presumed to be associated with traditional theory of mathematics and its teaching. Specifically, mathematics is a set of operations which are used to get correct solution to problems rather than tools of thought, the importance of getting the correct answer, the issue of teacher's control in the classroom and the development of students' extrinsic motivation. There were no items referring to the entity theory in this factor. The inquiry view of mathematics comprised of dimensions for which high scores were presumed to be associated with inquired-oriented beliefs of mathematics and its teaching. Specifically the view of mathematics as a tool of thought and not as a set of operations, the importance of students' understanding and not just the correct answers, beliefs concerning the incremental ability of students in mathematics and the development of students' intrinsic motivation. There were no items referring to students' autonomy in the classroom.

Teachers in this study appeared to be rather confident and enjoyed mathematics (M=3.83), while their traditional views of mathematics were moderate (M=3.14), and their inquiry based views were higher (M=3.78) than their traditional views but not very high.

As far as it concerns the relation between the three factors, the Pearson's product-moment correlation coefficient indicated a significant positive relationship between teachers' confidence/enjoyment and the inquiry-oriented beliefs of mathematics ($r=.276$, $p<0.5$) and also a significant negative relationship between teachers' confidence/enjoyment and traditional views of mathematics, ($r=-.247$, $p<0.5$). Even though the indices were small, it seems that teachers' scoring high on the inquiry based beliefs were more confident about teaching mathematics and enjoyed it more, while teachers scoring high on the more traditional views were less confident and enjoyed mathematics less.

Regarding the third question of the study concerning teachers' knowledge of teaching and particularly curriculum knowledge, Table 3 presents the percentage of teachers' by the number of questions in which they provided positive answers, out of the 8 items.

Number of positive answers	0	1	2	3	4	5	6	7	8
Percent of subjects	6.9	2.9	1	2	5.9	16.7	34.3	27	2

Table 3. Teachers' level of curriculum awareness (N=98, missing 2)

The results present a rather positive picture of these teachers' curriculum knowledge. A considerable percentage (63.3%) of these teachers seems to be well aware of significant issues and new trends (total of 6-8 positive responses) at the beginning of the implementation of the mathematics reform.

Teachers' positive responses ranged between 73%-88% in 6 of the 8 items. Particularly 88% were informed that an attainment target can be repeated in more than one grades and 82% were aware that the new mathematics curriculum included attainment targets, indicative activities, enrichment activities and assessment tasks; Also, 82% of the teachers were well-versed how the new curriculum deals with students' erroneous answers and 81% were informed that in the new curriculum addition and subtraction are perceived as conceptually connected therefore they must be taught jointly. Finally 76% of the teachers were aware that students are not expected to reach an attainment target in the same pace and 73% were notified of the importance of multiple representations in mathematics as included in the new mathematics curriculum. Teachers scored lower in two items of the questionnaire that referred to more detailed practices in the classroom. The first item refers to the use of enrichment activities in the mathematics class, which provides for differentiating instruction to cater for students of various mathematical abilities. The second item referred to the investigation stage, which is a stage of the didactical model suggested by the new curriculum. In this stage the students are expected to conjecture, investigate and discover the new mathematical concept.

DISCUSSION

The aim of this study was to examine teachers' beliefs, enjoyment/confidence and their acquaintance with the reformed curriculum at the outset of its implementation. Analysis of the data revealed the existence of three factors concerning traditional beliefs, teaching confidence and inquiry orientation. Similar to Stipek's et al. (2001) findings, the factor concerning traditional beliefs referred to the nature of mathematics, mathematics learning, control of the classroom and motivation. Items referred to the nature of mathematical ability were not included in this factor. Regarding teacher's inquiry-oriented beliefs another set of beliefs existed referred to the nature of mathematics, mathematics learning, the nature of mathematics ability and motivation. Items referred to students' autonomy in the classroom were not included in this factor. In this study teacher' inquiry-oriented beliefs found to be moderate, a result that policy-makers should take into consideration since, flourishing curriculum change will more probably occur when teachers' beliefs and curriculum reform goals are congruent (Handal & Herrington, 2003).

Similar to other studies (Manouchehri & Goodman, 2000; Stipek et al., 2001; Wilkins, 2008) teachers in this study who adopted more inquiry-oriented beliefs were more self-confident and enjoyed mathematics more than teachers who adopted more the traditional beliefs. More confident teachers may adopt beliefs and practices that require more decision-making and judgment.

Regarding teachers' acquaintance with the new curriculum the study reveals that teachers' curriculum knowledge (Ernest, 1989) is relatively high even though a small percentage of them have so far received in-service training. In line with the suggestions of other studies (Charalambous & Philippou, 2010; Manouchehri & Goodman, 2000) teachers should continue to receive systematic and sustained support targeting their knowledge of teaching mathematics (Ernest, 1989) in order to succeed in implementing it.

Curriculum implementation is not a process that translates directly into the mathematics classroom. Teachers are the key players in a successful implementation of a curriculum reform. Therefore their knowledge and beliefs should be identified, analysed and improved in the process of the curriculum implementation.

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FACTORS MOTIVATING THE CHOICE OF MATHEMATICS AS A CAREER AMONG MEXICAN FEMALE STUDENTS

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In this paper we present a report of an ongoing research focused on identifying the factors that motivated Mexican female students to choose mathematics as a career. The main body of data for the research was generated through semi-structured interviews. The preliminary results show that three are the main factors that motivated women in the sample to choose mathematics as a career: (1) to be aware of the fact that they are good at mathematics, (2) the influence of their mathematics teachers, and (3) the influence of their relatives. We suggest that these three factors promote the constitution of a mathematical identity, which in turn favors the choice of mathematics as a career.

Keywords: Career choice, female mathematics students, Mexican students, mathematical identity.

INTRODUCTION

There are several reports indicating that few tertiary students around the world are enrolled in science, technology, engineering and mathematics (STEM) related careers (European Commission, 2004, Organisation for Economic Co-operation and Development, 2008; Stine & Matthews, 2009). In particular, the recruitment rate of women in these areas is particularly low (see for example European Commission, 2009). Meeting the demand for scientists and engineers is a widespread concern because of the important role that these careers play in the development of modern society; however, there is also a global interest in producing a diverse and gender balanced scientific workforce. This situation has generated a number of studies that try to identify what factors attract and retain students (particularly women) to STEM careers (see for example Brickhouse, Lowery & Schultz, 2000; Herzig, 2004; Hill & Rogers, 2012; Mendick, 2005).

The situation in Mexico, where this study is being developed, is not different from that described above. The recruitment rate of female students in STEM careers, particularly in mathematics, is very low. The latest figures on the study of mathematics at the tertiary level in Mexico show that: (1) women represent 38% of the people studying a bachelor degree in mathematics; (2) women represent 24% of the people studying a master degree in mathematics; and (3) women also represent 24% of the people studying a PhD degree in mathematics (data taken from Barrera, 2012).

In spite of these problems to attract women to the study of mathematics, there are very few studies in Mexico (and in general in Latin America) focused on identifying the factors that may attract and retain female students to study math-related careers in this region. The latest Mexican study focused on this issue is that of Jiménez (2006), where some female researchers in mathematics were interviewed about the reasons that led them to choose mathematics as a field of work.

In this paper we report an ongoing research project focused on identifying some of the factors that may motivate Mexican female students to choose mathematics as a career. Specifically, we are trying to answer the following research question:

What factors motivates Mexican female students to choose mathematics as a career?

The main contribution of our work is to help to understand what motivates female students from Latin American countries to choose mathematics as a field of study. Our research could help to identify differences and similarities between the motivating factors to study mathematics among women from different regions of the world. We also believe that our research can produce pedagogical recommendations to promote the study of mathematics among young women in our country.

The manuscript is organized as follows: first, we will summarize the literature that we have consulted so far and clarify what is the function that this literature review has had in our research; then, we will explain the approach we have taken to collect and analyze empirical data; next, some preliminary results are presented, this is, some of the factors that seems to motivate Mexican female students to choose mathematics as a career; then, we will refer to the concepts of self and identity as theoretical tools to try to interconnect and make sense of the results obtained; finally we will present some conclusions.

A REVIEW OF RELEVANT LITERATURE AND ITS ROLE IN THE RESEARCH

When we began our review of the literature, we focus initially on mathematics education research journals, trying to locate studies that could explain why few women choose to study mathematics. In a second stage we extended our search to science education journals, trying to locate articles focused on studying what factors attract and retain female students to STEM careers.

The literature review had a dual role in our research. On the one hand, it allowed us to locate some of the methods used in the literature to identify the factors that may motivate women to study STEM careers; we used this information to design our own research method. On the other hand, the literature review was useful to identify hypotheses or possible explanations on why some women are attracted to this type of careers.

Regarding the methods used, we found that in some studies questionnaires are employed in conjunction with other instruments (for example in Holmegaard, Ulriksen & Madsen, 2012; Sjaastad, 2012), but most studies use open interviews to allow women to produce narratives about their experiences with mathematics (see for example Mendick, 2005; Piatek-Jimenez, 2008; Solomon, 2012). Through these personal narratives researchers try to locate activities and experiences that have led women to study mathematics.

With regard to the hypotheses or possible explanations for why some women choose to study (or not so study) mathematics-related careers, they are very different in nature. To explain why some women choose not to study mathematics, some authors claim that mathematics can be perceived as an unfeminine profession, resulting in a discrepancy between female identity and a mathematical identity (Piatek-Jimenez, 2008; Solomon, 2012). Another explanation for the low number of women in mathematics as a field of study is that there is discrimination against women in math-intensive fields and in the mathematics classroom —sometimes unconsciously— (Ceci, Williams & Barnett, 2009). There are authors who claim that the level of creativity required in some hard sciences, which is not socially favored among women, can be a reason why there is a low presence of women in these sciences (Hill & Rogers, 2012). There are at least two factors that have been identified as motivating and inspiring for women to study mathematics-related careers: (1) the confidence that individuals have in their own intellectual abilities (Eccles, 2007) and (2) the positive influence of significant persons, such as parents, teachers and friends (Sjaastad, 2012).

During our review of the literature, we also noted that some studies associate the process of choosing a career with the construction of an *identity* in young people. For example, Sjaastad (2012) uses as a theoretical tool the concept of *self*; he bases his discussion of the concept of self in the works of Higgins (1987) and Swann & Bosson (2010). The self refers to the attributes that a person believes to possess and the attributes the person would like to possess. One important thing here is that, the self is influenced and shaped by interpersonal relationships; as stated by Swann & Bosson (2010): “We know ourselves […] by observing how we fit into the fabric of social relationships and how others react to us” (p. 589).

In turn, Holmegaard, Ulriksen & Madsen (2012) relate the choice of a career with the process of defining oneself:

> “[T]he decision about which course of study to choose after finishing upper-secondary school is not limited to figuring out what could be interesting or promising; it is also about *defining* oneself, and making a decision about whom one wishes to become” (p. 4).

Similar to the theoretical position of Sjaastad (2012), Holmegaard, Ulriksen & Madsen (2012) conceptualize the constitution of an identity as shaped by interactions with others and the cultural context where the person is immersed.

The concept of *mathematical identity* or *identity as mathematics learner* can also be found in the literature on mathematics education (see for example Anderson, 2007; O’Hara, 2010). The construct of identity refers to “the way we define ourselves and how others define us” (Anderson, 2007, p. 8), and serves to explain what makes a person to feel like an able mathematics student and as a consequence get involved and engaged in mathematical activities. As we shall see, this construct could be helpful to explain some of the results of our study.

METHOD

After identifying in the literature some of the factors that motivate women to pursue a career related to mathematics, we conducted an exploratory study. A role of this exploratory study was to verify whether the factors located in the literature appeared in Mexican female students; another role was to try to identify other possible motivating factors not reported in the literature.

Description of the exploratory study

For the exploratory study a questionnaire was applied to 32 Mexican girls studying a bachelor degree in mathematics in the University of Veracruz, located in southeast Mexico. The questionnaire was applied during February 2012. The head of the mathematics department administered the questionnaire to randomly selected students. The selected students had different degrees of advance in their studies, and their ages ranged from 18 to 21. The questionnaire consisted of eleven open questions focused on their experiences with mathematics before entering the bachelor program and their reasons why they decided to study mathematics. It included questions like: what motivated you to choose this career? and, at what point in your life you decided to study mathematics and why? These two open questions were particularly helpful in identifying some of the reasons why these students chose to study mathematics. Among the reasons they expressed in the questionnaire are:

- Their mathematics teachers motivated them to study mathematics.
- Their parents influenced their career choice.
- Participation in competitions like math Olympiads motivated them to study mathematics.
- At some point in their lives they realized that they were good at mathematics and decided to study it.
- At some point in their lives they discovered some sort of mathematical applications and as a consequence they found the study of mathematics appealing.

With these explanations in mind, we designed an interview guide for semi-structured interviews. Through semi-structured interviews we generated the main body of data for the research. Next we explain in more detail how these data were produced.

Collecting data through semi-structured interviews

In this second stage of our study we interviewed 37 female students enrolled in the Higher School of Physics and Mathematics at the National Polytechnic Institute of Mexico, in Mexico City. As in the exploratory study, the selected students had different degrees of advance in their studies, but in this case their ages ranged from 18 to 22. All the authors of this paper participated in conducting the interviews. The interviews took place between August and September 2012, about a month after starting the school year. The student participation was voluntary and the interviews were conducted during downtime between lessons, in the gardens of the Institute. Our purpose was to create a casual and comfortable environment for the students. The interviews were audio recorded and their average length was 12 minutes.

The semi-structured interview guide contained a couple of open questions aimed at triggering students' narratives on the activities and experiences that motivated them to study mathematics. These open questions were: (1) can you name one or more experiences or activities of your past that influenced you to choose this career? and (2) was it difficult to choose your career, did you always know you wanted to study this or was there any particular incident that made you choose this career?

There are several similarities between the two samples we used in this study. On the one hand, both samples consist of Mexican women who are studying a bachelor degree in mathematics in a public school, this means that their socio-economic backgrounds should be similar (most likely belonging to middle class or lower middle class); another similarity is their age ranging from 18 to 22; also, both groups of students are studying in urban areas, although the students from Mexico City belong to a much more developed urban area. These similarities between samples made us consider that the explanations obtained in the exploratory study could be similar to the explanations provided by the second group of students. Thus, we expected that the above-mentioned questions could detonate narratives in which teachers, parents, or mathematical competitions were mentioned as factors influencing their choice. For this reason we prepared additional questions to deepen each of these factors, in case they appeared. For example: What did you like about your math teacher and her teaching?; How would you describe your experience in the mathematical competition (positive or negative) and why?; How your parents and friends reacted when you told them you wanted to study mathematics?

Data analysis

To analyze the interviews the audio recordings were distributed among all members of the research team. When analyzing the audio-recorded interviews, we focused on locating the instances where students mentioned any experience or activity that motivated them to study mathematics. Each of those instances was transcribed. To increase the reliability of the results, two researchers independently analyzed each recording; if there was any discrepancy in the interpretation of the interview, both researchers explicitly discussed how they were interpreting the content of the interview to reach consensus.

At the time of writing this report we had analyzed 10 out of 37 interviews. The results presented below are based on the analysis of those ten interviews.

PRELIMINARY RESULTS

The results presented next are the factors that ten Mexican female students mentioned in their narratives as motivating to study mathematics. Some students mentioned more than one motivating factor during the interview. We have classified these factors into seven categories. Figure 1 shows the frequency for each of those categories.

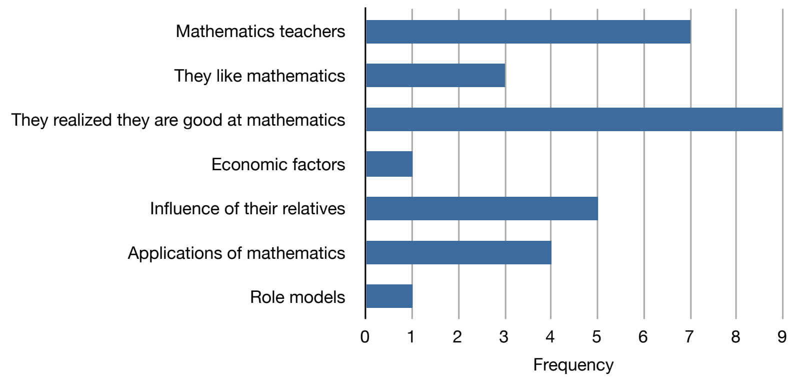

Figure 1: Some factors that motivate the female students in our sample to study mathematics.

Now we will explain in more detail each of the categories listed in figure 1:

Mathematics teachers. Mathematics teachers are the second most mentioned motivating factor among the students. There are several aspects that students highlight about their teachers: Some mentioned that they liked the way they taught, for example, by presenting step by step and detailed explanations of the topics; others mentioned that their teachers were enthusiastic and they transmit them their love for mathematics; other students, as illustrated in the following transcript, mentioned that their teachers assigned them extra activities and even invited them to some math-related extracurricular activities (like math competitions):

Student: The teacher taught me many things because I was interested; then, after finishing the class, she gave me more lessons, she taught me more things. Actually I went with her to several competitions.

They like mathematics. When asked what motivated them to study mathematics, some students simply respond: "because I like them". When asked why they like mathematics, one student said that she liked mathematics because of its exact nature, and because of the pleasure felt when solving a problem. However, some students find it difficult to clarify their reasons:

Interviewer: Can you explain why do you like mathematics?

Student 1: Well, not really. It's a delight, as someone who loves music.

Interviewer: Why do you like mathematics?

Student 2: I have no exact idea… I like them because… I don't know how to say it, I feel like it is a science.

They realized that they are good at mathematics. The main motivating factor mentioned by the students was that, at some point in their life, they realized that they were good at math. Different situations allowed them to recognize their mathematical skills: In some cases their teachers explicitly told them that they were good at math; some students noticed their mathematical skills because they did well in mathematical competitions or because they got good grades; some received social recognition when their classmates asked them for help to study or to solve their homework. In some cases there was also a self-recognition of their mathematical skills when noticing that their peers struggled to solve mathematical tasks while they solve them with ease.

Economic factors. There was a student who said that she would have liked to study a bachelor of finance at a private university, but due to financial constraints, she had to choose another math-related career at a public university. It was then when she decided to study a bachelor degree in mathematics.

Influence of their relatives. The family is an important source of motivation for students. There was a student who noted how proud his parents were of her because she was doing well in mathematical competitions; seeing her parents pleased and happy because of her performance motivated her to study mathematics. On the other hand, several students mentioned that someone in their families (their parents, their brothers, uncles) had a math-related job or degree, and that inspired them to study. See for example the next transcription:

Interviewer: What did your parents say when you told them that you wanted to study mathematics?

Student: What happens is that my dad he is a math teacher, then well… then he told me yes, it is a good choice. He encouraged me and asked me to make my best effort.

Applications of mathematics. There were four students who referred to moments in their school life where they found that mathematics could have many applications. The students claimed that this characteristic made them get interested in mathematics. The following is a transcript of one of these accounts:

Student: In high school a teacher played a video about how mathematics is everywhere, ... well it is also used in music and, I don't know, it called my attention to know that there are many applications and that at the end, everything has to do with mathematics.

Role models. When we talk about role models we refer to people who serve as an example and inspiration to students, but who are neither their teachers nor their relatives. One student remembered her encounter with an astrophysicist. Apparently he was a source of inspiration for her:

Student: My dad knows a person who is an astrophysicist, in fact he studied here [in the National Polytechnic Institute], he is now at NASA…I would like to study astrophysics, I really like the outer space, see, I want to see [...]

CONCLUSION AND DISCUSSION

In this project we are trying to answer the following research question:

What factors motivates Mexican female students to choose mathematics as a career?

Based on the previously presented preliminary results, we can say that there are three major factors that influence Mexican female undergraduate students to choose mathematics as a career:

(1) To be aware of the fact that they are good at mathematics.

(2) The influence of their mathematics teachers.

(3) The influence of their relatives.

Our results are similar to those reported in previous research. For example, Eccles (2007) mentions the confidence that individuals have in their own intellectual abilities as a motivating factor for women to study STEM careers; on the other hand, Sjaastad, (2012) recognizes the big influence that parents, teachers and friends can exert on young people for them to choose scientific careers. A result that we found in the study and we have not seen reported in the literature is that mathematical competitions can be a place where students can acknowledge or confirm their mathematical skills.

The three identified factors contain elements that help students to identify and confirm the ownership of attributes (such as the ability to solve mathematical tasks), which in turn could trigger the construction of an identity as mathematics learner (Anderson, 2007). When this identity is constituted, choosing to study mathematics appears as a natural and well-suited option.

For instance, the situation where their classmates asked for help to do their homework, can be interpreted as a process where female students realize that they have certain mathematical abilities; when the teacher assigns extra tasks to the student, in a way is implicitly communicating that the student is special: she possesses qualities that others do not have. In our view, a good performance in mathematical competitions can help students to confirm the ownership of such mathematical skills. These inputs that the female student receives may help to constitute the identity face known as *engagement* (Anderson, 2007), where students identify themselves as capable mathematics learners. On the other hand, noticing that some of your relatives have studied or perform some math-related job can support the creation of a mathematical identity compatible with one of the most important social groups to which you belong: your family. We believe that this type of influences from family members can encourage the student to imagine herself as a person that in the future could pursue a math-related career. Anderson (2007) refers to this aspect as the *imagination* face of identity.

We believe that it may be productive to continue exploring this theoretical perspective to support the development of our research. The next step in our research will be finishing with the data analysis and categorization of the results. Also, to delve deeper into the theoretical perspectives that connects the self and identity with career choice.

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INVESTIGATING TEACHERS' TRIGONOMETRY TEACHING EFFICACY

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The purpose of this paper is to share the process of investigating teachers' trigonometry teaching efficacy and determining their efficacy levels for a larger study. Firstly, Teacher Trigonometry Teaching Efficacy Scale (TTTES) was administered to sixteen teachers. Teachers generally scored the highest rank for their efficacy in the scale so they were not differentiated. Then, an interview protocol was designed to decide teachers' levels of efficacy from their experiences of teaching trigonometry. Teachers' answers were studied by using three indicators of self-efficacy: choices, effort, and thought patterns and emotions. Then, the researchers could determine teachers' trigonometry teaching efficacy.

Keywords: Teacher efficacy, trigonometry teaching efficacy.

INTRODUCTION

Teachers' confidence level of their ability to teach, teacher efficacy, has an important role in their teaching practices. Teachers with high teaching efficacy put more effort in teaching and use more diverse teaching strategies in class (Ghaith &Yaghi, 1997). The effective teaching methods may have positive effect on their students' learning and their desire to work on the subjects (Ashton, Webb, & Doda, 1983). So, it is important to study teacher efficacy and its relation to other variables.

Bandura (1982) developed social cognitive theory and suggested the concept of self-efficacy. Tschannen-Moran, Woolfolk-Hoy and Hoy (1998) defined teacher efficacy as the level of belief a teacher has in his or her ability to affect student achievement and they developed a model using Bandura's theory which emphasized the relationship between teacher efficacy and teaching experiences and it was widely used in literature.

The previous studies mainly investigated general teaching efficacy. However, the studies that investigate teaching efficacy for a specific topic such as trigonometry are rare in literature. In our study, we aimed to focus on trigonometry because it is one of the fundamental topics in high school curriculum with its relation to other topics such as complex numbers, derivatives. Also, it includes both algebra and geometry. Since teacher efficacy is related with student achievement and interest (Nelson, 2007); it may also be related with students' self-efficacy. Hence, studying the relationship between teaching efficacy and student self-efficacy in trigonometry would contribute to literature. However, administering a survey to study teachers' content-specific teacher efficacy was challenging because teachers would choose high self-efficacy options. For a larger study, the researchers needed to differentiate and group teachers according to their teacher efficacy levels. So, another approach was necessary to determine teachers' efficacy levels in order to study the relation between teaching efficacy and student achievement and self-efficacy. The focus of this paper is to share the process of development and the use of an interview protocol with a coding frame in order to determine teachers' levels of trigonometry teaching efficacy. Furthermore, the authors will also propose that the protocol and the coding frame may be used for identifying teaching efficacy levels for other topics.

TEACHER EFFICACY

Teacher efficacy has been defined as how competent a teacher feels in his or her ability to affect the performance of all students (Tschannen-Moran et al., 1998). In fact, a number of studies have concluded that teachers with high levels of efficacy differed significantly in their teaching practice from teachers with low levels of efficacy. Teachers with high efficacy tend to have greater levels of planning and organization as well as being more enthusiastic to teach (Allinder, 1994). Teachers spend more time teaching in subject areas where their sense of efficacy was higher (Riggs & Enochs, 1990); whereas teachers tend to avoid subjects when their efficacy was lower (Riggs, 1995). Specifically, high efficacy teachers demonstrated more effective teacher behaviors that led to higher student achievement (Ashton et. al., 1983). So, teacher efficacy is an important construct with its relation to teachers' teaching methods and it is necessary to understand and study it effectively.

Many teacher efficacy instruments have been developed in the last three decades. In 1984, Gibson and Dembo developed the Teacher Efficacy Scale (TES), and the scale included items related with general teaching efficacy- teachers' beliefs about their ability to affect students' achievement. The measure uses a 5-point Likert scale ranging from 1 (strongly disagree) to 5 (strongly agree). With the recognition that efficacy belief is context specific (Bandura,1982), researchers began to develop scales that focused on specific content areas, such as Science Teaching Efficacy Belief Instrument (STEBI, Riggs & Enoch, 1990). Riggs and Enoch changed the items on TES from more general teaching perspective to science specific items and they focused on the teachers' confidence level in teaching science. Similarly, Betz and Hackett (1981) developed Mathematics Self-Efficacy Scale (MSES). In addition, the authors were investigating teaching efficacy for a specific topic, trigonometry. Therefore, the authors adapted the Mathematics Self-Efficacy Scale (MSES) to trigonometry by changing items being specific to trigonometry teaching.

It is aimed to study trigonometry because of its importance in high school curriculum. It is a product of algebraic techniques, geometrical realities and trigonometric relationships. For most of the students in higher education, it is necessary to study trigonometry with its relation to other topics such as integral, derivative. Understanding of its basic concepts and application to the geometry and algebra is important for students to learn other mathematics topics effectively. However, studies have indicated that students had difficulty in understanding trigonometry (Akkoc, 2008) and they were not motivated to do it (Durmus, 2004). As teacher efficacy is one element that is related with student achievement and motivation, it should also be studied for such an important topic. However, there is limited research related to teachers' trigonometry teaching efficacy.

METHOD

As a part of a larger study we intended to differentiate 16 teachers according to their trigonometry teaching efficacy. For this purpose, Teacher Trigonometry Teaching Efficacy Scale (TTTES) which was adapted from MSES was used, but the results show that all 16 participants had high efficacy. With this result the researchers could not determine teachers' trigonometry teaching efficacy; in turn its relation with the student variables (achievement and efficacy) could not be studied. Therefore, the researchers developed an interview protocol with a coding frame to differentiate teachers as low and high trigonometry teaching efficacy. Among the sixteen teachers of the larger study, 13 of them voluntarily agreed to participate in individual interviews.

Sample

The sample of the larger study consisted of mathematics teachers (n=16) from various high schools in Istanbul, Turkey. In Turkish education system, the four-year secondary education is provided in two different types of schools, non-vocational and vocational. Non-vocational secondary education institutions consist of General High School and Anatolian High School. Anatolian High Schools admit students according to their high-stake test scores whereas General High Schools admit students with low scores or without any scores. These two types of institutions demonstrate some differences with respect to the number of students in classrooms, selection of teachers, conditions for admission, etc. All schools are required to apply the national curriculum.Therefore, teachers of different types of schools teach the same objectives as stated in the curriculum.

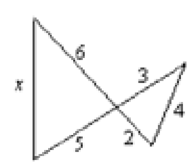
In this study, in order to control the effect of school, same type of schools (general high school) were chosen from similar districts of Istanbul with similar student profiles (such as low social economic status). Number of students who involved in the larger study was 571. Furthermore, among the 16 participating teachers, four of them were male and 12 of them were female. Ten of them graduated from mathematics department, three of them graduated from mathematics education, two graduated from physics and one from chemistry department. Teachers with non-educational degrees were certified to teach after a short period of pedagogy training. Teachers' years of experiences in teaching mathematics was high, as majority of them have been teaching more than ten years. Four of them had more than 15 years, three of them had 10-15 years, six of them had 5-10 years and three of them had less than 3 years of experience.

Teachers Trigonometry Teaching Efficacy Scale (TTTES)

This scale was adapted by the researchers in order to measure the efficacy level of teachers in teaching trigonometry. Mathematics Self-Efficacy Scale (MSES) which was developed by Betz and Hackett (1981) was used as the primary model. Betz and Hackett (1981) developed the MSES to measure college students' mathematics self-efficacy with greater specificity than previous instruments. This instrument has been used widely in research (e.g. Zimmerman, 2000). The MSES has 52 items which address some college mathematics topics such as algebra, calculus, economics, and statistics. Each item has a rating scale with 5 levels to show the confidence level of participants to solve the given problems for an item.

Since the purpose of the present study was to measure efficacy on trigonometry teaching, the teachers were asked about their confidence of teaching trigonometry problems. This adaptation of the items was supported by the suggestions of Bandura (1997) to assess self-efficacy. According to Bandura, the items should be aligned with the task being assessed and the domain which is analyzed.

For the adaptation, we needed to consider the Turkish curriculum for trigonometry. In order to determine the questions of the instrument, the objectives of national curriculum were used. In the curriculum, the trigonometry unit consists of six main sub topics: Trigonometric Functions, Graphs of Trigonometric Functions, Inverse Trigonometric Functions, Trigonometric Relations in a Triangle, Addition Formulas, and Trigonometric Equations. In other words, the mathematics problems for the instrument items were developed to cover all the sub-topics. There were 18 items in the instrument. The items were adopted from problems in several resources (MEB, 2005; Altuntas, 2007). Some examples from the TTTES are presented in Figure1.

- Graph the function $f(x) = 3\sin(2x)$.
- 

  Find the x value in the above figure.
- $\tan\left(\arctan\frac{1}{2} + \operatorname{arccot}\frac{2}{3}\right) = ?$

Figure 1. Items from TTTES

As in MSES of Betz and Hackett (1981), teachers were not asked to solve the problems but they only rated their perception of confidence to teach given problems. Each item has a rating scale with 5 levels ranging from 1 (I am not at all confident) to 5 (I am totally confident). The whole test can be seen in Sarac (2012). In our study, for this instrument, teachers generally marked the highest rank as their self-efficacy.

There were 16 participants and 12 of them got 90 and the remaining four got more than 80 out of 90. In the previous study (Betz & Hackett, 1981), when MSES was used with college students, the researchers did not report a similar situation. On the other hand, in our study teachers were asked about their confidence level to teach those problems rather than their confidence level to solve problems. It might also be possible that teachers thought to be tested about their profession and their knowledge of trigonometry. Hence, in our study TTTES was not effective in determining teachers' trigonometry teaching efficacy.

Individual Interviews

When we did not reach our purpose with TTTES, we searched for another way to get detailed information about teachers' efficacy. Instead of asking teachers about their confidence of teaching a given mathematics problem, we decided to conduct interviews with them about their teaching trigonometry experiences. Among the sixteen teachers, thirteen of them accepted to participate in an interview. Without asking direct questions about their confidence in teaching trigonometry, the interview was designed to talk with teachers about their experiences because ones' experience related to a task is one of the major indicators of self-efficacy (Bandura, 1997). Furthermore, as Philippou and Christou (1998) pointed out, "teachers' formative experiences in mathematics emerge as key players in the process of teaching since what they do in the classroom reflects their own thoughts and beliefs" (p. 191). Therefore, the researchers developed a semi-structured interview protocol with three questions:

1) Could you give information about your teaching methods for trigonometry?
2) Could you share your experiences related with teaching trigonometry in this year?
3) Could you share your experiences related with teaching trigonometry in previous years?

During the interview, teachers articulated their teaching trigonometry experiences and their teaching methods. They also shared the challenges they encountered and their methods of overcoming those.

Interviews lasted 40- 45 minutes. They were conducted and analyzed in Turkish but the researchers translated them into English for this paper. The interviews were audio-taped and transcribed. To be able to determine teachers' efficacy in teaching trigonometry as low and high, three indicators, *choices, effort thought patterns and emotions*, of teacher efficacy were utilized. These indicators were based on two sources: Bandura's self-efficacy theory and previous studies related with the features of high and low self-efficacy people (Pajares, 1996). The indicators and their use for the determination of teaching efficacy are explained in detail as follows:

Choices: People who have high self-efficacy engage in the activities more willingly and they tend to set higher goals to achieve. On the other hand, people with low self-efficacy tend to set incomplete goals and they feel incompetent (Bandura, 1997). In this study, some teachers reported that they really liked teaching trigonometry and wanted to teach trigonometry (they were coded as 1 for this indicator as willingly engage in the activity) while some stated teaching trigonometry to students was more difficult than other subjects (they were coded as 0 for this indicator as they were not willingly engage in the activity).

Teacher A: I really like to teach trigonometry. It is a wonderful subject, since I teach it by forming connections with analytic geometry. I teach it by helping students to connect it to the triangles and unit circle. (Engage willingly and set high goals. Choices indicator is present)

Teacher B: It is difficult to teach trigonometry because students do not have the necessary pre knowledge. It is so long that students get easily bored and once they get lost, they cannot continue. I am not able to help all of them. (Do not set high goals and do not wish to involve in teaching trigonometry. Choices indicator is not present.)

Effort: People with high self-efficacy put more effort on the job and they work harder. They show more self-regulated behaviours and use more effective strategies (Bong, 1997). They believe in themselves whatever the situation is. They attribute the success or failure to themselves while low self-efficacy people blame other factors such as crowded classrooms, intense curriculum (Rotter, 1966). Some teachers in the interview stated that they tried hard to teach trigonometry and strived to overcome the difficulty of students' lack of previous knowledge (they were coded as 1 for this indicator as they put more effort) while some stated that they could not do anything to teach the students who were not good at mathematics and unmotivated (they were coded as 0 for this indicator as they did not try to get over difficulties).

Teacher C: I strived hard to teach trigonometry since it is a very important subject. Firstly, I tried to teach the necessary prerequisite knowledge. Also, I gave homework to students to help them learn all the parts of it well. I talked individually with the students who are not so good at the subject and recommended them some extra work to close the gap with other students. (Putting more effort and striving hard to teach. Effort indicator is present.)

Teacher D: It is a difficult subject. Students' levels were low. It was necessary to complete their missing previous knowledge and to teach it at low level. However, it was not possible to teach all the previous knowledge because when I went back, the subject was messed up and I could not build it up. (Not striving hard for the students who do not have the previous knowledge. Effort indicator is not present.)

Thought Patterns and Emotions: People with high self-efficacy are more comfortable and they are less anxious (Pintrich & De Groot, 1990). While they talk about their experiences they use less negative words (Bandura, 1997) whereas people who have low self-efficacy concentrate on difficulties and use more negative words about their experiences. In this study, some teachers talked more on difficulties of teaching and blame students (they were coded as 0 for this indicator) while some concentrated on their efforts in teaching trigonometry (they were coded as 1 for this indicator).

Teacher E: This year it was good for me to observe that I could teach students some necessary knowledge in trigonometry. Also, I feel successful in that I helped them to study in 80 per cent. If I can help them to change their negative emotions about trigonometry, the remaining part becomes nice to teach. (Not concentrating on difficulties and not using negative words. She says that it is good to observe students learn. Thought Patterns and Emotions indicator is present.)

Teacher F: Teaching those [low achieving] students was very frustrated for me since I could not get any sign of learning from them. (Concentrating on difficulties, using negative words. Thought Patterns and Emotions indicator is not present.)

Teachers' answers to interview questions were evaluated by giving code for each indicator (0 or 1). The teachers who showed the features of the indicator were coded as 1 and the ones who did not show the features were coded as 0. With the summation of the codes for three indicators, the final score for each teacher was calculated. Teachers with 2 or 3 were categorized as having high trigonometry teaching efficacy and the ones with 0 or 1 were categorized as having low trigonometry teaching efficacy. The categorization of the teachers according to their interviews was checked by a mathematics education expert.

Among the 13 teachers, six teachers got 0 or 1 and were categorized as having low trigonometry teaching efficacy whereas 7 teachers get 2 or 3 and they were categorized as having high trigonometry teaching efficacy. Table 1 presents the scores according to teachers. Pseudonyms were used for each teacher.

Teacher	Choices	Effort	Thought Patterns and Emotions	Total
Oyku	1	1	1	3
Dilan	1	1	1	3
Cigdem	1	1	0	2
Hulya	0	1	1	2
Fahri	1	1	0	2
Kerem	1	0	1	2
Ozge	0	1	1	2
Hasan	1	0	0	1
Gaye	0	1	0	1
Ayca	0	0	0	0
Nermin	0	0	0	0
Hale	0	0	0	0
Melisa	0	0	0	0

Table1. Trigonometry Teaching Scores for the Indicators

Furthermore, as it was stated before, the differentiation of the teachers was used in a larger study (Sarac & Aslan-Tutak, 2012) to investigate the relation between teachers' trigonometry teaching efficacy and their students' self-efficacy. The students were grouped according to their teachers' efficacy level and the groups' self-efficacy levels were compared. Significant difference was found between two groups of students in terms of their trigonometry self-efficacy.

DISCUSSION

In this paper, we suggested a method of determining teachers' trigonometry teaching efficacy. Firstly, trigonometry teaching efficacy was attempted to be studied using TTTES in which teachers were asked about their confidence level to teach trigonometry for a given problem. However, we could not differentiate the teachers since teachers ranked the highest score for their trigonometry teaching efficacy for almost all questions. So, we decided to get further information about their teaching efficacy by developing an interview protocol.

In these interviews, teachers were not asked directly about their confidence level to teach trigonometry, instead they were asked to share their teaching methods and teaching process, assuming that teaching experiences were the major indicators for one's teaching efficacy. From their answers, we were able to determine teachers' trigonometry teaching efficacy. For instance, some teachers concentrated on the difficulties expressing negative feelings such as frustration of not being able to teach all students while some were on the more positive side expressing the enjoyment to teach all students. Therefore, teacher efficacy can be studied using the mentioned indicators- *choices, effort, and thought patterns and emotions*. So, this paper proposes a method for studying teacher efficacy for specific topics.

Teacher efficacy and student attitudes toward a subject found to be related (Nelson, 2007). By means of the interview protocol we could differentiate teachers according to their teaching efficacy and compare the students' of high and low efficacy teachers. In the larger study, teacher efficacy and student self-efficacy found to be related. The results of the larger study also support the effectiveness of the protocol to determine teachers' levels of efficacy.

Furthermore, the interview protocol and the coding frame may also be applied for other content-specific topics. These types of studies will also provide information about the effectiveness of the method. On the other hand, by using the findings of these interviews, TTTES can be improved by adding items targeting the indicators. In this study, we used binary scale for the indicators since our purpose was categorizing teachers in two groups. However, differentiated scales can be used for the efficacy levels for other studies. These efforts might yield into developing a new survey to measure teacher efficacy.

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UNCERTAINTY ORIENTATION, PREFERENCE FOR SOLVING TASK WITH MULTIPLE SOLUTIONS AND MODELLING

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In this study 235 ninth graders from ten German middle track classes were asked about their preference for solving tasks with multiple solutions, uncertainty orientation and treating tasks with multiple solutions in their everyday mathematical classes. Preference for solving tasks with multiple solutions and uncertainty orientation were assessed before and after a five-lesson teaching unit promoting modelling competency with either one solution or multiple ones as well as in the control group. The findings show that (1) preference for solving tasks with multiple solutions is connected with students' uncertainty orientation and treating these tasks in the classroom, (2) after the teaching unit, a group where multiple solutions were treated indicated the stronger preference for solving tasks with multiple solutions.

Key words: multiple solutions, affect, modelling, word problems, teaching methods

INTRODUCTION

The principles of high quality teaching mathematics include the use of cognitively demanding tasks, development of multiple solutions, reflecting on, comparing and discussing different solution methods (Silver, Ghousseini, Gosen, Charalambous, & Font Strawhun, 2005). For the recent 10 years, there have been first evidences that the comparison of different solution methods can improve students' mathematical competency, if students have a sufficient prior knowledge in the target domain (Rittle-Johnson, Star, & Durkin, 2009). This finding is in line with the outcomes of studies that compared teaching methods in different countries. Teachers in high-performing countries such as Japan demand from students to find more than only one solution and to discuss solution methods in the classroom. Teachers in the other countries often think that students will be lost in the variety of solutions and therefore do not practice them (Leikin & Levav-Waynberg, 2007).

There is still a lack of studies, which investigate emotions, attitudes and beliefs regarding to the use of multiple solutions. As students' improvement in the affect domain is an important goal of mathematics education, we focus in this study on students' uncertainty orientation, preference for solving problems with multiple solutions, and treatment of (or dealing with) these problems in the classroom. Further, we have investigated the possibilities to influence students' uncertainty orientation and their preference for solving problems with multiple solutions positively. A five-lesson teaching unit promoting multiple solutions while modelling was conducted and evaluated using a 3x1 experimental-control-group design.

This study was carried out in the framework of MultiMa-project (*Multiple solutions for mathematics teaching oriented towards students' self-regulation*) that has been funded by the German Research Foundation since 2011 (SCHU 2629/1-1). MultiMa aims to investigate students' dealing with multiple solutions while modelling, students' affect as well as the development of mathematical competency in learning environments oriented towards self-regulation (Schukajlow & Krug, in press).

THEORETICAL BACKGROUND AND RESEARCH QUESTIONS

Multiple solutions and modelling problems

The analysis of solving problems showed that there are three types of multiple solutions (see a similar approach by Tsamir, Tirosh, Tabach, & Levenson, 2010). The first type of multiple solutions can be conducted due to the variation in *mathematical* solution methods. The second type of multiple solutions can be developed if students solve problems with missing data. For solving these problems, they have to take assumptions about the missing data and thus, get different outcomes. The third type of multiple solutions includes the variation in mathematical solution methods as well as in different outcomes.

The missing data is one of typical features of modelling problems. The core of modelling activities are demanding transfer processes between reality and mathematics (Blum, Galbraith, Henn, & Niss, 2007). We illustrate different types of multiple solutions using the task "Parachuting". For calculation of hypotenuses while solving this problem, students can use as mathematical procedure either Pythagoras' Theorem or scale drawing. Further, in order to solve the problem, they have to take assumptions about some data such as wind power and can get different results using the same mathematical method.

Parachuting

When "parachuting", a plane takes jumpers to the altitude of about 4000 metres. From there they jump off the plane. Before a jumper opens his parachute, he free falls about 3000 metres. At an altitude of about 1000 metres the parachute opens and the sportsman glides to the landing place. While falling, the jumper is carried off target by the wind. Deviations at different stages are shown in the table below.

Wind speed	Side deviation per thousand metres during free fall	Side deviation per thousand metres while gliding
Light	60 metres	540 metres
Middle	160 metres	1440 metres
Strong	340 metres	3060 metres

What distance does the parachutist cover during the entire jump?

Affect and modelling

Stability (state vs. trait) and functionality (truths, feelings and preferences) are important characteristics of affective domain (Hannula, 2012). In this study, we focus on the personal trait "uncertainty orientation" and students' preference for solving tasks with multiple solutions.

The uncertainty orientation (Sorrentino & Roney, 1999) describes a person's typical ways of dealing with complexity, uncertainty, and abundant information (Hänze & Berger, 2007). Uncertainty-oriented persons are interested in complex situations and use these situations to gain the new knowledge. Persons with strong certainty orientation look for situations, which are already familiar. Huber, Sorrentino, Davidson, Eppler, and Roth (1992) found that students with uncertainty orientation learn more, if cooperative teaching methods were applied in the classroom. However, this result could not be confirmed in further studies (Hänze & Berger, 2007). One possible explanation of this inconsistence is that the level of structuring the learning material such as the type of tasks, influences the learning processes. The treatment of problems, in which all data are given, meets the needs of certainty-oriented students. The treatment of problems with missing data, whereas the development of multiple solutions is demanded, meets the needs of uncertainty-oriented ones. As the uncertainty orientation is a personal trait regarded to the dealing with uncertainty in different situations, we do not expect that solving the tasks with multiple solutions in the classroom can change this trait.

Preferences are closely connected with motivational constructs (Hannula, 2012). They can regard to the global objects such as learning, or to the specific situations, such as dealing with different kinds of tasks. The assessment of task-specific measures allows to collect information about students' affective dimensions and can help to answer questions that are specific for mathematics educations. Task-specific measures are more sensitive than traditional instruments and can be used for evaluation of short-term interventional studies, where significant changes in motivational traits using traditional scales could not be expected. In the study by Schukajlow et al. (2012), there was found that student-centred teaching method for fostering modelling competency improved students' *task-specific* enjoyment, interest and self-efficacy expectations. The nearby average relationship of $r=0.27$ between treating tasks with multiple solutions and students' preference for solving these tasks (Krug & Schukajlow, 2012) indicates that treating tasks with multiple solutions can influence students' preference in a positive way. One goal of this study is to replicate this result. Another goal is to investigate, whether treating multiple solutions while solving modelling problems influences students' preference for solving problems with multiple solutions positively. "Treating tasks with multiple solutions" means in this context that students work on problems, which demanded the development of more than one solution.

Research questions

The research questions of the study are:

1. How strong is a relationship between students' uncertainty orientation, preference for solving tasks with multiple solutions and dealing with this kind of tasks in the everyday classroom?
2. Does treating multiple solutions while solving modelling problems influence students' uncertainty orientation?
3. What is the impact of treating multiple solutions while solving modelling problems on students' preference for solving tasks with multiple solutions?

METHOD

Design and sample

235 German ninth graders (46% females; mean age=15.4 years, SD=0.62) were asked about their preference for solving tasks with multiple solutions and their uncertainty orientation before and after five-lesson teaching unit. Further, before the teaching unit the students were asked, how frequently they solved tasks with multiple solutions in their regular classes. In some German federal states, students, who completed the fourth grade, are assigned to the low, middle or high track classes depended on their performance level. The middle track classes were chosen for this study because in these classes the students of all three performance levels are to be found. Ten middle track classes (Realschule) from ten comprehensive schools (German Gesamtschule) and 8 teachers with at least two years' experience of teaching mathematics participated in the study. We assume that experienced teachers could better implement the instructions of the study.

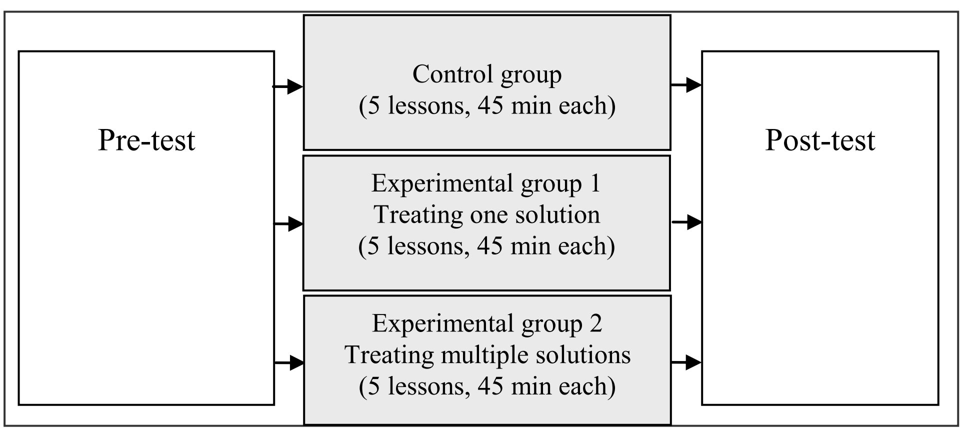

Figure 1: Overview of the study design

The sample consists of one control group and two experimental groups. The experimental groups needed a strong support by the research team, so six classes from three schools near the university were assigned to experimental groups. The remaining four classes were assigned to the control group. During the five-lesson period 105 students of the control group were not allowed to solve modelling problems – solving of which may lead to comparison of different solutions – as well as to the problems to the “Pythagoras’ theorem”. To control, whether teachers really used only these types of problems, all the tasks solved by the students of control group were collected and analysed. The task analysis showed that no problems that required multiple solutions were treated.

Three schools with two middle track classes each were assigned to the experimental groups. Each of six classes was divided into two parts with the same number of students in such a way that the average achievements in both parts did not differ and there was the approximately same ratio of males and females in each part. In one part of each class one solution of modelling problems (experimental group 1: “one solution”) and in the other part multiple solution of modelling problems (experimental group 2: “multiple solutions”) were treated. During the teaching unit, students of both experimental groups were taught using modelling problems about “Pythagoras’ theorem”. The topic “Pythagoras’ theorem” had already been treated using mathematical tasks without connection to reality before the MultiMa teaching unit, in order to foster the use of this powerful mathematical procedure and to prevent applying different solution methods.

To implement the treating of modelling tasks with and without multiple solutions, two teaching scripts were developed. Four teachers, who had to give lessons in experimental groups, received these scripts with all tasks to be treated and a detailed plan of the teaching unit. Further, they were instructed about the specific ways to promote multiple solutions vs. one solution. Each teacher taught the same number of student groups in the experimental group “one solution”, as well as in the experimental group “multiple solutions”, so the influence of a teacher personality on students’ learning did not differ between both groups. In each lesson that was provided in the experimental groups, one member of the research team was present to videotape and to observe the implementation of the instructions.

In both experimental groups the same methodical order was committed. Students solve a modelling task according to a special kind of group work (alone, together and alone again) (Schukajlow et al., 2012). A solution (or different solutions) of the first modelling task is presented in the first lesson by the teacher and in the following ones by the students. The teacher summarizes and reflects on the key points of each group. In the group “multiple solutions” the teacher emphasises the development of different outcomes by estimating the missing data.

In order to stimulate the development of multiple solutions in the group “multiple solutions” and to prevent the development of more than one solution in the other experimental group, two similar versions of each treated task were developed. In the group “one solution” students solved among other tasks the task “Parachuting”, where the data, needed to solve the task, were given. These data were the wind velocity and the altitude, in which the parachute opens. The question posed in the task was: “What distance does the parachutist cover during the whole fall, if the wind speed is middle?" The similar task in the group "multiple solutions" demanded the development of two solutions. The question posed in the task was: "What distance does the parachutist cover during the entire jump? Find *two* possible solutions" (see sample tasks by Schukajlow & Krug, 2012b).

Measures

The main difference between experimental groups was the demand to develop one solution or multiple solutions. We proved this key using students' questionnaires. After every lesson the students were asked about the number of solutions they developed for each modelling problem in this lesson. For example: "While solving the problem "Parachuting" I developed today (0: no solution; 1: one solution; 2: two or even more solutions)".

Students' self-perceptions were measured using a 5-point Likert scale (1=not at all true, 5=completely true) before and after a five-lesson teaching unit. A sample item was for uncertainty orientation (5 items) "I like unexpected surprises". The scale uncertainty orientation was adapted from the studies by Dalbert (1996) and Hänze & Berger (2007). Students' preference for tasks with multiple solutions (6 items) was developed using the theoretical background to multiple solutions. This scale consists of questions about three issues: (1) multiple mathematical solution methods ("While working on mathematical problems, I like different calculations leading to success."), (2) multiple outcomes ("While working on mathematical problems, I like to get different results") and (3) multiple solutions in general (e.g. "While working on mathematical problems, I like to use different solution methods.") Treating multiple solutions in the everyday mathematics classroom was measured using 6 items that are similar to the scale "preference for tasks with multiple solutions" (e.g. "In mathematics, we often work on problems that offer different solution methods"). Using of 6 items for measurement of each scale is one limitation of this study. The reliability values (Cronbach's Alpha) for uncertainty orientation were 0.86 and 0.87, those for the preference for tasks with multiple solutions 0.77 and 0.85 in the pre- and the post-test respectively, and the reliability value for treating multiple solutions in mathematics classroom was 0.81 in the pre-test. More information about the validity of the measures should be collected in the future studies. All measures were a part of the longer questionnaires. Pre- and post-tests took 30 minutes each.

RESULTS

Preliminary analysis

First, we compared the number of solutions developed by both groups, "multiple solutions" and "one solution" (Schukajlow & Krug, 2012a). We analysed students' answers using the t-test. The analysis shows that there are significant differences between the numbers of solutions that were developed in the respective groups ($T(138)=6.7$; $p<0.001$; effect size Cohen's $d=1.16$). Whereas the majority of the students in the group "multiple solutions" developed two and more solutions (mean=1.55, standard deviation SD=0.39), students in the group "one solution" reported on the development of one solution only (mean=1.14, SD=0.33). The analysis of the number of students' solutions in the experimental groups and that of the tasks treated in the control group indicates that it was possible to realise the instruction conditions as it was intended in the study.

Uncertainty orientation, preference for solving tasks with multiple solutions and treating multiple solutions in the everyday mathematics classes

To answer the first research question, we analysed the relationship between students' perceptions in the pre-test. There are low to middle statistically significant correlations between respective measures (cf. Tab. 1). Uncertainty-oriented students more frequently report on treating problems with multiple solutions in the classroom and on their preference to solve these problems than students without uncertainty orientation. The middle correlation of 0.46 was measured between students' preference for solving tasks with multiple solutions and the treatment of the tasks with multiple solutions in the classroom. As a correlation between treating multiple solutions in the classroom and uncertainty orientation is weak, it is possible that the treatment of such problems foster students' uncertainty orientation, but not significantly.

	UncO	**PrMS**	**MSC**
UncO	1	0.26*	0.17*
PrMS		1	0.46*
MSC			1

* The correlations are at least significant at the 5% level

Table 1: Pearson correlations between uncertainty orientation (UncO), preference for problems with multiple solutions (PrMS) and treating multiple solutions in the classroom (MSC)

The second research question pertains to the effect of treating modelling problems with multiple solutions on students' uncertainty orientation. Is it possible to change students' uncertainty orientation after the five-lesson teaching period and are there any differences between control and experimental groups? In order to test, whether the factor "type of intervention" influences students' uncertainty orientation in post-test, ANCOVA (covariate: uncertainty orientation in pre-test) was conducted. The ANCOVA revealed a significant effect of the pre-test score ($F(1, 213)=84.60$, $p<0.05$), but no effect of "type of intervention" on the uncertainty orientation ($F(2, 213)=1.53$, $p=0.22$). This result shows that treating multiple solutions while solving modelling problems does not influence students' uncertainty orientation.

The effect of the "type of intervention" on students' preference for problems with multiple solutions (research question 3) was also computed using ANCOVA (covariate: PrMS in pre-test). The ANCOVA showed effects of PrMS in pre-test on the PrMS in post-test ($F(1, 215)=56.43$, $p<0.05$) and a significant impact of the factor "type of intervention" on students' preference for problems with multiple solutions in post-test ($F(1, 215)=6.87$, $p<0.05$). Students in the group "multiple solutions" report in post-test with respect to pre-test on stronger preference for problems with multiple solutions compared with the students in the group "one solution" or in the control group. No differences in this scale were found between the group "one solution" and the control group.

		Control group	**One solution**	**Multiple solutions**
		M(SD)	M(SD)	M(SD)
UncO	pre	3.66(0.89)	3.61(0.94)	3.50(0.92)
	post	3.32(0.99)	3.49(1.03)	3.41(1.03)
PrMS	pre	3.22(0.85)	3.29(0.90)	3.35(0.73)
	post	2.98(0.92)	3.03(1.02)	3.56(0.87)
MSC	pre	2.96(0.78)	3.05(0.78)	3.04(0.77)

Table 2: Students' uncertainty orientation (UncO), preference for problems with multiple solutions (PrMS) and treating multiple solutions in the classroom (MSC) in the control and experimental groups

DISCUSSION

In the study reported here, students' uncertainty orientation, preference for solving tasks with multiple solutions and dealing with this kind of tasks in the everyday classroom were assessed. Between uncertainty orientation and students' preference for solving tasks with multiple solutions, a small statistically significant relationship was found. This relationship shows that students who were looking for unfamiliar situations like to solve the tasks that demand taking assumptions, allow choosing different mathematical methods and having different mathematical results. So, the results of the study support the assumption by Hänze & Berger (2007) that the level of structuring the material can influence students' learning. However, we did not gain any results, whether uncertainty-oriented students prefer solving well-defined tasks more than certainty-oriented students. It is possible that uncertainty-oriented students have more positive perceptions of solving any kind of mathematic tasks than certainty-oriented students. One important future research field is therefore to compare students' preference for tasks with multiple solutions with students' preference for clear-structured, well-defined tasks that have one right solution only, taking in account students' personal traits such as uncertainty orientation. Another important question concerns the connection between uncertainty orientation and the construct of identity. Uncertainty orientation may be a part of identity which influences our actions in learning situations.

The average correlation between the dealing with tasks required multiple solutions and preference for solving these tasks means that students who frequently report on solving the tasks with multiple solutions like to solve this kind of tasks more than students that solve them rarely. This finding confirms our previous results (Krug & Schukajlow, 2012) and indicates that treatment of tasks with multiple solutions can improve students' preference for solving this type of problems.

In order to prove the direction of the assumed connection between both measures, an experimental study was conducted. In the experimental groups modelling problems with and without multiple solutions were treated five lessons long. The analysis of the data showed that students of the group, where multiple solutions were treated, liked the tasks with multiple solutions more than the students who solved modelling tasks with one solution only, or than the students of the control group with respect to their pre-test. As no differences between the group "one solution" and the control group were found, the changes in the students' preference could not be attributed to the special kind of student-centred teaching method applied in the experimental groups. Also, we showed that a students' task-specific motivational construct regarding to multiple solutions can be changed positively. The positive changes in students' preference for tasks with multiple solutions can promote students' involvement in the content activities and, as a result, improve their performance. Investigation of relationships between students' preferences, emotions, beliefs and performance concerning different type of problems is an important future research field. Finally, we showed that the personal trait uncertainty orientation did not differ between the experimental groups and the control group. For changing this stable trait, a special long-term training program regarding different domains is needed.

As in this study we used new scales with a limited number of items, and as there are only few studies in the domain of mathematics that investigated students' preferences regarding to multiple solutions or their uncertainty orientation, a replication of our results in future research is essential.

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STUDENTS' MOTIVATION AND TEACHERS' PRACTICES IN THE MATHEMATICS CLASSROOM

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This paper presents five different families of social cognitive motivational constructs: efficacy, control, interest, values and goals. Two motivation theories will be developed further, namely achievement goal theory and self-determination theory. Research on the relationship between teachers' practice in the mathematics classroom and students' motivation, in terms of intrinsic motivation and goal orientation, will be reviewed. It seems like some aspects of mathematics teachers' instructional practices have a positive influence on both students' intrinsic motivation and goal orientation.

Keywords: Motivation, goal orientation, intrinsic motivation, teachers' practice

INTRODUCTION

The importance of motivation in mathematics education has been well documented (Hannula, 2006b; Pantziara & Philippou, 2007). It is believed that motivation is one of the energizing forces to learning and adaptive behaviour in school settings (Pintrich, 2003; Zhu & Leung, 2010). As Hannula (2006b) points out "To understand students' behaviour we need to know their motives" (p. 165). Theorists believe that motivation has a determinative role in students' success and failure in school (Cury, Elliot, Fonseca, & Moller, 2006; Pintrich, 2003).

Motivation is characterized by its complex nature due to its different definitions, different theoretical perspectives and the various ways that has been measured (Hulleman, Schweigert, & Harackiewicz, 2008; Pintrich, 2003). Motivation cannot be observed directly, but it can be manifested in cognition, emotion (affect) and/or behaviour. In this paper motivation is defined as "the preference to do certain things and to avoid doing some others (Hannula, 2006b). A vast number of theoretical perspectives have been developed in the realm of Educational psychology in order to better describe motivation. These theoretical perspectives can be seen as complementary to one another as each of them accounts for different aspects of motivation. Pintrich (2003) identifies five different social cognitive constructs that have dominated recent research on motivation in school settings. In this paper we will present these five basic families of motivational constructs. Two motivation theories will be developed further, namely achievement goal theory and self-determination theory, which integrates multiple social cognitive constructs. We will discuss research studies on how different aspects of teachers' instructional practices influence students' motivation, in terms of intrinsic motivation and goal orientation in mathematics.

MOTIVATIONAL CONSTRUCTS

One family of motivational constructs refers to self-efficacy beliefs and competence perceptions. Many different constructs are related with this theoretical family like self-efficacy, expectancy, perceptions of competence, self-worth, and self-determination (Bandura, 1994; Bouffard & Couture, 2003). Even though these constructs are defined differently they all focus on the same idea. Students who believe they are capable, and that they will do well, are more expected to be motivated, to have persistence and a more adaptive behaviour than students who believe that they cannot do well and therefore they will not succeed (Kaplan & Midgey, 1997; Pajares & Graham, 1999; Pantziara & Philippou, 2007). Research (Pintrich, 1999, 2003) also revealed that these self-assured students are more engaged in learning and thinking than students who mistrust their abilities.

A second family of motivational constructs are adaptive attributions and control beliefs (Pintrich, 2003). Basically these theoretical perspectives argue that students who believe that they have control on their learning and behaviour are expected to perform well and have higher achievement than those students who do not believe that they have the control of their learning. Similar to these notions are also entity and incremental theory of ability. Entity theory describes ability as steady and unalterable, whereas incremental theory characterizes ability as open to change (Cury, et al., 2006). Entity theory is posited to predict a poor set of outcomes, like low levels or persistence, performance and interest while incremental theory is posited to predict beneficial outcomes like high levels of persistence, performance and interest (Dweck & Leggett, 1998). In this family of research also belongs self-determination theory, which together with the fifth family of motivational constructs, which is achievement goal theory, will be developed further.

Self-determination theory is a model that integrates psychological needs and social-cognitive constructs (Ryan & Deci, 2000a). This model is built on the assumption that human beings have three basic psychological needs: the needs for competence, relatedness, and autonomy. Competence refers to the feeling of mastery and effectiveness in interactions with the environment. Relatedness reflects the feeling of being together with other persons in a secure community. Autonomy refers to being in control or being the perceived origin of one's own behaviour. When individuals are autonomous they experience themselves as volitional initiators of their own actions (Ryan & Deci, 2002). Cobb, Gravemeijer, Yackel, McClain and Whitenack (1997) use the concept of intellectual autonomy as a characteristic of a student's way of participating in the practices of a classroom community. They speak of students' awareness and willingness to draw on their own intellectual capabilities when making mathematical decisions and judgments as they participate in mathematics activities. The concept of need is useful because it allows the specification of the social-contextual conditions that will facilitate motivation. According to self-determination theory, students' motivation will be maximized within social contexts that provide them with the opportunity to satisfy their basic needs for competence, autonomy and relatedness (Ryan & Deci, 2000a, 2002). Self-determination theory makes a distinction between intrinsic motivation and extrinsic motivation. Intrinsic motivation refers to doing of an activity for its own sake and enjoyment. In contrast, extrinsic motivation reflects a behaviour that is undertaken in order to attain some separable outcome (Ryan & Deci, 2000a). Research from self-determination theory has demonstrated the importance of perceptions of autonomy and competence in adaptive behaviour, and the theory highlights the importance of providing some autonomy, choice, and control for students, in order to facilitate students' intrinsic motivation (Ryan & Deci, 2002). Research has shown that there is a positive correlation between students' feeling of autonomy and more positive feelings, better learning and performance in school (Deci & Ryan, 2000).

A third family of constructs related to motivation is interest and intrinsic motivation. Interest can be defined as a psychological state in which someone is engaged, and entirely absorbed by an activity (Hulleman, et al., 2008). A distinction in this area is made between personal and situational interest (Pintrich, 2003). Personal interest is a more stable and lasting disposition of an individual who is attracted, enjoys or likes to be involved in an activity for its own sake (Hulleman, et al., 2008; Pintrich, 2003). Situational interest refers to the psychological state of interest that develops through the interactions with a task's characteristics, like pictures, humor etc. (Hulleman, et al., 2008). Research on personal and situational interest has revealed that high levels of both types of interest are related with more cognitive engagement, learning and achievement (e.g. Pintrich & Schunk, 2002). Interest is one of the central features of intrinsic motivation in self-determination theory (Deci & Ryan, 1985). Students who are intrinsically motivated work with an activity for the enjoyment or challenge entailed. They experience high levels of interest. Research found that intrinsic motivation is positively related to a number of desired cognitive and motivational outcomes such as students' academic performance and self-esteem (Gottfried, 1985; Ryan & Deci, 2000b).

Another family of constructs refers to students' thoughts about the importance of a task. This family of constructs is reflected in expectancy-value theory. Theorist argue that individuals' choice, persistence and performance can be explained by their beliefs about how well they can do a task and how they value this task (Wigfield & Eccles, 2000). Expectancy beliefs refer to children's beliefs about how well they can do on forthcoming tasks in the immediate or longer future (Wigfield & Eccles, 2002). Task value beliefs are defined by four components: intrinsic interest; utility, importance and cost (Pintrich, 2003). Research has found that task value beliefs predict choice behaviour, such as the intentions to enrol in future math courses. Expectancy beliefs like efficacy or competence perceptions found to predict achievement as students are enrolled in the course (Pintrich, 2003; Wigfield & Eccles, 2000).

The fifth family of constructs is goals and goal orientation, and within this family there have been two main directions of research on student motivation. One direction focuses on goal content and the students' goal structures. These approaches assume that the students' goal structures are complex, and that the students tend to pursue multiple goals in the classroom. The goals are related to one another, and pursuing one goal might be necessary to attain another goal or different goals may be seen as contradictory (Boekaerts, 1999). Research shows that students' pursuit of social goals such as making friends and being responsible are related to effort and achievement (Pintrich, 2003). The second direction has focused on the nature of achievement goals or goal orientations. This has been one of the most active areas of motivation research on students' motivation. Achievement goal theory is concerned with the purposes of students' behaviour. Students with mastery goal are oriented toward learning and understanding as an end in itself. In contrast, a performance goal orientation refers to seeking to demonstrate that one has ability by outperforming others (Elliot, 2005). Recently, there has been a theoretical distinction between performance-approach goals, where the student is focused on outperforming others, and performance-avoidance goals, where the student is concerned with avoiding the demonstration of low ability or appearing incompetent in relation to others (Cury, et al., 2006). These goals have been related consistently to different patterns of achievement-related affect, cognition and behaviour. Mastery goals have been related to adaptive perceptions including feelings of efficacy, achievement, and interest (Anderman, Patrick, Hruda, & Linnenbrink, 2002; Cury, et al., 2006; Elliot & Church, 1997). Research on performance goals is less consistent, but students' performance orientation has been associated with maladaptive achievement beliefs and behaviour like low achievement, fear of failure and superficial cognitive commitment, i.e. copying, repeating and memorizing (Cury, et al., 2006).

HOW DOES TEACHERS' INSTRUCTIONAL PRACTICES INFLUENCE STUDENTS' MOTIVATION?

Most social-cognitive models of motivation assume that students' motivation is influenced by classrooms interactions, activities, practices and culture (Pintrich, 2003). Therefor the teacher's instructional practice has a crucial role in facilitating students' motivation. Within mathematics education there has been some research on the relation between different aspects of teachers' instructional practices and students' motivation in mathematics. In this paper we focus on two motivation frameworks, namely self-determination theory and achievement goal theory, and we will discuss research studies that have made use of these theories in order to understand or interpret how students' motivation in mathematics is influenced by teachers' instructional practices.

Students' needs and goals

Hannula (2006a, 2006b) claims that there is little room to meet students' need for competence, autonomy and relatedness in teacher-centred mathematics classrooms that tends to focus on the learning of routine procedural skills and individual work. More student-centred or reform-classrooms where the emphasis is on meaning-making and collaborative work would give students more opportunities to fulfil their needs. Hannula (2004) suggested the need for more research to examine how students' motivation is influenced by the mathematics classroom practice.

Wæge (2008, 2010) examined students' (age 16) motivation in mathematics in terms of needs and goals. The results showed a close relation between the students' feeling of competence, in terms of relational understanding (Skemp, 1976) and learning, and their enjoyment in engaging in mathematical activities. Students who developed a feeling of relational understanding experienced higher levels of enjoyment than students with instrumental understanding. The study also showed that there were particularly three aspects of the teaching approach that conduced toward students' feelings of competence and sense of autonomy during action: 1) instructional activities, such as projects, problem solving activities, and real life problems, 2) students' collaboration with each other, and 3) encouragement and acceptance of students' own strategies for solving problems. The three factors were closely related to each other. The results indicated that the instructional activities positively influenced the students' enjoyment in mathematics, because they fulfilled their needs for competence and autonomy. Further, the study showed that the students' need for competence and autonomy were realised into more specific goals. The students realised their need for competence into a general goal of mastery in mathematics and a more specific goal of developing relational understanding. The students were concerned about knowing what to do and why. The need for autonomy was translated into the more specific goal of using own thoughts and ideas and generating own solutions.

These findings are consistent with previous research on students' motivation in mathematics (Cobb, Wood, Yackel, & Perlwitz, 1992). Boaler (2004) describes how teachers at a high school employed a reform-oriented mathematics teaching approach in order to develop good relationships between the students and to reduce social and academic status differences in the mathematics classrooms. Teachers encouraged students to take responsibility for each others learning and they created multidimensional classes where many dimensions of mathematical work was valued, such as generating multiple solution methods, asking good questions, justifying and explaining their answers. The results show that students learned to treat each other in more respectful ways and they enjoyed mathematics more than students taught in more traditional approaches.

A study by Stipek and her colleagues (Stipek, Salmon, Givvin, & Kazemi, 1998) showed that three aspects of mathematics teachers' practices were positively associated with students' positive feelings as well as their learning: 1) a positive affective climate, which means that the teacher treated students with respect, listened to their ideas and valued all student contributions, 2) teachers' emphasis on learning and understanding and the encouragement of autonomy, and 3) teachers' providing substantive, constructive feedback. The more teachers focused on these aspects, the more students experienced positive emotions and enjoyed learning.

The findings presented here are consistent with self-determination theory and research on students' motivation, indicating that students' feeling of relatedness, autonomy and competence facilitate enjoyment and intrinsic motivation in activities. According to Ryan and Deci (2002), intrinsic motivation represents a prototype of self-determined activity. They suggest that there is a strong relation between intrinsic motivation and the need for autonomy and competence. Mathematics classrooms that support students' needs for autonomy and competence will engender their intrinsic motivation in mathematics. Contextual events that students experience as thwarting satisfactions of these needs will undermine their intrinsic motivation.

Students' learning orientation

One of the strengths of goal orientation theory in understanding students' motivation is that it considers how the role of the teacher and the instructional context might influence students' goal orientations. Thus a major tenet of achievement goal theory is that students' adoption of goals is partly influenced by the goal structures promoted by the classroom environments (Anderman, et al., 2002). Goal orientation theorists often focus on six categories when studying classroom motivational environment. The categories are often described by the acronym TARGET, which refers to task, authority, recognition, grouping, evaluation and time. Task includes activities, such as problem solving or routine algorithm tasks, and open or closed questions or tasks; Authority refers to the level of autonomy in the classroom; Recognition refers to whether the focus is on the learning process or the final outcome of students' performance; Grouping refers to whether the teacher divide the classroom into groups according to their performance or not; Evaluation refers to teachers' assessment of students' learning and whether they focus on grades and test scores or feedback as a means for improving students' learning; Time refers to the flexibility of the schedule.

Studies (Anderman, et al., 2002; Turner, et al., 2002) showed that aspects of mathematics classroom environments in which students adopted mastery goals differed from classrooms in which students adopted performance goals, in terms of avoidance goals. In mastery oriented classroom teachers focused on the process of learning and understanding (recognition), challenging students and learning from mistakes (evaluation). The instruction was adjusted to the developmental level and interest of students (task), and students were encouraged to generate own strategies and solutions (authority) and to collaborate with each other (grouping). Teachers also valued the time during the lesson referring to time allocation for different activities (time).

Pantziara and Philippou's study of 15 mathematics classrooms (2007, 2010) showed similar results. In order to analyse teachers' instructional practice they developed a list of six categories based on achievement goal theory and mathematics education reform literature (Stipek, et al., 1998). These categories were: task, instructional aids, practices towards the task, affective sensitivity, messages to students, and recognition. This list of categories has some commonalities with the categories proposed by Anderman et al. (2002). Both models included the categories Task and

Recognition. Anderman et al. (2002) incorporated also the categories Authority, Grouping, Evaluation and Time. Pantziara and Philippou (2010) included practices towards the task and the use of instructional aids, practices that Anderman et al. (2002) included in the category Task. Moreover, they (2010) refer to affective sensitivity and messages to students. Pantziara and Philippou (2010) found that in the classroom in which sixth grade students demonstrated the highest interest and self efficacy beliefs, the teacher used problem solving activities, visual aids, as well as open questions. New mathematical ideas were connected to students' prior knowledge and the teacher focused on understanding and worked with students' misconceptions. In addition, the teacher developed a warm environment in which the students' effort was recognised and positive comments were declared by the teacher.

These findings are consistent with previous research studies on the relation between teachers' instructional practices and students' goal orientation in mathematics (Nicholls, Cobb, Wood, Yackel, & Patashnick, 1990). Stipek et al. (1998) found that some of the same aspects of teachers' instructional practices that had a positive impact on students' intrinsic motivation also were a powerful predictor of students' goal orientation. Classroom in which teachers created a positive climate and in which they focused on substantive constructive feedback to students rather than test-scores were associated with a mastery orientation. These findings are consistent with a study by Cobb et al. (1992) indicating that students who experience a reform-based teaching approach are more likely to develop a mastery orientation in mathematics than students in more traditional classrooms.

The findings presented here are consistent with more general research on students' goal orientation, and they reveal that design principles and instructional practices suggested by goal orientation theorists (Pintrich, 2003; Stipek, 1996) promote students' mastery orientation in mathematics. These instructional practices are similar to ones promoted by the mathematics reform literature (Stipek, et al., 1998)

CONCLUSION

Pintrich's identification of five different families of social cognitive motivation constructs provides a structured overview of the different perspectives of motivation and research that has been done. Research studies on the relationship between teachers' practice in the mathematics classroom and students' intrinsic motivation and goal orientation indicate that some of the same aspects of teachers' instructional practices that have a positive impact on students' intrinsic motivation also positively influence students' goal orientation, in terms of mastery goals. It seems like a focus on learning and understanding, generating own solution strategies and a good affective climate positively influence both students' intrinsic motivation and learning orientation in mathematics. There is still a great deal of research needed to understand students' motivation for learning mathematics and how teachers' instructional practice influences their motivation.

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HANDLING NEGATIVE EMOTIONS IN LEARNING MATHEMATICS

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This study is part of a project that aims to investigate students' coping with negative emotions when faced with difficulties in mathematics. Frydenberg and Lewis's (1993) adolescent coping model was adopted to explore coping and The Coping with Mathematics Scale was developed to evaluate students' strategies for coping when faced with difficulties in mathematics (Ader, 2004). The aim of the present study was to determine the prevalent coping strategies of 490 middle school students using the data based on The Coping with Mathematics Scale. The results indicated that students used coping strategies focusing on solving the problem most frequently and significant correlations between use of various coping strategies and math anxiety were found.

Keywords: coping, mathematics anxiety

THEORETICAL FRAMEWORK

The present study is part of an ongoing project that aims to investigate students' coping with math anxiety and negative emotions when faced with difficulties in mathematics. Mathematics anxiety has been defined as the feeling of pressure that limits the use of numbers and solving mathematical problems in academic settings and everyday life (Richardson & Suinn,1972). It has been shown to lead to emotional symptoms, such as panic, fear of failure, self doubt, frustration, hopelessness, shame, powerlessness as well as sweating, nausea, stomach disturbance, difficulty in breathing and inability to concentrate. Mathematics anxiety turns mathematics into a source of stress for many people, frequently leading to detrimental cognitive consequences (Ashcraft & Ridley, 2005).

Mathematics anxiety being a source of stress for many students is thought to be overcome by some who can successfully cope with anxiety and negative emotions when faced with mathematical difficulties. The theoretical framework put forward by Frydenberg and Lewis's (1993) adolescent coping model was adopted to investigate this claim. It was assumed that as the prevalent coping strategies for math anxiety were unfolded, the obtained information could be utilized in math education to train math anxious students for more effective coping strategies (Frydenberg, 2004).

METHOD

Initially, The Coping with Mathematics Scale was developed to evaluate students' strategies for coping with difficulties in mathematics (Ader, 2004). The scale was found to be an effective tool to determine students coping strategies to overcome mathematics anxiety. Initially, items were generated by adaptation of items from Frydenberg and Lewis' Adolescent Coping Scale for contexts of dealing with mathematics. The opening study for the development of the scale was conducted with 751 students preparing to take the university entrance examination in Turkey. The final form of the scale in the first study comprised 36 items in three sub categories: 13 items in *coping focused on solving the problem*, 13 items in *non-productive coping* and 10 items in *coping with reference to others* (Ader, 2004). In the second study for the revised short form of the scale, data were collected from 174 adolescents. The psychometric characteristics of the 18 item short form of the scale were reported (Ader & Erktin, 2012).

The aim of the present study which was the third study of the project was to determine the prevalent coping strategies of students when faced with math anxiety and difficulties in mathematics using the data based on The Coping with Mathematics Scale. 490 adolescents from year 6 to 8 of a public primary school (aged between 11 and 15) took part in the study. Math anxiety levels of 293 of the students were also obtained.

RESULTS

The results indicated that students used coping strategies focusing on solving the problem more frequently than non productive coping strategies and coping strategies with reference to others. Yet use of coping strategies focusing on solving the problem were significantly lower among 8th graders in comparison with 6th graders. When the relationship among use of coping strategies and levels of math anxiety were considered, a strong negative correlation between use of problem focused coping and math anxiety, and a positive correlation between use of non productive coping and math anxiety were found. Building on the findings of this study, a pressing need for further studying coping strategies and the links between anxiety and coping through various other approaches (e.g. longitudinal studies) is highlighted.

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG9: LANGUAGE AND MATHEMATICS: ITS ORIGIN AND DEVELOPMENT

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Keywords: *Mathematical discourses; conceptual analysis; semiotic representations; mathematical practices; classroom norms; social interaction; group discussion; mediating artefacts; didactical environments; problem solving*

A BRIEF HISTORICAL OVERVIEW

In 1998, at the first Congress of the new-born European Society for Research in Mathematics Education at Osnabrück, Germany, there was a Working Group entitled 'Social Interactions in Mathematical Learning Situations' (Krummheuer, 1999). The same group was held in the Czech Republic (Krummheuer, 2002), and in Italy (Price, 2004). Most activities of the Working Group 'Language and Mathematics' were then initiated and developed, and have been maintained ever since through meetings in Spain (Duval, Ferrari, Johnsen-Høines, & Morgan, 2006), Cyprus (Morgan et al., 2007), France (Morgan, 2010), Poland (Cestari, Mercier, Ferrari, & Tatsis, 2011), and recently at CERME 8 in Turkey. When tracing the development of our Group by reading the texts in the Proceedings, we have come to see two major issues, the survey of themes, and the variety of methods and tools. For both issues, it becomes clear that language and mathematics research is not unique to our Group. It has important connections to other working groups. This is also reflected in the fact that some researchers at one conference may change to another group, and then perhaps later change back again.

Concerning the survey of themes one will notice that initially the emphasis was on students' mathematical learning. Fundamental to the first meetings of the predecessor of our Group (Krummheuer, 1999, 2002; Price, 2004) were the topics of classroom interaction and mathematical learning. These two foci have evolved towards the study of more general language and interaction phenomena that coherently draw on the assumption that the teaching and learning of mathematics is mediated by the role and use of language through particular discourses, registers, systems, and/or meanings. Thus, the term 'language' is taken as a term covering a whole range of studies. Classroom social interaction still is a strong theme in the Group as an object of study (Cestari et al., 2011; Duval et al., 2006; Morgan, 2010; Morgan et al., 2007). However, the Group has also come to include works were the object of study is language and, if it is the case that the mathematics classroom is the setting for investigation, social interaction is part of what happens when certain discourses, registers, systems, and meanings are orchestrated. Researchers in the Group want to understand why it is that teaching, thinking and learning happen as they do in contexts of mathematics education (not only schools and classrooms), and how one might expect language issues to have an influence on these processes. As a consequence of having broadened the survey of themes one can find a significant number of reports that do not summarize specific situations of interaction among participants in the mathematics classroom.

The variety of methods and tools has been the basis for very productive controversies, over which are more adequate and for what. Researchers from different theoretical traditions in the field have brought up the issue of how a common theme may be investigated in different ways. However, a characteristic of our Group is the dominance of qualitative approaches to the interpretation of data and findings, primarily from classroom based research (as can be seen below in the synthesis of the works presented at CERME 8). Throughout the meetings of the Group, there have been some studies that combine quantitative and qualitative approaches to their data and discussion of findings, but most of them give priority to the implementation of qualitative methods. The description, application and evaluation of analytical instruments have progressively gained representation. We see this fact as important in that a major effort is put into issues of rigor to unravel methodological complexity. Interesting discussions have been conducted on the choice of methods and the implications in terms of the resulting findings.

Despite all these positive comments concerning the development of the Group, much work is to be done, and it is not totally clear in which directions. In the final section, we will come back to what is reasonable to do next. We do not yet have, and cannot expect to have in the near future, a situation in which the European community of researchers on language and mathematics (education) can tell the rest of the community sufficiently refined solid finding emerging from our activity as a Group.

SYNTHESIS OF THE CURRENT PERSPECTIVE

At CERME 8 a total of 18 papers were presented in the Group. In addition four participants presented posters in the Poster Sessions, and two persons participated in the Group without being involved in a paper or a poster. The papers were grouped into six themes, corresponding to the six groups that were used for presentation and discussion during the Conference. As said before the topic *Language and Mathematics* is very wide and diverse and this is also reflected in the papers of the Group. Many of the papers could easily have fitted in other working groups and vice versa. Similarly, there are no clear borders between the six themes listed above.

Students' mathematical meanings and practices

This group comprise three papers, Barrier, Hache, & Mathé, Fernández-Plaza, Rico, & Ruiz, and Schütte. An important issue in this theme is that language is contextual and that learning mathematics means entering in specific ways to interpret the world as well as specific language practices. Language is a medium of the social negotiation of the specificity of mathematical practices and of the meaning of terms. Many mathematical terms are taken from everyday language and given a meaning in mathematics that not always coincides with the everyday meaning of the word.

The paper by Barrier et al. is concerned with the meaning of the word *circle,* whereas the paper by Fernández-Plaza et al. discusses the meaning of the word *limit* in calculus. In the paper by Schütte the topic is deictical language with pointing vs. the teacher acting as a model for learning the language of mathematics. Some questions raised in this session were what the role of language in the negotiation of particular meanings in mathematics can be, and which the difficulties are due to conflict between everyday and mathematical meanings of a word.

Semiotic systems and representations

This group comprise four papers, Priolet, Rønning, Swidan, and Nosrati. All four papers deal with issues of working with different semiotic representations and in particular issues to do with changing between semiotic representations. Common topics are mediation by semiotic representations and the use of artefacts. Moreover, the research behind all the papers in this group is based on carefully designed learning environments and can be said to have implications for pedagogical practice.

Priolet uses a framework based on four principles, one of them being conversion of semiotic representations, to work with small children solving elementary number problems. Rønning also works with small children but in his paper the focus is on different representations of fractions and decimal numbers, Swidan works with university students and how they make sense of the concept *antiderivative*, using a dynamic and multi-semiotic technological environment. Nosrati explores how students construct and choose their own semiotic representations when working with physical artefacts on a topic (group theory) which for them is an unfamiliar topic.

Group discussion, norms and communication

In this group are placed the papers by Morera, Planas, & Fortuny, Tatsis, Planas & Chico, and Iversen, four papers in all. A key issue in this group is how use of language positions students in relation to the learning of mathematics. The first three papers mostly deal with language in the sense of oral communication whereas the last one (Iversen) deals with students' mathematical writing.

In the paper by Morera et al., as well as the paper by Planas and Chico, the interest lies in identifying what the authors refer to as *Mathematical Learning Opportunities.* Both papers are concerned with the social aspects of mathematical learning and study interaction in whole group discussions. Tatsis is concerned with classroom social and sociomathematical norms and he is looking into interactional processes in order to identify factors affecting the establishment of such norms. Iversen is interested in investigating how students' mathematical writing in light of the concept *writer* *identity*. In the paper one particular student is chosen as a case to illustrate how he in different types of assignments constructs his identity as a writer of mathematics.

Diverse languages in the teaching and learning of mathematics

The concept *different languages,* used in the heading of this session could mean different things, and indeed this is the case. In the paper by Ni Ríordáin, 'different languages' means English and Irish. Ni Ríordáin discusses a potential connection between syntactical and semantic differences between Irish and English, and the learning of mathematics. She suggests that the syntactical structure of the Irish language in terms of sentence length, topical prominence and word order, may lend itself to easier interpretation of mathematical meaning in comparison to English. In the paper by Peters and Graham, the different languages that are studied are the formal mathematical language (mathematical register) and the informal language (natural register). Peters and Graham have studied mechanical engineering students' expression of their mathematical thinking when explaining a physical experiment involving three balls of different masses being rolled down a ramp. Based on the students' analysis of the system the tension between the everyday meaning and the precise mathematical definition of the terms involved is being carried out.

Use of tools and interactional strategies by teachers

The papers by Riesbeck, Schreiber, and Ingram, Baldry, & Pitt were all placed in this session. It may be argued that they do not have very much in common, an argument which is also reflected in the title of the session with two different aspects.

Use of tools, and in particular ICT tools, is discussed by both Riesbeck and Schreiber. Riesbeck studies pupils from preschool, year 1 and year 2 in their work with numbers and place value. The pupils work with different kinds of artefacts, from concrete materials to computers and interactive whiteboards, and Riesbeck shows how the pupils' language develops with the use of the different artefacts. Schreiber has worked with primary school pupils to produce mathematical podcasts. The pupils produce the podcasts, e.g. with the purpose of explaining a mathematical concept, in several versions where a new version is expected to be an improvement of the previous version. In this process it is of interest to investigate the development of mathematical knowledge as well as doing a semiotic analysis of the content. The paper by Ingram et al. is different in many respects. It is not about tools but it is about how teachers interactionally manage mistakes in the students' work. The structure of sequences of interactions in which a mathematical mistake occurs is described to reveal the conflict between the pedagogical and interactional messages.

Didactical environments, scaffolding and mediation

The two papers in this session, by Dias & Christinat, and Moulin & Deloustal-Jorrand, are based in the French Didactic Tradition. Both papers are concerned with oral language used between teacher and pupils and they both deal with a paradoxical question concerning the didactical contract. It appears that using a story in the classroom instead of a usual word problem may lead to dividing the problem into small parts, thereby reducing the complexity of the didactical situation. The papers give examples of two different approaches; one starting with a didactical situation which is looked at from a linguistic perspective, and one starting with a linguistic question which leads to a didactical approach.

SOME FUTURE DIRECTIONS

To expand the survey of themes and for our agenda as a Group to move forward, we see at least two directions: research on multilingual teaching and learning classroom practices, and research on the role and use of language in teacher education. The first direction is still weakly represented in the Group, though some papers have addressed multilingualism. The second direction would help to complement the mainstream research on teacher education that, in general, does not focus its attention on the language contexts in which the practices of teacher education take place.

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LIST OF PAPERS AND POSTERS PRESENTED IN WG9

Papers

Thomas Barrier, Christophe Hache, & Anne-Cécile Mathé. *Seeing-acting-speaking in geometry: A case study.*

Thierry Dias & Chantal Tièche Christinat. *A linguistic analysis of the didactical environment in support of the scaffolding concept.*

José Antonio Fernández-Plaza, Luis Rico, & Juan Francisco Ruiz-Hidalgo. *Meanings of the concept of finite limit of a function at a point: Background and advances.*

Jenni Ingram, Fay Baldry, & Andrea Pitt. *The influence of how teachers interactionally manage mathematical mistakes on the mathematics that students experience.*

Steffen M. Iversen. *Writer identity as an analytical tool to explore students' mathematical writing.*

Laura Morera, Núria Planas, & Josep M. Fortuny. *Design and validation of a tool for the analysis of whole group discussions in the mathematics classroom.*

Marianne Moulin, Virginie Deloustal-Jorrand, & Eric Triquet. *Reading stories to work on problem solving skills.*

Mona Nosrati. *Choice of notation in the process of abstraction.*

Michael D. Peters & Ted Graham. *An investigation into the tension arising between natural language and mathematical language experienced by mechanical engineering students.*

Núria Planas & Judit Chico. *The productive role of interaction: Students' algebraic thinking in large group work.*

Maryvonne Priolet. *Place of the conversion of semiotic representations in the didactic framework* R^2C^2.

Eva Riesbeck. *The use of ICT to support children's reflective language.*

Máire Ní Ríordáin. *A comparison of Irish and English language features and the potential impact on mathematical processing.*

Frode Rønning. *Making sense of fractions given with different semiotic representations.*

Christof Schreiber. *Primapodcasts - Vocal representation in mathematics.*

Marcus Schuette. *Linguistic intercourse with spatial perception. Comparative analysis in primary school, infant school and the family.*

Osama Swidan. *Perceiving calculus ideas in a dynamic and multi-semiotic environment - the case of the antiderivative.*

Konstantinos Tatsis. *Factors affecting the establishment of social and sociomathematical norms*

Posters

Candia Morgan, Sarah Tang, & Anna Sfard. *Studying the discourse of school mathematics over time: Some methodological issues and results.*

Anna-Karin Nordin. *Analyzing teachers' follow ups and feed forwards, seen as a way to enable students' participation in mathematical reasoning.*

Cecilia Segerby. *Textbooks and logbooks in mathematics.*

Anke Walzebug. *'Mind the gap'-Language based item difficulties in mathematics as a 'cultural gap'?* (not published in the proceedings)

SEEING – ACTING – SPEAKING IN GEOMETRY: A CASE STUDY

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The purpose of this paper is to describe and analyse the confrontation and changing processes of frequentation modes (seeing – acting – speaking) of 1st grade secondary school students (10-11 years old) during two geometric construction tasks. This work is based on a logic analysis of the mathematical concepts involved: the midpoint of a line segment and the circle.

INTRODUCTION

When we try to better understand how the students' "relationship to knowledge" changes in a learning situation in geometry, language interactions are often considered as preferential indicators of the reference framework underlying the students' geometry activity. In this case, the language remains subordinated to the analysis of the physical activity. As for us, wee try to consider language not only as a reflection of the conception of an acting subject but to take into account, in our analysis, both the language and the physical part of geometry activity. For this purpose, Bulf, Mathé, and Mithalal (2011) introduce the notion of *frequentation mode* which aims at accounting for the consistency of physical and language dimensions of activities in geometry. They insist on the fact that, on the one hand, language practices are not different from other practices, they have no hierarchy (in order to avoid language issues being pushed into the background of the action, especially in geometry) and, on the other hand, language practices are constituent parts of subjects' knowledge rather than mere reflections of pre-existent knowledge.

Frequentation modes of a geometrical object should articulate three dimensions at the same time: a way of *seeing* in geometry, which will form most of our theoretical horizon and will allow to make students' appreciation of a geometry activity more clear; the ways of *acting* taken by the subject and the instrument usage rules; the ways of the *speaking* and the subject's discourse on objects and actions. The word *seeing*, that we prefer here to the word *thinking,* even if they have analogous meanings, underlines the importance of visualization in geometry (Duval, 2005; Duval & Godin, 2006). As for the word *acting*, we have decided to focus specifically on the artefacts as a didactical variable which is determining in construction situations (Perrin-Glorian, Mathé, & Leclerc, 2013). At last, as for the word *speaking,* we shall intend to describe the specific *language games* (Wittgenstein, 1953) implemented by students and their changing process (Barrier, 2011; Mathé, 2012). Therefore, defining a frequentation mode consists of describing a geometry activity by a way of thinking and considering modalities of action consistent with a certain discourse characterized by its structure and the meaning given to the terms by the subject in the context. In this context, learning is described as a changing process of frequentation modes to a relationship in line with knowledge objects.

This approach seems to echo the hypothesis of a "discursive community" (Bernié, 2002; Jaubert, Rebière, & Bernié, 2003), a French approach analogous to the discursive or communicational one (Kieran, Forman, & Sfard, 2001). In a Vygotskian perspective, these authors introduce the notion of a school disciplinary discursive community: knowledge is tightly linked with the community which created it, learning in a school subject area is learning how to act-think-speak somewhat like experts. This means learning how to take position in a social universe characterized by interesting issues, specific objects, material and practices, in particular language practices, learning *Discourse practices* (Moschkovich, 2007).

However, we do not have the feeling to follow a radical *communicational* way of thinking (Sfard, 2001). Indeed, we are as interested in the dialogical and social aspects of discourse as in the semantic dimension of language, i.e. its ability to refer to external mathematical objects. The purpose of this paper is to show, from a case study, how a logic analysis of concepts can help to describe frequentation modes and their dynamics. We shall use the language of the first-order logic, i.e. the fundamental logic categories will be those of object, predicate and relation. This work is part of a larger project aiming at creating tools that could account for the geometrical practices both in the physical and language dimensions of these practices.

CORPUS AND METHODOLOGY

The extracts analyzed in the following pages are taken from data collected for another work dedicated to issues relating to the use of history of mathematics in the classroom. Detailed information on the context of data collection and on students' tasks is available in Barrier, de Vittori, and Mathé (2012). In this study, we selected some sequences of a lesson during which 1st grade students (10-11 years old) in a French secondary school had to draw a square on the playground using unusual artefacts: a chalk and a rope. The construction program given to the students and the expected figure are available in annex. The lesson lasts just under one hour and takes place outdoors, in the playground. Students are divided in groups of three or four and the teacher moves from one group to the other. The sequences reported have been filmed.

In this problem, students are asked to construct a midpoint and a circle. Starting from the logic analysis the concepts of circle, we shall attempt to bring out *a priori* the potential of frequentation modes which could be considered for the geometrical objects involved in the problem set. This will provide an analysis framework which will be used, during the progress of the sequence, to identify coexisting divergent interpretations of the situation and will help to better understand how they evolve. We will observe the students' gestures and procedures as well as the dialogues between them and with the teacher.

CONSTRUCTING THE MIDPOINT

A priori analysis

A line segment may be seen in different ways: it may be seen as a part of a straight line or as a pair of points. In the first case, the midpoint of a line segment is characterized by a binary relation between two objects, a geometrical object of dimension 1 – the line segment – and an object of dimension 0 – the midpoint (example of statement of the relation: the midpoint is the point of the line segment that splits it into two parts of equal lengths). In the second case, it can be characterized by a ternary relation between three objects of dimension 0: the end points of the line segment and the midpoint (example of statement of the relation: the midpoint is the point aligned and equidistant with the two end points). These two ways of seeing call for objects which differ in number and nature (in dimension). From a physical action standpoint, there are many possible construction procedures. In the first grades of secondary school, the more usual method involves using a graduated ruler (to measure the length of the line segment and to plot half of the length from one of the ends). The property explicitly studied in this procedure is the fact that the midpoint M of a line segment $[AB]$ fits the equality $AM = ½ AB$. Questions such as the alignment of the midpoint with the end points or the midpoint belonging to the line segment are hidden by using the graduated ruler. Another method will be used later: plotting the perpendicular bisector using a ruler and a compass. In the present situation, the rope can be used as an artefact to split the length and check the alignment. The construction requires to explicitly account for both alignment and equidistance properties. It should be noted that the construction program proposed (annex) does not explicitly require plotting the line segment involved. There are several procedures available. We shall describe three of them:

P1. The first procedure consists of plotting the line segment on the ground by stretching out the rope, identifying the ends if necessary and then folding the rope in two equal parts. The midpoint is obtained by plotting the length from one of the ends. In this case, the property under which the midpoint belongs to the line segment is no more explicit than in the ruler procedure, since the question of alignment (or of belonging to the line segment) is evened out by the plotting of the line segment that is made independently of the construction of the midpoint.

P2. If the line segment has not been previously plotted, the rope can be laid in a straight line on the ground and then be folded in two parts by moving only one of its ends, the other end staying in the same place. Therefore, the midpoint is placed at the end of the new line segment thus obtained. Theoretically, this procedure does not call for the issues about alignment or belonging to the line segment, but the question arises from a practical standpoint, since it is difficult to move half of the rope while the other end stays still.

P3. The third procedure explicitly accounts for the alignment property. It starts by laying the rope on the ground as described in the procedure P2 and by identifying the ends. Then the rope is folded in two equal parts and the new length is used to plot an arc of a circle, the centre of which is one of the ends of the line segment. The midpoint is obtained by determining, using the rope, the point on this arc which is aligned with the identified ends (the line segment can be plotted or not).

We consider that the distinction made between the different conceptions of the notion of midpoint, in terms of binary or ternary relations, in addition to the *a priori* analysis of possible procedures, may contribute to demonstrate the potential of frequentation modes which could be considered for the midpoint (of a line segment) object. The following *a posteriori* analysis should identify the frequentation modes in which the students stand, detect the possible coexistence of different frequentation modes and intend to better understand how the change towards a frequentation mode of the midpoint notion in line with the school expectations at this educational level operates.

***A posteriori* analysis**

In this paper, we shall focus on the physical and dialogical practices of a group of three students (E1, E2 and E3) and their interactions with the teacher (H) about the construction of the midpoint of line segment [*OE*]. The three points *O*, *E* and *I* are identified by a cross and by their corresponding letter on the playground. The line segment [*OE*] is not plotted and the three points *O*, *E* and *I* do not seem to be aligned. Then, the teacher intervenes and asks the students to explain how they have proceeded. E1 "shows" the way they used to construct the point. He starts by joining the two ends of the rope, joining both his hands to openly show the half-length obtained. This means that the group perceives the length constraints imposed on the construction of the midpoint. Then he lays a part of the rope on the ground. One end of the rope is placed in *O*, while the other end stays in E1's hand and the rope is laid so that it passes by *O* and by *I* (but it does not pass by *E*). Now, he folds the end he holds towards point *O*, without exerting any other pressure on the rope. It seems that this group has used procedure P2, in a more or less successful manner. The interactions proceed in the following way:

H: No, no, but ... have we got a means to check if it is the midpoint?

E1: Why, yes

H: What could we do?

E1: Plot a line.

H: A line? […] Check there... How, how did you place your rope to check that this is the midpoint? [Pause] How will you proceed?

E3: We lay it/ [E3 points his finger towards *O*]

H: Therefore, we put one end here and then the other end/ [E1 puts an end of the rope in *O*]

E3: We fold it

H: Yes, and the other end? You must stand...

E1: We fold it like that [E1 follows the procedure described above]

H: Well, this is not what I want

We can assume that H checks the control procedure which consists of using the rope as an artefact to check that the three points *E*, *O* and *I* are aligned. As for the students, they seem to be in a frequentation mode that is definitely different from the notion of midpoint. They focus on the distance constraints and they only consider global perceptive retroactions which cannot invalidate their construction strategy. This misunderstanding appears in the form of language interactions. For example, when H says "and then the other end" then goes on with "Yes, and the other end ", it seems that he expects an answer with something like a specification of the position where the other end of the rope should be placed. The students' answers are in the action field, to fold the rope in a certain manner, rather than in the place field. This extract shows how language interactions may be a place of confrontation between conflicting frequentation modes and an (attempt of) negotiation towards a shared frequentation mode. For example, when the teacher repeats E1's statement "we lay it there" by saying "Therefore, we put one end here and then the other end", he attempts to direct the students' look towards the ends of the line segment and introduces the end *E* as a reference of a position. Thus, he intends to lead the students towards an interpretation of the "midpoint" object defined by a ternary relation between three points. The technique implemented by the teacher aims at pointing out that point *E* should be taken into consideration. The purpose is to set a shared objects field from which construction language games could be compared. Nevertheless, students are not able to put on language indicators used by the teacher and they do not recognize the specific form of language game he wants them to play. Finally, this misunderstanding leads the teacher to artificially put aside the strategy of this group. Therefore he decides to introduce by himself the third point necessary to make them shift to the punctual standpoint:

H: [H puts his forefinger on point E] Yes but here, in relation to this point, is there a means to check that your point placed there will be the middle point, the midpoint of your line segment?

E3: Well, we plot er... the rope.

However, it is difficult for students to use the rope as a geometrical artefact to plot straight lines. So far, the rope has been laid on the ground in an approximate straight line, without exerting special pressure on its ends. Now, the teacher takes over a more important part of the problem. He uses the language to simultaneously set in action the three points the alignment of which is to be questioned and clarifies the fact that they must be linked by a specific relation.

H: If your point is the midpoint, how should these three points be?

E1: On the same straight line

H: On the same straight line, well then have we got a means to check your three points are really on the same line? What can we do?

E1: Oh no, there are like this! [E1 shows that the points are not aligned]

H: Well, how can we check then, how can we be sure it will be placed correctly? […] You cannot see how we can check the points alignment?

E3: Er… no

H: Well, your task will be... You have to find the way, just think, sort it out yourselves, find how to check that your three points are correctly aligned, that's all [H goes to another group].

This time the students see the necessity to align the points, thus focusing on the line segment, to the exclusive consideration of the lines and lengths by a punctual look, inducing a possible questioning on the points alignment.

In this first example, we have tried to point out the consistency between the modalities of physical action, the discourse and the way of looking at the figure for a given group of students, even when this consistency is disturbed by the teacher's intervention. Let us now present a second example.

PLOTTING A CIRCLE

A priori analysis

From a logical and mathematical standpoint, the circle can mainly be seen (for other characterizations of the circle, cf. Artigue & Robinet, 1982) as:

- a set of points characterized by a relation: the fact that they are at a given distance (radius) from a given point (centre). This representation corresponds to a plot using a compass or a rope, but also a "point-by-point" plotting (plotting multiple points at a given distance from the centre, then plotting a line if necessary, or linked line segments)
- a continuous line with constant curving. This vision of a circle is hardly operational except for freehand plotting (it can be combined with plotting a few points or few diameters then applying the point-by-point plotting described above), this characterization can also be used for checking a freehand plotting (or using artefacts if necessary)
- the given length line which contains the largest surface area (not quite operational, somewhat corresponding to the circle of a children's dance)
- a line with infinite number of axes of symmetry (not quite operational but it can be used to check during plotting)

It should be noted that the first characterization calls for a relation between objects of dimension 0 (points, including the midpoint which is "exterior" to the graph) whereas the three other points use properties applicable to a single object of dimension 1 (the line).

Of course, the rope can be used to plot circles (or arcs of a circle). This construction requires to follow the same preliminary steps as for plotting with a compass (this artefact is almost always used for plotting circles in a classroom): decision of the radius length to be used (if necessary, selection of the line segment to be plotted, or modalities of length measurement if the length is given with a numerical value) and of the centre around which the circle must be plotted. If the rope is used to plot, it might be difficult to hold one of its ends in a fixed point during rotation. With a compass, when the space between the legs has been fixed, the equidistance property of the points of the circle, or from the line to the centre, is accounted for by the stiffness of the artefact itself. In the context of plotting with the rope, this property is tightly linked with the fact that the rope must be held tight during the whole plotting process. It is physically felt by the student who makes the plot of the circle and it is the required condition to plot the circle. Plotting with a rope usually corresponds to plotting in the meso-space, whereas the compass is commonly used in the micro-space of the sheet of paper. This parameter induces different gestures: plotting with a compass requires hand work, while plotting with a rope requires moving the body and the arms and sometimes the intervention of two students is needed (one student keeps one end of the rope on the centre of the circle whereas the other holds the tight rope and draws the required circle with the other end).

In this case, the links between characterizations and modalities of construction clearly appear, as well as the links between the characterization and the nature of the circle object. The objects explicitly or implicitly handled and the nature of their relations (binary relation, property, etc.) differs from one characterization to the other. What is considered: the centre? the radius? Is the circle seen as a set of points? a line?

A posteriori analysis

In this part, we shall focus on the analysis of the sequence with the work of a second group, again made up of three students. The students are going to plot the circle with the line segment [OE] as a diameter.

The analysis of language interactions shows us that without their usual artefacts the students cannot instinctively adopt a mathematical frequentation mode. They suggest to draw "a round shape" and even a "normal round shape", i.e. a round shape which does not refer to mathematics but rather to what they usually use outside the specific approach of spatial issues raised in geometry. This justifies freehand plotting here. The standpoint on the circle adopted is that of a rounded shape made of a line (a closed line, characterized by its constant curvature and/or its symmetries for example). This is a global rather than a local point of view, since it calls for lines and not for points (and the centre of the circle is not evoked). The teacher's interventions can be seen as attempts to (re)-position the students in a frequentation mode of the circle object which is more in line with the school mathematical expectations, to guide them to express their practical concerns through a mathematical questioning of the properties of the objects and artefacts involved, using a language suitable for the school mathematical context. The background movement that the teacher attempts to bring about is stimulated by the questions: "A circle, what is it? What is a circle? ". Raised by the teacher, these questions call for a change in the students' way of seeing. The questions on the nature of the circle are not only or mainly aimed at obtaining a definition of the circle in return. The objective is to lead the students to "see" the circle as it is usually seen in the school mathematics context. Some of the answers given by the students may seem to be tautological ("A circle", " It's a circle", "Well, it's a circle") and useless in the context of knowledge. The students seem to be aware of it, but it is not the case if they are analyzed considering how the dialogue works and how the practices are inserted in the required context. These language interactions show a change in position. This movement is also revealed by the fact that, in other following answers, terms which are specific to mathematical vocabulary are introduced (centre, diameter and radius).

With the term "the-circle-with-a-centre-X-and-a-radius-Y" (used twice in an identical manner) and the explicit allusion to the «definition» of the circle, the teacher clearly introduces a formal dimension (Hache, in press), above all in relation to the situation of the exercise and the supposed frequentation mode of students. Besides, from the knowledge standpoint, the teacher, by using the words "centres" and "radius" for example, refers to the prevailing definition of the circle in the school context, i.e. the circle seen as a set of points placed at a same distance of a given point. As already mentioned, this characterization calls for a property which is made natural by using a compass in the usual situations of plotting. On the other hand, it differs from the instinctive characterization adopted so far by the students, which rather seemed to lie on the circle as a line, characterized by its constant curvature. In practice, shifting from one conception to another is not instinctive (Artigue & Robinet, p. 49), all the more as the characterizations appeal to students to have different looks on the figures (Duval & Godin, 2006). The last teacher's intervention in the above extract can be understood as an inducement to fit the way of seeing ("therefore, is formed by what?") with the way of speaking ("the circle with a centre O and a 3 cm radius").

CONCLUSION

The purpose of this paper was to describe and analyse the confrontation and changing processes of frequentation modes of 1st grade students (10-11 years old) during two geometric construction tasks. This work was based on a logic analysis of the mathematical concepts involved: the midpoint of a line segment and the circle. In both cases, according to the adopted standpoint, the mathematical concepts can be described from different categories of logic (property, binary or ternary relation) on objects different in number and nature. This analysis, although it was quite brief, seemed to be useful to consider the consistency and practical harmonization of the three dimensions "seeing – acting – speaking" we called for to describe the frequentation modes (Moschkovich, 2007, would maybe have said a *Discourse*). We could thus observe that the change in the way of seeing ("Oh no, there are like this!") of the students who worked on the construction of the midpoint was produced by the teacher's language action aimed at setting the objects *ends of the line segment* as references for some words in the language games specific to the geometry practice at school. This change in their way of seeing is associated with new possible uses of the artefact *rope* (physical dimension of geometry practice). As for the plotting of the circle, we attempted to show how the language practices could be linked with the plotting methods. It seems that the expression "a normal round shape" can be related to extra-school practices which justify the "freehand" plotting that we compared with a global vision of the circle as a rounded shape. If this approach is somewhat justified, the related vision is not that in use in mathematics at secondary school. On the contrary, the language game which calls for the expression "the circle with a centre O and a 3cm radius", and which is initiated by the teacher, introduces some elements required for invoking a punctual standpoint on the circle, in particular the centre of the circle. This centre, exterior to the line actually plotted, must be taken into consideration to implement the techniques which call for the equidistance relation.

We are just at the start of our research and we are not sure to be able to offer a pertinent discussion of these results. Nevertheless, we will try to situate it inside the today well-established *discursive framework* in mathematics education (Sfard, 2013). Our feeling is that the former semantic and dialogic perspective could be one way to consider both social and external aspects of language. Mathematical language games could be *outdoor games* (Hintikka, 1996), i.e. games involving the objects of the language one speaks. Analysis of communication quite often emphasizes interpersonal interactions. In this work, we think it necessary to integrate a specific focus on the interaction between students and external (even if dialogically constructed) mathematical objects, with the help of a logic analysis of the concepts at stake. Of course, all of this is nothing new. For instance, the Theory of Didactical Situation tradition in France has a long time ago pointed out the educational interest of the students-*milieu* interactions (Brousseau, 1997) and inside the communicational approach, Sfard (2001) clearly accounts for the “object-level aspects of discourse”. We only hope that this research, relying on logic analysis, could contribute to the content related dimension of language games understanding.

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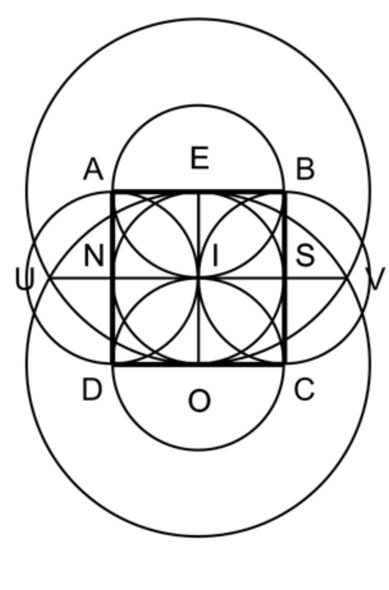
ANNEX

Construction program and example of the expected figure

Stretch out a rope the length of which must correspond to the side of the square to be constructed.
On the ground, mark its ends O and E and its midpoint I.
Plot the circle with a diameter [OE] and circles with radii [OE].
These two large circles are crossed in U and V.
Stretch out a rope between U and V.
Mark as N and S its intersection points with the small circle.
Points U, N, I, S and V are aligned in this order.
Plot the circles with respective centres E, O, N and S the radius of which should measure half the EO length.
These four large circles are crossed two by two in A, B, C and D.
These four points are the vertices of the square.

Follow these instructions and construct a square using the artefacts given to you.



A LINGUISTIC ANALYSIS OF THE DIDACTICAL ENVIRONMENT IN SUPPORT OF THE SCAFFOLDING CONCEPT

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Can the didactical environment be conceptualised as a constant evolving process due to all the protagonists' contributions such as: signs, actions, interactions, and language games? If the didactical environment during mathematical problem solving is without doubt composed of cognitive destabilizing elements, these must be compensated by resources required for supporting the learning process by adaptation. This is particularly important in the context of special needs education. In order to complete our studies of didactical environments, our different observations and analyses of interactions occurring during classroom situations, bring us to reflect on language acts. A linguistic analysis using the syntactic, semantic and pragmatic register is suggested. Its possible contribution to our understanding of didactical environments is discussed.

VERBAL SCAFFOLDING AND THE SEARCH FOR BALANCE

Our research explores various language interactions encountered during mathematical problem solving activities through a didactical dimension related to the notion of environment ("milieu") (Brousseau, 1990). Language phenomena that are responsible for the evolution of an environment suitable for the construction of mathematical knowledge in the specific context of special needs education are investigated. Hence our objective is to make the distinction between:

- What pertains to antagonism (the Piagetian unbalance) in mathematical problem solving activities: uncertainty, the masking of objects of knowledge, the need for verbal exchanges and the rupture of a didactical contract.
- What pertains to compensation or adaptation: for example by basing ourselves on the functions of scaffolding (Wood, Bruner, & Ross, 1976).

Our hypothesis is that the search for balance (or unbalance) by means of control or regulation of didactical situations is mainly possible by resorting to the scaffolding functions defined by Wood et al. (1976) The latter describes scaffolding as the resources employed by an adult or a specialist to help someone less competent (Wood et al., p. 89). After observing pupils and tutors in a problem solving activity, we identified six categories of scaffolding functions: 1. Recruiting; 2. Reduction in degrees of freedom; 3. Direction maintenance; 4. Marking critical features; 5. Frustration control; 6. Demonstration.

We expect that this model is coherent with the practice of special education teacher and it can be used with the notion of adaptation. During a learning situation in mathematics, these scaffolding functions may be correlated to linguistic processes. In our research project the following research questions are investigated:

1) Do special education teachers resort to all the six scaffolding functions during mathematical problem solving activities? Do they give priority to some scaffolding functions and why?
2) Can a linguistic analysis of discourse provide a better awareness of the progressive dimension of the didactical situations analysed through a semiotic process?

During this communication, preliminary results related to research question 1 and 2 will be presented and their implications discussed.

THE SPECIFICITY OF THE DIDACTICAL ENVIRONMENT IN SPECIAL EDUCATION

The concept of the didactical environment stems from the theory of didactical situations (Brousseau, 1998). According to this theory, the use of antagonistic situations (didactical situations creating an unbalance), are crucial for learning. The pupil learns thanks to a process of adaptation to an environment characterised by contradictions, difficulties and unbalance (Brousseau, p. 325). In these situations, learning is made possible thanks to successive adaptations of the didactical environment. In this model, regulations and retroactions provided by teachers are at the heart of didactical concern. Teachers have to anticipate the instructional components that will allow their students to be self-regulated and autonomous. One of the specificities that we observed in the context of special needs education, was that if the didactical environment contained elements producing a cognitive destabilisation, the teacher must compensate it by providing resources and educative alliances. It is necessary for the teachers to anticipate the student's diverse reactions. Indeed, some of the students with special educational needs can ignore the retroactions of the didactical milieu and never take active part in the learning process. The goal of the adaptive learning model is in such cases unachieved, unless complementary or compensatory resources scaffolding the adaptive learning process are provided. The challenge in teaching pupils with special educational needs is to control the progression of the didactical time, and at the same time the maintenance of a pedagogical and relational balance, crucial in the context of special education. The teacher has to submit complex situations to the students in order to involve different kinds of knowledge, without decreasing his requirement and without distorting the didactical environment.

3. LINGUISTIC ANALYSIS

To date, the study of the didactical environments (Brousseau, 1990) does not seem neither completed, nor complete. We wish to explore another approach of the distinctive elements of learning situations in the context of special needs education.

Based on the observations and analysis of classroom interactions during mathematics instruction conducted in our previous studies (Dias, 2007), a language analysis based on speech acts theory (Kerbrat-Orecchioni, 2001) is proposed in order to enhance our comprehension of the didactical environment. Three distinctive dimensions of discourse will be analysed: syntactic, semantic and pragmatic. As defined by Morris (1938) in his classic presentation of semiotics, syntax is described as the study of the formal relations and combinations of signs. Semantics refers to the relations existing between signs and objects to which the signs are applicable (their "designate"). Pragmatics describes the study of «the relations between signs and interpreters» (Morris, 1938, p. 6). These dimensions have been used by many scientists interested in language (linguistic, semiotic and communication). The theoretical framework for the linguistic analysis of didactical situations in our study refers essentially to Morris‘ work (1974).

Morris’ model assimilates semiosis to a dynamic process, similar to the pragmatic approach developed by Peirce (1978). It seems therefore particularly appropriate for gaining a better understanding of the various mechanisms involved in the interactions emerging during mathematical problem solving activities. This reference to behaviour suggests that the pragmatic dimension of discourse is essential in understanding the classroom’s dynamic. The syntactic dimension is a "hidden dimension" for the students. Indeed, only the rules structuring discourse are explicit. In Morris’ model all the signs do not necessarily refer to a real, tangible and perceptible thing. These "vehicles of signs" are mediators of knowledge in didactical situations.

The semantic dimension refers to the assignment of meaning(s) to the expressions of a discourse. In other words, this is the dimension where interpretation of discourse takes place. Because of the complexity of the relations existing between mathematical objects and their perception and comprehension, this is the main difficulty in mathematics instruction. It is in this dimension of discourse that the stakes of constructing meaning are the highest. Indeed, during mathematics instruction several actors interact with similar objects, which are not always interpreted in the same way (Dias, 2008). The verbal exchanges between the actors can lead to a consensus on the meaning of these objects, or not. This depends on the teachers’ skills in regulating the dynamics of the didactical environments (retroactions and antagonisms). If the teacher wants his pupils to interpret correctly his instructions, he must ensure that they are conveyed in the pupils’ comprehension zone. The objects present in the didactical environment, which are conveyed as signs to be interpreted, are essential elements of this comprehension zone. The main issue at stake in this semantic dimension is the construction of a shared understanding of reality (Lelong, 2004).

4. THE RELEVANCE OF A LINGUISTIC ANALYSIS

During communicative situations in the classroom, teachers often discuss with their pupils but also sometimes struggle with meaning in a "language game" (Durand-

Guerrier, Heraud, & Tisseron, 2006). The pragmatic dimension of discourse described in Morris' model, is based on the notion of "language in action". It defines the relation between language and its functional use by the speaker in communicative situations. The pragmatic dimension is essentially contextual, as it refers to the interactions between participants in communicative situations. It can therefore be described as having a psychosocial nature. This dimension is highly relevant in didactical situations in the context of special needs education. Indeed, if a teacher wants to change the behaviour or enhance the knowledge of his pupils, he must necessarily build a social cooperation with his pupils.

In our opinion, interactions are neither dependent, nor independent of their context of production. They are constructions of the context itself as well as manifestations of it. It seems to us that studying this pragmatic dimension of discourse is important to gain a more comprehensive understanding of didactical situations. In our linguistic analysis of discourse during interactions in pedagogical environments, the following references are used:

- The pragmatic register (Morris, 1938): analysis of the effects of the different utterances on the actors in the communicative situation
- The speech act theory (Austin, 1970; Searle, 1976). Austin and Searle distanced themselves from the syntactic and the semantic studies conducted by structuralists, which assume that every utterance implies an effect of this utterance, voluntary or not.
- The interactionist pragmatic (Kerbrat-Orecchioni, 2001), which studies the effects of the language acts during conversations and shows that the effects of some utterances are not consistent with their form. The interactionist analysis of language acts can highlight misunderstandings between teachers and pupils that can reinforce the disequilibrium present in didactical environments. This type of analysis seems particularly promising and could serve to identify several essential factors impacting on student-teacher interactions and their didactical relationship.

AN EXAMPLE OF CONTENT ANALYSIS OF A CORPUS

The example of content analysis presented in this section is based on an audio-recorded lesson, which took place in a self-contained classroom for pupils with special educational needs in the spring of 2011. The teacher had presented to her students a mathematical open-ended problem that they had to solve (Arsac & Mante, 2007). In order to solve this type of complex problems, pupils have to conduct an arithmetical investigation and use their knowledge of several mathematic facts.

The problem:

In a desert of 1000 km, you have to carry 3000 bananas with a camel. This camel can carry 1000 bananas on his back. We know that it consumes 1 banana per kilometre.

What is the largest number of bananas that you can get to the end of the desert?

Characterisation of scaffolding with Bruner's (1983) model

A content analysis in two steps is used in order to distinguish the exhaustive analysis of the teachers‘ verbal scaffolding and their linguistic interpretation.

Step 1: analysis of the teacher's interventions

The teacher's verbal interventions are systematically identified. Each language act is associated to a scaffolding function, based on Bruner's (1983) model and they are coded with the same numerals used by Bruner for each scaffolding function in his study.

When necessary, the audio-recorded conversations were re-heard to enrich the reading of their transcription. The non-verbal explicit acts (gestures and looks) could not be taken into account in the analysis, because the lessons were not videotaped.

Step 2: Sequencing scaffolding interventions

The transcription of the lesson is then divided into short episodes corresponding to communication sequences. This allows the identification of chains of scaffolding interventions that are used preferentially or at least recurrently by teachers. The goal is to establish a profile of teacher's monitoring of the progression of didactical time.

Episodes	lines	Scaffolding functions
1.	line 1 to 9	2-3-3-4-2
2.	line 9 to 19	1-3-3-4-2-2-(4)
3.	line 20 to 37	1-3-3-5-3-3-5-2-2
4.	line 38 to 47	1-3-3-2-(4)
5.	line 47 to 52	1-3-4-2
6.	line 55 to 63	1-4-(2)-6-5
7.	line 63 to 76	1-5-4-4-2-4-3-(4)-2

Table 1. Example of an analysis of content from a corpus

Our analysis shows a frequent use of function 3 (direction maintenance as sustaining the pursuit of the goal) during the first episodes of the activity. The teacher seems to compensate for the antagonism of the milieu created by the complexity of the mathematical problem that the pupils were asked to solve. Indeed, because there is no obvious solution to open-ended problems, pupils are confronted with mathematical retroactions synonymous with temporary failures. It seems therefore normal that the teacher tries to sustain her pupils' orientation on the task, in order to avoid that they abandon their search for a solution out of discouragement. Through this scaffolding technique she also aims at exerting a didactical control on the problem solving process.

An analysis of these episodes also highlights that the function 2 (mainly by simplifying the task) has a conclusive impact on communication. This may be a phenomenon very specific to mathematics teaching in the context of problem solving activities. Indeed, the use of this type of scaffolding is not anticipated in a learning situation which promotes pupils' search of a solution and the production of statements, whose validity is postponed. During the phase of *formulation* (Brousseau, 1990), only the retroactions of the didactical environment can foster the verbalization of knowledge. The teacher's simplification of the task does not play this role and stops the verbal exchanges between pupils.

In this extract, an evolution of the types of scaffolding functions used by the teacher can be observed in accordance with the progression of the problem solving procedure and the progression of the didactical time. Function 4 (identified mainly by pointing out critical features) appears progressively, as a manifestation of the teacher's will to stabilize the pupils' hypotheses and findings. In our opinion, this could be a phenomenon specific to problem solving situations in the context of special needs education. Special education teachers often observe that their pupils have difficulties in identifying the important elements in terms of knowledge after spending time trying to solve a problem. The profusion of information and the absence of identification of their domain of validity is a source of difficulties for pupils.

Scaffolding function 5 (Controlling frustration) appears belatedly in the succession of episodes. This confirms the teacher's willingness to loose progressively her control of the progress of problem solving. This could be described as a professional skill. It could be interesting to highlight it in a context of pre-service or in-service training for teachers.

The implicit character of language acts and disequilibrium in the didactical environment

This second type of analysis is related to our hypothesis that some of the teachers' interventions during mathematical problem solving activities are intended to scaffold pupils' learning, but are not perceived as such by the pupils themselves. Because of these miscomprehensions, elements creating disequilibrium and antagonism, which are not always consciously wished by the teachers, are introduced in the didactical environment. The following extract is an example of this type of perturbation.

20	teacher	this is not… Someone else ? Quentin ?
21	student	I calculated that for one banana the camel had to walk one kilometre, so I calculated 1000 x 1, which makes 1000
22	teacher	yeah
23	student	then I divided 3000 by 1000, which makes 3 camels.
24	teacher	yeah
25	student	then, as we know that one camel eats 1000 bananas, I calculated 1000 bananas for one camel, and then subtracted 1000 to 3000, which makes 2000

26	teacher	yeah
27	student	then as 2000 bananas were left, I sent another camel with 1000 bananas.
28	teacher	how much?
29	student	1000 bananas
30	teacher	ah yeah, 1000 bananas
31	student	then I did 2000 minus 1000 makes 1000 bananas
32	teacher	but your 1000 bananas stayed at the entrance of the desert ?

An analysis of the language acts identified as creating imbalance in the above mentioned extract yields the following findings.

line 22: interpretation of the answer «Yeah»

This first «yeah» is an assertion (statement presented as true). The illocutionary value is an affirmation, but the perlocutionary effect is to point out to the pupil that the teacher heard him and at the same time that he agrees with the content of his statement and enjoins him to continue his reasoning.

line 24: interpretation of the answer «Yeah»

With this second «Yeah», the teacher agrees with the operation performed by the pupil, but creates confusion because her illocutionary act validates the entire proposition. She intended to validate only the result of the division but not the pupil's proposal to use 3 camels. The assertion of this pupil's answer creates confusion, because it does not respect the constraints given in the instructions of the problem (only one camel can be used). Because of the teacher's assertion of this answer with a «yeah», the entire proposition of this pupil will be followed by his classmates in their reasoning during several rows of conversation. This will last some time, before the teacher realizes this confusion ad reminds her pupils that they have the right to use only one camel (line 59 of the corpus).

line 30:

The teacher uses the word «Yeah» for the third time. It has the value of an assertion: the locutor commits himself in recognising the truth of the proposal expressed by the pupil. However, this answer creates/sustains the disequilibrium because it was already used previously on two occasions with another linguistic value. Therefore, it is unlikely that the pupils will be able to interpret this intervention correctly on a semantic level.

line 32:

This intervention suggests that the teacher misunderstood the pupils reasoning. A true disequilibrium is created in the didactical environment and is followed by a time of latency showing the ambiguity of the communicative situation.

CONCLUSION

The preliminary analysis of this corpus highlights the impact of the verbal interactions on the didactical environment, and particularly on its antagonist characteristics. The interplay of interactions related to scaffolding functions contributes to the modification, enrichment, or complication of the didactical environment. It renders it either more accessible, or more opaque for the pupils. The teacher's use, or non-use, of the six scaffolding functions seems to depend from the specificities of the didactical environment, as well as from the didactical time.

The different scaffolding functions identified by Bruner seem to be located in the chronology of interactions and form sequences comprising a succession of scaffolding interventions. These sequences can be clearly identified and seem to be provided in a relatively stable way. Some functions are used more scarcely; probably by fear of a modification of the characteristics of the didactical environment, in particular the fear to make it an ally.

In our corpus, scaffolding seems to be related to the progress of the didactical time. The real function of scaffolding interventions depends on the perlocutionary effect of the language act. Thus, the pragmatic analysis of discourse that we adopted seems to contribute to a relevant identification of the real effects of verbal scaffoldings.

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MEANINGS OF THE CONCEPT OF FINITE LIMIT OF A FUNCTION AT A POINT: BACKGROUND AND RESULTS

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In this paper we present a description of previous work carried out by the authors on the general issue of designing and implementing a didactical planning for Spanish students from non-compulsory secondary education, 16-17 years old. The current research has as its aim to describe the meanings that students associate to specific terms from the language, such as, "to approach," "to tend,""to reach,""to exceed," and "to converge." Prior to the study, we reviewed the mathematical use of these terms and we contrast this with the colloquial use of the terms. From the semi-structured interviews used to gather information, we provide an analysis of the written data. It is important to highlight that students have contributed with a variety of meanings, in addition to those from the previous review.

INTRODUCTION

Since the academic year 2009/2010 we have been interested in investigating some problems related to the teaching and learning of the concept of the limit of a function at a point. This concept is important because it is necessary for the learning of the derivative and integral concepts and is more complex than the concept of limit for sequences. Furthermore, it is one of the key concepts that mark off the transition towards the advanced mathematical thinking. By exploring several textbooks we observed a large number of routine tasks about calculating the limit following an intuitive definition based on the idea of approximation. So we carried out an exploratory study about the intuitive meanings that students have about the concept of a finite limit of a function at a point. The students are given tasks using different representations such as verbal, graphic and symbolic representations (Fernández-Plaza, 2011). Some of the results have been presented before both at national and international conferences (Fernández-Plaza, Ruiz-Hidalgo, & Rico, 2011, 2012a, 2012b, 2012c).

Recently, we have gathered information by means of interviews in order to contrast our interpretation of the written records from students and to deepen the personal conceptions that students associate with the following terms from calculus: "limit," "to approach," "to tend," "to converge," "to reach" and "to exceed". The colloquial meaning of these terms has been shown to influence the understanding and this has been reported in several studies (Cornu, 1991; Monaghan, 1991). In (Fernández-Plaza, 2011), the effective use of these terms and other synonyms has been explored, but not the specific meaning implemented by students. By effective use of a term, we mean that students in fact use this term, and not a synonym. For example, for the specific term "to approach", a student may use "to get close" or "to approximate," among others. This does not count as effective use but it is related to “to approach.” Below, we describe the main achieved results so far.

MAIN ACHIEVED RESULTS

We summarize the most important results we have found out until the present moment. Firstly, we observed a persistence of misconceptions related to the limit as a non exceedable and unreachable value. This result is consistent with those from Cornu (1991) and Monaghan (1991). Here, we go deeper into the topic in the sense that, some students suggested a link between exceedability and reachability. We consider that this kind of misconceptions could arise from an overgeneralization of the particular case of monotone convergence.

Secondly, we discriminate between *process conceptions*, *object conceptions* and *dual conceptions* of the concept of limit. As *process conceptions*, we understand conceptions where the limit is closely related to a procedure about how to find it. With an *object conception,* the student is able to identify properties of the limit without depending on the process involved. Intermediate conceptions between these two are called *dual conceptions.* Thus when students were requested to discuss about the statement “The limit describes how a function f(x) moves when x moves to certain point,” the most of arguments could be classified as one of these three options depending on whether students interpreted the limit as “how” (process conceptions) or “where” (object and dual conceptions) a function moves.

Thirdly, we found conflicts with the arbitrary accuracy of approximation to the limit. Expressions such as “limit can be approximated *as much as you wish*” are taken to mean that some students think that accuracy is bounded in the practical process. We suggest that the ambiguity of the underlined expression could have made students do a crucial distinction between the potential infinite character of the process and its implementation in practice.

Finally, we pointed out the conflicts with the exact or indefinite character of the limit value.

Some subjects considered a limit as an exact number whereas others considered that the limit is an “approximate” number. We suggest, according to Sierpinska (1987), that the latter subjects do not know the exact value of the limit, but only approximations to it, that is to say, the limit is undetermined. The progressive improvement in the interpretation of these results gave rise to talking about *structural aspects*, such as object/process duality of the concept, exact/approximate character of the limit, potential infinite/finite character of the limiting process, reachability and exceedability of the limit. These structural aspects were used to characterize and establish connections between different conceptions about the concept of limit (Fernández-Plaza et al., 2012a). At the same time we tried to characterize the terminology used by students in their answers. We selected the terms “to approach,” “to tend,” “to reach,” “to exceed” and “to converge” among other reasons because they are used in the technical language and they describe different aspects of the concept of limit. Moreover, the influence of their colloquial meanings and everyday use on students' understanding have been reported in the literature. This problem leads us to three questions:

- Which are the different meanings and uses that these terms have in Spanish language?
- What is the terminology that students effectively use to explain their answers?
- What are the explicit definitions and meanings that students associate to these specific terms?

The treatment of the two first questions can be consulted in (Fernández-Plaza, 2011; Fernández-Plaza et al., 2011, 2012b). In the following section we are going to focus on giving answers to the third question.

DESCRIPTION OF THE CURRENT STUDY

We propose to describe how students explicitly define some specific terms from calculus in contrast with the colloquial and technical meanings of these terms. The chosen terms are "to approach," "to tend," "to reach," "to exceed" and "to converge."

Theoretical framework and prior research

We position this study in the research agenda of *Advanced Mathematical Thinking*, from the international group on the Psychology of Mathematics Education (Gutiérrez & Boero, 2006, pp. 147-172). There is no agreement to establish the transition from elementary to advanced mathematical thinking.

The educational stage analyzed assumes a period of transition in which students use elementary techniques to tackle mathematical contents whose development historically, epistemologically, and didactically has an advanced status.

Rico (2012) developed the notion of meaning of a school mathematical concept, based on reference, sign and sense. We analyze the systems of representation, formal aspects or references of the concept, and the phenomena that provides its meaning.

Three components constitute the basis of the meaning of a school mathematical concept:

- *Systems of representation* (sign), defined by a set of signs, graphics and rules, to express and highlight aspects of the abstract concept and to establish relationships with other concepts.
- The *conceptual structure* (reference) that comprises concepts and properties, the derived arguments and propositions and their truth criteria.
- *Phenomenology* (sense) that includes those phenomena (contexts, situations or problems), which are at the centre of the concept and provide sense to it. (Rico, 2012, pp. 52-53)

The mathematical language related to the concept of the limit of a function at a point includes the terms "to approach," "to tend," "to reach," "to exceed" and "to converge." We chose these terms, among other reasons because each of them refers in part to properties and modes of usage associated with the concept of limit, that is to say, the phenomena involved (see Fernández-Plaza, 2011, pp. 14-21).

Conceptual analysis of specific terms

By a conceptual analysis we understand the procedure that leads to establishing the mathematical use of the terms and we want to contrast this use with the colloquial use or use in other disciplines.

We describe the chosen terms below and we also include the colloquial meaning of the term "limit." Monaghan (1991, p. 23) notes that to a mathematician tends to, approaches, converges, and limit are interchangeable. In Spanish "*aproximar*" has two different meanings; the first one expressed by "to approach," (dynamic) and the second one expressed by "to approximate" (static) (Fernández-Plaza, 2011, p. 16).

The sentence "to tend toward a value" means "to approach gradually but never reach the value" (Real Academia Española [RAE], 2001) and expresses a very specific form of approach. Blázquez, Gatica and Ortega (2009) argue that a sequence of numbers approaches a number as a limit if the difference between the terms of the sequence and the limit decreases gradually, but they also argue that a sequence "tends toward a limit" if any arbitrary approximation to the limit can be improved by the terms of the sequence.

A study by Monaghan (1991) concludes that many students do not distinguish between "to tend" and "to approach" in a mathematical context. In a formal sense, to tend toward or to approach a limit is said of a sequence (e.g. the sequence 0.9, 0.99, 0.999… tends toward 1, but also that sequence approaches number 2) according to the definition of limit, but "to approximate" a limit is to give any of the terms of a convergent sequence ("0.999 approximates 1 with an error less than 0.01"). We justify the different distinction between these terms only for Non-University High Education because both of them are applied to the same object (a sequence).

The expression "f(x) tends toward L, when x tends toward a" may cause cognitive conflicts, as Tall and Vinner (1981) note, because x never equals a, so students may consider that f(x) never equals L.

"To reach" means colloquially "to arrive at" or "to come to touch" (RAE, 2001). We interpret the mathematical meaning of "reach" to be that a function reaches the limit if the limit value is the image of the x-point at which the limit is studied (continuity); by extension, the limit can be the image of any other x-value in the domain.

We see that "to exceed" means colloquially "to be above an upper level" (RAE, 2001), excluding the meaning "to be below a lower level". We will say that the limit of a function may be exceeded if we can construct two successive monotone sequences of images that converge to the limit, one ascending and the other descending, for appropriate sequences of values of x that converge at the point at which the limit is studied. The reachability or exceedability of the finite limit of a function can be easily interpreted as global or local concepts, but there is no logical implication of the two concepts.

The term "to converge" means colloquially "to come together from different directions". In mathematics, this term is equivalent to "to tend" and normally is applied to the limit of sequences and series and is not so often used in connection with the limit of a function at a point. We expected that students could invent a definition for this term in this new mathematical context.

Furthermore, the term "limit" has colloquial meanings that interfere with students' conceptions of this term, such as ideas of ending, boundary, and what cannot be exceeded (RAE, 2001). The term's scientific-technical use is related in some disciplines to a subject matter or extreme state in which the behaviour of specific systems changes abruptly (Real Academia de las Ciencias [RAC], 1990).

Prior Research

Monaghan (1991) studied the influence of language on the ideas that students have about the terms mentioned above, when the terms were used in connection with different graphs of functions and examples that school students verbally explained. We underline as a limitation of the approach adopted in this case, that the key terms that the students were asked to use were defined a priori, instead of enabling students to use their own words freely and spontaneously and to infer the appropriate nuances a posteriori.

In previous CERME proceedings there has been published papers related to the learning of the concept of limit of a function. The most relevant one in relation to this study is by Juter (2007) who investigated, among other aspects, how students interpreted the reachability of the limit in a problem solving context.

Method

A semi-structured interview was conducted in an ordinary classroom. The protocol of implementation was the prior request to the students to write their answers on the sheet provided, and the discussion of the answers was audio recorded. The subjects were organised into nine groups with 3-5 in each, in order to facilitate the interaction between the subjects and the researcher.

We focus on the following common question:

Describe how you understand the following terms: "to approach", "to tend", "to reach", "to exceed", "to converge" in the context of a finite limit of a function at a point.

In order to help the students to better express their conceptions during discussion, we showed them some graphics of functions so that some other characteristics of meaning of these terms could emerge, especially with the terms "to reach" and "to exceed".

33 subjects out of a total of 36 from the previous study (Fernández-Plaza, 2011) were selected. They were chosen deliberately, according to their previous answers and based on their availability. The subjects attended the second year of non-compulsory secondary school study (17-18 years of age) and they were all studying mathematics. They had received the instruction on the concept of limit according to the current curriculum.

Preliminary results and discussion

We are going to show some preliminary results from the analysis of the written records. Table 1 shows a classification of meanings and frequencies of the selected specific terms. According to these categories, we classify the definitions provided by the students by the codes (Ai, Bj, Ck, Dl, Em) (Note that each student produced at most five definitions). Only 2 out of 33 of the individual productions (the set of their five definitions) have the same code, therefore the differences are relevant (any two students could define 1, 2, 3, 4 or 5 specific terms in a different way).

We observe that 23 out of 33 individual productions establish some distinctions between "to approach" and "to tend." The most relevant differences of meaning between these terms are as follows:

- The possibility of not to reach or exceed the limit. In particular, some students used expressions such as "to approach more and more" to point out a potential infinite character of the process "to tend."
- The technical usage and the subjective view of the term "to tend."

Specific terms	Meanings	Frequencies
To approach	A1. To get as close as possible	14
	A1.1. Not to reach the limit	10
	A1.2. Not to reach and not to exceed the limit	2
	A1.3. To reach but not to exceed the limit	2
	A2. To establish the closest value to the limit	4
	A3. No answer	1
To tend	B1. To approach	5
	B1.1. Not to reach the limit	7
	B1.2. To approach more and more	4
	B2. Technical usage	8
	B3. Subjective	2
	B4. Other	5
	B5. No answer	2

To reach	C1. To arrive at or to touch the limit	27
	C1.1. Not to exceed the limit	3
	C2. To know the exact value of the limit	2
	C3. To know the value of f(x) for a given x	1
To exceed	D1. To surpass the limit of the function f(x)	19
	D2. To surpass the x-value.	7
	D3. To reach the limit and continue (To pass through the limit or x-value)	4
	D4. No answer	3
To converge	E1. The function is above the limit all the time	2
	E2. The function is below the limit all the time	2
	E3. To tend	1
	E4. To reach	1
	E5. The right and left-hand limits are the same.	3
	E6. The function takes the same value than the limit	1
	E7. Two functions or straight lines intersect at one point	9
	E8. Other	4
	E9. No answer	10

Table 1: Classification of meanings and frequencies about the selected specific terms

Below we exemplify answers from some categories in order to clarify their denomination. The other categories are denoted by a "representative" definition, for example, category A1 includes those definitions that express the idea or use the expression "to get as close as possible", so we do not consider it necessary to exemplify all of them:

- **Category B2: Technical usage.** An example of an answer is "*This term is used to indicate the value that x takes in a limit*". It does not state anything about the specific meaning of the action "to tend," that is, it is only a technical word; an agreement. Another answer is "*to tend to a number is to use the closest number to it, for example, if x→1 from the left, we use 0.9. From the right, we use 1.1.*" The term is used to describe a personal rule to calculate a limit.
- **Category B3: Subjective.** Examples of answers are: "*To approach to that number without being aware of it (without intending to obtain it)*" and "*To approach it as much as we want*" indicate a subjective aspect of the definition of the term "to tend."

From Table 1 we discuss the global results:

Most subjects (14 out of 33) consider "to approach" as *to get as close as possible*. Although it is relevant that 10 out of 33 subjects also consider that the function cannot reach the limit. Only 2 out of 33 admitted in addition that the function could in fact reach the limit but never exceed it. In general, "to approach" is considered as an intuitive and incomplete process.

However, the term "to tend" has some particular characteristics different from "to approach", such as a subjective view of its definition (2 out 33) ("*To approach it as much as we want*") or a technical usage (8 out 33) ("*This term is used to indicate the value that x takes in a limit.*"), that is it is an agreement in mathematics.

Regarding "to reach", most subjects (27 out of 33) consider it simply as *to arrive at or to get to touch the limit*. Only 3 out of 33 considered that the limit must not be exceeded. On the other hand, only two subjects considered that the limit is reachable if we can calculate the exact value, while only one subject stated that "to reach" is to know the value of f(x) for a given x, so there could be a possible identification between the limit and the image.

"To exceed" is basically *to surpass the limit or the x-value given* (19 and 7 out of 33), although some subjects (4 out of 33) gave more complete answers, in the sense that a limit or a given x-value are exceeded *if the function reaches them and continues, both above and below them*.

At the beginning, students recognised not to know the term "to converge" in the context of a finite limit of a function at a point. In fact, 9 out of 33 did not answer this question, so the researcher had to encourage them to write whatever they could imagine about any other situation and to invent a definition. Only one subject defined this term as "to tend", and the most frequent meaning (6 out of 33) was *two functions intersect at one point*, and 5 out of 33 define "to converge" as *the right and left-hand limits are the same*, a definition that could be considered suitable in this context. On the other hand, several subjects described situations where the *function is all the time above or below the limit*, that is to say, an asymptotic behaviour of the function, for example, $f(x) = 1/x$ converges to 0 when x tends to $\pm\infty$.

Preliminary conclusions

According to the discussion of the results from Table 1 and the aim proposed at the beginning; to describe how students define explicitly some specific terms from calculus in contrast with a previous conceptual analysis of these terms, we draw the following conclusions about their achievement.

Students interpret the meaning of the selected terms in many different ways, most of them extracted from everyday situations, so we agree with Monaghan (1991) and Cornu (1991) that conflicts between colloquial and formal language are still occurring.

The review of the use of specific terms has predicted partially the meanings that students were going to provide, above all the colloquial meanings. The technical use of the term "to tend" had been conjectured by Fernández-Plaza (2011, p. 36). The observed difficulty some students had to distinguish between "to approach" and "to tend" is consistent with Monaghan (1991) and Blázquez, Gatica and Ortega (2009). All the new meanings of these terms should contribute to enrich this review in order to increase its explicative power.

At the beginning, the term "to converge" had been considered unknown by students in the context of finite limit of a function at a point, but they were able to invent a possible definition for the new context.

It is relevant that exceedability and reachability of the limit are especially connected to the students' conceptions of the terms "to approach" and "to tend" according to Fernández-Plaza (2011, p. 40).

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THE INFLUENCE OF HOW TEACHERS INTERACTIONALLY MANAGE MATHEMATICAL MISTAKES ON THE MATHEMATICS THAT STUDENTS EXPERIENCE

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Making mistakes is an essential aspect of learning mathematics, both for the teacher and for learners. Teachers are generally encouraged to use students' mistakes as teaching points particularly in order to expose common misconceptions. However, teachers use a wide variety of interactional techniques to avoid directly and explicitly negatively evaluating a student's mistake during whole class interactions. This gives the implicit message that mistakes are embarrassing and problematic (Seedhouse, 2001). So, whilst teachers may explicitly state that making mistakes is part of learning mathematics, interactionally they are saying that they are problems and are to be avoided.

Making mistakes is an integral part of learning mathematics and many recent initiatives have advocated teachers using mistakes in their teaching, particularly to expose frequent misconceptions and mistakes to avoid. However, because of the interactional structure of the classroom, teachers often avoid explicitly negatively evaluating students' mistakes or mitigate their negative evaluations in some way. This avoidance of explicit and direct negative evaluation portrays the interactional impression that mistakes are to be avoided. Hence, the pedagogical messages surrounding the role of mistakes in learning conflict with the interactional messages given by teachers when they handle these mistakes.

In this paper, the structure of sequences of interactions in which a mathematical mistake occurs are described in detail to reveal the conflict between the pedagogical and interactional messages.

METHODOLOGY

This study uses a Conversation Analysis (CA) approach, particularly drawing upon the CA literature and studies on the preference organisation of repair in interactions. Conversation analysis is an unusual methodological approach in the social sciences in that it is not driven by research questions or a priori analysis. Rather, it studies the social organisation of 'talk-in-interaction' (ten Have, 1990) with the aim to explicate the structures that the participants themselves orient to in the interaction. This paper focuses on describing and explicating the interactional strategies teachers and students use when a mathematical mistake occurs in the interaction.

Repair can be defined as the treatment of trouble in speaking, hearing, or understanding (Schegloff, Jefferson, & Sacks, 1977). Trouble can take the form of a mistake or error, but is also used more broadly to include any difficulties occurring in the interaction (Seedhouse, 1996). In this article we will only be considering the treatment of mathematical mistakes and errors. The repair of trouble consists of three parts: the trouble source; the initiation of the repair; and the outcome of the repair. The person in whose turn the trouble occurs and the person who initiates the repair and the person who performs the repair may or may not be the same person. For example, the trouble could occur in Jane's turn, John could initiate a repair on the trouble in Jane's turn, but it might be Peter who actually performs the repair.

A great deal of research has been conducted into the structure of repair in different interactional contexts, in particular the preference organisation of sequences of interactions that follow a trouble source. The notion of preference is complex and is not used consistently within the CA literature. The term preference as originally used by Harvey Sacks (1995), refers to the structural features of turn organisation in interactions rather than the psychological meaning of preferring one thing over another. However, this distinction is difficult to make. Most authors focus on describing the common features of preferred and dispreferred responses rather than defining preference explicitly. These features include the markedness of responses, the frequency of types of responses and issues relating to face (Sifianou, 2012). Preferred responses are often unmarked, occur frequently and are not face-threatening. Dispreferred responses, on the other hand, are often marked, given hesitantly or delayed in some way, are seen rarely in interaction and can be interpreted as face threatening acts (Schegloff, 2007). Bilmes (1988) argues that there is an association between these features of responses but does not accept these as defining features. He argues instead that they indicate a reluctance by the speaker to give the response. This leads to many authors discussing how speakers 'avoid' dispreferred responses (Levinson, 1983; Mey, 1993). Bilmes argues that a response that would be noticeably absent if not given could be considered as an alternative indicator of a preferred response.

McHoul (1990) examined the preference organisation of repair in geography lessons and showed that the number of other-initiated repairs was greater than self-initiated repairs. This contrasts with ordinary conversation where there the frequency of self-initiated self-repairs is far higher (Schegloff et al., 1977) i.e. the same person in whose turn the trouble occurs initiates and performs the repair. However, even though other-initiated repairs appear more frequently in classroom interaction, self-initiated repairs are still preferred structurally.

When considering preferred and dispreferred responses in this paper, a response given by a student is considered a preferred response if it is treated as such in the turn that follows. Thus a mathematical mistake could be treated as a preferred response if the teacher accepts it without hesitation, or marking his response in some way. Thus, when teachers use mistakes as teaching points, the occurrence of a particular mistake might be the preferred response. Equally a mathematically correct and appropriate response could be dispreferred if the turn that follows treats it as such. In the data from the study used in this paper, there is an example where the teacher has asked for the mean, median and mode of a set of data and a student correctly gives the median and the calculation made to find it, but the teacher treats it as dispreferred and continues to talk about the previous student's calculation of the mean.

A conversation analytic approach focuses on the contextual information that the participants themselves see as relevant. In the interactions presented here, the roles of teacher and student are oriented to in the interactions and the context of the mathematics classroom is clearly relevant. Other contextual information, such as details of the sample such as the gender of the teacher of the nature of the school are not oriented to in the data and thus are not included in this paper except to say that all the classrooms were in England. The extracts used in this article are taken from a study of seventeen mathematics lessons involving four teachers from four different schools and students aged between 12 and 14 years. The lessons were video recorded and are naturally occurring. These recording were then transcribed using the Jefferson transcription system (2004), but for ease of reading only some of the details of this transcription are included in this paper. This article focuses only on the structure of repair of students' turns that contain a source of trouble that relates to the mathematics.

HOW MISTAKES ARE HANDLED IN WHOLE-CLASS INTERACTIONS:

When students give a preferred response, the teacher often positively evaluates this response immediately and without any markers or hesitations, using terms such as 'good', 'yes', 'ok' and 'that's right'. In the two examples that follow, the students' responses are correct responses and are treated as preferred by the teacher. The example below is taken from the third lesson recorded with Edward's class.

1	Kieren:	it could be either because like th- the chance is even.
2	Edward:	okay good. …

Edward lesson 3.

Teachers also often repeat the preferred response, which can occur with or without a positive evaluation:

3	Richard:	… ok so um you said that micro means what?
4	Sasha:	um millionth
5	Richard:	a millionth very good. …

Richard lesson 1.

However, when the students' response contains a mathematical mistake, the teacher tends to avoid direct and explicit negative evaluation. It is very rare to see an immediate 'no' to a student's response. Teachers generally use a wide variety of methods to avoid an unmitigated explicit negative evaluation, they are "doing interactional work specifically in order to avoid using unmitigated negative evaluation" (Seedhouse, 2001, p. 355). In other words, there is a dispreference for direct and explicit negative evaluations of students' mistakes.

Bald negative evaluations

In this study, there were only two examples where a mathematical mistake was treated as a preferred response in the turn that follows. These both occur in Tim's lessons:

21	Tim:	… if I did that in one operation, if I did that in one swift move, what am I actually doing
22	Nic:	dividing by five
23	Tim:	no

Tim lesson 4.

In the situation above, Tim has asked what dividing by 3 and then dividing by 2 would be if you did it as one operation. The task that the students have been working for the remainder of the interaction focuses on limits of sequences, and in this case what happens if you continue to divide by 3 and then divide by 2. The mathematics in this exchange was not central to the task that the students have been working on for the remainder of the interaction, which focuses on limits of sequences. The other occurrence is during an interaction about the probability of picking a particular cup out of a row of ten cups and Tim asks questions about probabilities related to the rolling of two dice. In neither of these exchanges, when incorrect answers resulted in unmarked negative evaluations in the teacher's responses, were the questions directly related to the overarching task. In other words, the mistakes made were not directly relevant to the learning objectives of the tasks within which they occurred.

In general, teachers use a wide variety of interactional strategies to avoid bald negative evaluations of students' turns. These strategies that teachers use to avoid negatively evaluating a student's mathematical mistake include:

Delaying or mitigating an evaluation

10	Sandy:	it's got a plus sign, you've got to like (.) take it off, but if it's like a minus you like (.) add it on
11	Edward:	(0.6) not quite, what is it about this that means that you have to take it away

Edward lesson 1.

In the extract from Edward's lesson there is a negative evaluations but it is delayed as it follows a pause of 0.6 seconds, and is not a bald 'no' but a mitigating 'not quite'. The initiation of repairs can take place over a series of turns, with the teacher offering multiple opportunities for a student to self-repair.

Initiating a repair

6	Tim:	... what fraction of that triangle have I actually shaded. Alex?
7	Alex:	um a half.
8	Tim:	have I shaded a half?
9	Alex:	no

Tim lesson 1.

In this extract from Tim's lesson, Alex has given the answer of a half which is not the answer Tim is looking for and Tim initiates a repair by rephrasing Alex's response as a question and returning the turn to Alex. There is no explicit negative evaluation of Alex's response and Alex is given the opportunity to repair (or alter) his response. By initiating a repair of the student's turn no explicit evaluation of the turn is made but it does indicate that there is some form of trouble with the student's turn and it offers the student the opportunity to self-repair.

There are many different strategies a teacher could use to initiate a repair, including repeating the question, or repeating the students' response (part or whole) but phrasing it as a question as in the extract above. Each of these does not explicitly evaluate the student's turn and offers the student the opportunity to self-repair, treating the student's turn as dispreferred. Many of these strategies can also result in other-repairs, where another student or the teacher themself provides the correct answer.

12	Drew:	is the range a hundred and seven- seventeen
13	Simon:	range a hundred and seventeen. the range is the biggest number take away the smallest number. the biggest number is a hundred and twenty five, the smallest number is eight, a hundred and twenty five take away eight. Ashley.
14	Ashley:	no because the the range is going to be in days absent so it'll be eight

Simon lesson 1.

The teacher could also offer an explanation of why the student has made a mistake without explicitly stating that their response included a mistake.

15	Richard:	… what names would you give to these things that are in the squares. I'm looking for the technical term for them. Chris?
16	Chris:	formulas
17	Richard:	formulas, you could say. formula often has equals in it, doesn't it like the formula for speed equals distance divided by time…

Richard lesson 4.

In the extract from Richard's lesson, Richard accepts the response but explains why it is not the response he was looking for before repeating the question.

The teacher could also accept the response given by the student but in revoicing it (O'Connor & Michaels, 1993) supply the correct answer:

18	Simon:	… quarter of a hundred and nine?
19	George:	twenty seven. point
20	Simon:	twenty seven point two five. twenty seven point two five going across. …

Simon lesson 4.

Questioning Strategies

Teachers have a wide variety of other strategies that they can use to handle mathematical mistakes whilst avoiding explicit negative evaluation. The teacher could, for example, begin a sequence of simpler questions that enable the students to be ‘led’ to the appropriate answer, a technique often referred to as ‘funneling’ (Wood, 1998). This does not mean to say that teachers do not negatively evaluate students’ responses, just that when they do they are usually delayed or mitigated in some way, indicating that they the previous turn was dispreferred. Direct, explicit and immediate negative evaluations are very rare.

THE ROLE OF MAKING MISTAKES IN LEARNING MATHEMATICS:

Mistakes and misconceptions have come to be seen as essential in the learning of mathematics. This is both from the perspective of the teacher and the learner. From the teacher’s perspective, they can be “a powerful tool to diagnose learning difficulties” (Borasi, 1987, p. 2) but can also give insight into how learners are thinking about the mathematics. This is particularly important where learners’ conceptions of mathematical ideas differ from those “generally accepted versions of the same ideas” (Sfard, 2008, p. 16).

It is through making mistakes or considering different conceptions that a great deal of mathematics has been developed (Tall, 1990). Burton’s (2004) description of professional mathematical behaviour includes making mistakes and using them to learn new things. Borasi (1987) offers examples of how mistakes and errors can be used within a mathematics lesson to initiate enquiry and enhance learning.

Traditionally mistakes and misconceptions were seen as something to avoid in the mathematics classroom, but Piaget’s influence and the influence of constructivist theories of learning began to challenge this and studies, such as those by Askew and Wiliam (1995) and Swan (2001) showed greater learning gains when mistakes and misconceptions were confronted and discussed than when they were avoided. From the perspective of the learner, making mistakes or confronting misconceptions and adapting understanding in light of these is fundamental to constructivist principals of learning.

Many authors emphasise the role of the classroom environment in the handling of students’ mistakes. The emphasis being placed on creating an environment where students feel safe to make errors and it is a classroom norm to discuss and explore errors (Schleppenbach, Flevares, Sims, & Perry, 2007). Several studies have introduced or developed rules for interaction to establish an environment where learners feel safe making mistakes and discussing them (Mercer & Sams, 2006; Ryan & Williams, 2001). However, it is important to note that student errors do not necessarily mean a lack of knowledge or understanding (Mehan, 1980).

O’Connor (2001) explores the decisions teachers need to make about whether they correct a mistake or not and raised some interesting questions about the role of the purpose of the activity and the role of the mistake within the interaction. This article is not concerned with these conscious decisions that teachers must make but instead looks at how the interactional norms of conversation and classroom discourse may influence how students may interpret the teacher's actions, both when a mistake is corrected and when it is ignored.

The conflict

By treating mistakes as dispreferred responses to questions, teachers are interactionally telling students that making mistakes is embarrassing and face-threatening and to be avoided. There is a conflict between the pedagogical message that making mistakes is an essential part of learning mathematics and the interactional message that mistakes are to be avoided (Seedhouse, 2001).

"Teachers are avoiding direct and overt negative evaluation of learners' ...errors with the best intentions in the world, namely to avoid embarrassing and demotivating them. However, in doing so, they are interactionally marking ... errors as embarrassing and problematic." (Seedhouse, 2001, pp. 368-369)

So, whilst teachers are not explicitly telling students that making mistakes or errors is embarrassing and problematic, the way that they handle these in whole class interactions implicitly tells the students that they are. The very responses such as partial agreement or mitigation, that may be intended to allow incorrect answers to contribute to learning, are also the ones that give the interactional message that these errors are to be avoided.

Conclusion

The purpose of this paper was to describe the interactional structure of repair in mathematics classrooms when mathematical mistakes are made. This has revealed a conflict between the interactional messages which treat mistakes as something to be avoided and the pedagogical message that mistakes are an essential part of learning. Teachers use a wide range of interactional strategies to avoid baldly negatively evaluating a student's turn, treating mistakes as dispreferred responses, and hence a source of trouble. However, this paper does not suggest that teachers should baldly negatively evaluate students' mistakes more frequently. Further research is needed to explore the influence of varying these interactional and pedagogical messages may have on the learning of mathematics. Indeed, the implications of raising teachers' awareness of this conflict needs exploration. It may be that this conflict is part of the norms of classroom interaction in the same way that teachers asking questions to which they already know the answer to is. The influence this conflict has on student and teacher interactional behaviour still needs to be explored.

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WRITER IDENTITY AS AN ANALYTICAL TOOL TO EXPLORE STUDENTS' MATHEMATICAL WRITING

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Learning to communicate in, with and about mathematics is a key part of learning mathematics (Niss & Højgaard, 2011). Therefore, understanding how students' mathematical writing is shaped is important to mathematics education research. In this paper the notion of 'writer identity' (Burgess & Ivanič, 2010; Ivanič, 1998) is introduced and operationalised in relation to students' mathematical writing and through a case study it is illustrated how this analytic tool can facilitate valuable insights when exploring students' mathematical writing.

Keywords: Identity, mathematical writing, analytical tools.

INTRODUCTION

In recent years the notion of *identity* has received increased attention in the field of mathematics education research. Despite of this Steentoft and Valero conclude that "identity is still only emerging in mathematics education research and is far from explored in full." (Steentoft & Valero, 2009, p. 76). As argued by Sfard and Prusak (2005) one of the key challenges in this endeavour is to provide operational definitions of the notion of identity itself.

How writers construct their identities in texts has been the focal point of several studies in the field of writing research. In fact in his overview of writing research Hyland (2009) notes this as one of the key issues to be addressed in writing research and point to the works of Ivanič (1998) as a major contribution in this context.

In this paper I will argue that Ivanič's notion of *writer identity* (Burgess & Ivanič, 2010; Ivanič, 1998) can in fact provide a useful way of operationalizing the notion of identity in relation to students' mathematical writing. In order to do so, I will initially introduce the notion of writer identity as outlined by Ivanič and elaborate on how this can be operationalised. Following this I will demonstrate the usefulness of these analytical tools by applying them to two different mathematical texts written by the same student. The center of attention will be how, and why, the student changes his way of representing himself as a writer of mathematical writing.

To sum up, the two research questions leading this paper will be: (1) *how can the notion of identity be operationalised in relation to students' mathematical writing, and (2) what kinds of insights can an identity perspective on students' mathematical writing provide?*[1]

THEORETICAL FRAMEWORK – THE NOTION OF WRITER IDENTITY

Ivanič distinguishes between four interrelated aspects or meanings of *writer identity* and characterizes these as "ways of thinking of a person's identity in the act of writing" (Ivanič, 1998, p. 23). As such the notion of writer identity is a *social* notion, which concerns the many ways in which writers position themselves through their use of semiotic resources.

Discoursal self. This aspect of writer identity is concerned with the impression that writers convey of themselves in a particular text. Writing mathematical texts is not just about getting the calculations right. A written text, mathematical or not, also leaves the reader with an impression of who the author is or perhaps wants to be. The discoursal self is constructed by the text characteristics of the particular text, but is closely related to the values, beliefs and power relations embedded in the discourses[2] that are present of the social context of the writers.

Authorial self or *self as author* as Ivanič prefers to call it, is the way writers appear as authors in particular texts. This involves in which ways, and to what extent, writers attribute the choice of content and form of the texts to themselves or to other authorities. Some writers of mathematics present the content of their writing as platonic truths while other writers present it as being the work of their own. As both the discoursal self and the authorial self are identities that are inscribed in particular texts they can both be analyzed using text analysis.

Autobiographical self. The third aspect focuses on the personal stories the writers bring with them to the act of writing. This involves norms, values and beliefs related to the writing of mathematical texts and as such this aspect will be shaped by the writers' previous encounters with writing events that involved mathematics. Obviously this aspect of writer identity cannot be disclosed through analysis of students' written texts alone, but can instead be further explored through interviews with the students.

Possibilities of selfhood. In any social context *possibilities of selfhood*, or *subject positions,* will be available to the writers in the act of writing. Students' mathematical writing in most cases takes place in the institutional context of school and is as such embedded in the various possible discourses of school mathematics. These discourses in turn shape the possibilities of self-hood available to the writers. This aspect of writer identity can be explored through classroom observations, studies of institutional texts such as curricula and by interviewing both students and teachers.

METHODOLOGY

Underlying the notion of writer identity is an understanding of writing as a *social act*. This fundamental assumption has methodological consequences both on the level of research design and for the use of analytical tools. In both cases the goal is to minimize the gap between text and social context by using context sensitive approaches when exploring students' mathematical writing. On the level of research design Lillis (2008) argues that this can be done by adopting ethnography as a research methodology. The two texts analyzed below are drawn from a one year long ethnographic longitudinal study that explored how upper secondary school students develop writing competences in the subject of mathematics.

Analytical tools for text analysis

Understanding students' mathematical writing as a social act has in this case led to a text analysis based on *systemic functional linguistics* (e.g. Halliday, 1978). A significant contribution as to how such analytic tools can be applied to mathematical texts is developed by Burton and Morgan (2000), Morgan (1998, 2006) and O'Halloran (2005). As noted by Burton and Morgan (2000, p. 430):

> Although sometimes seen to be peripheral to the main mathematical content natural language serves in the construction of the identities of the author and reader and of the epistemological and ontological assumptions underlying the writing.

Against this backdrop I will in this paper restrict the focus of the text analysis to the *mathematical language* of students' texts thus leaving out the *visual mediators* (Sfard, 2008) from the analysis although they too can contribute to identity construction in written mathematical texts.

Textual characteristics of the discoursal self

Characterizing writers' discoursal selves involves analyzing how the writers represent, or portrays, themselves in their mathematical texts. According to Hyland "[a]n example is the extent to which a writer takes on the practices of the community he or she is writing for, adopting its conventions to claim membership" (Hyland, 2009, p. 73).

An integral part of this would be to explore in what ways and to what extent writers make *use of mathematical vocabulary and conventional forms of mathematical language* (e.g. use of characteristic grammatical structures) (Morgan, 1998, p. 97), when we analyze how students try to claim membership to some sort of mathematical community in their texts. Another important aspect of how writers construct their discoursal self concerns their use of *personal pronouns* (Burton & Morgan, 2000; Rowland, 1999). Moreover, writers can direct the attention to themselves in their mathematical texts by *referring to their own actions and cognitive processes*. As with the use of the personal pronoun *I* this text feature can, I suggest, in a school setting be understood as students' way to accommodate to a demand of *showing their own understanding* of the mathematical issues or problems elaborated in the texts.

Textual characteristics of the authorial self

One way that writers can convey their authority is by using words or phrases that signal different kinds of *ownership*. As noted by Burton and Morgan (2000) this can be studied in mathematical texts by attending to the use of *modality*, which includes the use of *adverbs* (e.g. *almost, always, certainly, clearly, easily, nearly, potentially,* *possible*), *adjectives* (e.g. '*the derivative is easily calculated in this case.*') or *modal auxiliary adverbs* (e.g. *must, can, may, could*). Another, perhaps more subtle, way of expressing authority is by making it clear what is *not* in the particular text. Purposefully *omitting* more or less obvious parts of calculations or symbolic manipulations, and perhaps even underlining that this is the case, is a well known way of claiming authority in the field af mathematics. As the authorial self is closely connected to the writers' willingness to get behind claims or arguments that are put forward in the text, again another way of signaling authority in a text can be to explicitly express different kind of *choices* that has been made by the author. This could, for example, include choices regarding the mathematical content or style of the text (Hyland, 2009).

THE CASE OF CHRISTOPHER

Christopher is a student at a Higher Technical School (htx), which is one of the four upper secondary school programs in Denmark (grade 11-13). A review of the mathematical texts Christopher has produced as a student at htx shows that he has gradually found a way to construct his identity as a writer of mathematics that is quite *stable* with regard to how he represents himself in his texts. The extract of text shown below in Figure 1 is chosen because it illustrates how he typically does this[3].

The different assignments that Christopher works with during his time as a htx-student can roughly be divided into three categories: (1) *Home assignments* that typically consisted of a collection of separated word problems, (2) *Project assignments* that usually were constructed around one guiding problem, and (3) *Presentation assignments* that were tasks where the primary goal was to present some sort of mathematical topic or idea. The two text extracts below is taken from Christopher's responses to a home assignment (Figure 1) and a presentation assignment (Figure 2) respectively. In both cases the text was produced at the end of the school year and had to be handed in to the teacher who evaluated it afterwards

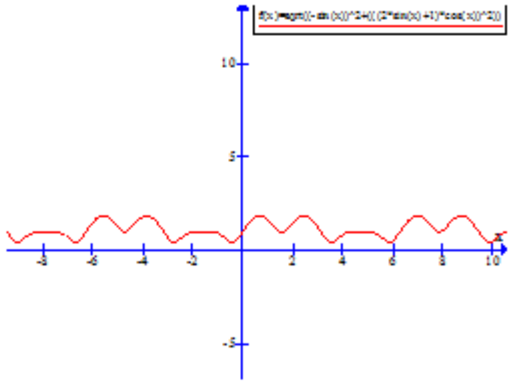
In several interviews I talked to Christopher about his perception of the different kinds of assignments in order to get some insight in to his *autobiographical self*. Home assignments are for Christopher associated with *training* and *standardization* and he describes these as *controlled* by the teacher. Regarding project assignments the converse seems to be the case. Christopher characterizes these with expressions like *independence* and *a possibility of putting the mathematics into a larger perspective*. In both cases the relations between type of assignment and the associated discourse seems locked for Christopher. This might be an important part of the explanation as to why he had continuously constructed similar types of discoursal and authorial selves in his mathematical texts in the last year of htx.

The *home assignment* consisted of a set of word problems or tasks, which had been posed at a written national examination in mathematics some years before. In the extract we see Christopher's answer to one of the tasks. The purpose was to determine the maximum speed of a moving particle whose motion is described by a vector-valued function. When we enter Christopher's text he has just found the derivate of the vector-valued function – the velocity vector. Notice how Christopher is clearly present as an acting agent in his own text.

Now where I have to find the maximum speed, I should start by calculating the length of my velocity vector, because the speed is defined like this.

$$|v(t)| = \sqrt{x'(t)^2 + y'(t)^2} \Leftrightarrow |v(t)| = \sqrt{(-\sin(t))^2 + ((2 \cdot \sin(t) + 1) \cdot \cos(t))^2}$$

By inserting it to Graph[1], I can get an overview of the various possible vertices.



This doesn't look all wrong, because I can imagine that the two big 'sharp' turns that appears in the first graph is what gives the two turns on this graph. The spaces between the two major vertices correspond to the major loop before the turn's returns on the original graph. By taking the derivative of the speed function I can find out where the vertices can be found.

$$|v'(t)| = \sqrt{(-\sin(t))^2 + ((2 \cdot \sin(t) + 1) \cdot \cos(t))^2} = \frac{2 \cdot \cos(t) \cdot ((2 \cdot \sin(t) + 1) \cdot (\cos(t))^2 - 2 \cdot (\sin(t))^2 \cdot (\sin(t) + 1))}{\sqrt{(4 \cdot (\sin(t))^2 + 4 \cdot \sin(t) + 1) \cdot (\cos(t))^2 + (\sin(t))^2}}$$

As this should be taking the derivative equal to zero, then the numerator has to be 0 before the whole fraction can and therefore I can just solve t in the numerator.

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The equation I solve is this:

$$2 \cdot \cos(t) \cdot ((2 \cdot \sin(t) + 1) \cdot (\cos(t))^2 - 2 \cdot (\sin(t))^2 \cdot (\sin(t) + 1)) = 0$$

--

I get 5 solutions, which indicates for which value of t there is a turning point:

$$t = 0{,}695 \quad t = \frac{\pi}{2} \quad t = 2{,}44 \quad t = \frac{3\pi}{2} \quad t = 5.88$$

-- By evaluating my graph I found out that t=0,695 and t=2,44 both were vertices. Those values have the same kind of vertices, and therefore, I can just insert one of the values in my speed function:

$$|v_{max}(t)| = \sqrt{(-\sin(0.695))^2 + ((2 \cdot \sin(0{,}695) + 1) \cdot \cos(0{,}695))^2} = \underline{\underline{1{,}865}}$$

Figure 1: Extract from Christopher's written answer to a home assignment

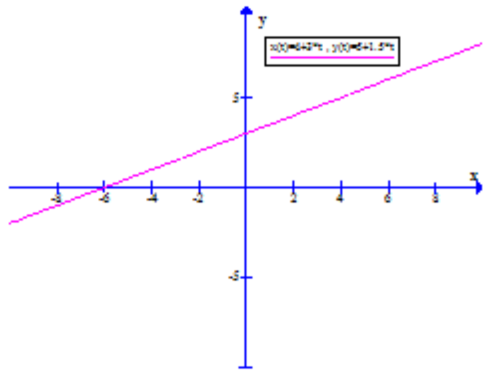
In the *presentation assignment* each of the students was assigned a specific mathematical topic, and their job was to present it in the best way possible to the other students. Christopher was assigned the topic *vector-valued functions*, and the text shown in Figure 2 is taken from a paragraph where he writes about *the straight* *line*. As we enter Christopher's text he has just sketched a graph of $f(x)=\frac{1}{2}x+3$ in a coordinate system.

1 As an example, we look at the line drawn earlier on. The line is passing through two
2 points A=(1;3,5) and B=(4;5). These points are used to describe a direction vector:

3 $\vec{r}=\vec{O}B-\vec{O}A \Leftrightarrow \vec{r}=\begin{pmatrix}4\\5\end{pmatrix}-\begin{pmatrix}1\\3,5\end{pmatrix}=\begin{pmatrix}3\\1,5\end{pmatrix}$

4 Using the point B as P_0 we can write the vector function:

5 $\vec{O}P(t)=\begin{pmatrix}x(t)\\y(t)\end{pmatrix}=\begin{pmatrix}4\\5\end{pmatrix}+\begin{pmatrix}3\\1,5\end{pmatrix}\cdot t=\underline{\underline{\begin{pmatrix}4+3t\\5+1,5t\end{pmatrix}}}$



6

7 If we want our parametric equation for the line to look like the original function for a
8 straight line y=ax+b, then we have to eliminate the parameter t.

9 Here, the prior example is used as a basis:

10 $f(t)=\begin{pmatrix}4+3t\\5+1,5t\end{pmatrix}$

11 To eliminate t, we isolate t in either the x-part or the y-part:

12

13 $x=4+3t \Leftrightarrow t=\dfrac{x-4}{3}$

14 This is then inserted to the y-part:

15 $y=5+1,5t \Leftrightarrow y=5+1,5\cdot\left(\dfrac{x-4}{3}\right)=\underline{\underline{0,5x+3}}$

Figure 2: Extract from Christopher's written answer to a presentation assignment

Construction of identities in the two texts

Due to limited space, the analysis will focus only on the most significant *differences* between Christopher's two texts. These are displayed in Table 1. Numbers in brackets refer to the line numbers in Figure 1 and 2 respectively.

Textual characteristics	**Home assignment text (Figure 1)**	**Presentation assignment text (Figure 2)**
Use of personal pronouns	Example – use of ‘I’ and ‘my’: ’*The equation I solve is this:*’ (13) ‘*… my velocity vector…*’(1-2)	Example – use of ‘we’ ‘*As an example, we look at the line drawn earlier on*.’ (1)
Reference to own actions and cognitive processes	Example – actions ‘*By evaluating my graph I found out that t=0,695 and t=2,44 both where vertices*’ (17) Example – cognitive processes ‘*This doesn’t look all wrong, because I can imagine that…*’ (6)	No explicit reference to own actions and/or cognitive processes
Signals of authority	Example – *omitting and modality* ‘*... therefore I can just solve t in the numerator.*’ (12)	No direct use of words and phrases that signals authority and/or use of modality
Use of conventional forms of mathematical language	Example – *‘spoken’* language: ‘*As this should be taking the derivative equal to zero, then the numerator has to be 0 before the whole fraction can and therefore I can just solve t in the numerator.*’ (11-12)	Example -‘*textbook’* language ‘*If we want our parametric equation for the line to look like the original function for a straight line y=ax+b, then we have to eliminate the parameter t*’ (7-8)

Table 1: Summary of significant differences between the two texts

So what kind of identities does Christopher seem to construct in his two texts? In the first text the discoursal and authorial selves are characterized by an explicit presence of Christopher appearing as an active and visible agent in his own text. It is Christopher himself as a person who appears as the source of the mathematical actions performed and the new knowledge presented. Christopher seems to construct an identity as *a writer who wants to make clear to his reader: (1) how he himself as a person is able to solve the task he is given, and (2) what he thinks while doing this.* It is a very different discoursal and authorial self we encounter in Christopher’s second text. His own clear presence as a person in the text is gone, and instead an identity as a *neutral mathematical communicator* using the voice of a mathematical textbook is constructed.

One of the most remarkable differences in the way Christopher constructs his identity in the two texts is the alternating use of personal pronouns. In an interview at the end of the school year I asked Christopher to comment on this difference between the two texts.

1	Christopher:	I believe it is when you are thinking that you are writing some mathematics to somebody else who understands mathematics, then you are kind of saying 'Now we do this' or … but sometimes I also say 'I' and I cannot really explain why. But in this case, I think I can …
2	Interviewer:	With these notes [*the presentation assignment text* [4]]?
3	Christopher:	Yes, because now it's not for <u>one</u> person, that I have to imagine that this is for, but generally for somebody who doesn't understand it, and then it might sound a bit stupid to say: 'I'm doing this' because it's not me, who does it. It's … generally you can do it like this. That's also why I wrote 'a vector-valued function is described' instead of 'I describe a vector-valued function'. It is, I guess, not me who has found it or … and sometimes I also just think that 'we' is just … It's just a better way of writing it than using 'I'. I think it sounds a little … In the home assignments … there I know that it's me who makes it, I mean calculates this stuff. So there I think it fits better than as for example in this paper, even though it is me who has done it. But that's not what's important for the purpose of this text.

In the interview Christopher gives three different reasons for his replacement of the personal and for him habitual, *I* with the more neutral *we*.

Community with the reader: The use of *we* indicates that both writer and reader are part of the same community, and Christopher does in fact himself point to the fact that an equal relation with the reader ('*... somebody else who understands mathematics…*') give rise to the use of *we* in his texts. At the same time he is aware that in other texts, which are only read by his teacher, he most often uses *I* and not *we* in his wording. This seems to leave him doubtful ('*...I cannot really explain why.*').

The source of knowledge: The use of *we* instead of *I* contributes to a linguistic move of responsibility away from the writer as a person, and this seems crucial to Christopher in this case. He recognizes, of course, that he is the author of both texts, but underline that in the case of the text on vector-valued functions this is not essential in relation to the purpose of the text ('*...it is me who has done it. But that's not what's important for the purpose of this text*').

Sounding in the right way: The third reason is affective in nature. Christopher states that '*It's just a better way of writing it than using 'I'.*' and continues '*I think it sounds a little ...*' referring to the use of *I*. What is meant by '*just a better way'* is of course not transparent, but it seems plausible that Christopher in this case, consciously or unconsciously, identifies with some sort of mathematical community, where the use of *we* would be regarded as more appropriate than the use of *I*.

CONCLUSIONS

In this paper one way of operationalizing the notion of identity in relation to students' mathematical writing has been presented. When producing mathematical texts students need to be able to decode and navigate in the purposes and possibilities of selfhood associated with writing in the mathematical classroom, and to construct identities that are perceived as *appropriate* in the given context. As pointed out by Morgan (2006, p. 239):

> While establishing appropriate identities is of importance to participants in any situation, it is of critical importance to students at all levels whose oral and written productions are to be assessed.

Different types of writing assignments offer different possibilities of selfhood to the students in the mathematical classroom and Christopher in the presented case reacts by constructing different kinds of identities in his two texts. For him *purpose* of the writing assignment and *audience* of the text seems to have a specific influence on the way he construct his identity as a mathematical writer.

In this way the case illustrates how varying the writing assignments of the mathematical classroom hold a potentially strong possibility of forming students' ability to develop their style of writing in mathematics. The analysis show that an identity perspective on students' mathematical writing can provide valuable insights about, what students want to achieve with their writing, how they understand the role and function of writing in mathematics education, and how this understanding can shape the way they express themselves in their mathematical texts. Such insights can be an important step towards providing better opportunities for students to develop linguistic and communicative competences in mathematics.

NOTES

1. The study presented in this paper is part of the research project *Writing to learn, learning to write – Literacy and disciplinarity in Danish upper secondary education* supported by the Danish Agency for Science, Technology, and Innovation. For additional information please go to www.sdu.dk/fos.

2. Throughout the paper I use the notion of discourse as Gee (1996, p. 131) defines *Discourse* (with a capital *D*).

3. In both text extracts the original layout has been changed slightly. Christopher inserts a line break when he uses the sign for material equivalence $\Leftrightarrow$. For the sake of saving space I have removed these line breaks simply letting the calculations continue horizontally. I do not analyze the layout of Christopher's texts in this paper.

4. My insertion in square brackets.

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DESIGN AND VALIDATION OF A TOOL FOR THE ANALYSIS OF WHOLE GROUP DISCUSSIONS IN THE MATHEMATICS CLASSROOM

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We present an analytical tool to characterise whole group discussions in mathematics classrooms. Following an exploratory study with 8th grade students, our interest lies in identifying and characterising interactional episodes and certain actions that support them. It has been confirmed (Morera & Fortuny, 2012) that the tool helps to detect what we call Mathematical Learning Opportunities –MLO. It shows not only what participants are doing in the course of their interaction, but also which the potential learning contents are. From a methodological point of view, the tool appears to operate in different didactical settings. We provide an example of its use in a teaching and learning setting of mathematical work mediated by a DGS instrument.

INTRODUCTION

Learning can be viewed as one of the ultimate reasons behind the actions of the students and the teacher in the mathematics classroom. In the context of whole group discussion, it is important to identify the various actions that have a role in framing the students' learning. Morera and Fortuny (2012) have argued that the identification of these actions and its development in whole group may serve to reach evidence of learning. While existing research has placed much emphasis on the presentation of pedagogies and practices that teachers can learn in order to improve the didactical effects of classroom conversations (Stein & Smith, 2011), developing systematic research on the potential of whole group discussions is a relatively new endeavour. Further investigation is needed to carry out an in-depth study of the nature of such discussions from an analytical perspective.

The goal of our research is to create and apply an analytical tool to identify and characterise: a) episodes of whole group discussion involving mathematical work, b) actions by the participants in these episodes, and c) opportunities in terms of potential contributions to the students' mathematical learning. This triple examination of episodes, actions and opportunities leads to a more global characterisation of whole group discussions in the mathematics classroom, under the assumption that whole group promotes episodes of mathematical learning. We are aware that the underlying notion of Mathematical Learning Opportunities –MLO– is itself critical in that one can imagine almost any situation to be a potential scenario for learning. What is scientifically useful for us is to view the sets of learning opportunities as an initial step toward the search for evidences of effective mathematical learning on the part of the students. This report provides an overview of the process that has been followed to create the tool, including examples with classroom data, and a summary of the current refinement being developed around methodological considerations.

Context of the research

For the research, ten class sessions of an experienced grade 8 teacher were chosen. A sequence of five inquiry problems was designed to study isometries in a collaborative way using DGS. After the students had solved a problem in pairs, the teacher led a 50-minute whole group discussion. DGS was involved in the students working in pairs and the whole group interaction. During the whole group discussions, the teacher and the students were observed and video-taped with three video cameras and several additional voice recorders to capture all actions in detail. The attention was focused on the interactions between the participants and the software as well as on all different series of significant actions produced by both the teacher and the students.

After data collection, qualitative data analysis software was used to organise and codify the whole group videos. The three authors worked together on carrying out the codification. The codification of videos was primarily expected to provide a general picture of the interaction. Since a very early stage of the research, the purpose was to create a well-established analytical tool to better understand how interactions between actions, which are the main components of an episode, support MLO.

EPISODES, ACTIONS, AND MLOs

A theoretical approach has been developed to help determine the relationships between the *actions* of an *episode* that support *mathematical learning opportunities*. In this section we explore the nature of these key constructs, and see them as highly interrelated. On the one hand, when a MLO occurs in an episode, it is appropriate to ask what actions are behind the creation of such opportunity. On the other hand, when certain actions take place in an episode, it is also appropriate to look for the possibility of emerging MLOs. In the interactional and critical approaches to mathematical learning (Planas & Civil, 2010), one of the main focuses is to understand how sequences of actions are to be enacted in ways that promote a sort of correspondence between actions and learning. Although each action is singular in that it is mainly enacted by one participant, its implications do not refer to concrete individuals but rather involve the different participants having a role in the discussion.

Identifying and characterising episodes

It is part of the tradition of interactional theories to take episodes as organised units of data. When deciding what to take into account in the identification of classroom episodes, the organisation of units of data was framed by criteria of topic cohesion. In the context of a whole group discussion, we focus on those episodes most likely to influence the students' learning by promoting qualitative shifts in the mathematical thinking over the course of collective participation around a concrete curricular topic.

The qualitative shifts were considered in the sense developed by Saxe and colleagues (2009) when talking about “the travel of ideas” in the classroom.

To characterise the selected episodes of the whole group discussion orchestrated by the teacher, we take the instrumental and the discursive dimensions. Concerning the instrumental dimension, we draw on the types of instrumental orchestration by Drijvers and colleagues (2010): *Technical-demo*, which refers to the demonstration of artefact techniques by the teacher; *Explain-the-screen*, which refers to whole-class explanation by the teacher, guided by what happens on the computer screen; *Link-screen-board*, where the teacher emphasises the relationship between what happens in the technological environment and how this is represented in conventional mathematics of paper, book and blackboard; *Discuss the-screen*, which involves a whole-class discussion about what happens on the computer screen; *Spot-and-show*, where student reasoning is brought to the fore through the identification of interesting student work during preparation of the lesson, and its deliberate use in a classroom discussion, and *Sherpa-at-work*, where a so-called Sherpa-student uses the technology to present their work, or to carry out actions requested by the teacher. We adapt this typology to broadly classify the nature of the episodes according to the didactical performance of a discussion in which the artefact is not necessarily given by a DGS instrument. Our types are the following: *Explaining the artefact*, *Explaining through the artefact*, *Linking artefacts*, *Discussing the artefact*, *Discovering through the artefact*, and *Experiencing the instrument*. The first three types are dominated by the teacher’s actions, while the last three types are dominated by the students.

Concerning the discursive dimension in the characterisation of episodes, we draw on types that have been grounded throughout the evolution of the process of our research. After having examined all the mathematical concepts and procedures in all the selected episodes (see Morera, Fortuny & Planas, 2012, for details of the discursive dimension of a particular episode), an effort was made to search for common patterns that might help to understand a generic development of the episodes and the particularities that are shared by most of them. Although any episode can be distinguished from other episodes of the same lesson, the search for discursive commonalities among them has allowed determining a linear sequence of types for general orientation: *Situation of the problem*, *Presentation of one solution*, *Examination of resolution strategies*, *Examination of particular and/or extreme cases*, *Consideration of cases*, *Examination of different solutions*, *Connections*, and *Generalization*. This linear pattern refers to commonalities arisen from episodes with a specific kind of mathematical problems that foster both particularization and generalization. A different kind of problems would have probably led to a different discursive characterisation. Nevertheless, the sequence above is expected to represent whole group discussion around a set of mathematical tasks that goes beyond the concrete geometrical problems on isometries in our study. As it will be shown later in this report, the characterisation of an episode by means of a series of discursive and instrumental types gives a complex picture of the interaction among participants, along with the contents and the direction of the interaction.

Identifying and characterising actions from the episodes

Episodes have been taken to be instrumental and discursive units of data. But if we go deeper inside their structure, we need to add a complementary characterisation based on the actions that constitute the episodes. In summary, we have first developed a major characterisation of episodes by means of instrumental and discursive types, and then have looked at them in terms of the actions that frame the two types.

To search for significant actions that include the participants' interventions and the instrumented acts related to the use of a particular artefact, we draw on the mediational approach to the teaching and learning of mathematics by Mariotti (2012). The emphasis on the relationships between humans and artefacts was initially fostered by our use of DGS environments, but has now turned to be generalised to any context of teaching and learning. We agree that knowledge is socially constructed by subjects involving the media because the participants collaborate to re-organise thinking with a different role than that assumed by written or oral language. The relation between different significant actions in an episode may provide opportunities to greatly enhance students' learning. Thus, we classify the significant actions within episodes as follows: Students' actions considered as *Thinking-Math Interventions*; Teacher's actions, considered as *Didactical Interventions*; and *Instrumented Actions* performed by the participants and centred on their use of artefacts. While the Thinking-Math and the Didactical Interventions are rather located in the discursive dimension, the Instrumented Actions are better thought of as related to the instrumental dimension. The relationship between the instrumental types (from the major characterisation of episodes) and the Instrumented Actions is highly complex and needs to be explored through the representation of the analytical tool. Similarly, the relationship between the discursive types (from the major characterisation of episodes) and the Thinking-Math and Didactical Interventions are also part of what is expected to be illustrated through the representation of the analytical tool.

Identifying and characterising MLOs

The identification and characterisation of episodes and actions is followed by the identification and characterisation of learning opportunities. We assume that the teaching and learning of mathematics take place in settings in which an important variation in the amount of MLOs exists. Moreover, we assume that such variation has an influence on the achievement of individual students depending on the quantity of MLOs in the specific settings of mathematical practice, together with the social conditions that qualitatively mediate such quantity. By either promoting or reducing the amount of MLOs, a change is made in the students' actual learning experiences. This is part of our argument for organising the search for mathematical learning in relation to the search for learning opportunities.

To make the notion of MLO operational, we take it on the form of opportunities for participation in a classroom discourse in which certain actions are oriented toward the discussion of specific *Mathematical Contents*, *Thinking Strategies*, and/or *Self-Regulating Activities*. In our study, actions by individuals are interpreted as potential contributors to the mathematical learning in groups. Through the three types of actions around 'contents', 'strategies' and 'activities', students develop their knowledge in interaction with other participants and build concrete relationships that help the classroom discussion to focus on the mathematics. Consequently, we see learning as a qualitative change in the contents, strategies and activities developed by a student (i.e. a learner) to become or keep being a participant of a community that has its own institutionalized repertoire, like it is the case with the mathematics classroom.

We structure the analysis of opportunities by establishing these three types that, in turn, may respectively favour conceptual (e.g., the notion of homothecy), procedural (e.g., the practice of conjecturing) and regulative (e.g., the norm of justifying) mathematical learning. The MLOs of an episode are also more broadly characterised through the types of actions that frame opportunities. This complementary characterisation is helpful in that it relates, for instance, the Thinking Strategy that is to be learnt with the students' and the teacher's actions that are enhancing the consideration of that particular strategy. Furthermore, two MLOs that are equally characterised as Mathematical Content and have the same conceptual content of reference can be distinguished by means of the complementary characterisation in terms of the actions in the episode that frame each MLO. The understanding of a MLO depends on both the contents for potential learning and the actions that contribute to making these contents emerge. On the other hand, the relationships between the MLOs and the actions in an episode point to the impact of the whole group dynamics on the students' learning.

THE RESULTING ANALYTICAL TOOL

The resulting analytical tool that has been created organises the characterisation of the whole group discussion throughout the characterisation of episodes, actions and MLOs. A complex representation of the whole group interaction is developed to inform about the richness of the discussion in terms of the amount of diverse MLOs. In this section, we explain the tool and then exemplify its effectiveness.

The representation of the episodes of a whole group discussion

For a concrete whole group discussion and as it has been argued earlier, the nature of the episodes is characterised through the instrumental and the discursive dimensions. Figure 1 illustrates the structure of the whole group discussion exemplified in this report. We claim that all the episodes identified in a whole group discussion can be represented in a two-dimensional matrix that suggests a coordinate system. Each episode is located in the system with two coordinates that determine its position, but the position is not uniquely determined as more than one episode may own the same two coordinates. The use of this representation allows whole group discussion to be interpreted in terms of a sequence of episodes with changing particularities concerning the use of artefacts and the interaction with the mathematical task.

Once the episodes have been defined and the whole group discussion has been structured as a sequence of characterised episodes, an in-depth study of the actions involved in each episode is required.

Whole-group discussion (Problem 3)	Situation of the problem	Presentation of one solution	Examination of resolution strategies	Examination and/or extreme cases	Consideration of cases	Examination of different solutions	Connections	Generalization	***(Discursive types)***
Explaining the artifact									
Explaining through the artifact							**(e_{11})**		
Linking artifacts		**(e_3)**							
Discussing the artifact			**(e_5)**					**(e_9)**	
Discovering through the artifact					**(e_7)**				
Experiencing the instrument	**(e_1)**	**(e_2), (e_4)**		**(e_6)**		**(e_8)**			
(Instrumental types)									

Figure 1: Representation of the episodes (e_i) involved in a whole group discussion

(Subscript *i* helps to follow the episodes chronologically)

The representation of the actions within each episode

The actions involved in the episodes are defined only by one element. As presented in the theoretical framework, the nature of this element differs depending on the agent (i.e. students' actions considered as Thinking-Math Interventions –TMI; teacher's actions considered as Didactical Interventions –DI; and Instrumented Actions –IA– performed by the participants and centred on the uses of particular artefacts). When an action has been characterised by considering the agent, it is codified to describe its nature in more detail. Apart from the chronological succession of actions, the relationships between them also have to be taken into account. Thus, we add oriented segments to connect the actions that are influenced by others. Finally, a structure is created to summarise all this information: a) the nature of all actions, b) the participants who are performing each action, c) the time sequence when the actions occur, and d) the oriented segments that relate the different actions (Figure 2).

We exemplify the use of the tool in the characterisation of the actions of one of the episodes in Figure 1. The episode comes from the 50-minute whole group discussion around the third problem of the sequence. The problem asks to find the centre of a rotation mapping two line segments given in the plane. We present the analysis of the first episode identified in the discussion. The episode presented (e_1) is characterised as "Experiencing the instrument". Two students are using the DGS technology to present their work and to carry out actions requested by the teacher. In the transcription, we can observe the participants involved and the different nature of their actions. Each action is linked to one of the three types presented above.

Student 1: We decided to do a perpendicular bisector between a point and its homologous. Then we tried this intersection and we realised that it coincided and that it was the right rotation centre.

TMI: To explain a procedure to reach a solution

Student 2: [While Student 1 explains their solution, Student 2 constructs the situation with DGS]

IA: To complete an explanation with visual construction

Teacher: Now that you know that this point is right, could you argue why?

DI: To ask for a mathematical argument

Student 2: If two points, when we rotate a piece, coincide, it means that they are at the same distance from that point. So if they are at the same distance, for example, the perpendicular bisector is the locus point of all equidistant points between the two original ones.

TMI: To elaborate on a deductive justification

Teacher: The locus.

DI: To reformulate technical vocabulary

Student 2: Ah, yes! The locus.

TMI: To correct technical vocabulary

Student 2: [While Student 2 expresses the justification, he uses visual DGS figures that are on the screen for his explanation]

IA: To draw on visual DGS figures

After having identified the types of actions involved in the episode, the situation is represented in a visual diagram that incorporates preliminary oriented segments (Figure 2). It is particularly important to have these segments triangulated.

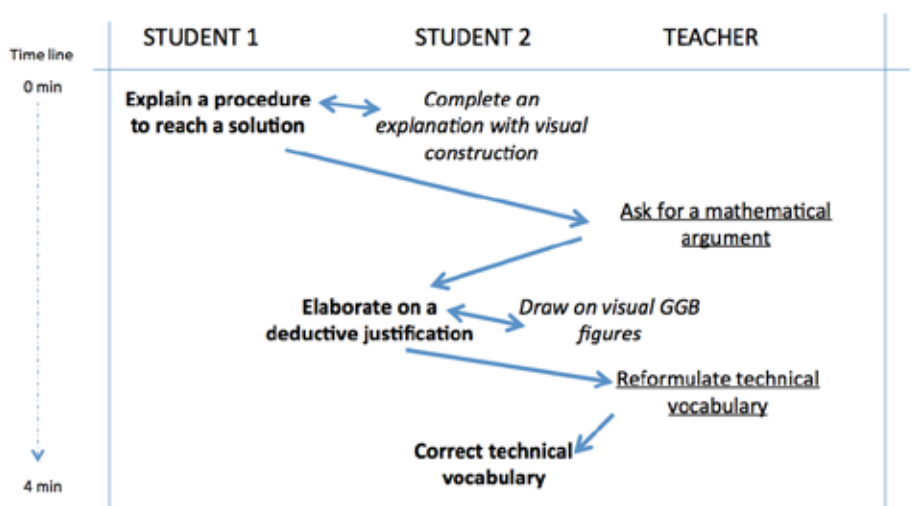

Figure 2: Visual representation of episode 1 (e_1)

The participants involved in this four-minute episode are two students and the teacher. If we focus on the nature of the actions, the students' Thinking-Math Interventions (bold) are central. We also observe that the Didactical Interventions (cursive) by the teacher are equally important. In his first intervention, he asks for an argument and in the second one, he makes a vocabulary correction of the expression "locus". The students' use of the software is crucial too. In the transcription, we see that they first use the software to make a construction and then, point to the screen and show the completed construction, which are considered Instrumented Actions (underline).

The representation of the MLOs within each episode

The final use of the analytical tool is to identify rich situations that can influence the learning process in a middle-term perspective. In the selected episode, three situations that can enhance the students' mathematical learning are interpreted in terms of MLOs. The first two rich situations are derived from relationships between Thinking-Math and Didactical Interventions, and the third rich situation is derived from relationships between Thinking-Math and Instrumented Actions. Following the characterisation of MLOs in three possible types, for this episode we find a dominance of Self-Regulating Activities. The contents of the interaction suggest the type of learning involved in the regulation of the task performance in the mathematics classroom. The following three MLOs are not based, for instance, on the mathematically correct performance of explanations, and justifications, but rather on the production of consciousness around the importance of such practices in the context of school mathematics.

- *MLO_1 –Importance of explaining through communicative skills and technology*

The fact that the students use software every time they want to explain or show something to the class, as occurs twice in this episode, is a significant relationship between the Thinking-Maths Intervention and the Instrumented Actions around the uses of DGS. The students complement their explanations by drawing on visual DGS figures: Student 1 explains the procedure to reach the solution and Student 2 develops a deductive justification. We consider that this situation gives importance to the dual explanation through a combination of communicative and technological skills. In further research, we will have to assess the possibility of being influenced by this episode when a student later uses the DGS to complement a written or oral solution.

- *MLO_2 –Importance of reasoning and justifying*

The fact that the students are asked for a mathematical argument after the presentation of incomplete solutions is illustrated in this episode by Student 1, who points out the importance of arguing the solutions for any mathematical problem. Moreover, in this situation, the correct justification is given by Student 2. This action also enhances the potential for learning because it provides specific knowledge (the correct argumentation) that may influence the understanding by other participants that have listened to the contribution made public by Student 2.

- *MLO_3 –Importance of correctly using the mathematical language*

The fact that the teacher corrects the mathematical expression “locus point” after its inappropriate use by Student 2 during his argumentation creates a rich situation in which the importance of using correct vocabulary is modelled. Student 2 reacts to the teacher’s intervention by correcting his expression and showing a more accurate use of the mathematical language. Thus the positive influence of the intervention is evident, and we can suggest that more students may be positively influenced by this situation.

In this episode we have identified three potentially rich situations that can be seen as MLOs and can enhance the mathematical progress of the students. With this example, we show that analysing the episodes with our tool provides an overall view of how the actions of an episode interact. This overview facilitates the detection of potential situations emerging as a convergence of different agents involved in the episode.

DISCUSSION

For the investigation of whole group interaction in the mathematics classroom, systematic methods are needed. We have presented an analytical tool to facilitate this investigation. We have summarised the design and validation of the tool, whose main objective is to identify rich situations that may enhance the students’ mathematical learning during whole group discussions. Its effectiveness has been illustrated by applying it to data from a transcript. The tool analyses whole group discussions from multiple perspectives. On the one hand, the framework of the instrumental orchestration by Drijvers and colleagues (2010) provides an episode-structure of the discussions. On the other, the findings about the actions involved in the episodes further support the humans-with-media theoretical approach that considers the subject and the tool involved in a mathematical activity (Mariotti, 2012). These results are consistent with our assumption of students’ progress being not only caused by verbal interactions, but also by artefact manipulations and nonverbal actions.

We are concerned with the problem of applying the tool to any setting of teaching and learning mathematics. The applicability of the tool can be expected to be improved by reconstructing some of the types for the instrumental and the discursive dimensions from a more global perspective. Nevertheless, it is not easy to broaden the definitions of these dimensions to include situations in which the artefacts are less visible and the tasks are less argumentative. At this moment, therefore, the tool still has limitations that constrain its generalisation. It has emerged from the analysis of discussions with particular teacher, students and tasks. Despite these limitations and others that may appear, we conjecture that the tool could be applicable, after minor adaptation, outside the scope of this study with a design experiment involving other contents and technical environments. It would be worthwhile to explore the effectiveness of the tool in various contexts such as discussions involving different teachers and students, other kinds of problems, and particularly other kinds of artefacts. Further research will help to refine the tool and carry out a careful analysis of its potential and reliability to explore whole group discussions in the mathematics classroom.

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READING STORIES TO WORK ON PROBLEM SOLVING SKILLS

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As part of a research on mediation through stories in the mathematics class, we present a didactic engineering (Artigue, 1988) built on two storybooks about problem solving. To characterise proper reading and understanding of mathematics problems, we define a reading-contract of the problem instructions, based on Brousseau's didactic contract (1998) and Eco's fictional reading contract (1994). We assume that the exploration of plots and alternative worlds in stories about problem solving, leads children to develop problem solving skills. Our study is backed by an experiment in a fifth grade class of a primary school. The storybook approach points out benefits of mediation through stories in problem solving.

INTRODUCTION

This work is part of a research exploring science's mediation through stories led by a multidisciplinary team from the S2HEP Laboratory (Sciences & Société: Histoire, Épistémologie et Pratiques). Based upon the heuristic power of stories (Bruner, 2003), this research points out some benefits of studying stories in experimental science classes. Following our results, we aim to extend mediation through stories to mathematics. In the context of the doctoral work of the first author, we examine problem solving in two complementary ways. The first one involves story-reading and the second one story-writing. With this guideline, we try to open a new field in the French Mathematics Didactics.

The main goal of this paper is to disclose benefits of mediation through story-reading in problem solving, especially on problem instruction understanding skills such as the selection of data. We focus here on a didactic engineering (Artigue, 1988), designed for 10 to 11 year-old children (primary school), involving the study of two French storybooks. We assume that the study of storybooks allows children to develop a proper reading and understanding of mathematic problems. More generally, we assume that stories have a heuristic potential that we can rely on to make children work on their problem solving skills (both understanding and solving).

We first present educational elements, with the analysis of curricula and mathematics textbooks, to frame our work. We then share items of our understanding of problem solving and mediation through stories. We introduce the engineering with the study of the storybooks. Finally, we highlight representative exchanges that occurred in the classroom and extracts by the children to expose some of the main results.

EDUCATIONAL ELEMENTS AND OBJECTIVES

Various curricula interpret problem solving as a key activity of school mathematics. It aims at improving reasoning and logical skills to bring meaning to mathematical objects. The French curriculum for primary school insists on the fact that problem-solving may "increase pupils' knowledge, reinforce sense's control and develop rigour and interest in reasoning" (Bulletin Officiel, 2008, p. 23). Indeed, problem solving implies devising a plan based on mathematical reasoning. Referring to Polya's (1945) four-phase model of problem solving, the first step is to understand the problem. We focus on this initial stage and try to figure out what a proper reading and understanding of the instructions for a mathematics problem must be. We consider instructions like specific texts because they are academic texts written as short stories. Following Jacobi (1987), they require specific techniques to be read and understood.

Having in mind Brousseau's didactic contract (1998), we question the existence of a specific (implicit or explicit) contract for reading and understanding problem instructions. We have analysed some French mathematics textbooks in order to see how school problem narratives are drawn up and how problem reading and understanding are taught. Using linguistic tools (Larivaille, 1974; Reuter, 2009), we have disclosed some limits in problem narrative structures. We have pointed out a lack of complexity in the narrative backing the problem. Even if they look like short stories, school problems are missing the main characteristic of a story: the disrupting factor. There is only one question to ask and only one way to solve the problem. The path through problem solving is mapped out (Moulin, Triquet, Deloustal-Jorrand, & Bruguière, 2012). Consequently there is no need for pupils to build mathematical reasoning. They can rely on automatic procedures without reading the instructions. Figure 1 illustrates two examples of stereotypical problems.

During two night walks, a bunny brings 57 carrots back home. During the second night, he brings 31 carrots back. How many carrots did he bring back on the first night?	To help his friend, a squirrel took 17 hazelnuts from his supplies. He said: "I still have 41 left." How many hazelnuts did he have in his supplies?

Figure 1. Examples of stereotypical problems

In the exemplified problems above, numbers are in the right order to answer the questions: 57 minus 31, and 17 plus 41. When you are expected to do a substraction the first number is bigger than the second one, and when you need to do an addition it is the opposite. Many problems are built using this structure. We can guess that this choice is made to help children to solve problems.

But this kind of structures allows children to go on with an "incidental solving" and prevent them from working on reading and understanding skills. Children need to practice those skills to be able to solve mathematics problems with a plan based on reasoning. They need to embed what we call proper reading and understanding.

Castellani (1995) says that good command of disciplinary support, as problem narratives, is crucial to acquire knowledge of the discipline. Since instructions are built as stories, we rely on Eco's fictional reading contract (1994) to build a "reading contract" for the problem instructions. We take the two following rules as the most important: 1) The world described in the instructions is not the real world but an idealized world to make calculations easier; 2) The instructions include what belongs to the story's context and what belongs to the data, so that a distinction is required.

We choose an approach based on storybooks in order to lead children to make these rules explicit. Stories have a heuristic potential, and relying on this characteristic, we assume that studying a story about problem solving can lead children to question their conceptions of such activity. One experimental assumption in our work is that the confrontation between the solving activity and the students' behaviour while doing it might create debates about problem solving in the class. Storybook reading and studying is not a usual approach but it fits in the French curricula: "They [children] have to interpret and build an argument in the literary field but also in the scientific field. They have to train in order to mobilize knowledge in complex situations to question, to search and to reason" (Bulletin Officiel, 2008, p. 10).

In brief, our main goal is to study characteristics of mediation through storybooks reading. So, we need to identify stories with a mathematics teaching potential (around problem solving activity), and we have to determine didactical and epistemological conditions for mediation through stories in the mathematics class.

THEORETICAL FRAMEWORK

Our choice to link literary work with mathematical work is mostly due to the fact that the French curriculum recommends language work in each discipline. Furthermore, mediation through stories produced interesting results in the experiment that we carried out in the science class. Bruguière and Herault (2007) show how pupils' first conceptions can be questioned through stories and Triquet (2007) points out that a narrative plot can be a powerful tool for knowledge building.

When referring to the stories' characteristics and educational potential, Bruner (1996/2008, 2003) claims that stories bring us to question behaviours. Additionally, Tauveron (2003) claims that understanding implicit data in a story involves cognitive and cultural work. A common point linking these works is that fictional stories suggest alternative worlds. Those worlds can lead the reader to have some "thought experiment" and question her/his own relation to objects (Triquet & Bruguière, 2010). In a mathematical story context, the "thought experiment" and the questioning must be about mathematical objects. Relying on this context, we made the following two assumptions: 1) If alternative worlds and implicit data of the story are linked with mathematics and problem solving activity, the reader has to begin a mathematical work to understand these elements; 2) Studying storybooks with a mathematical context and a plot about problem solving activity leads children to question and then to improve their problem solving skills.

To test the two assumptions above, we build an experiment based on the "didactical engineering" methodology (Artigue, 1988). This kind of methodology relies on an internal validation. As we previously did, it is necessary to establish some hypotheses. Then a plan is to be built to test each of them. The validation is made by the confrontation between an *a priori* analysis of the plan (with the corresponding epistemological, didactic and/or cognitive study of tasks) and an *a posteriori* analysis (with the corresponding lesson observations and children productions).

We also rely on Brousseau's Theory of Didactical Situations (1998) and especially on the concepts of environment, action phase, formulation phase and validation phase to build our engineering. For this particular work and hypotheses, our engineering consists in a plan of four sessions based upon two storybooks for children. We present the books in the next section. The two plots rely on mathematical problem solving and the characters' ways to solve it have an interrogative potential.

DIDACTIC ENGINEERING

So far we have established the general hypothesis that story-reading can be a way to make children improve their problem solving skills. In order to test this hypothesis, we build a didactic engineering involving work on a tale named *Le Problème* (Aymé, 1944/2002) and a play also named *Le Problème* (Lamblin, 2000). Aymé's *problème* is the following: "The woods of the town are 16 hectares. Knowing that each are is planted with 3 oaks, 2 beeches and 1 birch, how many trees are there in the woods of the town?" While Lamblin's *problème* is the following: "My daddy buys a strawberry pie and cuts it in 4 slices. Knowing that the pie's weight is 800g, how heavy is each slice?"

Storybooks' presentation

We need storybooks with a heuristic potential around problem solving. As shown by their titles, both stories are built on mathematical problems in a classroom. Children characters have to solve a mathematical problem given by the teacher. This is a necessary characteristic about the plots. The two of them rely on the mix-up between reality (characters' real world) and the mathematical world (the instructions). The characters merge some elements of the problem with their equivalents into reality and propose an unusual problem solving method. In Aymé's tale, the characters go to a forest to count trees because their problem begins with "the woods of the town" and ends with "how many trees are there in the woods of the town?" In Lamblin's play, the fact that the problem includes the words "my daddy" leads children to convert all elements in the instructions to make them identical to their reality.

The authors of the two stories used the possible mix-up between the sense and the signification of some words. This characteristic allows a work on the explanation of "games of language", and this explanation can be in turn productive of knowledge (Durand-Guerrier, Herault, & Tisseron, 2006). Because of this mix-up, characters make confusion between data and context elements in the instructions. Then rules of proper reading, as we defined them, are broken. In our engineering we want to make children work on these ruptures. Some elements, linked to mathematics and problem solving, explaining these ruptures remain hidden in the stories. So relying on Tauveron (2003), we assume that working on this mix-up may bring children to improve their problem solving skills as we want them to.

Hypotheses, tasks, and some methods

We worked with a class of 28 10-11 year-old pupils during four sessions of 50 minutes each. The tasks (shown below in this section) were given on four different sheets (one per session). In this way we could examine the evolution of the children's answers through the study of the story plots. The analysis of the answers complemented by the audio recording of the four sessions allowed us to explore some relevant characteristics and to support our hypothesis about mediation through story-reading.

We lead children to study the characters' problem solving activity in two ways: in relation to the plot and also in relation to the problem instructions. So, we came up with two specific hypotheses for this engineering:

> H1: Studying alternative worlds, linked to the mix up between the real and the mathematical world and working on implicit elements linked to the characters' reasoning, imply mathematical work about problem solving activity.
>
> H2: Fiction and "language games" allow a questioning on the nature of the different words included in the instructions and the referential function of words in mathematical problem instructions.

By going deeper into the analysis of the plot, children might link their problem solving activity to the way used by the characters to work on mathematical problems.

We focused on two activities, each one linked to a rule of our "reading contract for word problem instructions" presented in our objectives about problem solving.

The first activity relies on the study of the disrupting factors which are the same in both stories: the mix-up between the real word and the mathematical word. In Aymé's tale it occurs when a character suggests going into the forest to count the trees. In Lamblin's play it happens when one child asks the teacher if the problem is about an arbitrary dad or about her/his own dad. We can assume that studying the characters' reasoning and results can bring children to build judgment about the links between real world and its modelling in mathematic problem instructions. We suggest one set of three tasks to enable debates in the classroom. Our hypothesis is that arguments will evolve with the analysis of the two stories. So, each debate is preceded by individual work with stories' comprehension and questions. In the following lines we present the main task preceding each debate and the main question.

<u>First debate:</u> What can you say about the characters' methodology to solve the problem? What can you say about their result? We wanted children to build judgment

about the result and the methodology engaged and then to be able to argue about criticisms that can be addressed to the characters.
Second debate: What is the characters' reasoning (which is implicit in the story and relies on the links between the real world and the mathematical world)? We want children to debate about the relevance of this reasoning.
Third debate: After reading Lamblin's play, what can you say about the character's attitude (which is to make problem's instruction look like her/his real world)?

For the second rule of our "reading contract", we wanted children to work on the referential function of the words in the instructions. For them the goal of this set of activities is to identify data and context elements in some instructions and to learn their function in a problem solving activity. We assume that studying the characters' points of view, which rely on opposite attitudes toward the words included in the instruction, can bring children to understand these two notions. To test this hypothesis, we ask children to select data and context elements in Aymé's problem at three different times and we study the evolution of their selection.

Lesson 1 with Aymé's tale: After reading Aymé's tale, the children have to select in the instruction what they would use to solve the problem. Then, after solving it they have to select the elements they have actually used.
Lesson 2 with Lamblin's play: The pupils have to study how the characters modify the instructions and what are the consequences of those changes in the problem solving. Each change in the instruction brings a new word problem. We ask the pupils to solve all problems to see evolution (when changes concern data the solving is altered whereas when it is about context elements the solving is identical). The aim for them is to identify different natures of words: data and context elements. Then, we ask them to determine in Aymé's instruction what belongs to data and what belongs to context.
Lesson 3: We ask children to rewrite the problem instruction to work on data selection and context elements.

RESULTS

In this section we highlight some results showing how mediation, through the two storybooks, operates in the class. As we said in the previous section, we focused our analysis on two sets of tasks, each about one of the hypotheses (H1 and H2).

About the first hypothesis implying alternative worlds (H1)

Our first hypothesis is about the links between the real and the mathematical world. In the two stories alternative ways of solving and reasoning are brought up. We assume that the confrontation between the characters' solving and the children's behaviour will create debates about problem solving in the class.

During the debate session, one of the topics discussed was about the difference between "real problem" (in a real situation), "realistic problem" (inspired by a real situation) and "fictive problem" (on fictional elements)". Children in the class established a distinction among these three types. To go further with the first rule of our "reading contract", concerning links between reality and mathematical world, we can also say that children reflect on how the real world is included in word problem instructions. Between the two first debates, pupils' criticisms against the characters' methodology evolved in an interesting way. They included modelling and mathematical characteristics in their arguments, as can be seen in Table 1.

Before the first debate	Before the second debate
- The result is wrong because they counted the real trees of the forest. - The result is right because they were careful when the counted and other characters found the same result. - The methodology is wrong because a problem is unreal. - The methodology is wrong because you have to do calculations in order to solve a problem.	- The woods in the problem are not supposed to be realistic. By counting the real trees instead of doing a calculation, characters made a mistake. - A problem is not real (…) it is precise; in each part you know exactly how many trees there are. But in real forests there is not always the same amount of trees in each part. - The methodology is incorrect; there are too many places for mistakes because their forest is not the same as the one in the problem. They spend all day counting and doing calculation is faster.

Table 1. Evolution of pupil's criticisms between the debates

The evolution is due to the study of the tale. Pupils criticized the modelling of the world into the mathematical world. They realized that problem instructions are not identical to reality: the mathematical world is idealized, and overall simplified. They were able to critically reflect on the other storybook, reinvesting knowledge about links between mathematics and reality in mathematics problems for the last session.

About the second hypothesis implying fiction and games of language (H2)

Concerning the second rule of our "reading contract", which implies the characterization of data and context elements, we can say that the study of these two books made pupils understand these concepts.

During lesson 1, pupils were unable to make a selection of data in the instructions even after they solved the problem. 21 pupils (out of 26) were unable to distinguish between data and context in Aymé's problem. During the engineering, the confrontation between the characters' points of view and the study of the evolutions linked to the changes in Lamblin's story, helped children to separate data from context. We translate underneath an exchange that happened during lesson 2:

Student: At the beginning they are complaining that … on daddy is strawberry intolerant so they change it for apples.

(…) *Work about problem solving with apple instead of strawberry in the instructions.*

Tutor: Ok …so, does it change something to solve the problem?

Class: No.

Tutor: Why? Why do we change?

Student 1: Taste! Flavour!

Student 2: Context!

Tutor: Yes, this is a context element. It's like the presentation of data.

Other exchanges involving the same knowledge occurred during that time. After this work (Lesson 2), almost all children were able to distinguish data from context in Aymé's problem (only six of them made a wrong selection). They also succeeded in rewriting the activity and made context and data changes as they were asked.

CONCLUSION AND PERSPECTIVES

The work conducted in class confirms our hypotheses. The study of the two storybooks helped children to establish some rules of proper reading of mathematics problem instructions. Their answers show that questioning the plot, understanding the stories' problematic and comparing the characters' points of views allow children to develop problem solving skills. As we assumed, mediation through stories reading seems to be efficient in the mathematic class.

We are also working on mediation through story writing. We assume that stages in reasoning-making can interact with stages in story-writing. We link the action time, the formulation time and the validation time described in Brousseau's Theory of Didactical Situations (1998) with story-writing seen as an action description (action and validation), or an action anticipation (formulation and validation).

Regarding Scardamalia and Bereiter (1987, 1998), we assume that building a story including mathematical elements will lead children to build a reasoning linked to this story. These authors point out that interaction between the rhetorical space (within which the story is built) and the problem space (within the field of mathematics) leads the writer to improve knowledge and skills by processes of knowledge transforming. We rely on this framework to carry on our work about mediation through stories.

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CHOICE OF NOTATION IN THE PROCESS OF ABSTRACTION

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This paper focuses on the notation invented and used by students in a study concerning the process of abstraction. Participants were asked to identify the symmetries of a square, and to further investigate the various properties of the group formed under composition of the identified elements. Particular attention will here be paid to the differences in approach by academic track and vocational track high-school students in embracing the task, a difference which appeared to stem largely from the creation and use of notation for representing the members of the group.

INTRODUCTION

The underlying methodology of the project has been grounded theory (Glaser & Strauss, 1967) which emphasizes the generation of theory from immersion in collected data. This process has so far resulted in 22 clinical interviews with high-school students (aged 16-19) in Oslo (Norway), with roughly half of the participants from each of the academic and vocational high-school tracks. Both paired and individual work has been considered, with three of the interviews for each track involving two students working together. *Task based clinical interviews* have been used for data collection, which involve intensive interactions with the individual participant, extended dialogues between interviewer and interviewee and careful observations of the interviewee's work with 'concrete' intellectual objects (Ginsburg, 1997).

As a whole, the study from which this paper derives is an exploration of post-compulsory students' strategies for investigating and organizing abstract mathematical structures when engaging with an unseen task. More specifically it is a consideration of Norwegian students' encounters with the symmetry group of a square. As such a number of issues and questions have emerged, however this paper will concentrate on the differences in invention and subsequent use of notation by the two groups of students. Some of these differences will be discussed in light of APOS theory - as a framework for abstraction, and Duval's analysis of problems of comprehension arising from semiotic representations (Duval, 2006). The former thus provides an outline of the stages involved in moving from the concrete to the abstract, whereas the latter explores potential barriers in the transition between these stages. This study suggests that considering these theories together in an interwoven manner can be a useful step towards understanding certain students' seeming difficulties with abstract mathematics.

THEORETICAL BACKGROUND

Piaget proposed the term 'reflective abstraction' to describe the construction of logico-mathematical structures by an individual during the course of cognitive development (Dubinsky, 1991). Reflective abstraction is characterised by the fact that

> It does not proceed from a series of observations of contingent events or objects. Rather, it is a process of interiorizing our physical operations on objects. As we move sets of objects about ... we interiorize properties of mathematical operations rather than objects; we acquire implicit understanding of commutativity, associativity, and reversibility (Noddings, 1990, p .9).

This idea has featured prominently in Freudenthal's (1973) analysis of mathematical development, where he notes that "the activity of the lower level ... becomes an object of analysis on the higher level" (p.125). Other mathematics educators working in the Piagetian tradition have called this process 'entification' (Kaput, 1982), 'integration' (Steffe, Glasersfeld, Richards, & Cobb, 1983) or 'encapsulation' (Dubinsky, 1991). There are subtle variations within these definitions, but in essence they all describe "the process of forming a static, conceptual entity from a dynamic process" (Gray & Tall, 1994, p. 118). There is widespread recognition, both within cognitive psychology and mathematics education, of reflective abstraction as an essential focus for educators, a key to fostering understanding and development of intellectual thought (Simon, Tzur, Heinz, & Kinzel, 2004), and a crucial skill for a meaningful engagement with mathematics (Hazzan, 2005).

Much of the research on abstraction in mathematics has been carried out with a focus on young children, and the majority of Piaget's own work (on which much of the later research is based) was concentrated on the development of mathematical knowledge at the early ages (e.g. Piaget, Wedgewood & Blanchet, 1976; Piaget & Pomerans, 1978), rarely going beyond adolescence or basic mathematical structures. He did however suggest that the same ideas could be extended to apply for older subjects dealing with increasingly advanced mathematical concepts. This is the context most relevant to this study, and the chosen framework for analysis has therefore been 'APOS theory' (Asiala et al., 1996), which was specifically developed to address this less investigated avenue of abstraction in more advanced mathematical situations. APOS theory is based on Piaget's principle that an individual learns by applying certain mental mechanisms to build specific mental structures and subsequently uses these structures to deal with mathematical problem situations. APOS is an acronym for Actions, Processes, Objects and Schemas (Dubinsky & Moses, 2003, p. 403), and these can be described as follows:

> 1) A mathematical concept begins to be formed as one applies a transformation on objects to obtain other objects. A transformation is first conceived as an **action**, in that it requires specific instruction as well as the ability to perform each step of the transformation explicitly.
>
> 2) As an individual repeats and reflects on an action, it may be interiorized into a mental process. A **process** is a mental structure that performs the same operation as the action being interiorized, but wholly in the mind of the individual, thus enabling her or him to imagine performing the transformation without having to execute each step explicitly. Given a process structure, one can reverse it to obtain a new process or even coordinate two or more processes to form a new process via composition.

3) If one becomes aware of a process as a totality, realizes that transformations can act on that totality and can actually construct such transformations, then we say the individual has encapsulated the process into a cognitive **object**.

4) A mathematical topic often involves many actions, processes, and objects that need to be organized and linked into a coherent framework, which is called a **schema**.

As has been pointed out (Dubinsky & Mcdonald, 2002), these stages need not be linear or even exact descriptions of a learning process. However the theory as a whole provides a useful framework for analysis of qualitative data, and it is in this capacity that it has been and will continue to be used for the purposes of this study.

THE STUDY

Mathematical framework

The concepts that have served as a mathematical framework rest largely on group theory and will be introduced here in brief. Formally a mathematical group can be defined in the following way (Smith, 1998):

A set G equipped with a binary operation $\cdot$ is called a group if all of the following axioms are satisfied:

- Closure: For every a, b $\in$G their product a $\cdot$ b is in G.
- Associativity: For every a, b, c $\in$G it follows that (a $\cdot$ b) $\cdot$ c = a $\cdot$ (b $\cdot$ c).
- Identity: There is e $\in$G, such that e $\cdot$ a = a $\cdot$ e = a whenever a $\in$G.
- Inverses: Given any a $\in$G, there is b $\in$G such that a $\cdot$ b = e = b $\cdot$ a.

The group that has played a particularly central role in this study is the symmetry group of a square, often denoted D4, which is a member of the 'dihedral groups' (the groups of symmetries of a regular polygon). This group has been suitable for interview purposes for a number of reasons: It is unfamiliar to the students, and yet it involves a range of concepts that undoubtedly (implicitly or explicitly) form part of school mathematics, such as rotations, reflections, symmetry, identity and inverses, just to mention a few. Furthermore, the group has eight elements - a number large enough that it is not necessarily immediately obvious what these are, and yet small enough that within a reasonable time students can find these elements themselves by inspection, rather than being presented with them; the task is simultaneously attainable and challenging. Moreover, the group as a whole is not commutative. That is, a $\cdot$ b (a followed by b) is not *necessarily* the same as b $\cdot$ a (b followed by a) for two elements a and b in the group.

It should be emphasized that although group theory has been utilized as an underlying framework for the project, it has never been made explicit to the students. Rather it has provided points of reference along the way in the design of tasks and in the analysis of the data, much as APOS theory has been, and will continue to be, used as a supportive framework in the process of analysis.

The interview process

Interviews (lasting up to two hours including short breaks) were recorded on a laptop web-camera. The reason for this choice was that a laptop is such a familiar part of everyday life that it is far less obtrusive than a video camera being pointed at the participants. Moreover it is easy to carry around between interviews, and has left no improvement in quality of image or sound to be desired.

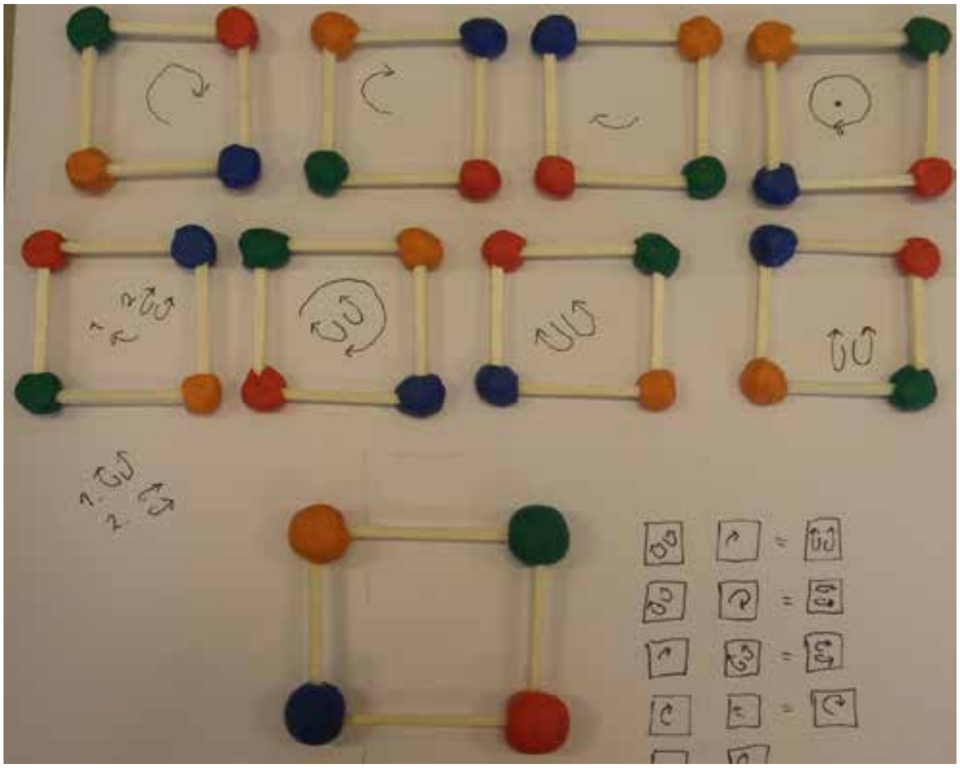

Figure 1

The interviewees had pen and paper available and were encouraged to think out loud and to write down their ideas. They were presented with a square with coloured corners (see figure 1), and were asked to ‘play’ with this figure under certain ‘rules’ which commanded firstly that it would have to end in the plane, and secondly that the symmetry of the figure had to be preserved. Each time a new move was performed, the students would record it by placing a smaller version of the square on the paper, showing the new position of the colours, and they would further describe, using notation of their choice, the motion that had been performed The square was always returned to its original position before the next move, and participants were asked to find as many configurations as possible. When they were happy with the number of possibilities found, the students were asked to combine pairs of moves, writing down what they did as they went along, and to comment on the results. The relationships between the elements of this new ‘system’ were discussed, during which the students were encouraged to lead the conversation, and they were subsequently given the freedom to explore any aspect of the system that caught their attention. For the initial record kept by one of the interviewees, see Figure 1.

PRELIMINARY RESULTS

The students' initial engagements with the square were mostly at a concrete level, with attention paid to its physical appearance and the position of the colours. However, while identifying all the elements of the group was largely an action based endeavour, investigations became increasingly abstract and towards the end of their interviews many students were touching upon concepts such as inverses and identity elements, identifying subgroups and looking for overall behaviours of the system. For example, many participants established that D4 as a whole is not commutative, and additionally tried to determine a pattern for this 'non-commutativity'. Several students considered questions such as: which elements commute? And perhaps more strikingly: When two elements fail to commute, what is the 'difference' between the outcomes? And, is this difference always the same for non-commuting elements? It transpired that this 'difference' is always one of 180 degrees. That is, if a· b=c and b· a=d (for distinct elements a, b, c, d in the group), then in order to get from c to d (or d to c), a rotation of 180 degrees is required, a result with which I was unfamiliar before the interviews. Indeed, while this and other student discoveries merit further elaboration, limitations of space prevent it. I mention them to illustrate clear shifts from physical actions and attributes towards abstract structures.

However before reaching this level of abstraction, participants had to choose some notation for the elements of D4 that they identified, so that this notation could later be used when operating with the members of the group (i.e. when combining two elements as shown in Figure 1). It is this specific aspect of the investigations that will be presented here, and it was largely in this process that a marked difference could be perceived between the academic and vocational tracks.

Figures 2a and 2b show the notation chosen by an individually interviewed academic and vocational track student respectively.

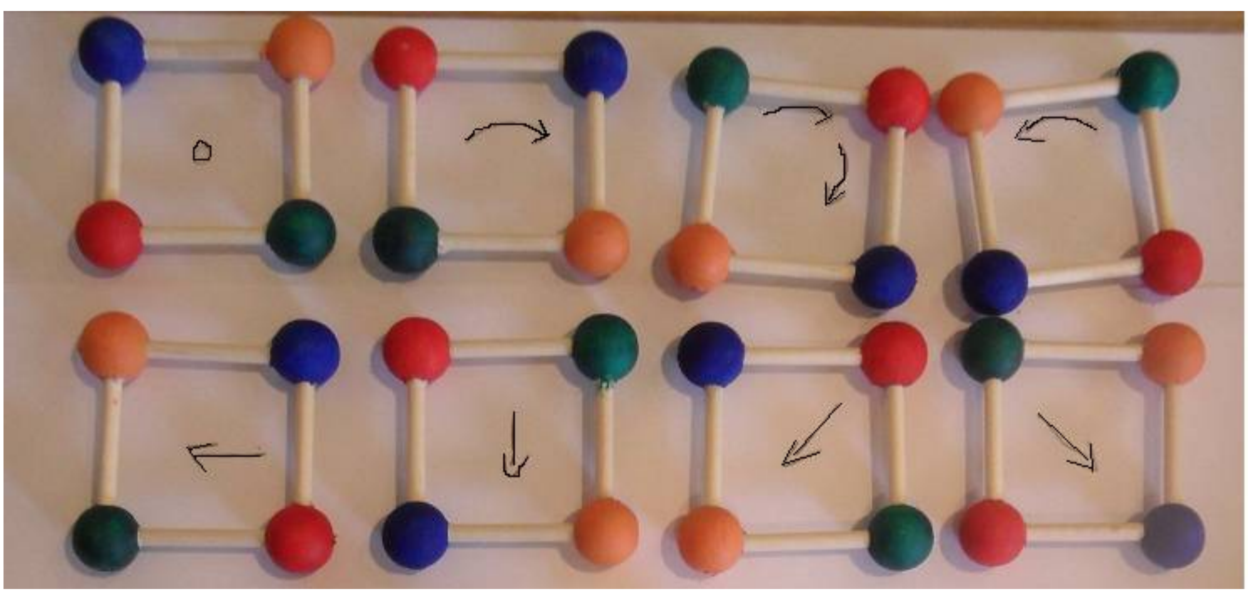

Figure 2a: Academic track student (individual interview)

The contrast is rather striking and serves to highlight an issue that is recurrent in the interviews. One might argue that the use of notation is not central in itself; however the problem seemed to be that when the students started combining elements, a good notational structure facilitated explorations of the behaviour of the system under consideration, which in turn appeared to ease the transition from the concrete to the abstract mode of thought. However the student with the notation shown in figure 2b was simply stuck. He could not remember what his symbols meant, and the wordy descriptions were too cumbersome to use.

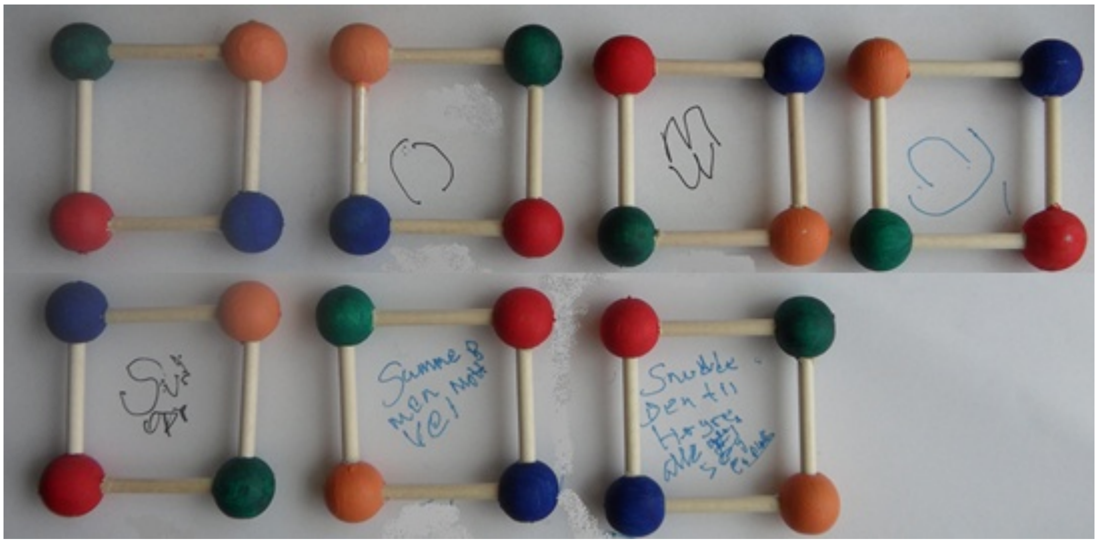

Figure 2b: Vocational track student (individual interview)

Similar differences were found in the paired interviews; the academic track notation was essentially identical to that in Figure 2a, and an example of paired vocational track notation can be seen in Figure 3. What the photos do not show however, is the amount of time taken by vocational track students in inventing their notation. In this context the paired interviews provided some useful insights - while the individual interviews were conducted largely in silence, the paired interviews involved negotiation (which was substantial for the vocational track) that was accessible to me as the interviewer. The following extract from a vocational track interview shows two students trying to agree on a way to describe a diagonal reflection.

Alex: For this one green and blue swap places.

Mona: hmm... what if you picked a description that was independent of the colours?

Jonathan: Nah... that will be hard...

Alex: flip it down to the right

Jonathan: maybe use the edges then

Alex: yeh, if we don’t use the colours and corners we have to use edges ... ok ... so if a person wanted to describe it to you ... what would be the easiest way so you'd get it [asking Jonathan]? For me it would be to take the top right bit and pull it down to the left bottom bit...

Jonathan: what right and left though? ... I like using a, b, c, d and draw a picture of a square and show what goes where.

Alex: yeh... but what if you don't have a picture of the square...

After further elaborate negotiations and various drawings, their final result was that drawn in the third square from the left on the second row in Figure 3 (The writing translates to: 'swap places with each other').

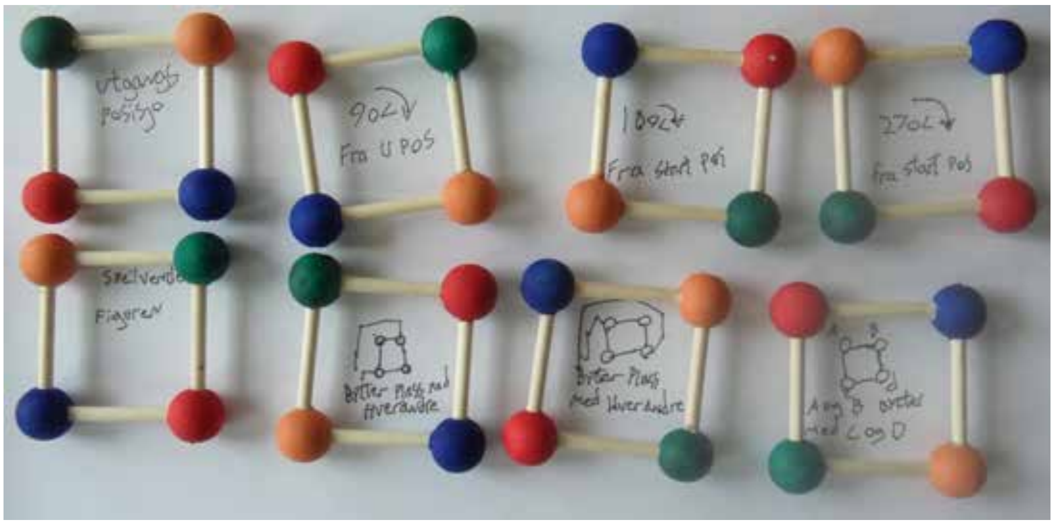

Figure 3

DISCUSSION

So what exactly are the differences between the chosen notations? Coherence and simplicity appear to be central issues. The academic track students tended to choose the same 'kind' of notation for the elements of D4 (such as seen in Figure 2a). This gave them an overview of the number of elements; for instance, once they had drawn one diagonal arrow, they naturally realised that the other 'diagonal move' would also give a member of the group. Moreover these simple arrows, where they continued to use them, facilitated further exploration of the system, particularly when pairs of moves were combined. The vocational track notation tended to lack both coherence and simplicity. As shown in Figure 2b, there was no 'single kind' of notation, but a mixture of drawings and words, which often hindered the identification of all the elements. Moreover, as shown in Figure 3, many students insisted on keeping a small drawing of a square as part of each notation. This underpinned their lingering dependence on the physical attributes of the concrete artefact, and additionally made the subsequent exploration of the system slow and cumbersome to keep track of.

It should however be noted that problems of notation were not limited to the vocational track students. The academic track participants were all fairly confident and coherent in the *invention* of notation; however it is important to distinguish between mere invention and invention followed by appropriate application. To illustrate this, the work in Figure 4 might be considered.

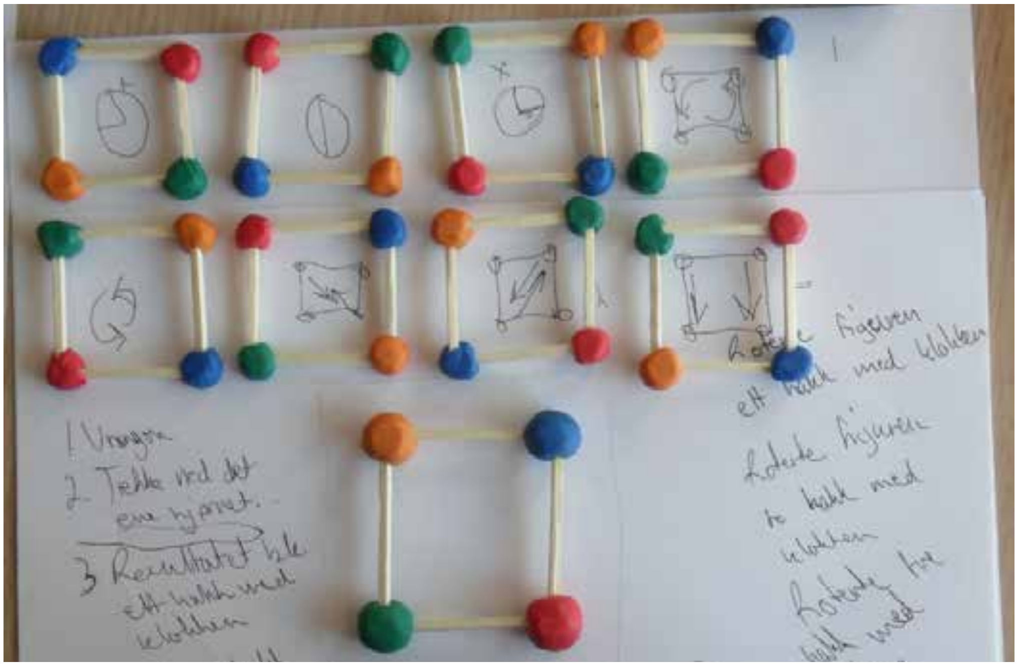

Figure 4

It can be seen that the student has succeeded in creating largely coherent and descriptive symbols for each element, however when he moves on to combining pairs of elements below, he does not think to *use* the symbols, but rather writes his operations and results out in words, a process which is time consuming and inefficient for exploring the abstract structures of the system further.

In the context of APOS theory, the issues described above are closely related to the first two steps of *action* and *process,* and how they can lead to *object* (AP → O). The preliminary observations and analysis of collected data suggest that at these stages serious barriers might occur, and that the speed at which these barriers are overcome depends largely on the invention and use of notation. (Once guided to overcome these difficulties, the explorations by both sets of students were rather impressive). The study thus indicates that it might be appropriate to consider APOS theory in conjunction with theories of semiotics, where for instance Duval (2006) contends that

> representations can … be signs and their complex associations, which are produced according to rules and which allow the description of a system, a process, a set of phenomena. There *the semiotic representations, including language, appear as common tools for producing new knowledge and not only for communicating any particular mental representation*. (Duval, 2006, p. 104, italics added)

Moreover he explicitly asks: “What cognitive systems are required and mobilized to give access to mathematical objects and at the same time make it possible to carry out multiple transformations that constitute mathematical processes?” (p. 104). Here it could be argued that a different way of posing this question might be: what cognitive systems lend themselves to facilitate student’s transitions between the stages proposed by APOS theory in the process of abstraction? One answer might be that signs and semiotic systems of representation are needed, not only to label objects at the action stage, but to promote the transition to thinking in terms of processes and encapsulating these processes into objects in their own right, or as Duval puts it: “to work on mathematical objects and with them” (p.107). Now whereas this idea has been researched in various ways in relation to the learning of basic mathematical structures by young children, there appears to be less focus on the interplay between theories of abstraction and theories of semiotics in more advanced mathematical situations.

The study presented here, though very much a ‘work in progress’, suggests that more careful consideration ought to be given to the semiotic resources drawn upon by older students, and that the ways in which these resources are used (or neglected) can give insight into barriers that students meet in the later stages of abstraction. Although further analyses will be necessary, these preliminary results suggest there is reason to believe that the students who struggled to invent and use notation appropriately during the task under consideration, might also struggle in other mathematical situations, as symbol use is crucial in the development of abstract mathematical proficiency, particularly in algebra where Norwegian students are repeatedly shown to perform poorly (Grønmo, Onstad, & Pedersen, 2010). And when considered against the backdrop of other research on students’ proficiency (or lack thereof) in abstraction and algebra (e.g. Breiteig & Grevholm, 2006) this study underpins a need to improve the situation. Initiatives and approaches such as suggested in the ‘Trip Line’, an algebra module designed for the high-school curriculum as part of part of ‘The algebra Project, Inc.’ (The Algebra Project, 2010), might well be worthwhile to pursue, as it unlike many current teaching plans pays explicit attention to the invention and use of symbols in algebra. It thus integrates Halliday’s (1993, p. 93) proposition that a good strategy for understanding human learning in general, would be to pay attention to “how people construe their resources for meaning - how they simultaneously engage in ‘learning language’ and ‘learning through language’”.

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AN INVESTIGATION INTO THE TENSION ARISING BETWEEN NATURAL LANGUAGE AND MATHEMATICAL LANGUAGE EXPERIENCED BY MECHANICAL ENGINEERING STUDENTS.

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This investigation is grounded within the concept of embodied cognition where the mind is considered to be part of a biological system. A first year undergraduate Mechanical Engineering cohort of students was tasked with explaining the behaviour of three balls of different masses being rolled down a ramp. The explanations given by the students highlighted the cognitive conflict between the everyday interpretation of the word energy and its mathematical use. The results showed that even after many years of schooling, students found it challenging to interpret the mathematics they had learned and relied upon pseudo-scientific notions to account for the behaviour of the balls.

INTRODUCTION

The purpose of this paper is to report upon a study which investigated how a group of first year undergraduate Mechanical Engineering students conceptually interpreted mathematics. The students were presented with a physical system which comprised of a ramp on which three different mass balls were rolled down. One half of the class was asked to predict where the balls would land relative to one another, the other half were given a composite photograph of the balls landing and asked to explain this behaviour. The students' analysis of the system was used to investigate the tension that arises from the everyday meaning attached to scientific terms and the precise mathematics definition. The concept and interpretation of the word energy was used to facilitate this study. This report first provides a brief discussion of some of the issues around the interpretation of mathematical language. It then goes on to discuss the investigation and finally, the conclusions draw together the theoretical and experimental aspects discussed in the paper. The conclusions also highlight the importance of the use of language in the learning and teaching of mathematics and why a conceptual approach to aid the interpretation of mathematics is important.

BACKGROUND

Mathematics, as an intellectual discipline, can be thought of has two interdependent aspects: the language element and the computational element. This interdependency is at the root of the difficulties experienced by many learners and indeed by mathematicians at all levels. The difficulties faced by learners can be eloquently summed up by Davis and Hunting:

> To put it in a simple form that highlights the students' dilemma: they need to know the language of mathematics in order to know what mathematics is about; conversely, they need to know what mathematics is about in order to know how to use the language. (1990, p. 117)

Natural language is used as the medium to convey and learn mathematics. The development of natural language relies upon forming a link between the three dimensional 'concrete' world we live in and the conceptual world of our minds. In terms of embodied cognition, one could say that perceptions are mediated by language to form conceptions, which in turn are used to interpret perceptions. Thus, as conceptions become more refined, perceptions can be interpreted in a more meaningful way. In the early stages of development a child learns to form the link between an object and the name given to it by the society the child lives in. This is evident when a child first learns to count in that a number name (a numeral) is attached to an object. The child also learns that once all of the objects have been counted, the last number indicates the quantity of the objects (the cardinal number).

Lakoff and Núñez (2000) suggested that this process of attaching meaning can be explained by the use of metaphors and conceptual metaphors. A metaphor defined to be the linguistic construction used to describe a subject in terms of another unrelated object and a conceptual metaphor to be where one idea is understood in terms of another unrelated idea. Lakoff and Núñez would explain the scenario of the child counting objects as an 'arithmetic as object collection' conceptual metaphor. As the child progressed more conceptual metaphors would be developed in order to sensibly interpret the world. This process of developing conceptual metaphors is reinforced by the use of metaphorical language extensively used in the teaching of mathematics. It is quite common when teaching children about numbers to employ the use of a 'measuring stick' metaphor to explain the relationship between numbers. In a similar way, the use of an 'arithmetic as motion along a path' metaphor is employed to explain negative numbers. The use of conceptual metaphors has been shown to be problematic within the classroom (Doritou & Gray, 2007).

The learner at some stage has to realise that concrete objects are not necessarily the subject of mathematical operations. For example, the arithmetic as object collection metaphor can prove to be problematic when the learner is faced with set theory where the natural numbers are described in terms of the empty set, that is, $\emptyset$ maps to 0, $\{\emptyset\}$ maps to 1, $\big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\big\}$ maps to 0, 1 and 2, since the empty set is the fundamental 'unit' and not a physical object. This highlights the problem that the learner can, in some instances, attaching a name to a concept at one stage in their mathematical development, but at a later stage the concept is modified but the familiar name is still used or a familiar concept is given a different name (e.g. the equation $y = 3x + 2$ becomes the function $f(x) = 3x + 2$).

THE BASIS OF COMMUNICATION

Two models of linguistic communication that stand in sharp contrast are the classical code model where sentences are comprised of sound-meaning pairs and the more recent inferential model where the sentence provides a semantic structure from which the meaning can be inferred (Sperber & Origgi, 2010). In order to communicate, according to the code model, the speaker and listener must interpret the sentences in precisely the same way. In contrast to this the inferential model acknowledges that ambiguity can arise from the listener misinterpreting the speaker's message. In this model, understanding the speaker's meaning is an inferential process using a common grammar which assigns semantic properties to the sentence and uses context to aid interpretation. As an example of the inferential model, Anderson (1997) studied two working environments: a garage and a postal service. He discovered that in many cases the workers would use words and phrases that did not conform to the standard natural language, but instead used abbreviated sentences and sometimes just words to make a request or convey an instruction. Since the workers were in the same environment, ambiguities did not arise.

Within a mathematics classroom environment it cannot be assumed that the learners possess the same lexicon as the educator, since in the context of mathematics, the language used is often inferential. In one respect the novice mathematician develops a form of proto-language where a lexicon of coded concepts is built where these concepts, to a certain extent, are independent of one another. It is not until a later stage that links are formed between these concepts and then, what could be termed, super concepts are created (Sfard, 1991). It could be said that the symbol reifies the concept in the sense that it can be manipulated and used to investigate both physical artefacts and abstract concepts. From the learners' perspective, the interpretation of these symbols can be a difficult and cognitively demanding process especially when the fundamental concepts rely upon a conceptual metaphor grounded in the 'real' world.

Nowak, Komarova and Niyogi (2001), in their discussion of cognitive development in humans, suggested that the brain developed as a modular system. They suggested that skills like reading and other more sophisticated object recognition processes that are constantly being used, develop their own modules based on ones in which similar processes already exist. Fodor (1983) suggested that modules must fulfill certain properties to a lesser or greater degree: they must be domain specific, they use encapsulated information, they are activated without conscious control (Fodor called them mandatory), their outputs are shallow, they are quick to process pertinent data, they have characteristic breakdown patterns and the neural architecture is fixed. Two of the key criteria suggested by him were domain specificity and encapsulation. Domain specificity is defined to be the class of information that the module is designed to accept or operate upon. Encapsulation refers to the notion that modules are not influenced by information or processes external to it. Pinker (1998) suggested that a module should be defined by the specific operations they perform on the information they receive.

For example in the case of mathematics; in order for a learner to make sense of vectors, some idea of what 'moving' means and what 'direction' means is crucial. The information gathered from the 'real world' experience activates certain modules which in turn enable a cognitive link to be made between the physical world and the mathematics used to interpret it (concept formation). At an intuitive level the concepts of speed and velocity are interchangeable and indeed in many mathematics problems the learner does not need to distinguish between them. The challenge comes when circular motion has to be studied. In this instance the object moving along a circular path is subject to continuous directional changes and hence must be described by velocity not speed. This creates a conflict for the learner who now either has to adapt a deeply ingrained concept to include the notion of changing direction or else has to separate the two concepts and rely upon semantic processing to correctly resolve any problems. This conflict was evident in an investigation carried by Peters and Graham (2009) where a group of trainee teachers were asked to explain the forces acting on an imaginary object suspended without any support in mid-air. This investigation demonstrated the cognitive conflict experienced by some teachers when attempting to use Newton's third law of motion to explain the problem. They could not identify the relationship between the object and the Earth in terms of forces and attempted to modify their learned concepts to allow for an object to be suspended without physical support.

The issues around the learning and teaching of energy are well reported at the secondary level of education (Millar, 2005; Sefton, 1998; Soloman, 1983; Trumper & Gorsky, 1993,). As Sefton points out there is no unique definition of energy and the one normally taught in schools is 'the capacity of a system to do work'. In this report energy is treated as a value which cannot change (first law of thermodynamics). In most cases, when teaching about energy, a transfer or a transform metaphor is used to explain how a system works rather than a process of conversion. This approach, along with intuitive ideas, leaves learners with the impression that 'energy is what makes things happen' (Ogborn, 1986).

When learners arrive at university to study, not only do they bring with them intuitive ideas of how the world works but also the conceptions or misconceptions developed from their previous educational experiences. In many cases, the students are able to solve mathematics questions procedurally but have great difficulty in solving problems at a conceptual level. As Vigeant, Prince, Nottis and Miller (2012) pointed out; one of the major differences between novices and experts is that experts are able to organize their approaches conceptually. In order to facilitate the journey from novice to expert, the need to develop a conceptually understanding is vital.

EXPERIMENTAL DESIGN

The group for this research comprised of 44 first year undergraduate Mechanical Engineering students. They had all studied mathematics to an advanced level during their secondary education. The investigation was carried out in the second semester of their first year. The group was divided into two subgroups and placed in different rooms. Once in their designated rooms they were seated so that they were unable to see their immediate neighbour's response.

Group A had a total of four tasks to complete. The first two were based on their observation of the first ball being allowed to roll down the ramp. The remaining tasks asked them to use their knowledge of mathematics to select pertinent equations which they could use to offer an explanation of the behaviour of the balls. Finally they were asked to draw conclusions about the important factors that determined the behavior of the balls.

In the first instance they were shown a small ball (tennis ball) rolling down a ramp (Figure 1). They were then asked to predict where a 'heavier' ball would land relative to the first one. Finally they were asked to predict where a third ball ('heaviest') would land relative to the other two. They were given the opportunity to request additional information. To facilitate this, the researcher had brought along a tape measure and a set of electronic scales which were initially hidden from view so the learners would not be given a false prompt that they required additional information. In order to record their responses they were given prepared sheets with the pertinent questions. A typical question was, "You have seen a ball being rolled down a ramp and made a note of where it landed. If another heavier ball was to be rolled down the ramp, would it land: (a) In front of the lighter ball or (b) behind the lighter ball?" The option 'land in the same place' was not given since the idea behind the investigation was to test the students' deep learning of interpreting equations but they could have stated, in the space provided for them to explain their answers, that the balls would land in the same place. The individual sheets were handed out at the appropriate time and collected in after each response. This was done so that as the investigation progressed the learners would be unable to modify their previous answers.

The second group, group B, was shown a composite photograph of all three balls leaving the ramp and landing in the same place (Figure 2). They were also given four tasks. The first task was to look at the photograph and offer an explanation as to why the balls landed in the same place. The remaining tasks were the same as group A in that they were asked to select equations that could be used to explain the behavior of the balls. These tasks were centred on them knowing the end result and having to provide reasons why this outcome occurred. In a similar way to the first group, their answer sheets were collected in after each task.

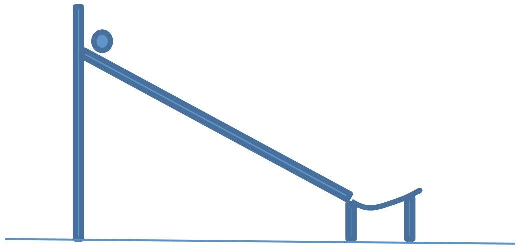

Figure 1: Ramp Configuration

Figure 2: Composite Photograph of Landing Balls

RESULTS

The results of the investigation were collated into a spread sheet. The student answers were coded with respect to the key words or phrases they used within their explanations. The most frequently used key words were then summed. If the student did not provide an answer it was coded as 'no explanation' and if they admitted to guessing or not knowing an answer, this was coded as 'don't know/guess'. An example of the method used for recording and analyzing the results is given in Table 1. This table shows the collated student responses from group A to task 4, part 1, which asked them to select equations from a list that could be used to predict where the balls would land relative to each other. Part 2 of this question then asked them to justify their selection.

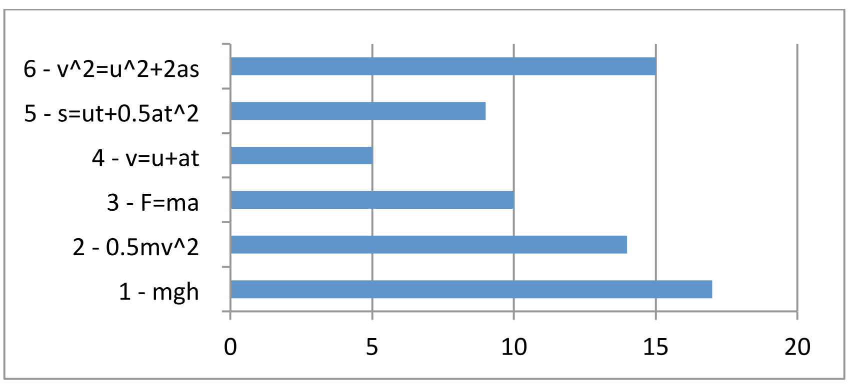

Table 1: Student responses to Task 4, part 1.

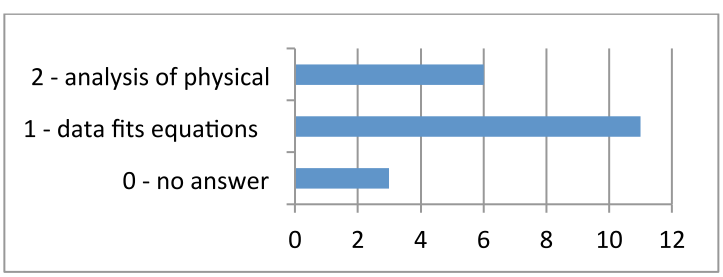

Table 2: Student responses to Task 4, part 2.

DISCUSSION OF RESULTS

It was evident from their responses that when studying mathematics and, in particular mechanics, they were used to a procedural approach where they used certain equations to analyse the components of a system rather than a conceptual approach.

One student wrote "The mass of the ball, the initial and final velocity and the displacement are the key values…." This suggested that the student was going to apply a series of steps, rather than taking the more holistic energy approach. The students answers to task 4 part 2 reinforces this in that when asked why they chose certain equations they could use to predict where the balls would land relative to one another, they chose the ones which would be required to calculate where the balls would land. The emphasis of the tasks was on the relative positions of the balls and not on calculating their positions. In order to be able to explain the behaviour of the balls the students should have focussed on the equations related to the conversion of energy from potential to kinetic. This would have required them to have a clear understanding of the concept of energy, its conversion from one form to another and be able to interpret the equations which described the relationship between potential and kinetic energy.

In their answers some of them talked about the energy of the ball as if energy was a physical thing rather than an abstract mathematical concept used to analyse physical systems. In effect these students had been taught in a way that the concept of energy had become reified and was a commodity that was physically transferrable between the different components of a system. This is not surprising since in every day conversations the word energy is used in a variety of contexts. For example, it is quite common to talk about how much energy a person gains from eating certain foods, or the cost of energy to heat a home. These, and similar statements, imply that energy is something that is consumed. In terms of using conceptual metaphors, the students employed a container metaphor in that energy was contained within a vessel and the only way it could be transferred was by emptying the contents from one container into another. For example, a typical response was "When the balls reached the bottom of the ramp, the potential energy was transferred to the kinetic energy of the ball…" Most of the students were aware of potential and kinetic energies and how energy can be converted from one form to the other. Although the equation for potential energy includes height, the students seemed unable to realise that this implied that for potential energy to exist a system is required. If one looks at the two equations, $PE = mgh$ and $KE = \frac{1}{2}mv^2$, it is evident that the common factor of m (mass) has no effect on the conversion of energy and consequently the velocity of the ball is only reliant upon the initial height from which the ball was released. Typical approaches when asked to justify their answers mathematically, were to calculate the

PE when the ball was stationary at the top of the ramp and use this to calculate a value for the KE and subsequently find the velocity of the balls. One student followed this procedure but still insisted that the mass was the major factor in determining how far the balls would travel. This insistence was linked to earlier statements made when they were asked to explain the behaviour of the balls without the use of mathematics. The empirical context of the investigation, that is a mechanics class, primed the students to think they were expected to calculate values (procedural approach), rather than to think conceptually. This was evident from the way many of the students wrote down a list of every equation they could remember involving forces (Newton's 2nd Law), displacement, time and acceleration equations (eg. $s = ut + \frac{1}{2}at^2$), energy (PE and KE) and attempted to fit the data they had to one or more of the equations rather than use the relevant equations to explain the scenario. One student wrote, "This equation covers all aspects of this experiment such as height of the object above ground level, acceleration due to gravity, etc to make the balls behave the way they do."

The second group were shown a composite photograph of all three balls leaving the ramp and landing in the same place. They were asked, using their knowledge of mathematics, how this could be explained. This form of the experiment involved the students in being able to 'reverse engineer' a system given the final outcome. The majority of responses from this group were similar to the ones given by the first group. For example, one student wrote, "The heavier ball has to overcome more friction and has more gravity acting on it, meaning it falls to the ground faster." Another student wrote, even after seeing the outcome, "If the ball has more mass it will gain more momentum on the way down the slope and so will travel further." When asked which were the most important factors influencing the balls, the height the balls were released from, the distance the balls travelled and their mass were considered the key ones. When asked to write down any conclusions regarding the behaviour of the balls, such things as the final velocity of the balls, the time of travel, mass, initial velocity and acceleration were considered the factors which determined where the balls would land.

CONCLUSIONS

This investigation explored students' interpretation of the word energy within the context of a physical system. The purpose of the investigation was to explore the tensions between the intuitive interpretation of the word energy and its mathematical interpretation.

The results indicate that after many years of education, students still rely upon intuitive explanations. It also revealed that in some cases students were unable to use an equation or set of equations to conceptually interpret a system. In this investigation, the everyday interpretation of the word energy has obscured, for many students the scientific notion of energy in the sense that they reify the concept of energy and employ a container metaphor to analyse a system involving energy conversion. Although this investigation centred around the interpretation of energy, the issues it raises can be applied to mathematics in general, for example, the interpretation of the word 'set', the language used in the teaching of the calculus (the theory of limits). In addition to the students' interpretation of the word energy, the language used within the task sheets would have some influence on how they approached the tasks. In order to assist students on their journey from novices to experts, educators must be careful in their use of language when teaching and place a greater emphasis on the interpretation and meaning of equations in order to develop conceptual learning.

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THE PRODUCTIVE ROLE OF INTERACTION: STUDENTS' ALGEBRAIC THINKING IN LARGE GROUP WORK

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We report preliminary results from a design research of six lessons in a mathematics classroom with students aged 15 to 16. The emergence of relationships between mathematical learning opportunities and students' interaction has been examined. The idea of Learning Opportunity Environments –LOEs plays a significant role in the study, as we consider it useful to better understand the social aspects of mathematical learning. Our results suggest, but do not demonstrate, progress in the students' algebraic thinking, as regards generalization processes that were fostered at different points of the interaction. We use data from the episodes to illustrate two major themes that are central to what takes place in large group work. More generally, these episodes bring up reflection on what LOEs are and what they are not, and for whom.

INTRODUCTION

Large group work has become a common practice with a modest amount of research compared to pair work. This report describes a study linking task-related interaction to mathematical learning in a large group. At the moment learning is not actually examined, but learning opportunities pretend to be demonstrated as having taken place. The motivation for the study is the increasing use of collaborative classroom settings, along with the need to refine scientific arguments around them. Following analyses of students interacting in small group more than twenty years ago (Cobb, Yackel, & Wood, 1991), we aim to examine the potential of large groups from the perspective of creating learning opportunities in the mathematics classroom. For the last two decades, research has pointed to the impact of social interaction on mathematical learning (Brandt, 2007). Other outcomes have pointed to unintended consequences of being involved in interactive settings (Gresalfi, Barnes, & Cross, 2012). We undertake to develop an analysis of large group processes that helps to understand the complexity of creating learning opportunities in the interaction with *many others*.

In what follows, we present our view of mathematical learning as a socially mediated process, and briefly introduce the methods applied in the research. In the last part of the report we comment on two episodes to illustrate major themes that point to the creation of Learning Opportunity Environments –LOEs. The episodes provide an indication of what we call "large group work". There is no data in this report to determine the extent to which learning took place across the large group or even to verify that others learned in response to comments that were made by the students who interacted. Moreover, although we focus the discussion on two episodes, we attempt to note complexities arising from scenarios with similar didactical and pedagogical orientations.

THEORETICAL FRAMEWORK

In this section we briefly present what we mean by social interaction and mathematical learning opportunities. Then we refer to the domain that frames the objects of teaching and learning in the design research, namely the transition from arithmetic to algebra.

The joint construction of social interaction and LOEs

Social interaction may lead to oriented actions for performance improvement in the teaching and learning of mathematics. The ideal of interaction as a cultural practice has survived to become a tool for potential learning and for group development so that it makes sense to interpret the notions of learning and learning opportunities as conceptually and empirically similar. Both refer to the favourable negotiation of circumstances toward the construction of knowledge. Cobb, Yackel and Wood (1991) suggested the value of thinking of learning at an operational level in terms of what we refer to as LOEs. We view students' learning as a product of their involvement in LOEs, and of their ability to optimize the interaction with others to improve mathematical discussion. In our study we focus on LOEs that may contribute to learning during the course of students' interactions despite minimum intervention from the teacher. Missed learning opportunities may occur due to the lack of active guidance provided by the teacher, but students have the potential to make the most of many other opportunities by themselves. LOEs still are an in-construction theoretical notion in that we need more examination on what they are, and more important, what they are not.

Our understanding of mediation points to the mutual influence of different realities in the accomplishment of specific goals and tasks. We see mediation as a culturally-based practice embedded in learning and group development, with the underlying assumption that individuals and groups are ready to (consciously) move their roles and positions in the interaction with others. In our work, mediation is explored in terms of the actions of participants in a mathematics classroom in which the creation of LOEs through peer interaction is expected to lead to the development of mathematical discussions. Moreover, our idea of mediation coincides with the selection of certain participants and issues that will receive more attention than others in the interaction. It is this discursive nature that raises a problem for the practice and theory of social interaction as a mediator of mathematical learning: what to do with and how to explain LOEs in which communication is not always successful, and/or is not well addressed in the direction of sharing knowledge. In response to this problem, it makes sense to develop studies that are built on the search for collective situations in which students make the most of a LOE to achieve learning, together with others in which they miss the opportunity at a given point of the interaction. In this report, we focus on the first type of situations.

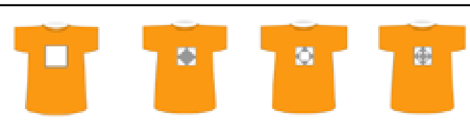
The difficulties of students in the transition from arithmetic to algebra

Algebraic thinking is the activity of doing, thinking and talking about mathematics from a generalized and relational perspective. In the transition from arithmetic to early algebra, one of the initial difficulties for students has to do with the learning of the language conventions underlying this mathematical domain. Algebraic symbols like an *n* become rather sophisticated, and in the resolution of a problem tend to be misinterpreted and misused. On the other hand, Lins and Kaput (2004) argue that an early favourable start to the learning of algebra is possible by leading the students to foster a particular kind of generality through the use of a problem-solving approach with generalized arithmetic. The reasoning required to solve certain problems can be expanded from concrete arithmetic situations to more complex situations that include the ability to use abstraction. We state that a problem involving several stages of algebraic thinking, from near to far generalization, helps students to adjust their reasoning from meaningful numerical cases to algebraic symbolism and mathematical abstraction. Thus three levels of reasoning may be recommended to represent a problem: a) concrete (considering small quantities), b) semi-concrete (considering big quantities), and c) abstract (using symbols).

Together with the three levels above, the exploration of visual growing patterns is another recommended way to introduce algebraic expressions (Warren, 2000). Many students experience difficulties with the understanding of geometric patterns as algebraic functions. Some of these difficulties come from the lack of an appropriate language to describe relationships between variables, the inability to visualize and complete patterns, and the complexity to connect verbal, visual and algebraic representations. Consequently, tasks that encourage visual strategies and relate number and geometric contexts are crucial in the early learning of algebra. For the design of the tasks in our research, we have considered the combination of concrete, semi-concrete and abstract levels of reasoning, and the combination of verbal, visual, numerical and algebraic representations to mathematically model regular situations of change.

RESEARCH CONTEXT AND METHODS

The investigation consisted of preparing and analyzing six lessons in a classroom with a group of students aged 15 to 16, and the teacher. The students were used to pair work and large group discussion. They were also used to problem-solving dynamics, to listening to each other, and to communicating their mathematical ideas. The research question was: *How does large group work contribute to the creation of mathematical learning opportunities in problem solving classroom environments?*



T-shirt Problem (lesson 4). Since 2009, a brand new T-shirt can be bought, with an added square in the drawing each year. Take a look at the T-shirts of 2009, 2010, 2011 and 2012 above, and find:

- How many white squares and grey squares does the 2015 T-shirt have?
- How can you express the quantities of white and grey squares for any T-shirt?
- How many white triangles and grey triangles does any T-shirt have?

Figure 1. Example of task from the problem sequence

To prepare the experiment, we elaborated a coherent and focused sequence of six word problems about generalization that helped to create a LOE. Coherence was based on the control of a progressive difficulty in the problems from the perspective of algebra, and also on how a problem was mathematically related to the next one in the sequence. The problems presented were challenging in many respects. For instance, the example in Figure 1 was intentionally complex from the perspective of the wording, in particular with regard to the expression “any T-shirt”. The T-shirts seem to be defined by the year, but it is not immediately clear if “any” means “any year T-shirt”.

For each fifty-minute lesson, one problem was presented by the teacher and then discussed by the students in pairs. The last thirty minutes were devoted to large group work. The teacher acted as a facilitator of the students’ interactions, and circulated around the room during pair work. Thirty minutes is a lot of time so that much pressure was on the quality of large group work, especially for the teacher to orchestrate it.

Data collection consisted of audio and video recordings of class discussions. We began by transforming audio and video files into transcripts. It took time to decide which type of transcription would better suit the aims of the research, while remaining an adequate representation of data with a double emphasis on the interaction and the mathematics. After having examined various options, we looked for key episodes in the videos and elaborated transcripts that illustrated interactional and mathematical features. To determine where transcripts of episodes begin and end, we gave priority to the mathematics. We identified large group moments in which mathematical practices were at the core of the discussion due to the existence of diverse meanings or the difficulty in understanding a mathematical reasoning. Thematic boundaries and learning opportunities may be differently perceived by different researchers, but the two authors’ agreement was guaranteed, along with a third researcher who intervened when it was difficult to reach agreement in the analysis of a specific episode.

Having constructed the set of episodes and reviewed the videos several times, we began a process of comparative and inductive analysis (Glaser, 1969) among episodes from the same lesson and then from the total set of lessons. We aimed to elaborate mathematical memos and interactional codes to mark changes in the students’ meanings, as well as changes in the direction of interactions. The direction of social interaction depends on whether participants direct their actions toward someone in particular, and whether such actions involve intentions concerning the interpretations under discussion. In an episode with practices of cross multiplication, for example, interactional codes may consider verbal actions by one student aimed at seeking others who share similar ways of making sense of cross multiplication. Other codes may be related to verbal actions aimed at helping each other to understand cross multiplication. There may be codes that point to students asking for clarification of ideas, and so on.

Any social interaction is a combination of interactional codes, and thus it is not possible to have key episodes that are univocally related to single codes. However, an exhaustive attribution of codes to episodes was not attempted. We gave priority to detecting one (possibly two) code(s) that influenced the evolution of the mathematical activity. Next we summarize the direction for two interactional codes that were constructed in advanced phases of the analysis in relation to key episodes. The following codes were developed in an on-going way as the analysis of new episodes contributed different codes and until there was a stable set:

- *Sharing responsibility* -A student follows up a mathematical explanation given by the peer in pair work and gives further information.
- *Expressing confusion* -A student reacts to a prior intervention by claiming lack of understanding with respect to a mathematical reasoning.

Throughout the analysis, interactional codes were completed with mathematical actions. Our use of the term mathematical action echoes the notion of mathematical practice by Godino, Batanero and Font (2007, p. 3): "Any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems." The attention to mathematical actions led to the elaboration of what we call mathematical memos. There was no limit on the length of memos. Nevertheless, we intended to summarize mathematical characteristics in the actions of students in an episode in about one paragraph (see next section). By providing codes and memos, we advanced toward the construction of themes in order to obtain cases of episodes. The creation of a theme involved specifying two components: i) the mathematical actions at that point in the lesson, and ii) the interactions that appeared connected to the realization of such actions. Emerging themes were expected to inform about learning opportunities in terms of relevant mathematical actions and interactions. Various themes were constructed and, as more data were analyzed, some of them came to assume a greater importance and constituted major themes. Some early themes were either discarded as the analysis progressed, or absorbed into more accurate descriptors. A final group of major themes comprises the preliminary findings of the study. Next we summarize two of these themes.

SHARING RESPONSIBILTY AND MAKING SENSE OF ALGEBRA

The excerpt below reproduces part of an episode in which two students are discussing the third question of the T-shirt Problem, "How many white triangles and grey triangles does any T-shirt have?" The T-shirt design pattern refers to an arrangement of squares inside each other. The midpoints of each side of the outer square are joined to make a smaller square inside it and so on. Jose and Gabriel worked together during pair time, and they are now explaining part of the solution in the large group.

Jose: If n is the number of the T-shirt, n equals the quantity of squares in that T-shirt. The first T-shirt has one square, the second has two, the third has three… But as said before, one square has no triangles. Then you need to take the number of T-shirts minus one, and multiply it by four because each square leads to four triangles. That's the total. If *n* is even, you divide the total by two and get the white and the grey triangles. When *n* is even, you get the same quantity of white and of grey squares. You've taken one out, and get the same quantity of white and of grey triangles.

Gabriel: No, this happens when *n* is odd, no... Yes, when *n* minus one is odd.

Jose: Yes, it is *n* minus one, odd.

Gabriel: When *n* minus one is odd. We made a mistake, when *n* is odd you get one more white square. It leads to the same quantity of white and grey triangles.

Jose: Yes. If *n* is odd, you divide it by two and get the number of the two types of triangles.

While in the first lessons the wording of the problems stresses one of the variables with the algebraic use of the letter n, the fourth problem requires the symbolic representation of the variables by the students. Jose uses the n to distinguish the location of the T-shirt, established at the start of his reasoning. To reach the unknown quantity of colored triangles, he gives the expression $4(n\text{-}1)/2$ for the special particular case of n odd. Despite the confusion between even and odd T-shirts, he progressively constructs the pattern with the support of algebraic language and visual thinking. Gabriel points to the issue of even and odd numbers having an influence on the adequacy of the pattern. The two students, however, seem not to be clear about the need to use n or n-1. In a former episode from the lesson, some students discuss whether the 2015 year T-shirt is the seventh in the collection, while some others consider it to be the sixth due to incorrectly interpreting the year subtraction, 2015 minus 2009. Such a difference also has to do with taking either 1 or 0 as the initial value for n. Gabriel looks at the drawing of the third T-shirt and concludes that the pattern by Jose works for the odd cases. He justifies his reasoning on the basis of the characteristics embedded in the general case given by the set of odd T-shirts.

In the end, both students seem to make sense of algebra by using the symbolic convention for the generic representation of natural numbers. But still another interpretation is plausible. We may see two students who are, at the moment of the episode, wrestling with their own ideas and who might want to resolve their personal understanding before attending to anyone else's answer or approach. If their focus is on individually reaching a solution, then it is quite conceivable that they will not, and will not want to, interact with anyone who might "spoil" their quest for untangling their ideas. Thus we wonder if learning would actually be taking place for Gabriel if he just accepts the reasoning posed by his peer. To generate a more precise understanding of the social aspects of the mathematical learning it is necessary to know more about the students who are involved in such learning, and the meanings given by them to pair and large group interaction with peers.

Sharing responsibility here represents situations in which a student follows up a mathematical explanation given by the peer in pair work and provides further information. The student feels that s/he should respond to what the peer says and does in some appropriate way. Gabriel and Jose share the responsibility of making sense of the algebraic pattern in the context of the T-shirt Problem. This code calls for responsibility based on joint efforts during pair work. For the different lessons, we often see students in the large group behaving as individuals still belonging to a pair structure. Students take more responsibility for what their peers in pair work say in the large group, compared to what other students say and do, for whom the main responsibility is expected to be assumed by the teacher. Although our analysis focuses on large group and all students showed different ways of participation, the influence of the pair work dynamics appears to be relevant in that it is a locus of responsibility. It can be argued that this sort of student-student collaboration would not be so present if large group discussion had not been preceded by pair work.

This episode illustrates how large group work can enhance learning opportunities by making public an error in relation to the algebraic use of n. The intervention by Gabriel makes the incorrect use of the variable explicit, which comes with a clarification of the connections among the value for n, the position of the T-shirt in the sequence and the appropriate pattern. By having the two students sharing responsibility for this discussion in the large group, the creation of a learning opportunity may be facilitated. If Gabriel had not paid attention to his peer, the mathematical error might have been overlooked. In other episodes, *Sharing responsibility* also acts as a mediator in the understanding and manipulation of algebraic expressions. In the same lesson and with respect to the same pair, when discussing the answer to the first question in the problem, Gabriel clarifies some of the words said by Jose concerning the connection between the value for n and the position of the T-shirt.

EXPRESSING CONFUSION AND LINKING REPRESENTATIONS

The episode partially reproduced below is an immediate continuation of the previous episode. Both take place in the fourth lesson around the solution of the third question of the T-shirt Problem. The following conversation helps to illustrate the fact that an expression of misunderstanding can become a resource to be exploited by students to optimize the creation of learning opportunities.

Teacher: Have you understood? Maria [a student], can you explain it?

Maria: Woops! I have understood nothing!

Jose: If they tell you that there are three T-shirts, one, two and three… This one, the third, has three squares because each year you have one more square.

Maria: It will have as many white triangles as grey triangles…

Jose: We'll talk about that later. Now, you have three squares in the T-shirt but one square does not generate triangles, the one in the middle. So you take one out.

Maria: Why do you take one out?

Jose: The one in the middle does not generate triangles. You take these two squares that do generate triangles. You multiply them by four. Each square generates four triangles [pointing to the third T-shirt]

Maria: Okay.

Jose: That's the total amount of triangles. Three squares minus one are two, multiplied by four is eight, the total. You don't know how many are grey and how many white. You only know that eight is the total. But the *n* is odd, and it's the same quantity for white and grey. You divide it by two and get the quantity of white and grey triangles.

A few minutes before, Jose had explained the pattern for the special particular case representing the odd T-shirts, $4(n-1)/2$. He was aware of the two special cases introduced by the particularity of a number being either even or odd. There was an initial algebraic approach to the explanation of the general pattern, that may be viewed as a mere symbol manipulation instead of an expression that shows relationships between the location of the T-shirt in the collection and the number of squares and triangles in the design. Such manipulation seemed to hinder Maria's understanding. Jose reacted by linking algebraic language with natural language and including references to the visual context of the problem. On a second attempt to clarify his explanation, he took the case of the third T-shirt, as a generic example to illustrate regularities that support his pattern. The episode suggests that Maria came to see the generality in the pattern through the particular case provided by Jose. The only evidence for this assertion is that Maria says "Okay" in response to the second part of Jose's explanation. We do not even know what Maria thinks about Jose's final contribution, and so we have no evidence for her understanding of that part. Of course, saying "Okay" is not evidence of understanding, and indeed in another episode we see how Maria says "Yes" in response to a question by the teacher and, two lines later, we discover that she has not understood what is under discussion. This is why our analysis points to the creation of LOEs but does not guarantee the identification of learning.

Expressing confusion here represents the actions of a student who reacts to an intervention by claiming lack of understanding with respect to a mathematical reasoning that has been exposed. Maria is the student who expresses such confusion. Although the term confusion rather suggests the individual dimension, it may arise from non-internal reasons such as people poorly communicating their ideas. A student may suppress information that the addressees need to make sense of what is being said. As part of an interaction, confusion is to be seen in terms of a collective challenge with people involved in the completion of a shared task. A student expressing confusion is not a guarantee of participants exploring the problem-solving processes. Such actions become unproblematic as long as participants are willing to help each other to understand mathematical contents by checking meanings. Understanding is facilitated when communication is seen as possible because participants are considered to be competent. Jose might not have provided examples for his mathematical thinking if he had interpreted Maria's intervention only as an expression of difficulties in the understanding. And vice versa: Maria might not have shown confusion if she had not seen Jose as sufficiently competent to follow her arguments.

We interpret here large group as a LOE in which learning opportunities are provided through making public the incompleteness of a mathematical reasoning. The fact that Maria publicly shares her confusion leads Jose to explain again his reasoning including, on this occasion, newer connections between visual reasoning and verbal patterns. Thus the opportunity to link different representations (verbal, visual, numerical, and algebraic) appears. The actions by the teacher are also relevant in the creation of this LOE. The teacher promotes the tacit demand from Maria, which becomes effective in the configuration of the turns that makes the mathematics evolve. In other episodes from this lesson, the study of the T-shirt sequence reinforces the opportunity to learn that there is more than one correct way to express the same relationship between two variables. This is a common learning opportunity arising from the six lessons: students realizing that two or more differently looking arithmetic expressions are equivalent and mathematically model the same realistic situation.

METHODOLOGICAL ISSUES AND PROSPECTIVE

The goal of this report was to describe how we have proceeded in the analysis of learning opportunities around the construction of algebraic thinking in the context of classroom large group work with other students and during problem solving. The identification of mathematical learning opportunities in episodes, which is in turn related to the identification of LOEs, is theoretically grounded on the recognition of opportunities as being influential in the promotion of mathematical learning. We assume that the awareness of learning opportunities in the classroom discourse is a condition to use them for learning. Thus the ideal of identifying students' learning is being first approached through searching learning opportunities. It is our position that learning becomes more or less fostered depending on the participation in LOEs and the exposition to learning opportunities. Consequently, learning can be understood as an increase in the exposition to such opportunities.

What are the criteria for us to claim the existence of certain learning opportunities? Even though we have not always evidence of students taking advantage of particular opportunities, and experiencing processes of learning something new (Jose, Gabriel, Maria... for instance, may be already aware of a mathematical knowledge and they may be merely reminded of it by another participant), we sustain the idea of identifying potential opportunities for learning as useful. This option brings up the substantial problem of reaching a multiplicity of learning opportunities, some of which do not necessarily contribute to the construction of mathematical learning. We see the current final set of multiple mathematical learning opportunities as the starting point for the second part of our study. We plan to explore connections between learning and learning opportunities. So far, our analysis of classroom episodes has consisted of two main dimensions that have been articulated through the identification of interactional codes and learning opportunities. It is interesting to include a third dimension that helps examine the ways in which the students are supported in their mathematical learning by means of the exposition to certain social environments. Such prospective might serve as a basis to more precisely conceptualize LOEs, as well as to construct a typology of mathematical learning opportunities.

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PLACE OF THE CONVERSION OF SEMIOTIC REPRESENTATIONS IN THE DIDACTIC FRAMEWORK R^2C^2

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This article is about the place that I have allocated to the conversion of semiotic representations at the heart of the didactic framework R^2C^2. Ten 3^{rd} year primary school classes were involved in this study: eight in France and two in the Czech Republic. This framework was tested in four of the eight French classes.

After having presented the theoretical framework, I describe the methodology, based essentially on the analysis of professional practice of teachers, using video recordings of sessions as well as interviews. Then I try to highlight to what extent the principle of conversion of representations can encourage the conceptualization process among pupils.

INTRODUCTION

Whereas in more than three classes out of four, the teachers declare that at least once a week they present their pupils to numerical problem solving (Priolet, 2000), the pupils' competences in this field remain very variable. This has also been reported by the General Inspectorate of National Education (IGEN, 2006).

Pupils' recurrent difficulties in learning numerical problem solving led me to focus this research on the observation and analysis of the teaching practices in primary school classes and on the implementation of a didactic framework.

THEORETICAL FRAMEWORK

As soon as I started this research on numerical problem solving in primary school (Priolet, 2000), I focused on the teaching practises. I wondered to what extent the pupils are effectively exposed to the conversion of semiotic representations, which is considered fundamental by Duval (1995) in the comprehension process enabling pupils to access mathematical objects. All mathematical objects, including numbers, are not directly accessible by perception or observable with instruments; their accessibility is possible by semiotic representations belonging to a system with its own meaning and functioning constraints (Duval, 2001).

Semiotic representations can be discursive (e.g. native and formal language) or non-discursive (e.g. figures, graphs, charts). Duval classifies these representations in registers and, according to him, the originality of the mathematical activity lies in the coordination of at least two registers of representation at the same time and in the possibility of changing between them.

He defines *conversion* as a transformation that consists in changing registers while keeping the same objects and *treatment* as a transformation within the same register. He considers as fundamental for the construction of a concept the ability to perform conversions between several registers of semiotic representation of the same mathematical object (Duval, 1995).

Each time a change of register proves to be necessary or that two registers must be mobilised simultaneously, we face a rise in the number of pupils' failures or blocks, at all teaching levels (Duval, 2006).

Novotná (2003) observed that if the graphic language is not presented previously by the teacher, it is rarely used by the pupils. She showed, in particular, the help that a chart can bring to problem-solving (Novotná, 2001). However, she warns against the risk of imposing such support, given the fact that individual differences exist in the manner of treating semiotic representations.

Apart from Novotná's work, my theoretical framework is also based on Glaeser's work (1973) which insists on the necessity of effectively putting pupils in situations to search solutions to a given problem. Of importance is also Brousseau's theory of didactical situations (Brousseau, 1990), in particular for managing the teaching and learning experiences by the teacher and for the concept of devolving some decision making to students. My approach also encapsulates contributions from the cognitive psychology by Vergnaud (1990) on the importance of the categorisation of problem-situations according to the mathematical relations at stake.

R^2C^2 DIDACTIC FRAMEWORK AND RESEARCH HYPOTHESIS

This research on learning and teaching mathematical problem solving in primary school leads me to venture the following hypothesis.

The learning of numerical problem solving can be enhanced if the pupil is exposed to four principles: P1: Looking for solutions to problems; P2: Networking of knowledge; P3: Conversion of semiotic representations; P4: Categorisation of problem situations.

The teaching of problem solving, when putting together these four principles and looking at their devolution to the pupil, may contribute to teachers' professionalization. It may develop teachers' reflective attitude on how they teach problem solving and especially on how they devolve to the pupil the responsibility for solving the problem.

Linking to the various theoretical works mentioned, and in order to test my research hypothesis I have conceived and tested a didactic framework given the acronym R^2C^2 (<u>R</u>esearch, networking (<u>R</u>éseau), <u>C</u>onversion, <u>C</u>ategorization). This framework is based on the implementation of the four principles (P1, P2, P3, P4), under the conditions of their coexistence, of their regularity and of their devolution to the pupil.

This article aims to analyse and describe more specifically the implementation of principle P3.

EXPERIMENTATION

Ten third year primary school classes (eight in France and two in the Czech Republic) were involved in this study. The experiment followed three main phases:

In the first stage, at the start of the school year, I observed and analysed problem solving sessions laid out by ten teachers. The observations of these sessions, known as initial sessions, were filmed (type n°1 recording). Then, all the pupils were given a pre-test made up of 12 numerical problems.

In the second stage, from January to March, four classes in France selected randomly among the eight were exposed to the didactic framework R^2C^2. Problem-solving sessions were carried out in these four classes that formed the experimental group. These were also recorded (recording n°2).

The four other classes, which formed the control group in France, continued the work planned by the teacher within the framework of their usual problem solving teaching. At the end of the academic year, the pupils were exposed to a post-test exactly identical to the pre-test.

Simultaneously to this experimentation in French classes, with the help of Jarmila Novotná, I extended the observations to include two Czech classes. The pupils in both classes were given the same test (translated into Czech language) as the eight French classes.

Artefacts introduced in the operationalization of the R^2C^2 framework

The implementation of the R^2C^2 framework includes the introduction and use of artefacts denoted «reference boxes» and «reference dictionaries». They lead teachers to implement the four principles (P1, P2, P3 and P4) in the teaching of the pupils.

Each pupil in each experimental class has a reference box that can be either an envelope, an exercise book or a cardboard box. The reference boxes are intended to support, without imposing it, the linking of verbal statements and various representations; operation, drawing, graph, text etc., in order to foster the conversion from one register to another (Duval, 1995). The basis of solved problems which will grow little by little will link to networking and categorisation activities. Then, each reference box becomes a reference for each class of problems (Vergnaud, 1990).

Another artefact, the reference dictionary was used in organising teaching and learning, as conceived by Brousseau (1990). A dictionary was developed collectively in each of the four classes of the experimental group as verbal difficulties arose.

DISCUSSION

Description and analysis of usual problem-solving teaching in the 8 classes in France. The place of the conversion of representations.

The observations of Type 1 sessions, as well as the self-reflective interviews, show that pupils were mainly exposed to the use of the textual and numeric register, registers between which conversions occur when doing problem solving. The iconic register was essentially mobilised during correction stages, under the teacher's control. It was introduced by the teachers when collecting pupils' work, or by a pupil showing at the board the iconic representations traced in his/her book during the resolution of the problem. In conclusion, and in the limits of this experiment, it turns out that teachers endeavour most of the time but not exclusively to put their pupils in a position to look for solutions to problems.

The responsibility for linking students' previous knowledge as well as for the conversion of representations mainly belonged to the teacher. However, during the sessions, I noticed that not every teacher practiced at the same time the networking and the conversion. No categorisation activities were noticed during these observations.

After the analysis of the usual problem solving teaching practices of these eight teachers I focus on the implementation of the P3 principle in the didactic framework R^2C^2 in the four classes of the experimental group.

Interpretation of the implementation of the principle of the Conversion of Semiotic Representations (principle P3 of the didactic framework R^2C^2)

In the Type 2 sessions, due to the implementation of the didactic framework R^2C^2, I notice that it is the teacher who defines the rule and it is the pupil who has got the responsibility to choose what to do during the searching stage: drawing, writing, directly calculating. Through the use of reference boxes pupils are invited to use the iconic register, on the one hand to elaborate on a drawn representation of the situation described in the problem title and on the other hand, to schematise and model the situation. Before coming back to more general considerations on the implementation of principle P3, I will deal with the recourse to drawings and then the recourse to the graph drawing.

Resorting to a drawing:

Pupils are in presence of a textual utterance which presents a situation to be solved. In referring to work from learning psychology, I consider that this situation, even in the case where it is about a daily life situation, must be reconstructed by the reader in order to elaborate a mental representation (Johnson-Laird, 1983). Resorting to the drawing seems to force the pupil to do a step by step reading of the textual utterance, or several readings which could suggest a better identification of the data. We concur with Novotná (2003) who develops the view point that the construction of the mental representation of the situation could be facilitated by resorting to a drawing which would allow less work for the memory.

Resorting to drawn graphs:

If one is interested in the representation of diagram types in the Vergnaud sense, one can consider (Novotná, 2003), that resorting to a drawn graph is going to facilitate the heuristic process thanks to the manipulation, in written form, of mathematical relationships.

In this didactic framework, the material form of the reference box a blank box planned for a diagram, invites the pupil to produce it. According to Novotná (2003), without a previous presentation by the teacher, the graphic language is rarely used by pupils. In the R^2C^2 framework, the use of the artefact reference box makes the appeal to the recourse to this kind of conversion explicit. Nevertheless, I drew the teachers' attention to the possibility to run the risk, as Novotná points out, to compel the pupils to fill in all the squares of the reference box.

After having considered more in detail the traces belonging to the iconic register I consider the implementation conditions of the P3 principle.

Problem 3 (translated):

Last Friday, the president of the Amicale Laïque association came to give us sweets to congratulate us with the good results we had at the Foulées Vertes. On the package was written '100 sweets'. That day, three pupils were absent from our class.

In order to share the sweets equally between all the children present on that day, how many sweets should we give to each pupil?

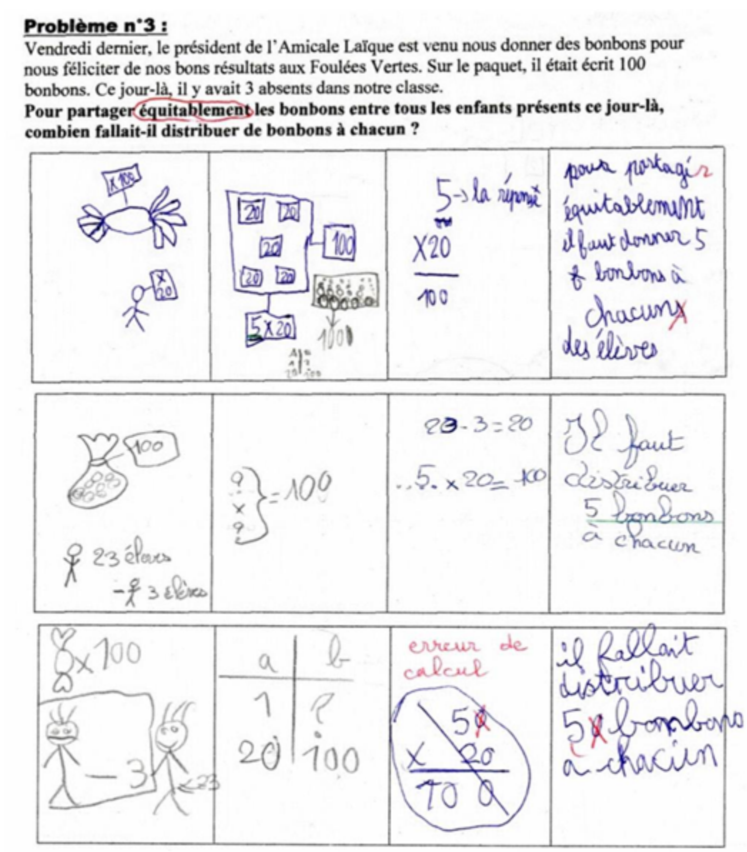

Figure 1: Resorting to the conversion of representations : Answers from three pupils in the same class of the experimental group.

I have chosen to illustrate the variety of the recourse to conversions of representations (calculation, drawing, diagram, text), through the work of three pupils, selected randomly, among those who had completed all the squares of the referent box (Figure 1). This selection must not hide the fact that the instructions in relation to the use of the artefact reference box specify that it is not imperative to fill in all the squares and then the recourse to the set of mentioned registers is not needed.

The regularity imposed in the use of reference boxes aims also to induce regularity in linking several registers. This regular implementation condition, mixing a variety of treatment tasks and register conversions, is mentioned by Pluvinage (1998). I am interested in the place of these operations in learning about problem solving. Duval (1995) points out the fundamental role played by conversion tasks between several registers of semiotic representations for the same mathematical object when constructing a concept.

The film recordings n°2 show a more personal involvement from the pupils, some involving themselves more in the conversion of representations, others not resorting to it because they proceed directly to problem-solving using an expert process. Others still feel authorised to use drawings or graphs. These observations show how the teachers take into account the conversion of representations (P3 principle) in the devolution to the pupils.

It also seems to me that the implementation of the P3 principle can have influence on the didactic contract (Brousseau, 1980) insofar as the pupil is now exposed to different representation possibilities accepted by the teacher. The analysis of Type 1 sessions revealed homogeneity in the initial practices of teachers, who demanded going through the normal presentation form solution/operation. This form, centred on a unique approach of resolution based on resorting to calculating mode, seems to me to encourage the use of the arithmetical operational technique during the learning stage (Priolet, 2008).

One can then consider that the fact of establishing, during Type 2 sessions, the use of different registers of representation modifies the didactic contract. However, the analysis of the documents that have been filmed, shows that some pupils produce an iconic type representation after having given the solution to the problem. One can see here again the effect of the didactic contract. The pupil wants to respond to the request to use different registers and fills in all the available boxes. It seems essential to consider, referring to Vergnaud (1997), that these drawings or graphs have a transitory status and that they are meant to be gradually forgotten as problems get mastered.

MAIN RESULTS

The quantitative analysis of the effects of the R^2C^2 didactic framework implemented in the four classes of the experimental group in France has been carried out by comparing the results of the pupils belonging to the control group to those of the pupils belonging to the experimental group during a pre-test and a post-test made up of the same twelve numerical problems. There were six problems concerning simple proportionality, one problem concerning multiplicative comparison and five problems requiring intermediate calculations. The score of each pupil was calculated by taking the number of attainments into account.

		Number of Pupils	Total score		Average		Difference
			Pre-test	Post-test	Pre-test	Post-test	
Groups	Control	65	304	364	4,68	5,60	0,92
	Experimental	72	315	457	4,38	6,35	1,97

Figure 2: Results from pre- and post-tests.

The difference (1,05) between the averages obtained at the pre-test and at the post-test (1,97 for the experimental group and 0,92 for the control-group) is significant:

$t_{Student} = 2,94 > 2,61$; ddl = 135 ; $p < 0,01$.

This study showed convincing results: the performances noticed in the control group increase by about one more problem solved, whereas the ones observed in the experimental group increase by about two more problems solved.

The pupils in the two Czech classes, not exposed to the R^2C^2 didactical framework, have also been given the same test with twelve numerical problems.

For these two Czech classes, the results of the test are better compared to those of the French experimental group (7,67 vs 6,35).

CONCLUSION

The main qualitative and quantitative results which emerge from this study come to reinforce the hypotheses I have stated.

The possibility of the possibility of using reference boxes encourages the pupil to the conversion (principle P3) between the textual, numerical and iconic registers, and thus favors the production of representations. I consider these productions, associated with a reading of the text and the drawing of diagrams, allowing to visualize the mathematical relationships at play, facilitate the construction of a mental pattern (Johnson-Laird, 1993).

The introduction of the artefact reference box makes it possible to consider a double networking: the networking of problems, according to underlying mathematical relationships (Vergnaud, 1997), which leads the pupils to an activity of categorization; the networking of diagrams which aims at the drafting of a model of these mathematical relationships. I can consider that the principle P3, thanks to the categorization and the establishment of a model, constitutes an element allowing to generate the process of conceptualization among pupils.

It should be mentioned that the results of the experimental group in France are inferior to those of the two Czech Republic classes observed in the framework of ordinary practices and not exposed to the R^2C^2 didactic framework. The pupils of the two Czech Republic classes considered it natural to work with similar problems, to try to use the register of representation best adapted to their problem solving strategy, to break down a new problem in several simple sub problems which they already knew how to solve (Priolet & Novotná, 2007). I notice similarities with the principles of the R^2C^2 didactic framework implemented in the four classes of our experimental group in France. That can explain the significant improvement compared to the four classes of the control group. However, the pupils in the French experimental group have only been exposed to the four principles of the R^2C^2 didactic framework for three months, whereas the practices observed in the Czech Republic were implemented each year and regularly throughout the school year. I attribute the superiority of the performances of the two Czech Republic classes to the long-term systematic work undertaken since the start of compulsory schooling.

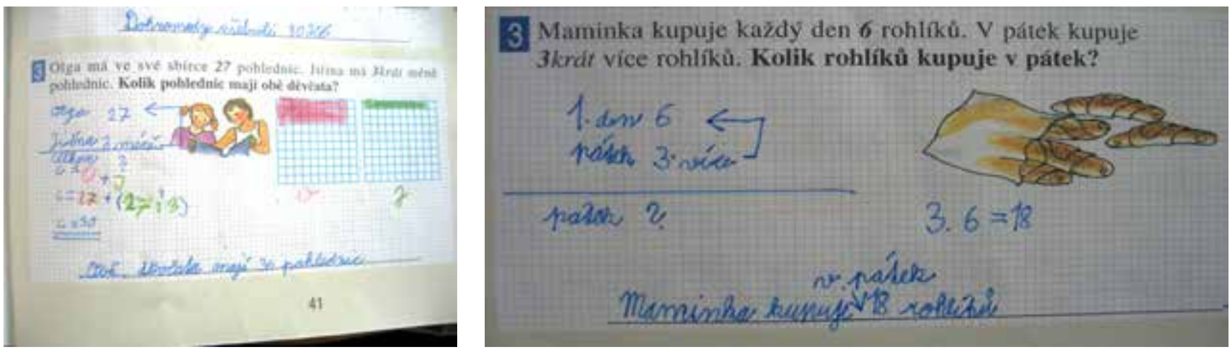

Figure 3: Czech Republic exercise books showing that pupils refer to different semiotic registers

The example below is extracted from a reflective interview and shows that the implementation of the P3 principle (the conversion of representations) had effectively brought to the teachers' awareness the consequences of the use of the presentation «solution-operation» that they imposed on their pupils. Indeed, this normal presentation, limited to textual and numeric registers leaves little place for conversions between registers.

> Teacher 5: I gave up the rigorous presentation «solution – operations» because I became aware that they thought they had to do operations with the numbers and that it translated by an operation and not necessarily by the right answer.

In conclusion, by a systemic approach, I can consider that the implementation of the R^2C^2 didactical framework favoured the process of conceptualization among pupils, however on approval that the principles are devolved to the pupil.

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THE USE OF ICT TO SUPPORT CHILDREN'S REFLECTIVE LANGUAGE

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The importance of language and learning with different artifacts in mathematics education is the focus in this paper. The language in the communicated situation or activity can be understood as a means of constructing meaning. The importance of context, different artifacts and communication in the teaching of mathematics is visualized through analyses of preschool, year 1 and year 2 in students´ work with numbers and place value. Through participation and interaction, through dialogue and using different kinds of artifacts, students develop new patterns of meaning that may facilitate their learning of mathematics.

INTRODUCTION

My interest is in how students in school learn or are taught number sense and place value problems. A socio-cultural perspective (Säljö, 2000; Vygotsky, 1986; Wertsch, 1991) on human development, reasoning and arguing calls for the use of linguistic and physical tools to analyze and make claims about events in the world. From this, I am interested in how people reason and argue and how they quantify and mathematize. In particular, how mathematizing and reasoning are connected when making a claim or by proving a point in a conversation involves people moving between everyday discourse and analytical languages.

In the new Swedish curriculum (Skolverket, 2011), students are expected to formulate and solve problems with mathematics, to formulate strategies and methods, to use and analyze mathematical concepts and relations between concepts and to use the language of mathematics in order to explain and justify what they have done. To see how different artifacts and communication influence the gaining of this knowledge is important for teachers.

Learning and knowledge about the surrounding world begins in what can be called the life world. Nevertheless, the dissonance between the environment inside and outside school is very distinct, especially with regard to mathematics as a school subject. Consequently, students´ interpretations of real life problems may become an obstacle for their learning in mathematics. The focus of this paper is how the language and physical tools in the communicative situation or activity could be understood as leading to the construction of meaning. The importance of context, tools and communication in the teaching of mathematics is visualized through analyses of first and second grade students´ work with number sense, place value and different tools.

In a socio-cultural perspective, the question is about how knowledge and information are learned and how this learning changes in new surroundings (Lave & Wenger, 1991). All activity, including the mental processes of thinking and reasoning, is mediated by tools and signs. Knowledge is developed through the cultural tools that are in the society (Vygotsky, 1978). Students in school have an ability to use artifacts that come from the real world. The artifacts are central because they change relationships between people and the world around them. For example, when teachers are introducing signs and symbols in the teaching of mathematics they try to mediate from a reality to a mathematical model. The teachers use language, such as pictures, symbols, signs, words and so on, when working with physical tools in order to get students to understand mathematical modeling. However, in spite of the use of these common artifacts, students have difficulties learning mathematics, because they are not able to use their knowledge and see mental pictures. In this study I describe different artifacts and discuss how students mediate with artifacts and with each other. In a socio-cultural perspective the mediation is very important. Therefore, the research question is: how does the interaction between students and their choice of language change when they use different artifacts in mathematics?

DIFFERENT KINDS OF ARTIFACTS

Wartofsky describes three different kinds of artifacts which are used in learning. These are categorized as primary, secondary and tertiary (Wartofsky, 1979).

Primary artifacts can be tools like axes, clubs, hammers, pins, computers, and mobile phones. They can be used directly in production processes and are based on utilizing people’s strength to solve different kinds of problems. However, understanding how to use those artifacts involves understanding the special surroundings where they are used. In this way, they help people to understand their world and to participate in the life that they live. In schools, there are many primary artifacts, usually called concrete materials that are used in the teaching and learning of mathematics. These include stones, bottle tops and Cuisenaire rods. Primary artifacts can be used without knowing how they help to complete the task that they are used for. They can also be talked about without knowing how they perform the teaching and learning job that we expect of them. However, this complicates the teaching and learning process when the purpose of the tools remains unspecified.

In the present day and age, it is necessary to understand how to use tools such as the computer and the Ipad in order to learn how to live in the world. Thus, communication and knowledge change with these new artifacts. To use a mobile phone, the user has to understand how to handle it and to do this he/she has to study the instruction manual. This requires having the knowledge to comprehend what is read and then translate this into an understanding of how to use the mobile phone. Secondary artifacts are tools that provide models for how to use tools and how to think about how to use those tools. They are externally embodied representations that collect and describe information. Secondary artifacts control decision making and planning processes. There are many of them in everyday life, such as recipes, drafts, designs and ways of classifying the world. In mathematics education, secondary artifacts are things such as mathematics textbooks. The pupils learn how to study a math problem from the textbook and also how to solve it.

However, when primary and secondary artifacts are used at the same time, the result could be that the teacher is teaching in one way and the students are learning in another. This can be seen in the following example.

In this example, the teacher provides a primary artifact such as base ten arithmetic blocks which the students are expected to use to illustrate whole numbers and decimal fractions. The instruction is:

> 2,1   1,9   0,8   0,01   1,25
>
> Use the material to represent these numbers, then have a friend write the amount that they see as numbers with decimal fraction parts. Discuss!
>
> Write your own five numbers and let a friend put it in a place value table. Discuss!

The students listen to the teacher and try to translate the information into specific actions. To do this, they must position the material (the primary artifact) appropriately, study the written instructions (the secondary artifact) and discuss what they are doing. In the second part of the activity, they still use a secondary artifact, the written instructions, but this time they produce new numbers which a friend must then illustrate with the blocks. The exercise could be changed so that the students would need to use new tools. For example, the students could use an empty number line on which they would have to put out their own numbers and find the difference between the numbers. Alternatively, they could work with computers to construct, change and compare the numbers and talk about what happens and why. When the students have been using a variety of tools and language they can start reflecting and talking about their knowledge of decimal fractions (Riesbeck, 2011).

Tertiary artifacts are tools that help people to construct new knowledge. They support people’s reflection and communicating skills. In order to learn mathematics students need not only to see and translate information but to know how to mediate knowledge with tertiary artifacts so that they develop their thoughts and representations of their understandings. Thus, it is important for teachers to understand how to move from having students use primary artifacts to developing their reflective skills from using tertiary artifacts.

For example, if students can construct geometrical figures using a computer program, they can think and talk together including critically discussing issues as they arise and finding new forms of arguing. Communication and language are very important to all human learning and it is through new language and tools we develop new knowledge. Through using language as a tool human beings can understand and construct new knowledge (Säljö, 2005).

Wartofsky (1979) suggests that people are dependent on artifacts in their doing and thinking. Primary artifacts help when people want to have a better understanding of the physical meanings connected to the world. On the other hand, secondary artifacts helps us in our communication and understanding of the world and the physical tools and the tertiary artifact is a thinking and creative tool.

THE DIFFERENT LANGUAGES CONNECTED TO THE ARTIFACTS.

The descriptions of the three kinds of artifacts suggest that all artifacts are needed for learning but the tertiary artifacts have extra significance, because of the sort of language associated with them, reflective language, and how this contributes to learning. When mathematics education expects students to learn only with books the language and thinking are connected solely to that of the secondary artifacts. Students can work with primary artifacts, such as the book doing the same type of exercises without thinking or reflecting (Johansson, 2005). They can become familiar with the problem in the textbook, if they can translate the text into actions, thus using the textbook as a secondary artifact.

However, when the goal is to learn mathematics, students need to talk about what they have read and they need to understand the new knowledge specified in the curriculum. In this case, it is necessary to work with tertiary artifacts. Pimm (1987) suggested that in any mathematics classroom there are many different languages. Students use conversational language which they have learnt at home or with friends. Students will also use a more mathematical language when they talk to each other while using concrete materials to learn and do mathematics. There is also the symbolic language of mathematics being used when students have to use concepts from mathematics and to argue and talk about how to solve the problems. There is also evidence of a reflective language when the students in the dialogue discuss and answer questions about why a specific answer is correct or if the solution is sufficient (Verschaffel, 2002).

The use of different artifacts will affect what languages are used. As Vygotsky (1986) put it, thought and language is a parallel project. In this perspective knowledge develops in interaction between people and between people and tools (Kozylin, 1998; Vygotsky, 1978). A dialogic nature to discourse, where students learn to question, argue, explain, justify and generalize will be achieved through the models provided by teachers and others in interactions with different artifacts (Riesbeck, 2008). Through using different kinds of reflecting questions, classrooms of mathematical inquiry which support language and thinking can develop (Hunter, 2007). Therefore, when students learn mathematics they also will use different languages depending on what kind of mediating tools they use.

The concepts of context, mediation and artifacts are central to the sociocultural perspective. The concept of context can be described as being the environment where the action takes place. Mediation implies that human beings interact with external tools in their perception of the world around them. Linguistic as well as physical artifacts are created by people to perform actions and solve problems. Using semiotic tools one can demonstrate how a linguistic element is connected to its meaning.

The semiotic triangle is made up of concept, symbols and objects which illustrate the possibilities for mediation. The semiotic triangle is used when analyzing interaction of the students when they are talking about numbers and place value.

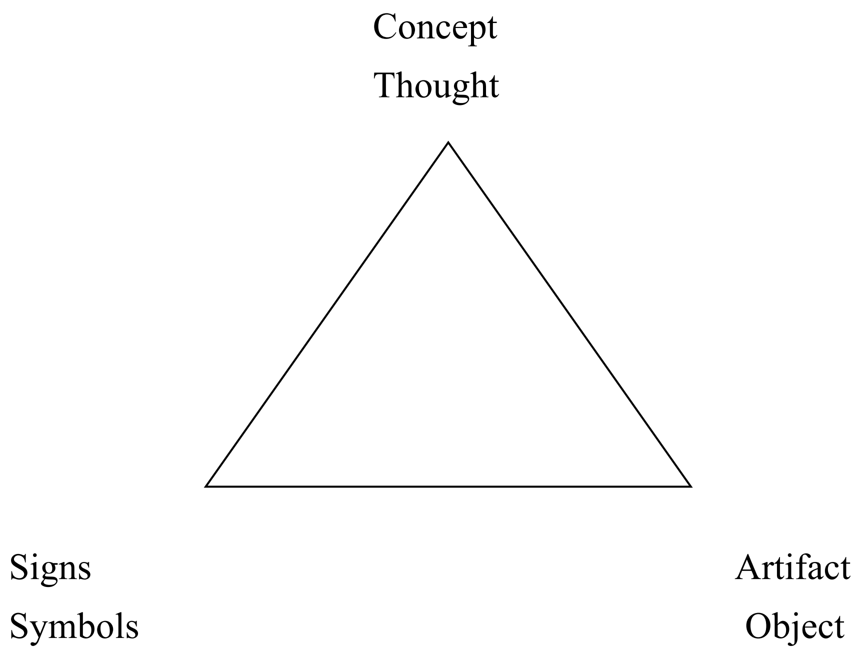

Figure 1. The semiotic triangle.

The semiotic triangle (Johnsen-Høines, 2000; Mellin-Olsen, 1984; Ogden & Richards, 1923) helps us to mediate from all three angles and is a tool to analyze the material in this study. This means that the teacher choose an artifact for the children when they for example work with blocks on the computer to understand place value and you analyze what kind of symbols or concepts there are when they are talking in this situation. What and in what ways are the children talking about math with different artifacts?

METHODOLOGY – THE STUDY

The study investigates the same set of pupils and their teacher as they moved between grade 0 (preschool class), 1 and 2 in a Swedish school for three years. Audio and video recordings were used to collect the discussions in interactive situations between teacher – children or child - child. In this paper a small part of the study is going to show how different tools can support different languages.

The children were six years old when the study began. The recorded lessons were about numbers and the teacher stood at the front of the class and used an interactive whiteboard. The sign 5 appeared on the interactive white board. The teacher and the children talked about this number five and the children told the teacher about pictures of different things. They went up to the board and moved pictures like rabbits, dogs and so on whilst they counted one, two, three, four and five. Then they sat in groups with concrete material talking and manipulating the material. The teacher followed this up by returning to the interactive whiteboard to show a film about the number five. During this year when the children were in preschool class, lessons using the interactive whiteboard and the teacher having a dialogue with the pupils about numbers appeared every week. The mediating process was about numbers and pictures and using the interactive board and the language was mostly everyday language.

In the following year when the children were seven years old, the teacher started one lesson about the little zero using the interactive board. The transcript is an excerpt from that lesson. In the transcript, L stands for the teacher whilst E stands for various pupils.

L: Do you remember the book we read about the little zero and the pictures we looked at? How was the little zero? How did the little zero feel in the beginning? Tell me!

E: It felt very lonely.

L: How did the little zero feel when he had said hello to everybody in the houses? What did all the other numbers say to him?

E: That he was nothing.

E: You are only a zero.

L: What happened later on? Where did the little zero go? The zero walked away to places where none of the other numbers could go. Where did the zero go?

E: Into the cave.

E: The zero was so upset.

L: Why?

E: He could not find a zero so he could count his money.

L: The zeros that could stand after the numbers so we can find big numbers. If a number stands for himself he isn´t worth so much. But if you put several numbers together they can be worth much.

In the first part of the teacher- pupil interaction the language is about the real world and they are talking about feelings for the little zero. But here the teacher works further on and begins talking about where to put the zero and what can happen with the value of the money. They are on their way to a more mathematical language. They were later on in the lesson sitting in front of the computer two by two, talking about where to put the zero and what numbers are in this place value table. Here they use a mathematical language.

In another situation the teacher and the pupils are working in a program showing blocks of one and blocks of ten. The pupils put the numbers under right block.

L: I put in ones and tens . What is the number?

E: The number is three.

L: Yes if you add it together. But I mean if you write it together as one number. What is the number?

E: The number is twelve.

L: How many tens are there in twelve?

E: One one and two tens.

L: But you can also write another number with those numbers.

E: Three

E: 21

L. How many tens are there in twenty?

E: Two

L: How many ones?

E: One

The teacher has several of this kind of lessons, interacting with the interactive whiteboard and the students. The pupils now are on their way to the language of mathematics. They get some help from the program on the interactive board and the discussions between the teacher and the pupils. In another lesson, the children were sitting in pairs in front of the computer working in the same program as they had done together with the interactive whiteboard but now they are going to use addition and subtraction. The numbers are 56+23. First they find 50, say that it is five tens and take five ten blocks on the computer to show. After that you can see the pupils say twenty and put two tens on the program to show each other and then they take in six and three unit blocks. They have a paper and they have to write down the answer and say how they got to the answer.

In the program on the computer the pupils show how they place the numbers under the right place value and talk about it. The different kinds of artifact change the pupils' use of language.

Another example from the class is when the pupils use Ipads. In pairs, they had to explain place value using mathematical words and using the program on the computer and then they are going to film with the Ipad. They practiced this together several times and when they were ready they used the Ipad as a video camera and filmed each other. This communication had to be perfect, so they worked hard and filmed each other several times until they thought it was good enough to show to the teacher and the rest of the class. The next transcript illustrates how one boy, was discussing place value in relation to addition and subtraction on the Ipad as the teacher was watching.

E: Hello, my name is Adam and I will show you how I´ve learnt place value.

E. The addition is 78+21.

E: I first take seven tens and then eight ones and then I take two tens and then it is 98 and after that I add one one and then it is 99.

E: The subtraction is 49-36.

E: First I take four tens and then nine ones and then I take away three tens and then I have 19 and after that, no I don´t know. We do it again!

L: But you have 19 and you should take away 6 ones.

E: But I made it in a wrong way.

They do it again. Another pupil is counting and talking in front of the Ipad.

E: I first take four tens and then nine ones and after that I take away three tens and then it is 19 and then…..

L: And then you have to take away six ones.

E: But I never put them in there.

L: You had 49 from the beginning and take away three tens.

E.: And take away six ones.

Most of the students were doing this in pairs and they were filming until they were convinced that everything was perfect. To use an Ipad filming the pupils' understanding of place value is one step to the tertiary artifact when the pupils learn to reflect on what they know and why they know it.

The different artifacts, interactive board, computer and Ipad, that the teacher used showed that the pupils were in different contexts and their language changed when they used different kinds of artifacts. If you use tools from daily life contexts or from mathematics contexts you get different language and learning. From a mathematical perspective the pupils show that they share the mathematical language and understanding of the signs and symbols of mathematics and they can in their talking use tens and hundreds.

DISCUSSION

If teachers are to support students using more abstract language and thoughts in mathematics, then there is a need to understand the impact of different artifacts in the classroom. When teachers use primary artifacts like concrete materials and computers, they have to think of the dialogue they want to develop. If the material is to be translated to a secondary artifact, then the students need to learn how to translate from the concrete material to the mathematical concepts. They also have to learn to write and reflect on their knowledge and ask questions like why is this answer right, or describe the process they have used and their decisions for using a particular model. The students must also actively engage with the language and the artifact at the same time through dialogue. Then it is possible for you to say that you have started to learn mathematics. Figure 2 sums up how using an artifact can support students developing their knowledge of mathematical concepts. Before arriving in the middle it is necessary to know things from the other blocks.

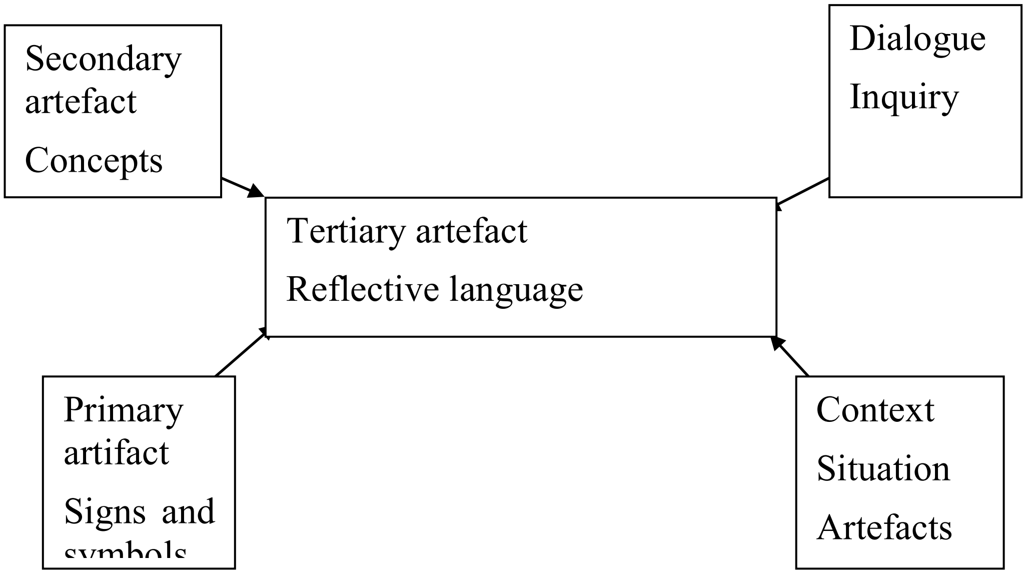

Figure 2. The reflective language as a tertiary artifact

In the middle is the tertiary artifact where the teacher and the students are reflecting and talking with the mathematical language in dialogue. Here they use signs, symbols and concepts from mathematics. In this study one can find the program the teachers and pupils are working with to support the reflective language.

The students are going to develop a language not to talk about their doing but talking about why the process or the product could be like that and they also document their new knowledge in mathematics.

Use of a tertiary artifact can be recognized in the qualities of language, tools and knowledge both from having a picture from the everyday world and also from the mathematical sphere. In this context, quality means that students communicate around a sign, a concept and a situation by looking critically at it, putting forth arguments for and against and document their own learning. To further their acquisition of new knowledge develops into an issue of learning to apply abstract and complex intellectual and practical tools, and to accomplish this the more essential it becomes to engage in communicative practices with different tools (Riesbeck, 2011).

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A COMPARISON OF IRISH AND ENGLISH LANGUAGE FEATURES AND THE POTENTIAL IMPACT ON MATHEMATICAL PROCESSING

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This paper presents some insights into the syntactical and semantic differences between Irish and English, and the potential impact on mathematical processing. Previous research suggests that learning mathematics through the medium of Irish at primary level education may enhance mathematical understanding (Gilleece, Shiel, Clerkin, & Millar, 2011; Ní Ríordáin, 2011). A key question being addressed here is whether characteristics of the Irish language potentially have a different effect on students' mathematical processing of text. No examination has been undertaken on the Irish language and its potential impact on cognitive processing. This initial analysis is striving to provide some useful insights for further investigation.

INTRODUCTION

Recent studies conducted in Ireland into Irish-medium education have demonstrated positive cognitive advantages in relation to mathematical and English reading attainment in comparison to students who attend all-English medium education (Gilleece, Shiel, Clerkin, & Millar, 2011; Ní Ríordáin, 2011). These investigations accentuate the positive benefits that can be reaped from being bilingual and are consistent with international findings in relation to the benefits of bilingual and immersion education (e.g. Bourton-Trites & Reeder, 2001; Clarkson, 2007; Genessee, 1987; Turnbull, Hart, & Lapkin, 2000). For Gaeilgeoirí (students who learn through the medium of Irish) in the transition from primary (Irish-medium) to second level (English-medium) mathematics education a significant relationship exists between their performance on the mathematical word problems through the medium of English and their Irish language proficiency (Ní Ríordáin, 2011). Gaeilgeoirí with high proficiency in both languages, and those who were dominant in Irish, performed mathematically better than their monolingual peers. This suggests that learning mathematics through the medium of Irish at primary level education may enhance mathematical understanding. A recommendation arising from my study was to review the Irish language with respect to the effect on the understanding and processing of mathematical text. This paper presents an initial analysis of the Irish language and its differences with the English language in relation to some grammatical features.

THE ROLE OF LANGUAGE IN MATHEMATICS

Language and communication are essential elements of learning and teaching mathematics (Gorgorió & Planas, 2001) and thus the language we initially learn mathematics through will provide the foundations to be built upon and developed within that language. Language is employed as a communication tool and facilitates the transmission of (mathematical) knowledge. We consider mathematical language as a distinct 'register' within a natural language e.g. Gaeilge or English, which is described by Halliday (1975, p. 65) as "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings." Each language will have its own distinct mathematics register and ways in which mathematical meaning is expressed in that language.

Vygotsky was one of the earliest theorists to begin researching the area of learning and its association with language. He concluded that language is inextricably linked with thought – "the concept does not attain to individual and independent life until it had found a distinct linguistic embodiment" (Vygotsky, 1962, p. 4). Although a thought comes to life in external speech, in inner speech energy is focused on words to facilitate the generation of a thought. If this is the case, it raises an important question – does the nature of the language used affect the nature of the thought processes themselves? The transition from thought to language is complex as thought has its own structure. It is not an automatic process and thought only comes into being through meaning and fulfils itself in words. Thought is mediated both externally by signs and internally by word meanings (Vygotsky, 1962). Communication is only achieved by the thought first passing through meanings and then through words. Therefore, language will play a significant role in the processing of mathematical text and the development of understanding (Hoosain, 1991).

Mathematics is not "language free" and due to its particular vocabulary, syntax and discourse it can cause problems for students learning and the development of understanding (Barton & Neville-Barton, 2003). A number of mathematics education researchers have identified characteristics of the English language that may impede mathematical learning (e.g. Austin & Howson, 1979; Durkin & Shire, 1991; Rudner, 1978; Wareham, 1993). In her review article, Galligan (2001) highlights the differences between Chinese and English in terms of vocabulary, word-order, use of prepositions and other grammatical features, and their impact on the understanding and processing of mathematical word problems. It raises the question of whether the language of learning impacts on mathematical ability and the processing of text. Other studies have highlighted a Chinese language advantage in relation to number sense (Fuson & Kwong, 1991), fractions (Bell, 1995) and logical connectives (Zepp, Monin, & Lei, 1987). It is important to note that when comparing mathematics processing in different languages and across different cultures, that there are many factors to consider e.g. social, political or pedagogical differences (Setati & Planas, 2012). However, what is of concern to the author is whether the Irish language potentially has a different effect on student attainment in mathematics. As Barton (2008, p.11) states "languages are examined not so much for their linguistic characteristics, but for their mathematical ones." No examination has been undertaken on the Irish language and its potential impact on mathematical cognitive processing – this initial analysis is striving to provide some useful *insights* for further investigation and some explanations for previous findings within the Irish context.

DEVELOPMENT OF THE RESEARCH

The research undertaken stems from the author's PhD studies (Ní Ríordáin, 2011). In this study the relationship between mathematics and language proficiency in Irish/English was examined, with psycholinguistic theories informing research practices. The findings provide support for Cummins Threshold Hypothesis in that Gaeilgeoirí with a high proficiency in both languages performed mathematically better than those dominant in one language and better than their monolingual peers. At second level education high competence in Irish was shown to facilitate the transition to English medium education (Ní Ríordáin & O'Donoghue, 2009). The study also found that Gaeilgeoirí encounter difficulties with the syntax, semantics and mathematics vocabulary of the English mathematics register (Ní Ríordáin & O'Donoghue, 2011). Gilleece et al. (2011) studied primary pupils' performance on national mathematics and English reading assessments and found that Gaeilgeoirí in 2^{nd} class (approx. age 8) and 6^{th} class (approx. age 12) in Irish-medium education are performing the same, or better, than their monolingual peers nationally in mathematics and in English reading.

More recent research investigating a bilingual approach (English and Irish) in third level mathematics education in Ireland, demonstrates that on average bilingual students performed better mathematically than those choosing a monolingual approach (Ní Ríordáin & McCluckey, 2012). The studies undertaken in the Irish context suggests that there are advantages associated with having two languages (Irish and English) for mathematics learning, and that learning mathematics through Irish may lead to cognitive advantages for these students. Therefore, this study goes some way in identifying some of the potential advantages that these students may experience by utilising Irish when engaged in mathematical problem solving.

METHODOLOGY

The list of differences between Irish and English are the consequences of a bibliographical review on several works around learning mathematics and language use. In particular, studies addressing mathematical processing in English (e.g. Austin & Howson, 1979; Galligan, 2001; Li & Thompson, 1981, Wareham, 1993) were of significance, as well as studies highlighting the key characteristics of the Irish language (e.g. Hickey, 1985; Mac Murchú, 1997). By utilising these studies I was provided with a foundation and framework for progressing with an initial analysis of the Irish language and its potential impact upon mathematical processing. Mathematics state examination papers (available in English and Irish) and mathematics textbooks (available in English and Irish) were analyzed in relation to the identified features arising from the literature review.

DIFFERENCES BETWEEN IRISH AND ENGLISH

In this section I summarise some of the key differences between the Irish and English languages, drawing on previous research. A comparison is undertaken in terms of some of the key features associated with the syntax and semantics of both languages.

Subordinate clauses and sentence length

In modern Irish, longer sentences are common, with the use of subordinate clauses. However, in many of the mathematical examples studied in current textbooks and state examination papers, shorter sentences are apparent. For example:

> *Tá poll in aice leis an mbun in umar sorcóireach oscailte d'uisce. Is é ga an umair ná 52cm.*
>
> Translation (T): There is hole near bottom in tank cylindrical open of water. It is radius of the tank 52cm.
>
> English (E): An open cylindrical tank of water has a hole near the bottom, with a radius of 52cm.

Shorter sentences lend to an easier understanding of mathematical text and are a desirable feature (Austin & Howson, 1979; Wareham, 1993).

Topic prominence

English is classified as a subject prominent language (Li & Thompson, 1981), whereas Irish tends to be a topic prominent language. For example:

> *Is slánuimhir é ceann amháin de na luachanna sin*.
>
> T: It integer is one of the values these.
>
> E: One of these values is an integer.

In this example the Irish reader's attention is drawn to the *integer*, whereas the English reader is drawn to the *value*. Therefore, the Irish reader is pointed to the topic of the sentence (Galligan, 2001).

Word order

Irish possesses the unusual word order Verb (V) – Subject (S) – Object (O), whereas English is classified as SVO (Galligan, 2001). For example:

> *Faigh comhordanáidí an dá phointe ina dtrasnaíonn na cuair* $y = f(x)$ *agus* $y = g(x)$ *a chéile.*
>
> T: Find coordinates the two points intersect the curves $y = f(x)$ and $y = g(x)$ each other
>
> E: Find the coordinates of the two points where the curves $y = f(x)$ and $y = g(x)$ intersect.

The placing of information and the unknown in sentences may have an impact on the ease of processing the sentence (Galligan, 2001; MacGregor, 1993). From the above example, English readers have a greater cognitive processing load in that they must hold in memory *co-ordinates of the two points* before reading the words *curves...intersect*. Whereas Irish readers are drawn to the key information of the sentence and this suggests a difference in mathematical processing in Irish.

Question structure

Irish has no words for "yes" and "no". The answer to a question contains a repetition of the verb, either with or without a negative particle (Hickey, 1985). For example:

> *An éisteann Seán lena mháthair riamh?* - "Does Seán ever listen to his mother?"
>
> - *Éisteann* - "Yes , he does"
> - *Ní éisteann* – "No, he doesn't"

In Irish, the question word tends to be placed at the start of a mathematical sentence and the syntactic structure is simpler when compared to English. Following Galligan (2001, p.117), "English question structure is more varied, and the change from the question to the answer requires changes to word structure and verb morphology."

Passive voice

Irish commonly uses the impersonal form (also known as the autonomous form) instead of the passive voice. For example:

> *Líonadh an umar le h-uisce.*
>
> T: One filled the tank with water.
>
> E: Someone filled the tank with water/The tank was filled with water.

The word endings 'adh'/'eadh' are used to indicate the passive (Mac Murchú, 1997), and therefore provides students with a strong cue when engaged in mathematical problem solving. English mathematical word problems have been criticised for difficult passive constructions (see Slobin, 1973 – reversible sentences), thus impacting the processing of mathematical text.

Redundancy

Rudner (1978) found that inferential and low information pronouns are sources of difficulty and hinder students' interpretation and understanding of English mathematical word problems. Mathematical text in Irish tends to be more wordy, thus impacting on reading time, but understanding may be clearer (Galligan, 2001). For example:

> *Cé mhéad soicind a bheidh caite nuair a bheidh aired 64cm ag an dromchla?*
>
> T: How many seconds will have passed when height 64cm has surface?
>
> E: After how many seconds will it be a height of 64cm?

Dialects

There are three dialects of spoken Irish – Munster, Connacht and Ulster. Some spelling conventions are common to all the dialects, while others vary from dialect to dialect (Mac Murchú, 1997). In addition, individual words may have in any given dialect a pronunciation that is not reflected by the spelling. Therefore, Irish can be a difficult language to interpret due to variation. Accordingly Irish students may need to engage more with a written mathematical problem due to the diverse nature of the Irish language. Consequently Gaeilgeoirí may develop stronger mathematical problem skills relating to comprehension and transformation.

Alphabet

The Irish alphabet consists of (Mac Murchú, 1997):

- Vowels **a, e, i, o, u**

  With an acute accent (sineadh fada) shows the length of the vowel **á, é, í, ó, ú**

- Consonants **b, c, d, f, g, h, l, m, n, p, r, s, t**

  The constant **h** serves as a notation lenition (bh, ch, dh, etc.) and as the h-prefix (ha, he, etc.).

Given that there are fewer characters used to write Irish, the meanings are more variable and hence context is more important when dealing with mathematical text.

Access to meaning

Orthography plays a key role in reading and processing mathematical text (Galligan, 2001). The nature of some of the Irish mathematics vocabulary allows readers to access the direct meaning of the words. For example, the word in Irish for velocity is ‘treoluas’ (direction speed) and parallel is ‘comhthreomhar’ (equal directionality). Many of the Irish words describe concepts/objects as opposed to just labelling them. Given that the more easily and quickly the meaning of words is activated, the simpler it is to process mathematical text. It may help to retrieve all the words associated with the concept thus enhancing the total cognitive structure (Galligan, 2001).

DISCUSSION

This initial comparison of the Irish and English language demonstrates that there are differences between the two languages. However, what is difficult to interpret is whether differences between the languages have a differential impact upon cognitive processing (Galligan, 2001). The syntactical structure of the Irish language in terms of sentence length, topical prominence and word order, appears to lend itself to easier interpretation of mathematical meaning in comparison to English. Accordingly, Irish may lend itself to easier mathematical word problem solving and the acquisition of enhanced processing skills. In particular it raises the question of whether Gaeilgeoirí have faster and more accurate access to mathematical text and accordingly strategies for arriving at a solution.

A significant insight from the analysis is that some Irish words assist in conveying meaning and/or permit the concept to be formed more readily. Similarly the sentence structure allows access to key information. Given that Irish readers are drawn to the key information of the sentence, and that this suggests a difference in mathematical processing in Irish, these surface features may aid mathematical problem solving, while providing a support for developing a deeper understanding of the word problem.

Also, context plays a key role in mathematics and in the interpretation of mathematical text. Given that the meanings are more variable in Irish and hence context is more important when dealing with mathematical text, this may lend to the development of better mathematical problem solving skills for Gaeilgeoirí. In particular, the author would suggest that it may lead to Gaeilgeoirí "reading more carefully and accurately because they have to rely on context more" (Galligan, 2001, p.126).

When comparing English and Irish, visually they do not appear to be significantly different. However, the syntax and semantics of a language plays a crucial role in interpreting mathematical text and developing meaning (Galligan, 2001). A key question arising from this analysis is do students who learn through the medium of Irish employ different processing strategies when interpreting mathematical text, and consequently does this impact on mathematical attainment and understanding?

I propose that proficiency in Irish and experience of learning mathematics through Irish may lead to cognitive advantages and enhanced processing of mathematical text (as outlined in previous sections) and accordingly may contribute to the development of this additive bilingualism. Additive bilingualism results when a second language and culture have been acquired without loss or displacement of an individual's first language and culture, and a positive self-concept is correlated with this form of bilingualism (Baker, 1996). However, subtractive bilingualism results when an individual's first language and culture are replaced by the new language and culture, usually occurring in a pressurised environment. As a consequence a negative self-concept may develop due to loss of culture and identity (Baker, 1996). Within the Irish context additive bilingualism is fostered through Immersion (Irish-medium) education at both primary and post-primary levels (Ní Ríordáin, 2011). High ability Irish bilingual students (additive bilingualism) display an enhanced meta-cognitive ability demonstrating flexibility in thinking and reasoning, self-correction, and an ability to select appropriate features for problem solving (Ní Ríordáin & McCluskey, 2012). This reinforces the point that bilingualism in Irish and English has the potential to enhance mathematical teaching and learning, while doing mathematics through the medium of Irish may contribute to enhance mathematical processing.

CONCLUSION

This paper presents an initial analysis of the comparison between Irish and English language and its potential impact on mathematical processing. Some promising insights are emerging, suggesting that students who learn through the medium of Irish may experience advantages in terms of processing mathematical text. Clearly further research is warranted in this area. Further analysis is needed into whether Irish language processing strategies have an impact on the way Gaeilgeoirí understand mathematical text and solve mathematical word problems. Moreover, do these processing skills transfer to a new language of learning and to the development of additive bilingualism? In particular, investigation is needed into how a particular language and its syntactical structure may impact on mathematical activity and reasoning (Morgan, Tang, & Sfard, 2011). The mathematics register is more than just the language medium of learning and therefore development of this study needs to incorporate other areas of study such as discourse analysis, socio-cultural aspects, linguistic theory, and semiotics. Moreover, there is a clear need for the development of descriptive tools/frameworks for the analysis of mathematical registers, in order to assess their impact on cognitive processing and mathematical reasoning, and to aid comparisons of different/multiple languages for learning mathematics. An in-depth study into the nature of the Irish language and its mathematical register could contribute to significant insights into the cognitive benefits reaped from being bilingual, whilst developing valuable analytical tools to be utilised in other contexts.

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MAKING SENSE OF FRACTIONS GIVEN WITH DIFFERENT SEMIOTIC REPRESENTATIONS

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This paper is based on observations of a group of 20 pupils in grade four in a Norwegian primary school. The pupils are presented with a task involving fractions. In the task the pupils are asked to judge the relative size of some simple fractions and also to identify equivalent fractions. The fractions in the task are linked to a context about milk boxes with different volume and the task also involves conversions between measuring units. These pupils are at a very early stage in their learning of fractions and my interest is mainly in inquiring into how they make sense of the situations they are exposed to, with special emphasis on semiotic representations.

Keywords: Fractions, semiotic representations, mediating artefacts.

INTRODUCTION

In a previous paper for CERME (Rønning, 2010) I have reported on a study of grade 4 children involved in the practical task of measuring out 15 dl of milk from boxes containing a quarter of a litre. The boxes were labelled 1/4 liter[1] and in that paper I discussed how the children interpreted the sign 1/4 and to what extent the interpretation had any effect on completing the measuring task. I showed that the presence of a measuring beaker as a mediating artefact to a large extent made it redundant to actually make sense of the sign 1/4. In this paper I report on a study of the exact same pupils, grouped in the same groups as before, working on a task that is mathematically similar to the measuring task reported on before but in terms of the representations used it is quite different. Here the children are presented with a task, given as a text accompanied by pictures, which they are asked to discuss and solve. My main research question is how the children make sense of the fractions given with different representations. I am in particular interested in how the children argue about the relative size of the fractions, how they argue about equivalent fractions, and how they handle fractions larger than 1. I will make connections to the situation described in (Rønning, 2010) and compare this to the situation described here.

THEORETICAL FRAMEWORK

In this study the notion of a sign is central. According to Steinbring a sign typically has two functions, a semiotic function; "something that stands for something else", and an epistemological function, indicating "possibilities with which the signs are endowed as means of knowing the objects of knowledge" (Steinbring, 2006, p. 134). What is special for mathematics, in contrast to other subjects in school, is that all the objects of study are abstract and that they can only be accessed using signs and semiotic representations (Duval, 2006, p. 107). Despite the abstract nature of the mathematical objects, mathematics is used as a tool to describe and make predictions about real life situations. A sign can therefore refer both to a mathematical concept as well as to a real life situation. In this study this dual nature can be exemplified through the sign 1/4, used as a representation for the mathematical concept "the fraction one over four", as well as for an actual quantity of milk, contained in a real or imaginary milk box. Using Steinbring's construct *The epistemological triangle* (2006, p. 135) the amount of milk in one box is the object or reference context and the concept is the idea of the fraction 1/4.

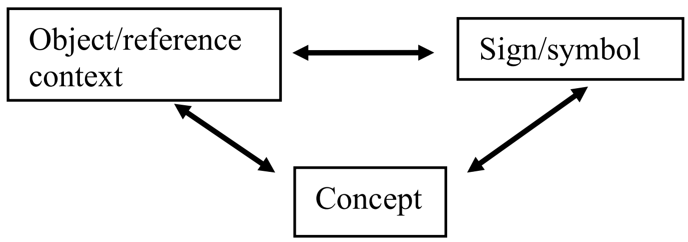

Figure 1: The epistemological triangle

In most cases a mathematical concept can have multiple representations with different characteristics. According to Peirce a sign can be an *icon,* which stands for its object by likeness; an *index*, which stands for its object by some real connection with it; or a *symbol,* which is only connected to the object it represents by habit or by convention (Peirce, 1998, pp. 13-17, 272-275). Successful learning of mathematics is often linked to the ability to switch between different representations of the same mathematical object. Being able to do this and keeping the connection to the same object is, according to Duval (2006), one of the most important obstacles to learning mathematics. In my paper iconic (depictive) and symbolic (descriptive) representations will play the main role. Depictive representations possess inherent structural features making it possible to extract relational information but they do not contain symbols for these relations. Descriptive representations also contain information about relations but to extract this information it is necessary to know the conventions embedded in the symbols (Schnotz & Bannert, 2003, p. 143).

Making sense of conventional representations can be thought of as creating strong links between the corners of the epistemological triangle. In the process of creating such links the learner often uses *hedge words* (Lakoff, 1973; Rowland, 2000) as an indicator of uncertainty, lack of a precise language, or as a search for approval from e.g. a teacher. Rowland groups hedge words into two main categories, *shields* and *approximators* (2000, pp. 60-61). In brief, shields indicate a vagueness on behalf of the speaker, that he or she does not guarantee the truth of the proposition to follow, whereas approximators indicate a vagueness in the proposition (that the speaker is aware of).

Much research has been done on learning with multiple representations and it has been claimed that using multiple representations will enhance learning. Susan Ainsworth has identified three main functions of using multiple representations;

complementary roles, to constrain interpretation, and to construct deeper understanding (Ainsworth, 1999, p. 134, 2006, p. 187). The first function is to use representations containing complementary information or supporting complementary cognitive processes. The second function is to use one representation to constrain learners' interpretation of another representation, e.g. to support the interpretation of an abstract representation. The third function has to do with encouraging learners to construct a deeper understanding of a situation.

In school mathematics fractions are in most respects taken to be equivalent to positive rational numbers. Many authors have described subconstructs of rational numbers and the list of subconstructs varies among different authors. Behr, Lesh, Post, and Silver (1983) claim that one can identify at least six different subconstructs: a part-to-whole comparison, a decimal, a ratio, an indicated division, an operator, and a measure of continuous or discrete quantities. According to Behr et al. the part-whole subconstruct is fundamental to all later interpretations and these authors also suggest a reconceptualisation of this subconstruct into what they denote as *the fractional measure subconstruct*. Previous studies about the learning of fractions have often been of a quantitative nature and focussed on the various subconstructs of rational numbers (see e.g. Behr et al., 1983). Recently there seems to have developed a stronger interest in looking at the learning of fractions through transformations between multiple forms of representation (see e.g. Ebbelind, Roos, & Nilsson, 2012). This study shares much of its theoretical foundation with my study.

METHOD

The 20 pupils in the class were grouped into four groups, each consisting of five pupils. Each group left the regular teaching in the class and came to a nearby room where I was waiting for them. I sat together with the children around a table and each child received a sheet of paper with the task written on it. They had not seen the task before or been given any information about what they were going to work on together with me. They could draw and write on the task sheet as well as on blank sheets that were available on the table. No concrete material was available. The episode was recorded by a video camera standing in a fixed position on a tripod. Each group of pupils got approximately 30 minutes for the task. The data for the analysis consist of the video recordings as well as the sheets of paper that the children used to write and draw on. Since I was actively taking part in the conversation I did not have the possibility to shift the position of the camera. It is directed towards the table and since the groups are so small it is to some extent possible to discern from the video the actual process when the children make drawings and link this to what was actually said at the same time.

To analyse the data I looked at the recordings and made summaries from the activities in each group, looking in particular for episodes that I considered important in relation to my research question. These episodes were transcribed, first in a style close to the spoken dialect and later to standard written Norwegian. For the purpose of this paper, parts of the transcriptions have been translated into English. Each section of utterances are given an internal numbering. All the children have been given English pseudonyms. The same child carries the same pseudonym in this article as in (Rønning, 2010). My analysis of the data is based on the pupils' utterances and their writings and drawings during the process, and through an interpretative process I will make some statements about the pupils' sense making of fractions. As an analytic tool I rely in particular on elements from semiotic theory and multimodal representations as well as theory about subconstructs of rational numbers.

In the task the pupils were presented with drawings of four different situations, each described by a figure as shown in Figure 2.

Figure 2: Situation A from the pupils' task

In the text it was explained that the pictures illustrated milk boxes, blue and red. It was explained that each blue box contained 1/3 litre of milk, and each red box 1/4 litre[2] of milk. Situation A showed three blue boxes, situation B four blue boxes, situation C four red boxes, and situation D three red boxes. The questions in the task are reproduced below.

- Which box, red or blue, contains most milk?
- Which situation, A, B, C or D, represents the largest quantity of milk?
- And which situation represents the smallest quantity of milk?
- Are there any situations with the same amount of milk?
- How many decilitres of milk are there in situation D?
- You need 15 decilitres of milk and you have boxes containing 1/4 litre, hence red boxes. How many boxes do you need?

COMPARING FRACTIONS

The question of finding out which box, red or blue, contains most milk immediately turns into a question of comparing the fractions 1/3 and 1/4, where the context with the milk boxes plays no role. The children introduce a new context where the fractions are represented by rectangular shaped figures, divided into parts in different ways. To illustrate 1/3 Fran draws a rectangle, divides it into three parts. Then she divides one of the three parts into two and shades this part to illustrate 1/4. Chris has a similar illustration of 1/4 as showed in Figure 3. An excerpt from a discussion in one of the groups is presented below.

Figure 3: Chris' illustration of 1/4

1.1	Fran:	Because, when it is, in a way, a third, then it is divided into slightly larger pieces, but when it is a fourth, it is smaller. So there is more space in a third.
1.2	Chloe:	If they get thirds, they get more, if they get fourths, they get less.
1.3	Chris:	The larger the number below, the smaller is the actual part.
1.4	Frode:	Yes, the smaller is the actual part.
1.5	Chris:	It is almost like this. Then it is almost as if we split this one [points to the shaded part of Figure 3], making it even smaller.

The children create iconic signs (Peirce, 1998) to link to a reference context having to do with dividing a whole into pieces. The numerator ("the number below", #1.2) is linked to the number of pieces and the size of each piece is linked to the size of the fraction in an inverse way. This is most clearly expressed by Chris when he states, "the larger the number below, the smaller is the actual part" (#1.3). When Chris links his statement about the numbers in the fraction (#1.3) to the illustration he is using hedge words, "almost like this" and "almost as if" (#1.5). I interpret these words to be *rounder approximators* whose effect is to "modify (as opposed to comment on) the proposition" (Rowland, 2000, p. 60). In his statement (#1.3) Chris shows no indication of vagueness, and therefore I interpret his statement (#1.5) not to be taken as an exact representation of a smaller fraction but as an approximation.

The signs 1/3 and 1/4 can be placed in an epistemological triangle dealing with the concept volume of one box where the blue and red boxes are placed in the reference context corner. Then the connection between sign and reference context is just a "communicative agreement" (Steinbring, 1998, p. 173). In order to reason about the relative size of the fractions it is necessary to introduce a new sign, the partitioned rectangles, that can function as a reference context for the signs 1/3 and 1/4. Then there is a structural reference between the sign and the reference context, "the connection between symbol and referent is indirectly mediated by syntactical and logical structures on the symbol level and the referent level" (Steinbring, 1998, p. 179). (See Figure 4. The sign > was not used in the interaction with the children.)

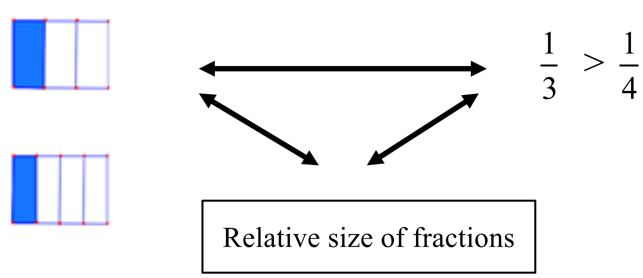

Figure 4: Comparing fractions

In the dialogue above Fran said that in 1/3 "it is divided into slightly larger pieces" (#1.1). In order to determine whether there are situations containing the same amount of milk a need to quantify the difference arises, *how much* larger is a blue box than a red box? Jessica and Ellie have suggested that situation A has the same amount of milk as situation C and in the dialogue below they justify their argument.

2.1	Jessica:	The blue one is in a way one more, I nearly said.
2.2	Ellie:	Then it is the same amount such that it is one more.
2.3	Jessica:	Yes. And then I think that the blues are almost one more than the reds.
2.4	Frode:	Yes …
2.5	Jessica:	One box more, yes there are three there [points to A] … and then you can in a way draw one more.

They agree that each blue box contains one third, and that the only possible solution, if any at all, is A and C.

2.6	Ellie:	But it is, the blue ones there [points to B], compared to the red ones, [points to C] there are five, and there are four, so it is like one more.
2.7	Frode:	OK.

The pupils have previously agreed that 1/3 is larger than 1/4 but how much larger? The answer given to this is that it is "one more" (#2.1). Also here the hedge words are interesting, e.g. when Jessica says that the blue box is "in a way one more" (#2.1) and "the blues are almost one more than the reds" (#2.3). This argument gives as a result that four red boxes are equal to three blue boxes. In this discussion I interpret the hedge words to be of the type that Rowland refers to as *adaptor approximators* (2000, pp. 60-61). Jessica attaches vagueness to the proposition (one more), not in the sense of a rounder (approximately one more) but in the sense that she does not have good way of expressing exactly how much larger 1/3 is than 1/4, so it is "in a way one more" . One may claim that Jessica, knowing that 4 is one more than 3 and that 1/3 > 1/4, the latter relation has a certain degree of "one more-ness" to it, albeit in a reciprocal way. The relation belongs, to a certain degree, to a category consisting of objects $\{a, b\}$ where a is one more than b (Lakoff, 1973).

The argument is developed further into concluding that four blue boxes correspond to five red boxes, when Ellie says that "the blue ones there, compared to the red ones, there are five and there are four, so it is like one more" (#2.6). It is not clear what is meant by *one more* but I interpret Ellie's utterance to mean that four blue boxes correspond to five red boxes. Her reasoning works in the particular cases given by situation B and situation C but not in general. If v_r denotes the volume of one red box and v_b denotes the volume of one blue box, and n_r and n_b denote the number of red and blue boxes, respectively, then the total volume will be the same if $n_r \cdot v_r = n_b \cdot v_b$, or $n_r = \frac{v_b}{v_r} \cdot n_b$. Hence, the relation is a multiplicative one, whereas the pupils suggest that an additive relation, $n_r = n_b + 1$, will give the same volume.

LACK OF FLEXIBILITY IN THE REPRESENTATIONS

In the second question of the task the pupils were asked to identify the situation containing the largest amount of milk. Prior to this it had been established that the blue box contained the largest quantity of milk and the pupils quickly suggest that situation B has the largest amount of milk. Fran starts to justify this suggestion but soon she begins to object to her own suggestion.

3.1	Fran:	Because there [in B] there are many blue ones. The more blue ones, the … But really, it is not possible because the blue one is one third and this one is four thirds, so it really is not possible.
3.2	Frode:	Not possible. Why not?
3.3	Fran:	Because then there is one too many. Is it.
3.4	Chris:	That is why A is the largest. There there are three thirds.
3.5	Frode:	In A, yes.
3.6	Fran:	Because it is not possible with four thirds. That is not possible. If it had been fourths, then it had been possible.
3.7	Chloe:	There is only room for three in one litre. If it had been written four it would be possible. That is why it is possible in C.
3.8	Fran:	It is not possible to take one more box than what in a way is there.
3.9	Chloe:	So it is possible with one fourth but not with one third.
3.10	Fran:	So really, it is A.

There seems to develop a reluctance to accept situation B in this context, when Fran explicitly says "four thirds, so really it is not possible" (#3.1). Looking at the previous attempts of explaining fractions (see e.g. Figure 3) this reluctance is understandable. The fraction is compared to a fixed quantity, the unit, in this case drawn like an almost rectangular figure. The concept of fraction is taken as the part-whole subconstruct (Behr et al., 1983), meaning that there is a certain number of parts, n, and one of those is "one nth". And it does not make sense to take $n + 1$ because the unit (the whole) does not contain more than n parts. "It is not possible to take one more box than what in a way is there", as Fran puts it (#3.8). Or, "there is only room for three in one litre" (Chloe in #3.7). Here 1 litre is the limit, the whole. This shows the limitations involved when looking at fractions only as parts of a whole and how the representation given by partitioning a given unit is not sufficiently flexible.

To make sense of the symbolic representation $\frac{1}{n}$ all the pupils use a representation of the type shown in Figure 3, i.e. a rectangular shape partitioned into n stripes. This is an example of a depictive (iconic) representation that enables the pupils to achieve insight into the descriptive (symbolic) representation. In this way the multiple representations can be said to have a constructive function (Ainsworth, 1999, 2006). However, the iconic representation also has a constraining function, and in this case a constraining function that goes too far. Ainsworth describes the constraining function as being of help in the meaning making stating that it entails using one representation "to constrain possible (mis)interpretations in the use of another" (1999, p. 134). In the excerpt above one can see that the constraining is too strong in the sense that it does not allow for the sign "four thirds". Situation B is a depiction of four blue boxes and this is expressed by Fran as being "four thirds". This sign is never expressed in symbolic form. Chris writes $\frac{3}{3}$ next to the three blue boxes in Situation A and in

Situation B he draws a curve around three of the boxes and another curve around the fourth box, and says that “A is the largest” (#3.4).

A multiple representation for Situation B could be as shown in Figure 5.

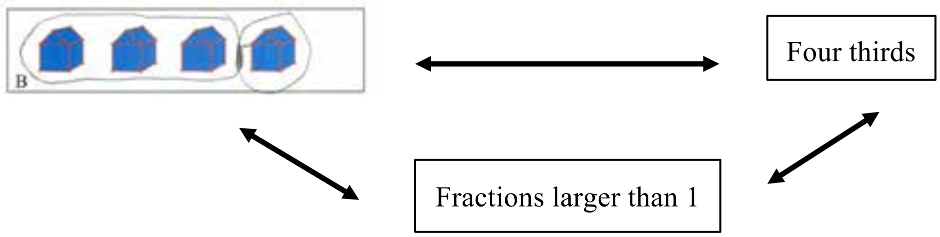

Figure 5: Reference context for fractions larger than one

Although the pupils use the verbal sign four thirds to represent Situation B, this is not accepted as a solution in this particular context, which we at an early stage in the conversation had agreed was about fractions. The epistemological triangle shown in Figure 5 for fractions larger than 1 is not established in the given situation. The pupils link the sign to the reference context but they cannot link this to a concept of fractions. Earlier they have linked the sign for the fraction (spoken or written) to the partitioned rectangle. The partitioned rectangle does not function as a link between the sign and the reference context in Figure 5 because the partitioned rectangle does not make sense in the case of four boxes of 1/3 each. Therefore there is a lack of flexibility in the representation, since one box corresponds to one stripe in the rectangle and there are only three stripes, so “it is not possible to take one more box than what in a way is there”, as Fran puts it (#3.8). Therefore, “really it is A [that has the largest amount of milk]” (#3.10).

In one of the other groups identifying the situation with the largest amount of milk presented no difficulties. Jessica read the question “Which situation, A, B, C, or D, contains the largest amount of milk?” and she immediately answers that it is B. This is repeated by Ellie and one of the other pupils. When asked why this is the case, Ellie says “because it is the largest number…“ . Ellie is interrupted by Jessica who gives the following explanation.

Jessica: Besides, you can see that it is most, and the reds are as many as the blues, so then you think, and here you said that blue was most, and it has to be blue which is most of red and blue there also.

Jessica accompanies her argumentation by pointing with her pencil to the task sheet and by combining what she says with the pointing that can be observed on the video I interpret her statement in the following way. She first verifies that the number of red boxes in C is the same as the number of blue boxes in B. Then she refers back to what we already had agreed on, that one blue box is more than one red box. Then she concludes that it must be the situation with four blue boxes (B) that contains the largest amount. I take it that she has tacitly assumed that none of the situations with three boxes could be a candidate for the largest amount.

Jessica's argument here does not need the concept of fractions larger than 1 so she does not face the same problems as the group with Fran, Chris and Chloe did. Jessica builds on previous knowledge that one blue box is larger than one red box ($v_b > v_r$) and for this the representations in Figure 3 can be used. Furthermore, by counting, it is clear that the number of blue boxes in B is equal to the number of red boxes in C ($n_b = n_r = n$). From this she concludes that $n\ v_b > n\ v_r$.

DISCUSSION

In this paper I have showed how representations that resonate strongly with the aspect of fraction as a part-whole relationship can be limiting and restrictive, in particular when it comes to dealing with fractions larger than 1. In a context, which in itself would be expected to be a familiar one (boxes of milk), one situation consisting of 4/3 litre was dismissed as being "not possible" and my interpretation of this is that because the task has to do with fractions, the pupils cannot accept four thirds, whereas they are happy to accept four fourths. It would have been of interest to pursue the statement "not possible" further by asking "what is not possible?" Instead I only ask "why it is not possible", assuming that we all have the same understanding of what "it" is. This can however, not be taken for granted.

Although mathematically the tasks for the pupils presented in this paper are quite similar to the ones discussed in (Rønning, 2010), there is one important difference. In the situation described in the previous paper the pupils had measuring devices available that functioned as mediating artefacts between the signs and the reference context. The presence of the mediating artefacts strongly reduced the need of making sense of the signs. Therefore some of the pupils read 1/4 as "one comma four[3]", and says "two comma eight" after having poured two 1/4 litre boxes of milk into the measuring beaker. One of the pupils saying "one comma four" is corrected by one of the other pupils who says that the sign / is not a comma but a slash (Rønning, 2010, p. 1018). The first pupil readily admits that she doesn't have a clue to what "one slash four litres" means but nevertheless the pupils in the group have no problems finishing the measuring task because they are guided by the measuring beaker where the scale has the function of an indexical sign (Peirce, 1998).

In the situation presented in this paper the depictions of the milk boxes in the task were not helpful in making sense of the fraction signs. Therefore the pupils created other signs that they could use but these signs turned out not to be sufficiently flexible to deal with the situations, in particular not in cases of fractions larger than 1.

NOTES

[1] Norwegian spelling of litre.

[2] In the text presented to the pupils the signs $\frac{1}{4}$ and $\frac{1}{3}$ were used.

[3] In Norwegian a comma is used for the decimal point so "one comma four" means 1.4.

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PRIMAPODCASTS – VOCAL REPRESENTATION IN MATHEMATICS

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After having investigated written and graphical-based communication I focus on vocal representations of mathematical contents. The mathematical podcasts were made by primary school children and therefore they are called PriMaPodcasts. The relationship to written and graphical representations, the special setting to produce the podcasts with primary pupils and an example is depicted in this paper. The different steps of production of the PriMaPodcast are classified in reference to the special relation of vocal and written communication. It is a first classification according to the linguistic model of Koch and Oesterreicher. Furthermore some interests of investigation concerning the use of PriMaPodcasts, the acquisition of competences, semiotic analyses and the use of digital media in teacher education are briefly described.

MOTIVATION

Our special interest in vocal representations of mathematical contents aroused from the investigation of written and graphical-based communication in the project 'Math-Chat' (Schreiber, 2010, 2013, see next section). A first conception was developed (Schreiber, 2011) to create podcasts about mathematical contents with primary school pupils, which was modified later (Schreiber, 2012a). In this project only audio-podcasts are produced, so the use of written and graphical elements is not possible and this kind of representation must be 'replaced' in some way by the primary pupils. Three different aims are crucial:

- The learning can be fostered because of the requirement of not using any written or graphical representation but only vocal representation for the description of mathematical contents.
- The level of the pupils' skills in reference to their mathematical expression, both spontaneous and planned, can be identified by analysing all products during the production of the podcasts.
- This is very promising for the purpose of investigation because it is possible to see what is used to replace the written and graphical representation when only vocal representation is possible.

Not only the product but also all steps in the production of PriMaPodcasts are of interest. There are seven different steps and the written and vocal representations are interwoven permanently. The written and vocal parts of this process are classified by the two-dimensional model of medial and conceptual orality and writtenness (Koch & Oesterreicher, 1985).

ANCESTOR PROJECT

The ancestor project ‘Math-Chat’ [1] was about the genesis of ‘mathematical inscriptions’ (Latour, 1987; Latour & Woolgar, 1986) in primary education: In an experimental situation, an internet-chat setting, the communication between pupils solving together given word-problems depended on the use of written and graphical representations. This setting offered insights into the learning of mathematics because mathematics depends on written forms of communication (Pimm, 1987). Writing constitutes an integral part of mathematical communication. Fixing ideas in a written form changes their status and makes them more explicit and conveyable, see Bruner's approach of the "externalization tenet" (Bruner, 1996, pp. 22-25). The focus on the medial written form of language in problem-solving situations in mathematics was the special issue of ‘Math-Chat’ (see Schreiber, 2005). In the ‘Math-Chat’ project a semiotic instrument for analysing inscription-based mathematical problem-solving processes has been developed. These processes are described as “Semiotic Process-Cards” (Schreiber, 2010, 2013).

PRODUCTION OF PRIMAPODCAST

According to the latest conception (Schreiber 2012a, 2012b) there are seven steps in the process of production of the so-called ‘PriMaPodcast’ (see Fig. 1):

1. Spontaneous Recording: The pupils have to answer a question or to react to a stimulus about mathematical terms like ‘infinite’, ‘greater than’, ‘less than’, or calculation methods like addition or subtraction or about geometrical objects like a circle or a cuboid. Their first spontaneous reaction is recorded with a voice recorder. This is like a brainstorming about the mathematical content and can be used in the further process because it is recorded and available for the pupils.
2. Manuscript I: The pupils have to plan an audio-podcast, which is to explain this content to others. For this they can and they should now take some notes and make a kind of manuscript for their recording. They are free to make it more or less detailed, to decide who is saying which part or to make more or less a rough draft.
3. Audio-podcast: The recorded audio-podcast should begin with the same question or stimulus as in the first step. It will be recorded based on the pupils’ manuscript. Depending on this manuscript it can be read out or the pupils speak freely.
4. Editorial Meeting: Two different groups meet here together with the teacher to give one another feedback to the created audio-podcasts. The feedback can be about the content, formal aspects or the choreography of the whole audio-podcast. These two groups decide with the help of the teacher if the created audio-podcasts are ready to be published or if there are changes to be needed.

5. Manuscript II: In connection with the editorial meeting the manuscript has to be revised. This way, the final version is initiated.

6. PriMaPodcast: The pupils take a final recording of their audio-podcast. Afterwards, it is ready to be published.

7. Publication: After the release of the PriMaPodcast, an administrator publishes them in a blog on the internet, http://blog.studiumdigitale.uni-frankfurt.de/primapodcast/ (in German). The advantage of publishing the PriMaPodcasts in a blog is the categorising of the several podcasts in main-theme and sub-theme categories which makes it easy to manage for a group of researchers.

Fig. 1: Steps to produce a mathematical podcast with primary pupils - PriMaPodcast

Similar to the ‚Math-Chat-Project', in which there is a link between medially graphical but conceptually oral communication, there is a link between the two kinds of communication in PriMaPodcast production, too. As it is explained below, orality and writtenness are present in the different parts of the production of PriMaPodcasts in different manners and both are interwoven.

CONFLICTING FIELDS OF WRITTENNESS AND ORALITY

To explain more about the interwoven use of oral and written communication in a medial and conceptual manner, I use the linguistic model of orality and writtenness of Koch and Oesterreicher (1985), well explained by Fetzer (2007, p. 79) as the "two dimensions of orality and writtenness (translated by the author: „Zwei Dimensionen von Mündlichkeit und Schriftlichkeit").

Koch and Oesterreicher (1985) have developed a model of communication that distinguishes between medial phonic and medial graphic communication and between communicative immediacy and communicative distance. The medial phonic and the medial graphic realization of communication are dichotomous, whereas the conceptual realization can be placed on a scale between the communicative immediacy and the communicative distance (see Fig. 2).

As an example a personal talk is not only medial phonic, there is an emotional closeness to the dialogue partner and thus conceptual oral. Writing in a diary is in a conceptual sense also oral, not formal and characterised by the 'closeness' to the reader, but it is medially graphical. An administrative directive in contrast is medially graphical and also conceptually an example for communicative distance. The language is strictly formalised in this case. If this administrative directive will be read aloud, it will be medially phonic, but conceptually it continues to be written. The language is still the same and characterised by distance and formalism.

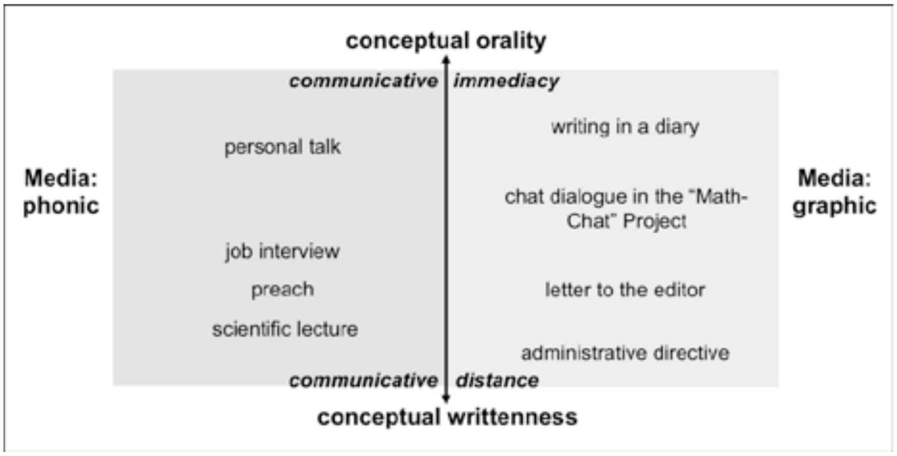

Fig. 2: Koch and Oesterreicher's model according to Fetzer (2007, p. 79; translation by the author)

EMPIRICAL EXAMPLE

A couple of PriMaPodcasts have been made in German and also a few in a bilingual German-Spanish and a bilingual German-English class. The PriMaPodcasts in German language are available in this blog: http://blog.studiumdigitale.uni-frankfurt.de/primapodcast/, the examples in Spanish and in English language are also available in blogs [2]. Further examples will be added to these two blogs. A blog in Greek and one in Turkish are planned.

In the PriMaPodcast which is presented here, the question was 'What is so special about a square in comparison to the other quadrangles?' This PriMaPodcast was developed by one girl and two boys in grade four and the recordings were led by a teacher-student. At first the transcript of the spontaneous recording is given [3]. After that the manuscript is depicted, which is written by one of the pupils as agreed with the others. Then the audio-podcast, which is based on this manuscript, is exhibited as transcript. The improved recording, the PriMaPodcast, is also documented as a transcript. It conforms to the manuscript, even though there are some different intonations and it is not fluently read in all cases. For my analysis this is of interest. All citations of the transcript are marked in squared brackets <like this>.

Spontaneous Recording:

At first pupil 1 (the girl) asks 'What is so special about a square in comparison to the other quadrangles?' <sp1> and pupil 2 answers spontaneously <sp2>, that he knows an answer. But it is pupil 3, who explains that there is a difference in terms of spelling <sp3>. He remarks that the capital letters of the two words are different. This assumption is correct even though his spelling of the second word is wrong. Pupil 2 takes the turn and has a content-related point of view and claims that the sides of a square are of equal length. To this he gives an example <sp4>. Pupil 3 adds another example to confirm the statement <sp5>.

sp1	00.00 p1	What is so special about a square in comparison to the other quadrangles
sp2	00.07 p2	Okay I know
sp3	00.10 p3	A square (*in German Quadrat*) is spelled with Q and the quadrangles (*in German Vierecke*) are spelled with F
sp4	00.14 p2	Squares have on each s. on each side a square is of equal length that means when one side is five centimetres the other sides of the square are five centimetres too
sp5	00.29 p3	And when it's one centimetre the other side is one centimetre too (*colloquial*)

You can see here that pupil 2 and pupil 3 both focus on the sides and on the shape of the square. With their utterances they do not actually compare the differences between a square and the other quadrangles. The pupils only describe the characteristics of a square. Other characteristics of a square, for example symmetry or angles, are not taken into consideration. Pupil 1 does not add another content-related argument either.

Manuscript:

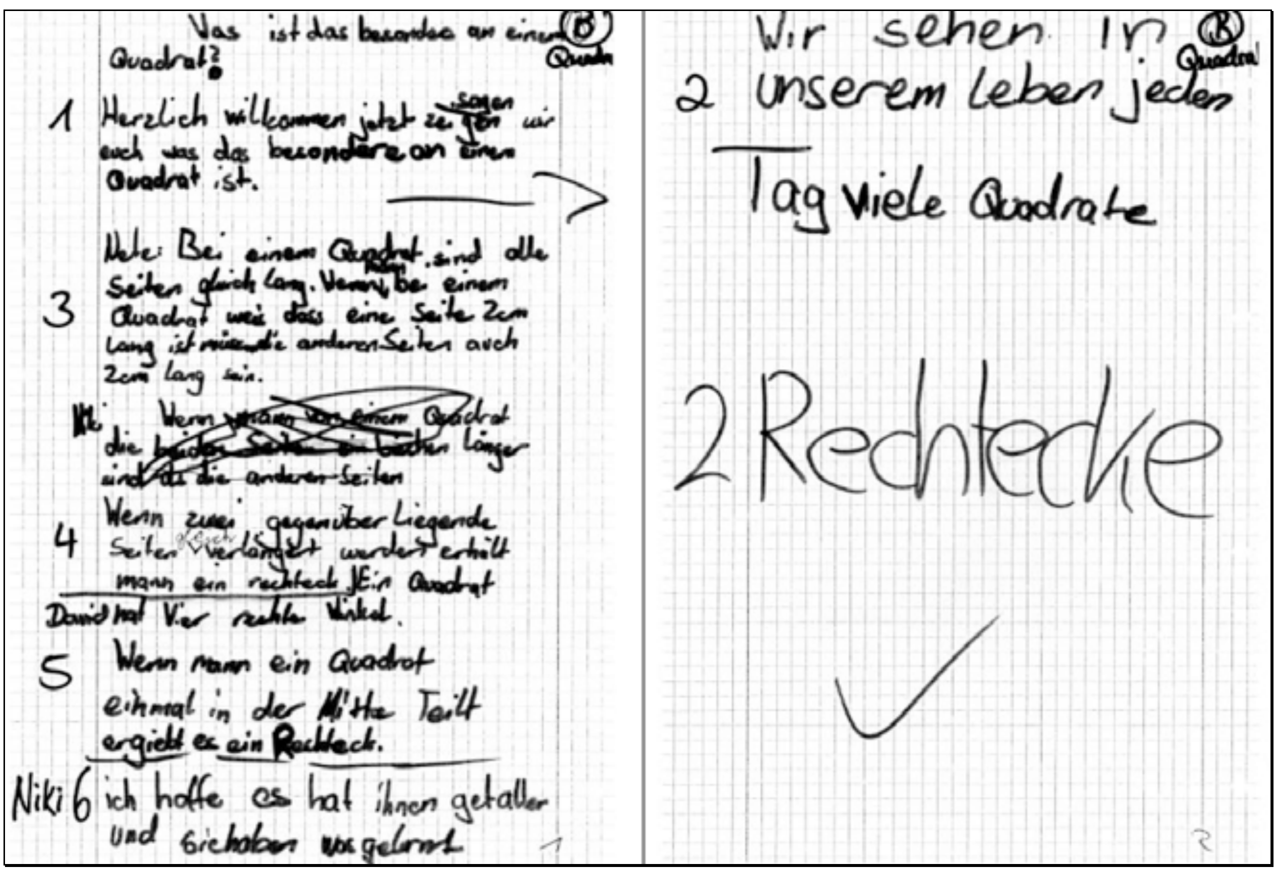

Fig. 3: Manuscript of the PriMaPodcasts (originally two pages in A4)

Here I give the translation of the manuscript (see Fig. 3):

What is so special about a square?	2 Every day we see many
1 Welcome now we are going to tell you what is so special about a square.	squares around us
3 Nele: A square has got four sides of equal length. When you know that a square has one side of 2 cm all the other sides have to be of 2 cm, too.	2 rectangles
~~When both sides of a square are a bit longer and the other sides~~	
4 When two facing sides are lengthened equally you get a rectangle.	✓
David 5 A square has got four right angles. When you split a square in the middle, you get two rectangles.	
Niki 6 I hope you liked it and you have learnt something	

Audio-Podcast:

At first they welcome the audience of their audio-Podcast by presenting their topic <a1>. Then they make clear, where you can find squares in the “every . day” <a2> world. After that, they take an utterance of the spontaneous recording and focus again on the length of the sides of a square <a3> with the help of an example. In utterance <a4> they explain how to transfer a square into a rectangle and utterance <a5> is another way of transformation. The right angles are mentioned in this context, too <a5>. With their statements in <a6> and <a7> they refer again to their audience.

a1	00.00	p1	Welcome now we are going to tell you what is so special about a square
a2	00.04	p3	Every . day we see many squares cede covers glazed tiles squared paper (*Hessian accent*) in our Maths school books around us .
a3	00.13	p1	A square has got four sides of equal length when you know that a square has one side of two centimetres all the other sides have to be of 2 centimetres too
a4	00.22	p2	When two facing sides are lengthened equally you get a rectangle
a5	00.28	p3	A square has got four right angles when you split it in the middle vertically or horizontally you get two .. rectangles (*Hessian accent*)
a6	00.37	p2	I hope you liked it and you have learnt something ..
a7	00.42	all p	By-yee

In this version, the pupils think of an audience of their work <a1>. This is different to the spontaneous recording and makes their work ‘public’. So it changes the status of the spoken text, which bases on the manuscript. With the first content-related statement they make clear what the question means in their “every . day” life <a2>. This shows their personal interest of the question. By taking an utterance of the spontaneous recording they focus another time on the length of the sides <a3>. And this shows that they are still aware of their first ideas. They do not only take over this idea, but they are able to generalise it and to add an example. For the first time, in utterance <a4> they make a difference and also a connection between squares and other quadrangles. It is not any quadrangle but a special one, a rectangle. They show the connection and the difference by using two modifications of the square: In the first case, it is a modification by changing the length of “two facing sides” <a4> and in the second case by splitting into two equal parts <a5>. Moreover, the pupils mention one more detail of the square by focussing on the angles <a5> and they are able to use the terms ‘vertically’ and ‘horizontally’ to explain the directions <a5>. In utterance <a6> they emphasize that their aim was to let the audience learn something.

PriMaPodcast:

After the editorial meeting the PriMaPodcast is recorded. It still bases on the same manuscript. Mostly all parts of the audio-podcast are in this PriMaPodcast version. The pupils only change details like leaving out the examples from every day life <pr3> and adding their names <pr7>.

pr1	00.00	p1	Welcome . Now we are going to tell you what is so special about a square
pr2	00.04	p3	Every day we see many squares around us
pr3	00.08	p1	A square has got four sides of equal length if you know that a square has one side of two centimetres all the other sides have to be of 2 centimetres too
pr4	00.17	p2	If two facing sides are lengthened you get a rectangle
pr5	00.22	p3	A square has got four right angles when you split it in the middle vertically or horizontally you get two rectangles (*Hessian accent*)
pr6	00.29	p2	I hope you liked it and you have learnt a lot.
pr7	00.33	p3	This commentary was given by Nele Niklas und David ..
pr8	00.37	all p	By-yee

In the PriMaPodcast version the pupils are aware of the imaginary audience <pr1>. This is clear because this version is for publication in the blog and the children were told this before. So their recordings will be made ‘public’ indeed. In this version the pupils read it out louder and more fluently than in the audio-podcast version. Leaving out the examples from every day life is an important change. Up to now it was of great interest for the children and they put emphasis on it by mentioning it at the beginning of the audio-podcast. But from their point of view it is not so important for the imaginary audience. All the other arguments are repeated <pr3; pr4; pr5>, which shows that the pupils see their relevance for the audience. The children were not asked to mention their names, but this seems of interest, too <pr7>. The pupils can identify with their PriMaPodcast and they finish it jointly.

CLASSIFYING IN THE CONFLICTING FIELDS

The communication in the 'Math-Chat' Project is medially graphical, but conceptually oral, because it is in a high range interactive, synchronous and little formal. The classification of the different phases of the development of PriMaPodcast consists of three parts: The spontaneous recording, which is medially phonic and conceptually oral too. The utterances are little formal and not very elaborated. The manuscript, which is the base for the later recording of the PriMaPodcast, is medially written but conceptually – depending on the kind of manuscript – it should be classified more to the vocal pole. The so developed PriMaPodcast is medially phonic and conceptually nearer on the oral pole than the manuscript.

The overlapping areas mean that, e.g., a manuscript for the PriMaPodcast can be conceptually more oral, than the medially graphical interaction in the Math-Chat project, which is in its part also conceptually oral. As well a PriMaPodcast can be conceptually nearer to oral than the spontaneous recording.

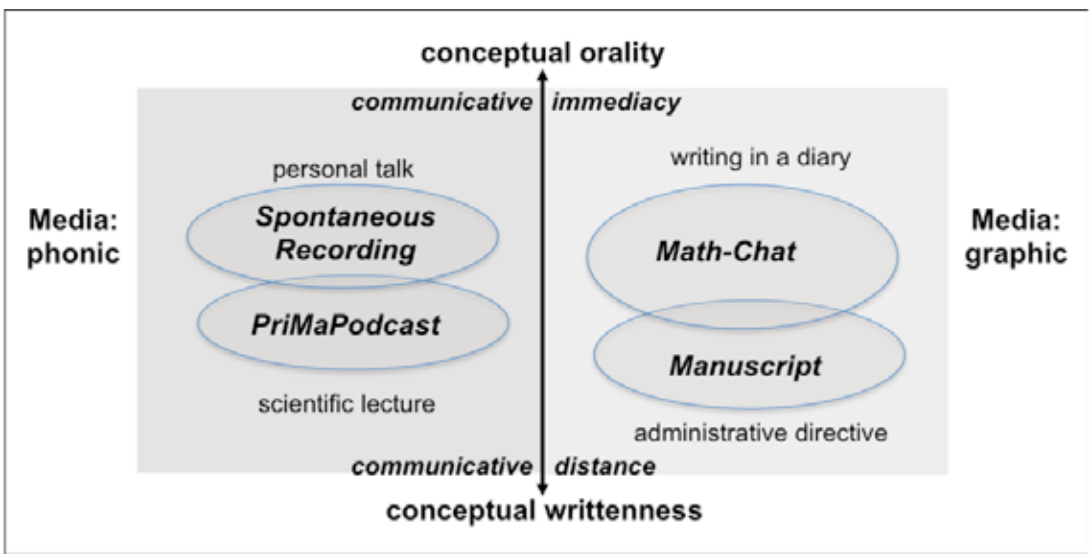

Fig. 4: Examples classified in the conflicting field of writing and speech (see also Schreiber, 2012a)

FURTHER INTERESTS OF INVESTIGATION

Here I want to highlight three aspects of interests of investigation:

- The acquisition of competence in mathematical learning
- Semiotic analyses of vocal representation
- Digital resources in teacher-education and mathematical learning

Communication is one of the mathematical competencies. It is overlapping with representation and also with argumentation. If we want to foster the acquisition of these competencies, we have to find a task in which they are required aligned and in variations. So for the empirical investigation there is a special requirement. I want to investigate this field by video analyses and explore, in what extent it is possible to foster this competencies acquirement. Especially the connection between written and vocal representation in the production of the PriMaPodcasts is of interest. This can help to identify the importance of the different models of communication.

As always in the field of reconstructive social investigation this investigative work should amplify the repertoire of methods: the vocal products in connection with the before created manuscript are subjected to a semiotic analysis. For this the analyses-method of the Semiotic Process-Cards is developed further to be used for vocal forms of communication. The goal of the more differentiated theory is to examine the semiotic aspects of interaction in mathematical classroom activities. Up to now they were limited to inscriptional aspects of mathematical communication (Schreiber 2010) and they are used for Gestures in this context (Huth, in press).

If it is true, that there is a deficit of use of digital media in primary mathematics classes (at least in Germany) like Mitzlaff (2008) and Ladel and Schreiber (2011) affirm, then we as educators in mathematics should close this gap. Especially for project-oriented mathematical education we have to create and to prove scenarios for all-day use in primary classes. This is what happens in the above described scenario of creating PriMaPodcasts. They can serve as a scenario to teach and learn mathematics and also as a possibility for investigative learning for the teacher students of primary education.

NOTES

[1] This study was supported by Müller-Reitz-Stiftung (T009 12245/02) entitled „Pilotstudie zur Chat-unterstützten Erstellung mathematischer Inskriptionen unter Grundschülern“.

[2] The blogs in Spanish language (http://blog.studiumdigitale.uni-frankfurt.de/primapodcast-es), and one in English language (http://blog.studiumdigitale.uni-frankfurt.de/primapodcast-en) are available but still in progress

[3] The audio-files are available here: http://blog.studiumdigitale.uni-frankfurt.de/primapodcast/

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LINGUISTIC INTERCOURSE WITH SPATIAL PERCEPTION COMPARATIVE ANALYSIS IN PRIMARY SCHOOL, INFANT SCHOOL AND THE FAMILY

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Many mathematical abilities as well as general cognitive abilities have come to be understood in the context of spatial perception. According to Newcombe and Huttenlocher (2006), a key aspect of improving spatial-geometrical ability lies in the linguistic coding of space – yet this linguistic intercourse with spatial perception remains unfulfilled. Research in the field of mathematics education indicates that this can be explained by a tendency to interpret situations in arithmetical terms, or to employ more deictically characterised, implicit forms of mediation. The following report presents a video-based empirical study, which attempts to reconstruct and compare linguistic intercourse with spatial perception in different places in which children learn – infant school, primary school, and family.

THEORETICAL BACKGROUND AND RESEARCH FOCUS

According to Hostetter et al. (2006), gestures of teachers are habitual and some of them tend to be conscious. Thus teachers are capable of actively influencing the gestures they produce. They can decide to increase their use of gestures or to stop using gestures altogether. Using this idea, the present study aims to establish how parents, educators and teachers construct the learning of spatial-geometrical mathematical content in linguistic terms. It also aims to identify and reconstruct differences in this respect that exist between the distinct places of learning. The goal of the study is to identify opportunities and risks through an integrated examination of linguistic intercourse with spatial perception, and to use it to develop a training concept to support parents and infant and primary school teachers in the fostering of spatial thinking. The research question is the following: Which kinds of linguistic intercourse with spatial perception can be reconstructed in the three distinct places of learning: family, infant school, and primary school?

The significance of perceptions of space in the learning of mathematics

The spatial world we live in requires the development of spatial perception. Almost every action of individuals takes place in the spatial world, and consequently cognitive or mathematical abilities are increasingly being understood in the context of spatial thinking. Here it is important to note that a broad competency in spatial thinking early in life not only has positive effects on a child's later capacities for thinking in spatial terms, but also on the proficiency in other areas of mathematics (Lüthje, 2010; Stern, Felbrich, & Schneider, 2006). Grüßing (2002), and Clements and Sarama (2011) have made observations that support the assumption that there is a positive correlation between spatial perception and mathematical achievement:

> Spatial thinking is an essential human ability that contributes to mathematical ability. … Further, mathematics achievement is related to spatial abilities …. (Clements & Sarama, 2011, p. 134)

These observations are supported by a similar negative correlation. Several studies have reached the conclusion that a lack of ability in spatial perception is linked to dyscalculia, and vice versa (Maier, 1999). Spatial perceptions determine thought and language. Spatial concepts are used in many of our everyday metaphors, for example "high-spirited", "on top of the world", "to take up a lot of space". Language is seeking here to describe abstract situations or feelings by using concrete concepts from the world of spatial perception (Fthenakis et al., 2009). Spatial ability is thus essential for children's development. Little attention is paid however to spatial-geometrical content in German schools. An inadequate system and an underestimation of its importance seem to have led to an aversion to the theme in the teaching community. For a long time, German geometry teaching has focused almost exclusively on the study of form, and on looking at the world from a two-dimensional perspective (Lüthje, 2010; Winter, 1976). Consequently, a number of authors have criticised modern geometry teaching:

> Spatial geometry is, as a rule, the learning of vocabulary, arithmetic and algebra. (Maier, 1999, p. 8, translated by the author)

Authors like Maier (1999) have demanded a restructuring of the current primary school curriculum that will make nurturing spatial perception a specific goal of geometry teaching. If we take the importance of focusing on ability in spatial perception in the development of children seriously, according to Fthenakis et al. (2009) we have to ask what the requirements are for teaching it in the places in which children learn. Henceforth we shall consider that the development of spatial-geometrical perception does not begin in primary school, but far earlier, in the family and in infant school, and that "effective learning depends on the support given to all those places where children learn and develop" (Fthenakis et al., 2009, pp. 38-39, translated by the author). Clements and Sarama (2011) have observed that, at earlier stages of children's education too, little or no time is devoted to geometry and spatial perception. They suggest failures in teacher training as a possible cause for this.

Language and the learning of mathematics

It is a fundamental goal of mathematics education to enable children to understand and experience the world with the help of a mathematical perspective. Since many mathematical objects and areas are represented in language, it is also a question of developing linguistic abilities that are linked to mathematics. In a number of approaches to mathematics education, language and communicative competence are accorded particular importance with regard to learning. Above all in primary school, according to Maier (2006), teachers often attempt to introduce concepts through visualisations with the help of illustrative models; that is to say, the form of "iconic" representations. There is an inherent problem here, because these visual techniques do not allow a comprehensive understanding of mathematical concepts. This is true above all because mathematical objects are abstract, and it is therefore difficult to access these objects sensually through enactive and iconic visualisations. This dilemma can only be resolved through the use of language and symbols, underlining the importance of language in the learning of mathematics:

> As fundamentally abstract concepts, [mathematical concepts] can only be legitimately handled and represented on a linguistic-symbolic level. (Maier, 1986, p. 137, translated by the author)

According to Maier (1986, 2006), intensive verbal communication is an indispensable educational tool when introducing new mathematical concepts. Pimm (1987) goes one step further, and considers mathematics as a social activity that is structurally closely linked to verbal communication. From this standpoint he uses the metaphor "Mathematics is a language?" (ibid., p. XiV) to raise the question of whether mathematics may be considered as, if not a natural language, a unique style of language. His concern is to structure the concept of mathematics partly in linguistic categories, with the principal goal of rendering the teaching and learning of mathematics easier to explain (Pimm, pp. 14-15). Returning to the area of geometry in the light of this idea, it is worth referring to Newcombe and Huttenlocher (2006), who have remarked that the linguistic coding of space is a central competency with relation to spatial thinking. Although the literature places great importance on linguistic intercourse with mathematics, particularly with relation to spatial perception, so far I have noted the following: in German schools, spatial-geometrical content barely appears to feature, whilst outside of school in the family a linguistic intercourse with space is frequently neglected. At the time of writing the author is unaware of any experiments that have been carried out on spatial perception in German infant schools.

Support in learning situations containing an element of mathematics

In infant school, primary school and in the family, children are involved in processes of negotiation of meaning through which they become increasingly familiar with mathematical content. Tiedemann (2012) interprets these processes as "support". She draws a link to Bruner's (1983) concept of a support system whereby children are involved in repeating models of interaction, which leads to their becoming increasingly autonomous actors in their cultural environment. A crucial point here is that support can be understood as an interactional phenomenon produced by adult and child together. Whilst the adult assists, the child orients the adult's utterances and actions by his or her reactions in the situation of negotiation of meaning. Tiedemann (2012) adapts this approach for mother-child mathematical discourses, and reconstructs different aspects and methods of support. Therefore, linguistic intercourse with spatial perception in primary school, infant school and the family as described in this study can be reconstructed and understood as "support". One can explore, in the three different places of learning, forms of support that can be observed in processes of negotiation of meaning relating to spatial perception.

Research gap and focus of the study

Anderson (1997) investigated the parent-child interaction of 21 four-year-old children and their parents in situations containing elements of mathematics. An incidental result of her study is relevant to the present paper. In Anderson's study families were given multilink blocks, a child's book, blank paper, and preschool worksheets, and a fifteen-minute audio recording was made of each family as they began to try and solve the puzzles. It was found that the parent-child interaction included a wide range of mathematical approaches of various levels of sophistication, showing counting as the core mathematical skill. Interestingly however, she could not find virtually any explicit verbalisation of spatial description in the interaction between parent and child. This suggests that verbalisations of spatial descriptions were compensated for or replaced by deictic gestures in order to resolve the situation.

In relation to recent research on indicative gesture, the most important works in English include above all Alibali (2005), Alibali and Nathan (2007) and Hostetter et al. (2006). Following these authors, gesture can be differentiated to support both teacher behaviour and interaction with spatial perception. Alibali and Nathan (2007) emphasise that gesture is a means of communication which can be employed by teachers, like language, to support the understanding of their pupils. Therefore, on the one hand gesture can be used to clarify something that has already been portrayed in language. On the other hand, according to Alibali (2005), deictic gesture is used to convey information that has not been linguistically expressed. From an educational perspective, this fact seems to contain a crucial problem. Approaches to mathematics which focus on the senses are problematic because mathematical objects are abstract, and it is difficult to access them sensually through visualisations. A deictically characterised form of support however lends itself to precisely the visual approaches which cannot communicate mathematical concepts.

If we take learning to be a process of dialogue, which may only be described in the course of coordinating the mental activities of at least two individuals, and whose components, in the sense of genetic interactionism (Miller, 1986), are anchored in the activities of the collective, all participants in the dialogue must adapt and re-adapt their interpretations of situations in order for processes of negotiation of meaning to successfully make progress. Usually only a small functional adaptation of individual interpretations is required from the participants in the dialogue to move the interaction forward. This kind of temporary adaptation of meaning is called "working consensus" (Krummheuer, 1992, p. 25; on the concept see Goffman, 1959, pp. 9-10). According to Krummheuer and Brandt (2001), learning can be understood in this light as the increasingly active participation in the asymmetrical processes of the negotiation of meaning, so that the conditions of autonomy of the adult can be translated to the children. The question is whether this kind of short-term working consensus and opportunities for learning through gaining autonomy can still be realistic goals if, during negotiation of meaning in asymmetrical interaction, aspects of mathematical issues are expressed by the adult through deictic gesture instead of language. These gestures could result in a vague situation where the subject of negotiation is often implicitly obscured, and where the children are barely given any opportunity for more active participation ("Implizite Pädagogik", in Schütte, 2009, p. 191; or "Implicit Pedagogy" in Schütte & Kaiser, 2011, p. 247).

Even when learning is anchored in the mental constitution of the individual, and the individual participants are accorded every decisive mechanism and process in Piaget's sense of genetic individualism (Miller, 1986), a further problem emerges. If language is understood in this way, most learning does not take place in processes of negotiation of meaning between participants in a dialogue: instead, learning processes are portrayed by the individual constructs of the individual participants. It is not only that processes of negotiation of meaning remain vague when certain aspects are not linguistically expressed. Opportunities to access processes for creating mental constructs can also be hampered and restricted, since deictic gesture is far from being unambiguous or self-explanatory. In addition, it can be concluded with Huttenlocher and Newcombe (2006) that the use by adults of deictic gesture as a substitute for vocalisation will give children less opportunity to learn the linguistic coding of space.

In some mathematical topics gesture plays an especially large role in teaching. Unlike in other mathematical areas, negotiation on spatial information seems to demand the use of gesture. Conclusions drawn by Acar (2011) also point towards a neglect of spatial perception in everyday family interaction. Looking at various play-situations, Acar shows that parents approach building games – which belong in the realm of spatial perception from a mathematical point of view – in such a way as to make them exercises in arithmetic. They thus focus the support they give to the child on counting the wooden blocks, whilst spatial-geometrical application remains lacking.

METHODOLOGY AND METHODS

The basis for this study is qualitatively oriented, both in and outside a school environment; more precisely, it represents an interactionistic approach of interpretive education research in the field of mathematics education (Tiedemann, 2012). Video recordings form the empirical basis for the study. The research questions will be adapted according to an innovation in perspectives: the research will simultaneously look at children of similar ages (5-7 years) in primary school, infant school and in families. A comparison of these different places of learning takes the different kinds of development in children into account, and at the same time gives a comprehensive base for proposals regarding the development of spatial thinking. It is assumed that the individuals under observation, in view and in spite of the potentially new situation (study materials, camera, etc.), nevertheless resort to routines of behaviour and interaction that they have previously found to be successful. On this basis the observed situations are taken to represent the everyday in the schools, infant schools and families studied (Schuette, 2009). In this sense, the study that forms the basis for this paper aims to help initiate a shift in the focus of research, or at least to reveal possibilities for such a shift. With reference to Krummheuer and Naujok (1999), the present paper aims to identify conditions under which teaching can be altered; it also attempts to discover possibilities for altering behaviour in adults that may be habitual and is not rationalised by the presence of a fixed aim.

Methods of data collection

The study is planned as a cross-sectional video-study. Data are collected in infant and primary schools as well as the respective families of the infant and primary school pupils. In the infant school, on two respective visits two different play and discovery exercises were carried out with one teacher and two children. Similar play and discovery exercises were carried out, also over two visits, in the families of the children. In the families the situations were played out with a parent and the respective children; in the primary school, however, the procedure was different, since the children are taught by teachers with mathematical didactical expertise. This meant everyday teaching was investigated in relation to the central ideas of space and form. Teachers were given thematic areas around which to independently plan and deliver a lesson, without any further external input. The families of the primary school pupils were given tasks fundamentally similar to those given to the families of the infant school pupils, only adapted to suit the older age-group. The characterisation of components of spatial perception is based on a model by Maier (2001, pp. 71-72), which is founded on: (1) spatial perception, (2) visualisation, (3) mental rotation, (4) spatial relations and (5) spatial orientation. On the one hand these components can be categorised such that the "outside position" refers to the participants in the dialogue; that is, whether an inside or outside position is adopted. On the other, we can differentiate between static and dynamic mental processes. Every play and discovery situation was divided up into three tasks. The first situation could be categorised as a building situation, and was developed on the model of the spatial geometrical game "PotzKlotz". The skills demanded were spatial geometrical areas, spatial relations, mental rotation and visualisation. The thematic focuses of the three parts of the exercise given in the infant school and in the families of the infant school pupils were as follows: (1) working together to build a copy of a model, (2) building from instruction, (3) building a model and then reconstructing it. For the families of the primary school pupils the focus was slightly different: (1) building from instruction, (2) building a model and reconstructing it, (3) comparing buildings. An example of building from instruction for the families of the infant school pupils is given below.

PotzKlotz - Building and Discovering Together (2 Players)

You need some building cards and 5 building blocks. Choose one of the cards, and work together to build the structure that is shown on it. Repeat 3 times.

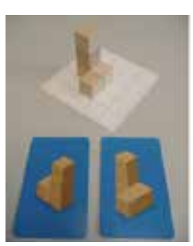

Figure 1. Building from instruction

The second play and discovery situation can be understood as an adaptation of Piaget and Inhelder's "Three Mountains" test (1971, p. 251, cf. Grüßing, 2002). The situation demands skills of spatial orientation. The three foci for infant school and infant school families were: (1) reconstruction from a photograph, (2) orientation from a given position, (3) search of positions from a given orientation.

Methods of data analysis

The analysis of the interaction units in the video-recorded situations aligns itself with a reconstructive-interpretive methodology and a central element of the Grounded Theory research method, namely the methodical approach of comparative analysis (Strauss & Corbin, 1996). In the analysis of the linguistic intercourse with spatial perception, video sequences are analysed with the help of interactional analysis (Krummheuer & Brandt, 2001). The interactional analysis allows the constitution of negotiations of meaning in interaction between individuals to be reconstructed. In order to reconstruct additional common structural features in the linguistic intercourse with spatial perception in the different institutions and with different adults, the video-recorded situations are systematically compared using the comparative analysis methods.

RESULTS

Analysis of the linguistic processes of negotiation in the families yields interesting results. Similarly to the results in Schütte (2009), the mothers' behaviour can be described with the help of the theoretical concept of "Implicit Pedagogy". As they work through the tasks, the mothers hardly ever use conceptual terms like "block", "build", or "building" (in relation to learning environment 1), instead using indexical expressions such as "here", "that", "there", etc.

6-9	<N:	that's how that is supposed to look…can we do that/
13	<N:	can we put that on that, too\ can you put it on that/
20	<N:	that's right\ where should we put the others now/
66	I:	first let's put that on the ground

It might be assumed that the young children guess these concepts and begin to use them. This would correspond to the procedure in the Year 4 classes, analysed by Schütte (2009) in a range of different aspects, in which the achievement of the goal of introducing a mathematical concept appears to lie above all in its naming. However, if one examines this through the theoretical lens of Bruner's language acquisition approaches (1983), there is a clear difference to the processes of negotiation in the Year 4 classes. According to Bruner, children require a linguistic model, whose linguistic behaviour they can begin imitating. Over time they then lastingly integrate linguistic patterns, concepts, etc. into their own patterns of linguistic behaviour, actively using them more and more by themselves. In this way, children also imitate the linguistically reduced behaviour of the mothers described above, leading to a negative spiral in the linguistic level of negotiation. It could be argued that the children here are very young, and that a more elaborate linguistic negotiation is therefore neither appropriate nor to be expected. But something interesting occurs in the breaks between tasks, where the mothers are no longer directing the actions in the learning environment: suddenly, they start to use concepts, and their linguistic behaviour becomes more elaborate.

51 <N: now let's find another new one/ ok then <u>squash</u> the <u>blocks</u> again

How can this behaviour be explained? If we stay with the theory of Implicit Pedagogy, it can be concluded that the mothers seem not to want to "give away" concepts to their children when they perceive that they are in a learning environment. It seems important that the children guess these concepts, and are then the first to use them, which would contradict the above-mentioned language acquisition theories. Similar behaviour was observed in Year 4 teachers during the introduction of new mathematical concepts. In this case, the behaviour appeared as specific to teachers. But these parallels to the linguistic structuring used by the mothers indicate that this understanding of teaching, which could be characterised as a vulgarisation of active discovery learning, should not be ascribed to a professional pedagogical teaching approach. Rather, it emerges from an everyday perception of teaching and learning, that is, that in all kinds of learning processes, one should not "give away" concepts to children. In relation to the discovery of mathematical patterns and structures, this is surely correct. It is not possible, however, to simply "discover" concepts or special linguistic features or conventions. For this purpose, children require a person offering an advanced interaction by which they can orient themselves, and who serves as a (linguistic) model.

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PERCEIVING CALCULUS IDEAS IN A DYNAMIC AND MULTI-SEMIOTIC ENVIRONMENT- THE CASE OF THE ANTIDERIVATIVE[1]

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The present case study was designed to analyze the objectification processes of making sense of the antiderivative concept when it is being studied graphically in a dynamic and multi-semiotic technological environment. This study is guided by sociocultural theory, which considers artifacts to be fundamental to cognition and views learning as the process of becoming aware of the knowledge that exists within a culture. The case study focuses on two seventeen- year- old students. In the course of the discourse micro-analysis I identified three essential foci in the objectification processes 1) objectifying the relationship between a function and its derivative 2) objectifying the relationship between a function and its antiderivative 3) objectifying the vertical transformation of the antiderivative graph.

INTRODUCTION

The integral concept is considered to be central to learning calculus and I cannot imagine any curriculum in calculus not containing the integral concept. The integral consists of two essential concepts – the antiderivative and the definite integral. The fundamental theorem of calculus connects these two concepts. Therefore, significant learning of the integral concept must encompass the learning of the conceptual aspects of both these concepts (Thompson, Byerley, and Hatfield, in press). Thompson and Silverman (2008) suggested to introduce the accumulation function as a tool to connect these two concepts. Thompson et al. (in press) have proposed a didactical sequence that serves to emphasize the conceptual aspects of the integral concept when taught to college students as the accumulation function, and using technological tools. Following Thompson and Silverman (2008) and Thompson et al. (in press), I propose a learning unit composed of five sessions, in order to improve the understanding of the processes involved when high school students are learning the integral as accumulation using graphical means. This learning unit is based on graphic and numeric signs and dynamic interfaces, which consider the graph of the accumulation function as central in conceptualizing the integral concept. Yerushalmy and Swidan (2012) have analyzed and identified processes of making sense of the definite integral as accumulation in a dynamic and multi-semiotic environment among high school students.

Few research studies have examined the learning of the antiderivative concept in general, and among high school students in particular. For example, Haciomeroglu,

[1] This case study is part of a PhD study under supervision of Prof. Michal Yerushalmy, university of Haifa - Israel.

Aspinwall, and Presmeg (2010) have examined the mental processes and images used by three university students to create meaning for graphs of the antiderivative function. Berry and Nyman (2003) have studied college students' understanding of the relationship between a function and its antiderivative function while drawing graphs of functions from graphs of the derivative by using a sensory calculator (CBR[2]). The scarcity of studies dealing with the process of learning of the antiderivative concept by high school students and the usefulness of graphical tools such as graphic interfaces in teaching the conceptual aspects of calculus is what motivated me to designs the current study. This study aims to analyze the processes involved in high-school students' learning of the antiderivative concept graphically as accumulation function, using graphical and dynamical tools.

THEORETICAL FRAMEWORK

According to sociocultural theory artifacts of any kind are central and play a fundamental role in cognition. It has been claimed that, within the social use of an artifact to accomplish a task, shared signs, which relate to the artifact, are produced and may be related to the content intended to be learned (Bartolini Bussi and Mariotti, 2008). The relationship between an artifact and knowledge is expressed by culturally determined signs. The relationship between an artifact and accomplishing a task is expressed by signs such as gestures, speech and drawing.

Signs in general, and mathematical signs in particular, play two roles. Radford, Bardini, Sabena, Diallo, and Simbagoye (2005) define these roles as "social objects in that they are bearers of culturally objective facts in the world that transcend the will of the individual. They are subjective products in that in using them, the individual expresses subjective and personal intentions" (p. 117). Berger (2004), who studied the functional use of mathematical signs, suggests a twofold interpretation of the meaning of signs and objects: a personal meaning, "to refer to a state in which a learner believes/feels/thinks (tacitly or explicitly) that he has grasped the cultural meaning of an object (whether he has or has not)," and a cultural meaning, "to the extent that its usage is congruent with its usage by the mathematical community" (p. 83). In the context of using artifacts, Bartolini Bussi and Mariotti (2008) describe the relationship between the personal and the mathematical meaning as a double semiotic relationship: "On the one hand, personal meanings are related to the use of the artifact, in particular in relation to the aim of accomplishing the task; on the other hand, mathematical meanings may be related to the artifact and its use." (p. 754). Adopting these terms, I define the double semiotic relationship as the semiotic potential of an artifact, and assume that the potential is defined with respect to a particular design and pedagogical goals.

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[2] Calculator based ranger is a motion sensor device that can collect and represent real-world motion data, such as distance or velocity.

Learning in this setting means participating in an active process that leads to making sense of the elements, and bringing about an encounter between personal and mathematical meanings. In other words, to learn something, the learner must attend to the existence of the knowledge within the culture and become aware of its existence (Radford, Bardini, and Sabena, 2007). The attention and awareness processes of an existing mathematical object require engagement in a mathematical activity to grant meaning to the object. (Radford, 2003) called this process an objectification process. Objectification requires making use, in a creative way, of different semiotic tools such as words, symbols, and gestures available in the universe of the discourse (Radford, 2003). Semiotic tools play a central role in the objectification processes of knowledge. To examine the objectification processes of learning the antiderivative concept graphically, I intend to answer the following question: How do the personal meanings that emerge from accomplishing a task with a dynamic artifact take on mathematical significance for a pair of high school students?

THE MATHEMATICS, PEDAGOGY AND SEMIOTICS OF THE ARTIFACT

The artifact used in this study was *The Calculus Integral Sketcher* (CIS) (Shternberg, Yerushalmy, and Zilber, 2004) (Fig. 1). As a multi-semiotic system, the CIS contains different types of signs that can be grouped into two categories:

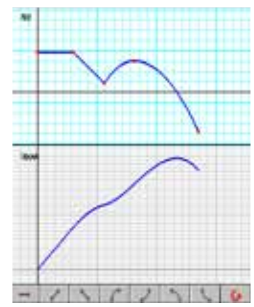

Fig. 1

1. ***Cartesian Graphing system:*** Two Cartesian coordinate systems, one above the other, coordinated horizontally. The curve in the upper Cartesian system signifies a function *f*. The curve in the lower system signifies the values of $\int_a^x f(u)du = g(x) - g(a)$ where the derivative of *g* is the function *f*.
2. ***Iconic Graphing tools***: Students choose the graph in the upper Cartesian system by pressing an icon at the bottom of the CIS. Schwartz and Yerushalmy (1995) describe this set of icons as a mediating language for modeling consisting of segments from a wide range of single variable functions.

Sketching the graph of a function *f* in the CIS is carried out by choosing an icon from the icons applet, placing it into the upper Cartesian system, and manipulating it by dragging the segments or their end points. Fig. 1 shows a curve drawn using multiple icons (linear and nonlinear), and its antiderivative graph *g* is produced by the CIS in the lower system. The tool allows dragging *f* freely and the *g* changes accordingly. Dragging the *g* is permitted only vertically (shifted by a constant) and therefore *f* does not change.

THE DESIGN OF THE STUDY

This case study is part of a longitudinal study that aims to analyze the learning processes of the integral concept graphically. This study explored about one hour of learning by Mohamed and Ahmed, two seventeen-year-old students in the mathematics class taught by the author. The episode took place in the computer lab at the students' school. At the time of the episode, the students had already acquired the concepts of function and derivative, but not that of the integral[3]. The students were familiar with performing differentiation symbolically. The two students shared a computer, and the researcher introduced them briefly to the interface.

To explore the processes of making sense of the antiderivative graphically, I asked the students to interpret the mathematical relationships between the graphs that appeared on the computer screen. To cover the majority of the cases of one variable functions, I asked the students to create graphs using either a single icon or multiple icons. The students were given the following instructions:

> To complete the task, you will use the Integral tools of CIS. You are invited to create graphs of single icon and multiple icons. Your task is to come up with a conjecture and explanation about the mathematical relations between the upper and lower graphs. You can work as long as you want, until you feel that you can sketch a graph without any tool that will appear in the lower window for a given function graph.

The students were video-recorded and their computer screens were captured. The video recording was achieved by software which captured the footage in two different windows: the computer screen and the students' body. The researcher was present as an observer and available to provide technical and miscellaneous clarifications.

DATA ANALYSIS

I used *attention* and *awareness*, Radford's (2003) categories of objectification of knowledge, to analyse the evolving processes of personal and mathematical meanings. I identified *attention* as a declaration about the existence of a mathematical relationship between objects in the semiotic system. In the present study, declarations about the existence of mathematical relationships tended to be based on visual considerations. Justifications and interpretations based on mathematical considerations of the mathematical relationships students had noticed were defined as *awareness*.

I present here the second round of analysis. The first round consisted of reiterative watching of the video, concentrating on students' actions with the tools, their repeated gestures, argumentation, and interpretations. The second round involved searching for and classifying the transcripts into three main categories which were taken from the data: 1) objectifying the relationship between a function and its derivative 2) objectifying the vertical transformation of the antiderivative graph 3) objectifying the relationship between a function and its antiderivative. Therefore, I collated chronologically the utterances and gestures in the discourse that were related to these categories.

[3] They had encountered the integral symbol and its computational uses in their physics and electronics lessons

OBJECTIFYING THE RELATIONSHIP BETWEEN A FUNCTION AND ITS DERIVATIVE

The attention of the relationship between the lower graph as a function and the upper graph as its derivative

12 Mohamed: [while they drag the graph of *f*] Constant function parallel to x-axis [initially (Fig. 2) appears; then, as *f* is dragged vertically, there appears on the screen a constant function in the upper Cartesian system and a linear function in the lower Cartesian system (Fig. 3)]

13 Ahmed: This is a function [the lower graph] and that is its derivative [the upper graph].

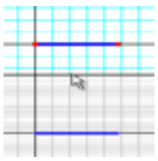

Fig 2

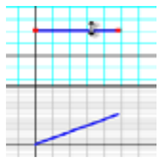

Fig 3

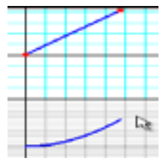

Fig 4

Fig 5

Fig 6

The students create a semiotic system, which contains two constant zero functions by the first icon  of the CIS [12]. When they drag *f* in the upper Cartesian system vertically they create a new semiotic system which contains a constant function in the upper Cartesian system and a non-constant linear function in the lower Cartesian system [12]. Immediately, the students declared a correlation between the two graphs – function and derivative. This suggests that the semiotic systems they create help them to pay attention to the correlation between the graphs, while their dragging of *f* vertically and the corresponding change of *g* suggests that the students become aware of the correlation between the slope of the linear function and the y-value of the constant function. This is shown in the next transcript.

The awareness of the relationship between 'g' graph and 'f' graph

18 Ahmed: The slope is increasing [tracing the graph in the lower Cartesian system (Fig. 4) with the mouse] increasing, increasing

19 Mohamed: You must consider from here to here [trace the interval (0,1) on the x-axis in the lower Cartesian system]. How much is the slope? From here to here [indicates with the mouse on the x-axis from the origin up to one, then goes up vertically to reach the graph in the lower system] it is about one.

20 Ahmed: It is about one [pointing with the mouse to a point its y-value is one on the upper Cartesian system] … how much here?

21 Mohamed: From here to here are two three… Ahh… It is increasing… the slope will behave like this [gestures with his hand (Fig. 5)] its slope is increasing [makes a gesture with two fingers, emulating the linear function (Fig. 6)]

In this excerpt, the students create a linear function in the upper Cartesian system and the graph of a quadratic function appears in the lower system. Ahmed starts to make sense of the connection between the two graphs. He describes the tangent slope of the quadratic by tracing it on the graph with the mouse and by using rhythmic speech "increasing, increasing, increasing" [18]. The words [How much is the slope? From here to here] and the gestures [indicates with the mouse on the x-axis from the origin up to one, then goes up vertically to reach the graph] performed by Mohamed in [19] suggest that he is describing the rate of change of the quadratic function at several points. In [20] Ahmed starts to describe the connection between the two graphs for a specific value of x that the two graphs share. They correlate the tangent slope value in a point where its x coordinate is one in the lower graph to the y-value of the function in the upper graph for the same x coordinate [20]. Mohamed's gestures and words suggest that he is becoming aware of the behaviour of the tangent's slope of the quadratic function. The gestures performed by Mohamed – the first gesture signifying the tangent slope of the quadratic function and the second gesture signifying the linear function graph in the upper Cartesian system- together with Pointing by the mouse to the point whose y-value is one on the upper graph. Suggest that the students are becoming aware that the behaviour of tangent slope value of the lower function on points is the same behaviour of the y- value of the upper graph at the same x-coordinate [21].

OBJECTIFYING THE VERTICAL TRANSFORMATION OF THE *g* GRAPH

The attention and awareness to dragging the *g* graph vertically

36 Mohamed: There is no difference between the slopes of the graphs when we drag them up and down. We get the same slope if we drag it like that [drags 'g' vertically – (Fig. 7)]

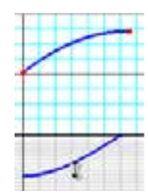

Fig 7

37 Ahmed: What is the value of this point? [The intersection point with the y-axes]

38 Mohamed: It is not the matter of the value of the point…The matter is considering the slope of the lower function, for example the slope is one [makes a motion with the mouse like a tangent line at a point on the lower graph] here is one [points toward the coordinate point in the upper graph]. The slope here is two [makes a motion with the mouse like a tangent line at a point on the lower graph and simultaneously he points toward a coordinate point in the upper graph]

39 Ahmed: I don't agree with you

40 Mohamed: It is correct. The slope of the lower graph isn't changing even if you drag it [dragging 'g' vertically], the slope will not change

The students drag *g* vertically. They observe that *f* is not moving. Mohamed observes that the shape of *g* does not change when they drag *g* vertically. Mohamed demonstrates his awareness of this fact by using the word 'slope'. His vertical dragging action, the words he is using, the gestures he is performing on the lower graph and the pointing gestures he is performing on the upper graph in [36] and [38] suggest that he is 'seeing' the tangent's slope of *g* in different points and coordinates the tangent slope value to y-value of points on the upper system. He is repeating this process for other graphs which differ by a constant.

OBJECTIFYING THE RELATIONSHIP BETWEEN *f* AND *g*

First phase: Objectifying the function change

In this excerpt the students are discussing ways to describe how the tangent slope is behaving. While Ahmed claims that the slope of the tangent to the left of the minimum point is decreasing (Fig. 8) Mohamed is explaining how he sees the increase of the tangent slope.

62 Mohamed: It is minus two; minus one [y- values of *f* from left to right] it is increasing [points to *f*].

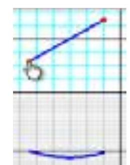

Fig 8

63 Ahmed: It is the derivative [points to *f*]

64 Mohamed: This graph [*f*] describes the slope of the function [*f*]

65 Ahmed: Wait a moment… here it is a function [*f*]… how much is the slope? [Tracing the left side to the minimum point on [*g*], it is decreasing until it becomes zero

66 Mohamed: It means increasing

67 Ahmed: Correct! It is increasing [tracing *f* until the intersection point of *f* with the x-axis]...when it becomes zero [*f*] it gets this point [he connects, with the mouse, the intersection point in the upper graph with the minimum point of *g*]

68 Mohamed: When the slope is zero [traces with the mouse the minimum point of *g*] it intersects the x-axis [indicates the point where *f* intersects the x-axis]

The semiotic system of graphs, the gestures and the words in [62] suggest that Mohamed is considering, for the first time, the graph of *f* and reading two consecutive y-coordinates, as a way to explain the increase of the tangent slope. While, Ahmed still maintains the view - the graph in the lower system is a function and in the upper system it is a derivative - which mentioned and analysed in the last excerpt. Mohamed's declaration in [64] suggests that he has seen the connection between the y-values of the graph in the upper Cartesian system and the tangent slope value of the lower graph. In [65] Ahmed is checking Mohamed's claims by tracing the lower graph up to the minimum point. His tracing gesture imitates the behaviour of the tangent of the graph. The mismatch between Ahmed's tracing gesture and his statement "It is decreasing until it becomes zero" suggests that Ahmed has a difficulty describing the process where a negative tangent slope is increasing [65, 66]. In [67] Ahmed leaves the 'g' graph and concentrates on describing the upper graph, noticing that the intersection point between the upper graph and the x-axis corresponds to the minimum point in *g* graph. In line [68] Mohamed explains this correspondence by identifying the slope of the minimum point as zero. This explanation suggests the role the word "slope" plays in correlating both graphs and in conceptualizing the idea of the antiderivative.

Second phase- objectifying the rate of change of 'f'

The students use the fourth icon  to create an increasing function graph with a decreasing rate of change in the upper Cartesian system (concave down Fig. 10). For *g*, they obtained an increasing graph with an increasing rate of change (concave up) in the lower Cartesian system. The students are confused about having two increasing graphs where one has an increasing rate of change *g* and the other, a decreasing rate of change *f*. In the next excerpt, I analyse the processes and the emergence of the new semiotic means that enable them to solve this matter.

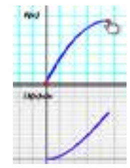

Fig 10

101 Ahmed: [Fig. 11 appears on the screen] It is the derivative [*f*]

102 Mohamed: It is zero. Okay? Here it is two… If the value is two what does it means in the integral? The slope of the integral function at this point is two… What have we got here? Three. It is decreasing at three.

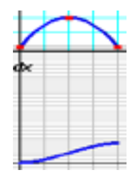

Fig 11

103 Ahmed: It is decreasing. The slope is decreasing

104 Mohamed: Look here [he deletes the right branch of Fig 11 and drags the right red point diagonally to get Fig. 10] [Silent for 10 seconds] I got the idea…wait a moment… when x is one the value is three [indicates the point (1,3) in the upper graph] Therefore, the slope is also three when x is one. [Points to lower graph] here it is four [points to the point (2,4) on the upper graph] thus the slope here is also four [makes a gesture like a segment on the lower graph]. Here its value is five plus [points to the point where x is 3 in the upper graph] thus its slope is five plus.

After they encountered difficulties interpreting the concavity of the integral graph [Fig. 10], the students made a new graph, using two different icons in the upper graph [Fig. 11]. The left icon is increasing graph and concave down, and the right icon is decreasing graph and concave down [Fig. 11]. Initially Ahmed signified the function graph as derivative [101]. Mohamed adapts Ahmed's claim as he is determining three y-values of the function graph. The students are well aware that each y-value in the function graph is the value of the slope of the tangent in the integral graph [102], and they are using the y-value of the function and the tangent slope of the integral function graph as means of semiotic mediation to explain the mathematical relationship between the two graphs. The use of these means is well illustrated in [104], where Mohamed interprets the relationship between the function graph and the integral graph.

DISCUSSION

Through the semiotic lens I determine three essential foci of objectifying the antiderivative concept. The first focus is objectifying the relationship between a function and its derivative. The constant function icon plays a central role in objectifying the relationship between the graph in the lower Cartesian system and the graph in the upper Cartesian system. The vertical dragging of the constant function graph performed by the students allows them to create an additional similar semiotic system, while linking the graphs according to the cultural meaning of the antiderivative helps them notice the difference in the similar semiotic systems. Once they notice that, the connection between the antiderivative graph slope value and the y-value of the constant function becomes apparent. Changing the semiotic system to include a linear function and a quadratic antiderivative produced changes in the semiotic resources the students used to objectify the relationship between the function and the antiderivative graph. Utilizing additional semiotic resources allows me to suggest that the slope of the graph in the new situation is not obvious to the students, unlike in the constant function case. In order to make this obvious they use rhythmic speech to describe the behaviour of the tangent's slope, and gestures to objectify the change of the tangent slope. The second focus considers the objectification of the vertical transformation of the antiderivative function. Dragging the antiderivative function graph vertically by the artifact and noticing that the function graph is fixed help the students become aware of an essential mathematical element of the antiderivative concept. The little variety of the semiotic resources used in this focus allows me to claim that objectifying the mathematical fact that the antiderivative of a function is a family of antiderivative functions is actually quite simple. The third focus considers the objectification of the antiderivative idea. That is, the y-value of a point on a function's graph is the tangent's slope value of the antiderivative function's graph for the same x-value. The current focus consists of two phases: objectifying the change of the function and objectifying the rate of change of the function. The first phase emerges through the students' confusion in describing the behaviour of a tangent slope of a decreasing graph with an increasing rate of change. This confusion has helped them leave the lower Cartesian system in favour of the top one. I assume that this happened because of the linearity of the function graph in the upper Cartesian system. The second phase does not refer to the shape of the anti-derivative function behaviour only, but does refer to how the shape of the graph behaves. This phase emerged when the students created two contrasting graphs, one concave up and the other concave down.

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FACTORS AFFECTING THE ESTABLISHMENT OF SOCIAL AND SOCIOMATHEMATICAL NORMS

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The present paper initially examines the concept of norm as it has been implemented in mathematics education. A sample excerpt is then used to demonstrate that the analysis of norm establishment would be better accompanied by the consideration of the face-saving needs of the participants. The relevant sociological and linguistic theories are also presented in order to provide the background for that claim.

INTRODUCTION

One of the main current trends in mathematics education research puts the emphasis on collaborative learning environments. In line with sociocultural views on learning been directly related to the societal contexts, numerous studies have been made. It is noteworthy, however, that there are voices that call for attention in the mere acceptance of any teaching/research approach, just because it belongs to the mentioned trend: "the current obsession in mathematics education with group work and discussion is not, and cannot be, a panacea" (Mason, 2004/2008, p. 1). The present paper is the product of the author's interest in the interactional processes that constitute or assist the establishment of taken-as-shared mathematical meanings (Cobb, Wood, Yackel, & McNeal, 1992). In the relevant literature a number of theoretical constructs have been used in order to assist the researchers to describe and interpret the observed phenomena. Among these, I am mostly interested in the concepts of the social and sociomathematical norms as they have been introduced for mathematics education by Paul Cobb and his colleagues (e.g. Cobb & Yackel, 1996; Yackel & Cobb, 1996). These concepts have been used quite extensively in studies that vary from classroom interactions (McClain & Cobb, 2001) to collaborative student pairs (Tatsis & Koleza, 2008) and from kindergarten (Tatsis, Skoumpourdi, & Kafoussi, 2008) to university (Yackel & Rasmussen, 2002). My aim here is to examine whether these concepts suffice to account for the establishment of mathematical knowledge. The answer to this question may be of interest to the researcher or the teacher who wants to interpret the participants' actions while they interact in a mathematical environment. I have to stress though that I do not aim to offer a novel approach in interpreting interactions; I merely wish to examine the complementarity of particular existing theories.

SOCIAL AND SOCIOMATHEMATICAL NORMS

The concept of norm is derived from the broader notion of *prescriptions* which are: "behaviours that indicate that other behaviours should (or ought to) be engaged in. Prescriptions may be specified further as demands or norms, depending upon whether they are overt or covert, respectively" (Biddle & Thomas, 1966a, p. 103). A similar concept to that of the norm is the *obligation* (Voigt, 1994) which connects the various routines of the classroom and regulate the students' and the teacher's actions. Different typologies for the norms have been proposed by sociologists, each based on different criteria; Morris (1966) groups the various criteria into four sets:

(a) distribution of the norm (extent of: knowledge, acceptance and application of the norm),

(b) mode of enforcement of the norm (reward-punishment, severity of sanction, enforcing agency, extent of enforcement, source of authority, degree of internalization by objects),

(c) transmission of the norm (socialization process, degree of reinforcement by subjects),

(d) conformity to the norm (amount of conformity attempted by objects, amount of deviance by objects, kind of deviance).

The above typology was not created to be implemented as a whole; it is rather the researcher's decision on which criteria to use. For example, the distribution of the norm is related to, among others, the extent of knowledge of the norm by subjects (those who set the norm) and by objects (those to whom the norm applies). Thus, in the case of (mathematics) education, a researcher who is interested in the distribution of a norm within a classroom may focus on the objects (students), while one who is interested in the distribution of a norm within the mathematicians' community may focus on the subjects. Yackel and Cobb (1996), while observing an 'inquiry-oriented' mathematics classroom have made another distinction, between social and sociomathematical norms. The former are related to the general structure of classroom activity and some examples are: "explaining and justifying solutions, attempting to make sense of explanations given by others, indicating agreement and disagreement, and questioning alternatives in situations in which a conflict in interpretations or solutions had become apparent" (Cobb & Yackel, 1996, p. 178). The latter are specific to mathematical activity and relate to which contribution counts as "a different mathematical solution, an insightful mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation" (Cobb & Yackel, 1996, p. 179). The authors relate the process of establishing norms to the process of constructing mathematical beliefs and values, thus attempting to connect a sociological with a psychological approach. Similar to the concept of norm is that of the *metadiscursive rules* "that is, mostly tacit navigational principles that seem to underlie any discursive decision of the interlocutors" (Sfard, 2002, p. 324). Although Sfard considers norms as a sub-case of rules[1], she offers some useful (for the teacher and the researcher) characteristics of the metadiscursive rules:

> [M]etadiscursive rules may *evolve* over time (as opposed to the object-level rules of mathematics, which, once formulated, remain more or less immutable). Metarules are also made distinct by being mainly *tacit*, and by being perceived as *normative* and value-laden whenever made explicit. Finally, metarules are *constraining* rather than

deterministic and are *contingent* rather than necessary. (Sfard, 2008, p. 202, emphasis by the author)

Finally, one could mention the notion of the *didactical contract* introduced by Brousseau (1997), which refers to specific habits of the students that are expected by the teacher and vice-versa. The responsibility of knowledge construction is seen as shared between the teacher and the students and the didactical contract assists that process. However, a familiar didactical contract can also create problems for the students, especially when they enter a situation where the contract changes considerably, e.g. by entering the university.

ROLE THEORY AND THE NOTION OF FACE

All the approaches mentioned in the previous section are to some extent related to a sociological theory. What is common in them (and in other approaches of the same tradition) is the focus on the interactional aspects of the activities that take place in educational settings. The human agent (teacher, student or researcher) is present, but the actions of these agents are usually analysed under the lens of the mathematical activity, e.g. examining the changes in students' knowledge or beliefs about mathematics. What seems to be missing is a consideration of the fact that all participants in any interaction can be viewed as some kind of actors, and sometimes the whole 'scene' is set before them. People's behaviours, seen under this interpretive model, form the research ground for *role theory* as set by Biddle and Thomas (1966b) and exemplified by Goffmann (1971, 1972). The central concepts in that theory are *face* and *role performance*. Face is defined as "the positive social value a person effectively claims for himself by the line others assume he has taken during a particular contact" (Goffman, 1972, p. 5) and can be further categorised into positive and negative: positive face is related to a person's need for social approval, whereas negative face is related to a person's need for freedom of action (Brown & Levinson, 1987). The participants not only have these needs, but recognise that others have them too; moreover, they recognise that the satisfaction of their own face needs is, in part, achieved by the acknowledgement of those of others. All these affect the role performance which is defined as "all the activity of a given participant on a given occasion which serves to influence in any way any of the other participants" (Goffman, 1971, p. 26). For example, when a student enters the mathematics classroom s/he is primarily concerned about maintaining face; in order to achieve that s/he has to deploy various linguistic and non-linguistic strategies. The same is the case for the teacher. Each communicational move may be initially characterised as face saving or face threatening for the speaker and/or the listener. A request or a question from the teacher for example, is considered a face threatening act; a student's wrong answer on the other hand, may or may not be face-damaging, depending on the classroom's established norms. The effect of the consideration of face can be seen in the use of vague language (Rowland, 2000), which in turn affects the 'quality' of mathematical propositions, thus the learning of mathematics itself. But why would someone use vague language, especially in the mathematics classroom? A comprehensive list is given by Channell (1985, 1990, 1994, as quoted in Rowland, 2000, p. 67):

- giving the right amount of information;
- deliberately withholding information;
- saying what you don't know how to say;
- covering for lack of specific information;
- acknowledging and achieving an informal atmosphere;
- expressing uncertainty;
- downgrading the importance of something as to highlight something else;
- expressing politeness, especially deference;
- protecting oneself against making mistakes.

The above list provides, among others, direct links with the concept of face (positive and negative) and its protection against possible threats. It also stresses the point that I made at the beginning of the section that participants in any interaction, including students in the mathematics classrooms, may be viewed as persons with aims similar to those of any other person. Sometimes, for example, a student might feel the need to not convey the information requested by the teacher; and this might be due to some (inter)personal reasons, related to saving face. In the next section I will present an example of the implementation of these theories; my aim is to demonstrate their compatibility with the concepts of social and sociomathematical norm.

AN EXAMPLE

Mathematical concepts are (or should be) clearly defined; the same is the case with the rules of logic, which assist the mathematicians in expressing conjectures, proving theorems and generally conducting research in pure mathematics. The rules related to these processes can be referred to as object-level rules (Sfard, 2008). However, the interactions taking place in a mathematics classroom or in any other research setting are governed by metarules and norms, whose main characteristics, as they have been already described, are quite far from being fixed. I believe that what provides these norms with their flexibility is the fact that human agents are involved in their establishment. The needs of these agents may evolve over time, thus the norms have to be renegotiated or even replaced. For example, turn-taking – or, more generally, who has the right to speak – may be of high importance in the first years of schooling, but, as students mature, the norm of the quality of one's contribution should be valued. This could be seen as an example of a social norm (on turn-taking) being transformed into a sociomathematical norm.

Two issues need to be addressed at this point. The first is that an established norm does not necessarily have to be a desirable one; in other words, a norm might hinder – instead of assist – mathematics learning. A characteristic example is the 'mathematical differentiation norm' (Tatsis & Koleza, 2008) according to which mathematics is comprised of distinct, non-overlapping fields, such as algebra and geometry; there is also a distinction between mathematical and practical solutions. In the following excerpt two female student teachers have just found the answer to the following problem: *Figure 1 shows a triangle in which three lines are drawn to one or the other of the opposing sides from each of two vertices. This divides the triangle into 16 non-overlapping sections. If 14 lines are drawn in the same way, how many non-overlapping sections will the triangle have?* The interaction takes place in a laboratory setting and the researcher has taken a passive stance throughout the interaction by choosing not to intervene or assist the student teachers.

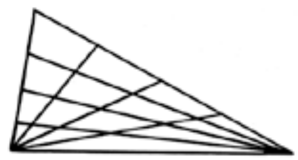

Figure 1: Number of non-overlapping sections

134	Student A:	Okay, if we draw three segments we have four triangles. But this is practical. You may say that in three, if you draw three line segments from the one side and also from the other, it created four line segments, four triangles, with four sections each. Right? So, in total the overlapping will be four times four, 16. If you draw 14 line segments you will create 15 triangles with 15 overlapping sections. So, okay, does this have 100 % proof? And if someone tells you: I with 60? It will… 61 triangles that will be created, with 61 non-overlapping sections. So, indeed it will be 16 here, so indeed it'll be 225 here, so indeed 3600 and so on.
135	Student B:	Good, if we justify about 15, let the other one with the 60 bother for that.
136	Student A:	Do we have an eraser or something?
137	Student B:	Why, does it necessarily have to be like that?
138	Student A:	What?
139	Student B:	A formula?
140	Student A:	We are not talking mathematically, that's all, okay? And what I say is correct, to say: since with three is the immediate larger in 16…

The two students agree to write down the solution, but when student B starts writing, the following exchange takes place:

149	Student A:	It can't be. Something else must be happening here. You keep writing, keep writing.
150	Student B:	What was Thales's Theorem about? It doesn't fit.
151	Student A:	For parallels, angles… let it go. It can't have theorems, here it's a practical issue.
152	Student B:	If it's practical, the way found is correct.

The above excerpt can be seen under many different lenses, but for the purpose of the present paper I focus only on the norms which are evident from the students' contributions. The most apparent is the norm of cooperation, according to which the students are expected to work together to solve the given problem; this is evident by the use of the first plural form 'we' in most utterances. What is also evident throughout this brief excerpt is the students' need to 'mathematise' either by creating a formula or by proving their conjecture that the number of non-overlapping sections equals to the number of lines drawn increased by 1 in the power of two. These are manifestations of the norm related to the mathematical efficiency of a solution as described by Yackel and Cobb (1996). Finally, we may observe the norm of mathematical differentiation mentioned before, especially in [134], [140] and [151]-[152]. At the same time, there are some utterances that can't be fully interpreted by the concepts of norms; a characteristic case is [150] where student B asks a question which she immediately withdraws. One may wonder why she did not care to wait for her colleague's response. At a first glance, mentioning a theorem can be related to the norm of mathematical efficiency, interpreted as: "all propositions should be justified".[2] The student, struggling between a 'practical' solution (since they had to draw few cases to justify their answer) and a 'theoretical' background that characterizes school geometry, comes up with one of the most familiar theorems. She has the need to contribute to the common task of writing an efficient solution and she offers the best she can at the given moment: Thales Theorem. By proposing something so novel at that moment she is exposed in front of her colleague, but mainly in front of the researcher, who was present. This is a typical face threatening act; and student B in order to minimize the potential damage to her positive face (in case her suggestion is proved to be inadequate) she immediately adds that "It doesn't fit". Student A, who seems to know the theorem, quickly rejects the plausibility of the theorem and they move on. Such exchanges are frequent in educational settings where many students feel that their positive face will be damaged in case of a wrong answer; and sometimes they choose to be silent and this has serious implications for their learning, but also for the establishment of the whole learning community.

Another remark on the given excerpt may come from the observation that student A seems to be in charge of the process; she expresses verbally the solution in [134], leaving to student B the role of merely writing it down, even when she thinks that their solution might not be complete [149]. The small excerpt contains some face empowering verbal acts, such as "what I say is correct" [140] and some face threatening acts for student B in [149] and [151]. Tatsis and Koleza (2006) while analyzing the interactions of three meetings of pairs of preservice teachers, observed the following roles:

(a) The dominant initiator: makes many suggestions, rarely asks for the partner's opinion and always tries to maintain face; demonstrates a low level of conformity to most social and sociomathematical norms, sometimes adjusts a norm by his/her acts. Whenever in a difficult position, attributes it to external factors (e.g. the difficulty of the task or even the inability of the partner).

(b) The collaborative initiator: makes many suggestions, asks for the partner's opinion, gives information whenever necessary and – most of the times – tries to maintain face; is ready to withdraw a suggestion but only if the opposing one is strongly grounded; generally demonstrates a high level of conformity to the social and sociomathematical norms established.

(c) The collaborative evaluator: makes relatively fewer suggestions compared to the previous roles, always gives information (whether asked or not) and tries to maintain face when s/he believes that it is not against any norm; thus, s/he shows a high level of conformity to the norms established and his/her acts demonstrate a high level of uniformity and facilitation to the partner's acts.

(d) The insecure conciliator: makes few suggestions and does not try to maintain face in an explicit way; shows a low level of conformity to most norms as s/he accepts his/her partner's suggestions without evaluating them; demonstrates the highest level of facilitation to the partner's acts. Whenever in a difficult position, attributes it to uncontrollable factors, such as ability or task difficulty.

The small presented excerpt does not provide sufficient amount of information to lead us to a valid characterization of the two students' roles. However, we may infer that student A is an initiator, and possibly a dominant one. Student B's behaviour, on the other hand, provides indications that she is a collaborative partner, but we cannot conclude whether she is an initiator or an evaluator.

Generally, in role theory, role performance is conceived as highly situated, thus the roles described before aimed to interpret the participants' act in the particular context (collaborative problem solving). What is important, however, is the fact that these roles are described in relation to the social and sociomathematical norms which are established. The importance of this fact lies in the realization that, while the role of the teacher is seen as central in the process of establishing the desired norms (Sfard, 2008), the actual process is much more complex due to the face needs of the participants (who fulfil these needs by performing various roles). I will return to this point in the final section of the paper.

The second issue that needs to be addressed is that a norm does not have to be accepted by all members of a community to be considered as such. Actually, as I already mentioned, the distribution and the conformity are just two attributes that can be used to describe a norm. This idea is not taken up by many contemporary researchers, who usually see a norm as a characteristic of the whole community of the classroom.[3] The first implication of this view is that other, 'minor' norms, which are enacted by few participants, are not given much attention. The second implication is that it creates an assumption that all norms which are desirable by the teacher are efficient for all students. Let me elaborate on this, by using as an example the norm of what counts as an insightful mathematical solution (Cobb & Yackel, 1996). It is quite apparent that the notion of 'insightful solution' is a rather subjective one, and even if the teacher establishes some metarules on that, there will still be some students for whom these rules will be inadequate.

CONCLUSIONS – AN ATTEMPT TO RECONCILE

In the preceding sections I have attempted to briefly present the basic characteristics of the concept of norm and then role theory with its relevant concept of face. I have also demonstrated how these concepts seem compatible, or even more, complementary, since they can be jointly applied in the analysis of interactions. A question might then arise: is such an approach justified by the theoretical assumptions of the relevant theories? The answer is positive, and it can be derived by examining some of the references in the papers mentioned before; they seem to have a common interest in the importance of interactions and many cite the theory of symbolic interactionism (Blumer, 1969). According to that theory, the role of symbols, especially language, is vital for the process of interactions; it is through symbols that people establish shared meanings and define the situation they are involved with. The person is not treated as a passive receiver of society's influences, but as an active participant who takes part in the formulation and negotiation of shared knowledge during the process of symbolic interaction.

The concept of norm is a fine tool for the researcher at the initial stage of the analysis, or for the teacher who wants to establish a feasible didactical contract with the students. However, once the researcher reaches the micro-level of utterances, s/he will probably notice that some of them do not seem to fit to any established norms; the researcher may also encounter some utterances that were not supposed to be heard in a mathematics classroom (e.g. because they are vague), which at the same time seem to be effective from a communicational point of view. The teacher might have more pragmatic worries concerning the enactment of the norms that should be established. How shall s/he deal with utterances like: "The maximum will probably be, er, the least 'll probably be 'bout fifteen." (Rowland, 2000, p. 1)? This utterance clearly violates the sociomathematical norm of clarity; however, this consideration cannot assist the teacher in establishing that norm, unless s/he becomes aware of the fact that the student might be using these linguistic forms in order to protect his face. Thus, acknowledging the face-saving needs of the participants might be of great importance in the mathematical classroom and particularly, in the establishment of social and sociomathematical norms.

NOTES

1. According to Sfard (2008) a rule is considered a norm only if it fulfils two conditions: it must be widely enacted within the discursive community and it must be endorsed by almost all members of that community, especially those considered as experts.

2. This can be also seen as an example of a negative influence of an established norm.

3. However, there is research (Planas & Gorgorió, 2004) indicating that in multi-ethnic mathematics classrooms students adhere to different sets of norms, which in turn results in learning obstacles for some of them.

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STUDYING THE DISCOURSE OF SCHOOL MATHEMATICS OVER TIME: SOME METHODOLOGICAL ISSUES AND RESULTS

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The project "The Evolution of the Discourse of School Mathematics"[1] seeks to investigate how the nature of participation in mathematical discourse expected of students in England has changed over the past three decades. We have chosen to use the examinations taken by almost all students at the end of compulsory schooling (aged 16 years) as our lens. The General Certificate of Secondary Education (GCSE) examination is widely used as a gatekeeper for further education and employment as well as being a key accountability measure for schools. It thus has high-stakes consequences for schools, teachers and students and, as such, has a strong influence on the curriculum and pedagogy experienced by students through much of their secondary school career (Broadfoot, 1996).

The project rests on a theoretical orientation that understands doing mathematics as participating in mathematical forms of discourse (as a consumer and producer). Hence our analytic approach focuses on discourse, examining the forms of language and other communicative modes involved in engaging with the examinations. In our poster at CERME7 (Morgan & Sfard, 2011), we presented a preliminary analytical framework for identifying characteristics of examination texts. The overall structure of this framework was based on Sfard's characterisation of mathematical discourse as consisting of vocabulary and syntax, visual mediators, routines and endorsed narratives (Sfard, 2008). This was augmented by questions to be posed to the texts and detailed analytical tools also derived from Sfard's communicative theory of mathematical activity and from multimodal social semiotics and systemic functional linguistics (Halliday, 1978, 1985; Kress & van Leeuwen, 2001; Morgan, 2006).

A major challenge in beginning this project, however, was to develop this framework into an analytical tool with a sufficient degree of delicacy to distinguish between texts and sufficient robustness to enable it to be applied efficiently and reliably. See Tang, Morgan and Sfard (2012) for an account of the process of development of the framework, focusing in detail on the analytical categories *specialisation* and *objectification* of discourse. Since beginning the project we have come to realise more clearly the challenges it poses and the complexity of the variables involved. Time is a crucial variable defined by our original aims. We have therefore sampled

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examination papers from eight points in time between 1980 and 2011, chosen following a review of curriculum and assessment policy and practices in order to capture key changes during the period. This sample consists of 31 question papers containing a total of 558 questions. On the one hand this is a large sample of questions, yet, as the questions range across all parts of the school curriculum, the number of questions on any topic area is relatively small and, even within a single area of the curriculum, the mathematical focus of questions set in different papers can vary considerably. In attempting to look at variation over time, therefore, it has been necessary to find a way of looking that enables comparisons across the full range of the curriculum.

In this poster presentation we will exemplify a small part of the developed analytic framework, focusing on the specialised mathematical nature of the discourse. The fundamental question with which we interrogate the data is: to what extent and in which ways do the examinations demand that students participate in specialised mathematical discourse, using and producing texts with characteristics of such a discourse. We will present indicators used to address this question, exemplifying their application to examination questions from different years and topic areas, together with quantitative results showing variation in this aspect over time.

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ANALYZING TEACHERS' FOLLOW UPS AND FEED FORWARDS, SEEN AS A WAY TO ENABLE STUDENTS' PARTICIPATION IN MATHEMATICAL REASONING

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In an on-going study, mathematics teachers' communications with students are investigated, with an interest in how teachers' acts respond to, interact with and take forward students' mathematical reasoning. To analyze this, an analytical framework, combining follow up and feed forward, separately used in earlier studies (Boistrup, 2010; Brodie, 2010) will be proposed. The follow ups and feed forwards are seen as providing students with opportunities, in different ways, to participate in mathematical reasoning. The framework is to be used for studying teacher-student interaction as well as interaction in whole class discussions.

INTRODUCTION

Mathematical reasoning is seen in many international frameworks as crucial for becoming mathematically proficient. My interest lies in how teachers can enable students to participate in mathematical reasoning and thereby develop their competence to justify ideas and conclusions, to create arguments and improve their conceptual understanding.

A Swedish study showed (with several exceptions and some variation) that students, in general, are offered limited opportunities to develop their competence in reasoning (Bergqvist et al., 2009). Generally, activities in the classroom focus mainly on routine tasks and mathematical procedures.

The purpose of this poster will be to propose a basis for an analytical framework that will be used in an ongoing study with an interest in mathematical reasoning.

THE STUDY

The aim of the study is to examine the teachers' use of follow up and feed forward as a means to enable student participation in mathematical reasoning and the possible effect it has on students' reasoning. Follow up is a notion used by Brodie (2010) to describe teachers' responses to students' contribution. Follow up is aiming to respond to, interact with and take forward student contributions. Feed forward is a notion that has been used by Boistrup (2010) as an act related to future acts by the student and/or the teacher, which can create opportunities for mathematical reasoning.

Research questions

- How does the use of different types of follow up and feed forward enable student participation in mathematical reasoning?
- Does the use of follow up and feed forward change when the teachers participate in a collaborative development project?

- How do the different types of follow up and feed forward affect students' reasoning?

METHODOLOGY AND ANALYTICAL FRAMEWORK

Data is gathered from school development projects, seeing collaborative learning as a means to improve teaching. Lessons are video recorded in such a way that the teacher's actions, when interacting with individual students and in whole class discussion, will be captured together with chosen students' interaction with each other.

For analyzing the follow up and feed forward provided by the teachers, notions from earlier studies will be combined in an analytical framework. Some follow ups (Brodie, 2010) attempt to transform the student contribution. It could be by asking the student to elaborate on his/her idea or by the teacher contributing something new which the student uses or by the teacher making her own mathematical contribution. Other follow ups maintain student contribution, without any intervention, in the public realm for others to elaborate on. To the analytical framework will also be added notions presented by Boistrup (2010), for example feed forward as guiding and challenging. The above-mentioned types of follow up and feed forward will, from a socio cultural perspective (Lerman, 1996), be seen as enabling student participation in mathematical reasoning and learning.

Students' ability to reason, as a possible effect of the follow up and feed forward given by the teacher, will be analysed adopting a multimodal perspective where the use of different communicative modes will be taking account for (see e. g. Boistrup, 2010).

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TEXBOOKS AND LOGBOOKS IN MATHEMATICS

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This poster presents the main themes found in a literature review about logbooks and their impact on helping students to gain more information from reading mathematical textbooks. Previous research in years 3 and 4 has shown connections between reading skills and mathematical skills. Logbooks are suggested to overcome these difficulties.

In Sweden, recent research has shown that a dominant practice in mathematics education involves students working individually in textbooks (Johansson, 2006; Myndigheten för skolutveckling, 2007).

To gain meaning from textbooks, students need to read and locate specific information. However, not all students possess the skills necessary to do this and struggle to gain anything from their textbooks except the location of the exercises (Weinberg & Wiesner, 2011). According to Johansson´s (2006) analysis of Swedish mathematical textbooks, the textbooks lacked important parts and did not provide possibilities to reflect and ask questions on the content of the text or to discuss the role of mathematics in daily life and society.

There is a clear need for students to use and develop the mathematical language used in the textbooks. One way for students to do this is to use logbooks to reflect on what they are reading in textbooks (Borasi & Rose, 1989). However, little research has been done that shows how logbooks can be used to support students´ reading of their textbooks.

Consequently, I suggest the need to research the following questions:

- In what ways, can writing in mathematics support the students´ understanding of what they read in their textbooks?
- How can writing in logbooks contribute to communicating about mathematics more generally?

READING AND WRITING IN MATHEMATICS

Möllehed´s (2001) study in years 4-9 showed an interrelation between mathematics word problem solving performance and reading comprehension skills. In particular, vocabulary is considered to be a key factor in reading comprehension (Westlund, 2009). In reading mathematics textbooks this may be especially important because the vocabulary can be difficult for the students to comprehend because there are terms that can be found only in the mathematics. Some of the mathematical words can also have a different meaning in the natural language, for example odd and volume (Lee, 2006; Myndigheten för skolutveckling, 2007).

In year 4, mathematics is considered to become much more difficult for Swedish students when many new concepts are introduced and the amount of text in the tasks increases and becomes more advanced (Myndigheten för skolutveckling, 2007). Therefore, I would suggest that the research study be situated in a year four classroom, when most students are about 10 years old.

In the mathematical logbook, students can express and reflect on their feelings, process, knowledge, and beliefs about mathematics (Borasi & Rose, 1989). According to Lundberg and Sterner (2006), the logbooks can also become mathematical dictionaries for the students, where they continuously supplement new words and explanations in order to develop their mathematical language.

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG10: CULTURAL DIVERSITY AND MATHEMATICS EDUCATION

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Keywords: *Inclusion; exclusion; politics; agency; participation; dialogical self; foreground; ethnomathematics; critical mathematics education; non-school mathematics*

The work and the discussions that animated the activities of WG10 sought to position mathematics education within the realms of culture, society, the political and the historical. Participants in WG10 presuppose that mathematics education refers to more than the encounter between an individual and a mathematical object. The process of teaching and learning mathematics occurs within a culture that both influences and is influenced by the workings of mathematics education. This way, the group was interested in understanding learning and teaching mathematics in culturally diverse schools, classrooms and other educational settings, by analysing the variety of sites where mathematics education takes place and the differences in the organization and structure of practice in such contexts. On the other hand, it is assumed that research is not an innocent activity producing a diagnosis of the state of mathematics education practices or proposing solutions to the problems of practice. Rather, it is an active participant in shaping, discursively, the possibilities of seeing and inventing practice. It was the focus of the group also to cover research and its discourses, and the way in which such discourses contribute to the formation of particular subjectivities and ideologies in and through mathematics education. As a way to address both the teaching and learning of mathematics and the research done on this issue, the group privileged theories and methodologies that explicitly connect what is usually called the micro level—people interacting with mathematics—with the macro level—culture, society and the political. Inter-disciplinary perspectives including fields such as socio-cultural and discursive psychology, cultural studies, history, anthropology, linguistics, sociology, discourse studies, political sciences, psychoanalysis and philosophy are among the most prolific theoretical frames used by the participants of WG10 to attain such a goal.

Against this background, the diversity, both methodologically and theoretically, of the 17 papers and 3 posters submitted allowed participants to tackle a set of problems that, although highly influential in the teaching and learning of mathematics, often remain understated in mathematics education research. These problems were brought forth by an emphasis on the sociopolitical dimension of mathematics education, and by the use of broader theoretical frames than the ones provided by mathematics, didactics and psychology. The theories used by participants ranged from modern influences such as the works of Bourdieu (e.g., Johansson & Boistrup) and Bernstein (e.g., Bohlmann, Hinkelammert, Rhein, & Straehler-Pohl) to more postmodern approaches such as the philosophy of Žižek (Straehler-Pohl & Pais) and Foucault (Valero). We also had works using discourse analysis to investigate the practices of school mathematics (e.g., Doğan & Haser, Che & Bridges), as well as social psychology, in particular the notion of dialogism (e.g., Abreu, Crafter, Gorgorio & Prat; Machado & César; Newton & Abreu). Some of the papers took advantage of philosophical thought to engage in a theoretical discussion about the possibilities and limits of mathematics education and its research (e.g., Kollosche; Valero; François & Pinxten). On the other hand, we also had participants taking advantage of theories already developed within mathematics education. This was the case with ethnomathematics and critical mathematics education, two fields concerned with the sociopolitical dimension of mathematics education, and which have been in the last two decades gaining importance within the field. Some of the participants based their research within one of these two approaches (e.g., Domite; Campos, Jacobini, Wodewotzki & Ferreira; Fernandes; Martins; Hochmuth), and some sought for an articulation between the two (e.g., François & Pinxten).

While most of the participants take advantage of empirical classroom material to develop their argumentations, there were a significant number of papers dealing with mathematics education from a strictly theoretical perspective (e.g. Valero; Kollosche). The majority of the papers and posters presented, however, focus its analysis in diverse classroom episodes. We had the opportunity to discuss classroom situations involving immigrant students in Catalonia (Abreu, Crafter, Gorgorio & Prat), low income students in Turkey (Doğan & Haser), a combination of the two in a German context (Straehler-Pohl & Pais) and deaf mathematics students in Portugal (Borges & César). Another set of papers focused on the relations between school and out of school mathematics, whether we are talking about adult education (Fantinato; Johansson & Boistrup) or the relevance of the family context to the learning of mathematics (César, Machado & Borges; Newton & Abreu). Together, these works offered a fruitful basis for comparing and discussing the cultural, social and political roles that mathematics education is asked to perform in diverse contexts.

The method we use to organize the work in the working group emphasizes discussion and collaboration between the participants. Since CERME 7, WG10 adopts a way of presenting papers where it is not the author of the paper who presents it, but another member of the group. That is, each of the authors has the task or presenting another individuals paper, and to elaborate a set of comments or questions in order to initiate a general discussion. The presentation should not exceed 5 minutes—something hard to do when it is the author of the paper presenting, but fairly easy when another one is doing it—allowing 25 minutes for discussion. This proved to be a good strategy to guarantee an informed and enthusiastic discussion between the participants. Furthermore, this form of organisation helped promote links, searching for connections and working out synthesis and contradictions across papers.

The nature of the papers presented in this group allowed for a discussion of mathematics education beyond a strictly didactical approach. Participants in WG10 are interested in investigating not only the particularities of the teaching and learning of mathematics, where an individual subject encounters a mathematical object, but also the broader cultural, social and political context that crucially influences what is happening in a classroom. Bringing these dimensions into play requires not only broader theoretical frames (as the ones we indicated before), but also a broader questioning of the societal role of mathematics education, as well as our role as researchers in addressing pressing social problems as the worldwide failure in this school subject. During the five days of work, we had the opportunity to discuss some of the political problems involving mathematics education and its research. As a suggestion for further discussions within WG10, we list some of the questions that occasioned lively discussions among the participants:

1) Is failure a necessity of schooling and, if so, how can we develop research taking this into consideration? This was a recurrent question among participants, and one addressed in some of the papers. If we accept that current schooling needs failure as an inherent sign of its success, that is, if we assume that schools today have an important economic role in the way they exclude and select people, how is it possible to map the role of research when addressing this reality? Can an educational enterprise solve problems that are, in their very

nature, political and economical ones? Perhaps more and better research in mathematics education is not enough when addressing the persistent problem of failure in this school subject.

2) What kind of theories and methodologies are needed to address the political dimension of a classroom? If we recognize that there are sociopolitical factors that influence what is happening in mathematics classrooms, how to develop research that can take into account these broader societal dimension? What kind of theories can allow us to "see" the political operating in the classroom? These questions emerge after one realizes that many of the problems that both teachers and students experience when teaching and learning mathematics are not didactical ones, in the sense that they can be approached by better ways of teaching and learning mathematics, but political ones, having to do, among other things, with the economical organization of schooling.

3) What do we gain from investigating experiences of success? What do we gain from investigating experiences of failure? Mathematics education research is often interested in reporting experiences of success. After a problem has been identified, an analysis is carried out and a way to solve the problem is proposed. This approach has been, however, criticized for not having in consideration the gap between the aseptic environment of research and the real, messy, nature of schools. Can it be that more important than reporting experiences of success is to investigate failure, namely, the reasons why well-elaborated strategies tend to fail when implemented in the normal, daily work of schools? On the other hand, how can we, then, maintain faith in the possibility of change? Can successful participatory research in real school settings be a catalyst for change? If so, is the next step, then, to report on moments that contributed to succeed or to reflect on those moments that failed?

4) Is mathematics in reality or in our gaze? This question aroused when discussing the (ir)relevance of real-life problems for the learning of mathematics. Some of the research in ethnomathematics or in critical mathematics education claims for the importance of connecting school mathematics with real world situations. It is argued that such connection can make mathematics more meaningful for students, as well as allowing them the contact with what some authors call "mathematics in action", that is, mathematics as it is used in broader society. The discussions have problematised how researchers assume that turning mathematics more "real" can allow students to learn it better. Researchers "see" mathematics in reality

because they have a “trained eye” that allows them to identify as mathematics something that, for a student, is just a part of her or his routine. What are the effects of imposing our “mathematician view” on aspects of reality (like the construction of an indigenous house, in the case of ethnomathematics) that have yet not at all been related to mathematics for the students?

5) How to grasp students’ identity or “self”? These question emerge from the set of papers we discussed dealing with social psychology. Participants raised the question of how to be sure about the fidelity of students as well as teachers’ speech? How to interpret the speech of the participants in our research? Shall we go in depth into their inner self meanings, or shall we posit the discourse within the sociopolitical arena and interpret it as a symptom of broader ideologies?

Each paper, in one way or another, tackled some of these major issues, which were recurrent in our discussions. We list them here because we think they deserve reflection from a research community that too often leaves these broader (political) questions unaddressed. It is our wish that the next WG10 continues exploring these burning questions, together with the development of methods and theories that can allow us to tackle the political diversity of school mathematics.

List of papers

Celso Ribeiro Campos; Otávio Roberto Jacobini; Maria Lúcia Lorenzetti Wodewotzki & Denise Helena Lombardo Ferreira	SOCIAL-POLITICAL INTERFACES IN TEACHING STATISTICS
David Kollosche	LOGIC, SOCIETY AND SCHOOL MATHEMATICS
Elsa Fernandes	THE EMERGENCE OF AGENCY IN A MATHEMATICS CLASS WITH ROBOTS
Guida de Abreu, Sarah Crafter, Núria Gorgori & Montserrat Prat	UNDERSTANDING IMMIGRANT STUDENTS’ TRANSITIONS AS MATHEMATICAL LEARNERS FROM A DIALOGICAL SELF PERSPECTIVE
Hauke Straehler-Pohl & Alexandre Pais	TO PARTICIPATE OR NOT TO PARTICIPATE? THAT IS *NOT* THE QUESTION!
Inês Borges & Margarida César	DEAF STUDENTS AND MATHEMATICS LEARNING: PROMOTING INCLUSION AND PARTICIPATION
Karen François & Rik Pinxten	MULTIMATHEMACY
Margarida César, Ricardo Machado, & Inês Borges	SCHOOL AND FAMILY INTERPLAYS: SOME CHALLENGES REGARDING MATHEMATICS EDUCATION

Maria C. Johansson & Lisa Björklund Boistrup	IT IS A MATTER OF BLUENESS OR REDNESS: ADULTS' MATHEMATICS CONTAINING COMPETENCES IN WORK
Maria Cecilia Fantinato	MATHEMATICS *FOR LIFE* OR MATHEMATICS *OF YOUR LIFE*: A STUDY OF THE RVCC PROCESS OF PORTUGAL
Maria do Carmo S. Domite	ETHNOMATHEMATICS AND TEAHCER EDUCATION: REASONING OVER THE MEANING OF STUDENTS' PREREQUISITE AND THE TEACHER'S LISTENING.
Nina Bohlmann, Johannes Hinkelammert, Felix A. Rhein & Hauke Straehler-Pohl	REFLECTIONS ON RECONTEXTUALISING BERNSTEIN'S SOCIOLOGY IN TEACHERS' INSTRUCTIONAL STRATEGIES
Oğuzhan Dogan & Çiğdem Haser	THE GAP BETWEEN MATHEMATICS EDUCATION & LOW-INCOME STUDENTS' REAL LIFE: A CASE FROM TURKEY
Paola Valero	MATHEMATICS FOR ALL AND THE PROMISE OF A BRIGHT FUTURE
Ricardo Machado & Margarida César	DIVERSITY, DIALOGISM AND MATHEMATICS LEARNING: SOCIAL REPRESENTATIONS IN ACTION
Richard Newton & Guida de Abreu	THE DIALOGICAL MATHEMATICAL 'SELF'
Sónia Martins	'REGIME OF COMPETENCE' IN A SCHOOL PRACTICE WITH ROBOTS

List of posters

S. Megan Che & William Bridges	SINGLE-SEX MATHEMATICS CLASSROOMS IN PUBLIC SCHOOLS: A CRITICAL ANALYSIS OF DISCURSIVE ACTIONS
Reinhard Hochmuth	SOCIO-POLITICAL ISSUES IN THE CONTEXT OF DIFFERENT CONCEPTUALIZATIONS OF SUBJECTS
Richard Newton & Guida de Abreu	"IT DOES MAKE YOU FEEL A BIT HOPELESS": PARENTS´ EXPERIENCES OF SUPPORTING THEIR CHILDREN'S SCHOOL MATHEMATICAL LEARNING AT HOME

UNDERSTANDING IMMIGRANT STUDENTS' TRANSITIONS AS MATHEMATICAL LEARNERS FROM A DIALOGICAL SELF PERSPECTIVE [1]

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This paper examines the transitions that immigrant students experience as mathematical learners in schools and investigates conceptual and methodological tools that contribute to the understanding of the transition processes that promote or hinder students' success in the host culture schools. From a psychological point of view, the study of the transition processes requires a consideration of the person's subjective experience. To help our discussion we will borrow from Dialogical Self 'I' positions theory to examine interview data. We will examine a case study, that of Felipe, an immigrant student who saw himself as liking mathematics and doing well in the subject in Chile, until the moment he realized that as an immigrant student in Catalonia he failed maths.

INTRODUCTION

This paper examines the transitions that immigrant students experience as mathematical learners in schools. Our aim is to investigate conceptual and methodological tools that contribute to the understanding of the transition processes (Zittoun, 2009) that promote or hinder students' success in the host culture schools. For Zittoun (2009) "Transitions can be defined as processes of catalysed change due to a rupture, and aiming at a new sustainable fit between the person and her current environment" (p. 410). Often, a starting point in the study of processes of transition is external criteria imposed by researchers. From a psychological point of view, however, the study of the transition processes requires a consideration of the person's subjective experience. In Zittoun's words "If an event is studied as a rupture that is likely to bring a person or an organism to engage in changes, then the organism or the person under study must perceive the event as a rupture, in some respect" (p. 412). If rupture is a central feature of the analytic process of studying shifts one must re-cast or re-think what is meant by the rupture by addressing sense making such as markers of identity (me/my), relationships (reference to others), disruptions in time, distance from the experience and hypothetical thinking about alternative trajectories.

We wish to tackle what is meant by a perceived rupture. This is a methodological challenge as most of the time, research has focused on transitions that occurred in the past and therefore the person under study has to be able to recount past events and experiences and how they made sense of these. In addition, the researcher has to have the methodological tools to examine the processes. To help our discussions about 'points of shift' in interview data and transitional ruptures we will borrow from Dialogical Self 'I' positions theory (DST) (Hermans, 2003) to look at verbal shifts whereby participants go from non-elaboration, to suspension of the voice to a re-engagement. The reason to borrow from this theory is that it allows us to explore the shifts and dialogues between I positions that are voiced by the individual when recounting how they experienced certain events and trajectories in their lives. Drawing on James' (1890) concept of the "I" (self-as-subject) and "Me" (self-as-object) and on Bakhtin's (1973) metaphor of the "multi-voiced" (or polyphonic) novel, the DS Dialogical Self is conceptualized as a multiplicity of "identity positions", self (internal positions) and positions that speak for significant others (external positions).

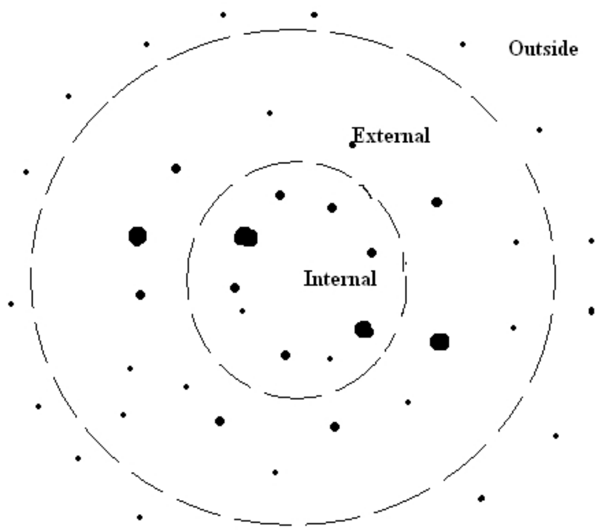

Figure 1: Positions in a multi-voiced self (Hermans, 2001)

The "I" is always positioned in space and time, and these locations result in movement and dialogue between "I" positions. (For more detailed descriptions of the theory see Hermans 2001, 2003, and how it can be applied to immigrant transitions of young people in schools Britain see Abreu, O´Sullivan-Lago & Hale, 2012, Prokopiou, Cline & Abreu, in press). It is precisely the potential to examine the shifts and dialogues in "I" positions, both overtime and across practices (in this case mathematical practices) that makes us believe that examining "I" positions will enable the identification of:

(i) what is/was experienced by the self as rupture;

(ii) which strategies were used to protect the self (e.g. suspending, giving up studying mathematics);

(iii) what was experienced by the self as restoring continuity (e.g. engaging in transitions processes).

THE CASE OF FELIPE

In this paper we will examine the case study of Felipe, an immigrant student who saw himself as liking mathematics and doing well in the subject in Chile, until the moment he realized that as an immigrant student in Catalonia he failed math. The data examined was obtained in an interview carried out in Catalonia by one of the authors of this paper. This is one of the most common methods to elicit memories of one´s life trajectories. In our research we have used a specific type of psychological interview that attempts to elicit both narrative and episodic data. Felipe was born in 1993 and spent his early childhood in Chile. In 2008 he moved to Spain (Barcelona) where he continued his education. He arrived in Barcelona during the middle of the academic year and was recommended by the school to go back and repeat the year. He was educated in a private school in Chile but went into a state school in Spain. Felipe provides a useful case study around which to explicate theoretical ideas of rupture and 'I' positions because he has a positive mathematical identity before the transition to a new country, which is disrupted by the move. It is this conflict and tension to the 'I' as a good mathematics student that we explore in more depth here.

ANALYSIS

When Felipe arrived in Barcelona he was surprised to find, like many other immigrant students, that mathematical operations and strategies were not the same as he has experienced in his first country of education (Gorgorio & Abreu, 2009). In this first extract Felipe attempts to reconcile the rupture to his sense of self as a good mathematics student:

> Felipe: I couldn't believe it, because I told him it's not that I couldn't, **I failed with sickening grades** and he tells me [his previous maths teacher], but that's not possible, you had good grades and this and that. It's that there was a big jump in level and said that on top of that, it was a big mistake to go to a public school because achievement dropped a lot and I couldn't, **I just couldn't fit in my idea was to study the sciences and because of math I couldn't really do it and now I'm hanging** here in humanities, I don't know.

Felipe attempts to bridge his pre-transition self with his 'new' self by talking to his previous mathematics teacher who is surprised that his student is struggling when he had been successful in Chile. Felipe provides himself with a number of plausible explanations for struggling with mathematics in Spain: a change in level of mathematical difficulty, the move from private to state school and in another part of the interview, learning a new language in the form of Catalan.

The realization of the "**I failed maths**" is the marker of the disruptive moment for the self, and takes center stage in our analysis, as failing math results in a disruption of what Felipe has planned as his academic career.

Felipe: I wanted to study something related with biology, with cellular biology or with genetic modification or whatever, but related with that and then (???) possibilities it's that now **I see myself without options**, my grades are really low and it seems really hard…

The disruption to the self has far reaching implications beyond the here-and-the-now in the student's life. Felipe's failure to maintain his I position as a good mathematics students forced him to re-think his future career. Theoretically this is prolepsis in action. The notion of prolepsis borrowed from Cole (1995) can be used to explore the way the imagined future, which is framed by past experiences, mediates and constrains the world of the present. Felipe's imagined future is constrained by his past transitional move to a new country and his subsequent failure to succeed in mathematics learning in a new system of education.

What strategies were used to protect the self

Felipe tries retrospectively to make sense of what happened to him and in doing so recounts strategies to protect the self from the disruption. At the early stages of arrival, he saw the self as foregrounding I-positions, such as 'I as closed', 'I as afraid of voicing my problems', as the 'I did not know the other' (the teacher, and the colleagues). Not knowing the new cultural others made Felipe worried that making his problems visible will have the same reaction as in Chile (ridiculing the self), and was a reason for withdrawing. Retrospectively he realizes this would not have been the case, but this knowledge about the new cultural others comes later to be useful for the self at arrival.

Interviewer: You said, I mean, you said, hey, you work differently here. Did you five them some kind of signal, did you let them know?

Felipe: The thing is, **the first few months I think you get here like a little closed because**, well, you're new and **you're nervous**, **you don't know people**, and on top of that they have a very different lifestyle, at least her in Sant Cugat....

Interviewer: How?

Felipe: Well, people are more, for example, **if a kid in Chile**, for example, if he had said he didn't understand or something, **they'd start to make fun** and other stuff, but here I saw, I don't know, people are like, ok, no problem, you know?

Interviewer: Well, this is good, this is a good thing

Felipe: But that's good, but, well, **I didn't think that way** and **I just kept quiet**.

Interviewer: You were a little…

Felipe: **I kept quiet out of fear of looking bad or I don't know, because I was embarrassed**, because **I'd been here for a very short time** and **I didn't know**...

The process of withdrawing offers the self some protection from the new and unknown (as also discussed in Abreu & Hale, 2009). Felipe has assumed that ways of thinking and behaving in Barcelona would be similar to Chile and so feared that people would make fun of him. It took him time to realize the differences in the way people would react to him. He shows retrospective awareness of the importance time plays in the process and returns to the subject when asked his views about the teachers and students in here (Barcelona) and there (Chile).

Interviewer: And what do you think of the teachers here, the relationship between the teachers with the students in, for example, a math class. How do you see them? Does it seem like over there, like a private school in general, or like public?

Felipe: No, I think the relationship between the teacher and students is the same but the thing is this relationship grows in strength as time passes, right? And what you know about the teacher, if you arrived and you met him two days ago, it's difficult to get close to him in a way where he'd help you with something you don't understand or don't know (…) and so one doesn't know he's going to react or anything, he's an unknown person for you and it's hard to take the step to…

Another strategy for protecting the self during transitional rupture was to suspend aspects of the self which were vulnerable.

Interviewer: And you've stopped doing, I mean, you've changed, in the year, the two years you did ESO here, if you change the way you do operations, for example, or did you continue with the way you did them in Chile?

Felipe: The truth is **I got off track right away**, I think **I stopped** and left it behind, that's what I tend to do, the truth is when I don't get something, I leave it aside and I say there'll be time later and I focus on other things, and in the **end I left it aside** and **I just ignored it totally to not cause any trouble**, and I saw it really high, you know, and so I didn't understand and then the time comes when, damn, if you don't understand a second-grade equation, you're stuck there.

As Felipe recounts "I stopped", "I left it aside", "I just ignored it totally", and he makes sense of this suspension in terms of "not to cause any trouble". Suspending is a strategy he describes as part of the way he reacts to this type of situations "that´s what I tend to do". It may well be that this is a common strategy for the self to deal with a rupture and gain time to work out transitions. However, later he realizes that suspending his engagement with the mathematical learning has specific consequences for future learning, as he realizes that "if you don't understand a second-grade equation, you're stuck there".

What was experienced by the self as restoring continuity (e.g. engaging in transitions processes)

Felipe's final phase of his initial post-transition rupture is to let go of his 'I' as a good mathematics learner self.

Felipe: Exactly, I mean, it was a question of grades because, like I told you, if a teacher is impressed because I have good grades or even if my classmates, because **in the end I told them that I no longer was doing math**.

Interviewer: Illogical.

Felipe: Yes, because **I've always liked it, liked it… from since I was little** and then of course the sciences use all that, right?

Felipe implies here that either the teacher or his classmates could have played a role in restoring his 'I' as a good mathematics student self-identity position. In the end Felipe lets go of the mathematics identity developed in Chile and re-positions his relationship with school mathematics, with the 'I' position he acquired during the period of rupture, "I no longer was doing math". This resulted in Felipe following a different pathway in his studies, "I took another option". For his baccalaureate "maths" was replaced by Latin. However, his desire to follow a scientific career remained and contact with older friends already at University started pointing out options of re-engaging with mathematics at University…

Felipe: … in the end I took another option… and (it was a) disaster… I left maths… and now I do latin… but some friends of mine… older that are already at the uni tell me things… they said to me that they give you some options, that you can do maths as a propedeutical subject… or biology… and if this is true, yes, this is a possibility… is a good alternative… but I'm not well… but if this is so…

When Felipe reflects on his approach to his learning following his transition to Barcelona, he returns to the issue of irreversible time to explain his experiences. He used the example of his younger sister to suggest that his transition could have been more positive and successful if they had been addressed by significant others, such as teachers and parents, at an earlier time.

CONCLUSIONS

Using dialogical self "I positions" as a conceptual and methodological tool in this paper we examined the ruptures an immigrant student experienced in the transition across school mathematical practices from one country to another. We have used a case study approach to explore changes in mathematics identity positions when an identity position, in this case the "I as a good mathematics student" is under threat.

We first examined the student descriptions of "I-positions" that reflected a rupture to the self. Then we examined the "I-positions" involved in coming to grips with a new and unfamiliar sense of self (e.g. being no longer good at maths) and trying to provide the self with explanations for the change. This involved three specific dialogical self strategies: withdrawing of self, suspending of the self, and a letting off the old mathematical self go. This process also meant the student had to engage with a distinct change to their imagined future, as the imagined "I-position" as studying a scientific subject at University became uncertain (not possible). We argue that the transition process involved dialogical self work to restore some continuity in the self. In this case continuity was not afforded by the new mathematical I position (I as failing mathematics), thus the perceived alternative, was to give up school mathematics. This refers to a specific time in the experience of the student, and it is interesting to consider two aspects on Felipe´s reflection. One aspect refers to his reflections on the way "time" impacts on the dialogical self work. As he stresses that in the initial stages the key others (teachers and colleagues) are unknown to the self. Thus, the self dialogues are still drawing on images from the past (schools in the home country). The other aspect refers to Felipe reflection that significant others parents and teachers can play a role on the impact of time by becoming aware and addressing issues at the earlier stage. To briefly think about future research: we know quite a bit about the disruption to the self and learning post immigration transition but there is still little practical guidance for significant others on how to help their children and students in this process.

NOTES

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REFLECTIONS ON RECONTEXTUALISING BERNSTEIN'S SOCIOLOGY IN TEACHERS' INSTRUCTIONAL STRATEGIES

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In this paper we discuss the complex process of developing an in-service teacher-training program that aims at promoting teachers' explication strategies. Therefore, we critically examine this process from the sociological perspective that has provided the theoretical basic tenets for our work. Following Basil Bernstein's sociology of education, these basic tenets are, 1) that the recontextualisation of knowledge in schools creates a social order, 2) that the hierarchising of meanings that permeates this process is constructed in social arenas outside school, and 3.) that a mathematics education for social justice requires the explicating of hierarchies of meanings in school. An analysis of one teacher's realization of instructional explication strategies will build the grounds for this reflection.

INTRODUCTION

In the last decades, the amount of research on the social and political dimensions of mathematics education has significantly increased. One of the theories that is often applied as a frame of reference is Basil Bernstein's sociology of education. In this paper we want to present and reflect first attempts to recontextualise research-findings from our research-group (e.g. Gellert & Hümmer, 2008; Gellert, 2009, Gellert & Straehler-Pohl, 2011; Straehler-Pohl & Gellert, 2012) that stem from this theory in classroom practice. A crucial component of a pedagogy that bears the potential for social change on the school level is to make the criteria for evaluation explicit and visible for all learners, while keeping a high level of conceptual demand (e.g. Morais, 2002; Gellert, 2009). At the same time, we maintain that this abstract claim is not a general rule in the sense that it can be applied in the same way at any place at any time. Rather, the meanings of *explication* and *high level of conceptual demand* call for a sensible recontextualisation respecting the particularities of actual educational contexts. Recontextualisation of pedagogic theory into practice can never be a direct transfer that is independent of the particularities of the target-context. Making our research findings that operate on a high level of abstraction accessible for teaching practice therefore calls for a cautious reflection on the potentially emerging boundaries within particular contexts. For example, research all across the world has shown that in contexts of low-ability expectations or in contexts of low-class (while the former are not seldom a result of the latter) teachers tend to draw too extensively on students' supposed experiential environments (e.g. Hoadley, 2007 for South Africa; Straehler-Pohl, Fernandes, Gellert & Figueiras, forthcoming for Spain). In order to transform these tendencies, it does not suffice to declare the reduction or abandonment of real-world contexts in mathematics classrooms with lower-class learners. The documented cross-contextual similarities in the teachers' discourses should rather make us aware, that there might be structural reasons within *and* across the teaching-contexts that reinforce these teachers to teach in the way they do. We try to target exactly these institutional contexts with students who are supposedly *inferiorly able* and *socially disadvantaged*. Thus, our didactical aim is to help teachers to comprehend their interactional routines and transform them in a more empowering way. We sought to approach this by designing an in-service training for practically promoting a pedagogy that we theoretically have identified as favorable for students of marginalised backgrounds. These students have to be regarded as a 'risk-group' concerning their further educational and vocational opportunities and whose social participation is massively jeopardised. We argue, that this is not solely a matter of more or less effective learning but has to do with a differential distribution of different kinds of knowledge to different social groups. With the *pedagogic device*, Basil Bernstein provides a conceptual frame, how this process *is structured* and in turn *structures* pedagogic practice. The pedagogic device enables us to understand, why it is important for teachers to develop instructional explication strategies in order to do more than just lifting the mathematics achievement of at-risk students to a basic-competence-level, but making accessible a code that is a carrier of *power* within *and* outside the mathematics classroom.

THE PEDAGOGIC DEVICE

The pedagogic device comprises three rules that constitute the basis for any pedagogic discourse: the distributive rules, the recontextualising rules and the evaluative rules. All three rules are hierarchically interrelated with the distributive rules being at the top. *Distributive rules* regulate the formation of systems of meaning through the production of specialised knowledge and thus hierarchise different forms of knowledge. According to Bernstein (2000), the distributive rules constitute a social arena in which meaning hierarchies are negotiated and determined. Following Durkheim, he differentiates between two general forms of knowledge. On the one hand, there is *esoteric* knowledge that has a more distant relation to a material base and is thus less context specific. *Mundane* forms of knowledge, on the other hand, have a closer connection to a specific context due to their proximity to a material base. On the level of the distributive rules esoteric knowledge is categorized in a higher hierarchical position than mundane knowledge. This is important for the level of the *recontextualising rules* where the reproduction of knowledge is regulated. Here, meaning hierarchies are again negotiated but in dependency on the distributive rules. While, from a theoretical perspective, the recontextualisation is reliant on, but not determined by the distributive rules, empirically, the recontextualising rules seem to reproduce the hierarchies agreed upon on the distributive level. The re-structuring of categories of knowledge by recontextualisation does not follow the intrinsic rules of the original discourse but takes place according to a specific logic of transmission. Thus, meanings are again hierarchised and structured, but in a way distinctly different from the original discourse. In contrast to the higher-level rules, the *evaluative rules* do not regulate how forms of knowledge are hierarchised but make the hierarchical relations visible by means of evaluation. Evaluative rules become visible for example in standardized assessments or in teaching practice. In the context of pedagogical practice, evaluative rules evoke a transformation of knowledge into consciousness of the individual, that is to say knowledge is reproduced. In order to allow students to perform successfully in school, the intrinsic rules of the pedagogic device need to be explicated. We conclude that, *within* the classroom, teachers cannot effectively challenge hierarchies of meanings that operate *outside* the classroom. *Within* the classroom, their agency is restricted to the level of the evaluative rules: They may or may not realise a pedagogic practice that explicates the hierarchies that are the outcomes of the pedagogic device in its present state; they may give access to dominant discourses or to dominated discourses.

Gellert (2009) makes us aware that keeping particular *rules of the game* implicit and leaving it to the students to independently make their way from the mundane to the esoteric (or fail at it) is a part of the common sense on teaching mathematics. However, if teachers aim at offering all students the same chance to successfully partake in socially valued forms of classroom activities, they need to make the outcomes of the device transparent, so that the hierarchy of esoteric and mundane forms of meanings and knowledge can become visible for all students. Gellert (2009) suggests that the rules that need to be made explicit permeate the pedagogical practice in mathematics classrooms on the following levels:

1. which area of mathematics is taught (algorithms, tasks drawing on real-life experiences, heuristics etc.),
2. if and in what way school mathematics is related to academic mathematics or to everyday knowledge,
3. what constitutes a successful participation in mathematics lessons and which criteria a student's contribution in class has to fulfil.

In one way or another, all these levels contain a relationship between the mundane and the esoteric. Consequently, we encourage teachers to depart from the common sense of leaving certain rules of the pedagogical game implicit in order to enable more students to become successful learners in the mathematics classroom. When we take into consideration the research, exemplarily referenced above, we need to be aware that implementing a pedagogy that explicates the hierarchy of the esoteric and the mundane without disempowering and alienating students and teachers is a sensitive and long-term endeavour.

PROMOTING INSTRUCTIONAL EXPLICATION STRATEGIES

Our project's goal was to design an in-service teacher-training program where teachers develop strategies and practices of explicating implicit rules of school mathematics. This training is explicitly not aimed at providing materials, influencing teachers' beliefs or transmitting knowledge, but seeks to build on teachers' existent interactional routines. Brought to consciousness, we argue that these routines can provide the basis for the development of effective explication strategies. As Bernstein

(2000) makes us aware, the move from decontextualised meanings (e.g. sociological theory) to contextualised meanings (e.g. pedagogic practice) is never a matter of direct transfer but inevitably follows a recontextualisation process, where relations of meanings are re-ordered and re-negotiated (see recontextualising rules above). In order to realize such a negotiation process and to integrate it in the concept of the teacher-training we involved experienced teachers in the designing process. This is in line with research results indicating that participants of such programs accept and apply the suggested ideas more likely when other teachers partake in the designing-process of the program (e.g. Lipowsky, 2010). We invited six teachers who work with 'at-risk students' in their daily practice to join us in the process of planning the teacher-training program. Over a period of four months we convened three meetings that took a whole weekend each. At the beginning stood an introduction to the general problematic of implicitness of evaluative criteria and the challenge to explicate them, inspired by recent sociological research in mathematics education (e.g. Dowling, 1998; Cooper & Dunne, 2000; Gellert, 2009;). Based on this introduction, a discussion was initiated in which we could relate our rather theoretical perspective to the teachers' everyday practical experience and vice versa. This discussion resulted in the choice of four domains that all of us consensually regarded as crucial for explicating the dominant code: a) verbal modes of expression, b) modalities of documenting learning processes, c) everyday context problems, and d) mathematical games.

a) The domain 'verbal modes of expression' shall enable learners to recognize and realize utterances within a school mathematics register in delineation of an everyday or common sense register;

b) In the domain 'modalities of documenting learning processes' the main aim is to make individual learning processes visible by opening up discussions about evaluative criteria;

c) The *right amount* of reality to take into account when dealing with 'everyday context problems' is not a fixed quality depending on the given problem sui generis. Rather it may vary with different contexts (e.g. "problem of the week" might differ from standardized tests). The ability to recognize and realize this right amount is a crucial condition for achievement.

d) The domain 'mathematical games' is concerned with the use of games in the mathematics classroom and aims at revealing the boundary between *a logic of play* and *a logic of school mathematics* in order to enable learning for those who tend not to recognize this boundary.

In each domain we figured out possibilities of implementing interactional explication strategies in the mathematics classroom. The teachers then tried to put these ideas into practice. First attempts of this recontextualising of the theory into the context of the teachers' particular situations were videotaped in order to attain two objectives: Firstly, we aimed to reflect on the potentials and boundaries that emerged. Secondly, sequences of the videotapes shall be used in the in-service teacher-training program for the purpose of illustrating and reflecting on the recontextualisation.

ANALYSIS

In the following we will present and analyse a case study on Paul, one of the six teachers, who took part in the project so far. We will start with a brief introduction of Paul, based on our shared experiences in the run-up of the lesson. Paul gave this lesson in order to provide us with videotaped illustrative material for the planned teacher-training program. Then we will briefly present the mathematical game Paul chose for his lesson and finally analyse his approach on explicating implicit rules of school mathematics from our perspective.

The case of Paul

Paul works as a teacher in the 8th and 9th grade (in the age of 13 to 15) at a "Förderschule", a school that exists beside the regular school system and brings together children with socio-emotional 'disorders' and cognitive learning disabilities. Additionally, a high migration rate and a low socio-economic structure in the feeding area affect the school's daily routine. Paul teaches all subjects in his classes. He did not study mathematics as an academic subject. We experienced Paul as a very calm and patient person, who does not let himself be disturbed when being challenged or questioned. We would describe him as 'down-to-earth' with a close contact to the students' realities. His motivation for taking part in the project is rather a general openness and the motivation to improve himself as a (mathematics) teacher. Paul chose the domain of mathematical games for his lesson.

Paul's game: Nummero

For his lesson Paul chose the game "Nummero". In this game each participant selects a number between one and one hundred. S/he keeps the number secret. The other participant will have to find out this number by asking questions about it. There is a fixed set of allowed questions that are printed on game-cards, such as "Is it an even number?" or "Is the number between … and … ?" (the students have to fill the gaps on their own). On the basis of the given answer, the asking participant can eliminate an amount of impossible numbers by crossing them out on a hundred board (see Fig. 1) in order to keep track of the numbers already eliminated. The participants alternate with asking a question. Whoever finds out his opponent's number first is the winner of the game.

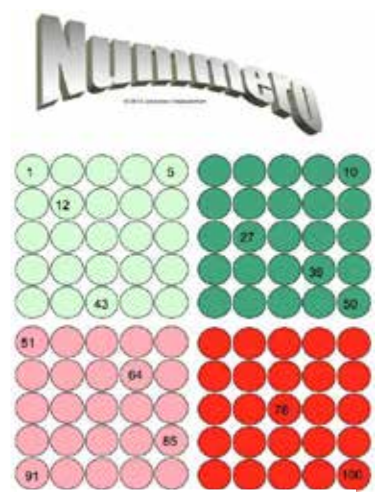

Fig. 1: Nummero

Discussing the game

As Paul and the research team considered the game as not challenging enough for his own 8th and 9th grade classes, Paul asked a colleague in a 5th grade (students are in the age of 10 to 11) for permission to take over one lesson. While his colleague agreed, he expressed serious concerns that the game would be too demanding for his students. As it was not his own classroom, Paul agreed on choosing another, much less demanding game. However, this game proved to be free from any impulses for mathematical reasoning and hence was entirely unsuitable for the declared purpose of explicating the hierarchy of a logic of play and a mathematical logic within the mathematics classroom. Finally, we presented Paul's colleague the choice to either let Paul proceed with the more challenging game or to cancel the lesson. Paul seemed pleased to have the opportunity to try to prove his less optimistic colleague that the students can do more than he expected.

Paul's lesson: Data and Analysis

Interaction I: Paul opened the lesson by announcing that they will play a game. He emphasised that besides playing, his students are supposed to "learn something". In a short conversation Paul introduced a hundred board (see Fig. 1), where he asked some students to fill out missing numbers. He announced that he would explain the rules of the game by means of playing a first round with the whole class against the teacher. He showed them a piece of paper on which he had noted a number that they were supposed to find out. Thereupon Paul asked a student, Sven, also to choose a number. Doing so, Sven wrote down '86' and fixed it in a way that his classmates could see it, but not Paul. After this, Paul introduced the game-cards with the questions. The first card enabled Paul to ask for a specific figure contained in his number. Accordingly, Paul questioned if 'five' is contained what the students negated.

Paul: Well, then I can cross off some numbers that are not possible any more. · …· So the five is not contained. Then, of course it can't be the fifteen either, can it? · *He starts crossing 5 and 15 by drawing a line*·

Students: No. [...]

Paul: And twenty-five, thirty-fi-, forty-five they are all impossible. [*He crosses out the column from 5 to 95]* Okay. And… but the fifty-one isn't possible either, is it?

Students: No. [...]

Paul: Fifty-one is impossible. Exactly. Then I'll cross it as well. *[He starts crossing the row from 51 to 59, and then, asked by a student, he also crosses 50]*

Analysis: The teacher chose a trajectory from a school mathematics content to the mathematical game. Thus he first drew the attention to the hundred board and just then shifted to the rules of the game. We interpret this as a sign of sensitivity for providing a dominant space for the mathematical side of the activity. When taking the first game-card, the instructions on the card triggered off a conversation between the teacher and the students, in which the teacher offered an insight into his thoughts. He asked his students questions in order to involve them in the game and simultaneously to share his ideas. This decision to openly discuss a strategy is characteristic of a school mathematical logic. In everyday contexts, competing participants of a game would not be expected to uncover their thoughts or even to help each other. However, even though Paul brought up the situation of sharing thoughts on strategies, he did not delineate it from the logic of play since he did not contrast his approach to playing games outside the classroom. In addition, he did not initiate a discussion about the meaning of crossing a whole row or column, but only verbalized the crossing out of single numbers. Thus the school mathematics' aim of reasoning ran danger of fading behind the need to proceed in the game.

Interaction II: The students picked a card saying "Is the number greater than…?".

Sven:	Is the number greater than... seventeen?
Paul:	Than?
Sven:	Is the number greater than... fifty-seven?
Paul:	Than fifty-seven. No, it is less. Okay, now you have to cross out your numbers. […] The number is less than fifty-seven.
Sven:	[*shows with his pen somewhere between 46 and 47:*] Up to here.
Paul:	No, less! Listen. Is the number less… what did you ask?
Sven:	Than fifty-seven.
Paul:	Is the number greater than fifty-seven. And I said no, means it is less.

Sven then started crossing out numbers that are greater than 57, but left out the 88 and instead, falsely crossed out 51 to 56. A discussion on which numbers need to be crossed out emerged between the teacher and the students.

Analysis: The discussion between the students and the teacher revolved around the question whether the numbers less or greater than '57' need to be crossed out. This resulted in a strong emphasis on a game-procedure, while the reasoning on strategies (e.g. why to choose 57) completely faded into the background. Emphasising the dominance of a mathematical logic, Sven would have had to explain his strategy behind choosing the number '57'. However, the teacher did not ask Sven to explain but instead instantly requested him to cross out his numbers. Thus, at this point, the implicit expectations within the mathematics classroom of unfolding mathematical strategies in order to 'improve the success' in playing was not made visible to the students. Instead, the emphasis was on following the playing routine. However, we can interpret the discussion about the meaning of 'less than' and 'greater than' as an effort to keep a mathematical frame of reference by emphasizing a mathematics *content*.

Interaction III: Paul's next card asked whether his number was located on the light-coloured half of the hundred board, which was approved by the students.

Paul:	My number is located on the light-coloured half, right? • …• Well, then I can cross all dark.
Sven:	On the dark • *pointing on the dark-coloured half*• • …•
Student:	(...) on the dark side?
Student:	On the light side.
Paul:	Have a look, you wrote down a number for me. And this one is located on the dark, isn't it?

Analysis: The students seemed to have difficulties with deciding whether the teacher should erase the light-coloured or the dark-coloured half. The teacher repeatedly pointed out the number chosen by the students, thus trying to help them make their decision. At this point, again, the discussion focussed on a question exclusively tied to game-procedures, i.e. which part of the hundred board can be crossed out. Following a logic of school mathematics, it would have been fruitful to discuss what "dark side" and "light side" meant mathematically and how this knowledge could promote making a decision without searching the particular number in the hundred board. However, the students were not asked to explain or to justify their decision for one half of the game board. The focus on game-procedures again seemed to emphasize the logic of play while neglecting a school mathematical logic, thus giving the students a certain impression of how they are expected to think and behave in the mathematics classroom. That the hierarchy of these two discourses is actually the other way around is left implicit.

DISCUSSION

In this paper, we aimed to critically reflect on the potentials of recontextualising Basil Bernstein's sociology of education in interactional strategies for teaching that systematically strive for an explication of rules and criteria. Usually they remain implicit in mathematics classroom discourse, which hinders students from disadvantaged backgrounds to recognize what counts as a legitimate contribution and a desirable learning outcome. While research produces more and more evidence on how these processes occur in regular classrooms, our intention was to analyse intentionally created teaching-contexts in which explication is a conscious point on the teacher's agenda. In order to be able to evaluate the capacities of interactional explication strategies on a broader scale, this appeared to us as a necessary first step. We expect that confronting ourselves with the boundaries that emerge in such a process helps us canalizing our efforts.

Our analysis of Paul illustrates that reflecting on explicitness and its sociological relevance and bringing it to consciousness is a step into the right direction, however it proved to be a very first in a bigger number of necessary steps. The least we can say about the success of Paul's intervention is that it released the students from excessively lowered expectations. Still, we cannot conclude to having provoked a full suspension of a discourse of low expectations. For example, Paul's strong insistance on procedural aspects of the game is characteristic for such a discourse. Even though Paul has sensitized himself in the meetings and in our discussions to the importance of explicating the dominance of the logic of mathematics *within* the mathematical game by an emphasis on reasoning, he frequently missed out on situations that bore the potential for such an explication. We tend to see this contradiction not as a result of an inadequate realization of the aims that Paul set himself but rather as a result of the very structural intricacies that inevitably arise when using situations that *usually* are characterised by *implicitness* to *explicate* something. Taking up Bernstein's concept of classification, using games in order to explicate structural rules of school mathematics means juxtaposing a weakly classified activity with a strongly classified activity in order to explicate the boundaries that characterize the latter. This inevitably creates a situation, where the strong classification of the latter is challenged. As Bernstein (2000) makes us aware, classifications create a "*psychic system of defence*" (p. 12). A challenge of classifications initiates a threat to this system. So it appears quite comprehensible that the more and more Paul and his students proceed in the game, the more and more he loses track of explication and the more and more he is drawn towards a logic of play. However, we do not see this process as inevitable and conclude by giving up our belief in teacher agency. Rather, the analysis of the case of Paul has provided us with new insights, of which we can now effectively take advantage in the upcoming teacher-training program. Further, the existence of video-taped material gives us the opportunity to contextualize our findings for the participating teachers and thus going one step further in the recontextualisation of Bernstein's sociology in interactional explication strategies. In order to overcome a "psychic system of defence", one needs to be aware of it.

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DEAF STUDENTS AND MATHEMATICS LEARNING: PROMOTING INCLUSION AND PARTICIPATION

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Including students who need special educational support in mainstream schools brought new challenges to teachers. Assuming an interpretative paradigm and an intrinsic case study design, we focus on the adaptations so that two 12th grade Deaf students could learn mathematics with their classmates. The participants were these Deaf students, their classmates, and their mathematics and special education teachers. Data collecting instruments were observation, interviews, informal conversations, students' protocols and documents. Data were analysed through a narrative content analysis from which inductive categories emerged. Results focus on five interactive patterns used in classes. They facilitated the mathematics communication and learning of the two Deaf students and their inclusion.

INTRODUCTION

The sound cultural diversity that characterises Portuguese schools grew in the last decades (César & Oliveira, 2005). It brought additional challenges and responsibilities to teachers (César, 2009, in press). They are expected to rethink the curriculum and their practices having in mind students' characteristics, needs and interests (Allan & Slee, 2008; César & Ainscow, 2006). Vulnerable minorities need special attention and inclusion should fit the different characteristics of the countries and cultures (Timmons & Walsh, 2010). National and international policy educational documents stress the need to promote a more inclusive education (ME, 2008; UNESCO, 1994). But despite legislation, students needing special educational supports still face barriers (César & Ainscow, 2006). Deaf students experience high underachievement and school dropout rates and they are one of the smallest groups at Lisbon University (Almeida, 2009). Due to their specific communicational characteristics their hearing peers and teachers undervalue their (mathematical) performances (Borges, 2009; Borges, César, & Matos, 2012).

The designation of inclusive education is often used. It assumes different meanings. All of them concern social justice, equity in the access to school achievement and the promotion of students' participation in school activities, namely in mathematics (César & Ainscow, 2006). But the focus and the ways to achieve these goals are different. Ainscow and César (2006) designate these differences as "a typology of five ways of thinking about inclusion" (p. 233). We assume a position connecting the last and third ways of thinking. The aim is to achieve a quality education for all (the fifth way of thinking). But to achieve this, we need to pay close attention to all vulnerable groups and to promote equity (the third one). As Clapton (2009) claims, we need a transformatory ethic of inclusion, rupturing with the previous concepts of disability and inclusion. Thus, we follow the recommendations from UNESCO (1994). We conceive school inclusion/exclusion as a contribution to students' life trajectories of participation which are shaped by inter- and intra-empowerment mechanisms developed – or not – through school practices (César, 2013).

Adapting the curriculum to each and every student (César & Santos, 2006; Rose, 2002), respecting and valuing their participation in different cultures (César, 2009, 2013, in press), and allowing them to give a meaning to school knowledge (Bakhtin, 1929/1981) is more striking when it comes to mathematics. This subject is associated with high academic underachievement, rejection, negative social representations and low positive self-esteem (Machado & César, 2012). Giving a meaning facilitates knowledge appropriation and the transitions between contexts, scenarios or situations (Abreu, Bishop, & Presmeg, 2002; César, 2009). It involves reflecting on classroom practices, including the nature of the tasks, working instructions, interactive patterns, didactic contract, evaluating system, regulatory dynamics and inter- and intra-empowerment mechanisms (César, 2009, 2013, in press). Thus, every teacher can use the curriculum as a vehicle for inclusion or as a lever for exclusion (Rose, 2002).

Policy educational documents point to mathematics communication as one main goal (e.g., Abrantes, Serrazina, & Oliveira, 1999; NCTM, 2000). Elaborating and testing conjectures, producing sustained argumentations, establishing connections, or being critical about mathematical issues regarding society are significant aspects in mathematics learning (Alrø, Ravn, & Valero, 2010; Matos, 2009). A communicational common basis is needed, creating intersubjectivities and making mathematical messages understandable (Borges & César, 2011, 2012; Borges et al., 2012). Students need learning opportunities, support and adaptations that are adequate to their uniqueness, including their cultural diversity, that facilitate meaningful mathematics learning and proficiency. Assuming a historical-cultural approach and knowing students' zone of proximal development (Vygotsky, 1934/1962) facilitates the promotion of transitions from their solving strategies and ways of thinking into more formalised mathematical conceptualisations (César, 2009; César & Santos, 2006; Roth & Radford, 2011).

Like Sfard (2008), we assume learning and thinking as communicating. Thus, social interactions play an essential role in mathematics education and teachers' practices need to promote students' participation and their engagement in school mathematics activities. Investigating adaptations performed when Deaf students participate in mainstream classes assumes relevance as Deaf experience particular communicational barriers that often compromise their school achievement (Borges, 2009). They must communicate and think mathematically and be able to make transitions regarding their (mathematical) knowledge, abilities and competencies.

METHOD

The problem that originated this research regards the barriers to communication and to the access to the mathematical cultural tools (Vygotsky, 1934/1962) that Deaf students experience when included in the mainstream educational system. This work is part of a broader study (Borges, 2009). In this paper we focus in two of the four research questions: (1) What adaptations are preformed by this teacher in this 12th grade class that includes Deaf and hearing teenagers?; and (2) What changes are performed by the hearing students in their communication while working and interacting with these Deaf students? These questions do not focus on mathematics learning directly, but we observed mathematics classes. We studied the participation of two pre-lingual profound and severe oralist Deaf students in their 12th grade mathematics classes: Dário and Artur (false names). They were achieving cases as they had the expected age and planned to go to university. The disclosure of successful cases contributes to a more inclusive education (Allan & Slee, 2008; César, 2009; César & Santos, 2006).

We assume an interpretative paradigm (Denzin, 2002) and an intrinsic case study design (Stake, 1995). The participants were these two Deaf students, their classmates, their mathematics teacher (Mariana) and their special education teacher. Mariana had taught other Deaf students in previous school years and she was particularly sensitive to their needs. The data collecting instruments were: participant observation (audio recorded and registered in the researcher's diary); interviews; informal conversations; students' protocols; and documents. The observation included the attendance of one class per week (November to June, a total of 17 classes). The numbers in the codes refer to the observed lesson – 1 to 17. The contents were mainly functions. Data treatment and analysis used a narrative content analysis (Clandinin & Connelly, 1998), starting with a floating reading. More in-depth readings included the search of interactive patterns. Inductive categories emerged (César, 2009), such as the interactive patterns used in mathematics classes (Borges, 2009).

RESULTS

The analysis of some episodes and empirical evidences allowed for the recognition of five interactive patterns used in these mathematics classes: (1) spatial regulation; (2) working rhythm regulation mechanisms; (3) reinforcement schemes; (4) tutorial co-construction; and (5) clarification of doubts. These patterns played an important role in the inclusion process and in mathematics knowledge appropriation.

Spatial regulation

A teacher can walk around the classroom, be in a backlit position, and speak while writing on the blackboard or consulting a book without stopping a hearing student from following his speech, such as in the examples illuminated by Machado and César (2013). But as an oralist Deaf student uses lip reading as his/her main way of communicating with hearing people, a simple rotation of the face, a misarticulated word or a too speedy sentence breaks the communication. Mariana's ways of acting illuminate she was concerned with these details: "Mariana mentions the number of the lesson and dictates the summary. (…) She repeats near Dário (…) slower. She does the same near Artur" (15th lesson, May 13, 2009, p. 137). These were essential features for these two Deaf students. A less rigorous diction or the omission of syllables turn lip reading into an impossible task. This was an essential move to allow these Deaf students' access to mathematical cultural tools (Vygotsky, 1934/1962). Their hearing classmates also used adaptations in communication and so enable peer interactions: "Núria, who arrived a little late, asked Dário about the summary. He does not understand and she repeats only the word summary, rotating completely her face towards him and speaking the word a little slower" (15th lesson, May 13, 2009, p. 138).

In the communication between Deaf and hearing people, the oral information can be complemented with gestures and/or other visual aids. Besides the blackboard, we observed the use of technologies like the viewscreen, the interactive board, and the graphing calculator. In one class, while using a computer program that allows visualising the image of a graphing calculator in the interactive board, we registered: "Mariana starts giving instructions about the definitions of the calculator, exemplifying in the projection in the interactive board. Students repeat the procedures in their calculators" (11th lesson, April 22, 2009, p. 109). Thus, the oral instructions were complemented with the use of the virtual calculator, facilitating mathematical learning. These resources are useful for any student but they are particularly important for the Deaf, as sight is their privileged means when communicating, and as Sfard (2008) underlined, communication is a main mediating tool for mathematics learning. These complements benefit the hearing and Deaf students as intended by the inclusive education approach (Borges, 2009).

Working rhythm regulation mechanisms

In these mathematics classes some working rhythm regulation mechanisms emerged and played an important role in students' engagement in these school mathematics tasks. This teacher used them often. Those mechanisms were similar both for Deaf and hearing students. But they were used much more often with the Deaf, as she knew that the communicating characteristics of the Deaf may exclude them from what is going on in the class, particularly in whole group discussions and, as stated by Borges (2009), they also get distracted more easily.

Mariana [to Artur]: Haven't you done [exercise] b?

Artur: That's for homework.

Mariana: For homework? Oh, you are always watching the clock! Then, write it down. Your homework is the [exercise] 300, Paragraph b, c, and d; Test 9, Page 14. [The bell rings. Mariana speaks to the whole class] Finish [exercise] 300 and do Test 9. [Mariana goes near Dário and repeats the homework] (7th lesson, March 4, 2009, p. 77)

Besides the instructions given specifically to the two Deaf students and to the whole class, the teacher chose to ask students about their progression in their solving strategies to promote their working rhythm. Thus, instead of telling them to work or to be quiet, Mariana led the students' attention to the mathematical tasks and alerted them, in a subtle way, whenever they needed to work faster.

Another mechanism to regulate the working rhythm had to do with teacher's positioning. By moving around between the students' desks while they were doing autonomous work, Mariana got closer to them and saw how they were progressing. This way of acting was more frequent with the two Deaf students as she wanted to keep them in a similar rhythm and to know if they were struggling with any difficulty. This particular attention to students' performances is mentioned in other researches as an essential feature for school achievement, particularly in mathematics (César, 2009; Machado & César, 2012). Sometimes Mariana remained longer next to a particular student to be sure s/he would keep working. This often happened with Artur who would easily get distracted: "(…) Artur starts talking to his right side. Mariana walks by and says «Well?» and stays next to him following his work, preventing him from being distracted again" (6th lesson, February 11, 2009, p. 65).

It was curious to see that the hearing classmates would also regulate Deaf students' working rhythm. Sometimes the classmate that shared Artur's desk brought his attention back to work, illuminating the peers' role in students' performances, as also stated by César (2013). The intersubjectivity they developed enabled her to do so only using non-verbal language: "Artur has "his head in the clouds" and Melissa taps him on his shoulder and, without saying anything else, he understands the message and returns to work" (17th lesson, June 3, 2009, p. 158). From what we have observed, Artur did not feel embarrassed or displeased with these small remarks. Their special education teacher (SET) also mentioned this: "Artur accepts perfectly (…) the criticism, quotation marks, of him being inattentive, not very concentrated (…)" (SET, interview, p. 14). Thus, with the help of his mathematics teacher and peers, Artur's working rhythm improved as well as his mathematical performances.

Reinforcement schemes

The mathematics teacher introduced simple, discrete and efficient reinforcement schemes. For instance, she would confirm the steps used in a particular solving strategy. The teacher could say: "Mariana [to Dário]: That's it" (1st lesson, November 26, 2008, p. 17). Other times students requested these reinforcement mechanisms:

[Artur asks if what he has done is correct. Mariana says it is]

Artur: Did I get away with it?

Mariana: You did. [Laughs] (3rd lesson, January 14, 2009, p. 39)

Classmates also used reinforcement among themselves, encouraging each other. Sometimes, after seeking together for an answer they would share the pleasure of finding it, as also stated in other researches (César, 2013; Machado & César, 2012). In this episode Artur and his classmate, after discussing about the correct option for a multiple-choice exercise, participate in the general discussion:

Mariana: (…) therefore the answer is…?

Melissa and Artur: It's D.

Mariana: It's D. [Melissa and Artur celebrate by hitting each other's right hand in the air – a "high five"] (3rd lesson, January 14, 2009, p. 38)

After discussing the task, these students felt confidant to answer to a question asked to the whole class. Some authors claim this is a clear sign of their participation (e.g. Sfard, 2008). These Deaf students often volunteered to answer to questions during the general discussion. This illuminates how they felt included in the mathematical activities. This celebration illuminates a well-accomplished socialisation. Artur participates in a teenagers' typical way of acting. We infer a high level of inclusion in their peer groups, desirable in an inclusive education (UNESCO, 1994).

Tutorial co-construction

As observers we often saw the elaboration of an answer or solving strategy including an interaction engaging two persons (teacher/student or student/student) or the whole class, in general discussions (teacher/class). This illuminates the central role played by social interactions in mathematics learning as stated by Roth and Radford (2011). Mariana's interventions were mainly questions or suggestions – which constitute an interactive pattern that characterises her practices, and was also mentioned in other researches (Machado & César, 2012). There was a clear effort to avoid giving the answers to students. She preferred to give students time and space (César, 2009, 2013) so that they could find the answers on their own, allowing them to mobilise and develop their mental tools (Vygotsky, 1934/1962).

Mariana: What is the first thing that you have to do here? U_n tends to what value?

Artur: This is really confusing.

Mariana: It may be confusing at first but then the conclusions are the same. Remember what we did a while ago. (…) It tends to…?

Artur: They become really small.

Mariana: It tends to…?

Artur: -5, no?

Mariana: No. (…) Try to find it using the calculator. [Mariana goes near Artur and helps him constructing the graph in the calculator] (4th lesson, January 21, 2009, pp. 45-46)

The teacher does not contradict Artur when he says that this content is confusing. She tells him that it can be confusing only at the beginning, which implicitly conveys the message that she believes he will understand that topic if he goes on trying. Implicit messages are very strong elements in (mathematics) learning as well as in students' commitment/rejection towards it (César, 2009; César & Santos, 2006). Implicit messages are essential regarding these students' processes of inclusion.

Another detail is the improvement of Artur's mathematical communication, an essential feature according to Sfard (2008). He states the succession tends to "really small" values. Mariana, without criticising him, repeats the question asking for an accurate answer. When Artur guesses one value, Mariana could have given him the answer. But she suggests he should try to find the expected number in his graphing calculator. She continues to push Artur to find the answer by himself and, once again, that brings an implicit message: she believes he can find the answer on his own and improve his mathematical performances. She believes that he can learn – an essential aspect to achieve students' engagement as also stated by César (2009, 2013) and Roth and Radford (2011).

Clarification of doubts

As we mentioned before, during the moments of autonomous work the mathematics teacher used to walk around between the students' desks. By doing so, two patterns of doubt clarification emerged, according to who initiated this interactive pattern: the teacher; or the students.

> Dário raises his arm. Mariana does not notice it and goes near Artur to become aware of his progress [regarding his work]. Dário lowers his arm. Mariana clarifies another student's doubt and when she has finished Dário raises his arm again. Mariana goes near him and confirms what he has already done and the next step as Dário asks her if his idea is correct, or not. (2nd lesson, January 7, 2009, p. 29)

We can infer a safe class culture characterised by tolerance and the absence of a competition level harmful for the students' learning. When Mariana is unaware Dário had requested her help, she first walks and goes near another classmate. Thus, Dário lowers his arm, waits, and when she is available he calls her again. This happened without any manifestation of unpleasantness and he keeps on working while waiting.

During the observations we also realised that students shared what they knew and co-constructed their answers in a similar way as described by Roth and Radford (2011). Usually they only requested their teacher's help when they could not go further on their own. The teacher respected these clarifications of doubts among classmates: "[Mariana] comes back, near Dário, who is talking to Melissa about the exercise. She waits until Melissa finishes her explanation and only then she participates in their discussion" (5th lesson, February 4, 2009, p. 57). With this kind of acting the teacher encourages autonomy (an essential competency for students who are preparing themselves to go to higher education or to start working), promotes mutual help and respect, creating more inclusive spaces and times.

Sometimes individual clarification of doubts could originate useful contributions:

> [Dário looks a little longer to the resolution in the blackboard, while he bites a nail and says to Núria, with a worried look]
>
> Dário: I didn't understand! [Mariana is explaining something to Alexandra and when she returns to the blackboard she adds the rule for deriving the exponential [function]. Dário makes a face that seems to tell us that this detail was what was missing for him to

understand the solving strategy] (9th lesson, March 25, 2009, p. 90)

The doubt of Alexandra gave rise to a collective enlightenment. It led the teacher to infer that remembering the exponential function derived rule was probably going to benefit other students. Looking at Dário's facial expression, she was right.

FINAL REMARKS

The results illuminate a well-accomplished inclusion process regarding these two Deaf students, both as mathematics students and as youngsters in peer groups. This class constitutes an example of what is recommended by UNESCO (1994). Deaf people tend to have some difficulties due to the communicative characteristics associated with profound and severe deafness. But, in these cases, they participated in their peers' extra-classes activities, like going to esplanades or to the movies which illuminates their inclusion process and socialisation, as stated also by César and Santos (2006) in another research.

The main adaptations this mathematics teacher introduced in her practices had to do with her spatial positioning and the care with the speed and articulation of words. It was intended for Dário and Artur to participate in the mathematics classes. Knowing the access to the Portuguese oral language is a further challenge to the Deaf, she often passed by their desks, making sure they were progressing in the tasks with an adequate rhythm, and that they were not getting blocked by linguistic barriers, such as an unknown word, or a new mathematical designation. Perhaps because it was a 12th grade class, we were unaware of changes in the nature of the tasks or in their instructions. Nevertheless, there was clear concern, both from the teacher and the classmates, to make Dário and Artur feel like legitimate participants, and to respect their characteristics, interests and needs. The importance of legitimate participation in mathematics classes is also underlined by César (2009, 2013) and by Machado & César (2012).

It was interesting to observe that the hearing classmates adopted ways of acting and communicating with Dário and Artur similar to the ones used by their teacher: being careful with the articulation and speed of the oral speech, turning their face to them, simplifying the vocabulary whenever it was needed. This way of mimicking the teacher's adaptations draws attention to the importance and influence of the role of the educator as a facilitator (or blocker) of a more inclusive education, as also mentioned by César and Santos (2006). The horizontal interactions played several roles: contributed to the development of the students' autonomy; promoted mutual help, facilitating the inclusion of the Deaf (and other) students; helped developing aspects related to the socialisation and to create a relaxed and healthy way of Dário and Artur experiencing their deafness. In short: as mentioned by Sfard (2008), communication played an essential role in mathematics learning, particularly in formal educational scenarios. We would like to stress that the inclusion of these two Deaf students in a mainstream class was not only beneficial for them. Their presence in the classroom asked for a particular care with communication and that also facilitated hearing students' mathematical learning. The diversification of ways of communicating made this experience a very rich one for them all in what regards socialisation and citizenship.

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SOCIAL-POLITICAL INTERFACES IN TEACHING STATISTICS

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The objective of this article is, based on the assumptions of Critical Statistics Education, to value the social-political interfaces in teaching Statistics by modeling projects. For this, we present a practical case, one in which we address the issue of corruption in Brazil, placed in the context of the teaching of index numbers, within the discipline of statistics in an undergraduate course in Economic Sciences.

INTRODUCTION

Political awareness and the discussion of social issues related to a student's reality are the main goals of Critical Education (CE) at any level of schooling. In our view, as in the opinion of the main organizers of this theory, such a goal can be pursued regardless of the syllabus of the subjects. We understand that educators can build adaptations to embrace themes that facilitate discussion of political and social issues which are relevant to the student's reality.

Critical Statistics Education (CSE), as presented by Campos (2007), shows the possibilities of integration and combination of objectives between these approaches by connecting the fundamentals of Statistics Education (SE) and CE. In this context we take the foundations of CSE to highlight and deepen the possibilities of enhancing the socio-political interfaces within the teaching of undergraduate courses in Statistics. For this purpose, we present a fragment of the theoretical basis of CSE and show how it is possible to obtain positive results with this integration, by using a mathematical modeling project that we had developed within the discipline of Statistics and applied in an undergraduate course in economics.

Another aspect that seemed relevant in this approach was the ability to assemble multiple media types within this modeling environment. Jacobini (2011) shows us that this congregation helps to motivate students to immerse themselves in a research environment. He also pointed out that it widens and diversifies the universe of opinions on the issues worked, enriching and stimulating classroom discussions.

CE, SE, AND CSE

As a development of critical thinking, CE has emerged in opposition to traditionalism in the educational system, and its foundations can be credited mainly to Jurgen Habermas in Germany and Paulo Freire in Latin America. Freire's work, which proposes emancipatory knowledge, inspired Giroux (1997), who extended the idea of democratization and politicization of education, within a vision of the teacher as an intellectual transformer: "essential for the category of intellectual transformer is the need to make the pedagogical more political and the political more pedagogical" (p. 163). Giroux emphasized a perspective from which "critical reflection and action become part of the fundamental social project to help students to develop a deep and abiding faith in the struggle to overcome economic, political, and social injustices and further humanize themselves as part of this struggle" (p. 163). Thus, Giroux defended the use of "pedagogies that incorporate political interests that have an emancipatory nature, i.e., the use of pedagogical forms that treat students as critical agents, making knowledge problematic, using affirmative and critical dialogue, and arguing for a qualitatively better world for all people" (p. 163).

Skovsmose, in turn, incorporated these concepts, and progressed in the development of CE. He stated that "it is essential that the issues relate to fundamental social conflicts and situations and it is important that students can recognize problems as their own problems" (Skovsmose, 2004, p. 24). Centered around the question of democracy, Skovsmose emphasized that this issue must be present in mathematics education and thus he worked towards a Critical Mathematics Education, in which working with modeling projects is valued.

Mainly developed since the 1990s, SE was conceived in a context of unease, trying to question and reflect over problems related with the teaching and learning of this discipline. This education was ignited by the difficulties that students have in thinking or reasoning statistically even when they show calculation skills. Seeking to differentiate the pedagogical problems presented by Statistics from those presented by the teaching of Mathematics, several authors converged on the idea that the teaching of Statistics should focus on the development of three specific skills: statistical thinking, statistical reasoning, and statistical literacy.

Statistical literacy has been well characterized by Gal (2004), who emphasized two interrelated components:

a) people's ability to interpret and critically evaluate statistical information, arguments relating to data from research and stochastic phenomena found in different contexts;

b) people's ability to discuss or communicate their reactions to this statistical information, along with their interpretations, opinions, and understandings.

According to Gal, in order to develop these abilities the educators must promote activities such as dialogues and discussions to encourage the students and their ideas when faced with real-world data containing statistical elements.

Statistical thinking is linked to the idea of evaluating the statistical problem globally, understanding how and why statistical analyses are important. Thus, statistical thinking is related to the ability to identify the statistical concepts involved in the investigations and problems dealt with, including the nature of data variability— the uncertainty, how and when to properly use the methods of analysis and estimation, etc. According to Chance (2002), this capacity provides the student with the ability to explore the data in order to extrapolate what is given in the texts and to generate new questions beyond those indicated in the research.

Pfannkuch and Wild (2004) have gone further in the study of this ability and have identified five types of thought which they consider essential for Statistics:

a) Recognition of the need for data: proper obtainment of data is a basic requirement for a correct judgment in real situations.

b) Transnumeration: changing registers of representation to facilitate understanding.

c) Variation consideration: to observe the data variation in a real situation in order to guide the strategies for studying them.

d) Reasoning with statistical models: refers to thinking about the global data behavior.

e) Statistical contextual integration: identified as a fundamental element of statistical thought. Results must be analyzed within the problem context and they are validated in accordance with the knowledge related to this context.

In order to develop these types of thought, Falk and Konold (1992) believe that the students shall be guided to make an internal revolution in their ways of thought, leaving behind the idea of seeing the world in a deterministic way.

The way in which people reason with statistical concepts composes what is generally called statistical reasoning. According to Garfield (2002), to reason statistically means doing appropriate interpretations of a certain data set, to represent or summarize the data correctly, to make connections between the concepts involved in a problem, or to combine ideas involving variability, uncertainty, and probability.

The development of statistical reasoning should lead the student to be able to understand, interpret, and explain a statistical process based on real data. Ben-Zvi (2008) emphasizes the importance of this capability. He states that all citizens should have it and that it should be a standard ingredient in education.

As we can see by the brief descriptions above, there are many commonalities between the three capabilities, especially between statistical thinking and reasoning. The cited authors categorize thinking and reasoning quite similarly and, in attempting to explain them, show a convergence of cognitive and conceptual aspects among them. In our view, rather than highlighting the differences among these capabilities, statistical educators must undertake further research on how to develop them in students. Campos (2007) gave some suggestions for classroom works:

a) work with real data and relate it to the context in which it is involved;

b) encourage students to interpret, explain, criticize, justify, and evaluate the results, preferably working in groups and discussing and sharing opinions.

We understand that these suggestions can be supplemented to address major aspects of CE, and with this in mind, Campos, Wodewotzi and Jacobini (2011) suggest the following actions:

a) problematize teaching, work on statistics through projects contextualized within a reality consistent with the student's;

b) promote debates and dialogues among students and between them and the teacher, assuming a pedagogical democratic attitude;

c) thematize the teaching by prioritizing activities that enable the discussion of important social and political issues;

d) use technology in teaching, valuing skills of instrumental character;

e) adopt a flexible pace for developing the themes;

f) discuss the curriculum and the pedagogical structure adopted.

According to Campos (2007), by adopting these actions in the educational process, we will be practicing a CSE that goes against the traditional teaching model, which follows an alienating path by assuming a false political neutrality.

Campos (2007), Campos, Wodewotzi and Jacobini (2011), and Campos, Jacobini, Wodewotzki and Ferreira (2011) have argued that working with mathematical modeling projects comprises an appropriate pedagogical strategy to carry out CSE. With modeling we create motivation, facilitate learning, give meaning to the contents worked, value the concepts' applicability, and develop students' thinking that is critical and transformative of their reality, and we also promote the understanding of the political and social role of statistics. We understand that CSE worked through mathematical modeling is an efficient way for the articulation between theory and practice, and favors the breakup of arbitrary boundaries between disciplines, allowing a broader and more effective scope.

DESCRIPTION OF THE SETTING

In an Economic Statistics discipline, taught in an Economic Sciences course by the first author of this paper, one of the program contents is that of Economic Indices, which include index numbers and others socioeconomic indices. Throughout the discussion of these indices in the class a question arose about the possible existence of a corruption index. As this is a controversial subject and has generated much debate, we proposed an activity be held related to the subject, organizing groups, and selecting topics for each group to prepare a presentation. The 30 students in the class were divided into six groups and chose the following topics:

1) Misuse of public funds

2) Bids and overbilling

3) Movies about corruption

4) Books about corruption in Brazil

5) Corruption on the Internet

6) Corruption Index

It was suggested that students prepare a brief report on the research topic, in addition to a presentation, preferably with the aid of the Power Point program that might contain snippets of audio and/or video. After a period of two weeks the groups performed the presentations, which were followed by a session of questions and discussions. After the last presentation there was an intense debate and the students decided to undertake some actions to repudiate the corruption problem:

a) to disclose the sites to fight corruption on their Facebook pages;

b) to support the protests against corruption marked for the international day against corruption (December 9);

c) to sign a petition against the unfair increase of the parliamentarians' salaries [1].

DESCRIPTION OF PRESENTATIONS

Group 1 - Misuse of public funds

The group presented a definition of corruption and quoted the DE$VIÔMETRO, a kind of monitor or counter of the misuse of public funds, available on Facebook [2]. A study by FIESP (São Paulo's Industry Federation) was presented, entitled "Corruption: economic costs and proposals to combat it" which projects an average cost of corruption in Brazil of 2.3% of GDP [3]. Some graphs and tables were shown giving data on corruption in Brazil. The group displayed a video from YouTube, wherein a state Representative denounces corruption [4].

Group 2 - Bids and overbilling

This group explained what overpricing is and mentioned a report issued by the Federal Police, reporting that overbilling of about R$ 700 million was found in 303 public works inspected. They showed a chart comparing some market prices as opposed to inflated prices. A video, available on YouTube was shown by the group, pointing out some suspected fraud in bids [5].

Group 3 - Movies and documentaries on corruption

This group presented a brief analysis of three films, which are summarized below. An excerpt of each movie was presented, and a debate enriched the explanations.

"Quanto Vale ou É por Quilo?" (How much is it worth or is it by weight?) Directed by Sergio Bianchi, this 2005 production features a parallel between the slave trade in Brazil and the fake NGOs that exploit poverty.

*"Brasília 18%"***.** Directed by Nelson Pereira dos Santos, this 2006 movie is a fiction about the disappearance of a parliamentary staff member. With a plot full of corruption, the film reveals a disbelief in the punishment of crimes committed by politicians.

"Tropa de Elite 2" (Elite Troop 2)**.** Launched in 2010, this film, directed by José Padilha, shows the clash between the Special Operations Battalion (BOPE) and drug trafficking in Rio de Janeiro. The film shows the involvement of police and corrupt politicians with the militias that exploit poor communities in Rio.

Group 4 - Books about corruption in Brazil

This group chose two books to analyze and do a presentation. These works, summarized below, report corruption cases that occurred over the last 20 years.

"A Privataria Tucana" (The Toucan Pirate-Privatization)**.** Written by Amaury Ribeiro Jr. and edited by Geração Editorial, this book exposes the irregularities in the privatization of state enterprises that took place in Brazil during the 1990s.

"Sanguessugas do Brasil" (Brazil's Bloodsuckers)**.** Edited by Geração Editorial, this book was written by Lucio Vaz in order to report the misuse of public money and to recount the history of corrupt politicians and businessmen.

Group 5 - Corruption on the Internet

The goal of this group was to show how the issue of corruption is addressed on the Internet. The group presented several examples, like news sites, denunciation sites, and others [6].

Group 6 - Corruption index

The group mentioned Transparency International, an NGO that fights against corruption [7]. This organization produces an annual report which highlights the indices of perceived corruption in several countries around the world. The group explained that the index is composed of a combination of studies linking corruption to data collected from various institutions. The corruption perception index (CPI) of 2011 is based on 17 data sources from 13 institutions and their scale is from 0 to 10, where 0 means that a country is perceived as highly corrupt and 10 indicates the country is very transparent and not corrupt. According to the CPI, Brazil is in 73rd position and has a score of 3.8, considered by the index to be very corrupt.

ANALYSIS

In this project, the statistical content worked on was an index calculation. Realizing its complexity, the group that presented this issue only superficially addressed the methodology for calculating this corruption index. They discussed the possibility of creating a simpler and more objective index. However, it was understood that the lack of data on levels of corruption in the countries complicates this task. We understood that the study of corruption index deepened students' knowledge about this type of calculation. However, we must emphasize that the objective of the activity was to involve the students in a discussion of a socio-political problem rather than to deepen their procedural knowledge of Statistics.

For the three skills cited by SE's theoretical foundations, we had observed that working with the real situations involved in the corruption index calculation allowed students to have an overview of the problem. This overview manifested itself when we proposed a problem (to develop an index calculation) and the students needed to research a calculation methodology, as well as find a way to obtain the necessary data and, in addition, to proceed with an analysis of the results. As we said on the description of SE theory, statistical thought is related to the idea of making a global evaluation of a problem, and understanding how and why statistical analyses are important. In this way, like we said, statistical thinking is connected to the capability of identifying statistical concepts involved in the investigations and in the problems worked. This was accomplished in this activity when students were able to see the difficulties that surround the complexity of this index and followed some statistical tools used in its determination. Thus, we understand that this activity advanced their development of statistical thought.

In the theoretical part of this work we had observed that the development of statistical reasoning should lead students to be able to understand, interpret, and explain a real data-based statistical process. The group that researched the corruption index experienced this process, including doing their interpretations and explanations about the studied content. The other groups were able to follow the reports and presentations, as well as take part in the discussions and debates. Thus, we believe that this activity favored the development of statistical reasoning, especially that identified by Pfannkuch and Wild in the (a) item.

As we described in the theoretical part of this work, statistical literacy manifests itself by someone's ability to discuss or communicate their reactions to statistical information, and render their interpretations, opinions and understandings. In this context, we understand that the group who presented the corruption index work had advanced in the development of statistical literacy, as they had elaborated a report which contained statistical data and calculation methodologies of the index. They also made a theme presentation, putting their arguments into words, based on statistical information, using their own vocabulary to do so.

Concerning the other groups, we could observe in the reports and in the presentations the use of typical expressions and terminologies from statistics. Also, we observed some graphs and tables presented, which makes us believe that a contribution had been made to advance this literacy.

We would like to point out that there is no objective methodology capable of measuring the development of the three capacities yet. Thus, the only thing we can do is to observe whether we are stimulating their development or not, based on the theoretical indications that we had presented.

Regarding CE, we understand that we had highlighted it in many ways. Both in the analysis of films and of books, at various opportunities students were placed face to face with the problem of human poverty, of social and racial discrimination. The discussions showed a sense of rebellion and rejection of humiliating and degrading situations that were shown by groups 3 and 4. The social problem of unequal distribution of wealth in the country was wide open to students, who reacted angrily. The problem of misuse of public money for illicit enrichment by politicians, businessmen, and their allies was highlighted in the presentations of groups 1 and 2, generating intense debate and discussions to bring to the forefront the idea that people can and should unite themselves against rampant corruption. Full of indignation for the many examples of the misuse of public money that were presented, with the presentation of group 5, students could realize that there are tools to combat the crimes committed by those engaged in corruption, especially if people united themselves around actions ranging from public protests to the drafting of petitions that could become law by popular initiative against politicians and corrupt businessmen. Students realized that social networks and the Internet can be powerful allies for this purpose.

As for CSE, we see that throughout the project we were on the path traced by the theoretical considerations foreseen by Campos, Wodewotzi and Jacobini (2011), because:

a) we had problematized the course, working topics related to Statistics through contextualized projects, linked to a reality consistent with that of the student;

b) we had encouraged the debates and discussions among students and between them and the teacher, thus taking a pedagogically democratic posture;

c) we thematized the teaching and focused the activities that emphasized the debate of several important social and political issues;

d) we used a technological basis in the education and valued the skills of an instrumental nature to the student who lives in a highly technological society;

e) we adopted a flexible pace in implementing the work presentations.

FINAL CONSIDERATIONS

In carrying out educational activities related to the project described here, we sought to show one possibility of working at the insertion of CSE into the content of the discipline of Statistics in an undergraduate course. In this context, we emphasized the interfaces involved in the proposed socio-political themes, which emerged from the pedagogical environment experienced by the teacher. Our interest in reporting this experience was to show that opportunities to introduce a theme related to social and political problems occur at various times in the pedagogical process, and it is up to the educator to take advantage of these situations to encourage critical, investigative, and challenging thoughts among the students who excel when they are placed face to face with a social problem that involves their realities.

It should be emphasized that throughout the project execution reported here we observed that the diverse media used as a research source (books, movies, and Internet), combined with the possibility of integrating several technological devices (the use of software, audio, video and slide projections) enriched the themes as they showed the reality closest to the student, increasing their involvement and causing their reaction, maximizing the enhancement of the analyses and discussions.

Without losing focus on the statistical contents, we believe that adopting the CSE in a multimedia environment can greatly enrich the education process. Thus, we give the students the opportunity to better understand their own reality and to find the paths that can lead them to actions against the unjust and unequal system in which they live. Thus, we understand that the teacher performs a much broader role and makes their education more meaningful, interesting, and true.

NOTES

1. www.avaaz.org/po/petition/Fim_do_aumento_salarial_parlamentar_acima_da_inflacao

2. www.facebook.com/Desviometro

3. http://www.fiesp.com.br/indices-pesquisas-e-publicacoes/?tema=competitividade

4. www.youtube.com/watch?v=q21rM03_R18

5. www.youtube.com/watch?v=pE96Q1m1I50

6. www.transparencia.org.br/; www.facebook.com/queroofimdacorrupcao ; www.mcce.org.br/ ; http://topicos.estadao.com.br/corrupcao

7. www.transparency.org/

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SCHOOL AND FAMILY INTERPLAYS: SOME CHALLENGES REGARDING MATHEMATICS EDUCATION

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Policy documents and research regarding mathematics education stress the need to overcome underachievement and students' rejection. They also claim about the essential role played by families in order to achieve this. This work is from FAMA – Family Math for Adult Learners project. We assumed an interpretative paradigm and developed a case study in Portugal. The participants were mathematics teachers (N=28), 8th graders (N=108), and their families (N=52) who answered to a questionnaire. Some were selected for an interview and for focus groups. Data were treated through descriptive statistics and a content analysis. Results illuminate different expectations from these groups and also the lack of possibilities to act as legitimate participants felt by poorly literate parents from vulnerable minorities.

INTRODUCTION

Portuguese educational statistics show the most important criteria to predict students' marks is their parents' level of literacy (César, 2013). This clearly illustrates the essential role played by the cultural background and by family support towards schooling. It underlines the privileged life trajectory of participation (César, 2013), particularly in school, that is experienced by students whose families accomplished a long schooling path. These families usually participate in the mainstream culture (Favilli, César, & Oliveras, 2004). The majority of the school contents, vocabulary, tasks and examples, both used by teachers and in the textbooks, are well adapted to students from towns in the seaside – the richest and most developed part of the country – and from the mainstream culture. Many other students participating in vulnerable cultures – usually socially undervalued – often experience school underachievement and early school dropouts (César, 2009, 2013). Thus, at schools affected by poverty, whose students' families did not accomplish a long schooling and participate in cultures far away from the mainstream, the need for regulatory dynamics between school and families is particularly intense (César, in press).

Families play a very important role in students' life trajectory of participation (César, 2013), in and outside school, and in their engagement in school activities, namely in mathematics (Borges, César, & Matos, 2012; César, 1987, in press; Sheldon & Epstein, 2005). Regarding life trajectories of participation, some families expect their children to accomplish university studies and that school will play a main role in their professional opportunities, while others believe their children's future is much more related to working experiences than to school knowledge. Thus, expectancies regarding schooling, achievement, and future opportunities to get better life conditions vary a lot. Another important issue is that parents get involved in mathematics activities in different ways according to their mathematics knowledge, and their previous experiences (Díez-Palomar, 2008; Green & Hoover-Dempsey, 2007). Their positive self-esteem, the way they perceive their empowerment, or if they assume themselves as legitimate participants (César, 2009; Lave & Wenger, 1991), believing they have a voice that can be expressed and respected, or as peripheral participants who are often silenced and have less opportunities to express themselves and be listened to, also shape the way they help – or feel unable to help – their children when they are studying mathematics at home. As mathematics is one of the most rejected subjects, presenting high levels of underachievement and playing a decisive role for students' vocational choices and future professional lives, it is also the subject that is more affected by parents' levels of literacy and by their previous experiences as mathematics learners.

The development of negative social representations about mathematics and themselves as mathematics learners is connected to students' and parents' experiences of school failure and underachievement (César, 2009, 2011). Changing these negative social representations is an important step in order to promote students' access to mathematics cultural tools and also to school and social achievement and inclusion (César, 2009, 2013). It is also an essential move in order to construct their identities and to allow them to experience learning opportunities based in equity (César, 2009, 2013; Cobb & Hodge, 2007).

METHOD

This work is part of FAMA – *Family Math for Adult Learners* – an international project involving countries like Spain, France, United Kingdom, Switzerland, Portugal and Italy. We focus on the Portuguese data and in the third specific goal: to support people from socially vulnerable groups who are part of marginalized social settings, particularly immigrants, persons affected by poverty and women. We began by collecting accounts from three groups: teachers, students, and their families. Data collection lasted from November 2010 until March 2011. FAMA team agreed that in countries collecting data from mainstream schools, students should be mainly 13/14 years old. In Portugal this corresponded to students attending the 8th grade. Compulsory education has long ago regarded students until they were 15 years old (AR, 1986), i.e., those considered in this study.

We assumed an interpretative paradigm (Denzin, 2002) and an intrinsic case study design (Stake, 1995). FAMA project included multiple cases (one in each country participating in it). The criteria to choose the schools were that they should be multicultural settings, from poor backgrounds and including students and families from diverse vulnerable minorities. In Portugal we selected a school in the surroundings of Lisbon. But as no school had as many mathematics teachers as we needed (at least 25 per country), we used other schools with similar characteristics, regarding the mentioned criteria, and in the same region, in order to collect the teachers' questionnaires. In the Portuguese case study the participants were 28 teachers, 108 students, i.e., all 8^{th} grade classes from that school, and 52 family members. The majority of the family members were parents. There were many more mothers (87%) than fathers (13%) acting as responsible for students' schooling (*encarregados de educação*), and more female (75%) than male teachers (22%) [3% did not state their gender]. This is usual in Portuguese schools. Regarding students who answered to the questionnaire, 46% were male, 53% were female and one did not mention his/her gender.

All the participants answered to a questionnaire, specific for each group: teachers, students, and families. Then, we selected 5 teachers (all those teaching the 8^{th} grade), 22 students, and their 22 family members for an interview. The criteria to select students were: (a) only children, others with only older or younger brothers/sisters, and those who had them both; (b) high, medium and low achievement, particularly in mathematics (it also meant different ages); (c) different parents' schooling levels; (d) diverse cultural backgrounds; and (e) gender. We interviewed all parents from the selected students. Finally, we had a focus group for teachers (N=5), three for students (N=3, N=6 and N=8, in a total of 17 students), and another three for families (N=3, N=5 and N=5, in a total of 13 family members, two of them being the mother and the father of the same student). Due to external constrains we could not have in the focus groups all those we interviewed. We must underline the huge effort these families made in order to participate. All the interviews and focus groups took place in the school and they had to conjugate their schedules with ours. As many of them had temporary and badly paid jobs, they were not allowed to leave earlier or to ask for a free mid-day. Thus, many had to use a holiday day or meet us late, at night. This is a sign of their engagement in this work and in their children's schooling. We could get 17 students and 13 family members for the focus groups, although none of them explicitly told us s/he did not want to participate. Those who did not participate were sick or had last minute issues that did not allow them to participate at the scheduled time and place. As focus groups involved several students or parents at once, they were not as easy to reschedule as interviews.

Data treatment and analysis was based in descriptive statistics (questionnaires) and in a narrative content analysis (Clandinin & Connelly, 2000), based in successive readings, regarding the open questions of the questionnaires, the interviews and the focus groups. The FAMA team previously decided the six categories of analysis. Thus, they were deductive categories, based in the literature review. They were the same for all countries and case studies: affect, cognition, teaching and learning, contents, participation, and structure. All interviews and focus groups were fully transcribed. This allowed for an in-depth analysis, trying to understand these persons' life trajectory of participation (César, 2013), particularly at school and at these students' homes. For each category we marked each part of the transcripts that illustrated it, and also if they contradicted other parts of the interview or focus group. We focus on the participation category. In order to maintain their anonymous participation, as recommended by ethical principles, we used a code system that still allows readers to pair students and families: the number of the students and respective families are the same (e.g., Mother 3 is the mother of Student 3). Confronting different papers (César, 2012; César & Machado, 2012) and the National Report (César, 2011) this allows for a better comprehension of the participants life trajectories of participation (César, 2013).

RESULTS

Regarding participation, teachers mainly stated: (1) parents' lack of time to come to school; (2) their wishes regarding marks and students' progression, but their lack of concern about their (mathematics) knowledge; and (3) the discouraging role played by complains in what regards parents' participation in school. These three points are illustrated in the next excerpts, in the order we mentioned them.

> 12 T3 – (...) It's difficult to bring parents to school because sometimes they aren't even at home. (…) In this sense and in these cases, I believe that school has to take the first step and it must help parents realise that it's worthwhile coming to school and know what's going on… Or even know it from home. To keep informed about what's going on with their children… Ah… (…) But there are very complicated situations! In which family's engagement is set up in a very difficult way in school (…). (Teacher 3, Interview, Turn 12, pp. 2-3)

This first excerpt also mentions a very important topic: the school should do the first move, should be aware of parents difficulties and should turn coming to school into something appealing. Other researches by César (in press), César and Oliveira (2005) or Sheldon and Epstein (2005) also illuminate the role played by school and teachers in parents' engagement in their children's schooling. We would add that it is also needed to empower parents so that they feel themselves as legitimate participants (César, in press). Otherwise, when they feel afraid of talking to teachers because those ones have a better argumentation and know school rules better, or when they feel helpless in changing their children underachievement and/or way of acting, they will avoid coming to school, as mentioned in other studies (César, 2009, 2013; Favilli et al., 2004).

> 14 T5 – That is… it's a subject… a very worrying [subject]!... Because… the… family engagement in school when "our son is getting well", when he is praised and shines at school, it's an easy engagement. That doesn't happen when "our son" is… acts the other way around!... i.e., when we are called to school for those reasons we should never be called, we tend to get away from school… Ah… How do we fight against this? I don't have the slightest idea… (…) I don't have

any answer… [The tone of voice denotes a smile]

15 R – And the school? Are there any dialogue spaces between the family and the school?

16 T5 – The school tries to! The school tries to… But… Ah… It's complicated… There's this need, but in practical terms, that need is… Then, there are too many barriers: or it's because the timetables aren't compatible ones, or because we've two hours to talk about five hundred subjects… (Teacher 5, Interview, Turns 14-16, pp. 2-3)

This excerpt states once again that teachers' and school practices facilitate parents' participation. But this teacher accounts that he does not have any possible moves to attract parents to come to school when there are difficult situations that need to be solved in a collaborative way. Once again he points up the lack of time, both from parents and teachers. Thus, what comes from this excerpt is also that there is a gap between discourses – parents should participate in schools – and practices, as timetables, spaces, or what are regarded as the most important issues to be addressed, do not include parents' participation in the school life. Overcoming this gap is possible, through the development of regulatory dynamics between schools and families, as illustrated by César (in press). But this means a huge effort from the directive board of the schools and also from teachers and families. Schools clearly have to do the first moves.

The focus groups allowed for a more in-depth understanding of the difficulties experienced by parents regarding their participation. All teachers mentioned the importance of parents' schooling when they are helping their children studying mathematics. But they also mentioned that the economical background also played an important role, as private lessons (*explicações*) are expensive but are also quite common in Portugal. What is interesting is that no teacher mentioned that the need for those private lessons also meant that schools were not working so well, i.e., there is a lack of a more critical approach to the educational system itself.

7 T1 – (…) But probably a student from a superior socio-economical context has more potentialities regarding private lessons, regarding extra work, that allows, isn't it? The parents' schooling itself can also allow, according to having time or not, to help children at home, while here probably it isn't so… it's not like that. There are many parents who during the day are working and are away and – isn't it? – they're trying to earn some money, and they can't… (Teacher 1, Focus Group, Turn 7, p. 2)

Participation was also discussed in what regards the times teachers try to contact some parents, particularly those whose children are in trouble and who do not answer the phone when they realise that it is a call from the school. Thus, participation is also shaped by expectancies, as shown by César (2013) or Green and Hoover-Dempsey (2007). If parents expect teachers to tease them, they tend to ignore the call and they do not contact the school. Thus, teachers need to find ways to ask parents to come to school also to listen to nice comments and not only that their children are in trouble, as mentioned by Favilli and his associates (2004), or by César (in press).

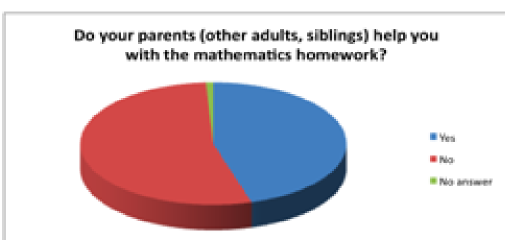
51 T2 – That depends because – and now I'm putting things a bit my way – there are parents that we contact and they simply have the school number in their cell phone memory, and they just ignore it, they don't answer that call. They answer if it's an unknown number.

52 R – Hum, hum.

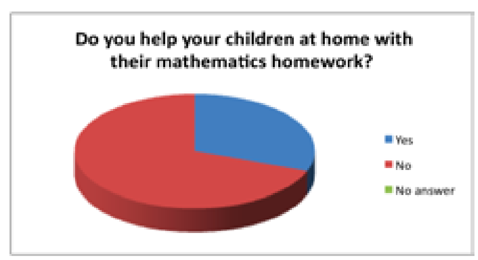
53 T2 – That reveals a lot, regarding some of the parents. Others call us regularly. This I'm reporting as director of the class (…). (Teacher 2, Focus Group, Turns 51-53, p. 5)

When they discussed possible ways of overcoming this problem, they mentioned the emails, or a platform in the internet. But due to the poverty that affected many students' families they also claimed that many of them did not have internet at home and these solutions did not prove to be efficient ones.

Regarding participation students never mentioned if their parents went or did not go to school. What they mentioned was if they participated in their mathematics work when they were doing it at home. Fifty-eight students (54%) state that their parents do not help them (see Graph 1). But even more parents (N=36, corresponding to 69%) claim that they do not help their children (see Graph 2). This is also mentioned in the interviews and in the focus groups: many parents do not feel competent to help their children when they are doing mathematics activities at home.



Graph 1 – Students' questionnaire



Graph 2 – Families' questionnaire

In the interviews it was clear that those whose parents did not help them reported that their level of schooling and time were the main motives to explain their lack of participation in their mathematics activities.

21 S6 – They don't remember anything from school! They try! But they don't always succeed. Sometimes I have to be the one who teaches them… [Ironic smile] (Student 6, Interview, Turn 21, p. 2)

In these cases students have to rely on themselves to study mathematics at home, unless they go to private lessons or to an after school centre of studies. It is mentioned by those whose families help them that this is an advantage. Some account that their older siblings have an easier way of explaining mathematics –

using what they call “a teenager language” – and others stress that it is very good to have a family that studies with them because many colleagues do not have this privilege.

42 S15 – Because I understand well what they [referring to his brother and his parents] mean… because sometimes – how can I say that – they explain more with “teenagers’ language”.

43 R – And at school, you don’t get that?

44 S15 – I also get it at school. Sometimes. Also in the books, there are contents that are less explained, let’s put it like this… If this is the case, he [referring to his brother] helps me better. To understand it better. (Student 15, Interview, Turns 42-44, p. 3)

This excerpt illuminates how important it is to have family support when studying mathematics, as it also allows students to understand better the contents that are less clear or less explored in the textbooks. This support is not only important for students who do not use the textbooks to study. It is important for all of them, as shown in the last excerpt below. Thus, we can understand better the Portuguese educational statistics results which expressed that the most important criteria for getting high marks were parents’ levels of schooling (Rodrigues et al., 2010).

32 S7 – I don’t know… It’s so cute. To see that I can count on them for this! To see they’re interested! About my school path, about… about me! They’ve interest! Isn’t it? It’s good to know I’ve someone that I can count on this issue, about school, that I don’t have, as I know that most of my colleagues have to do it all by themselves, they have no one at home that helps them and so I value my family very much! (Student 7, Interview, Turn 32, p. 4)

Summing up, students who can count on their families to work with them when they are studying mathematics at home are supported in what concerns knowledge. But the important point is that this participation in their school life also makes them feel more secure emotionally and more able to develop their own abilities and competencies. As this study was developed in a setting affected by poverty, some students valued a lot their families and the chance they had, as they were able to share their home activities with them each time they felt any kind of difficulty regarding their studies. But the differences between those who have support and those who cannot count on it are also visible in the focus groups:

185 R – Yes. And when you have homework and you have some doubts what do you do?

186 S4 – I don’t do. (…)

188 S7 – I ask for help. (Students, Focus Group 2, Turns 185-186 and 188, p. 8)

These very short but also very quick answers explained it all: those who have family support when they are studying mathematics, when they experience a difficulty they ask for help and they get helped. But those who cannot count on their families to study mathematics, they simply do not do their homework or they stop studying it.

When confronting students' accounts to their families' accounts it is important to be aware that they confirm each other. For instance, Mother 6 told us that she experienced a lot of difficulties regarding mathematics and her son had reported that sometimes he was the one who had to teach his parents (see Student 6 excerpt above). Mother 6 is already attending the 12th grade, but she failed in mathematics and this subject makes her feel uneasy, as she stated:

> 2 M6 – I'm going to be honest: mathematics it's not and it never was my strongest [subject]. My mathematics is delayed since my 10th grade. I still did not accomplish the 12th grade, but it is still delayed, mathematics is not my strong [point]. We try to help him in what we are able to, in what we remember, me and my three sisters [Says her sister's name] already finished her university graduation recently, but… it's not our strong point. He does things on his own and he learns by himself. It doesn't come from us. (Mother 6, Interview, Turn 2, p. 1)

Father 7 also as a very similar discourse to the one we reproduced above (see Student 7 excerpt). He also states that having family support when they are studying is a very important issue.

> 4 F7 – Or when I pass and ask, '[Says the name of his daughter], do you need anything?'. She sometimes says yes, other times she says no... Ahm... But whenever is possible, yeah, I do help [her]. (…)
>
> 18 F7 – ... mathematics is a subject... peaceful! Thus in [says his daughter's name]'s evolution.
>
> 19 R – And in your view, should the family or is it profitable that the family gets engaged in their children's mathematics learning?
>
> 20 F7 – I think so, I mean... It's helpful we become involved in all their process of education, right? Educative. Ahm... Both in mathematics and in other subjects whatsoever. I think it's worthwhile. For everyone. And if they need help when they ask for help, a… In part it's good, and it's great to know that we can help. (Father 7, Interview, Turns 4, and 18-20, pp. 1-2)

This account illuminates two important features of this father's participation: (1) he is attentive to his daughter's needs and sometimes he asks her if she needs anything; and (2) he feels confident about his competencies to help her. His daughter showed a similar high positive self-esteem and calm way of facing school evaluations and daily work. Thus, confronting families and students' accounts allowed for a better understanding of their life trajectories of participation, in and outside school.

FINAL REMARKS

Promoting parents' participation in their children's home mathematics activities and in schools is a complex issue. Despite their wish to help their children their low schooling levels associated to their lack of time and confidence as mathematics users makes them feel unsecure and uncomfortable when their children ask for their help. Teachers are aware of their difficulties and their rejection of mathematics activities, but they do not realise how much effort some of them put into helping their children.

Parents' abilities and competencies play an essential role in their participation. Some of them assume themselves as legitimate participants and that facilitates their children's development of a positive self-esteem. Their children also show higher expectations towards their own performances. But many parents feel unable to help their children and act as peripheral participants. In these cases schools need to be able to develop regulatory dynamics (César, in press), particularly in order to empower families and to distribute power, allowing them to have a more active role in their children's schooling, and promoting equity regarding school achievement.

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THE GAP BETWEEN MATHEMATICS EDUCATION AND LOW-INCOME STUDENTS' REAL LIFE: A CASE FROM TURKEY

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The purpose of this study was to trace the reflections of cultural differences based on social class in the elementary mathematics education in Turkey. Critical discourse analysis was conducted to examine these possible reflections. By researching mathematics education from a critical perspective, this study aimed to contribute constructing a starting point for socially responsible mathematics education. Elementary mathematics curriculum, textbooks, classroom practices, and teacher interviews were the main data sources. The discourse analysis of mathematics education contexts implied that elementary mathematics discourse replaced the 'real life' in mathematics problems with the life of middle and upper middle classes and ignored the low-income students' cultural backgrounds.

Keywords: Elementary Mathematics Education, Cultural Differences, Real Life Problems, Critical Discourse Analysis

THEORETICAL FRAMEWORK AND PURPOSE OF THE STUDY

The difference in educational outcomes of students coming from different race, ethnicity, class, gender, and language backgrounds were commonly researched and discussed in international education community. For many years, researchers have highlighted that public education systems did not produce equal outcomes for all students. Significant differences in students' achievements, graduation rates, and university attendance were continually observable among groups classifiable by race, ethnicity, class, gender, and language background (Gregson, 2007).

These highlighted differences have directed critical education researches to examine how class, race, and gender are represented and struggled over in schools, in education programs, textbooks, and teaching practices. The deficiencies in the representations of class, race, and gender in education materials were underlined as one of the blockades to their accessibility to high quality education (Gutstein, 2006). This paper specifically focuses on the representations of different social classes in elementary mathematics education discourse.

'Class culture', in the context of this study, was used to examine whether different cultural/social practices of different social/socio-economical classes were valued in a different way so as to provide a ground for inequalities in the mathematics education or not. 'Class culture' issue was examined with the help of the concept of 'cultural capital'.

The literature on cultural capital and its relationship to educational inequality was inspired largely by the work of Pierre Bourdieu. As described by Bourdieu (Bourdieu, 1977; Bourdieu and Passeron, 1977), cultural capital is the vehicle through which background inequalities in students' life are translated into differential academic rewards. The cultural capital theory argues that the culture transmitted and rewarded by the educational system reflects the culture of the dominant class. To acquire cultural capital, the student must have the capacity to receive and decode it. The acquisition of cultural capital depends on the cultural capital transmitted by the family. Consequently, the higher the social class of the family, the closer the culture it transmits is to the dominant culture and the greater the resultant academic rewards (Bourdieu, 1977; Bourdieu and Passeron, 1977).

'Cultural capital' concept's major insights on educational inequality is that students with more valuable social and cultural capital become more successful in school than do their peers with less valuable social and cultural capital. This perspective is very useful in attempts to gain a better understanding of how class influences the transmission of educational inequality (Lareau & Horvat, 1999).

In the light of cultural capital literature, the main purpose of this study was to investigate how social class differences are addressed in elementary mathematics education in Turkey. Parallel to this aim, this study intended to extend the understanding of the relationship between mathematics education and social/cultural differences and of teachers' views about this relationship.

METHODOLOGY

In line with the purpose of the study, the following research questions were formulated:

- To what extent are elementary mathematics curriculum, elementary mathematics textbooks, and elementary mathematics classroom practices in Turkey free from cultural values?
- How are the different cultural values and practices of different social classes reflected and addressed in the elementary mathematics curriculum, textbooks, and classroom practices?
- What are the perceptions of mathematics teachers about the relationship between mathematics education and dominant cultural values?

Since how power relations are produced, maintained, and challenged through texts and the practices is one of the main concern of CDA (Locke, 2004), it was used as a research methodology of this study. The study employed critical discourse analysis in investigating these research questions and included curriculum documents and implementation, textbooks, and teachers interviews as data sources.

This study investigated the elementary mathematics curriculum (from 6th to 8th grade), elementary mathematics education textbooks (from 6th to 8th grade), an elementary mathematics classroom [one 7th grade classroom (28 students at the age of 13)], and views of an elementary mathematics teacher.

Participant School and Classroom

The study was conducted in a public school in the second semester of 2009 – 2010 academic year. Although the participant school was not far from the city center, the district was composed of Gecekondus (poor quality houses constructed without any proper plan and infrastructure and occupied by very low income families) and lower middle class and middle class apartments. According to school's Strategic Planning Report, most of schools' students were living in Gecekondus near the school district. The families were coming from rural areas of Ankara and near cities, and they maintained their close links to their home towns. While most of students' fathers were working at temporary jobs with minimum wages, their mothers were generally house wives.

The socio-economic conditions of the students in the participant classroom reflected the socio-economic conditions of the district in which school was placed. Most of the fathers were working as driver or construction worker with minimum wage. Only three of the mothers were working as officers (office staff in a state institution), the others were housewives.

Data Collection

There were three main data sources for the investigation of these research questions. The first data source of the study was the curricular materials in elementary mathematics education, such as elementary mathematics curriculum, guidelines/booklet /guide book, textbooks, and teachers' reference books for 6th, 7th, and 8th grades published by the Ministry of National Education. The second data source was the practices and interactions including explanations, examples and questions, teacher-students interactions, homework, and projects in a mathematics classroom which were observed in the second semester of 2009-2010 academic year. The third data source was pre- and post-interviews with the participating teacher. Mathematics teachers' views about relationship between critical issues and mathematics teaching were explored through these interviews.

Data Analysis

The analysis of the data started with the construction of a sample code list after the classroom observations were completed. The content of the curriculum and textbooks was analyzed with this preliminary code list. This code list was composed of textbooks and classroom examples which had reference to daily life situations. This pre-analysis resulted in some improvements in the code list: Three main categories emerged in this pre-analysis. Classroom observations were coded with this improved code list. It appeared that this improved code list was sufficient to analyze the context of the classroom observations. A part of this code list was provided in Table 1.

Up to this point, all data of the study was coded by only the researcher. After the construction of this final code list, the whole data was coded by both the researcher and another elementary mathematics education researcher to address the reliability concerns. The general objectives, specific learning outcomes, the vision of the elementary mathematics curriculum and sample activities and lesson plans in the curriculum, the homework projects, the examples and questions in the textbooks, the content of the classroom activities and teacher-students interactions were analysed with this final code list by two researchers.

While coding, the researchers read each material individually and coded the critical expressions in the documents. After coding individually, they compared their codings to see whether they were parallel or not. The researchers reached over 85% agreement in assigning codes. When there were differences between the results of coding, researchers discussed the existence of a critical meaning in the statement and decided whether to include these statements in data analysis or not. The controversial statements about which researchers did not convince each other were not presented as a finding of this study. These controversial statements constructed 4% of total data coded.

Theme	**Codes/Categories**	**Examples**
Class Culture	Family Life	Mert, his wife and their two children participate different activities such as theater, cinema, and exhibitions every weekend. Cansu takes a lot of photos in her visit to Çanakkale Cemetery on summer holiday with his family Esen family redecorates their house that they newly bought
	Adult Life	Mr. Ali cares for his health. He decided to buy a summer house to escape from stressful pace of everyday life Mr. Okan wanted to get a camera and investigated the prices of different trademarks and models of cameras Mr. Hasan wants to rent a car before for his summer vacation
	Children Life	The following paragraph describes the help of 6.grade students to a poor school. Pınar is going to French course in half of the month Sengul will buy fruit juice, chocolate and candies from shop for her birthday party

Table 1: Code List for the Analysis of the Data

The pre- and post-interviews with the teacher were also analysed with the help of the second coder. Additional to emerged categories (listed in the table as family, adult, and children life), analysis of teacher's interviews was composed of her comments about these categories. The questions in the pre-interview were basically about teacher's general views about the relationship between mathematics education and dominant political, cultural, and patriarchal views and values. For example, one of the pre-interview questions was about the relationship between politics and mathematics education: "When you consider mathematics education in general, such as the explanations in curriculum, examples in textbooks or activities in your classrooms, how do you assess the neutrality of these contexts in terms of political and cultural views?". Post-interview questions, on the other hand, basically addressed teacher's views about my specific inferences from the curriculum, textbooks, and classroom observation analyses. For instance, one of the post-interview questions was about the gap between mathematics problems and students' cultures: "Context of some of the textbook examples seemed considerably far from the students' real life, for example, students were expected to behave as a manager in a bus company in one of the examples. What is your perception of such examples? Can you assess your examples in the classroom in terms of this perspective?".

RESULTS

The main issue investigated in the study was whether different socio-cultural practices of different social classes were valued in a different way so as to provide a ground for inequalities in the mathematics education or not. Possible findings would also answer the question "Whose lives were presented in mathematics education discourse?"

The analysis started with the curriculum's emphasis on (i) the importance of the relationship between mathematics education and students' real/daily life and (ii) the importance of the solving real-life problems. The vision of the curriculum indicated that mathematics should be taught based on concrete and finite-life models. In addition, raising students with the ability of using mathematics in their daily life and appreciating mathematics as an important tool in real life was defined as the prior objectives of curriculum. I concluded that 'solving real life problems' was placed at the heart of elementary mathematics teaching. However, I also concluded that the questions 'what is real life' and 'whose life will be served as real life' were not satisfactorily answered in the program. I pointed out that these unanswered questions would imply that lower social classes which had a limited/restricted voice in both social life and educational organizations can face with the risk that their silence will continue in mathematics classrooms.

The analysis of curricular materials showed that this risk turned into reality in the context of the textbooks. The daily life problems in the textbooks were analyzed under three main headings: (i) The problems based on life/activity of whole family; (ii) The problems based on off-class activity of children/students; and (iii) The problems based on life/activity of an adult character. Based on the findings describing a families', children' or adults' life, I tried to draw a profile of 'daily/real life' in the problems. This constructed daily life profile was examined for whether it reflected a 'daily life' of specific social classes or not.

The analysis of textbooks specified that 'the family' in the problems were the ones who participate in different activities such as theatre, cinema, and exhibitions. At every weekend, characters visited grandfather, grandmother, or eldest of the family in national holidays, travels different resorts/seaside during the summer vacation, and builds/bought a new house or redecorates the existing one.

'The child' in the problems was the one who would go to different courses through the year, such as a language course or a musical instrument course, or a sport course, participate different out-door activities with his/her friends such as going to swimming or taking a trip, go to shopping for his/her birthday party, have a computer and a bookcase, and s/he would organize with his/her small size classroom either a trip to touristic destination or a campaign to help for a poor school.

Finally, 'the adult' in the problems was the one, who would care for his health either by buying a summer house to escape from stressful pace of everyday life or becoming a member of a sports club, and investigate the prices of different technological devices. In line with these descriptions, I concluded that the 'real life' in mathematics problems was replaced with the lives of middle and upper middle class individuals.

The replacement of 'real life' with the middle-class social life in the textbooks could be considered as the consequences of the asserted deficiencies in the curriculum. When all the expressions about 'real life' in the curriculum and the replacement of 'real life' with the middle-class social life in the textbooks were considered, it seemed that it hid complicated realities, especially for the working-class students.

Moreover, observations of classroom practices implied that the questions solved in the classroom were very similar with the textbooks' examples in terms of the class culture they represented. The families who saved up money to make investment, who loaned a credit from a bank to meet your needs, who went for a holiday or who paid their automobile insurance were not similar to the families of participant students. However, detailed description of classroom practices indicated that the focuses of problem solving procedures were mainly centred on the basic calculations and the cultural contexts of the problems were near to vanish. Classroom observations implied that although the influences of class culture were reduced in the classroom with respect to textbooks, there was still no reference to the lives of participant students in the classroom activities.

Lastly, teacher interviews pointed out that participant teacher's views about class culture issue were parallel to her views of politically neutral mathematics. Although my analysis of textbooks and classroom practices provided counter examples, the teacher believed that different cultural values and practices were respected similarly in schools. In the light of teacher's responses, I also tried to discuss the warnings of cultural capital literature about the possible inequalities that the difference in cultural resources would bring to classroom whether teachers are aware of these inequalities or not. Teacher's answers also underlined two reasons for limited usage of real life examples in the classroom: (i) Her view that the effects of real life examples is generally overestimated and applicable to only successful students; and (ii) Her view that finding a common concept for all students in the classroom is very difficult task.

CONCLUSION

To conclude, the overall context of elementary mathematics education replaced the 'real life' in mathematics problems with the life of middle and upper middle classes. 'Real life' problems in mathematics textbooks were not prepared to serve appropriately for working class students and they provided middle class students a cultural advantage. The congruence between the life of middle class students and the life presented in mathematical problems could make these problems easier for them in comparison to their lower class peers. When it was considered that middle and upper middle class students already had economic advantage, this cultural advantage would increase the achievement gap between lower classes students and them. It could be claimed that more working class students would be compelled to failure in this unequal cultural and economic conditions. The possible failure was not the only result of this middle class domination for working classes students; they would also have very limited opportunities to comprehend their live conditions through mathematics and so limited chance to take action against these conditions. While mathematics education had the possibility to make them aware of their lives, this possibility was vanished in the realm of middle class culture.

Based on the findings of this study and with reference to current critical mathematics education literature, the following implications could be stated for teachers, teacher educators, curriculum developers, textbook writers, and policy makers.

First of all, the emphasis of the curriculum on the importance of 'real life' and 'problem solving' should be protected; however, there should also be specific directions about integrating the life, culture, and problems of different social classes - especially the lower classes- which had a limited/restricted voice in both social life and educational organizations. Although transforming curriculum is necessary for transforming curricular materials, such as textbooks and workbooks, it will be not sufficient. Textbooks are one of the bridges between curricula and students (and also teachers). Therefore, there is a need for special consideration/attention to address their deficiencies. Integrating the daily life of working class students into mathematics textbooks and focusing on their social and economic problems in mathematics problems will also be helpful for conducting socially just mathematics.

Neither reforming curriculum nor revising textbooks would promise the positive changes in classroom practices. Transforming classroom practices is strongly linked to the change in teachers' perceptions about the relationship between mathematics education and critical issues. Teachers' perception of neoliberal view of education as a ‘common sense’ and their perception of mathematics education as politically and culturally neutral would be one of the reasons of the reproduction of social inequalities through mathematics education. Therefore, educating in-service and pre-service mathematics teachers about class- and culture-sensitive mathematics would be a barrier for this reproduction. Although mathematics teacher curricula emphasis teachers’ content, pedagogic, and pedagogical content knowledge, there are not sufficient courses or content in other related course addressing the role of mathematics education in social justice. Teacher education programs should cover possible links between mathematics education and cultural diversity, gender equity and social justice. Teacher education students, as future teachers, ideally could be able to use strategies for identifying their students’ social and cultural environment to make them cope with their problems.

This study tried to portray current standings of mathematics education in terms of class culture. Although I pointed out some alternatives above, a comprehensive answer of what could be done to establish improvements in these issues was not the scope of this study. In the light of critical mathematics education literature, there can be an attempt to construct a class- and culture-sensitive mathematics curriculum. Investigating the possible impacts of such program on students’ attitudes towards mathematics and towards these critical issues will greatly enrich the related literature.

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ETHNOMATHEMATICS AND TEACHER EDUCATION: REASONING OVER THE MEANING OF THE STUDENTS' PREREQUISITE AND THE TEACHER'S LISTENING

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This paper investigates an approach between ethnomathematics and the mathematics learning processes in the scholarly context – however it does this from an ethnomathematician's perspective, not that of a cognitive psychologist. I have been developing research in the area of teacher education with the objectives of: a) recognizing how much mathematics teachers are aware of the movement/literature on teacher education and, b) searching an understanding how and to what extent they are available/able to appreciate and legitimize the first/previous knowledge of the students. With the help of the results from this research, I have focused on two aspects of mathematics education processes - the notion of the student's prerequisite and the notion of the teacher's "listening" (Freire, 1996) – that play a key role in the mentioned ethnomathematics approach and mathematics learning processes.

INTRODUCTION

Ethnomathematics, as a movement in the contemporary world, brings out, according to Vergani, one approach with the most promising currents of critical and transdisciplinary in today's thinking like sociolinguistics, cognitive linguistics, phenomenology, biology of knowledge, semiotics, symbology, holistic paradigm and complexity (Vergani, 2003, p.127). Such recognition from an intense researcher who has been for a long time involved in ethnomathematics studies – as a link and holder of current critical-holistic thinking – means a lot to mathematics education both in terms of historic comprehension of knowledge and of political-practical-theoretical tools for other movements on this perspective. Indeed, the importance of ethnomathematics´ recognition resides in the fact that, thanks to that approach, today many of us can get space to exercise the struggle against formal disciplines superiority – which, assuming the character of knowledge, exclude the rest – those who do not participate in the formal (academic) sphere.

However, if on the one hand it has been accepted to recognize such potential of Ethnomathematics in the political-philosophical sphere, it has been a consensus, among educators involved in these studies, that to take ethnomathematics as a way/method to school education is a highly complex proposal. In a certain sense, this paper aims to developing the thinking about how this might be achieved through the analysis of teacher educator responses to ethnomathematical problems.

In fact, ethnomathematics has been, on the one hand, very successful in developing itself in education as a way to explicit/legitimate spatial and quantitative relationships implicit on the know-how of one group, revealing – from technique to meaning - the differences, from a social-ethnic group to another, in respect to mathematical relationships. On the other hand, even to D'Ambrósio (1990), the biggest concern in the educational point of view, as well as the essential step to ethnomathematics´ diffusion is to take it to the classroom, it is possible to say that the movement of ethnomathematics as a pedagogical practice is still crawling. Why is it so? What happens in the school operational dynamics that could make ethnomathematics assumptions difficult to be incorporated?

An attempt to answer the question above may be the fact that in the school educational environment, some educators seem to be indifferent to the influence of culture in the understanding of mathematics ideas. Such concern really seems to be a waste of time and effort, important only to anthropologists or, at best, to mathematics education researchers who have not discussed earlier and/or closer on the psychology of mathematics education studies (Meira, 1993). Indeed, from our search for such value among mathematics educators (research results shown ahead), some of them seem indifferent to distinctions of social class and culture, while others seem to wish the elimination of these distinctions. And, naturally, among the latter, there is the questioning if what is worth preserving can be re-built/transmitted by teachings via school.

Although I have chosen this approach to start a discussion about ethnomathematics, I do not intend to leave the subject in this rather negative and explanatory tone. After all, I really believe that the teacher should treat school education via cultural patterns of behavior and knowledge, both because, agreeing with Fasheh, it helps the student to become more attentive, critical, appreciative and more confident to face the mathematical relationships the teacher wants to develop as well as help them to build new perspectives and search new alternatives, "and, we hope to help them to transform some existing structures and relationships". (Fasheh, 1982, p.8)

ETHNOMATHEMATICS: A POINT OF VIEW

From my point of view, one of ethnomathematics founding basis is the belief that different mathematical relationships or mathematical practices can be generated, organized and transmitted informally, like language, to solve immediate needs. And like an operational means of doing, in the center of the know-do processes of a community, mathematics is part of what we call culture. From this point of view, I not only consider ethnomathematics as the area of study that reflects on the cultural roots of mathematical knowledge, but also as the set of quantitative and spatial relationships, generated in the heart of the cultural community, which compose, frequently, what has been theorized as mathematics.

This ethnomathematical perspective is related to the understanding of the meaning of culture, which has passed by innumerous interpretations over the last century, a plot of signs with which people mean objects, happenings, situations and other people around them – and, each individual possessing the code moves easily in the universe of his/her culture, acts on the certainty of having his/her behavior confirmed by the group (Silva, 1993, p.28). In this sense, the relationships involved/built in both fields – cultural and mathematical are structured, naturally in different levels of epistemological complexity but, certainly, including mathematical objects in the (cultural) plot "of signs with which people mean the objects" (Silva, 1993, p.28).

From what was considered, facing the question "Ethnomathematics, how to interpret it?", it can be recognized as a line of mathematical education research that investigates the cultural roots of mathematical ideas, indispensable to a better understanding/meaning of one of the education areas – the mathematical education – and the assumptions generating its construction such as, contact with other areas of knowledge, cultural contact, values among others. Ethnomathematical studies, somehow, try to follow the path of anthropology, searching to identify (mathematical) problems from the "other one" knowledge, in the sense of understanding the knowledge of the "other one". In terms of school, for ethnomathematics the teacher would find meaning in the teaching and learning action if he/she takes as starting point the group's cultural patterns – not an easy task, because - as mentioned - it is as if he/she were searching to identify and interpret an amount of meaningful (ordained) symbols to "another one" different from him/her.

And, once again, what is this? What different manifestation can we meet in the "other group" that is both essential in terms of cognition and difficult to be incorporated? Based on my experience – with an ethnomathematical view on mathematical education and the teacher's education context – we frequently meet situations in which different inclinations and different choices manifest – all of them conditioned by cultural values. And, recognize certain aspect of things as a cultural value consists in taking it into account in the decision making, that is, in being inclined to use it as something to take into account upon choice and orientation we give to problem solutions, to ourselves and to others. The following situation reveals this condition well:

The indigenous teacher Maximino, of Guarani-Kayowa ethnicity, when asked by participants of the Studies Research Group in Ethnomathematics-GEPEm-FEUSP about the nature of arithmetic operations to the Guarani-Kayowa, reveals very well the value put into counting, which by itself is already differentiated from the universal way. Maximino Kayowa explains:

> “[...] a family invites another family to have lunch at their home... and when the wife asks the husband “How many people are coming?”, he can answer like this: “they are four and four means the father, the mother, two sons (counting like one) and two daughters (counting like one). “The same sex they are one”. Maximino continues: “maybe the husband answers three, what means the father, the mother and four children, if they are the same blood they are one. (FEUSP, 11 de abril de 2002)

From the discussion so far, we can recognize that whereas ethnomathematical view searches for detachment necessary not to explain all perceived relations linked to academic/universalized mathematics, maybe we always have to question ourselves - when the discussion context is ethnomathematics - about the existing relation between my knowledge and values and those of the others and about which relations should be established or are established between collective and individual knowledge and values.

This way, facing the question “Ethnomathematics, how to interpret it?”, the answers should have more in view the question “How to interpret the(ethno)mathematics to be worked at school?” - it would be more valuable to mathematical learning that they did not come specially from discussions of mathematics philosophy, but of mathematical education. That is, we do not want to interpret it as a set of disciplines and/or a scientific activity, but as a social-cultural product, opening the way to talk about cultural diversity, difference, interculturality.

FOCUS OF INTEREST: TEACHER EDUCATION

As indicated on the title, one of the focuses of interest in this work is related to teacher education from the ethnomathematics perspective. And from this point of view we have tried to call the attention of educators to the fact that in this immense volume of investigations about teacher education – most of them already concerned not only with the student’s intellectual needs but also with education’s social functions - “the student has not been out of the teacher education proposals, but he/she has not been in either”. (Domite, 2000, p 44).

In some way, we must encourage the teachers to want to understand more and more, and deeper, the school where they work and the students they receive, that is, generate bigger availability to formulate questions “school, who are you?”, as well as “who are our teachers?” and “who are our students?”. Being able to recognize beforehand who is part of the group, what they know and how they know, can make the teacher notice the potential of taking into account student’s culture in the pedagogical doing process.

From the mentioned perspective - a perspective closer to the wishes of ethnomathematical studies - some initiates in the teacher education have been precious, specially those ones inspired by Freire’s and Schön’s original ideas.

Freire brings and opens to (school) education the proposal of situating the educational action in the student's culture. To Freire, the consideration and respect to the student's "first knowledge" and "the culture that each one brings inside him/herself are the goals of a teacher who sees education under the liberating point of view" (Freire, 1967), that is, recognize it as means to generate a structural change in an oppressive society – although Freire states that (school) education does not reach that objective immediately and, even less, alone. Schön, in turn, brought to educators the assumption that it is from the teacher's reflection about his/her own practice that transformations can happen, suggesting to the educator to take the teacher to ways of reflection operation in action and of reflection about action (Schön, 1987). According to the author, it is from reflection about our own practice that transformations can happen.

From what was considered and trying to understand what can be done on the interface of teacher education and ethnomathematics, I came closer and closer to Paulo Freire's studies, choosing him as basic theorist to answer to my questionings. My highest intent is to propitiate a transformation of the relation we have, as teachers, with ignorance about **who our students are**, **what they know and how they know about them.**

With these concerns and since one of the basic presuppositions of ethnomathematics is in focusing/identifying/legitimizing the quantitative and spatial relationships based on the knowledge of the "other one", I have been developing research in the scope of teacher education with these objectives: a) recognizing how much teachers are aware of the movement/literature on teacher education in the mathematics educational field; b) searching an understanding of the conceptions of the teachers on education and culture and/or how much they are available/able to appreciate and legitimize the first/previous knowledge of the students; c) problematizing issues/processes that emerge in the social reality of a classroom, in which the knowledge of the student becomes (by force of circumstances) the axis of the teacher's concern; d) understanding the possible connections between Ethnomathematics and the movement of Teacher Education and, e) to better understand what the ethnomathematics educators would like to see in the movement of teacher education.

This research tried to collect information on the basis of two proposals. The first (Part I) was constituted of interviewing mathematics in-service teachers and postgraduates, supported by questions about teacher education. The second proposal (part II), and here is the main focus of this research, was to request the manifestation of the investigated individuals, based on the confrontation with a situation that is distinct from those of regular standards. It is worth highlighting here that this second part of the research was born from my consideration, beforehand, about the lack of deepness of the answers to be given to the third question of part I - that is, I suspected that the set of ideas that are there had not had an impact on their professional lives' histories. The prepared script is as follows:

Part I - Interview with mathematics in service teachers and postgraduates **from these questions:** 1) What have you heard about Teacher Education?; 2) Write/explicit some ideas, challenges or suggestions you have seen or heard related to teacher education and, 3) In your opinion, what are the main features we, teachers, have to have/develop when we decide to put in the center of the process of teaching-learning our student's previous feelings, attitudes, opinions, culture and knowledge?

Part II - How would you go forward/continue the lessons like these that were presented to teacher Mário and teacher Janaína (two cases coming out in two different public school classrooms in São Paulo city).

Situation 1 - teacher Mário begins, in one of his 5th grades, a conversation with his students on the calculating division, by asking: Teacher: How do you calculate 125 divided by 8? José (student), who sells bubblegum at the traffic light downtown, starts speaking: José: We are more or less 10 "guys", almost all day long, some boys and some girls. Then, we divide like this: more for the girls, who are more responsible than the boys, more for the taller ones than the smaller ones". Teacher: Give us an example, José. For example, how was the division yesterday or the day before? José: Ah! Like this ... there were 4 girls, one of which is small; 6 were tall boys and 2 more or less small. Then we were 12 and the gums were 60. Then, it was given half and half, a little more for the girls. The small girl ended up with 3 and the others with 6 or 7, I do not remember well... The boys... **Now you have to put yourself**	**Situation 2 -** teacher Janaina begins, in one of her Adult Education Course classrooms, a conversation with the students on the percent calculation: Teacher: What do you know about percentage? How do you do the calculation of a percentage? Luiz: Even today I needed to make a calculation... 35% of 195 and I did like this... 19 + 19 + 19 and then plus 9,5. It's 30 plus 27 ... more or less 10. Teacher: How did you get 19? Tell us a little about your way of calculating. Luiz: Ah! I do not know why I did it like this... every time that percentage appears I divide by ten because somebody taught me this way, and I add the times that it appears... like this... 30% I add three times, 40% I add four times. Teacher: And how did you get 9,5? Tell me the way you thought to do this. Luiz:I know that one has to divide by two when it is 25% or 35% or 45%, but I do not know why I do this **Now you have to put yourself**

in teacher Mário's position and continue the class...	**in teacher Janaina's position and continue the class...**

So far, questions related to Part II of the research were examined with the analysis of 48 answers, all in-service mathematics teachers. Among these 48 teachers, 28 are public school teachers with more than 10 years of experience, 11 with less than 10 years (3 of them also in private schools) and the last 9 are also postgraduates.

In order to analyze the answers, we took into account two types of constraints: first, the teacher`s perceptions (the one who should continue the "started class") on the situational processes of teacher-students and students-students interactions, and second, how and to what extent the content of such situations have been perceived by the subject teachers and what he/she does in order to take it into account the scholar instructional movement and/or do not distort it. Then, the passages collected under such conditions - grouped and regrouped by similarities of the teachers` attitudes - lead to the configuration of three thematic axis.

The first axis comes from the teacher's desire to transform the real situation into an exercise or mathematical problem - looking towards the teaching of a mathematical content. One of the teachers reacted like this: "Very interesting José, very interesting! But let us think about the division in equal parts...". The second reveals the reflective/interrogative teacher getting into a problematization process trying to recognize what happens in the confrontation between universalized mathematical knowledge and the contradictions that emerge from reality. One representative of this group/axis reaction: "If we take into account the contextualized knowledge of students like José and Luiz, are we contributing to a more meaningful learning of mathematics?". The third axe represents the teacher's beliefs, values, collectivity and power relationships, somehow related to the pedagogical practice. One of the teachers: "This is a terrible political issue... our students selling bubblegum at the traffic light... what are those children doing there?"

And, what have we learned from this ongoing research? How does this add to our understanding of the teachers' attitude issue of taking into account the "first knowledge" of students in a classroom? I would say that my first comprehension of this process – from this research on - is directly focused on a process developed by the teacher – his/her **listening –** the development of his/her availability to listen to the student. As Freire points out, the teacher should develop an "opening to the speech of the other, to the gesture of the other, to the differences of the other, and (...) this does not mean, evidently, that listening requires from those who really listen his/her reduction to what the other speaks (...) this would not be listening, but self-effacement". (Freire, 1996, p. 135).

Listening to the students, according to Freire, is really, speaking "with" them, while simply speaking "to" them would be a way of not listening to them. And here is a great challenge: our **listening** as (mathematics) teachers. Usually, we, as teachers, educated by the so called traditional school, are not prepared to listen – and then, to speak "with" the other – once our teachers' pedagogical practices were almost always related to "**explanations"** or presentation of **questions already formulated by them.** In fact, as Freire & Faundez state, "the educator, in a general way, already brings the answer when nobody has asked him/her anything"! (Freire & Faundez, 1986, p.53).

The observation of these attitudes by the teachers – taking onto account the student's "first knowledge" and "listening" – became one of the aspects that we try to insert/explore/include in our investigations on (ethno) mathematical education and teacher education. In both cases, I believe I can declare that a lot of mathematics learning difficulties occur due to the lack of emotional and intellectual involvement of students in the preparation of the problems they solve. On the one hand, the teacher seems not to consider that the student, adult or child, has a conception of one aspect of knowledge that resulted from his/her learning history – first knowledge - and, it is this knowledge, as it is, that will do the filtering between him/her and new knowledge. On the other hand, the issues that instigate the mathematical thinking action and that can lead to mathematical problems posing is not shown to the students. And then, if everything is defined and ready, how can what the students have to say about mathematics be important? Is it worth listening to them?

The second attitude to be developed in us, mathematics educators, when the purpose is to train teachers in order to take into account the student's "first knowledge" in a classroom, is to rethink with and among teachers **another notion of prerequisite** – like that knowledge that serve as filter/support to the learning of (new) ideas in mathematics.

In general, the idea of **prerequisite** traditionally employed in mathematical education is like a basis of logical order, indicated by the mathematician, as a *necessary* fact to the knowledge of the item to be studied. **Prerequisite** in this new vision refers to teacher's efforts in understanding how the student understands this or that (mathematical) idea, how he/she makes meaningful relations around a mathematical idea/content – how such mathematical knowledge is to the student... how he uses it, manages.

If we return, for example, to teacher Janaina's classroom and really take as a starting point the way the adult student Luiz uses the idea "every time that percentage appears I divide by ten and I add the times that it appears... like this... 30% I add three times, 40% I add four times..." (because somebody taught me so), we would have something valuable to start a meaningful process in terms of percentage. What would be necessary to be awakened to the teacher in terms of attitude? Naturally, even if it is a poorly elaborated attitude for us mathematics teachers, we should observe that from the information - acquired in social/familiar environment- what Luiz brings bout the subject is full of memory, symbols and reasoning.

Luiz's intervention elucidate, somehow, the new meaning and role of what we have called pre-requisite, especially when we are in the field of action of mathematics teaching in primary school. The meaning is in showing/discussing the value of taking the knowledge the student already brings/manages/uses for the construction of mathematical relations traditionally expected by conventional school education.

From our point of view, the understanding of a new vision, by teachers, of **pre-requisite** – as what the student knows how to use, whatever logic/rationality and terms of this use – should be one of the aspects to be especially inserted/explored/included in the Ethnomathematical and Teacher Education investigations.

CONCLUSION

This article's intent was to look back at problems and solutions of the middle way between Teacher Education and Ethnomathematics, as well as put some problems for us to reflect about this possible interaction. Trying to reflect on such interface, I realized that we need to be alert to three important points – or three changing proposals.

The first is the fact that there are innumerous problem-situations and solutions from the non school context - which results in transit by different areas of knowledge and are validated/shared by experience – which the mathematics we learn in the school context does not allow us to notice, maybe due to our tradition of always valuing one kind of mathematics – the mathematics built at the academy, in general, free of contexts.

The second, directly linked to the first, when we notice such problem-situations as rich situations in terms of mathematics teaching, the construction of a *bridge* between this set of (mathematical) ideas and that one systematized by the school is in danger due to inter-relations between thinking and emotion, thinking and traditions, thinking and religion, thinking and the myths that lead to unexpected situations because of the tendency of language to take different meanings. In fact, for this *bridge* to happen it is, a lot of times, necessary a translation between the speeches by careful attention to meanings, to representations and, a lot of times, to linguistic elements.

Third, if our objective with a research of this kind is to develop a teacher education curriculum in which we problematize questions/processes that emerge from the student's socio-cultural reality, we recognize that the foundation for a research and/or a mathematics teacher education to act in this direction requires an incursion in literature focused not only on Mathematics, like in Anthropology, Sociology, History, Psychology and, specially, in the research production about Teacher Education in the educational field.

And finally it was possible to notice that preparing mathematics teachers to the development of the student's concerns and "first knowledge" is not incompatible to orienting them to teach mathematics – on the contrary, this can be one of the aspects to be developed in the teaching process.

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MATHEMATICS *FOR LIFE* OR MATHEMATICS *OF YOUR LIFE*: A STUDY OF THE RVCC PROCESS OF PORTUGAL

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This text aims to present some results of a post-doctoral project, which sought to understand pedagogical practices in contexts of adult education in Portugal, in the area of Mathematics. The study was based on a theoretical framework that reconciles literature from the fields of Adult Education, Ethnomathematics and Experiential Learning. A qualitative and multi-sited research was conducted, following the dynamics of the adult education centers, the professionals who work with adults and particularly the formers of an area titled "Mathematics for Life", which was part of the process of Recognition, Validation and Certification of Competences. The practice of the RVCC process proved to be challenging and complex, both to professionals and to adults involved.

INTRODUCTION

Throughout my journey as a researcher in the areas of Adult Education and Ethnomathematics, I have focused on some issues, such understanding the quantitative and spatial representations of a group of low educated adults (Fantinato, 2004), or how the ethnomathematical approach can contribute to teacher education. Among other subjects, I have investigated the extent to which contact with cultural diversity, which is characteristic of adult education, stimulates openness to unconventional forms of mathematical reasoning among teachers. I have studied as well, how adult Mathematics teachers create spaces of dialogue between different types of mathematical knowledge in the classroom (Fantinato, 2008). These issues were present during the achievement of my postdoctoral research[1], which sought to understand principles and practices of adult education both in Brazil and in Portugal, and in particular to grasp Mathematics teaching practices and their forms of interaction with students´ diverse mathematical knowledge and their construction processes.

This text aims to present some results of the development of this project, during the stage carried out in Europe. I have chosen here to perform an analysis, from the ethnomathematical perspective, of the Portuguese process of recognition, validation and certification of competences - named RVCC process - focusing on its contradictions and on the challenges faced by professionals of "Mathematics for Life" (ML) area.

THEORETICAL FRAMEWORK

From the moment of its entry into the European Community in 1986, politics of education and training of adults in Portugal have come under the influence of the

Lisbon Strategy[2] and its values of "lifelong learning". One of the consequences of seeking integration into European models was the creation of the *Centros Novas Oportunidades*[3] (CNO), which from 2005 to May 2012, brought together the many activities related to adult education in Portugal, most of them located in school groups. The main activity performed in CNOs was the RVCC process, which was intended to assign a final school certification to adults.

The RVCC process is a practice on recognition of acquired experiential learning, and as such, according to Canário (2006), brings two ideas constituting its essential foundations: the notion that a person learns from the experience and the principle that one should not teach people what they already know. It assumes that "people are lifelong producers of their own knowledge, and that this knowledge driven from experiential learning processes, may be the subject of recognition, validation and certification[4]" (Cavaco, 2009a, p. 150). The RVCC process works on the principle of recognition of skills. This is a process that

> is based on a set of methodological assumptions [such as Balance of Powers, (auto) biographical approach] that allow demonstration of competencies previously acquired by adults throughout life, in formal, non-formal and informal contexts, which unfolds in the construction of a Reflective Portfolio of Learning oriented to a Key Competencies Referential. (ANQ, 2007, p.15)

A competence approach involves recognizing and valuing the knowledge acquired, especially in informal and non-formal contexts, reflecting the life learning of adults, differing, thus, from approaches that focus on the acquisition of subject content in formal learning contexts. These are life skills that enable adults to "understand and participate in knowledge society, mobilizing through them the knowing, the being and the know how to solve problems that the changing current world confronts them constantly" (ANEFA, 2002, p. 9). In this perspective, in the RVCC process competencies appear "as emerging from the action, giving them a finalized, contextual and contingent character" (Canário, 2006, p. 41).

Experiential Education, one of the theoretical orientations that guide the practice of recognition of experiential acquired learning in Portugal (Canário, 2006), considers experiential learning as "a local use knowledge, that the individual shares with the other members of the community to which he belongs [...] comprises the dimensions of knowledge, know-how and know being (Cavaco, 2002, p.39).

The research line of Ethnomathematics (D'Ambrosio, 2001) has also addressed the issue of knowledge constructed within contexts of life, more specifically, of mathematical knowledge built by different cultures, namely "how social groups are aware of their needs and on what terms they use their local math to address them" (Moreira, 2009, p. 66). Ethnomathematics:

> Has pioneered research and studies that seek to understand the different modes of youngsters and adults mathematical reasoning, that result from a cultural background

built predominantly in contexts of domestic and professional life, without excluding previous school experiences. (Fantinato & De Vargas, 2010, p. 37).

I consider that, despite some crossovers between this theoretical approach and Experiential education, Ethnomathematics can provide a differentiated analysis of some principles of the ML component of the RVCC process, as well as a new look at the (mathematical) experiential knowledge of adults to be certified.

RESEARCH CONTEXT AND METHODOLOGICAL PROCEDURES

Research carried out was of a qualitative nature, using the modified analytic induction approach (Bogdan & Biklen, 1994). The project was developed in several CNOs and therefore the investigation can be considered as multi-sited. I have carried out a descriptive study, with a concern for the meaning given by participants. Throughout the investigation I used different instruments for data collection: document analysis, participant observation and semi-structured interview. Analysis and data collection were developed alternately. Because this was a qualitative study, I considered my own transformations in the process as a way of defining the object of investigation.

In a first moment of investigation, I carried out an analysis of the official documents of the National Agency for Qualification of Portugal (ANQ), which provided theoretical and methodological policies and guidelines for adult education in this country, at the time research was conducted. Particularly, I studied the Referential of Key Competencies of Adult Education (ANEFA, 2002), for primary and secondary level, trying to locate where these documents addressed the area of Mathematics. Since I did not find in the document for the secondary level explicitly mathematics[5], I focused on the analysis of one of the four areas of Referential for basic level, the one titled "Mathematics for Life". This document is structured into three skill levels, called "B1, B2 and B3, taking as reference the correspondence with the cycles of the Primary School, although they can not be identified with them" (ANEFA, 2002, p.11), and serves as a parameter to the RVCC process.

Fieldwork itself happened from November 2011 to March 2012, when I scoured several CNOs located in the metropolitan area of Lisbon and nearby municipalities. I watched practices and conducted interviews with directors, coordinators, and especially *formers* of “Mathematics for Life", using *snowball sampling technique*, one respondent indicating another, which indicated another, and so on, until I had enough information. Four of the Formers of ML selected for interview were graduated in Mathematics, and one had his graduation degree in Engineering. These professionals had all previous experience as Mathematics school teachers, and by the time data was collected, two of them, besides working with adults, were also teaching Mathematics to regular students. However, despite their experience as classroom teachers, in carrying out professional practices of RVCC process with adults, they would identify themselves as *formers*. The different designations are consequence of different professional roles, while interacting with adults or with school children.

The interviews took place at informants´ workplace, at a time that was convenient for both the interviewer and interviewees. Later they were transcribed and analyzed, based on the categories that emerged from the process. The analysis gave priority to challenges and contradictions of daily experience with RVCC process by professionals responsible for the area of mathematics. Therefore, this paper focuses on the interview data with ML Formers.

UNEARTHING THE PROCESS STEPS

The dynamics of the RVCC process comprises several steps. Upon arriving to CNOs, adults are received by technician for diagnosis and referral, which evaluates their ability to perform the RVCC process and indicates the level that should be integrated. After the RVCC professional has evaluated the adults in an individual interview, they go to next stage, when their work will be oriented by Formers from four different areas, one at a time, among them, “Mathematics for Life”. The *decoding sessions* take place at this moment, whose main objective is:

> to try to show people what they have, or can speak in terms of subjects, to meet, matters which make, mathematically speaking [...] shopping, budget management, administrating a house, or a division of a house, and all that that implies. Or something that people can make at work, and then can write about. And then try to make them realize that with these situations, they can get validate competencies in the four units that compose the Referential (Mariana[6], ML former).

The initial work of the ML former is to show the adults what issues they might indicate in terms of their life experience and from a mathematical point of view, so they can later articulate in the written form their life story, where the situations described demonstrate competencies that can be validated according to the Referential "Mathematics for Life" (ANEFA, 2002). During the first decoding sessions, trainees have access to the Competencies Referential. One of the challenges of ML former is the adaptation of this official document text - written in formal language - into a form that is understandable to adults. Educators try to seek this challenge by elaborating tools for their work in the process. Mariana built a sheet entitled "What competencies from your daily life you can demonstrate?”, which features more accessible and contextualized questions, elaborated consonants concepts that are at the Referential. One of the questions in this form asks if the adult usually relate the number of installments of a home, a car or an appliance with all the loans and the interest rates one will pay, and asked to describe life experiences that demonstrate this competence. Leandro, another ML former, when starting the decoding sessions, usually passes a movie that shows journalists interviewing fair dealers about their daily mathematical knowledge. He said it was a film "that dismantles everything," and that helps adults perceive that his role as former is identical to the role of those journalists, who go identifying in fair dealers daily practices some mathematical contents: simple rule of three, series sequence, "contents that are taught here in the eleventh year" (Leandro).

The sessions of "Mathematics for Life" are developed in small groups of about ten people. Another former, Celina, believes that this group sharing favors the exchange of experiences, and so “they can catch a life example. One says one example, another remembers another, another remembers another, it generates a chain reaction” (Celina).

Throughout the sessions adults go on writing successive versions of their autobiography, which are evaluated by formers and returned to adults with their comments on the competencies achieved and on the ones which needs to be developed, until reaching a final version that proves the required competencies for validation in the area of ML. This written explanation and the making of a speech about the activities they perform day by day is one of the main difficulties of the RVCC process, especially for adults with little schooling, because "the ability to construct the discourse on action, namely the formalization of the action, is a skill that is developed and is perfected at school, which allows us to understand the difficulties that low educated adults have in this area " (Cavaco, 2009b, p. 763).

RECOGNITION, VALIDATION, CERTIFICATION: PROCESS ACTIONS AND CONTRADICTIONS

When they started working in CNOs, professionals involved in RVCC faced the challenge of developing a work that was entirely new to them and to which they were unprepared. Although some of our respondents have participated in 2008 in a formative action proposed by ANQ[7], where some principles of this new methodology were transmitted, this training above all represented the learning of new terminologies and new roles (Fantinato & Moreira, 2012).

Likewise, for the professional in the field of Mathematics, there is a change of term that corresponds to a change in function, and Leandro makes explicit in his words:

> In the area of RVCC I am a former. I am a mathematics teacher at school. Yes, because there are different terminologies. For example, in the RVCC one does not give lessons, there are sessions, it is not the teacher, it is the former, more from the perspective of guiding. And in the diurnal course it is the usual methodology: classes (...) Methodology is completely different. During diurnal course student learns, in nocturnal course, especially in RVCC, student already brings the knowledge, he will show the competencies he has. (Leandro)

Leandro's speech indicated that there was a difference between the teacher´s work, who follows a school model - plans, teaches, evaluates - and the ML former, who "gives examples of things that adults will be able to look at their daily life" (João, director of CNO). Former Fernanda also declared not being prepared for the process practice:

> Usually I give lessons, students then show what they learned of what I taught. When I got here, they've brought their knowledge and I had to look at what they knew, what they already knew [...] (Fernanda)

Leaving therefore the acting as teachers to assume the role of formers, these professionals, to "ensure adequate performance must develop specific skills, rather distinct than those requested when they were exercising their functions as regular teachers" (Cavaco, 2009b, p. 700-701).

Despite not knowing Ethnomathematics, it is observed that ML formers, while developing the RVCC process, seem to be stimulated to look at the diversity of "ways of knowing and doing of the various cultures and an acknowledgment of how and why groups of individuals [...] perform their practices of a mathematical nature, such as counting, measuring, comparing, classifying" (D'Ambrosio, 2009, p. 19). When talking about their work, these professionals mentioned the everyday mathematics and valued adults own knowledge, built in their life experiences. They recognize, for example, that "the adult who comes from the rural area has very different apprenticeships than those learned by an adult that comes from the urban environment" (João).

However, although RVCC practice promotes "readiness for dialogue" (Freire, 1997) of the teacher to diverse mathematical knowledge, it carries an intrinsic contradiction. Recognition of the diversity of adult knowledge does not imply its validation, since it follows a single referential as standard. The formers, even recognizing mathematical competences of adults, could only validate those that were listed in the Referential. As says João: "We can only validate the competencies that are on the Referential, what you did in your life that can represent that competency. This is hard".

The story presented by former Rodrigo, describing his conversation with a man who raised birds, and who at first said he did not recognize the presence of Mathematics in his life history, shows some aspects of the complexity of the process:

> "So you but if you raise birds, you do not have to buy rations, food for birds, what quantities do you buy? How much do you spend? Because if you make a table with such things, how much you spend per bird, quantities, and etc. All of this has to do with Math" [...] He understood. Then went home, and came up with things done. But what he wrote, that did not translate any special competence. However, after talking with him, I realized that the man used to build his your own cages of birds. He used to make the place where the birds were, then had to take measures, had to buy wood, had to go in search of prices, had to buy the network to put around. I told him: "This is where the competence is, when you take these measures, and do all those things, the competencies are there." That was when he realized what he had to do. So he came up to me with a concrete example of how

> he had done there with the cages backyard. Because he had to buy the wood costing I don't know how much, took those measures, and so and so. So I could validate those competencies. Because before, I could not do it, even though he had such abilities […]. (Rodrigo)

Rodrigo´s report exemplifies the role of ML former, as a mediator between previous acquired knowledge and the competences listed in the Referential. But how to accomplish this if the Referential presents highly technical language, next to school mathematics? For example, how can a low educated adult identify in which situations of his life he can infer "laws of formation of numeric or geometric sequences, using mathematical symbols, namely designating expressions[8]"?

Another contradiction of RVCC process is the fact that it presupposes the use of the written mode of presenting the autobiography, which contrasts with the presence of mental calculation skills, from the so-called "oral mathematics" (Carraher, Carraher & Schliemann, 1988), common among low educated adults. This represents another challenge for ML formers. As Mariana says: "There is a phase where they say: 'Ah, but I do it in my head,' and I say, 'Yes, but what you do by heart, write, place, show, that it can be a proof' ". As for former Rodrigo, mental calculation usually reveals itself along with other calculations, and he realizes that "people who have this skill get the result sooner than others." However, despite recognizing this ability among adults, which indicates "some facility in mathematics," he warns: "I can not validate only because the person knows how to do good mental calculation."

The practices of recognition, validation and certification of acquired experiential learning among adults, involve therefore a great complexity, related to the act of valuing knowledge resulting from the action and to the search for establishments of links between them and theoretical knowledge. I agree with Cavaco, when she says that "theoretical knowledge and knowledge deriving from the action have different natures and irreducible differences, making it difficult and artificial any process that intends to merge or overlap them" (Cavaco, 2009a, p. 150).

Undoubtedly, "assessment is another element that contributes to this complexity in recognition devices of acquired experiential knowledge" (Cavaco, 2009b, p.766). Throughout RVCC process, adults are evaluated for their successive written autobiography. According to coordinator Leticia, this is a "dramatic" time, where professionals need to be careful how to address adults, because they are confronted with what they were not able, and can drop the process: "they leave, because they do not feel comfortable with someone who has more skills, they do not feel good about themselves. Because they were not able" (Leticia).

The adult going through the process receives from the CNO team several suggestions for improvement, until he reaches an autobiographical text that presents the competences needed to be validated and, thereby, can obtain the corresponding school certification. If needed, adults are referred for additional training, where formers work more individually with each adult in their needs. Even so, at the end of all these stages school certification of the expected level is not always obtained, and there is the alternative of the adult receiving a partial certification, that is, only in some areas of RVCC.

MATHEMATICS *FOR LIFE* OR MATHEMATICS *OF YOUR LIFE*?

The proposal of "Mathematics for Life" Referential (ANEFA, 2002), used as a parameter to validate the mathematical competencies of adults who perform the RVCC process, presents some seemingly contradictory characteristics, which are a reflection of the inherent contradiction of validating knowledge built into everyday life by checking their equivalence to school mathematics knowledge. To what extent (mathematical) knowledge built in the diverse contexts of everyday life can be translated into a mathematical language seen as universal and unique? A discussion on bridging the gap between local knowledge and school knowledge, that is, transfer of knowledge from one context to another, has been in effect one of the critical points of the debate on the educational implications of the ethnomathematical approach (Pais, 2011). Power relations and values assigned to the different types of mathematical knowledge are present in this process of "translation".

The experience of the RVCC process is not immune to these contradictions. For adults with little education, to recognize that they built apprenticeships in contexts of life and that this learning can have be valued as a school certification, is certainly a challenge, which becomes even greater in the case of a socially valued area such as Mathematics. The contradiction of guidelines from RVCC process appears in the name of the area "Mathematics for Life". Is this a mathematics applied to everyday life, or recognition of the present mathematics in daily practices? Former Fernanda explains the little adequacy of this name, mentioning her professional experience with the process: "It seems that we are giving them the math and then people apply it. But that's not what we do here. Life, that already has mathematics, is that we seek".

NOTES

[1] Postdoctoral held in two phases, the first in Brazil between March and September 2011, in the Faculty of Education, Universidade de São Paulo (supervisor Maria do Carmo Domite), and the second in Portugal, between October 2011 and March 2012, at the Institute of Education, Universidade de Lisboa (supervisors João Pedro da Ponte and Darlinda Moreira), with a grant from Fundação para a Ciência e a Tecnologia (FCT).

[2] The Lisbon European Council (2000) set policies to boost the economy, end unemployment, stimulate competitiveness and reduce the differences in schooling and technology inclusion among adult populations of the member states of European Union.

[3] After May 2012 the *Centros Novas Oportunidades* (New Opportunities Centers) came to be known as *Centros para a Qualificação e o Ensino Profissional* (Centers for Qualification and Vocational Education), and lost many of its functions of adult education and training.

[4] Every translation from Portuguese in this paper will be our own..

[5] This Referential covers three interdisciplinary areas; content area of mathematics can be found, implicitly, in the area titled Sociedade, Ciência e Tecnologia (Society, Science and Technology).

[6] The names adopted herein are fictitious, in order to protect the identity of informants.

[7] Training held in the Algarve, which lasted for three weekends.

[8] Level B3 evidence criterion, from "Mathematics for Life" Referential (ANEFA, 2002, p. 24).

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THE EMERGENCE OF AGENCY IN A MATHEMATICS CLASS WITH ROBOTS

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In this paper we will discuss and analyse students' participation in a mathematics class when using robots to think about the mathematical concepts. Our aim is to highlight the emergence of students' agency in the interplay with robots and discuss the role of agency in participation and in the learning of mathematics.

INTRODUCTION

Learning mathematics can be understood as participation in social practices (Lave e Wenger, 1991), where people get engaged in solving problems and making sense, use mathematical representations, concepts and methods (Boaler & Greeno, 2000). As such, analyzing students' participation in mathematics classes becomes important when we want to understand and discuss learning as a phenomenon emerging from participation in social practices. In this paper, we are not interested only with whether or not students participate but how they are given opportunities to participate and how they take advantage of them.

Both Situated Learning Theories and Critical Mathematics Education discuss the learning process as something in which the person must act in order to learn. Alrø and Skovsmose (2002), for instance, explore the idea of learning-as-action in which action can be associated with terms like aim, decision, plan, motive, purpose and intention. That is, "action presupposes both an involvement of the person and an openness of the situation" (p.43). On the other hand, school mathematics classes which allow students to engage in practices of negotiation and interpretation, using physical and discursive tools and resources, provide learning scenarios in which students participate by adapting to the constraints and agreements of it.

Through the project DROIDE II[1] – we have created learning scenarios in which robots are physical artefacts with which students think during school mathematics practices. Our aim is to understand how students produce meanings and develop learning of topics and mathematical concepts when robots are mediators' artefacts.

In this paper we will analyse students' participation in mathematics classes, amidst the scenarios created where robots had a central role, discussing the role of agency in the learning of mathematics.

ON AGENCY

> An individual's agency refers to the way in which he or she acts, or refrains from acting, and the way in which her or his action contributes to the joint action of the group in which he or she is participating. (Gresalfi, Martin, Hand & Greeno, 2009, p.53).

According to our understanding, individual agency relates dialectically with structure, and it is a dynamic competence of human beings to act independently and to make choices (Andersson & Nóren, 2011). Agency is not born with the person.

Giddens (1984) argues that agency is the capability to make the difference and that it relates exclusively to humans. Artefacts are only "allocative resources that influence social systems only when incorporated in processes of structuration." (p.33). Latour (1991) offers a concept of agency that is not restricted to human actors. He coined de term *actant* in order to allow the association of agency with humans. Doing this, he opens the space to conceive agency as something that can be allocated not only to humans. Indeed, if agency is the capability to make the difference then the machines also can exhibit forms of agency. In actor network theory, agency is not restricted to humans but is also attributed to technologies (Rose & Jones, 2005).

Material Agency

Pickering (1995) made a distinction between human agency and material agency. Humans are active and intentional beings. Human agency has an intentional and social structure and intentionality is what makes the difference between human and material agency. Physical artefacts are essential for the modern world. People manoeuvring in the field of material agency "capture, seduce, download, recruit, enrol, or materialize that agency, taming and domesticating it, putting it at [their] service, often in the accomplishment of tasks" (Pickering, 1995, p.6). Human agency is itself emergently reconfigured in its engagement with material agency.

> There is no way that human and material agency can be disentangled. Or else, while agency and intentionality may not be properties of things, they are not properties of humans either: they are the properties of material engagement. (Malafouris, 2008, p. 22).

Pickering (1995) made a distinction, when he developed the terms conceptual and disciplinary agency, in his socio-historical analysis of a case of research in mathematics. Mathematicians "exercise conceptual agency when they engage in decision making, exploration, and strategizing" (p. 53).When they decide to use an established method, agency is turned over to the discipline. According to Pickering (1995) what generally happens in physics and mathematics is "a dance of agency" that combines the conceptual agency with the disciplinary agency, or the conceptual agency with the material agency.

Critical Agency

Alrø and Skovsmose (2002) presented the idea of critical learning of mathematics. This idea entangles the notion of agency. Although the word "agency" was absent in

Skovsmose's work until 2012, it has been implicitly present in expressions such as intentionality, action, reflection and choice used in his work (Anderson & Norén (2011).

Greer and Skovsmose (2012) think of *Critique* in terms of "critical agency" signifying that critique refers to both reflection and action. The concept of critical agency entangles the dialectic between reflection and action. And the genesis of critical agency lies on the ability and dispositions to imagine that things can be different (Skovsmose & Greer, 2012).

When we want to discuss the critical learning of mathematics, action appears as a fundamental notion and can be associated with expressions such as aim, decision, motive, and intention. This presupposes an engagement of the person. On this conception of learning, intention and action are strongly connected as well as in the notion of human agency.

Alrø and Skovsmose (2002), with the aim of discuss in dept the notion of critical learning of mathematics, also introduced expressions such as intention, reflection and critique. They tried to clarify the dialogic basis of critical learning (of mathematics) in terms of intention and reflection: "While intention refers to the involvement of persons, reflection refers to considerations carried out by persons" (p.157). To participate in a dialogue, in a way that promotes learning, presupposes engagement and this engagement can be characterized in terms of the notion of intention:

> Intentions-in-learning are essential for the students' ownership of the learning process. […] In order for the dialogue to continue, it is important that the intentions of the participants are continually modulated and adjusted to each other. (p. 157).

But students in the classroom have others intentions than those related with the learning of mathematics. They may have intentions of being noticed by the teacher or to sit next to their best friend, and the like. These are the underground intentions. But these intentions are intentions too and consequently they are part of the acts, but not of the learning acts (of the official classroom activity).

There are, according to Alrø and Skovsmose (2002), the resources of intentions. Intentions are formed based on experience, impressions, prejudices, preferences, etc. These resources of intentions are called dispositions. The disposition of the person is the raw material for their intentions and is seen as individual but also as a feature of the person's culture. When we think about learning-as-action, the notion of dispositions is used to describe the source of motives that students have to engage in the learning process.

> "The student's dispositions for learning are thus indicative of the factual possibilities that the student holds for the school system and of the student's interpretation of these possibilities. Correspondingly, the student's dispositions make up a heavily structured

> framing condition for intentions-in-learning. The students expose such intentions in patterns and according to their notions about learning and going to school." (p. 160).

Peoples' intentions reflect, to a large extent, their foreground or the foreground of a group of people to which they belong. Foreground can be understood as the way people interpret their possibilities given a certain political, cultural and economic context and their social position in it. Thus, the foreground refers to the person perspective (Skovsmose, 2005). As a result, the foreground of people should be considered when interpreting their actions. The background must also be taken into account. Both influence what a person might want as well as person's possible actions. Both represent resources of intentions (including underground intentions).

METHODOLOGY

In this article we adopt a qualitative mode of research, because our aim is to develop an understanding of human systems. In this case the system composed by a technology-using teacher and his or her students and classroom (Savenye & Robinson, 2004).

To use Situated Learning Theories as a theoretical foundation implies some methodological assumptions such as assuming that investigating is to participate in a wide range of practices in which the investigation occurs (Matos & Santos, 2008). That was the positioning assumed by the researcher involved in the data collection. As a participant in the research, the researcher also learned. So, the participant observation was a central strategy and acquired the status of data collection methodology.

The data collection was made in two months of the school years 2010-2011. We chose to work with two classes of 7th grade students (ages between 13 and 15 years old) studying functions. There was an initial session, at Droide Lab in University of Madeira, where students had their first contact with the robots. A video cam was used, focused on a group. Four 90 minute classes were recorded (also with a video cam focusing in a group).

The analysis was made based on the video transcriptions and on the field notes taken by the researcher and teachers involved in data collection. The units of analysis included person, activity and the contexts where the activity took place.

We have been trying to find patterns of interaction, among students as well as among students and teachers, using Greeno's (2011) ideas about students' interactions because it allows us to explain students' participation in school mathematics' practices and to make visible the positioning students assume concerning agency and accountability (Fernandes, 2012). But we have also been trying to understand students' intentions in order to discuss critical agency. On this paper the findings will focus on a particular case study of a student that will be referred as 'He'.

A SCHOOL MATHEMATICAL PRACTICE WITH ROBOTS

During the initial session, students went to University of Madeira, to the DROIDE Laboratory, to assemble and program robots. When pupils assemble the robots, particularly when they 'dress it', they put so much of their living experiences and of themselves on the robots. There is a personification of the robot.

On the following day, back to school, they worked on a worksheet the 'notion of a function'. The aim was for students to work with robots, oriented by the questions of the worksheet, and understand, learn and define the concept of function. It took two 90 minute classes. The worksheet had a closed and very scholar structure. The innovation was the inclusion of the robots to think about the mathematical concepts involved. Each group of students received a worksheet and, even before robots were distributed, the teacher asked them to read attentively the issues on the proposal.

The task consists in analysing two graphics depicting two robot trips. Firstly, students had to make a description of the robot trip having the starting point as a reference. The second question was about the robot`s programming in order to realize the trips, if possible. The graphics presented were the following:

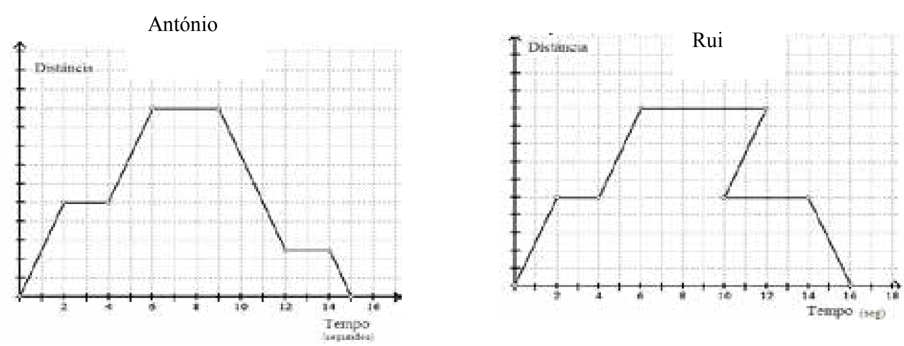

Figure1 – Graphics presented on the worksheet

The school mathematics practice analyzed can be characterized by the resolution of mathematical questions in groups, in which students had to discuss every task, to describe the processes that leaded them to results and finally students' presentation of the results and conclusions to the rest of the class. The wider group discussion was mediated by the teacher.

'He' was one of the 10 boys that had failed the preceding year and during this school year only had a marginal participation in mathematics classes. Since the moment he began to work with the robots, 'He'`s posture, in mathematics classes, changed. 'He' was the one handling the robot in the group, programming it and checking the programming results. It seems that robots made 'He' able of imagining that things can be different.

'He'`s group read the graphic concerning António`s trip (on the left side) with few hesitations. After analysing António`s graphics and programming the robot to make that trip, experimenting it on the floor and verifying if it is well done, they came back to the desktop and asked teacher's help.

He: In the second graphic we don`t really have to do anything, right?

Teacher: Why do you say that? What do you mean by "don't have to do anything"?

He: We already analyzed Rui's graphic and we can't program it.

Teacher: And why can't you?

He: We can't because there's no command that allows us to make the robot go back in time.

Teacher: But where in the graphic do you see that the robot has to go back in time?

He: Right here teacher (He pointed the graphic to the 12s moment), at 12 second the robot was at a distance 10, but also at a distance of five, because the robot went back and time does not go back. It can't be at two places at the same time. We can't program it because it isn't possible.

'He' was very much convinced that this programming wasn't possible. Even so, he couldn't convince his colleagues that were not, at the time, able to see his point. After discussing his point of view with the teacher (may be with the intention of showing her what he can do with mathematics now) he left his colleagues to proceed with the task of programming the second trip even knowing it wasn't possible, as he pulled aside and started writing. After some time, the teacher went back to the group and asked if they already had reached to a conclusion. One of students in the group replied:

Pe: Yes we did. We can't program it. We only made it until here (pointing on the Robotics Invention System programming interface, to the path until the 12 second).

The inclusion of the robots motivated 'He', opened a space of factual possibilities that 'He' held and made him committed to the resolution of the working proposal. He convinced the teacher but not his colleagues. Probably due to the way other students saw 'He' in terms of mathematical knowledge. He was a student with marginal participation and maybe because of that his mathematical explanation wasn't accepted by the group. It was not supposed that 'He' was accountable to the solution of mathematical question due to his trajectory in mathematics classes along all the school year until robots arrive.

'He''s questions to the teacher were very useful for two reasons: (i) in order to include the teacher in the accountability system, as the person who could convince the others students of the group, once he wasn't been able to convince them (ii) in order 'He' shows to the teacher what he was mathematically capable of doing (accountable for). These are 'He''s underground intentions. After solving every other question of the worksheet, which included writing the condition needed to allow for a correspondence to be a function, students had to comment on the following sentence "The correspondence presented by António is a function. Rui's correspondence isn't a function"

'He' again asked the teacher a question, for what he already seemed to have the answer, showing once again what they had been able to achieve making himself accountable to the idea.

He: Teacher, can we say that Rui's graphic isn't a function because there is one single time corresponding to two distances?

Teacher: And that's what can't happen for a trip to be possible?

He: Yes it is. For a trip to be possible, it can't be at the same time at two different places. Rui's robot at 10s is at the distance of 5 and 10.

'He' was the 'motor' of this group for the 'good' resolution of the mathematical question proposed, displaying his conceptual agency, that was emergently reconfigured in its engagement with material agency. Using the robots by which he showed great interest since the first session, seemed to be the starting handle to operate the change on 'He' participation on mathematics class. In this episode we can see some reflections of 'He', probably provoked by the introduction of the robots. He was able to explain why the correspondence was not a function in terms of the robots' functioning "[the robot] can't be at two places at the same time". The robot, associated to the notion of function, was part of the shared repertoire of this class. The students used that sentence every time they have to justify that a correspondence is a function and after they 'translated it' to the situation they had to solve.

Two classes after, students worked on the notion of "proportionality[2] as a function". The first question of the task was to compare the speed of two robots and to discuss if time and distance were in proportion. They have to program the robot to move forward for 1second and then measure the distance covered by the robot; and then to do the same thing for 3 and 6 second. After that they have to calculate the quotient between distance and time and then to conclude about proportion. The group of 'He' has done for 1 second, and after doing for 3 seconds, 6 seconds, twice, 'He' said:

He.: It cannot be. The robot moved again 110 cm. I'll ask the teacher.

Fi.: I can't understand why you say it's not correct. We have already measured twice and it gives the same value and not 138cm as you want.

He.: You only have to do calculations. Do it!

Fi.: What calculations?

He.: 46 times 3…

Fi.: is 138.

He.: It is because of that that I say that it should be 138.

Pe.: Ah! Now I understand… we have to multiply the first value by the second

He.: Exactly. If in one second 'Tank' moved 46cm, times 3 is 138 e for 6 is 276… and this is not what we obtained…measuring.

In this episode we can see the difference of posture from the group colleagues to 'He'. 'He' was accountable to and accountable for by the colleagues. The different participation of 'He' on mathematics class gave him mathematical authority. Now colleagues listen and ask him to explain what he is saying.

[After that, teacher arrived to the group, listened all the students' complains about what is not going well, and said:

Teacher: Assuming that you measured correctly, try to think about reasons to justify the difference between what should be and what it really is.

In the students' discussion about the reasons for that, emerged justifications such as: the ground is not completely plan, robots' batteries had spent, braking time is different for different robots, etc.

It is interesting to see that students that had had a marginal participation now are able to reflect on mathematics. Robots brought to several students motives and disposition to engage in the learning process. This engagement made the difference on the kind of participation assumed by those students. 'He' and several other students, most of them 'disposable', in Skovsmose's (2005) sense, held that possibility from the school system.

During this work with robots, teachers were invited to organize a workshop to some classes of the school. We (researcher and teachers) decided to invite some students to do it for the colleagues. 'He' was one of the invited students. When we were in a meeting to prepare the workshop, 'He' asked:

He.: Teacher, is it possible to know for which classes we'll do the workshop?

Teacher: It's not important. There are colleagues from the school.

He.: But … I would like to know if 8º2 (the class he belong the preceding school year) will be…

Teacher: No. 8º2 will not be.

On the following day 'He' did not come to the workshop.

On the following mathematics class teacher asked him why he didn't come to the workshop. First he said that the alarm clock didn't went off and then, after a pause of some seconds he added that he wasn't interested in teaching robots to school colleagues that he didn't know well. When researcher talks with him (informal interview) he explicitly recognizes that it will be interesting to teach the colleagues that 'know more than me …because they are on the 8th grade'.

'He' seems to have the expectation of showing his colleagues of the preceding school year class, where he was not been successful, that now he is. This scenario did not occurred, so 'He' did not activated intentions to participate on the workshop and gave up. This intention of 'He' reflect his foreground. Analyzing his attitude without taking it into account can generate a conflict between teacher and 'He'.

FINAL CONSIDERATIONS

The way pupils were involved in the construction of robots was a very important aspect of the process. Students personify the robots putting a lot of their experiences on it. The result of all this was that students felt ownership over the robots. This fact makes them deeply engage in practice and the results are visible in terms of participation and consequently of learning.

To introduce robots on this learning scenario made agency to emerge on students that usually have a marginal participation. These students acted on the field of material agency, brought by robots to the created scenarios, captured this agency and placed it in service of the task they have to carry out (Fernandes, 2012). Human agency was reconfigured in its engagement with material agency (Pickering, 1995). 'He''s work with robots changed his participation in mathematics class and also his group colleague's participation.

Some students, such as 'He', saw in robots the possibility to imagine that it was possible to be successful on mathematics class and they grabbed it. Thus, robots displayed, in students, intentionality to engage on the learning process and made them to reflect on the learning process.

Analysing students' participation without paying attention to their background and foreground can lead to narrow view of the learning of mathematics.

NOTES

1. The research reported in this communication was prepared within the project DROIDE II – Robots in Mathematics and Informatics Education funded by FCT under contract PTDC/CPE-CED/099850/2008.

2. After several tests we verified that the time that the robot needs to reach the standard velocity, as well as, the braking time, are negligible. So we can assume that, to the end of this question, time and distance covered vary in proportion.

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MULTIMATHEMACY

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This paper starts from two statements based on a literature review. The first one concerns the learning process and states that learning is situated and socioculturally contextualized. Learning happens in the space of the background and the foreground of the learner in his or her particular environment of experience. This statement is based on the Vygotsky and the cultural psychology approach (Cole, 1996) and on the work of Vithal and Skovsmose (1997). The second statement concerns the drop out of schools. Based on the international comparative research on mathematical skills we claim that the drop out of school of many groups of children (OECD, 2010) has to do with the insufficient learning system at school that fail to fit with the daily background knowledge of the children.

INTRODUCTION

In Pinxten & François (2011) we introduced the concept of multimathemacy (after multiliteracy) to discuss the political agenda of ethnomathematics. We argued that multimathemacy should be the basis of the curriculum in order to guarantee optimal survival value for every learner. We described multimathemacy is an educational perspective that invites the teaching of different cultural insights on counting, proportional thinking, mapping or spatial organization in preschool knowledges. We argued that this view offers bridges between academic mathematics and cultural knowledge traditions for schooling. In this paper we will further elaborate on the theoretical framework and on the learning theories that support our statements on the concept of learning (mathematics) as a situated and socioculturally contextualized process.

Learning mathematics is a particular subset of learning. Hence, it is relevant to look at the learning theories, which are available so far. Since we focus on learning in /of different cultural groups or populations in this paper, we went looking for an inclusive theory of learning. That is to say, one that is sensitive to context, culture and social differences. This means that we hold, as an *a priori*, that learning is a process that happens not only in the brain or even in the organism of a single individual. Rather, we see it as a process of change in the individual in interaction with the social, cultural and environmental contexts. Looking at learning in this way we were driven almost necessarily to the sociocultural learning theories of Vygotsky and other neo-Vygotskian authors, lately synthesised in the cultural psychology theory, namely by Michael Cole. Learning is always 'situated' learning (Lave, 1988; Lave & Wenger, 1991), and through manipulation of the contexts of learning in teaching settings, the process of learning can be influenced substantially.

Secondly, and consistent with the first choice, we focus on the characteristics of all parties in the process, when devising a curriculum and learning strategies. That is to say, the curriculum developers (in this case, the mathematicians trained in Academic Mathematics), the teachers (basically of the same background) and the pupils have their own mental setups when entering the learning process in a mathematics classroom.

We take this seriously and investigate what the input of all of them amounts to. When developing a curriculum and teaching procedures we will take all of these into account.

The basic reason why we feel obliged to go along this track (apart from mere ideological preferences for this or that societal model) is the fact that the drop out in the schooling is consistent and at the same time rather specific. In the overview report of the OECD (2010) two of the key factors are described as "-continuing disparities in scholastic achievement between first and second generation immigrant students and their native peers; -lower *scholastic achievement* and graduation rates for indigenous populations in countries with long history of migration" (OECD, 2010, p. 14, italic in the original). Data from the PISA 2003 and 2006 (OECD, 2005; 2010) show that on average across all participating countries native students perform better in mathematics than first and second-generation immigrants (OECD, 2010, p. 24). This pattern is particularly troubling as it appears that native students perform better than the second-generation immigrants who are born and raised at the same country. At the same time – and part of the explanation of second-generation immigrants' situation – figures indicate that a student coming from a low socioeconomic status is "twice as likely to be among the low achievers" (OECD, 2010, p. 25). Recent studies (Zhao, Valcke, Desoete, Zhu, Sang, and Verhaeghe, 2012) reveal that a large proportion of mathematics performance can be predicted from contextual variables, one of them being the link between the SES of parents and mathematics performance. Of the predictors of mathematics performance at age 10, the effect size of mother's education level is higher than that of father's education level. Mother's educational level is also related to the mathematics score in primary education (Zhao et al., 2012). Zhao et al. (2012) assume that mother's higher educational level implies that mother expects her children to take more responsibility at home and in relation to their thinking and learning.

Concerning indigenous students the challenges identified across all countries (having indigenous populations that pre-date the arrival of European settlers viz. Australia, Canada, New Zealand and the United States) are the following: "difficulty in accessing and receiving the level of early childhood education and care recommended; lower levels of literacy and scholastic achievement; lower rates of graduation; proportionally higher representation in vocational education and training streams than their non-indigenous peers; and lower rates of participation in tertiary education in many of these countries (OECD, 2010, p. 26).

This is some of the most important reasons why the executive board of the OECD states that educational systems have to become more effective and more equitable. These international research findings are confirmed by national research results, e.g. the Council of Australian Governments (COAG) states that indigenous people are the most educationally disadvantaged group within Australia. Their educational outcomes are substantially lower than non-indigenous students. For example, in 2006, 45,3% of the indigenous Australians had completed the 12th grade, compared to 86,3% for non-indigenous Australians (Howard, Cooke, Lowe, & Perry, 2011). If we know that Australians who have not completed the12th grade are less likely to have the same opportunities as those who do, we can speak in terms of inequality and violation of human rights.

We want to understand what is going on, and our proposal is that the learner's perspective is not enough in the focus of mathematics educational programs so far.

PISA research (OECD, 2005, p. 190) shows that high performance in mathematics education consistently links with high scores in reading and science knowledge. At the same time, low performance in school is uniform for a second group of the school population. The gap between both groups seems to consolidate or even widen, rather than narrow over the years. We interpret these results here as corroboration of our main thesis, namely that cultural and social differences between learners do count in education. That is to say, when pupils perform poorly in the dominant language of the mainstream culture in which they participate (which is more or less different from their home language) and in the dominant world view (for which the same can be said), then schooling which disregards in a general way these differences will presumably yield larger gaps between subjects of the dominant social and cultural groups (i.e., middle class white groups) and others. We want to understand what is going on, and our proposal is that the learner's perspective is not enough in the focus of educational programs so far. Here, we concentrate on mathematics education only, starting from a general focus on learning in the first place.

THE LEARNER'S PERSPECTIVE

The learner is not a mere receptive or passive party in our view. Hence, learning theories, which 'situate' the learner and the learning process in contexts will carry our attention, and we will disregard the other ones. In a very general sense, we follow Cole's (1996) synthesis in this respect.

In Vygotsky's (1934/1962) intriguing approach of almost a century ago, learning was first and foremost understood as a dialectical process between a learner and his or her environment. In other words, it was not identified merely with the processes inside the head of the learner or even at the edge of it. For example, in the very powerful stimulus – and response theory (behaviorism in its many versions) learning is studied as the result of the processing of (controllable) stimuli by means of the responses they trigger in an individual. Neither is it equated with a particular form of adaptational action on the part of the individual in his or her biological maturation cycle (as was the case in Piaget's learning theory: Piaget, 1972). Vygotsky and his school broke away from these approaches and situated the learning processes plainly in the field of interaction between a learner and the physical, social (-historical) and cultural environment or set of contexts.

Such a focus has tremendous consequences for education. First of all, it entails that characteristics of both the learner and the environment matter in the curriculum and in the learning procedures. If the pupil is unable to grasp the point of the learning process, then failure will probably ensue. But if the context is too poor, too far removed from anything understood or recognized by the pupil or in any other way 'foreign' to the pupil's knowledge categories, then failure to learn will also be the result. In line with Bakhtin (1986), Bruner (1984) maintained that you always create or hear about a narrative in terms of your life experiences and background. Giving meaning and creating knowledge of the world is relative and it is dependent on the individual's past and present experiences.

Secondly, it then becomes important to look for types of matching between the student's mental setup and background knowledge and the challenges and possible inputs in the context. The latter could be hidden, openly offered, presented as triggers or otherwise entered in the interaction process with the learner. It is clear that learning procedures are in focus here.

Finally, evaluation of learning output stops being the assessment of the pupil's responses only. It clearly and equally involves the assessment if the success or failure to induce learning by the contexts of the pupil as recent research reveal that a large proportion of mathematics performance can be predicted from contextual variables (Zhao et al., 2012).

When we put the learner in the focus, it follows that we need to 'flesh out' the individual learner a bit more to go beyond the trivial. We side with a cognitive theory of the learner, claiming that some parts of the metaphorical 'black box' can be filled in a hypothetical, but nevertheless dependable way without losing scientific credibility.

In terms of mathematics education recent research in this area was done by Scandinavian colleagues in the research group of Skovsmose (Alrø, Ravn, & Valero, 2010). Skovsmose, who coined the concept of Critical Mathematics Education (CME) situates mathematics education within a broad social and political context. Indeed to Skovsmose mathematics teaching and learning could aim at developing democratic competencies. This is why CME is concerned with mathematics education for all –independent of color, gender and class. CME is concerned with the practical application of mathematics – being an advanced technological application or an everyday use. It is also concerned with the democratic setting of a classroom situation, with the life in the classroom, and with the critical voice of pupils. A mathematics class has to be a space of learning where ideas are presented and negotiated. Indeed to Skovsmose and Borba (2004) CME is concerned with the development of critical citizenship.

In this social and political embedded learning process, any learner is a subject within historical, social and cultural contexts, from which he or she brings into the learning situation previously gathered concepts, problem solving strategies and learning procedures. These are summed up under the label of 'preschool knowledge'. Obviously, the contents of this category are primarily defined by the worlds of experience of the child: the peer groups, the family, and the physical and sociocultural environment of the child. Hence, street children will differ in their preschool knowledge from Amazonian Indians, from city dwellers in Western Europe, from Aborigines and Torres Strait Islanders in Australia, or from peasant children in rural China.

A further sophistication introduces the distinction between 'background knowledge' and 'foreground knowledge'. It was Vithal and Skovsmose (1997) who emphasized the concept of foreground – besides the notion of background. Where the background means what children bring to the classroom, foreground is to be understood as "[T]he set of opportunities that the learner's social context makes accessible to the learner to perceive as his or her possibilities for the future" (Vithal & Skovsmose, 1997, p. 147). Skovsmose (2005) also emphases the political and cultural situation as an important aspect of the foreground, since they provide – or blocks – the opportunities for the learner. It makes the political nature of the learning process explicit because it has to do with the student's possibilities in future life, not the objective possibilities as formulated by an external institution but the possibilities as the student perceives them.

The learner brings a 'background of knowledge' into the learning situation: he or she already appropriated knowledge, which is relevant for the issues or the problems presented in the school setting. For instance, each child has a mental map of the environment, which will allow to cover the distance between school and home cultures in a rather efficient and safe way. At the same time, the child has a 'foreground of knowledge', which is the set of extensions of the knowledge that is appropriated together with the competencies to enable further learning, like understanding the school culture, management of problem solving techniques regardless of concrete contexts, and so on are examples of this. The child is actively learning in a classroom while making use of this 'mental setup'.

LEARNING AND CULTURE IN THE MATHEMATICS CLASSROOM

Every couple of years the OECD assesses the quality of education throughout the world. These researches yield regular reports (the PISA reports), which give an overview of the success and failure ratio of children in and through schooling in mathematics education. A recurring point in these reports is that (lower and higher) middle class pupils have a high success rate, whereas lower social classes and minority groups are showing poor results (OECD, 2010).

Our hypothesis reads as follows: the common mathematics curriculum and teaching procedures start from the point of view that mathematicians (belonging to the group of so-called Academic Mathematics) define the school program in basic lines. The understanding seems to be that mathematics is what Academic Mathematics says it is. Obviously the immense sophistication of this type of knowledge and its proven worth in scientific and technological research feeds this status. The pupils then try to master that, equipped as they are with their particular sets of background of knowledge and foreground of knowledge. The systematic and clearly not decreasing failure in school mathematics of the groups mentioned, can hence only be blamed on the academic mathematics group who set out the rules of the curriculum.

When we take the stand that learning is in actual fact always situated learning and that the learner brings his or her background of knowledge and foreground of knowledge into the learning situation, we can draw the conclusion that the fact of disregarding the background of knowledge and the foreground of knowledge in the learners can explain why their performance is consistently poor. At the very least we can explain why more schooling does not automatically yield better results in mathematical education, especially for particular groups (as is shown in the PISA reports). But in order to do that, we have to take one more step.

Learners have a background of knowledge and a foreground of knowledge. But what about mathematicians, and their product of thought, i.e., Academic Mathematics? We propose the hypothesis that mathematical knowledge in Academic Mathematics has implicit categories, worldview notions, intuitions, and the conceptual frame, which can be argued to be compatible with or translatable into that of a particular group of learners to a larger or smaller extent. Concretely, we suggest to investigate whether the middle class western subject's background of knowledge and foreground of knowledge is more easily translatable, accessible, or more closely overlapping with the worldview and categorization of Academic Mathematics than is the case with North American Indians, or lower class local groups and immigrant groups in Western Europe.

What we do by forwarding this hypothesis is not denying the tremendous worth of Academic Mathematics as a way of thinking and as a formidable tool for science and technology. Neither do we fall prey to a simplistic relativism, denying the high level of sophistication of this discipline. Instead, we claim that Western Academic Mathematics, like any human product, has its roots and that in learning Academic Mathematics these roots may show their relevance.

The structure of the Indo-European languages distinguishes between verb and noun forms. With this distinction corresponds a differentiation between things/states and operations/processes in the conceptualization of the perceived reality. Intuitively, mathematical thinking sophisticates these deep structural linguistic and cultural differentiations (Pinxten, van Dooren, & Harvey, 1983). Hence, the emphasis on geometric figures (with a thing-character) and their constitutive forms, on sets and their elements, on operations (of multiplication and so on) performed on entities (a number, a series, etc.). The point we want to make is that formal thinking elaborates the intuitive world view which is given in language and in folk knowledge (Atran, 1990). When investigating other cultural traditions we learn that Athapaskan and Cherokee languages, like Classic Chinese are ‘verb languages’. That is to say, the noun category is inexistent or at least not substantial, corresponding to a view on reality as basically a world of events (Whitehead, 1906).

Again, regardless of the great achievements of Academic Mathematics, it is our conviction that it will be important to take these preschool differences into account for mathematics education. The dropout rates (cf. the PISA reports; OECD, 2010) might well be better understood in the light of these differences in preschool competences, to be found in the learner’s background of knowledge. Following that line of reasoning, we must then conclude that it is likely that neglecting the background of knowledge of the child will yield lack of insights or more difficulties with the Academic Mathematics – inspired mathematics curriculum and learning procedures. Alternatively, we advocate to systematically involve the child’s background of knowledge and foreground of knowledge in the educational process. By necessity this means that mathematics education (and hence also introduction to Academic Mathematics concepts and theories) will have to take into account the different backgrounds of knowledge and foregrounds of knowledge of varying cultural traditions.

FINAL REMARKS

Learners always bring their background knowledge into the learning process. We proposed that Academic Mathematics is culturally embedded, that is to say, Academic Mathematics has its own (implicit) categories, worldviews and applications. The challenging question remains how to close and thus to overcome the gap between on the one hand the diverse Communities of the learners and on the other hand the communities of the learning institution (school cultures, curriculum, teacher, …). In order for teachers to deal with the diversity of backgrounds of knowledge, they should have knowledge of these cultural backgrounds, traditions, languages, practices and mathematical practices the learner (can) bring into the learning context. Teachers should have an anthropological perspective on the learning processes, on the school culture, and on the diverse cultures the learners bring to school. That is why we propose to use anthropological studies in the learning process in general.

As an alternative to the monolithic approach to mathematics we can now pave the way for our option for multimathemacy. Multimathemacy is an educational perspective that invites the teaching of different cultural insights on counting, proportional thinking, mapping or spatial organization in preschool and out of school knowledge and this view offers bridges between academic mathematics and cultural knowledge traditions for schooling.

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IT IS A MATTER OF BLUENESS OR REDNESS: ADULTS' MATHEMATICS-CONTAINING COMPETENCES IN WORK

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In this paper, a social and critical perspective on mathematics education is operationalized through the concepts of habitus, field, and foreground. As part of an on-going project, we have made a tentative analysis of how reliance on colours in the field of workplaces can be seen as signs of mathematical aspects of a person's workplace competence. The analysis of this initial qualitative case study suggests that making this kind of mathematics explicit contributes to an understanding of what mathematics may "become" in work. Our assumption is that this point of departure, in the long run, can contribute to a deeper understanding of mathematics' function in workplaces, in society, and in school.

INTRODUCTION

Adults' mathematics containing-competences are in focus in a new project led by professor Tine Wedege, Malmö University. The purpose of the project is to describe, analyse and understand these competences – including social, ethnic and gender related aspects – in relation to demands made on students' qualifications in formal vocational education. One underlying aim for the project is to reverse the often assumed direction of knowledge distribution to be mainly from school *to* work to also include the process of school learning *from* work. Consequently, the name of the project is "Adults' Mathematics: In Work and for School." Taking part in one of the project's qualitative studies, we have conducted two out of twelve observations in workplaces within the sectors of nursing/caring and vehicles/transport. The observations will be followed by life history interviews, later in the project.

WORKPLACES AND MATHMATICS

The fact that mathematics at work appears to be different from what is taught in schools is well described in the literature (e.g. FitzSimons 2012; Hoyles, Noss, Kent & Bakker, 2010; Wedege, 2000, 2004). In work, mathematics is often integrated in machines, technology, routines and competences (Wedege, 2004). Often adults show resistance towards mathematics. One example is given in Wedege (2000), who interviewed and observed people who referred to mathematics as something they didn't know or didn't use. This was in contrast to the fact that they told stories of, or participated in, activities where it was a straight-forward process for the researcher to construe them as containing mathematics. In contemporary workplaces, constantly evolving technical development and division of labour are changing the conditions of work. This could be seen as a widening gap between conception and execution of mathematics containing work tasks (Jablonka, 2003). The objective of this paper is to explore the concepts of habitus, field, and foreground as tools for investigating mathematics-containing situations in work and how they are connected to the fields of workplace mathematics and school mathematics. The term mathematics-containing competences is used by Wedege (2000), and relates to the relation between everyday knowledge and school knowledge. In this paper the term is used as a broad interpretation of mathematics in order to understand differences between work and school. What is considered as mathematics in one field may not be viewed similarly in the other. Furthermore, we want to discuss the relation between the individual and the social and how different practices can be understood where mathematics is concerned. As Valero and Zevenbergen (2004) describe it:

> ... practice is social because it is historically constituted in complex systems of action and meaning in the intermesh of multiple contexts such as the classroom, school, the community, the nation and even the globalized world. (p. 2)

ANALYTICAL TOOLS: HABITUS, FIELD AND FOREGROUND

Our understanding of the concepts of habitus, field and foreground includes how they are related to each other and to mathematics in work and at school.

Habitus

When connecting individual experiences to structural and social factors a theory that takes both levels into account is required. One account of this view is presented in Salling Olesen (2008). He writes about how individual workers "carry their specific social history with them into the workplace, and they embody the experiences learned in the workplace in their subjectivity" (p. 123). In this paper we adopt Bourdieu's (1991) concept of *habitus* to approach the dialectics between the individual and the social structure. This dialectics is built into the concept of habitus. It is conceptualised as a system of dispositions grounded in biographical experience, which is by its nature purely individual. Members of a certain class or group can, however, share ideas and habitus in the form of a class habitus but the essence of habitus is something unique for each individual (Bourdieu, 1991). In other words, habitus can carry both conscious and unconscious dispositions to act in the social world. Habitus is a complex system consisting of preferences and motives for individuals to act in the social world. An explanation of habitus, given by Bourdieu, is that of having a sense of how a given situation should be tackled. Then the individual has a "feel for the game" (Bourdieu, 1995).

Habitus and field

The above mentioned structures are described by Bourdieu (1991) as social fields and he argues that the socialized body with its habitus generally carries the same history as the social field where it is acting in establishing an infraconscious corporal awareness. In a certain field people are willing to play the game with the rules set by the field. A field has its tensions due to the battle between the newcomers trying to find a position in the field and the established actors who dominate the field (Bourdieu, 1991). Bourdieu (1989c, quoted in Grenfell & James, 1998) states:

> the relation between habitus and field operates in two ways. On the one side, it is a relation of conditioning: the field structures the habitus, which is the product of the embodiment of immanent necessity of a field (or of a hierarchically intersecting sets of fields). On the other side, it is a relation of knowledge or cognitive construction: habitus contributes to constituting the field as a meaningful world, a world endowed with sense and with value, in which it is worth investing one's practice (p.16).

Two fields: school mathematics and workplace mathematics

Wedege's (2000) examples of how one person can argue that s/he does not know mathematics, while simultaneously actively applying mathematics in work illuminates workplaces and schools as two different fields. In the preface to the Swedish translation of the book "Les règles de l'art: Genèse et structure du champ littéraire", Broady (cited in Bourdieu, 2000) explains how the concept of field can be understood, which is that social fields have similarities with physical force fields. However he warns that one should be careful when using the concept of field. A school, for example, can be seen as a field for teachers and staff members while it is not a field in the same sense for students who are passing by this field. We will return to this below, in the discussion. The following statements can be construed as differences in the rules of the game of mathematics, as a part of other rules set by the field. Wedege (2000, p. 197) lists several differences between the fields of "task-driven" school mathematics and workplace mathematics. Table 1 highlights two of them. (See also FitzSimons, 2012.)

Work	**School**
There are often different solutions to a task. Accuracy is defined by the situation and the veracity can be negotiated.	There is only one correct solution, the accuracy is defined by the teacher and the veracity it not something that can be discussed.
Reality is the reason for using mathematical ideas and techniques. Solving tasks has practical implications	The task operates as a pretext for using mathematical ideas and techniques. Solving tasks has no practical consequences.

Table 1: Differences between the fields of school mathematics and workplace mathematics

Habitus and foreground

Skovsmose (2012) describes how the notion of foreground was first developed in 1994, and he elaborates on his earlier explanations. Drawing on Skovsmose, we understand students' foregrounds as the opportunities society makes visible and possible for individuals. Skovsmose (2005) suggests that it is important not only to focus on students' backgrounds but also their foregrounds. "Foregrounds are not panoramic and coherent pictures of possibilities" (Skovsmose 2012, p. 2). Nevertheless, foregrounds have "objective" elements such as social, economic, political and cultural parameters. Still, foregrounds should not be understood as an objective affair. Instead, foreground is formed by the person's interpretations of the possibilities the circumstances provide. "In this sense the foreground becomes a complex mixture of subjective and external factors" (Skovsmose, 2012, p. 2). Foregrounds are dynamic rather than stable. Construing foregrounds should be seen as an on-going process and Skovsmose (2012) suggests that it is relevant to talk about *foregrounding* (as an action). Wedege (2011) suggests a local integration of Skovsmose's notion of foreground with Bourdieu's theory of habitus. When investigating the theoretical relevance for, and the advantages of, the integration of habitus and foreground, she concludes that they should be complementary because both frameworks are based on the idea of action and dispositions to act, and, furthermore, both are rooted in a critical perspective. As Alrø, Skovsmose, and Valero (2007) note:

> Learning-as-action can only take place on the grounds of the person's *dispositions*, that is, on the person's readiness to find motives to engage in action. Dispositions can be seen as the constant interplay between a person's *background* and *foreground*. (p. 7, italics in original)

METHODS AND TRANSCRIPTS

The research design of this study is a multiple case study (Bryman, 2008), with comparative elements. In this paper, the comparison is related to previous research and our own experiences from school. Things taken for granted by those in the field could be noticed by an outsider and consequently made the object of attention. In this process, an important feature is reflexivity (Hammersly & Atkinson, 2007), and the assumption that researchers affect situations, while at the same time being affected in return. In the research design of the qualitative part of the project, the video-recordings described in this paper, are the first step of data-gathering and analysis. One way of interpreting a setting that is not ours can be to notice what is not the case, or what could have been the case in our own settings. In other words, to be attentive to hypothetical situations without letting go of the actual situation: "Doing critical research also means to explore what is not there and what is not actual. To research also what is not there and not actual means to investigate what could be" (Skovsmose & Borba, 2004, p. 210).

Earlier studies show that mathematical elements in workplace settings are subsumed into routines and are highly context-dependent. Moreover, the present mathematics is constituted through a variety of semiotic resources (e.g. written texts, symbols, speech) including artefacts (e.g. tools). Here we draw on multimodal social semiotics (Van Leeuwen, 2005; see also Björklund Boistrup & Selander, 2009). This has consequences for our transcripts, which are made multimodally. In Table 2, we identify Time, Body (what people do including resources and artefacts), Speech (what people say and how they say it), and Gaze (where people look). The significance of including resources other than speech in the transcripts also has connections to habitus as an embodiment of the field (Bourdieu, 1995). The films are transcribed in the software Videograph.

A MATTER OF BLUENESS OR REDNESS: EXAMPLES OF ANALYSIS

Here we start by introducing a part of the transcribed observation from the garage.

The garage

The first transcript starts when a mechanic, "Anton", has just finished a check-up of antifreeze/cooler by using a measuring device and comparing the value shown on the device with a value on a certain scale. Without letting his focus go from the work or the car he says: "So washer fluid is the next check-up to be done" (our translation). In the transcript, *A* stands for Anton and *R* for the researcher.

Time	Body	Speech	Gaze
14:19	A: Walking to water tap with bottle in hand	A: It is very concentrated so it has to be mixed with water. R: Okay	A: Looking at researcher
14:31	A: Pouring water into the bottle	R: How should it be mixed? What is the – how much water should there be, so to speak	A: Looking at the bottle
14:33	A: Stops pouring water and walks back to the car	A: I don't know exactly but it R: But how exact is it then, how precise is it? A: Well it is…	A: Looking at R
14:55	A: Pouring the mixed liquid into the tank in the car	A: It is the colour [inaudible] that is how exact it should be. But it ought to be quite blue anyway	A: Looking into the tank of washer fluid. Looking at R

Table 2: Transcript from the garage [selection]

The transcript from the garage shows that the mechanics Anton was going to refill the tank with washer fluid. What cannot be seen in the transcript is that the tank in the car contained a little wash fluid already. In order to complete the task, he had to estimate the volume required to fill the tank. He also had to be aware of which concentration the liquid should have, compared to the one in the tank. He used concentrated washer fluid that was to be diluted with water. For this purpose Anton had to estimate or measure both how much there was already in the tank, how much concentrated washer fluid he needed to add, and how much water to add in order to get the proper concentration when he had filled the tank. He had been told by his boss that the concentration should be 50/50. As observed by Wedege (2000), accuracy in the workplace field is defined by the situation. This could mean that the blueness may need to be changed due to conditions such as outdoor temperature. In some way the blueness functioned as a security for him and he said: “It ought to be quite blue anyway,” by which he probably meant that rather a little bluer than running the risk of freezing washer fluid. This also corresponds to Wedege’s suggestion of tasks in the work place having practical implications. During another part of the observation, Anton answered a question about where he had learnt to estimate the volume of another liquid. After a little thinking he responded by talking about his early interest in cars and motor bikes. He didn’t learn it from school, however, he said. The competence seemed to be so well integrated into his habitus that he did not even think of himself as having learnt it, but rather as if he was born with it, or at least as being a natural part of his body and mind. In the terms of Bourdieu (1995), his socialized body with its habitus carried the same history as the field where it was embodied. He was actually so confident that he did not follow the advice from the boss to make a concentration of 50/50. Anton relied on his capacity of judging the blueness accurately as he relied on it when measuring the other liquid. Maybe his early interest in cars and motor bikes, and his competence, could be seen as a part of his foreground for choosing this kind of work.

The hospital

At another workplace, a semi-emergency unit at a hospital, we were invited to follow a nursing aide, Anita, in her work. During our observation she made a check-up on her patients. When the round was completed Anita went to a computer to put the values into the digital hospital record. While doing this she took a chart with colours from her pocket (Figure 1).

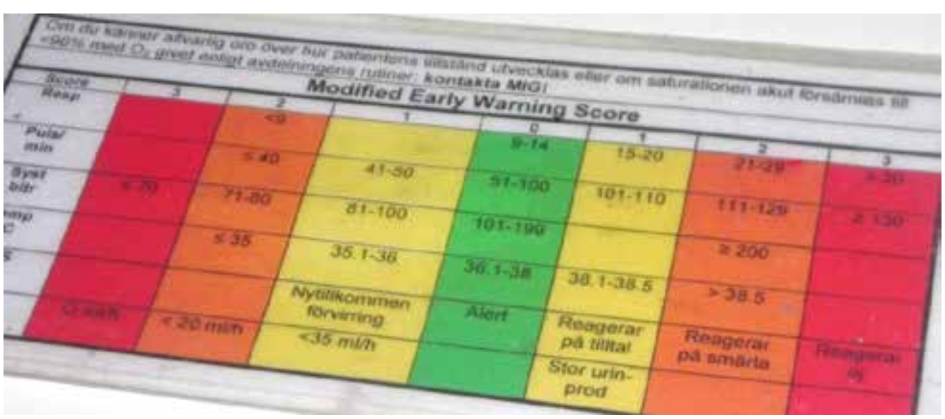

Figure 1. Chart with colours

The chart (Figure 1) was coloured outwards, from green in the middle (0), then yellow (1), orange (2), and, finally, red (3) at either end. In the green, middle column, the ranges of values were considered normal. Each column was given a score, and a certain total score needed immediate attention from a doctor. The check-ups were done every fourth hour and included examinations like blood pressure, pulse/heart rate, temperature, volume of urine, and oxygen saturation. All the time, she had to change her priorities due to other more tasks that had to be taken care of. Therefore she kept the values from the patients in mind. When there was time she wrote them down on a piece of paper.

During our observation one patient needed a further check-up of the oxygen saturation. When the blood sample was taken by a doctor, Anita had to interrupt her work and go to a digital laboratory to get an analysis of the blood sample. After the sample was put into a measuring machine, she did not wait for the result, which would automatically be filled out in the patients' hospital record. She explained: "This will take one minute, but one minute is a long time so I will not wait for it." During the observation Anita was handling multiple tasks simultaneously. She also continuously compared the values given by the patients' supervising monitors and with questions to the patients or with taking the pulse manually. Although she was gathering numeric values they were surprisingly absent in her documentation. The values seemed to be a part of a bodily consciousness of the field and nothing worth mentioning. The expression "to have a feel for the game" (Bourdieu, 1995) could be said to be a relevant description of Anita's work. The nursing aides even had their own expression for this. They called it "the third eye", which meant just knowing what to do. Only once did she use values/figures explicitly; to encourage a patient whose values were much better than previously. For nursing aides like Anita, the critical feature was that the values should not enter the red sections of the chart.

Similarities and differences with regards to mathematics containing competences and the context

The observations were made in two very different settings. The second observation was in a clean, hygienic area, with a majority of women, mostly clad in white, taking care of humans. The first observation was in a more dirty area, with a majority of men, clad in blue, taking care of cars. The situations observed could be viewed as being very dissimilar, taking place within such different settings. Nevertheless, they have a commonality in that they rely on devices and colors, even using the colors as a security system. The mechanic relies on the blueness of his solution to determine the concentration of the washer fluid, and the nurse concentrates on her patients' values not getting into the red zones of the chart. One difference in the reliance on colours is that the mechanic's use of the degrees of blueness may not be easily communicated to others. Each mechanic has to make their own judgement of the blueness. In the case of the nursing aide, the well-defined colour system was presumably implemented to facilitate communication within different levels of organization, and thereby increase the security for patients. Workers in both case studies used measurement in potentially critical situations, but there was an absence of explicit numbers which, it seemed, was replaced by a focus on colour.

DISCUSSION

We have made a tentative analysis of how reliance on colours could be seen as signs of mathematical aspects of a person's workplace competence. As analytical tools we have chosen the concepts of habitus and field from Bourdieu's theoretical framework combined with Skovsmose's notion of foreground. When the observed workers, in the next step, contribute their mathematical life histories, a more profound construal of habitus will be added to the analysis. We think that our chosen tools will then have greater application. We suggest that the tools can make visible individuals as agents construing their foregrounds and directing them towards certain fields. This may not be a straightforward affair. However, the individual's habitus and foreground, as analytical tools, can shed light on the possibilities for him/her to affect the rules of the field, and also its mathematics-containing activities.

As a possible consequence of what we have observed so far, vocational education needs to be careful when contextualising tasks. It should consider that the rules of our school field are embodied in our habitus and can easily be taken for granted. At the same time, school can be seen as an arena where students try to work towards their intended foregrounds. Some may already have experience from the field of work. A task like the washer fluid concentration could not be solved at all with school mathematics. The volume in the tank is unknown as well as the concentration. Therefore, the volume and the concentration to be added are also unknown. The only parameters known are that the final result is to be a full tank with a concentration of 50/50. The context that we as mathematics and/or vocational teachers may produce to inspire and help the students could instead make it more difficult. Students may need to decontextualize the task to compare it with the real world they know or in order make sense of the contextualisation and deal with the mathematics integrated in it. There may also be a gap between conception and execution in both fields, although noticing the colours in the workplace could be seen as a way of securing the execution. In a mathematics classroom there will most probably be no colours to rely on. From our experience it is not unusual that the mathematics, instead of being highlighted, is embedded in a language or a kind of mathematical 'conjuring' that is not familiar to all. For students with experience from the field of work, their understanding may be linked to the mathematical rules of that field, with its practical applications. Sometimes there may even be no practical implications in the workplace. The washer fluid could, for example, have been diluted in advance using more mathematical measurements. Yet, the social rules, formed by the structure of the field, and the agents in the field may be equally strong as the mathematics rules. As noted above, in school there is often only one correct solution and its veracity is not open for discussion. However, in the case of the hospital observed, the accuracy could not be negotiated either. Yet, in a mathematics classroom, construing and constructing charts would most probably not be done under time pressure, while simultaneously handling other tasks. In a modern society, fields may to some extent be floating when new professions grow and individuals are mobile, and these may contribute to the changing conditions and rules in the fields of work. In this we see a challenge for schools to investigate the rules of their own field, and to understand the rules of workplace fields in order to make mathematical tasks accessible to different students. Our aim with the forthcoming research is to contribute to this understanding. Habitus and field, connected to foreground, could be useful as analytical tools for this purpose.

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LOGIC, SOCIETY AND SCHOOL MATHEMATICS

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An example of Aristotelian logic, which I show to be paradigmatic for Scholastic logic and school mathematics, is analysed for its social functions. This analysis builds upon a socio-historical interpretation of the meaning of early logic and shows its dialectic religious, epistemological and political dimensions: On the one side, logic can be used to appease, to emancipate and to amplify thought; on the other side, it might lead to fear, subjection and intellectual constraints. The combination of both allows logic and mathematics to become an instrument of power. Mathematics education, then, is the institution in which the acceptance of this instrument of power is cultivated.

AN EXAMPLE OF ARISTOTELIAN LOGIC

Research on the social functions of school mathematics tries to answer the question in which dimensions society and school mathematics interact. While Ole Skovsmose, Roland Fischer, Philip Ullmann and others have contributed in answering this question, many issues have not yet been addressed. This paper is a contribution to a deeper elaboration on the nature of mathematics and its impact on society. One of the main characteristics of mathematics is logic. This paper examines the social function of logic in the mathematics of German secondary education, using an example of Aristotelian logic. The discussion will address questions on a historico-cultural as well on an educational level: *Which social functions did logic serve in ancient Greece? What social functions does it serve in contemporary school mathematics?*

The Greek philosopher Aristotle is considered the founder of logic since he compiled, formalised and analysed rules for speaking and thinking in the 4th cent. BC. In philosophy, his *oeuvre* is a milestone. His logic processes the transitions in speaking and thinking which ancient Greek has gone through and which is expressed in the work of his predecessors, such as Anaximander, Parmenides, Socrates and Plato. But what had caused these transitions; which purposes does the new kind of thinking serve? The socio-cultural analysis of the four laws of thought which Scholasticism has identified in the work of Aristotle (e.g. Schopenhauer, 1813/1903, § 33) will provide answers for these questions. The four laws of thought are:

1. Law of identity. *Everything stays the same, nothing changes*. The principle of identity is tautological as long as it is read descriptive. Read prescriptive, i.e. as the rule to speak and think in a way in which everything stays the same, the principle of identity provides our speech and thought with concepts that are reliable in the sense that they do not change their nature with the speaker, the location or time. Already at the beginning of the 6th cent. BC, Anaximander of Miletus, probably a student of Thales, had argued for the existence of something "infinite"

which is “indestructible”, “deathless” and “imperishable” (Aristotle, trans. 1930, 203b). Half a century later, Parmenides seized the idea, described it with similar adjectives and called it *truth* (*aleteía*; Parmenides, trans. 2009, pp. 14–23).

2. Law of excluded middle. *Everything is or is not; there is no other way.* The law of excluded middle divides our statements into two categories, e.g. truth and the false, and leaves no other option.
3. Law of excluded contradiction. *Nothing is and is not at once.* The law of excluded contradiction demands a decision between the categories. Together with the law of excluded middle, it forces the statements of speech and thought into an antagonism introduced by Parmenides: the antagonism of being and not being, of true and false, leaving no room for a state in between or beyond the extremes.
4. Law of sufficient reason. *Everything but one thing has a reason and is defined by it.* Distancing himself from mythological thought, Anaximander claimed that everything but one thing has a reason which is its destiny (Anaximander, trans. 2007, p. 35). Therefore, he is often considered the founder of scientific thought. The law of sufficient reason is a method to decide over true and false on the one hand, and provides a scheme with which to order ideas on the other hand.

RELEVANCE OF THE EXAMPLE

The four laws of thought of Aristotle are relevant for philosophy, mathematics and contemporary school mathematics. The discussion of syllogisms constitutes the core of Aristotelian logic. It addresses the question which forms of conclusion are certain. Certainty here means that the truth of a statement can be shown with necessity and out of sufficient reasons. As the laws of thought are essential for the discussion of the syllogisms, they form an essential part of Aristotelian logic. Certainly, the logic of Aristotle was not uncontroversial among his contemporaries: it competed against the old myths and against alternative philosophical schools such as the school of the Sophists represented by Protagoras. However, it was most successful as it attracted most attention in Ancient and later philosophy.

Euclid’s *Elements*, the first systematic compilation of mathematical concepts, theorems and proofs, and *the* reference of mathematics for two millennia, are strongly influenced by Aristotelian logic (Wußing, 2009, p. 191). This influence does not only consist of proofing as the method for the validation of mathematical statements. Aristotelian logic also influences the whole architecture of Euclidian mathematics: Its concepts are defined as tight as possible to avoid any variations of meaning and to provide them with unalterable identities; as sharp as possible to allow a clear inclusion or exclusion of every phenomenon; and as connected as necessary to be able to verify the properties of the concept. Today, mathematics has the very same architecture. Alternatives, such as polyvalued logics, logics without the law of excluded middle and others exist, but they are hardly used in mainstream or even school mathematics. Thus, school mathematics is based on Aristotle’s logic:

- The concepts of contemporary school mathematics and their properties stay the same all through the curriculum. Usually, they are presented as universals which do not evolve or vary with culture or through history, leaving no room for an individual reading: Even and odd numbers, circles and so forth are its invariants.
- The concepts of contemporary school mathematics are structured in a way to either describe a phenomenon or not; they follow the antagonism of being and not being: A natural number is either even or odd; it cannot be both or something else. In fact, every classification follows this rationale: A straight line either shares points with a circle or not; and if so, then it shares either one point or two.
- The concepts of school mathematics refer back to their origin by which they are defined. Every super- and subordination, e.g. from the polygon down to the right-angled triangle, every implication, every transformation of terms and equations leans on the assumption that the truth of the new is already hidden in the original.

Although the relevance of the example of Aristotelian logic for the purpose of this paper has been illuminated, its social impact still has to be examined. The work of Klaus Heinrich, whose intention was to examine the "suppressed of philosophy" in its genesis (Heinrich, 1981, p. 10), provides a basis for a cultural analysis of early logic. He looks for radical changes in the thinking, feeling and activities of a society and asks for their reasons. His original perspective on Aristotelian logic is insightful as he succeeds in a fruitful combination of theory of science and cultural history. The following look at the birth of Ancient logic focuses the social circumstances that allowed Aristotelian logic to establish; it considers logic the intellectual manifestation of an experienced practice. What brings the thinkers of ancient Greece to cultivate that new form of thinking? Which were the social pitfalls and contradictions, the concerns, troubles and wishes that lead philosophers to logic?

LOGIC AND RELIGION

Before the birth of philosophy and science, only the myth offered explanations of the word, especially of its most alarming phenomena: the fatal threat of age and disaster. Hesiod's *Theogony* and Homer's epics describe the mythical world of gods who personify death, hunger, storms, floods, earthquakes, droughts and epidemics. These gods, and thus the phenomena they personify, were understandable as they were imagined as human-like creatures, who knew friends and foes, intrigues and murder. The strongest influence on the nature of gods was assumed to be their descent: It is regarded as an unalterable trait, as a fate that cannot be escaped as in the curse of the house of Atreus (Heinrich, 1981, p. 99). The myth allows for a comprehensive orientation in the world; it is an early philosophy of nature. However, this orientation was threatened as democracy developed. The open discussions on the market place, the agora, did not only address politics but also philosophy, ethics and religion. Thus, traditional convictions were called into question, leading to a confusion of world views (Vernant, 1962/1982, p. 52). Socrates was the incarnation of scrutiny and died for it. The ongoing collapse of religious beliefs as well as political and economical structures led to a lack of orientation and security, and called for a new and possibly safer set of convictions.

However, Anaximander's disengagement from myth is only half-hearted. It is true that his world view goes without supernatural beings. But one the one hand, Anaximander sticks to the idea of the fateful power of descent as he claims that everything has a reason which is its destiny. This connection becomes clearer in the original word *archē*, which means not only reason or cause but also origin, descent and birth. On the other hand, Anaximander still believes in an ever-reliable existence, which he himself considers "divine" (Anaximander, trans. 2007, p. 37; Heinrich, 1981, pp. 60ff). Parmenides, in turn, presents his logic as a divine revelation. Calling the idea of the everlasting *truth*, he founds essentialist philosophy, which imagines a world of imperishable truths and dominates a large part of Antique philosophy (Vernant, 1962/1982, pp. 131f). In Heinrich's reading, the four laws of thought that had developed with Anaximander and Parmenides are an abstract derivative of the mythical religion – a world view that preserves the ideas of descent and the imperishable but avoids the debated existence of Hesiod's and Homer's gods. The spirituality of the laws of thoughts consists of the belief in the imperishable which opposes the threat of passing away and of the conviction that everything is connected and determined by ancestry. The law of identity is the imperative to belief in the imperishable. The laws of the excluded middle and excluded contradiction create the need for a decision between being and not being, between the true and the false. It is the mixtures and alternatives of being and not being, becoming and passing away, that are excluded. Therefore, Heinrich recognises early logic as a doctrine of salvation: "'Do not be afraid', for there is an existence which is not affected by fate and death" (my translation; Heinrich, 1981, pp. 45f).

LOGIC AND KNOWLEDGE

Apart from the promise of salvation, the presented example of logic offers a paradigm to order and explain the phenomena of our world. Its rules are easy and familiar; they provide a supportive and socially accepted 'machinery of thought' which allows thinking to approach complex fields on well-concerted ways. This potent form of thinking is the answer to the confusion caused by the collapse of traditional religious, moral and political beliefs and it is the impetus of philosophy. An illustration of the latter is Plato's Socrates who, in the dialogue with Theaetetus, stated that "suffering from confusion shows that you are a philosopher, since confusion is the only beginning *(archē)* of philosophy" (Plato, trans. 1921, pp. 155f; incorporating the translation in Heinrich, 1981, p. 31).

The price for the extended range of thought is the limitation of the thinkable. It is a main point in the *Dialectic of Enlightenment* by Horkheimer & Adorno that logical thinking loses sight of everything that does not fit into the antagonism of being and not being. Spinoza's logical ethics and the *Tractatus* of the young Wittgenstein, who wanted to trace language back to logic, show the limits of logical thought. Historical and cultural alternatives prove that logic in the form of the four laws presented is not the only form of fruitful thinking. Homer's epics, which count to the earliest records of ancient Greek writing, present meaning in analogies instead of logically ordered, in colourful images that connect the phenomena and emphasise commonalities. Even in the boom of early logic, Greek philosophy has critical schools such as the one of Heraclitus, who confronts the belief in the imperishable and true with the idea that everything is in flux and nothing persists, that "one cannot step twice into the same river" (Heraclitus, trans. 1979, p. 53). Today, many indigenous cultures understand and approach the world in a way which is based on the ideas of flux and mixtures instead of on the stasis of the unalterable and separated (Little Bear, 2002).

LOGIC AND POLITICS

Aristotelian logic also has a political dimension, which Xenophanes points at when he calls it a 'technique of reasonable speaking' (*logikē technē*): It is a tool for public speech, a tool for convincing and discrediting. Ancient Greek city states were usually governed by a democratically organised military aristocracy which represented a large part of the bourgeoisie. Politics was performed on the market place, where orators had to win majorities for their political campaigns. Politically powerful were those who performed best in convincing the people and discrediting the opponents (Vernant, 1962/1982, pp. 46–68). With his work, Aristotle wants to give directions for a convincing speech. He wants to show "what we must look for when refuting and establishing propositions" (Aristotle, trans. 2006, § 1). The question for truth and its origin, the rejection of the indefinite and unsteady, the exposure of inconsistencies, the installation of antagonistic options and the demand to decide are rhetoric weapons provided by philosophy. The teaching of such tools to the politically ambitious military aristocracy provided an income for many philosophers and explains their interest in logic. Thus, early logic constitutes a basis for democratic decision making. It provides socially accepted rules for argumentations in political discourses and helps to reduce more repressive forms of power. Nevertheless, even a democratically legitimated administration needs tools to control the population. Philosophy develops techniques which allow taking advantage of the logical laws of thought. As the acquisition of these techniques requires prosperity, logic has been an instrument of power for the aspiring middle class since its very beginning. As intended by Aristotle, the popular acceptance of the laws of thought subjects the masses to a form of speaking and thinking that they have nothing to set against.

THE DIALECTICS OF LOGIC

The cultural-historical analysis of the example of Aristotelian logic reveals its dialectic nature and allows for a comparison of the benefits and constraints it brings. On an epistemological level, it has already been pointed out that the laws of thoughts *ex-*

pand the range of thinking while *narrowing down* its focus. On a political level, early logic constitutes a *tool of power* which allows a less violent form of government but is *restricted* to one social class. Apart from that, the dialectics of logic become visible on a religious level: The belief in an imperishable truth might appease people who fear changes, especially the passing away. The emotionality with which Anaximander's infinite and Parmenides' truth is defended against any variation indicates a defence of this pacification. As Heinrich points out, even the *non* in Parmenides' "non being" (*mē einai*) translates with an undertone of menace (Heinrich, 1981, pp. 45f). Parmenides presents his logic explicitly as the salvation from an insane form of thinking – a form which "changes ways", which "considers being and not being the same and yet not the same" and which believes in "becoming and passing away". Parmenides' poem compares to a religious conversion using defamation and dictation: Those who think differently are said to be "double-headed", "helpless", "erratic", "drifting", "deaf" and "blind", "lost in confused wonder" and "unable to make decisions" (my translation; Parmenides, ed. 2009, pp. 17–23). The reader is told what to think and consider: to avoid the dissidents and follow logic.

However, the Frankfurt School points out that the belief in the unalterable only appeases some people while it frightens others: A world whose essence is invariant is a world that people cannot contribute to, that cannot be affected and must be endured. For Parmenides, humans are not the measure of all things (as Protagoras claimed to point out that everything is characterised by how we perceive and use it) but things *are by themselves*, beyond any human impact (Heinrich, 1981, pp. 32ff, 42). Consequently, the dead truths can only be approached in a quest for unveiling their dead and timeless mysteries. The dying Socrates believed in that when he told Phaedo that "the philosopher desires death" which is the ultimate "separation of soul and body" and frees the philosopher "from the dominion of bodily pleasures and of the senses, which are always perturbing his mental vision" and hindering him "to behold the light of truth" (Plato, trans. 1892, pp. 64–65). It is the paradox of essentialism that it forms an alliance with death to defend the passing away.

There is a touch of irony in the fact that logic cannot justify its basic laws in argumentation but has to build on myth and demand obedience as seen in the works discussed above. After his postulate of the law of excluded contradiction, i.e. that "it is impossible for anyone to suppose that the same thing is and is not", even Aristotle scolds: "Some, indeed, demand to have the law proved, but this is because they lack education; for it shows lack of education not to know of what we should require proof, and of what we should not" (Aristotle, trans. 1933, p. 1006).

Considering the actuality of the dialectics of logic, it has to be admitted that today such a pathetic statement concerning the laws of thought would hardly be expected. In Western society these laws are widely consolidated and accepted whereas in ancient Greek they had to be defended against alternatives. Right in the genesis of what came to be Aristotelian logic, in the time when the laws of thought had to fight for their place in the world, the social background of logic comes to light. Therefore, ancient culture can tell us about disputes whose consequences we have learnt to accept without question. Indeed, it can hardly be claimed that the cultural-historical analysis of the example of Aristotelian logic does not affect our age: On the one hand, the exclusion of the changing, of alternatives and mixtures as well as the orientation on ancestry are integral parts of Aristotelian logic, regardless of the social circumstances of the time; even today, they open a field of possibilities for religious, epistemological and political use. On the other hand, the fear of passing away, decision making in democracy and the organisation and validation of knowledge are still challenging our culture (although they might appear less urgent due to the techniques we already have developed to deal with them).

Logic and mathematics have been enormously influential for philosophy and science as we know it today. In the age of Enlightenment, when science had to find its meaning and place in society, logic and the empirical method were the points of orientation and left a formative imprint on modern thought. René Descartes was a pioneer of this process. He was convinced of "the great superiority in certitude of Arithmetic and Geometry to other sciences" and argued that "in our search for the direct road towards truth we should busy ourselves with no object about which we cannot attain a certitude equal to that of the demonstrations of Arithmetic and Geometry" (Descartes, 1684/1990, pp. 224f). Mathematics is considered a prototype of science as it can be more logical than any other science: Its objects can be alienated from our world as far as necessary to fit the logical form and the validation of its assertions does not need any experiments but relies on logical argumentation alone. Mathematics has become modern not by turning towards empiricism but by petrifying its unique status. The foundational crisis of mathematics was connected to the critique on the idea of an unalterable truth. When the crisis was silenced by David Hilbert by negating all connections between mathematics and the world we live in, Hilbert installed mathematics as the science of pure structures: as a science that commits itself thoroughly to the order of logic (Hilbert, 1922).

LOGIC, MATHEMATICS EDUCATION AND SOCIETY

School mathematics is only a part of what is called 'mathematics' today – a part with a long tradition of specialised content and its presentation. As school mathematics builds on a long tradition and has little structural need for contact with contemporary academic research, it is not surprising that it hardly includes any new mathematics or any new philosophy of it. Especially the idea of 'truth' is more classical than modern, more Platonic than Constructivist. School mathematics excludes any contents that could threaten the classical idea of truth, e.g. non-Euclidean geometry, paradoxes of set theory or alternative logics. On the contrary, it adds only those parts of mathematics to its Euclidean core which approve the power of logic: Calculus and probability theory demonstrate how even the infinite and chance can be mastered by logic.

Experience shows that German students stop calling a task 'mathematical' when it no longer has a unique solution. Obviously, school mathematics follows the law of identity in allowing only one true and right answer, excluding any variation, individual interpretation and ambiguity. How school mathematics effects students can be traced in school books, e.g. in Brückner's (2008) representative school book for the 7th grade of high school. In there, we do not only find an overwhelming dominance of tasks that allow for only one true and right answer and tasks that ask for the "truth content" of statements or for a decision between "true or false", but also a task in which the reader is told that an algorithm "was applied wrongly". The fact, that the authors provided this task with the emphatic title "Caution, mistake! Watch out!" demonstrates how uncommon it is that the mathematics school book provides something else than 'the truth'. This taboo of the alterability of mathematics enables students to believe that school mathematics is about true and right answers and that, in general, every task that can be formulated mathematically has a distinct solution. Anything that could lead students to a different belief is excluded from the mathematics classroom. As school mathematics is the only mathematics students know, it prepares students to believe that mathematics is generally able to provide unique and unquestionable answers for any question. This turns mathematics into a tool of power with the help of which the public can be convinced of originally questionable decisions.

The same school book presents the following text without any further explanation, but with a sketch showing a circle and a line of each type (p. 142; my translation):

> Lines and circle can have different locational relations.
> Secant: a line that cuts a curve (g_1)
> Tangent: a line that touches a curve (g_2)
> Passant: a line that avoids a curve – the passing line (g_3)

Somebody not knowing these terms could rightly ask: *Can a tangent be a passant as it does not intersect but passes the circle undamaged? Can a secant be a tangent, i.e. can you cut without touching?* Only the reader familiar with the idea of classification could know that it is forbidden to place a line in more than one or in none of the three categories and that the definition quoted above is 'meant like that'. Expecting the excluded middle and the excluded contradiction in the contents of mathematics is an *unspoken* prerequisite for its understanding. Thus, mathematics education does not only cultivate an unreflected and politically problematic idea of truth which gives power to those able to use mathematics; it also provides this power unequally. Those who understand the latent order of mathematics gain the possibility to perform well and become confident in the use of mathematics while those who do not understand its latent order are excluded from its power. Not talking about the logic architecture of mathematics leaves a large part of the students without any chance of understanding it and keeps the circle of mathematicians as exclusive as possible.

Although I presented only two examples for connections of school mathematics, logic and society, experiences from the classroom and students' comments on mathematics education (Jahnke, 2004; Motzer, 2008) indicate further connections, drawing even closer to the dialectics of logic presented before. On the one side, there are students who dedicate themselves to mathematics passionately, who understand and use its order, and who are able to explain and argue more convincingly than others. On the other side, there are students who are frightened by mathematics, who consider it lifeless and unapproachable, who – in spite of great efforts – do not understand its order, and who must accept the explanations of the teacher and capable students unchallenged. In an extreme case, we can speak about two different types of students: some who enjoy mathematics and logic and use it to order and present their knowledge persuasively, and others who – anxious and confused – distance themselves from mathematics and trust in the guidance of the mathematically educated. This mechanism separates a logically emancipated elite from its subjects. Particularly, this mechanism influences the students' relations to mathematics. It determines how mathematics is perceived and how people react to it. Research on mathematical world views shows that such a predisposition might indeed feature a view of mathematics as an inerrant decision-maker, the involvement in which is frightening and too demanding (Leder, Pehkonen, & Törner, 2002). In consequence of this mechanism, mathematics serves as an instrument of power which is trusted by the majority of people and whose rationale is no longer questioned. Mathematics education acts as an institution which unconsciously installs this instrument of power. This study show that the social impact of mathematics education is not a matter of the style of teaching but connected to mathematics itself; for – since its very beginning – mathematics was the manifestation of a tool of power called 'logical thought'. The question on how to face this social impact of mathematics is open to debate.

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DIVERSITY, DIALOGISM AND MATHEMATICS LEARNING: SOCIAL REPRESENTATIONS IN ACTION

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Learning is a dialogical process in which diversity plays a major role. This research illuminates the connections between collaborative work, social representations and mathematics learning. We assumed an interpretative paradigm and developed an action-research project. The participants were eighth graders (N=21), a teacher/ researcher and two other observers. Data was collected through an instrument to evaluate students' abilities and competencies, tasks inspired in projective techniques, questionnaires, observation, informal conversations and students' protocols. We did a narrative content analysis from which inductive categories emerged. Paula's life trajectory of participation is a paradigmatic example we discuss through the analysis of her social representations and her performances in some mathematics activities.

INTRODUCTION

Freire (2003) stated that education is a political act, based on social interactions and participation, or in the lack of possibilities to participate. Thus, it is dialectical and dialogical (Marková, 2005), as there are always conflicts that need to be addressed. However, according to dialogism, conflicts contribute to knowledge and progress. Education should celebrate and respect students' values and cultures, facilitating their participation in school activities, particularly through their (cultural and social) practices. How teachers interpret curricula and put them into practice, as well as the social interactions they promote while students are solving mathematics activities also play a major role in their performances and achievement (César, 2009, 2013). In order to express and respect each student's diversity, allowing them to have access to achievement, social interactions must contribute to the distribution of power, i.e., to students' empowerment (Apple, 1995). As Alrø, Skovsmose and Valero (2005), we assume diversity as a resource for learning as it brings into classes different experiences and perspectives related with knowledge, ways of thinking, solving strategies, abilities, and competencies. Diversity is connected to the participation in different cultures, and to the use of different symbolic resources, cultural tools and artefacts (Vygotsky, 1934/1962). Thus, in multicultural settings, being aware of the importance of culture, social interactions, and diversity is even more important, as they shape – and are shaped by – students' (mathematics) performances (César, 2009, 2013; Cobb & Hodge, 2007; Machado & César, 2012).

Some educational policy documents highlight the need of taking (cultural) diversity into account in order to promote equity and achievement in mathematics (NCTM; 2007). Learning meaningful mathematics and being able to internalise knowledge, after attributing it a sense, is a way to develop some abilities and competencies, which are powerful to avoid exclusion, as they are needed in a society that is increasingly complex and technical. Intercultural and inclusive practices are essential to allow students who participate in socially undervalued cultures to become legitimate participants while developing mathematics activities, particularly in formal educational scenarios, like school (César, 2009, 2013; Lave & Wenger, 1991; Machado, 2008). In Portugal, a new construct coined by César (2013) has gained significance in mathematics education: students' life trajectory of participation, in and outside schools. The construct of trajectory stresses the time dimension in learning – and in life – as it includes past, present and future, i.e., direction, path. The trajectories are connected to participation, or the lack of opportunities to participate, i.e., the lack of power. However, learning is a lifelong process. Thus, these trajectories of participation are life trajectories. They begin when we are born and only end up when we die. For teenagers, school, family and their peer groups play a main role in their life trajectories of participation.

Students often experience feelings of rejection, frustration, and construct negative social representations (SR) about mathematics that shape their performances (Machado, 2008). Knowing SR is essential to promote students' access to mathematics achievement and to avoid school dropouts (Machado & César, 2012). The problem that originated this study was the existence of students' negative SR about mathematics and their impacts in their (under)achievement and in their school and social inclusion. The research questions we address are: (1) Which are these eighth graders' SR about mathematics at the beginning of the school year?; (2) Which changes do we observe in those SR during the school year?; and (3) What are the impacts of collaborative work on students' SR about mathematics, in their mathematics knowledge appropriation and in their mobilization and development of abilities and competencies?

THEORETICAL FRAMEWORK

SR are a dynamic, multi-faced and dialogic construct (Marková, 2005). They give us powerful insights about how the other interprets the contexts, scenarios and situations that s/he is experiencing. Thus, SR play an essential role in mathematics learning process, particularly in formal educational scenarios (Machado & César, 2012). SR are constructed since we begin interacting with other people and they are shaped by our life experiences but also by the *media* or the values of a particular culture. As many students construct negative SR about mathematics it is important to develop classroom practices that facilitate changing these negative SR into more positive ones. It means that teachers should create spaces and times that facilitate students' transitions between the different cultures in which they participate (César, 2009, 2013), like the school culture and their home culture(s), also facilitating students' mobilisation of abilities and competencies, particularly in mathematics classes.

Teachers should also reflect upon the importance of power relations (Apple, 1995) and they should distribute power among them, the students and other educational agents (César, 2009). Similar to the appropriation of knowledge, the internalisation of empowerment mechanisms happens on the inter-individual plane and the intra-individual plane. Accordingly, César (2013) coined two new constructs: inter- and intra-empowerment mechanisms. She states that there are inter-empowerment mechanisms, used by those who have more power. Later, students are able to internalise them and they become intra-empowerment mechanisms. She illuminates the importance of these inter- and intra-empowerment mechanisms in students' mathematics performances and in their life trajectories of participation, particularly for those whose cultures are more distant from the school culture. Perret-Clermont (2004) designates as thinking spaces those where students can interact dialogically sharing their ways of thinking, solving strategies, doubts and fears. They are also assumed as a security and trust space and time, where students (inter)act as legitimate participants and power is distributed (Apple, 1995; César, 2009, 2013). This aspect is very important because if we want to develop equity among students (Cobb & Hodge, 2007), we need to allow them to assume their own voices, instead of keeping some of them silenced, namely in mathematics classes.

In multicultural settings collaborative work, particularly in dyads and small groups, plays an essential role (César, 2009, 2013; Machado, 2008; Tielman, den Brok, Bolhuis, & Vallejo, 2012). To develop collaborative practices one needs to negotiate a coherent didactic contract promoting autonomy, accountability, critical sense, respect for each other's cultures, and focusing on students' ways of acting and reacting. The teachers' practices change and teachers face much more challenges, as they have to work outside their comfort zones (Skovsmose, 2000). They should act as a mediator rather than someone who possesses all the knowledge. Then, they need to elaborate, adapt, and/or select tasks with different natures regarding students' characteristics, interests, and needs. Teachers should give working instructions that facilitate students' engagement in mathematics activities while working in their zone of proximal development (ZPD) (Vygotsky, 1934/1962) and respecting their own cultural mental tools (César, 2009, 2013). This also means reflecting upon the evaluation system and using diverse means of evaluation, conceiving this process as an auto-regulatory mechanism regarding meaningful learning.

METHOD

This study is part of a Master thesis developed within the *Interaction and Knowledge* (IK) project. The IK main aims were studying and promoting collaborative work in formal educational scenarios, and promoting a more inclusive, intercultural and high quality mathematics education (César, 2009). This project developed three research designs: (1) *quasi-experimental* studies; (2) action-research projects; and (3) case studies (for more details see César, 2009). We assumed an interpretative paradigm (Denzin, 2002) and developed an action-research project (Mason, 2002) at a very multicultural school near Lisbon whose region is affected by unemployment and poverty. The participants were the students (eighth graders, N=21), the teacher/researcher and two other observers. Students worked collaboratively during the whole school year, mainly in dyads but also in small groups (e.g., during the project work, in Statistics or in Functions). Data was collected through questionnaires (Q) (beginning and end of the school year), tasks inspired in projective techniques (TIPT) (beginning of the 1st and 2nd terms; end of the 3rd term), an instrument to evaluate students' abilities and competencies (IACC) (first week), participant observation (written in the researchers' diary - D) and students' protocols (collected the whole school year). Such data was then treated and analysed through a narrative content analysis (Clandinin & Connelly, 1998), performed in a successive and in-depth way. Inductive categories emerged from this analysis (for more details see Machado, 2008). They allowed us to trace students' life trajectories of participation, in and outside school, as well as to illuminate the inter- and intra-empowerment mechanisms that facilitated students' mathematics learning (César, 2013).

RESULTS

According to the epistemological and pedagogical principles of the IK project the practices we developed included a different type of practices since the first week of the school year. Teachers did not teach any mathematics contents in that week. They used three instruments (TIPT 1, Q1, IACC) to gain access to a deeper knowledge about students' characteristics, interests, and needs. The TIPT 1 allows teachers to know students' SR about mathematics. The Q1 gives information regarding students' lives (about parents or other relatives, hobbies, future job, among other aspects), and their life trajectory of participation in school. The IACC allows knowing the abilities and competencies students are able to mobilise and the ones they need to develop. This information is very important for teachers. It allows students and teachers to participate in adequate practices (e.g., nature of the tasks and working instructions) regarding students' cultural diversity, i.e., respecting their own characteristics, avoiding deficit interpretations, assuming that learning and development are a lifelong process. Teachers' practices should contribute to promote students' knowledge and development, even when the educational system needs improvements. We assume that collaborative practices, based on dialogical interactions, also allow teachers to promote (academic) positive self-esteem and to avoid (subtle) ways of rejection of mathematics and forms of exclusion.

We are going to analyse a paradigmatic example (Paula, pseudonym) that illuminates other students' life trajectories of participation during that school year. She was 14 and lived in a low socio-economical area. She and her family were Portuguese but she had characteristics that put her away from the mainstream culture: she did not have a computer at home and did not spend time in social networks, she did not wear trademark clothes, and she did not have any sort of gadgets that were usual for many Portuguese teenagers. Still, she was similar to many other students in her school and she felt accepted by her peers. The expectations her family had towards schooling were low, as they did not believe a school diploma would provide for her living in the future. Thus, she did not put any effort or trust in school and in her performances, particularly in mathematics. She was polite, but did not engage in school activities.

Thus, she experienced subtle forms of social and school exclusion. Her socialisation among her peers included extra-school activities such as going to the cinema, to the shopping centre, to the esplanades, or to the gardens near her school. Thus, she had the same kind of life experiences as her classmates.

Paula was doubling the eighth grade. Her SR about mathematics was a very common one: a difficult and boring subject. As she reports: "It is the subject in which I have more difficulty and I think it's less interesting" (Paula, Q1, September 19, 2006). She also added, "(…) [mathematics] is tiring" (Paula, Q1, September 19, 2006). This illuminates a strong rejection and a negative self-esteem regarding this subject probably shaped by her previous school experiences.

> For me mathematics is a subject in which a lot of attention is needed and a teacher who knows how to explain it well and who doesn't transform mathematics into something even more boring (*mais secante*) than it already is; and it is also a set of themes that include numbers. (Paula, TIPT 1, September 19, 2006)

She constructed a negative SR about mathematics because she associated it with a boring activity and something that needs a lot of attention. She also related mathematics to numbers. Another important feature of her answer is that she never describes mathematics as important for daily life or to get a better job, as stressed by many students (Machado, 2008) and by the *media*. This was probably shaped by the low importance of school in her life trajectory of participation and by the lack of intra-empowerment mechanisms. She participated in a culture with different life goals, ways of thinking about their future, and their role in the society. For her and her family, poor people only went to school because it was compulsory. Their lives were not going to be improved by school knowledge and studying many hours was a waste of time.

When we developed collaborative practices our aim was to support students' (mathematics) knowledge appropriation and their mobilization and development of abilities and competencies. This was our way of avoiding exclusion, of promoting equity and inclusion. In order to achieve this we needed to promote inter-empowerment mechanisms to facilitate students' access to achievement, particularly in a school system that was not tailored for them. Our commitment was precisely to avoid exclusion and to provide these students with a high quality education.

The majority of these students did not study in a regular basis outside school. One of the practices we developed to improve students' study was our conception of the homework: (1) they received some homework to do once a week, always at the same weekday; (2) what counted for the evaluation was doing or not doing the homework; and (3) if they were not able to do it or to finish it they had to explain why that happened. At least once a term the homework needed the collaboration of family and/or friends in order to collect data needed to solve it. This was one of the inter-empowerment mechanisms we put into practice in order to promote an intercultural education, that valued family's contributions to school learning. Different solving strategies used to solve the homework were discussed in class. This facilitated students self-regulation of their study, and facilitated the internalisation of inter-empowerment mechanisms as students realised we were valuing their contributions. It also promoted respect towards diversity and improved students' (academic) positive self-esteem – essential steps towards more positive SR about mathematics and about themselves as mathematics learners.

During the first term (September to December) Paula only solved one in seven homework. As she stated in an informal conversation registered in the researcher's diary, in the first weeks of the school year she preferred "Not doing it rather than being confronted with another failure" (D, September/October, 2006). After some talks with her teacher and observing what was going on with her colleagues she decided to do her first homework (see Figure 1).

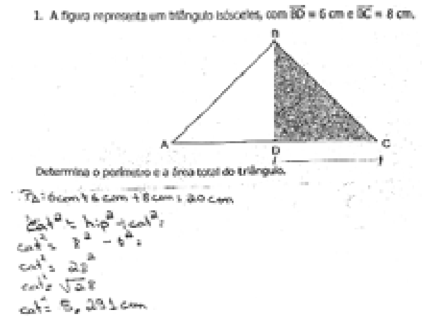

Figure 1: Paula solving strategy in her homework (October 19, 2006)

The task was composed by two parts, which included calculating the perimeter and the area of the triangle. As we can observe in Paula's solving strategy she was not able to mobilise the knowledge related to the Pythagoras theorem. Thus, although using it to calculate half of the base of the triangle, she did not realize that what she did to calculate the perimeter was wrong. As some other students also used this solving strategy, the teacher decided to discuss it when he gave the homework feedback to the students. He started by asking Paula to explain her reasoning. She went to the blackboard and drew the triangle. She explained that she calculated first the measure of DC and then the perimeter. However, as soon as teacher asked why did she calculate the measure of DC if she only used two values (8 and 6), she said: "Oh sorry! I am so dumb! I thought wrongly!" (D, October 19, 2006). Then, she explained what it should had been done and with other students' help she was able to solve the task. In that term Paula only did this homework. In the following terms she did the majority of them (2nd term - four in six; 3rd term - two in three), which illuminates that the changing process regarding participation is slow, particularly regarding outside school mathematics activities, i.e., the ones in which students feel more insecure. Thus, it takes time to become part of students' life trajectory of participation. Once the teacher was able to empower Paula, particularly through the way he acted in class, promoting inter-empowerment mechanisms, she progressively became more engaged in mathematics activities and then she was able to internalise and use some intra-empowerment mechanisms.

The next task (see Figure 2) was connected with the previous one but also with some daily life situations. It includes two questions in which students: (1) have to calculate the area occupied by the four triangular faces in the figure; and (2) have to calculate the cost of ink cans knowing that 1 litre of ink costs 1,59 Euros and 2 litres of ink allow them painting $10m^2$. Paula and Carolina started to do the first question in which Paula took the leadership. They calculated the measures they needed.

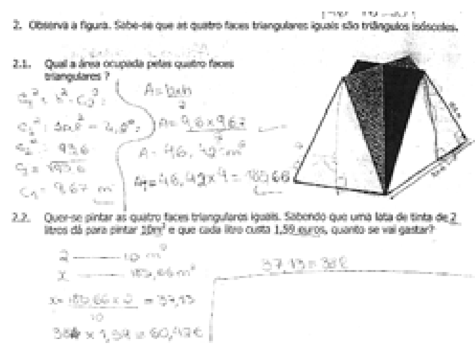

Figure 2: Paula and Carolina solving strategy (October 26, 2006)

In the second part Paula did not know what she was meant to do. She was confused by so many data. Thus, Carolina took the leadership and explained Paula the solving strategy that she suggested. This was needed because the students knew that in the whole-class discussion, every student could be asked to explain the solving strategies of her/his dyad. When Carolina did the last step – to calculate how much was the cost – Paula interrupted her and said that she was doing a mistake, because in a store we do not buy 37,13 litres. We buy 37 litres or 38 litres (D, October 26, 2006). After Paula's comment, this dyad finished the second question. This episode illuminates that the role of the most competent peer, while working in their ZPD (Vygotsky, 1934/1962), may change during a mathematics school activity (Paula, Carolina, Paula). This is important if we want students to become more confident, autonomous, developing a positive self-esteem and some intra-empowerment mechanisms (César, 2013). It also illuminates that Paula was able to connect a daily life situation with the school mathematics, i.e., she was able to do what Abreu, Bishop and Presmeg (2002) called transitions between two contexts and cultures (home and school). This is particularly important for students whose home culture(s) is/are more distant from the school culture, and for those who value school knowledge less regarding their future.

At the end of school year, Paula and her colleagues answered to TIPT 3. While analysing Paula's drawing (see Figure 3), we observe that she changed her SR. Several aspects illuminate this change: (1) the teacher has a happy face and he is closer to the students than to the blackboard. Moreover, what the teacher is saying is understood by all the students; (2) there are not any chairs and tables, which illuminates the freedom of moves students connect to the autonomy collaborative work promotes; (3) the blackboard has a smiling face, i.e., it became a nice place where she felt secure and confident; and (4) the mathematics contents (Pythagoras theorem and special cases of multiplying binomials) are connected with two different experiences: (a) related to the Pythagoras theorem, was when she started to change her SR about mathematics and started to engage in mathematics activities; and (b) it was the content she disliked the most and in which she experienced more difficulties (Paula, Q2, June 15, 2007). By then she already realised that it was better trying than not doing anything because otherwise her colleagues and her teacher could not do anything in order to facilitate her knowledge appropriation (D, June 15, 2007). For Paula, this was a very important step and corresponded to the internalisation of some inter-empowerment mechanisms, using them as intra-empowerment mechanisms in future situations.

Figure 3: Paula's drawing in TIPT 3 (June 15, 2007)

This illuminates the importance of teacher's ways of acting and reacting, and of the development of inter- and intra-empowerment mechanisms (César, 2013). These mechanisms allow students to become confident and to participate in mathematics activities, feeling that each one has a space and a time to be a legitimate participant (César, 2009, 2013; Lave & Wenger, 1991). The Portuguese educational system has no optional subjects until the 9th grade, and does not fit the needs of many students. Nevertheless, teachers' practices can make a difference and avoid exclusion.

She also chose to write. Her writing corroborates what we observe in her drawing. Paula stated very clearly that by then mathematics was real nice and funny.

> In this term for me mathematics was super, mega, deep, huge, intensely fun… (Paula, TIPT 3, June 15, 2007).

During that school year, Paula had Level 2 (1st term) and Level 3 in the other two terms (Level 1 is the lowest and Level 5 is the highest mark; you fail if you get Levels 1 or 2). She was able to progress to the 9th grade without any negative marks. She stated that "The mathematics classes could be always like this… for next school year" (Paula, Q2, June 15, 2007), which illuminates the important role that collaborative work had in her life trajectory of participation, during that school year. It also illuminates that by then she realised that being at school and learning could be important for her future and to get a better job. However, above all she felt much more confident in her abilities and competencies, and she was able to mobilise them in many more contexts, scenarios and situations, as we observed and she also told us in several informal conversations.

FINAL REMARKS

Dealing with diversity is more challenging and meaningful if we assume it as vehicle for inclusion and learning, namely in mathematics. Teachers need to know students' SR about mathematics since the beginning of the school year to decide about the tasks and working instructions they will use in classes. Gathered with other information from the IACC, a questionnaire and the observation it shapes decisions about the first dyads. Collaborative work, in dyads and small groups, associated with a coherent didactic contract and working instructions can be used as a meditational tool to facilitate the change of students' SR. It also facilitates students' engagement in school activities, their mathematics knowledge appropriation and the mobilization and/or development of abilities and competencies, as it promotes dialogical social interactions between students, and also with their teacher. Developing practices that shape more intercultural and inclusive scenarios in which students have voice(s) and the power is distributed facilitates their legitimate participation in mathematics activities, promoting a high quality mathematics education. It also allows students to internalize inter-empowerment mechanisms and then using them as intra-empowerment mechanisms (César, 2013). Thus, their life trajectories of participation are also changed and include more possibilities of choice.

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'REGIME OF COMPETENCE' IN A SCHOOL PRACTICE WITH ROBOTS

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This article analyses and discusses some topics that highlight the social nature of learning (Lave & Wenger, 1991). In particular it analyses how the 'regime of competence' (Wenger, 2008) is defined in a school practice, in which students work with robots. The learning scenario presented here followed a project work methodology and involved two primary school classes. The research reported in this paper follows a qualitative approach and participant observation was a central strategy in data collection. People in action constituted the unit of analysis. Relevant considerations about how competence is usually defined in the school context emerged from the discussion of the results.

INTRODUCTION

In the context of school, learning is often understood as an individual process that results from the act of teaching and in which it is best to disengage it from other students' activities. Based on this assumption, classrooms are often organized as a place where students shall pay attention to the teacher and focus on 'making' exercises, while ignoring from the distractions of participation in the outside world. This model is inadequate when we take learning as a social phenomenon (e.g. Fernandes, 2004; Lave, 1996; Lave & Wenger, 1991; Santos, 2004; Wenger, 1998). Aspects of mathematics and mathematics education, largely ignored in the past, have gained a renewed interest within our scientific community in the last two decades. The classroom as a social context with different backgrounds, beliefs, agendas and expectations of its players has been a focus of much research in mathematics education (e.g., Atweh, Forgasz & Nebres, 2001; Boaler, 2000; Valero & Zevenbergen, 2004).

Lave's studies (1996) of the acquisition of mathematical competence based on practices involving adults in workplace situations, specifically, within tailoring apprenticeships, led her to argue that cognition is located in particular forms of situated experience, not simply in mental contents. In the work that Lave developed together with Wenger (1991) it is clear that the focus moves away from cognition to a new approach that is related with learning concerns. Lave and Wenger (1991) clarify the relations that they consider essential between participation and learning: learning is a deepening process of participation in communities of practice.

By understanding a community of practice (Lave & Wenger, 1991) as a social learning system, Wenger (2010) locates learning in the relationship between the person and the world. This is a relation of participation where the social and the individual complement each other and where meaningful learning requires both participation and reification to be in interplay. Over time, this interplay creates a social history of learning that gives rise to a set of criteria and expectations by which participants recognize membership. In other words, participants define what counts as ‘competence’. According to Wenger (2010) this competence includes being able (and allowed) to engage productively with others in the community and to use appropriately the repertoire of resources that the community has accumulated through its history of learning (p. 180).

Our aim in this paper is to discuss some ideas that enhance the social nature of learning by focusing on the ‘regime of competence’ (Wenger, 1998) of students from two primary school classes working together in a project work with robots [1].

EXPERIENCE AND COMPETENCE

Following Wenger (1998), the term participation describes the social experience of living in the world in terms of membership of social communities and active engagement in social enterprises. To participate is also to belong. It is both personal and social. It is a process that combines doing, talking, thinking and feeling. It involves the whole person including her/his body, mind, emotions and social relations. Engagement in social settings involves a dual process of making meaning, which results from the interplay between participation and reification (Wenger, 2010, p. 179). Wenger (1998) uses the word reification to refer to the process of giving form to our experience by producing objects that freeze this experience into “thingness” (p. 58). By reification, we create points of focus around which the negotiation of meaning is organized. Although participation and reification mean different things, we cannot conceive one without the other. They complement each other (Wenger, 2010). On the one hand we engage directly in activities, conversations, reflections and other forms of personal participation in social life and on the other hand we produce physical and conceptual artifacts such as words, tools, concepts, methods, stories, documents and others forms of reification that reflect our shared experience and around which we organize our participation. The interplay between participation and reification creates a social history of learning by which participants define a ‘regime of competence’ (Wenger, 1998).

Wenger (2010) defines a ‘regime of competence’ as a set of criteria and expectations by which the members of a community recognize membership. In this sense, communities are seen as social configurations in which their members experience competence and are recognized as competent (Wenger, 1998). Therefore, the term competence has no meaning, when disconnected from a particular practice. What is seen as competence is constructed and defined within the community. “It is by its very practice – not by other criteria – that a community establishes what is to be a competent participant, an outsider, or somewhere in between” (Wenger, 1998, p. 137). To be competent includes the understanding of what is important and what matters in the community, reflecting the accountability to the joint enterprise.

According to Wenger (1998) this competence is not merely the ability to perform certain actions, the possession of certain pieces of information, or the mastery of certain abstract skills. Competent membership presupposes three features: **Mutuality of engagement**: the ability to engage with other members and to respond to their actions in order to establish relationships in which this mutuality is the basis for an identity of participation. **Accountability to the enterprise**: the ability to understand the enterprise of a community of practice and take some responsibility for it and contribute effectively to its pursuit and to its on-going negotiation within the community. **Negotiability of the repertoire**: the ability to make use of the repertoire of the practice to engage in it. This requires enough participation in the history of the practice to recognize it in the elements of its repertoire and it requires both the capability and legitimacy to make this history newly meaningful.

The interplay between a 'regime of competence' and the experience of meaning allows learning by practice. According to Wenger (1998) there are moments when competence may drive experience and other moments in which the opposite occurs. In order to achieve the competence defined by the community, competence may drive experience when newcomers transform their experience until it fits within the regime. On the other hand, the members of an existing community also have a need to transform and increase their experience. However, new experiences may lead to the need to redefine the enterprise and require to add new elements to the repertoire of their practice. When one or more members have had some experience that currently falls outside of the 'regime of competence' of a community to which they belong, they may very well attempt to change the community's regime so that it includes their experience. Thus, the meaning of their experiences is negotiated with the community of practice. They can invite others to participate in their experience and then seek to reify it for the others. If they have enough legitimacy as members, they will have an impact on the 'regime of competence' of the community and participate in the process of creating new knowledge. Learning can be thought of as a process of continuous interaction between experience and competence, "whichever of the two takes the lead in causing a realignment at any given moment" (Wenger, 1998, p. 139). A certain tension between experience and competence is what promotes learning.

METHODOLOGY

This paper takes *learning* as the phenomenon of study. The research sought to understand how the use of robots could contribute to the development of mathematical and other competencies, and to the appropriation of mathematics concepts by primary school students. To do this, a learning scenario was designed, which involved two primary school classes (2nd and 3rd grade, 24 and 16 students, respectively) from a school in Funchal – Madeira island – Portugal. In this learning scenario the children worked together with robots. In this section, we will describe the learning scenario and the methodological options. This establishes a connection between the nature of the phenomenon under study and the theoretical background.

Learning Scenario

We conceptualize scenarios as "stories about people and their activities" (Carroll, 1999, p. 2). Scenarios have some characteristic elements, such as a context, a setting, the agents or actors and their goals. It includes a sequence of actions and events to be developed in order to achieve certain goals. These goals are changes to be accomplished by the actors in the circumstances of the scenario. The scenario's narrative is fundamentally a description of people accomplishing tasks, pursing goals and using technologies to achieve those goals. The learning scenario was constructed by the research team, the teachers from both classes and by their students. At the beginning, the research team presented, to both teachers, a draft of the learning scenario to be implemented. That initial draft was discussed and modified according to teachers' and students' ideas. In this process students had the opportunity to express their own options. These were very important for them and for the success of the project. Between the working sessions, teachers often contacted the researchers to convey students' opinions and expectations. The learning scenario was developed in two stages: the first between May and June 2011 and the second between April and July 2012. The scenario's activities followed a project work methodology.

In this project, students worked with two kinds of Lego robots: RCX and NXT. The programming environment is a very intuitive icon-based drag-and-drop programming language for both robots. It was designed for an easy introduction to programming. Students simply build up their program block by block by choosing program blocks that work with the motors and make the sensors react to inputs. In this way, they could create programs that range from simple to complex. Students and teachers had never worked with robots before.

Students worked in heterogeneous working groups with students from both classes. Teachers had to support students in their work. Researchers sought to support students and teachers and to take advantage of situations that could contribute to facilitating the emergence of mathematical concepts. Based on that intention, researchers assumed a questioning attitude towards students' work in their practice with robots.

In the first part of the scenario's implementation, students had to construct robots and define their physical and emotional features. Their creations would become characters in a play-story written by them all. After writing the story, students had to program their robots in order to perform their roles in the play. The initial goal was to accomplish those tasks in order to make the robots as characters on the play.

In the second part of the scenario's implementation, students, teachers and researchers decided to produce a film. The written story was used as its storyline. Students established new tasks to produce the film and created teams to accomplish those tasks. Each student chose in which team(s) he or she wanted to work. In this paper we will focus on the teams that decided to do the programming of the robots.

Methodological options

The nature of the research related in this article is qualitative and it was given particular relevance to the process and not to the product of the developed activities (Bogdan & Biklen, 2006).

Taking the phenomenon under study – learning –as connected to participation in specific practices, it became important not only 'to observe' but also 'to participate' in the activities in which students were involved. In fact, assuming a situated perspective of learning as a theoretical framework has certain methodological implications: Investigation is participation in the constellation of practices in which the research occurs (Matos & Santos, 2008). This was the position of the researchers involved in data collection. Participation was also learning. Thus, participant observation was a central strategy of data collection. The challenge was to maintain a genuine participation and to be able to reflect on it (Matos & Santos, 2008). In this research there was a close connection between the phenomenon under study and the theoretical framework. The unit of analysis in this research was constituted by people in action and it was analysed in the dialectic between the theoretical framework and the practices that were observed, experienced and reflected and that instantiate empirically the problem under study (Matos & Santos, 2008).

The study involved semi-structured interviews with some participants in order to clarify some aspects of the practice that raised doubts or were unclear when the data was analysed. The working sessions were video and audio recorded with a focus on students' interactions. Not every phenomenon could possibly be recorded so researchers wrote down what occurred in the form of extensive field notes. Soon after, these notes were analysed in order to note patterns of behaviours, events and phenomena to be investigated in further observations.

WHAT DOES IT MEAN TO BE COMPETENT IN THIS PROJECT WITH ROBOTS?

A 'regime of competence' is constructed as a shared process of definition of a community's joint enterprise (Wenger, 1998). In this study, the student's joint enterprise was to write and implement a play-story in which robots were the characters (Martins & Fernandes, 2012). That joint enterprise allowed opportunities for students' engagement across distinct forms and levels. In that process of engagement, the competencies of each member were jointly constructed. Students defined what was important for each one to know and they had to develop abilities to make a connection with what each one didn't know. In the project work with robots students often needed to establish a division of tasks. That aspect contributed to the mutual recognition of competencies in the on-going practice. In that process students assumed responsibility for distinct aspects of the joint enterprise. The choices they made were intrinsically linked to their individual preferences but were also in accordance with what was considered as important to pursue within the community's joint enterprise (Martins & Fernandes, 2012).

Some students were responsible for specific tasks within the practice. Once students agreed on what was important to do in order to accomplish a specific task, they jointly gave legitimacy to those students who were responsible for it. Students revealed an ability to understand the community's joint enterprise and to be responsible for it. This revealed their accountability. In this sense, we can argue that competences are neither merely individual nor abstractly communal. They implied a negotiated definition of what the community is about. What makes engagement possible is a matter of diversity and constant negotiation of meanings, reflecting the way participation occurs in on-going activities (Wenger, 1998). As we saw earlier, new tasks were jointly defined in order to be able to produce the film. In our, analysis, we will focus on the tasks that were carried out by the programming teams. Those two teams (NXT team and RCX team) had to learn to program the robots in order to correspond to the remaining teams' solicitations when the shooting of the film began.

R and M were two students who worked in the RCX team. In the extract presented here, these students were working together on the same computer, although they had two computers at their disposal. They were trying to program the RCX robot, T-Rex [2]. These two students were initially disappointed because they were having some problems uploading the program to the robot. One of the researchers, Res, helped them to solve this problem. Throughout this process students' difficulties were discussed. The researcher noticed that the program they were trying to upload was very long and that the students were using the programming blocks without having the notion of the action that the robot would execute after programming. The researcher challenged them:

Res: I want you to program the robot in the computer. Don't ask him to do too many things…. Then you can write in a paper what the robot is going to do when you run the program.

Later, the researcher found that the students were programming on distinct computers and the programs were clearly 'shorter'. M continued programming the T-Rex and the other student, R, was programming another robot. The researcher asked M:

Res: So, can you tell me what the robot is going to do when you run the program? [Student whispered:]

M: I can't talk now. I don't want that R hears what I'm saying.

Res: Why?

M: Because we are doing like this: I´m programming my robot and R is programming his robot. Then we are going outside. The programs will run and we have to discover the programming from each other´s robots… looking to the robots.

The researcher continued observing. Afterwards, the two students went to the courtyard to test the programs they've made. In the courtyard the researcher realized that students began to identify the blocks they used in programming in the robots' actions.

This extract shows how students couldn't upload the program to the RCX brick at the beginning and how they were disappointed. Some mistakes were made in that moment and students experienced some conflicts, advances and retreats. However, all those components have proven to be learning opportunities. In the interactions between the students and the researcher, relations of mutuality were established, in which they jointly defined what was important to learn. To achieve the competence that was defined by the community, students transformed their experience of programming in order to fit within the 'regime of competence'. The researcher expected that students could achieve a better understanding of the program blocks that they used, when she challenged the students to write on paper the robot's actions. Both students revealed accountability and tried to develop this understanding about programming. However, instead of writing the actions of the robot on paper, they assumed that knowing how to program also meant recognizing the program blocks in the robot's actions. This was a big step in the learning of programming.

We will now analyse an episode with the other programming team. Seven students who were working on two computers constituted the NXT programming team. In the first session as a team, three students programmed Lama and Spider and the remaining students programmed two twin dogs. In the following session, the students had an additional computer available. The researcher suggested to swap the robots. All students agreed except H:

H: I was in the working group that built the robot Lama. In the previous session I was working in Lama's programming and I want to continue doing it. I don't want to program the other robots.

Res: But which is your team now?

H: NXT programming.

Res: So there is not a specific team responsible only for Lama's programming, right?

H: No it doesn't.

Res: If you have programmed the Lama and the Spider I think you now must change, for all of you have the opportunity to program all NXT robots.

H: Then I will not do anything.

The other groups worked on programming Spider and Lama and H returned to his group, which did not yet choose any robot. Later a student from H's group, A, asked:

A: Can we program one of the twin dogs?

Res: Of course. H said to me he didn't want to program a dog…

A: He doesn't want but we want to. We don't want to stay without doing anything.

After that, the researcher noticed that H still did not participate in programming the robot. This was very different from his behaviour in the past working sessions, in which he was always very involved in the activities. The researcher noticed that the students that were programming the Spider and the Lama were having some difficulties in doing it. She asked H:

Res: Can you help your colleagues with the Spider and Lama's programming? I think they're having some problems with that, but I can't help them at this moment. I really think that your team' colleagues are having some difficulties with the dog's programming too and they do need some help.

H: Can I?

Res: Of course. But don't program the robots alone, without teaching them. You have to find a way to make them understand how to program. I think you can do it.

After this moment H started to support all of the three programming NXT teams.

This episode demonstrates that H was not revealing mutuality in his engagement with the practice of programming in the beginning. What the researcher and his colleagues expected from him was not the same as what he considered important. That ambiguity ended up compromising his engagement in the on-going practice. After the researcher recognized H as competent to program any NXT robot he was committed to help other students with this task. The legitimacy that the researcher recognized in H was a turning point in H's participation in the practice of programming NXT robots. In fact, granting the newcomers legitimacy is important because they become aware of what a community regards as competent engagement (Wenger, 1998).

CONCLUDING REMARKS

Learning mathematics implies knowing mathematical concepts and using them to solve problems. When students were programming, several mathematical concepts connected with time and space have emerged, such as: positioning, orientation, duration, trajectories, direction and movement. However, this project work with robots gave students the opportunity to experiment with many other important aspects that we identify as being crucial to students' mathematics learning. These include developing and evaluating inferences, testing hypotheses and investigating conjectures, communicating coherently mathematical ideas and evaluating arguments and strategies of others.

In the school practice under analysis, mutuality was the basis of membership's recognition by the community and thus, the regime of competence was defined in relationships based on mutuality (micro-level). Those relationships were guided by the negotiation of meanings, by the constant division of tasks and by the accountability in achieving the goals that were jointly defined in the learning scenario. Students have done that and had that opportunity because teachers and researchers gave them legitimacy to do so. These aspects led to opportunities of alternative approaches to tasks and to opportunities to use different artifacts. These approaches and artifacts helped the community to define the competencies of participants, that is, the definition of the community's 'regime of competence'. The analysis shows how school practices with these features provide learning opportunities for all involved, in which the errors and conflicts are taken as natural and can be recovered as special situations for learning to occur.

In school practices of a more 'traditional' nature, competence is often conceptualized as being good in making/producing something and little emphasis is placed upon the process of doing and the relationships that have developed between people when they were doing it. As we have discussed, being competent is closely linked with what is recognized individually and collectively as competence in a particular practice, revealing the accountability within the joint enterprises. It implies not only being recognized as competent but also having legitimacy to participate meaningfully in the constant negotiation and definition of what we want to achieve.

What is defined as 'regime of competence' at the macro-level by the Portuguese Minister of Education is defined in a document named "competencies book". That document defines which competencies must be developed at school and what kind of experiences (e.g. projects) teachers must provide to their students in order to develop those competencies. The way teachers recontextualize this document to their practices is quite different for each one of them and in the most cases they conceptualize competence as being a good student, having good classifications in the exams and being good in following the school's norms. In this experience of working with robots the regime of competence defined in the micro-level matches the regime of competence defined in the macro-level and the conditions that promote that success was connected with the nature of the proposed tasks and the artifacts that allowed student's engagement and mathematical concepts emergence.

NOTES

1. The research reported in this communication was prepared within the project DROIDE II – Robots in Mathematics and Informatics Education funded by Fundação para a Ciência e Tecnologia under contract PTDC/CPE-CED/099850/2008.

2. T-Rex, Lama, Spider and two twin dogs are robots created by students that became characters in the play.

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THE DIALOGICAL MATHEMATICAL 'SELF'

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Mathematical identity is a growing area of contemporary research. This paper uses dialogical self theory to support a notion of mathematical identity as being made up of multiple positions. These positions make up the mathematical 'self'. This paper focuses on a single case drawn from a wider study of parental mathematical identity. It demonstrates the application of dialogical self theory to show how mathematical experiences influences I-positions, how social and cultural factors shape social positions, and how the dialogical mathematical self can be seen to consist of multiple positions which vary both spatially and chronologically.

INTRODUCTION

A growing body of literature is beginning to look at how mathematical experiences are incorporated into, or reflected in, a person's mathematical identity. For instance, studies have investigated mathematical identity in US classrooms (Boaler & Greeno, 2000; Esmonde, 2009), in parents of primary school children in the UK (Abreu & Cline, 2003; McMullen & Abreu, 2011), in both parents and children in the US (Esmonde et al., 2011), and on pupil and teacher identities in the UK (Crafter & Abreu, 2010) and Spain (Gorgorió & Prat, 2011).

From a sociocultural perspective Martin (2007, p.150) defined mathematical identity as:

> …the dispositions and deeply held beliefs that individuals develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics to change the conditions of their lives. A mathematics identity encompasses a person's self-understandings and how they are seen by others in the context of doing mathematics

These generally incorporate notions of positioning and positionality that can be linked back to the work of Harre and van Langenhove (1991). This defines positions as a discursive mechanism through which "people locate themselves and others within an essentially moral space" (Harre & van Langenhove, 1991, p.396). In other words, in terms of mathematical identity, through dialogue people form positions for themselves regarding mathematics.

Much of the research on identity previously mentioned used notions of positioning. Advancements in psychological and educational research shown currently in dialogical self theory present an opportunity to take research involving positioning and mathematical identity further. This affords an opportunity to better comprehend parental mathematical identity. In doing so it may begin to enhance our understanding of how identity shapes parental involvement and activity, itself a key factor in mathematical attainment in UK primary school children (Duckworth, 2008).

THEORETICAL FRAMEWORK

Dialogical self theory provides a systematic approach to understanding identity through positioning of the 'self' via participation in sociocultural activity. It originated in the ideas of Hubert Hermans and builds upon the notion of 'I' and 'self' presented by the American philosopher and psychologist William James as well as the idea of polyphonic 'voices' within a personality provided by Russian literary critic Mikhail Bakhtin (Hermans, Kempen & van Loon, 1992).

When discussing his theory of the self, James distinguished between the self as object and the self as subject (Hermans et al., 1992). The self as object is characterised as 'Me' and the self as subject as 'I'. 'I' is the self as knower, a sense of personal identity, whilst 'Me' is the self as known (Hermans, 2001) and 'Mine' is our mental 'belongings' (e.g. my daughter) (Hermans & Hermans-Konopka, 2010).

Through his analysis of Dostoevsky, Bakhtin developed the notion of the polyphonic novel (O'Sullivan-Lago & Abreu, 2010). Bakhtin argued that the characters in Dostoevsky's work have separate, distinct voices. He termed this a polyphony of voices. It is through this mechanism that the novel is told, rather than a single voice of the author. Bakhtin applied this to the study of personality (Hermans et al., 1992). He suggested that dialogical processes involved the interaction of different voices (Hermans, 1996).

In dialogical self theory these two theoretical foundations are combined. The 'self' is narrative, and therefore temporal, and so evolves over time. It is evident in the stories we tell and the ways in which the past shapes the present 'I'. It is also spatial in the sense that the 'I' shifts depending on the context we find ourselves in.

As with the research outlined earlier, dialogical self theory utilises Harre and van Langenhove's (1991) concept of positionality. Indeed, positioning is central to the temporal and spatial nature of dialogical self theory (Raggatt, 2011). The self is made up of a variety of I-positions that alter as the self moves and evolves across time in different contexts. Hermans (2001) compares I-positions to characters in a story, each of which has a separate background that shapes its voice, producing a narrative, storied self. An example of an I-position might be 'I as a student' or 'I as a father'. I-positions can easily support or conflict with each other.

Dialogical self theory is culturally embedded (Hermans & Hermans-Konopka, 2010). I-positions require dialogue and mediation. In this way dialogical self theory can be linked to the sociocultural theories of Vygotsky, Leont'ev and Wertsch (Hermans & Kempen, 1995). In dialogical intercourse in sociocultural contexts, we reflect on the labels given to us by others. These can become incorporated into the self as social positions (Raggatt, 2011).

As suggested earlier, research into mathematical identity is growing in popularity yet very little research has utilised dialogical self theory to look at mathematical identity using the notions of I-position and social positions.

METHODOLOGY

In order to investigate dialogical constructions of the 'self', twenty-four parents (16 mothers and 8 fathers), all of whom had children attending UK primary schools, took part in a semi-structured episodic interview with the first author. This approach, established by Flick (1997), asks respondents to produce opinions and narrative episodes linked to a series of pre-selected questions. Episodic interviewing is based upon the belief that narrative is a mechanism through which people understand and make sense of their experiences (Flick, 1997). Episodic interviews have been used previously to investigate processes of identity formation in mathematics (Crafter & Abreu, 2010) and in cultural contact zones of immigration (O'Sullivan-Lago & Abreu, 2010).

Interviews were conducted in the parent's home, digitally recorded and then transcribed. The respondents produced a number of narratives and opinions that were then subjected to analysis. This analysis focused on studying narratives to ascertain the positions generated by the parents. The study of narratives to elicit positions has been widely used to study mathematical identity (e.g. Boaler & Greeno, 2000; Crafter & Abreu, 2010; Esmonde, 2009; Esmonde et al., 2010; Gorgorió & Prat, 2011).

The dialogical self analysis followed three stages by investigating I-positions, social positions and multiplicity. These were based upon the approaches taken from a number of different authors.

I-positions

O'Sullivan-Lago and Abreu (2010) studied I-positions in their work on identity in cultural contact zones. In their study, I-positions were identified through the coding of episodic interview transcripts. Using a similar approach in this study, codes were applied to segments of text where the interviewees positioned themselves whilst discussing experiences or opinions associated with mathematics and mathematical activity.

Transcripts were first open coded before being studied for patterns and commonalities. This resulted in the merging of some open codes into larger pattern or thematic codes.

In this stage of analysis coded segments for the 'self' generally, but not exclusively relied upon 'I' in the first person (I, we, me, us), for instance "*I like numbers. Alright, I like numbers, I like adding things up*" was coded as 'I as enjoying mathematics'.

Social positions

The second stage of analysis addressed the role of the sociocultural environment on mathematical identity. This studied the 'voices' or social positions that could be seen in parental narratives.

This style of analysis followed the approach of Aveling and Gillespie (2008), which focused on the relationship between I-positions and the sociocultural environment.

Their coding of interview data was designed to discover the social origin of I-positions. It focused on reported speech and what they termed ‘echoes’, which are “utterances that are not attributed to others, but that nonetheless seem to have a distinct social origin beyond the speaker” (Aveling & Gillespie, 2008, p.6).

Following this approach, this stage of analysis began by open coding segments of data and then combining and refining codes through a recursive cycle. Here codes reflected actual specific voices and more generalised individuals. Codes could also be often connected to I-positions. For instance ‘I as good at mathematics’, an I-position, could be a general voice ‘I as more successful at mathematics than others’, or a more specific voice ‘I as more mathematically successful compared to my friends’.

Multiplicity and dialogical positioning in the mathematical ‘self’

Multiplicity within the mathematical ‘self’ was the subject of the final stage of analysis. It was ascertained through comparing the different positioning of the ‘self’ within individuals. This showed not only the number of positions but also the variety of similar and different ‘self’ identifications. Next, the positions within each individual were studied in terms of chronology to see whether positions remain fixed or shifted over time. Finally, by looking at the context in which they occurred, positions were compared spatially.

ANALYSIS

Because of the difficultly of presenting the wealth of data analysed in the project within this short paper, the analysis therefore focuses upon the results of a single parent within the sample. Ian was a father with two daughters, Megan in Year 4 (8-9 years old) and Louisa who attended reception (4-5 years old). Ian constructed fourteen positions related to mathematics and was broadly representative of the sample as a whole.

In first stage of analysis, focusing on I-positions related to mathematics and mathematical activity, Ian displayed twelve different I-positions as shown in Table 1. This shows Ian built a mathematical ‘self’ consisting of a range of I-positions associated with mathematical experiences, aptitudes and behaviours.

I-position
I as replicating my own upbringing
I as supporting a work ethic
I as a competent user of mathematics
I as confused by mathematics
I as good at mathematics
I as not good at mathematics
I as a novice and learning mathematically from my child
I as apprehensive of mathematics
I as enjoying mathematics
I as feeling supported by my parents
I as not interested in mathematics
I as regretful of mathematical activity

Table 1: Ian's mathematically-related I-positions

For instance in Excerpt 1 Ian positions himself as 'I as a competent user of mathematics'. Here the I-position is underlined and preceded by a number to specify its position in the text.

Excerpt 1: Ian - Interview	
Dialogue	I-position
Something, it keeps, (1) <u>I wasn't brilliant at it at school but I've got a pretty good grasp of anything like that.</u>	(1) <u>I as a competent user of mathematics</u>

In the second stage of analysis, which investigated the social and cultural influences on positioning, Ian constructed two distinct social positions. These are shown in Table 2. These social positions suggest that Ian saw himself as similar to his daughter Megan in terms of mathematical ability and attitude, and that his views of mathematics reflected the society in which he grew up and lived.

Social positions
I as similar to my child
I as reflecting the influences of my social environment

Table 2: Ian's mathematically-related social positions

A social position can be seen in the next example. Here I-positions are underlined and preceded by a number whilst the social position is presented in italic and preceded by a lower case roman numeral.

In Excerpt 2 Ian again constructs competent ability positions associated with mathematics and uses these when comparing himself to his daughter Megan. Here we see the links between I-positions and social positions. It also shows how Ian positions his daughter using his own experiences.

Excerpt 2: Ian - Interview		
Dialogue	I-position	Social position
I wouldn't say that it's her strongest subject really at school. (i) *She, she's very, very like I used to be at school I think.* *(1) <u>If I could be bothered to do it I'd do it and I'd be good at doing it.</u> (2) <u>If I couldn't be bothered to do it, I'd just wing it and get through it and be like that.</u>*	(1) <u>I as good at mathematics</u> (2) <u>I as a competent user of mathematics</u>	(i) *I as similar to my child*

Excerpt 3 shows how the environment in which Ian lived influenced his attitude towards the importance of mathematics. It shows that he saw a value to mathematics in order to be successful in society. This can be labelled as a somewhat vague social position of 'I as reflecting the influences of my social environment'.

Excerpt 3: Ian - Interview	
Dialogue	I-position
Researcher: Do you think it is important to be good at maths? Ian: Yeah definitely, (i) *You've got (pause) everyday life you use it and you are going and obviously it's a big thing in the structure of what you are going to do when you get older. It gets you certain jobs and things like that. You're always going to, you're always, it's the same as like English and things like that, you're always going to need it aren't you.* And you're better off being good at it than bad at it.	(i) *I as reflecting the influences of my social environment*

The third stage of analysis looked at multiplicity. The fourteen different mathematically-related positions exhibited by Ian hint at the multiplicity of positioning that is central in dialogical self theory. This polyphony of positions acts to construct a unique and complex mathematical 'self' for Ian. Multiplicity was a feature across the entire sample, as shown in the range of positions (6 to 26) and the mean number of positions (15).

The parents within the sample showed varying amounts of spatial and temporal shifts in positioning. It was noticeable that some parents presented a greater degree of stability over time and across contexts than others. Stability tended to increase in parents that were mathematically confident or who saw themselves as 'good at maths'.

When studying Ian's I-positions it is evident that many appear contradictory. For instance Ian positioned himself as 'I as a competent user of mathematics', 'I as confused by mathematics', 'I as good at mathematics' and 'I as not good at mathematics'. As we see when comparing Excerpt 1 and Excerpt 4, Ian's mathematical position changed depending on context and across time. In Excerpt 1, when discussing helping his daughter Megan with her homework, he positioned himself as a competent user of mathematics, even though he acknowledged that he "wasn't brilliant at it at school". When asked directly what he thought of mathematics, shown in Excerpt 4, Ian saw mathematics as a school activity that he was not good at and which filled him with apprehension and dread.

Excerpt 4: Ian – Interview	
Dialogue	I-position
Researcher: What do you associate with the word mathematics?	
Ian: Probably Numbers, that's probably the first thing that comes into your brain, numbers. (1) <u>I don't know because I was rubbish at maths at school (laugh).</u> (2) <u>You probably think, "Oh god not maths again". Numbers, dread, homework really.</u>	(1) <u>I as not good at mathematics</u> (2) <u>I as apprehensive of mathematics</u>

In a different context, shown in Excerpt 5, this time playing darts with his own father, Ian positioned himself as 'I as good at mathematics'. This stems from an ability to quickly perform mental multiplication. A change in context and activity produced a re-positioning towards mathematics. Again, he uses this to compare himself to his daughter.

Excerpt 5: Ian - Interview		
Dialogue	I-position	Social position
And I always say the same thing to her, because growing up when I was 9, 10, 11, 12 me and my dad used to play darts all the time. (1) And I learnt all my numbers from playing darts, growing up with my dad. So anything that's multiplied into 16, treble this, treble that, double that, I got it straightaway. (i) *And she's like that.*	(1) I as good at mathematics	(i) *I as similar to my child*

Excerpt 6 shows a change in positioning over time. Ian positioned himself first as enjoying mathematics at secondary school. This is opposite to the position he reconstructed of his primary-level education. Again uses this to support an identification of himself and his daughter.

Excerpt 6: Ian - Interview		
Dialogue	I-position	Social position
Err yeah, well (pause) (1) probably more from secondary school really because I probably started getting it at secondary school whereas sort of (i) *when I was sort of Megan's age I was probably exactly like she is.* (2) As can't be bothered, don't matter.	(1) I as enjoying mathematics (2) I as not interested in mathematics	(i) *I as similar to my child*

In the case of Ian, as with the whole sample of parents, it is possible to see chronological instability over time as remembered events and experiences lead to the creation of different mathematical I-positions. Similarly there is spatial instability as different activities result in different I-positions.

DISCUSSION AND CONCLUSION

The case presented in this paper is typical of the findings of this research project. It shows a highly complex mathematical 'self' that is made up of a series of different, and often contrasting, I-positions. These positions can also often be seen to reflect social positions emanating from the surrounding social and cultural environment. The mathematical 'self' can also be seen to shift and alter depending on the mathematical context and also change across a lifetime.

The analysis also shows the potential for using dialogical self theory to better understand mathematical identity, linking a context-specific activity (mathematics) to a context-specific theory of identity (dialogical self theory). A limitation of this research is that it only focuses upon the identity of the parent and does not address the identity of the child. Through involving children in further research it should be possible to use dialogical self theory to better understand the mathematical 'self' of both parents and child. In turn this should allow greater understanding of the ways in which experiences shape identity, and identity shapes mathematical activity. This could reveal the extent to which the 'self' is shaped by the 'other'. It could also study the degree to which a child's mathematical identity reflects the identity a parent holds or constructs for their child. Understanding these kinds of elements is crucial if we are to support parental involvement in children's mathematical development, a major contribution to mathematical attainment in the UK.

NOTES

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TO PARTICIPATE OR NOT TO PARTICIPATE? THAT IS NOT THE QUESTION!

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We take an experience of educational failure in school mathematics to illustrate the liberal capitalist ideology at work in educational institutions. This will be done by means of confronting the official discourse, which posits inclusion and equity as fundamental goals of mathematics education, with its actualization within a secondary school, whose student-intake can be labeled as marginalised or underprivileged. What normally runs well within the official discourse, when actualized in a specific practice, often encounters a series of obstacles that end up perverting the official intention. Usually research strives for identifying such obstacles under the imperative to eliminate them. This is assumed to ensure the full actualization of the official aims. However, we are instead interested in understanding these obstacles since they stand for the symptomatic points, which allow one to grasp the ideology manifest in current educational practices.

INTRODUCTION

International organisations (e.g. OECD), professional institutions (NCTM, 2000) and researchers (see e.g. Atweh et al, 2011; Herbel-Eisenmann et al, 2012; Gellert, Jablonka & Morgan, 2010, for very recent editions and conference proceedings) have been positing mathematics education as a key element for the development of a social just and equitable society. It is assumed that a quality mathematics education will allow people to become active participants in a world where mathematics formats many of the decisions that influence our lives. As a result, the main task of mathematics education research has been the development of teaching and learning strategies that can allow a meaningful mathematics for all. The fact that failure in school mathematics persists worldwide is seen by researchers as a contingent occurence of a system that officially aims at equity and freedom (Pais, 2012; Pais, Fernandes, Matos & Alves, 2012). As such, researchers are often interested in describing successful experiences, showing how the obstacles to the learning of mathematics can be overcome, instead of analysing episodes of failure (Gutiérrez, 2010). This propensity to report successful experiences is supported by a broader ideology that Lacan (2008) characterized as *evolutionism*: the belief in a supreme Good, in a final goal of progress which guides its course from the very beginning. In the case of mathematics education, the supreme goal is "mathematics for all", and research is set on eliminating the obstacles standing in the way of this goal (Lundin, 2012; Pais & Valero, 2012).

In this paper we present a study of educational failure. We settle our investigation in a secondary school that can be labeled as *marginalised* or *underprivileged*, and analyse two classroom episodes that led to students' exclusion from learning mathematics. If we followed the evolutionistic thesis, we were expected to formulate strategies to overcome the problems that led to students' failure. These could be formulated in terms of teacher education (e.g. a different way of interacting with the students), the curriculum (e.g. more challenging tasks), or the classroom organization (e.g. project or work group instead of blackboard centred and individual work). However, we will instead analyse this classroom episode *as it is*. This is because we are not interested in providing solutions for the problems of practice, but to pinpoint the ideological injunctions at work in the way teachers and students interact in the classroom. By analysing things as they are (instead of how they should be), we seek to make visible the incongruences between the official discourse and the life experiences of students and teachers.

We will focus our analysis in the way students decide to participate or not in the activities proposed by the teachers. The frame is set in a way that failure cannot be attributed to anything else than student's individual choice not to participate in the classroom activities. However, as we shall see, this is a false choice since participation in classroom activities also leads to failure. By analysing these cases, we are lead to conclude that current educational practices in underprivileged mathematics classrooms initiate students into patterns of decisions not to participate.

THEORETHICAL APPROACH

As a point of departure for our analysis we claim that schooling in current capitalist society *needs* failure as an integrative part of its economy (Baldino & Cabral, 2006; Bowles & Gintis, 1977; Pais, 2012). Failure in school mathematics is not an empirical phenomenon that can be solved through better research and the proper crew, but a necessary feature of existing schooling. As such, it becomes paramount to report not only experiences that tell stories of success on the local level, but also episodes that evince how failure is being built at the heart of an educational system that has inclusion and democracy as its self-legitimising principle.

As a way of conceptualizing educational failure, we find support in the work of Slavoj Žižek, who, in the last two decades, has been recovering the outdated notion of *ideology* as a crucial concept with which to understand the dynamics of our current capitalist society. Ideology operates in the discrepancy between the official discourse—which exalts the supreme goals of democracy, equity and inclusion—and its actualization into what Žižek (1997, p. 93) calls a life-world context. What, at the level of the enunciated content, runs smoothly—practically nobody within mathematics education research contests the supreme goal of mathematics for all—when actualized in a specific practice (in our case, school practice) often encounters a series of obstacles which ends up perverting its official intentions. This way, the motto "mathematics for all" functions as the necessary ideological double concealing the crude reality that - under the veil of meritocracy - mathematics is not for all.

Ideology simultaneously conceals its "motives" whilst making them actual and effective. It is in this sense that Žižek (1989, p. 34) says that ideology always appears in its sublated form, that is, its injunctions make effective what it "officially" conceals. When it is claimed that everyone should be provided with a meaningful mathematics education, this official claim conceals the obscenity of a school system that year after year throws thousands of people into the garbage bin that the school system itself erects. This happens under the official discourse of an inclusionary and democratic school. It is in this discrepancy between the official discourse and its (failed) actualization that ideology is made operational. Within the official discourse, what is *necessary* is the abstract motto of "mathematics for all", all the exceptions to this rule (the ones who fail) being seen as contingencies. However, from the critical/dialectical discourse we are deploying here, what is *necessary* is precisely the existence of those who fail, the abstract proclamation being a purely contingent result of the frenetic activity of individuals (researchers, teachers, politicians) who believe in it. The antagonistic character of social reality – the crude reality that in order for some to succeed others have to fail – is the *necessary* real which needs to be concealed so that the illusion of social cohesion can be kept.

One of the ways of achieving the societal demand of mathematics for all is by implementing ability-streaming at the transition to secondary school. According to the official rhetoric, the stratification of streams shall allow the effective design of classes specifically for students with difficulties in mathematics (or more generally for students with difficulties with so-called abstract thinking). Apparently, students are confronted with the choice of participating or not in the official discourse, by means of active engagement in the classroom activities. However, as Žižek (2006, p. 348) puts it, "[t]his appearance of choice, however, should not deceive us: it is the mode of appearance of its very opposite: of the absence of any real choice with regard to the fundamental structure of society". In our case, this appearance of choice to participate in classroom activities disavows the absence of any real choice regarding the possibilities these students have of pursuing a valuable education. The system initiates students to blame failure on their own choices in order to keep the appearance of a free and equal school system.

METHODOLOGICAL APPROACH

Mathematics education postulates equity as a crucial democratic value and as a final goal of schooling. This goal is understood to be achievable by some kind of evolutionary process that just demands *more* research and *more effective* research to overcome exclusion in mathematics education. Instances of exclusion are understood as dysfunctional within institutions that are supposed to promote inclusion. By locating our research in the critical research paradigm, and by focus on the *functionality* of exclusion within educational institutions, we seek to show how instances of exclusion are functional in 'keeping the system running'. This is, as pointed out by Popkewitz (2007), a fundamental task of a critical research:

> To make the naturalness of the present as strange and contingent is a political strategy of change; to make visible the internments of the commonsense of schooling is to make them contestable (p. xv).

In this article, our problem consists in analysing the functional moments in the exclusion of students from mathematics classroom in a marginalised social environment.

Our critical investigation has been initiated by the discussion of already available data from the "Emergence of disparity" project, in which one of us is engaged [1]. Data-collection used videography, as this project had its main focus on the social interactions that discursively produce mathematical knowledge and consciousness. The focus was rather on the symbolic relations that were produced through interactions than on how interactants perceived this production. Extensive interview-material that systematically captures students' voices on their perception of their own social exclusion and educational marginalization is therefore not available.

However, what we are trying in this paper is to provide a *theoretical reading of a social reality* instead of a 'valid' reconstruction of individual motives and perceptions of this reality. This is in line with the critical orientation outlined above. Therefore, we used a key-incident-analysis (Kroon & Sturm, 2000) as a methodology to reconstruct case studies from the available videotaped material with the intention to explicate our theoretically driven reading of the data. Key-incidents are concrete incidents in the data that researchers deliberately chose to "make explicit a theoretical 'loading'" (Erickson, 1986, p. 108).

These key-incidents were selected from a data-set including videotapes of the first fourteen consecutive mathematics lessons in September 2009 at a low-streaming secondary school in Berlin, Germany. The students were in the seventh grade, in the age of twelve to fourteen. Before the summer holidays they have visited different primary schools and all have finished it with a recommendation to attend the lowest of three available ability-streams in secondary school. The school is settled in a neighbourhood that in the public discourse is often referred to as a ghetto. The students in the classroom are between the ages of twelve to fourteen. They can be considered as underprivileged given the social segregation that results from where they live, by their background as members of a cultural minority, by knowing the instructional language only as a second language learner, and by the institutional selectivity of the German streaming school system. Notwithstanding all these difficulties, the official discourse is one of inclusion and equity, with efforts being made by the school staff in order to make mathematics meaningful and valuable for the students.

ANALYSIS - TWO CASES OF RESISTANCE

In an exemplary report on the mathematics classroom under analysis, we (Straehler-Pohl & Gellert, 2011) have described the pedagogy enacted as one that

> in order not to overcharge – infantilizes students and – in order to enable classroom management – objectifies students. [...] Learning in such mathematics classrooms adds to the underprivileged conditions that these learners face. (p. 198).

Classroom interactions are set up in a way that they hardly provide opportunities to acquire mathematical knowledge. Rather, students acquire a "consciousness of one's own ignorance" of mathematics (Straehler-Pohl, forthcoming, p. 19). Even though the cognitive demands set by the teachers were excessively low, students continuously failed on these demands and seemed to have no concerns demonstrating their failure. We concluded that most of the students did not fail because of an incapability of meeting the requirements set on them. Instead, they demonstrated an awareness of the fact that participation in this kind of mathematics education won't bring them back on the road towards participation in a meritocratic society. Failing on tasks can be seen as an integral part of the local classroom culture: For the students, it ensures that they demonstrate to their peers that they do not naively believe in the fallacy of the mathematics classroom. Simultaneously, students' failure on the tasks serves to reassure the teachers that they had been "right" in the excessively low choice of cognitive demands. Thus, taking the decision to participate in the classroom activities means to take part in the construction of one's own marginalisation. It implies not to make use of the opportunity to take a decision not to participate in a senseless and discriminating activity. The local classroom culture can thus be described as a *discourse of learning not to participate*. We will now provide two examples of students who chose to resist to this discourse of learning not to participate and report on the consequences it had for them.

The case of Melinda

Melinda's resistance is characterized by a total refusal of the teachers' authority (most of the times two teachers are present in class). In the beginning of the first math class in this new school, each of the students was required to complete the sentence, "I am feeling ___, because ____". Though just having had rare chances to get to know the second teacher, Melinda articulated the following: "I am feeling bad because today we have class with this teacher [pointing at the second teacher]". During the course of the mathematical activity (performing "887-339" at the blackboard), Melinda spent quite some time talking to Mariella, her classmate, in a foreign language, which was mostly ignored by the teacher, though two times the teacher spoke out an admonishment in a rather calm voice. When Mariella was demanded to finish the task at the blackboard, Melinda shouted at her: "what are you doing bitch?". Though understandable quite loud and clear, this interruption remained unsanctioned. However, a few minutes later, Melinda "collected" (word of the teacher) her third calmly spoken admonishment and was thrown out of the classroom for the rest of the day. The following day, math class took a similar course, resulting in Melinda being thrown out. The third day, Melinda did not appear anymore. She had been expelled from school. As she was still in the age of compulsory education, she would have been directed towards another low-streaming school in the neighbourhood.

The teacher conceived Melissa’s failure as her own personal choice. From teacher's and Melissa's classmates' perspective, she had the opportunity to choose to participate in the classroom activities, but refused to do so. But is this truly a choice of participation?

The case of Hatice

On the third day Hatice, who already was known as a truant to the teachers, appeared in class for the first time. In class, Hatice was quietly doing the calculations demanded of her by the work sheet (such as 9700-300). Hatice was among three students who succeeded in finishing their work sheets. The next time Hatice appeared in class, she completed three work sheets in twenty minutes including 186 simple multiplication exercises. The fourth sheet - that was given to Hatice "as a repetition" (words of the teacher) - claims on top of the page that, "it is now getting harder and harder", and concludes at the bottom that, "if you solve all the problems correctly, you are the king of computations". When Hatice came back to her seat and started filling in the solutions on the work sheet (see Fig. 1), the second teacher asked her to "read the instructions first". However, there were no instructions for the first 54 exercises. Ignoring Hatice's confusion, the teacher commanded, "read!". It was not before task no. 7, that there was an instruction. Hatice did not show up anymore during the following lessons.

It is getting harder and harder - number range up to 10.000. Respect the transitions!

1. 4509 + 2 =	3490 + 20 =	8970 + 40 =
3804 + 7 =	7180 + 70 =	1990 + 50 =
7205 + 9 =	2570 + 90 =	5980 + 60 =
2. 3400 + 760 =	2780 + 45 =	3247 + 80 =
8900 + 480 =	1460 + 52 =	1199 + 20 =
7600 + 690 =	9690 + 69 =	7881 + 40 =
3. 2800 + 754 =	7997 + 4 =	5600 + 2800 =
6700 + 986 =	4995 + 9 =	4800 + 2400 =
8900 + 211 =	1999 + 8 =	1900 + 2700 =
4. 5000 – 50 =	3200 – 60 =	4200 – 400 =
8000 – 80 =	6100 – 90 =	7300 – 900 =
2000 – 10 =	9300 – 70 =	8100 – 600 =
5. 8000 – 8 =	5200 – 7 =	2482 – 6 =
2000 – 4 =	1700 – 4 =	8113 – 5 =
6000 – 5 =	9800 – 1 =	3685 – 9 =
6. 3200 – 23 =	8000 – 45 =	9010 – 40 =
4400 – 98 =	3000 – 21 =	6020 – 80 =
7900 – 11 =	5000 – 83 =	4030 – 60 =

7. Write all the exercises in the form of packages and compute them!

4000	4500	1740	2375	+	2	20	200
6000	5800	7690	9764	-	3	30	300

When you have solved all the tasks correctly -

then you are the king of computations.

Fig. 1: Worksheet (translated)

Discussion of the two cases

It seems straightforward to read the case of Melinda as case of resistance, which is characterised by a clear and absoulute stance of opposition. She makes very clear that she is not about to acknowledge the teachers' authority and thus is not about to participate in any of the activities imposed by the teacher. Instead she uses them to stage her resistance. For the teachers, entering a dialogue with Melinda would require taking a step up towards her and to give space to her voice. The teachers are not willing to do this, but still they also do not use their authority to react oppressively. They just rarely raise their voices and never shout, even though at times this would appear quite comprehensible. Admonishments were not characterised explicitly by threats of punishment, but by some sort of countdown leading towards physical exclusion from the class. The message is clear: Melinda's resistance is her own choice; she gets the opportunity to decide to participate in the classroom activity herself. Thus, Melinda is agentive in her own exclusion. Apparently, she took the opportunity to decide not to participate in the meaningless classroom activity, thus it seems as if she herself was responsible for the consequences: i.e. expulsion from the school.

The case of Hatice (when being present) is quite contrary. By apparently taking the decision to participate in the activity seriously and with dedication, she goes through it so fast, that the meaninglessness of the whole activity becomes visible for all those participating. When we read Hatice's behavior as a case of resistance, then we have to identify a different target than mere authority. Through her dedication, Hatice seems to be participating in a different activity than her classmates: It seems as if struggling with the work sheet is an integral part of the game that the teachers and the students are playing. Thus, just finishing it, as if it was no demand at all but just a practice of mechanical routines posits her outside the activity. Thus, Hatice also makes use of the opportunity to decide not to participate in the classroom activity. Hatice's resistance is thus not targeted at the superficial power-relations that guarantee the teachers' authority, but it is targeted at the underlying tacit rules of the game. It seems as if this does not remain unnoticed by the second teacher: instead of complimenting her for carrying out (correctly) three times as much calculations as her peers, she *invents* some illusive instructions to slow down Hatice. It seems as if the teacher signals her that she should rather participate in playing the game in the way her peers do. The message communicated to Hatice is that the way she participates is not considered to be legitimate. As a matter of course, Hatice's succeeding absenteeism is not taken as a challenge to question the organisation of classroom activities. Instead it is constructed as a matter of "truancy", as if it was a personal attribute.

DISCUSSION AND FINAL REMARKS

Apparently the "choice" that students face regarding school mathematics is one between participating in the classroom activities and refusal to participate. However, the argument of this paper shows that in certain mathematics classes, the choice is not an “individual” choice between participating and not participating, but between two modes of “non-participation”. On the one hand, participating in classroom activities that contribute to an understanding of one's own ignorance of mathematics. This implies participating in one's own stigmatization and one's own exclusion from access to socially valued vocational and educational opportunities. On the other hand, the alternative is a straightforward non-participation by abandonment or exclusion from the school system. Regarding the first option, although the majority of students explicitly participate in the classroom activities, the narrow-mindedly mechanical and arbitrary activities guarantee that the outcomes of this learning will not provide students the right skills and knowledge to open up further educational or vocational options. Thus, students' decisions to participate in classroom activities results in their non-participation in further education, in very much the same way as the direct decision not to participate. As such, the choice is a false choice, since in either ways students are paving the way of exclusion from a consensually valued form of life. At best, students can postpone the materialisation of an already determined exclusion.

We claim that this report reveals that we do risk a lot when we keep on considering educational failure as the unpleasant obstacle on the didactic road towards salvation. While the particular instances reported here make it easy to blame the teacher (and her pedagogical actions), we want to stress that it is not only the teacher, who *organizes* meaningless activities, but also the majority of students who actively *participate* in the *game of failure*, however demandless and meaningless the activities are. Together, the teachers and the students continue a system, where failure *is* a necessity and a predictable result of the process. An extensive interview with the teacher indicates that the reasons neither lie in the teacher's individual pedagogical ineptitude nor in a lack of professionalism; it is rather, the result of long years of experience in one and the same school that constantly and apparently inevitably produces failure. The reasons also neither lie in the students' cognitive inability nor in their bad behaviour; it is rather, a result of six years of school, showing them that *they are not the ones who profit from making use of their agency*. Thus, instead of choosing the path of resistance in claiming their right for a meaningful, demanding and empowering education, the students prefer to play the game of failure. Thus, the *necessity* of educational failure is a *result,* namely a result of the systematic organization of segregation for the supposed sake of the objects of segregation. To re-turn necessity into a contingency and thus, as a changeable educational phenomenon, we would need to deconstruct the naturalness of segregation.

A meritocratic legitimation of the school system presupposes that schools are places where equal students meet freely, and where some kind of "invisible hand" guarantees that the competition of individuals' egotisms works for the common good. What such an approach makes invisible is that such merit is possible only by the demerit of others, i.e., the notion of personal merit is only possible as long as others fail. This is the liberal capitalist ideology at work, by means of making individuals recognize their choices as their own, as free choices that they took – especially when these choices imply failure. Finally, a failed student is robbed of the ownership of all the work s/he produced: All the time he spent in school, is not his or her own, since he will not receive the diploma at the end of the year. However, without producing all this work, it would be impossible for him or her to keep living within the system [2]. Thus, it becomes an imperative that individuals must realise failure as the result of a wealth competition among equals, and repress the traumatic truth that they fail so that others can succeed. Schools need this subversive supplement in order to retain their indispensable role in maintaining our democratic and inclusive society. In order for school to be the most important ideological apparatus, to function as a credit system (Vinner, 1997), it is not productive for it to be presented as an exclusionary institution. In order to perform well in the role of credit systems, schools need to be presented as inclusionary and emancipatory places, places where phenomena such as exclusion and failure are seen not as necessary parts of the same system which purports to be trying to abolish them, but as contingent problems, malfunctions of an otherwise good system.

NOTES

1. For more details, see Knipping, Reid, Gellert and Jablonka (2008).

2. We should note here that school represents one of several possible systems. It represents the dominant system within our society and the fact that school is compulsory for everyone ensures that no one remains unaffected by the dominance of this system.

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MATHEMATICS FOR ALL AND THE PROMISE OF A BRIGHT FUTURE

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That "Mathematics is for all" is a recent statement in mathematics education research. Nowadays it is closely linked to the promise of social, economic and cultural development. It has become a commonsense statement positions mathematics education centrally in achieving a bright future. I examine the statement using an analytical strategy grounded on the cultural historical analysis of Michel Foucault and the examinations of the school curriculum as social epistemologies as presented by Thomas Popkewitz and collaborators. Three distinct but related elements allow me to examine the ideas on which "mathematics for all" emerged during the 1980's. The elements point to the relation between the statement and the construction of exclusion for groups of students, particularly those of "low socioeconomic status".

THE COMONSENSE OF "MATHEMATICS FOR ALL"

That people's scientific and mathematical knowledge is important for the future well-being of society has become an idea that plagues commonsense. Parents, politicians, employers, teachers, and even children themselves have all learned the lesson well. Who would question the need for scientists, engineers and mathematicians for producing the great technological progress that has made the world reach this high point of advancement —and decay, as well? The idea that mathematics —as well as science— with all its applications in technology is the motor for achieving the promises of Modernity and that, therefore, their teaching and learning are central to the constitution of massive school systems is as old —or new— as the end of the 19th century. While towards the end of the 19th century those who had learned mathematics were struggling to find a place for its teaching in the classic curriculum of many European countries (Howson, 1974), now at the beginning of the 21st century the school curriculum without mathematic is unthinkable. Even more so, recent policies in various countries start over privileging mathematics at the expense of other subjects. The involvement of mathematicians with education was during most of the 20th century concerned with securing that an intellectual elite of highly competent youth would feed the production of mathematicians at universities. The issue was not a matter of quantity, but of quality (Jurdak, 2009): Enough but few would be educated in the finest form of human thinking. With the configuration of contemporary universities, those people would provide a highly valued knowledge to other disciplines in the pure sciences and the growing amount of applied scientific and engineering fields. They were the ones to educate mathematics teachers. They were the ones to produce the highly desired and needed knowledge for the progress of society.

Nevertheless, the idea that mathematics should be for all —for each one and everybody, not only for all those who engaged with the study of mathematics— is an even more recent idea that is made intelligible at the end of the 20th century. The grid of conditions on which the statement of "mathematic for all" emerges is more complex than the natural evolution of education in an increasingly technological society or a response to the demands of the knowledge society. The statement is produced on the grounds of the advancement of mathematics education research as a field of scientific inquiry on the teaching and learning of mathematics and all its associated social and cultural phenomena. While until the 1970's the beginning of systematic reflections about the teaching and learning of mathematics were focused on securing the excellence of those few who had an interest in the subject, the 1980's started to be the time of the democratization of mathematics, of mathematics for all to empower all, and, more recently in the 2000's, of mathematics for all to secure the national competitiveness in the global knowledge economies. Such formulated statement is an expression of a desire, of a normative vision that should guide policy proposals and, above all, research-based interventions to provide access to as many as possible to the important and valued knowledge. Such statement is however not an innocent declaration of good wishes for the future. The aim of this paper is to explore how at least three connected events during the 20th century set the ground on which it becomes possible for mathematics educators and now for the public in general to desire a "mathematics for all".

For such exploration I start by locating my analytical strategy within the area of cultural studies of schooling and the curriculum and making available some of the main concepts in my theoretical toolbox to examine the school mathematics curriculum. I then deploy my tools on three elements which, I will argue, make part of the grid on which nowadays the statement "mathematics for all" can be thought. I conclude with some remarks on the usefulness of a cultural historical approach in the study of mathematics education.

ANALYTICAL STRATEGY: DECENTERING THE COMMONSENSE

The appropriation of the work of Michel Foucault in education has prolifically nurtured critical studies of schooling and the curriculum (e.g., Popkewitz & Brennan, 1998). In mathematics education it has been appropriated to think about the constitution of learners and teachers as subjects within the web of power of the institutional discourses of mathematics education practices (e.g., Walkerdine, 1988; Walshaw, 2004). Such studies have provided insightful interpretations of students' and teachers' identity formation in terms of their process of subjectivity in the practices of mathematics education. Walshaw (2004) claims that thinking mathematics teachers' practices with the analytical strategies and concepts of

Foucault help her decentring the essentialist assumptions that other types of literature build around teachers' knowledge, believes or experiences as being the fundamental core of their identity as mathematics teachers. Although important in a quest for understanding the socio-political constitution of mathematics education practices, such approach that still focuses on some of the traditionally defined actors and elements of the classic didactic triad of mathematics education research is, from my view point, not enough to decentre the many essentialisms that the discourses of mathematics education have established.

Recent studies inspired by the work of Foucault deploy analytical strategies to explore the epistemological functioning of mathematics education discourses and their effects of truth in generating ways of thinking about mathematics education. Knijnik and collaborators (Duarte, 2009; Knijnik, 2012; Knijnik & Wanderer, 2010) have been not only examining ethnomathematics philosophically, but also providing cultural histories of statements that navigate as truths and have become the common sense of mathematics education. That "we need to bring reality to the classroom" as a pedagogical strategy, that "we need to use concrete materials for teaching", or that "we have to promote mathematics for all" are not simply the accumulated knowledge that comes from applying research into the improvement of practice. Such statements epitomise culturally and historically inscribed forms of thinking about mathematics education. The work of Foucault and his strategies to perform a "social epistemology" (Popkewitz, 1991) of mathematics education as part of the school curriculum is an important move in order to:

> "[…] place the objects constituted by the knowledge of schooling into historically formed patterns and power relations. Epistemology provides a context in which to consider the rules and standards by which knowledge about the world and "self" is formed. Epistemology also provides the means to investigate distinctions and categories that organize perceptions, ways of responding to the world, and the conceptions of "self." Concurrently, social epistemology locates the objects constituted by the knowledge of schooling as historical practices through which power relations can be understood. Statements and words are not signs or signifiers that refer to and fix things, but social practices that generate action and participation." (Popkewitz & Brennan, 1997, p. 293)

In other words, thinking mathematics education with Foucault in terms of making a social epistemology of mathematics education as part of the school curriculum allows us evidencing the way in which mathematics education and mathematics education research practices together and inseparably generate concepts, distinctions and categories that regulate the possibilities of thinking and being in/with mathematics as a privileged area of knowledge in the school curriculum.

It is important to my examination of the taken-as-truth statement that "mathematics education is for all" Foucault's analysis of the involvement of fields of academic inquiry and knowledge in the production of the ordering of Modern life (Foucault, 1971), and of the organisation of different technologies of governmentalisation

(Foucault, 1997; Lemke, 2001). As I have argued elsewhere (Pais & Valero, 2012; Valero, 2008), understanding mathematics education as political enlarges the research gaze to notice and indeed focus on the way mathematics education research is implicated in producing and organizing what is conceivable as "mathematics education". In my exploration in this paper I build on Popkewitz' (Popkewitz, 2008) insights in how educational sciences have provided the tools for fabricating the cosmopolitan child through being a cornerstone of the planning of social life for the promise of a better and brighter future. In this paper I connect the statement of the need of a mathematics education for all for creating a brighter future with the way in which educational sciences in the 20th century have produced the elements for the reasoning making possible such statements.

One important effect of the school (mathematics) curriculum as a process of governmentalisation is its operation as mechanisms of classification of people. As technologies that embody the norms of reason and reinscribes them in populations and in individuals, the mathematics curriculum operates inclusions/exclusions. Popkewitz (2008) argues that any cultural thesis about the subjects of schooling effects abjections. Abjection is the way that exclusion is generated as the effect of defining the norm for inclusion and its hope for those who are not part of that norm. The statement "mathematics education is for all" functions as a discursive device that declares the necessity of making of success in mathematics learning the norm. While the statement apparently sounds as the expression of an intention of exclusion, it operates simultaneously the exclusion of those who do not comply with the norm (Popkewitz, 2004). The mathematics curriculum as a technology of the self effects in children's mind, bodies and conduct the compliance with the norm, and thus operates inclusions and exclusions. This way of thinking brought me in my inquiry to focus on the doubleness of the words "mathematics for all". If mathematics for all embeds those for whom learning mathematics is not possible, I looked at how, historically and in the formation of the social sciences and mathematics education research there were forms of identifying for whom learning mathematics is not possible.

This means that an examination of the statement that mathematics is for all as part of the logic of the curriculum as governmentality techniques invites to consider how the formulations of inclusion in relation to education and mathematics education identify those for who are not meant to be included and be successful in school mathematics. In other words, the emergence of mathematics for all as an important statement in in the current functioning of the mathematics curriculum emerges on the grounds of many children who had been abjected and for long time had been excluded from having a chance of success in school mathematics. Thus the examination of the statement of mathematics needs to be for all goes hand in hand with the examination of statements about those who do not succeed. In this case I will concentrate on the relationship between statements such as students with low socio-economic status do not succeed n mathematics and the statement of mathematics is for all.

My analytical strategy involves visiting a number of interconnected spaces that without any linear or strict logical connection, but rather with discursive resonance, map different aspects of the statement under examination. A rhyzomatic analytical move (Deleuze & Guattari, 1987) of how this idea is made thinkable becomes a strategy. I also move in the connection of ideas in time and space. As mathematics education research is thought as an international field of inquiry, and probably because for many of its practitioners mathematics is still conceived as a universal activity, then as much as there are particular reinscriptions of these ideas in national or local contexts, at the same time there is a tendency to an abstract, internationalized discourse about what those ideas are and how they could be played out. In keeping my eye on the ideas that circulate across nations I try to make evident how a field of inquiry generates truths that seems to be transferable from place to place and from time to time, contributing in this way to the reification of mathematical ability as a human ability and right that equates with reason, and with that installs one unified logic of being.

EDUCATION, SCIENCE AND THE SOCIAL QUESTION

The social sciences and educational research can be considered as expert-based technologies for social planning. In the consolidation of Modernity and its cultural project in the 20th century, the new social sciences were seen as the secular rationality that, with its appeal to objective knowledge, should be the foundation for social engineering. The invention of statistical measurements in the social sciences generates constructs that help identifying the ills of society that science/education needed to rectify. This is an important element in how educational sciences address the differential access of different children to the school system. Constructs such as students' "socio-economic status" —later on expanded to school's and communities socio-economic status— emerged in the 1920's in a moment where the newly configured social sciences were addressing *The Social Question*, that is, the growing problems with crime, poverty, alcohol abuse, sexual abuse, school underachievement, etc., of the growing population in urban centers (Popkewitz, 2008) caused by immigration and the urbanization of many other types of populations. In Europe and in other societies, the association between the religious and normative grounds of educational thinking and the emergence of educational sciences made possible to articulate salvation narratives for facing the social problems for which education was a solution (Tröhler, 2011). How "the Social Question" was addressed in different countries in the turn between the 19th and the 20th century allows to see the different religious, political and economic rationalities behind making of education the motor of modern development (e.g., Tröhler, Popkewitz, & Labaree, 2011).

Despite particular inscriptions in space, there is a common thread in the way in which the social sciences were devising technologies to deal with difference from the values of a dominant class that at that time has already established itself as the norm for measurement of all types of deviation. The “nurture–nature” debates —on whether it is the innate, genetic charge of an individual what is determinant in people’s “right” development, or whether it is what the environment makes available the determinant of one’s development— emerge hand in hand with the dilemma of how to govern a growing unfit population, and the pressures of democratization of education to tame the masses and make them productive, well trained work-force. Measurements of intelligence, measurements of achievement and measurements of socio-economic status were and still are technologies to provide the best match between individuals and educational and work possibilities. The double gesture of educational sciences of, on the one hand keeping a rhetoric for the importance of access to education, and on the other hand reifying difference by constructing them as a fact, inserts human beings in the calculations of power.

MATHEMATICS AND PROGRESS

The emergence of the connection between people’s mathematical qualifications and social progress can be traced to the end of the 19th century. During the second half of the 19th century, mathematics teachers in different countries struggled to make mathematics part of school the classic school curricula. Its place was relegated to vocational and military forms of education (See Howson, 1974 for the case in England). In the first number of the journal in 1899, Laisant and Fehr (1899), envisioned the important mission of the journal in contributing to the international, systematic and serious reflection on and study of mathematics education. They recognized the importance of the *preparation* of teaching staff, a group of “teachers deeply engaged with their mission, consecrated to it with all their devotion, instruction and intelligence” (p. 1, my translation). During the second industrialization, a time of tremendous scientific advancement, teachers had realized that *betterment* is always possible, no matter which pedagogical strategies teachers had used. The justification for the need for betterment was formulated as follows:

> “The future of civilization depends greatly on the direction of mind that the new generations will receive in relation to science. Within the scientific education, the mathematical element occupies a dominant place. From the point of view of the pure sciences or from the point of view of the applications, the 20th century that is about to begin will place demands which nobody must or can avoid.” (Laisant & Fehr, (1899)

In the times of the Cold War, a similar argument emerged, however the justification was related to keeping the supremacy of the Capitalist West in front of the growing menace of the expansion of the Communist Soviet Union (Kilpatrick, 1997). Nowadays, professional associations argue that the low numbers of people in STEM fields can severely damage the competitiveness of developed nations in international, globalized markets (e.g., National Academies, 2007).

The narrative that connects progress, economic superiority, and development to citizen’s mathematical competence is made intelligible in the 20th century. The consolidation of nation states and the full realization of the project of Modernity required forming particular types of subjects. The mathematics school curriculum in the 20th century embodied and made available cosmopolitan forms of reason, which build on the belief of science-based human reason having a universal, emancipatory capacity for changing the world and people. The 'homeless mind' (Berger, Berger, & Kellner, 1974 cited in Popkewitz, 2008, p. 29; Popkewitz, 2008) that school mathematics has operated is a type of individuality where the subject is set in relation "to transcendental categories that seem to have no particular historical location or author to establish a home" (Popkewitz, 2008, p. 30). In this way, subjects are inserted in a logic of quantification that makes possible the displacement of qualitative forms of knowing into a scientific rationality based on numbers and facts for the planning of society (Poovey, 1998). Thus, from the turn of the 19th century to our days the mathematics curriculum is an important technologies of the self that inserts subjects into the forms of thinking and acting needed for people to become the ideal cosmopolitan citizen.

MATHEMATICS FOR ALL IN RESEARCH

That high achievement in mathematics is a desired and growing demand *for all* citizens is a recent invention of mathematics education research. In the move between the years of reconstruction after the Second World War and the Cold War, where school curricula was modernized with focus on the subject areas for the purpose of securing qualified college students (Rudolph, 2002; Thompson, 1959), mathematics education in the decade of 1980s faced the new challenge of democratization and access. The "Mathematics Education and Society" session at ICME 5 in Adelaide is seen as the first formal session in an international mathematics education conference to have publicly raised the need to move beyond mathematical elitism towards inclusion of the growing diversity of students in school mathematics. The well-documented systematic lack of success of many students in school mathematics was posed as a problem that mathematics education research needed to pay attention to and take care of. Mathematics education researchers, the scientific experts in charge of understanding the teaching and learning of mathematics as well as of devising strategies to improve them, took gradually the task of providing the technologies to bring school mathematics to the people, and not only to the elite. The idea of succeeding in mathematics as an issue of equity was made intelligible in a historical grid of events at the end of the 20th century. The identification of mathematical achievement with the wealth of nations is a result, among others, of the growing series of reports that produced comparative information on educational achievement and development (e.g., Baker, Goesling, & LeTendre, 2002; Heyneman & Loxley, 1982). Such reports can be seen as performances of the comparative logic of Modernity that operates differential positioning, not only among individuals but also among nations, with respect to what is considered to be the desired and normal level of development and growth.

"Mathematics for all" can be seen as an effect of power that operates on subjects and nations alike to determine who are the individuals/nations who excel, while creating a narrative of inclusion for all those who, by the very same logic, are differentiated. As Popkewitz (2004) points, the rhetoric of inclusion in mathematics education embeds in itself the exclusion of those whose forms of being are distant from the norm.

BEYOND THE PROMISES OF A BRIGHT FUTURE

I argue that it is on the grounds of at least these three elements that the it is possible to examine the ways in which during the 20^{th} century the statement "mathematics education is for all" came to enter the discourses of school mathematics. The mathematics curriculum as a technology of governmentalization installs the framework within which it is possible to think of school mathematics. Such ideas that organize populations as well as subjects work on the doubleness of a statement such as "mathematics for all". While apparently declaring the necessity of inclusion, the statement draws clearly the boundaries for who are not part of the "all" included in the statement. In other words, mathematics for all implicitly states that mathematics is not for all, and that those for whom participation in school mathematics is not a possibility need redemption. The redemption and help should of course come from the successful application of the technologies devised by mathematics education research.

My analytical move allowed pointing to at least three rhyzomatically connected nodes. First, the social sciences and educational sciences in particular have during the 20^{th} century built statistical measurements that put forwards constructs such as "socio-economic status". Such measurements identify and reify the people who do not fit with the norm. Second, the association of mathematics with the narrative of progress during the 20^{th} century builds on the operation of the homeless mind citizen of Modernity. Third, mathematics education research brings together the fear for those who do not comply with the norm with the narrative of a bright future with mathematics. Mathematics education research and its statement on the need to make mathematics education fro all incorporates the excluded, the ones who have to be brought into the calculations of power. It redeems them by making low socio-economic students part of the didactical efforts of mathematics education.

I do not intend to say that being one of the persons who does not achieve as expected because of one's differential position of socio-economic status is simply an unimportant "social construction". My intention is to offer a way of denaturalizing the commonsense of a statement such as "mathematics is for all". The denaturalization makes visible the network of historical, social and political connections on which the fact that differential students' social and economic positioning is related to differential mathematical achievement.

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SINGLE-SEX MATHEMATICS CLASSROOMS IN PUBLIC SCHOOLS: A CRITICAL ANALYSIS OF DISCURSIVE ACTIONS

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This poster will present the results of a critical discourse analysis of classroom lesson sessions from middle grades public mathematics classes. The teacher of interest for this particular poster session is responsible for coeducational, all girls, and all boys classes. The data for analysis derive from 10 instructional sessions for each of these class types, giving a total of 30 lesson session transcripts for analysis. The authors employ a critical theoretical frame, from a feminist perspective, to the discourse analysis. Findings indicate that, at a macro-level, the academic rigor of the learning environment is consistent across the three class types. However, momentary utterances by teacher and students indicate more nuanced differences in the socio-cultural classroom environment.

POSTER CONTENT

Study Context

The participating school of focus for this presentation is a public middle school with approximately 700 students in the southeastern U.S. More than half of the students at this middle school receive subsidies for meals, indicating a lower-than-median socio-economic status. In 2011, this school received ratings of Average on the Annual Report card; in 2012, however, the school was given a grade of A (the district as a whole received a grade of B). Thus, there are mixed indications about the academic 'quality' of this school. Single-sex classes in mathematics (among other academic subjects) have been an option at this school for several years. There are some 8th grade students at this school who have had single-sex classes in at least one academic subject for four years. This presentation focuses on one 6th grade mathematics teacher (male) who teaches coeducational, all girls, and all boys mathematics classes. The research focus for this study—a critical analysis of the discourses in this teacher's classes—is part of a larger, federally funded project investigating classroom environments in single-sex mathematics and science public school settings.

Methodology

We are using critical discourse analysis, which "highlights the ways power relations work implicitly through language, demonstrating how language practices serve to reproduce and perpetuate power hierarchies and how language practices may be used for intervention and control" (Curran, 2008, p. 82). Within this critical classroom discourse analysis, we incorporate a thematic analysis (Lemke, 1990) as well as an analysis of discourse moves (Chapin, O'Connor, & Anderson, 2003; Krussel et al., 2004). The thematic analysis reveals semantic relationships in classroom discourse, while an analysis of discourse moves focuses on actions teachers and students take to participate in or influence discourse.

Findings

This case study of one educator who teaches three different class types is part of a large research project involving several other teachers, students, and classes. The data were collected during the spring of the 2011-2012 academic year and thus are under analysis. We have collected more than 100 lesson sessions in addition to data related to student academic performance, student academic self-concept, and classroom academic rigor. The 30 lesson sessions from the teacher in this study are the first to undergo discourse analysis at the utterance level; we focus on this teacher because he is one of two mathematics teachers participating in this study who taught all three different classroom types. Though are analyses are ongoing, preliminary indications point to the importance of this level of fine-grained analysis because of the ephemeral nature of momentary utterances; their transitory nature, however, masks the potential impact of such utterances on students' constructions of social norms (including gendered norms and academic norms). Discourse analysis is a way of capturing those fleeting moments in time that have potential for impact disproportionate to the quantity of time it takes to utter the statement.

POSTER STRUCTURE AND ORGANIZATION

The poster will be organized into three columns; transcript excerpts illustrating the emergent discursive themes will occupy the middle column. Illuminating excerpts will be included from all three class types (coeducational, all girls, and all boys). The first column of the poster will house the research statement, relevant scholarship, and methodology sections. Articulation of the discursive themes, a summary of the findings, and a discussion section will occupy the third column.

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SOCIO-POLITICAL ISSUES IN THE CONTEXT OF DIFFERENT CONCEPTUALIZATIONS OF SUBJECTS

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The explication of a socio-political dimension within issues related to mathematics education in school, university and research is crucial for a full understanding of subjective experiences by students, teachers and researchers. Different conceptualizations of subjectivity within society by Radford & Roth, Brown and Holzkamp are discussed and compared with respect to their inherent concretizations of specific societal assignments.

BACKGROUND

In a series of papers (Pais & Valero, 2011, 2012) claim that the political dimension is often neglected in the research on mathematics education. Furthermore they advocate the thesis that this is not only true for the so-called (quantitative and qualitative) "mainstream" in educational research but also for "critical approaches" following the paradigm of the cultural-historical school for example. This situation in research corresponds to the observation that socio-political issues typically do not play an important role in subjects' reflections on individual experiences in the contexts of studying, teaching or researching, either. In (Pais & Valero, 2011), the authors argue that the disavowing of a political dimension "is precisely one of the strongest limitations for bringing equity and quality together."

The poster was rooted in the conviction that the socio-political dimension does not only come up in the structure of schools and universities as "credit system", "production of failure" or as the crucial ideological state apparatus in the reproduction of capitalism (Althusser, 2001). Beyond that, a socio-politically informed reading of school and university mathematics requires an adequate conceptualization of "subjects". Categories are needed that allow in particular to analyze one's own experience in the stream of life within a capitalistic society, its intertwined cognitive and emotional aspects like feelings of anxiety, anger or vague indispositions. The categories should enable subjects to relate themselves to the societal context such that its contradictions can be reflected, and that restricted actions which remain within a reproduction of those contradictions may potentially be overcome.

THE POSTER

The starting point of the poster was the observation that the understanding of the historical specifity of society and consequently that of subjectivity is debatable. The concrete goal was to work out differences in conceptualizing subjectivity between the following three approaches taking into account their inherent concretizations of societal assignments:

a) the cultural-historical approach sensu (Roth & Radford, 2011),

b) the Lacanian approach sensu (Brown, 2008a, 2008b) and

c) the subject-scientific approach sensu (Holzkamp-Osterkamp, 1991; Holzkamp, 1991).

In particular, from a) to c) an increasing conceptual abstractness of applied historic-specific societal categories could be observed and related consequences for the categories that allow to analyze subjective experiences in educational contexts were discussed. Details has to be presented elsewhere.

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"IT DOES MAKE YOU FEEL A BIT HOPELESS": PARENTS´ EXPERIENCES OF SUPPORTING THEIR CHILDREN'S SCHOOL MATHEMATICAL LEARNING AT HOME

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Parental support of children's learning in mathematics has long been an aim of policy makers. However, a number of studies have suggested that factors exist which inhibit parents from playing an active role in their children's mathematical education at home. Using a sociocultural perspective I report findings from a PhD project investigating how parents support their primary school children's mathematics. Here we draw on narrative-episodic interviews undertaken with 24 parents of primary school-aged children. The parents were asked to recount their experiences of learning mathematics and of doing school mathematics with their children. This data was thematically analysed in order to develop an understanding of parents' personal mathematical histories, their perceptions and representation of primary school mathematics, and their experiences of completing school work at home with their children. From this it became clear that the majority of parents in the sample faced a number of problems and impediments to supporting their children's conceptual development in mathematics. Examples of these experiences, such as divergent understandings, curriculum changes and motivation are presented. It was also found that parents appeared to utilise a wide-range of strategies and approaches to overcoming these barriers. Illustrations of these strategies, for instance parental guidance and evaluating understanding are discussed. Dangerous

The content of this poster is displayed in discrete sections that:

1) Introduce the area of study, for example outlining the findings of Desforges and Abouchaar (2003) regarding the significance of parent-child interaction in academic outcomes for primary-aged children.

2) Highlight the four key research questions of the wider study, namely:

- How do parents support children's development of conceptual understanding of primary school mathematics?
- What techniques and strategies to parents use to support their children's mathematical development?
- What barriers do parents face in supporting their children's mathematical development?
- In what ways do teachers and schools shape parental experiences and teaching practices?

3) Outline the characteristics of the study participants and the main method of data collection, namely episodic interviews (Flick, 2000).

4) Discuss the thematic analysis (Braun & Clarke, 2006) undertaken on parental interview transcripts.

5) Describe the key findings that emerged from the study concerning:

- Parents' representations of mathematics
- Parents' representations of parent-child school mathematical interaction at home
- Parents' representation of communication with school shaping parental mathematical practices

6) Concludes by showing how the findings of this study link to contemporary research stressing the importance of parents' past experiences mediating their current activity (O'Toole & Abreu, 2005) and the different barriers parents face supporting their children (McMullen & Abreu, 2011).

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG11: COMPARATIVE STUDIES IN MATHEMATICS EDUCATION

Eva Jablonka (Sweden/UK)

Paul Andrews (UK/Sweden)

Richard Cabassut (France)

Christine Knipping (Germany)

Keywords: *Comparative curriculum analysis; validity of comparison; language and mathematics education; transition between school sectors; school-based teacher education; mathematics teacher beliefs; mathematics teacher guides; mathematical modelling; number sense; affect in mathematics education*

SUMMARY OF THE GROUP'S ACTIVITIES

The working group adopts an eclectic perspective on comparison, preferring to conceptualise it as a fundamental process of research rather than limit it to cross-cultural or cross-national forms. The twelve papers presented and discussed in the working group focused on both cross-cultural and cross-contextual similarities and differences across a wide range of aspects and levels of mathematics education. Moreover, the relatively small number of papers presented allowed for very full and professionally fruitful discussions of the issues they raised.

Group participants came from a range of institutions in Australia, Estonia, Finland, France, Germany, Sweden, the United Kingdom, the United States of America and Vietnam, while co-authors unable to be present represented institutions in Belgium, Chile, Latviaand Spain. The papers were arranged in five broad interlinked themes in order to facilitate discussion of the general issues of interest outlined in the call for papers. Indeed, all presentations provoked discussion of methodological and substantiverelevance to comparative education research in general and to mathematics education research in particular. In the following summary, only presenters' names are given for papers that were multiply authored. A full list of papers and all authors is provided on the last page of this report.

Theme 1: Comparing curricula – goals and methodologies

Three papers were grouped under this theme:

A cross-national standards analysis: quadratic equations and functions (presenter: MelikeYigit, Purdue University, USA)

This paper compared four culturally different countries' – the Caribbean, China, Turkey and, as reflected in the Common Core State Standards for Mathematics, the USA - curricularpresentations of quadratic equations and functions. Analyses, undertaken against three dimensions; content, mathematical reasoning and cognitive level, found much cross-cultural variation in systemic expectations.

Modelling in French and Spanish syllabus of secondary education (presenter: Richard Cabassut, IUFM, Strasbourg University, France),

This paper compared the curricular presentation of mathematical modelling in France and Spain. Drawing on the anthropological theory of didactics (ATD) it highlights well differences in the two systems' conceptualisation of modelling, not least with respect to its position as either explicit or implicit curriculum knowledge.

Analyzing mathematics curriculum materials in Sweden and Finland: developing an analytical tool (presenter: TuulaKoljonen, Mälardalen University, Sweden).

This paper compared teacher guides, as textbook support for teachers, from Sweden and Finland. Exploiting an analytical tool developed in science education, the study examined how and to what extent the guides could support teachers' professional learning. The results indicated considerable differences in the ways in which teacher guides are conceptualised and, therefore, prone to supporting teacher learning.

The issues emerging from the general discussion focussed on methodological questions of curriculum analysis and on levels of curriculum. Reviews of the role of teachers' guides (whether as support for a textbook or an official curriculum document) and of the level of autonomy of curriculum agents (textbook authors, schools, Head teachers) for each of the participants' countries provided interesting insights into the diversity of organisational forms and steering mechanisms. This diversity was acknowledged to be an important factor when choosing an appropriate methodology or level of analysis. Linked to this was the general issue of whether it is appropriate to use frameworks developed in one system for analysing data from another. It was noted, for example, that a framework exploiting verbs in curriculum documents, as placeholders for types of mathematical activity, would not work with documents in such activities are presented through nouns (such as *modélisation*, *Modellbildung*). Also, it is important to acknowledge that curriculum documents may vary considerably in length, not least because different cultures present such matters in varying ways, with different levels of specificity reflecting cultural communicative norms. Differences in level of detail alsoindicate the space left for autonomy of curriculum agents.

Theme 2: Characterising and comparing classroom instruction from three vantage points: 'neutral' observer, teacher practice and learners' experience

The development of foundational number sense in England and Hungary: a case study comparison (presenter: Paul Andrews, University of Cambridge, UK)

Drawing on framework derived from the literature, this paper compared how a case study teacher in each of England and Hungary addressed, when teaching a lesson on number sequences to grade one students, students' acquisition of foundational number sense. The analyses highlighted substantial differences in the opportunities the teachers incorporated into their everyday teaching activity.

School-based mathematics teacher education in Sweden and Finland: characterising mentor-prospective teacher discourse (presenter: MalinKnutsson, Örebro University, Sweden)

This paper compared how Finnish and Swedish school-based mentors and primary teacher education students undertake their roles in professional review meetings. The analyses indicate substantial differences in the ways that typical Swedish and Finnish teacher education discourses play out, with Swedish tending to focus more on the student teachers' learning, while the Finnish attended more to children's learning.

Comparing mathematical work at lower and secondary school from the students' perspective (presenter: Niclas Larson, Linköping University, Sweden)

Drawing on key concepts from the anthropological theory of didactics and Bernstein's theory of pedagogic discourse, this paper compared how students experience and perceive theirmathematics education at lower and upper secondary school, and found substantial differences linked, in particular, to the *instructional* and *regulative* discourses of the two traditions of mathematics teaching.

An important issue raised in the discussion concerned the role of the researcher as a cultural insider: How is this managed when examining data from both culturally familiar and unfamiliar cultures? The importance of being aware of blind spots as well as of the danger of imposing categories and taxonomies that arise in one teaching culture onto another was recognised. Another issue raised, was the need to understand the power relations between participants during data collection, and of acknowledging culturally and micro-culturally constructed interpersonal relationships. Further, cultural authorship of theoretical frameworks was discussed, and openness towards the empirical field was seen as important.

Theme 3: Modes of constructing data, developing analy-tical categories and pursuing validity in comparative studies

The validity-comparability compromise in crosscultural studies in mathematics education (David Clarke, ICCR, University of Melbourne, Australia)

Thistheoretical paper argued that the use general classificatory frameworks in cross-cultural research maynot only misrepresent aspects of mathematical performance, schoolknowledge and classroom practice valued by thecommunities under scrutiny but sacrifice validity in the interest of comparability. Seven dilemmas are presented to highlight different ways in which these tensions may play out.

The problem of detecting genuine phenomena amid a sea of noisy data (presenter: Christine Knipping, University of Bremen, Germany)

This second theoretical paper argued that those engaged in qualitative comparative mathematics education research should acknowledge two key principles; that analyses should attend both to the explanation of phenomena rather than data and the common discourse necessary for anytheoretical scientific discourse. These principles were illustrated by examples drawn from several well-known comparative studies.

Some 'dilemmas' in comparative studies were discussed in more depth, such as the advantages or disadvantages of exploiting too inclusive or too distinctive categories. As to the distinction between data and phenomena, participants agreed that the issue might be of greater complexity or more obvious in comparative studies than in other research. In this respect it was acknowledged that all data are, essentially, constructed and that data and phenomena codetermine each other; and that in cross-cultural research criteria for what constitute data must hold legitimately across settings.

Theme 4: Strategies and goals of comparing learners' emotional and affective experiences

Comparing the structures of 3rd graders' mathematics-related affect in Finland and Chile (presenter: Laura Tuohilampi, University of Helsinki, Finland)

This paper compared third grade students' mathematics affect structures in Chile and Finland – a comparison new to the comparative field. Exploiting a survey instrument developed in Finland the study found both similarities and differences in the affect structures yielded by the students in the two countries, highlighting how cultures construct students' affective responses to mathematics.

Boredom in mathematics classrooms from Germany, Hong Kong and the United States (Eva Jablonka, King's College London, UK)

Drawing on interviews with students from six classrooms in each of Germany, HongKong and the United States, this paper explored students' elaborations of the notionof boredom in relation to their classroom context, identifying substantial differences in the ways in which the concept is construed cross-culturally.

The dialectics between practice (e.g. classroom practice) and emotions arising from this practice was discussed, and different theoretical frameworks were acknowledged. The question as to what extent emotions can be seen as social constructs was discussed with reference to some examples, as for example 'shame'. Further, it was discussed whether trying to explain students' emotions as deriving from their general cultural context (rather than as deriving from the micro-culture of the mathematics classroom context) represents a form of cultural essentialism. As a consequence of the view that labelling emotional experiences in fact constructs what we call the emotion, a discussion about the role of predefined labels for emotions in questionnaires arose, as these might trigger a response that one thinks one has experienced it in the labelled way.

Theme 5: Modes and principles of using quantitative methods in comparative studies

Mathematics teachers' beliefs in Estonia, Latvia and Finland (presenter: MarkkuHannula, University of Helsinki, Finland)

This paper, which exploited a multidimensional survey instrument, compared the mathematics- and mathematics teaching- related beliefsof Estonian, Latvianand Finnish teachers, with a key variable being the language of instruction employed in teachers' schools. The analyses indicated both cross-national and cross-lingual differences in the ways that teachers' beliefs played out.

Re-examining the language supports for children's mathematical understanding: A comparative study between French and Vietnamese languages (presenter: HienThi Thu Nguyen, University of Pedagogy,HoChiMinh City, Vietnam)

This final paper, drwing on data from French-speaking Belgium and Vietnam, examined how the ways in which languages articulate numbers influence third grade children's mathematical understanding and performance. The analyses indicated that aspects of the Vietnamese language facilitated students' performance in some domains, when compared with French-speaking students, but not others.

It was noted that when imported instruments fail to be reliable in new contexts, this is an important research outcome. One approach to overcoming this problem would be to include open items to allow the voice in the new context to be heard. Further, examples were given of concepts exploited in studies that may have different salience or vocabulary in different cultures. For example, effective or efficient teaching is an acknowledgement of a local discourse. The relationship between qualitative and quantitative approaches to data gathering and analysis in comparative studies was another focus of the discussion. Questions raised included: How does one mode inform the other? How do they fit our objectives? What is the impact of cultural proximity (Finland and Sweden) or distance (Belgium and Viet Nam) on research design and expected outcomes? Another emerging issue concerned the sensitivity to cultural minorities in larger-scale studies: Are the cultural groups under scrutiny clearly defined?

A CROSS-NATIONAL STANDARDS ANALYSIS: QUADRATIC EQUATIONS AND FUNCTIONS

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With the advent of the Common Core State Standards (CCSS), scholars are eager to understand its implications for their work. In this study we compare the characteristics of the CCSS for Mathematics (CCSSM) with other countries' mathematics curricula. In particular we investigated how quadratic equations and functions are introduced in the curriculum documents of four different countries: the Caribbean, China, Turkey, and the U.S. These were analyzed under three dimensions: content, mathematical reasoning, and cognitive level. The results show that all the standards introduce the foundational concepts of quadratic functions, but with varying procedural and conceptual expectations.

INTRODUCTION

The newly released *Common Core State Standards (CCSS*) have aroused wide interests in the field of education. As these standards state,

> The standards are designed to be robust and relevant to the real world, reflecting the knowledge and skills that our young people need for success in college and careers. With American students fully prepared for the future, our communities will be best positioned to compete successfully in the global economy. (Common Core State Standards Initiative [CCSSI], 2010)

Implementing these new standards has become a matter of concern for teachers, researchers, policy makers, students and parents. The standards also have implications for other countries, since international students and scholars are an increasing part of the post-secondary education population in the United States. As a group of international researchers studying mathematics education in the United States (we derive from the Caribbean, China, and Turkey), we set out to compare the *CCSSM* against the standards and teaching plans with which we are most familiar. We designed, therefore, a cross-national study to investigate the similarities and the differences of learning expectations for students among these four economically, socially, culturally, and geographically diverse countries. These countries also vary in terms of the structure and history of their educational systems, as well as the implementation of their intended curricula. However, the Caribbean, China, Turkey, and the U.S. all communicate, in various written forms, what and how students are expected to learn mathematics (these differences will be discussed subsequently); for simplicity we refer to these documents as *Written Learning Expectations for Mathematics* (WLEMs).

In this study, we compare, in particular, the WLEMs related to quadratic equations and functions, irrespective of grade level. In so doing our goal is not to rank nations, but to ascertain the strengths and weaknesses of different educational systems and provide a basis for improving the teaching and learning of mathematics (e.g., Cai, 2001; Porter & Gamoran, 2002; Robitaille & Travers, 1992).

BACKGROUND

Standards Analysis in Previous Studies

A few studies have focused on curriculum standards analysis; however, most of them only look at a single aspect of these standards, such as the coherence (e.g., Schmidt, Wang, & McKnight, 2005), format (e.g., Reys, Dingman, Nevels, & Teuscher, 2007), or content (e.g., Reys, 2006).

Schmidt et al.'s (2005) examination of the coherence of mathematics and science standards revealed that, compared to the six high-achieving countries in the *Trends in International Mathematics and Science Study* (TIMSS), each grade in the various U.S. national standards devoted instructional attention to many more topics and these topics stayed in the curriculum for more grades. Reys et al.'s (2007) report summarized the format of state-level curriculum standards and graduation requirements for high school mathematics. The result showed that states are varied with respect to required mathematics credit hours and courses for graduation. In work conducted by the Center for the Study of Mathematics Curriculum (CSMC), Reys (2006) reported results of comparing state-level K-8 mathematics curriculum-standards documents in three content strands: number and operation, algebra, and reasoning. The report highlighted how different topics within these strands were sequenced and emphasized across the different state documents.

Only one research study examined the multiple aspects of the standards: clarity, content, mathematical reasoning, and negative qualities (Raimi & Braden, 1998). Because this study aimed to critique the standards, not all the categories were suitable for international comparison.

Significance of Quadratic Equations and Functions

Solving quadratic equations is one of the most conceptually difficult topics in the secondary school mathematics curriculum (e.g., Vaiyavutjamai & Clements, 2006), and educators should be aware of these difficulties and be prepared to help students to confront these challenges. Even though quadratic equations play an important role in secondary school curriculum around the world, studies concerning teaching and learning quadratic equations are quite rare in algebra education research (Kieran, 2007; Vaiyavutjamai & Clements, 2006).

In addition, although quadratic functions are one of the most important concepts extending beyond linear functions in the secondary school mathematics curriculum, students have struggled to understand quadratic functions (Ellis & Grinstead, 2008). Students' struggles with quadratic functions are based on a few key areas, including inflexible connections between multiple representations; rigid views of graphs as whole objects; incorrect interpretations of the role of parameters; and tendencies to incorrectly generalize from linear functions (Ellis & Grinstead, 2008). These findings indicate that educators should be aware that students must move flexibly and develop connections among different representations in order to establish a functional understanding.

METHODOLOGY

Theoretical Framework

The process of building a robust framework within which to perform a cross-national comparison led us to a three-dimensional framework: content, mathematical reasoning, and cognitive demand. For the content category we chose to follow the definition of content as "coverage," and our analysis involved a comparison of the topics and concepts that each set of standards requires in relation to quadratic equations and functions.

The second dimension of the analytical framework is mathematical reasoning in relation to quadratic equations and functions. This part of our framework is based on the recommendations of NCTM's (2009) Reasoning and Sense Making document. This document emphasises *Reasoned Solving* with algebraic symbols, which means that "problem solving with equations should include careful attention to increasingly difficult problems that span the border between arithmetic and algebra" (p.34). This ensures that students can see that algebra extends arithmetic reasoning and is a more powerful approach when solving more challenging problems alone.

In terms of functions, the document suggests three essential elements:

- *Using multiple representations of functions.* Representing functions in various ways, including tabular, graphic, symbolic (explicit and recursive), visual, and verbal; making decisions about which representations are most helpful in problem-solving circumstances; and moving among those representations.
- *Modeling by using families of functions.* Working to develop a reasonable mathematical model for a particular contextual situation by applying knowledge of the characteristic behaviours of different families of functions.
- *Analyzing the effects of parameters.* Using a general representation of a function in a given family (e.g., the vertex form of a quadratic, $f(x) = a(x - h)^2 + k$ to analyze the effects of varying coefficients or other parameters); converting between different forms of functions (e.g., the standard form of a quadratic and its factored form) according to the requirements of the problem-solving situation (e.g., finding the vertex of a quadratic or its zeros) (p. 37).

The third dimension, cognitive level, focuses on how students are expected to learn concepts, what they are expected to do in the process, and how they are to demonstrate their understanding. We chose to use the Bloom's Taxonomy, which is a classification of levels of learning objectives that educators set for students. It was initially proposed by a committee of educational psychologists chaired by Benjamin Bloom in 1956. In this paper, we used a revised version of the taxonomy created by Anderson and Krathwohl (2001). Bloom's Taxonomy divided educational objectives into three "domains": *Cognitive, Affective,* and *Psychomotor*. Within these domains, learning at the higher levels depends on having attained prerequisite knowledge and skills at lower levels. In this study, we only attend to skills in the cognitive domain. There are six levels in the taxonomy, moving from the lowest order processes to the highest: (a) *knowledge*—Exhibit memory of previously-learned materials by recalling specifics and universals, methods and procedures, and patterns, structures, or settings; (b) *understand*—Understand facts and ideas by organizing, comparing, translating, interpreting, giving descriptions, and stating main ideas; (c) *apply*— Solve problems in new situations by applying acquired knowledge, facts, techniques, and rules in a different way; (d) *analyze*—Examine and break information into parts by identifying motives or causes. Make inferences and find evidence to support generalizations; (e) *evaluate*—Present and defend opinions by making judgments about information, validity of ideas, or quality of work based on a set of criteria; and (f) c*reate*—Create new product or point of view (Krathwohl, 2002). We also referenced a list of action words of the taxonomy categorized by other scholars (Center for University Teaching, Learning, and Assessment [CUTLA], n.d.).

Data Collection & Analysis

The quadratic functions and equations WLEMs from all the different countries in the study were collected for analysis, and the Turkish and Chinese WLEMs were translated by the members of the research team from these respective countries. Researchers who performed the translations used contextual clues and examples wherever possible to help maintain fidelity in the meanings of WLEMs.

For data analysis in the content dimension, team members recorded all the distinct topics and concepts related to quadratic equations and functions. For the mathematical reasoning dimension each country's WLEMs were coded for unique opportunities for mathematical reasoning according to the framework criteria. A narrative was compiled for each country and a cross-WLEM analysis was conducted. For the cognitive level dimension of the framework, we used Bloom's Taxonomy action verb analysis to code each WLEM into Bloom's categories. During the data analysis process, we wanted to make sure that the number of verbs that are used represented the actual number of tasks in each WLEM (a WLEM could include multiple tasks), and this rule was applied to all of the countries. To do this each WLEM was read in its entirety to determine the actions that students would be required to undertake (e.g. "Be able to plot the graph of a quadratic function, and know its properties by exploring the graph." (Chinese WLEMs, 2011, p. 21), has two verbs, which indicate two tasks were required) and each WLEM was coded based on the verbs that dictated its learning goals. Each set of WLEMs was coded by two members of the research team. Discrepancies were discussed and resolved by the team.

DATA INTRODUCTION

Caribbean WLEMs: The most recent Caribbean WLEMs document was written in 2008, created for 16 English speaking countries in the Caribbean region and is 47 pages long. It provides a rationale for the mathematics included, general and specific course objectives, a summary of how the syllabus is organized, and detailed descriptions of the exams that would be used to assess students (including the distribution of content to be covered and the details of the topics covered). The mathematical objectives listed in the WLEMs for quadratic functions list specific skills that students should acquire and gives a few example problems. In the general objectives, students are given esoteric mathematical expectations that are very broad. Some examples of the general learning expectations include: learning to treat algebra as a language and a way of communicating, appreciating the role of symbols and algebra techniques in problem solving, and being able to reason abstractly (*Caribbean Examination Council* (CXC) syllabus, 2008).

***CCSSM*:** The *CCSSM* WLEMs is a 93 page document. It consists of two main sections: Standards for Mathematical Practice and Standards for Mathematical Content. In this study, we focus on the content section, which is organized by grades (K-8) and organized by content strand at the high school level (Number and Quantity, Algebra, Functions, Modeling, Geometry, and Statistics and Probability). It provides descriptions of what students should know and provides one or two example equations or problems where appropriate.

Chinese WLEMs: The *Full-time Compulsory Education Mathematics Curriculum Standards* (FCEMCS) was enacted by the Ministry of Education of the People's Republic of China, and published by Beijing Normal University Press in 2001. It is a 44 page document, which includes three stages: *stage one* (grades 1-3), *stage two* (grades 4-6), and *stage three* (grades 7-9); four areas of mathematics content: numbers and algebra, figures and geometry, statistics and probability, and integration and practice; and four types of objectives: knowledge and skills, mathematical thinking, problem solving, and emotion and attitudes. The latest edition of the *General High School Mathematics Curriculum Standards* (GHSMCS) was published by People's Education Press in 2011. This high school mathematics curriculum consists of two types of courses: compulsory and elective. The compulsory course contains five modules: *Math 1-5*. For the high school standards, we only focus on the module *Math 1,* which is a 5 pages long document and covers the WLEMs in relation to quadratic equations and functions

Turkish *WLEMs*: The high school mathematics curriculum (grades 9-12) has two main areas: mathematics and geometry. The mathematics curriculum is 364 pages long, and the geometry curriculum has 207 pages for grades 9 and 10, 127 pages for grade 11, and 176 pages for grade 12. In the WLEMs, every concept was designed according to four sub-sections: *subfields*, *gains*, *hints for the activity*, and *explanations*. These sub-sections indicate what students must learn about a concept by the end of instruction, examples that are helpful for teachers to teach a concept, and explanations that are recommendations for teachers to use in their instructions. These standards are the requirements to be achieved by all students, and students must learn those standards in order to go a grade further.

FINDINGS

Content

In terms of content for quadratic functions and equations, the Caribbean WLEMs provided an introduction to basic principles, skills, and processes. Students are expected to be able to factor quadratic expressions, solve abstract quadratic equations and context specific ones in word problems, use symbols to denote quadratic functions, and be able to graph and interpret quadratic functions. Two optional specific objectives require that students find and understand the significance of the axis of symmetry and find the number of roots for a particular quadratic equation. The content range among *CCSSM*, Chinese and Turkish WLEMs are similar to those of the Caribbean, but there are also notable differences. For *CCSSM*, the content on quadratic functions requires that students solve quadratic equations using different methods, using multiple representations (graphical, tabular, and symbolic), comparing and evaluating the characteristics of quadratic functions with other families of functions; and modelling real or natural patterns with quadratic functions and also problem-solving by using quadratic functions in realistic contexts. *CCSSM* also emphasizes the relationship between different types of functions. The Chinese and Turkish WLEMs also include fundamental properties of quadratic equations and functions (e.g., zero roots and extreme values) and basic problem solving means (e.g., graphing, factoring, and completing the square). Additionally, there are some minor differences: Chinese WLEMs are more specific on the properties of functions (e.g., direction of the opening, monotonicity, domain and range, and even/odd functions) whereas Turkish WLEMs also embed their mathematical content in specific examples and have made problem solving a foundational part of the WLEMs.

Mathematical Reasoning

Reasoned solving of equations. The Caribbean WLEMs clearly require students to be able to solve quadratic equations and solve a pair of equations when one of them is a non-linear equation. The *CCSSM* high school algebra standards require students to create quadratic functions that describe numbers and relationships, to solve quadratic equations in one variable, and to understand solving equations as a process of reasoning. Both *CCSSM* and Chinese WLEMs emphasize the connection between equations and functions; that equations can be solved by using graphs of the corresponding functions. The Turkish WLEMs emphasize the connections between the roots and coefficients of quadratic equations.

Using multiple representations of functions. The Caribbean and Turkish WLEMs have similar requirements with regard to the form of representations. They both expect students to use graphical, symbolic, and tabular representations while the use of technology is not mentioned. The *CCSSM* are the only WLEMs among the four countries that mention verbal representation. They also state that students should be able to graph functions both by hand and with technology. The Chinese *WLEMs* emphasize plotting graphs of quadratic functions and suggest using technology, such as the calculator, to analyze the graph and to find the solution of the corresponding equation.

Modelling by using families of functions. There is no explicit mention of modelling with quadratic functions in the Caribbean WLEMs, with the exception of the general objective that students should "appreciate the usefulness of concepts in relations, functions and graphs to solve real-world problems" (*CXC*, 2008 p. 24). The *CCSSM* has a set of modelling WLEMs embedded in other WLEMs that appear throughout the entire high school section. For example, there is an emphasis on modelling skills in the domain of building functions, which requires the ability to describe a relationship between two quantities in a contextual situation (F-BF). The Chinese WLEMs not only require a deep understanding of the concept of modelling through various examples, but also the ability to represent functions in an appropriate form (e.g. graph, table, and equation) in a real world context. The Turkish WLEMs do not have specific standards on modelling. They provide many examples, but in these examples, the model is usually given.

Analyzing the effects of parameters. The Caribbean and Chinese WLEMs include understanding and using the form $f(x) = a(x - h)^2 + k$, although this is optional for Caribbean students. All the *WLEMs* require a general representation of the quadratic function except for the *CCSSM*. However, the *CCSSM* mentions using various equivalent forms. Uniquely, it requires students to represent the quadratic equation and its solution in the form of complex numbers $a \pm bi$ for real a and b. The Turkish WLEMs emphasizes the use of $f(x) = ax^2 + bx + c$. All the WLEMs emphasize the study of the properties of functions through graphs (e.g., finding the zeros, symmetry, intercepts and extreme values) while they have slightly different emphases on problem solving methods. The Caribbean WLEMs focus on analyzing the graph, with symbolic approach being optional; the *CCSSM* focus on factoring and completing the square; the Chinese WLEMs suggest various methods, including factoring and completing the square, root-formula, and dichotomy (by using the calculator); the Turkish standards do not have specific requirements for problem solving methods.

	Knowledge	**Understand**	**Apply**	**Analyze**	**Evaluate**	**Create**	**Total**

CCSSM	0	5 extent, interpret, estimate, observe	14 choose, complete the square, factor, graph, solve, write, show, sketch, relate, calculate, determine, combine	0	1 properties	0	20
Caribbean	0	0	10 determine, draw, factorize, interpret, sketch, solve, use	0	0	0	10
Chinese	7 Know	7 determine, know, obtain, represent, understand, learn	11 apply, draw, examine, find, plot, represent, solve, use	2 experience	1 investigate	0	28
Turkish	1 Identify	1 Identify	11 calculate, find, show, sketch, write	0	0	1 organize	14

Table 1: Action Words for Bloom's Taxonomy in Cross National Comparisons

Cognitive Level (Bloom's Taxonomy)

With respect to cognitive level, we analyzed 45 WLEMs of quadratic equations and functions across all the countries and found 72 tasks (Table 1). The results show that Bloom's *apply* category is most common across all the WLEMs; 46 tasks were identified in this category. In the Caribbean WLEMs all action words alluded to the *apply* category. The *CCSSM* has 14 such tasks, the Chinese WLEMs has 11, as does the Turkish. We also found some distinctive characteristics among the WLEMs. As indicated above, the Caribbean WLEMs has only one category of action words, *apply*. The *CCSSM* has none of the categories: *knowledge, analyze* or *create*. The Chinese WLEMs includes all categories except for *create;* being the only standards to address the *analyze* level. Additionally, the Chinese WLEMs emphasizes the foundations of *knowledge* and *understanding*. Turkish WLEMs do not have any action words at the *analyze* or *evaluate* levels but are the only WLEMs to include the *create* level.

DISCUSSION

One of the limitations of this study was the different representations of learning objectives used in the different WLEMs of the project countries. While there were similar content coverage goals, what was expected of students, how the content emerged, and the role of the teacher in deciding when the content was to be taught varied across WLEMs. Also, several issues emerged during the coding of WLEMs translated from Turkish and Chinese that led us to infer that some of the original intent of the WLEMs was lost in translation. Besides identifying action verbs using Bloom's taxonomy we also had to read the each WLEM completely to determine whether the verb was categorized correctly or if the verb needed to be placed into a different Bloom's level depending on the requirements from the WLEMs. In the categorization, there is also the possibility that the same verbs in different WLEMs might imply different actions, and one action in different WLEMs might be represented by multiple verbs. Therefore, the precision of our results might be limited by the translation of WLEMs. In addition, the conclusions that we can make about these four different *WLEMs* are only with regard to quadratic functions and equations. Further studies would be needed to make larger comparisons regarding all of the content taught in high school mathematics.

We believe our study may facilitate discussion between mathematics education colleagues internationally. Particularly, since the *CCSSM* is newly released, we hope that the similarities and differences identified above will help US-based mathematics educators and teachers, not only understand the *CCSSM* more fully but implement it more effectively. Of course, no WLEMs gives a complete picture of the curricular opportunities for students to learn about quadratic equations and functions; they cannot predict the quality of instruction or the level of achievement that students experience in their post-secondary careers. However, we can conclude that the learning expectations for these concepts are comparable across the four countries and are also diverse in many aspects.

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THE DEVELOPMENT OF FOUNDATIONAL NUMBER SENSE IN ENGLAND AND HUNGARY: A CASE STUDY COMPARISON

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Foundational number sense - being able to operate flexibly with number and quantity - is a predictor of later mathematical achievement. In this paper, drawing on lessons on number sequences to grade 1 children, we examine how two teachers, one English and one Hungarian, construed locally as effective, created opportunities for children to develop foundational number sense. The Hungarian teacher, in ways typical of that country's mathematics teaching tradition, offered frequent and coherent opportunities for students to develop foundational number sense. The English teacher, working in a tradition whereby interactive technology increasingly mediates classroom discourse, offered few and less coherent opportunities, masked by the teacher's frequent attention to display features of the technology.

INTRODUCTION

Described as a "traditional emphasis in early childhood classrooms" (Casey et al 2004: 169), the quality of young children's number sense is a key predictor of later mathematical success, both in the short (Aunio and Niemivirta, 2010) and the longer term (Aunola et al, 2004). Consequently, particularly as number sense deficits tend to lead to later difficulties (Jordan et al., 2007), the development of children's number sense "is considered internationally to be an important ingredient in mathematics teaching and learning" (Yang and Li 2008: 443). Importantly, without appropriate intervention children who start school with limited number sense are likely to remain low achievers throughout their schooling (Aubrey et al., 2006).

Defining foundational number sense

Our understanding of number sense is that it comprises two related but distinct concepts that the literature typically conflates. The first, which we label foundational number sense (FNS), concerns the number-related understandings children develop during the early years of formal instruction. The second, which we have labelled applied number sense, draws on the first and concerns the number-related understanding necessary for people to be mathematically competent and functionally effective in society. In this paper, while remaining mindful of the latter, we focus on the former.

For the purpose of this paper we define FNS as the ability to operate flexibly with number and quantity. It develops over time "as a result of exploring numbers, visualizing them in a variety of contexts, and relating them in ways that are not limited by traditional algorithms" (Sood and Jitendra, 2007, p. 146). It comprises, so our understanding of the literature informs us, several elements, among which are:

Awareness of the relationship between number and quantity (Berch, 2005; Clarke and Shinn, 2004; Van de Rijt et al., 1999; Griffin, 2004).

Understanding of number symbols, vocabulary and meaning (Clarke and Shinn, 2004; Van de Rijt et al., 1999; Malofeeva et al. 2004; Yang and Li, 2008).

Systematic counting, including notions of ordinality and cardinality (Gersten et al., 2005; Griffin, 2004; Malofeeva et al. 2004).

Awareness of magnitude and comparisons between different magnitudes (Gersten et al., 2005; Griffin, 2004; Ivrendi, 2011; Jordan et al., 2007; Malofeeva et al. 2004).

An understanding of different representations of number (Ivrendi, 2011; Jordan et al., 2007; Yang and Li, 2008),

Competence with simple arithmetical operations (Berch, 2005; Ivrendi, 2011; Yang and Li, 2008).

An awareness of number patterns including recognising missing numbers (Berch, 2005; Clarke and Shinn, 2004; Jordan et al., 2007).

In this paper, as an introduction to comparative research in the field, we examine the question, what opportunities do two teachers, in different cultural contexts and considered locally as effective, create for their students to acquire FNS?

DATA COLLECTION AND ANALYSIS

In addressing our question, we compare excerpts from two lessons taught to grade one children in England and Hungary. The lessons from which they were drawn were each part of a wider collection of videotaped lessons gathered independently of each other. The English lesson derived from the second author's PhD case study examination of primary mathematics teachers' enactment of whole class teaching. The Hungarian lesson derived from a study of exemplary Hungarian primary mathematics teaching undertaken by the third author to inform curriculum development activities in England. Both teachers were construed against local criteria as effective in the manner of the Mathematics Education Traditions of Europe project (Andrews, 2007). Thus, the two data sets were not only serendipitously available but also appropriate for comparative analysis. Both teachers, with microphones, were video-recorded in ways that would optimise the capture of their actions and utterances. Both data sets entailed repeated observations of a small number of case study teachers over a period of several months in order to ensure a sense of the typical lesson. The Hungarian lessons were supported by a home-based English-speaking colleague providing a contemporary translation augmented by the first author's sufficient understanding of Hungarian to be able to follow much of the discourse of a mathematics classroom. In other words, while the two sets were collected independently, they were gathered in similar ways and amenable to similar analyses. The excerpts analysed here were based on the teaching of number sequences. This topic was chosen because, among the various FNS-related components, it was the only one addressed explicitly in both sets of lessons. Moreover, our belief was that number sequences would provide more opportunities for the incidental teaching (Radwan, 2005) of the other FNS components. With respect to analysis each excerpt was viewed simultaneously and repeatedly by all three researchers. This led to our seeing not only that each excerpt comprised three distinct phases, which frame our analyses, but also which components of foundational number sense were addressed, both implicitly and explicitly.

RESULTS: THE ENGLISH EXCERPT

This excerpt derives from a lesson taught to 6 and 7 year olds on number sequences and patterns. It began with the teacher, Sarah, reminding her children about previous work on sequences, how odd and even numbers create an alternating pattern in the number system, and informing them that today they would be looking at some more sequences and number patterns. The 5 x 10 grid below was displayed on the interactive white board – a resource typical of English classrooms - with the cells containing the first six even numbers shaded. Sarah asked if anyone could explain the pattern. One child responded by saying that they all end in 0, 2, 4, 6 or 8, after which Sarah commented that they are all even. Next she asked about the column patterns and, after a suggestion that the numbers in each column end in the same digit, Sarah accepted the suggestion that the column pattern goes odd even odd. At this point, exploiting the whiteboard's software, she displayed the odd columns in one colour and the even in another.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	33	24	25	26	27	28	29	30
31	32	43	34	45	36	37	38	39	40
41	42	53	44	55	46	47	48	49	50

Commentary: In these first few minutes several FNS categories appeared to have been addressed. For example, Sarah's explicit emphases seemed very much geared towards inducting her children into an awareness of number patterns alongside a clear expectation that children recognise number symbols and vocabulary.

Next, Sarah put up another 5 x 10 grid but with no coloured cells. Beneath the grid was the following:

1, 4, 7, 10, 13, __, __

She asked how other numbers in the sequence could be found and invited her class to look at the number grid. Having evoked no responses Sarah tapped, in turn, each of the five numbers to change the colour of their cells and create the image below. At this point she commented that the grid looked different from that of the previous problem and asked *have I done it wrong?* Different students offered, tentatively, both negative and positive responses, which went without comment.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	33	24	25	26	27	28	29	30
31	32	43	34	45	36	37	38	39	40
41	42	53	44	55	46	47	48	49	50

Next, Luke raised his hand and the following ensued:

Sarah	Luke?
Luke	Sixteen
Sarah	Why sixteen?
Luke	Because you're adding on three
Sarah	Because it's adding on three isn't it (she taps the cell to change the colour)... What's going to be the next one? Isla?
Isla	Nineteen
Sarah	What's going to be the next one? Ian?
Ian	Twenty-two
Sarah	Twenty-two (she taps the cell). And the next one? (more hands go up this time) Rachel?
Rachel	Twenty-five

Commentary: In the above is evidence of different aspects of FNS. The four students were clearly extending the sequence given them. Also, the assertion that they were adding on three indicated an engagement with simple arithmetical operations and, of course, they were still being made aware of the relationship between symbols, their vocabulary and meaning. By now the grid looked as below.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	33	24	25	26	27	28	29	30
31	32	43	34	45	36	37	38	39	40
41	42	53	44	55	46	47	48	49	50

Sarah	Twenty-five (she clicks on the cell, which becomes red) Can anybody see any colour patterns coming out of this?
Boy	It looks like a bit like stairs!
Sarah	Rosie? (Rosie says something inaudible so Sarah walks to the middle of the room to hear what Rosie says)

Sarah	Yes, they're going diagonally aren't they. Yes when we extended this pattern we started to see that. There's a diagonal pattern made by the squares coloured in. Now, if Rosie's right then this one's (she taps 34, which turns red) going to be in our sequence. I'm going to fill it in and then count on three each time. One, two, three (she taps 28, which change it to red). One, two, three (she taps on 31). One, two… (she points to 34, which is already red, and faces the class)
Children	Threeee (a few children shout out)

Commentary: In this closing episode Sarah appeared to be encouraging systematic counting from an ordinal perspective as well as further opportunities for children to recognise number symbols, vocabulary and meaning. Thus, over the whole excerpt, Sarah seemed to have addressed four of the seven FNS categories.

RESULTS: THE HUNGARIAN EXCERPT

This excerpt was taken from a sequence of lessons focused on children's coming to know and work with integers to 20. The lesson began with Klara, having written on the board prior to the start of the lesson, the configuration shown below.

–	–	–	–	–	–	–	–	–
3	7	6	10	_	_	_	_	_

Having been invited to do so, the class read out the numbers in unison as Klara pointed to each in turn. Next, moving from left to right, she invited volunteers to explain how each number could be derived from the one preceding it. Students volunteered that the first operation was add four, followed by subtract one and add four. With each offering Klara wrote the operation underneath, as shown below, before inviting predictions as to what operation would be expected next. Eventually, after several contributions, the table was completed.

–	–	–	–	–	–	–	–	–
3	7	6	10	_	_	_	_	_
+4	-1	+4						

Commentary: During this period it seems to us that Klara had encouraged several aspects of FNS. Firstly, an introduction during which children were invited to recognise and read the numbers on the board was focused on the recognition of number symbols and their vocabulary. Secondly, in the ways in which successive numbers were identified, Klara was addressing simple arithmetical operations. Thirdly, the episode was explicitly focused on an understanding of number patterns.

Next, Klara produced some cards, each of which had a letter written on it. She announced that she was going to ask questions, the answer to each would be one of the numbers in the sequence. Each correct answer would yield a letter to spell a word that would tell the class where it would be going in the story of this particular day. The following reflects the first minute of next five minutes of discourse.

Klara	So my first statement is… please look only at the numbers on the board… I am thinking of the largest one-digit number. Balasz?
Balasz	Six (Pupils protest)
Klara	Look at the sequence again, and please correct yourself.
Balasz	Seven
Klara	Look at the number line… Ferenc?
Ferenc	Nine
Klara	That's right. So I will give you a reward for the nine (Klara placed a card with the letter Í on the board above the number nine). The next number I am thinking of… You mustn't look behind you (Referring to a picture on the back wall) is the value of the black stick in our collection. Perszi?
Perszi	Eight (Pupils protest)
Klara	(to Perszi) Look around, the others don't agree with you… Mara?
Mara	Seven
Klara	Let's see who's correct. (They all look at the back wall, where they can see the members of the Cuisenaire rod collection and their values)
All	(In chorus) Mara was correct (at which point Klara placed a card with the letter Á above the seven).

The lesson continued with Klara asking a different form of question for each number in the sequence. These included statements like, a number two smaller than nine, the largest two digit number, the smallest one digit number, a number whose digits add up to 4, and so on. In each case at least one child was involved in publicly responding to the questions posed. Eventually, as shown in figure 4, the following emerged with only the number ten left without its corresponding letter.

B		Á		B		_		Í		N		H		Á		Z
3		7		6		10		9		13		12		16		15
	+4		-1		+4		-1		+4		-1		+4		-1	

Commentary: In this phase several FNS categories were addressed. The largest one-digit number discussion encouraged not only recognition of number symbols, their vocabulary and meaning but also awareness and comparison of magnitude. In considering a Cuisenaire rod, Klara addressed not only an understanding of different representations of number but also an awareness of the relationship between numbers and quantities. In fact, every statement seemed to address an element of FNS. For example, a number two smaller than nine is a different representation of seven. Similarly, several statements involved simple arithmetical operations.

Having identified all the letters bar the one linked to ten Klara passed responsibility to her students and invited them to offer statements appropriate to that number. This led to the following:

Mara:	It is the bigger neighbour of the number 9.
Ildikó:	It is the smallest 2-digit number.
Csaba:	It is the smaller neighbour of the number 11.
Gabor:	The sum of its digits is 1.
Judit:	Even number.
Klara	Have we got anything else?
Zsolt:	It is the sum of the 1 and 9.
Klara:	Yes, the sum of the 1 and 9… and who knows the letter in my hand?
All:	Sz (the juxtaposition of s and z in this manner is, in Hungarian, an alphabetic letter with a sound similar to the s in sun)
Klara:	Yes, and where are we going today?
All:	Bábszínház. (Puppet theatre)

Commentary: In this final episode was further evidence of Klara's promotion of FNS. For example, both understanding of different representations of number and recognition of number symbols, their vocabulary and meaning was, we believe, implicit in all contributions. Simple arithmetic could be seen in Gabor's statements that the sum of ten's digits is one and Zsolt's suggestion that ten is the sum of one and nine. Awareness of number patterns was implicit in Judit's even number suggestion, while awareness of magnitude was implicit in Mara's bigger neighbour of nine and Eva's smaller neighbour of eleven.

DISCUSSION

In this paper we have attempted to show how opportunities for students to acquire FNS played out in two culturally different classrooms. Apart from explicit focus of the excerpts – number sequences - neither teacher appeared to focus explicitly on the development of FNS. However, through an analysis of such snapshots we can examine the extent to which the development of foundational number sense is an integral, albeit implicit, component of children's early school experiences of mathematics. Moreover, since both teachers were construed locally as effective, such snapshots may offer insight into how teachers, in different cultural contexts, have been conditioned - by their experiences as learners of mathematics, their professional training and subsequent career opportunities - to address such issues.

The analyses above, summarised in table 1, indicate both similarities and differences in the ways in which foundational number sense was addressed. In respect of similarities both teachers addressed several categories, with Klara addressing six of the seven categories and Sarah four. Both encouraged, throughout their respective excerpts, students' recognition of number symbols, vocabulary and meaning. Both encouraged the awareness of number patterns and missing numbers and both exploited simple arithmetical operations. In respect of differences Klara addressed three categories, the relationship between numbers and quantities, comparisons of magnitude and different representations of number that Sarah did not, while Sarah was seen to address systematic counting when Klara did not.

	Sarah's episodes			Klara's episodes		
Relationships between numbers and quantities					X	
Number symbols, vocabulary and meaning	X	X	X	X	X	X
Systematic counting			X			
Comparisons of magnitude					X	X
Different representations of number					X	X
Simple arithmetical operations		X		X	X	
Number patterns and missing numbers	X	X		X	X	

Table 1: the distribution of the categories across the excerpts' episodes

However, while it is clear that both teachers encouraged various aspects of FNS as part of the incidental learning of their lessons it seems to us that Klara's was a more didactically complex encouragement than Sarah's. Put crudely, Klara addressed, on average, four categories of foundational number sense per episode while Sarah addressed barely two. Klara encouraged mathematical reasoning, while Sarah seemed to subordinate such reasoning to an examination of the coloured patterns on the interactive whiteboard. This latter expectation, it seems to us, appeared not only a distraction from children's learning of mathematics but indicative of a more general problem for teachers working in technology enhanced classrooms (Muir-Herzig, 2004; Wang and Reeves, 2003). Moreover, if number sense develops gradually as a result of exploring and visualizing numbers in different contexts (Sood and Jitendra, 2007) then Klara's practice seems more likely to succeed than Sarah's.

Interestingly, both teachers' practices find some resonance with earlier studies of mathematics teaching in the two countries, albeit at the level of the upper primary classroom rather than the lower primary. Andrews (2009) found Hungarian teachers exhibiting didactical sophistication in their encouragement of cognitively demanding but coherent learning outcomes. The same study found English teachers exhibiting relatively unsophisticated didactical practices in their promotion of substantially more modest and less coherent goals. That is both Klara's and Sarah's practices appeared commensurate with that of their compatriot colleagues working in later phases of schooling. Thus, the limited evidence of this study indicates that teachers working in the first years of schooling, teachers defined locally as effective, behave in ways similar to their compatriots and that the Hungarian tradition seems more likely to facilitate a secure FNS than the English. That being said, clearly more research in this particular field is necessary if we are to understand more fully how teachers induct their learners into this essential mathematical prerequisite.

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MODELLING IN THE FRENCH AND SPANISH SYLLABUS FOR SECONDARY EDUCATION

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In this paper we present the exploratory phase of an ongoing comparative study on the teaching of modelling in France and Spain. The study aims to describe the place of modelling in the curricula. In this exploratory phase we will use some tools of the Anthropological Theory of the Didactic (ATD). We will discuss the rationale, theoretical framework and methods of this study. Then we will determine if modelling is designated to be taught in France and Spain. We will try to give some conditions that could explain the place of modelling in the syllabus. We will formulate propositions for the next step of this research.

RATIONALE, THEORETICAL FRAMEWORK AND METHOD

Blum and Ferri (2009) point out that "mathematical modelling (the process of translating between the real world and mathematics in both directions) is one of the topics in mathematics education that has been discussed and propagated most intensely during the last few decades. In classroom practice all over the world, however, modelling still has a far less prominent role than is desirable"(p.45). For us (as in Cabassut, 2009) a modelling cycle is the process of solving a real world problem by translating it into a mathematical one (called model of the real world problem), then solving this mathematical problem before translating the mathematical solution back into solution of the real world problem, and validating this solution.

In the previous quotation of Blum and Ferri (and in the whole article) we note the gap between the importance of modelling in mathematics education research and the impact of modelling in classrooms. With this in mind, we compare here the role of modelling in French and Spanish curricula. To compare the teaching of modelling between France and Spain we will use some tools of the Anthropological Theory of Didactic (ATD) described in (Bosch & al., 2006) that points out the important role of institutions in teaching. In modelling, we can refer to everyday life or to sciences, from which the real world problem comes, and to mathematical knowledge from which the mathematical problem comes. How is the transposition of these two references undertaken in the teaching institution? Is modelling designated to be taught in France and in Spain?

We present the exploratory part of our on-going work. We concentrate initially on the question: in the curriculum,is modelling designated to be taught? This question will be answered in the second section. In further research we aim to see if modelling is taught, for this aim we will need to analyse textbooks, resources, interviews teachers, etc. Our study is limited to the secondary school curriculum because a similar study about primary school was previously made between France and Germany by Cabassut and Wagner (2011).

In the third part of this paper we will try to justify the conclusion of the second part using the levels of determination proposed by ATD.Dorier (2010) gives a description of each level:

> "Chevallard takes into account the level of the Pedagogy, i.e. the general teaching principles included in the description of the curriculum of an institution. Then the level of School, takes into account how the general curriculum is structured, the division into disciplines, the time allocated to each, the fact that teachers are mono- or pluri-disciplinary, etc. The next level deals with Society, that is to say, the institutional organisation of the educational system in a country or a region, the most general level of the curriculum, etc. The highest level has to do with Civilisation, it takes into account variations between different cultures, like western versus eastern culture" (p.12).

Each level could help to explain the place of modelling in the syllabus.

To analyse the different curricula we refer to the official texts produced by the corresponding ministries of education of both countries and select those sentences where some words semantically related to modelling appear. From these collected citations we try to answer the question: is modelling designatedto be taught? In order to analyze levels of determination, we will use the same method but we will enlarge the reading to more general text, as European Parliament recommendations, PISA results, etc.

We have chosen a bi-national research team because "the comparative method seems to be a major tool in clinical questioning, making it possible to break with the apparent naturality of observations in each country, which encourages the constitution of multinational teams of research" (Cabassut, 2007 p.2431).

IS MODELLING DESIGNATED TO BE TAUGHT?

The French secondary school

In France, secondary school lasts from grade 6 (11 to 12 years old) to grade 12 (17 to 18 years old) and, in contrast with primary school, mathematics teachers are subject specialists. Secondary school is organised in two parts. There is a common school from grade 6 to grade 9, *collège*, with the same curriculum throughout France. After this schools are differentiated (vocational, technical or general, *lycée*). We will consider the curriculum of both the *collège* and the *lycée*. The contents of the first year of *lycée* (corresponding to grade 10) are common for all students. In the final two years (grades 11 and 12), students choose between literary, scientific or economic branches. Compulsory education ends at age 16, corresponding to end of grade 10 for pupils who have never repeated a school year.France is a centralised country and the same official texts edited by the Ministry of National Education (MEN) describe the curriculum and are applied everywhere. There are two kinds of official texts. The first describes the content to be taught for every year of the curriculum and is found in the newspaper of the Ministry of Education (Bulletin Officiel de l'EducationNationale [BOEN]). The second presents resources or advice produced by the MEN. For mathematics these texts are often produced under the responsibility of the body of General Inspectors of Mathematics (Inspection Générale de Mathématiques) that monitors mathematics teaching everywhere in France. The present curriculum was introduced in the *collège* from 2006 to 2009 and in the *lycée* from 2010 to 2012.

Curriculum of general education (from grade 6 to grade 10)

In France, the common base of knowledge and skills (BOEN, 2006) considers that

> "the main elements of mathematics are acquired and exercised primarily by problem solving, especially from realistic situations. […]. On leaving compulsory school, the student must be able to apply the principles and processes basic math in everyday life, in his private life as in his work. […]. The student must be able [...] to model so elementary, to understand the link between natural phenomena and mathematical language which applies to it and helps to describe it." (p.6-9).

The introduction to the *collège* syllabus asserts that

> "through problem solving, modelling of some situations and progressive learning of the demonstration, students learn little by little what a real mathematical activity is: to identify and to formulate a problem, to conjecture a result by experimenting on examples, to build an argumentation, to check the results by assessing their relevance for the studied problem, to communicate on a research, to give form to a solution" (BOEN, 2008, p.9).

For problem solving, the relations with everyday life or other subjects, and particularly sciences, are mentioned. After this introduction the mathematical content of the syllabus is described by mathematical domains: data organization and functions, numbers and computing, geometry, magnitudes and measures, and later analysis, statistic and probability, algorithmic, arithmetic...

For grade 10 the objective "is to train students in the scientific process in all its forms to enable them to model and to engage in research activities [...], to make a critical analysis of a result, of a process [...], to communicate in written and oral form" (BOEN, 2009, p.1). In the detail of the content of different mathematical domains we find references to modelling and models. For example, regarding problems related to first degree equations, the syllabus recommends that "each time the different stages of work have to be identified: setting equation, solving the equation and interpreting the results" (p.29), which is a reference to a kind of modelling cycle.

Scientific branch (grades 11 and 12)

This branch develops scientific education in grades 11 and 12. Moreover, in grade 12 pupils have to choose one speciality among earth and life sciences, physics and chemistry, or mathematics, with a supplementary syllabus. The introduction to the mathematics syllabus of the scientific branch specifies that "activities […] should lead students to: search, experiment, model, […] explain a process, communicate results in a write and oral form" (BOEN, 2010a, p.1). Different mathematical domains such as analysis, probability and geometry, refer to modelling and models (for example: "Diffusion model of Ehrenfest: N particles are distributed in two containers, and at each instant, a randomly selected particle exchange container" (BOEN, 2011a, p.18). We have also found nine explicit connections to the science syllabi. For example, in relation to sine and cosine was "progressive sinusoidal waves, mechanical oscillator" (p.6) or, in relation to probabilistic independence was "heredity, genetics, genetic risk" (p.12). We think that such relations with the science syllabi encourage students to understand how mathematics is used to model science. In the mathematics specialism syllabus of the final year, the study of the situations considered in the context of this course leads to a modelling work and places students in a position to undertake research.

Economic and literary branches (grades 11 and 12)

For students following the literary branch mathematics is optional, although, as with the scientific branch, those pupils who choose to apply mathematics have to be trained "to develop the following skills: to implement independent research; to conduct reasoning; to have a critical attitude towards their results; to communicate in writing and orally [...] to experiment and to model" (BOEN, 2010b, September 30, p.1). Different mathematical domains (algebra, analysis, statistics and probability, geometry) allude to modelling and models. In the economic branch curriculum, particularly in the last grade, we find, as expected, some explicit references to modelling. In this branch "teaching is based on problem solving. [...] The study of such situations leads to modelling work, and places students in a position to research". Examples of these problems are given, including "workflow, simple problems of graph partitioning under constraints: the traveling salesman problem management, road or air traffic, scheduling sports tournaments… modelling of inter-industry trade (Leontief matrices)" (BOEN, 2011b, p.10).

The Spanish secondary school

In order to study the teaching of mathematical modelling in secondary education in Spain, we focus on the official syllabus from the Ministry of Education. Spain is divided into 18 autonomous regions; each of these regions also has an official syllabus of secondary studies. Each regional syllabus must be framed within the national one, which is why, for this work, we centre our attention on the syllabus published by the Ministry of Education at BoletínOficial del Estado (BOE).

The structure of secondary education in Spain is regulated by Education Law (BOE, 2006, p.17158-17207). It comprises two stages: compulsory secondary education (from grade 7 to 10) and high school (grades 11 and 12) from which, after passing a test, students can access university. In contrast with primary education (grades 1 to 6), secondary teachers are subject specialists, with each subject taught by a different teacher. To analyse the content of every stage, we will focus on the study of the syllabi which regulate both the compulsory secondary (BOE, 2007, p.31680-31828) and the non-compulsory high school (BOE, 2008, p.27492-27608).

Curriculum of compulsory secondary education (grades 7 to 10)

All curricula refer to eight core competences or essential skills that students must reach across all subjects. These are communication skills, mathematical competence, competence in the knowledge and interaction with the physical world, information processing and digital competence, social and civic competence, cultural and artistic competence, learning to learn competence and autonomy and personal initiative.

In the introduction to the section on mathematics it is mentioned that, historically, mathematics “has been used by scientists of all times to build models of reality” (BOE, 2007, p. 31789). The objective is that students, at the end of compulsory secondary education, should “be able to use [mathematics] to think critically about the different realities and problems in today's world” (BOE, 2007, p. 31789). To achieve this objective, they recommend that the content is presented in a problem-solving context. Thus, problem solving stands as the cornerstone on which to work the mathematical content of the curriculum. In this introduction we find some explicit references to modelling, particularly in the domain of geometry “learning of geometry should provide continued opportunities for [...] modelling” (BOE, 2007, p. 31790), and also in the domain of relationships between variables from tables and graphs (analysis function) through which “students are intended to be able to distinguish the characteristics of certain types of functions in order to model real situations” (BOE, 2007, p. 31790). Finally, the syllabus offers methodological guidelines focused on mathematics as a discipline. For example, they recommend working on open situations as this allows students with higher levels of cognitive development to be able to “conceptualize progressively contents in order to ask questions about what is sought” (BOE, 2007, p. 31803), while these open situations can also serve to support and reinforce students with difficulties.

Spanish high school (grades 11 and 12, from 16 to 18 years old) comprises two years of study divided in three branches: literary, scientific and artistic. Mathematics is only compulsory for scientific students during their first year. Below we discuss aspects of the mathematics offered to students in the science and literary tracks; students in the artistic high school do not have to study mathematics.

In its introduction to the content for science-oriented students, the syllabus specifies that mathematics “gives rise to the necessity to solve practical problems [...] and [mathematics] are supported by their ability to [...] model real situations” (BOE, 2008, p.27574). Among the seven general objectives of mathematics for scientific students, no explicit reference to modelling appears. However, the need to “use scientific research strategies and skills specific to mathematics [...] for general research and explore new situations and phenomena” (BOE, 2008, p.27575) is stressed.

During the first year of high school, mathematics (Matemáticas I) is divided into four domains: algebra (and arithmetic), geometry, analysis and probability (and statistics).

The only explicit reference to modelling appears in the domain of analysis, where pupils must be “able to model situations and phenomena with known graphics” (BOE 2008, p.27575). In its section on evaluation the syllabus asserts that pupils should be able to “solve problems drawn from social reality and nature involving the use of equations and inequalities, and must interpret the results” (BOE, 2008, p.27576).

During the second year of high school, we find no mention of inquiry based learning, problem solving or modelling in the mathematics content. However the evaluation criteria suggest that teachers should “intend that students manage information drawn from various sources and use available technologies [...] model situations, [...] extract information, make interpretations [...] and process mathematic data” (BOE, 2008, p.27577).

For students of the **Literary branch** mathematics is not compulsory, although in the syllabus for Mathematics Applied to Social Sciencesit is emphasized that it should be worked from a practical point of view rather than from a mechanical point of view, going “beyond the mechanical resolution of exercises that requires only the immediate application of a formula” (BOE, 2008, p.27606). In order to understand the use of mathematics “activities arising should encourage the possibility of applying mathematical tools to analyse social phenomena particularly relevant, such as cultural diversity, health, consumption, coeducation, peaceful coexistence and respect for the environment” (BOE, 2008, p.27605). In the two years of high school, mathematics (as applied to social sciences) is subdivided into three domains: algebra, analysis and statistics (and probability). Regarding the domain of algebra, in the evaluation criteria we found that pupils must “use appropriate techniques to solve real problems giving an interpretation of the expected solutions” (BOE 2008, p.27606). In general, teachers of mathematics have to show students how to “deal with real life problems, organizing and codifying information, developing hypotheses, selecting strategies and using both the tools and modes of argumentation of mathematics to face new situations effectively” (BOE, 2008, p.27606).

ANSWER TO THE QUESTION AND DISCUSSION

In the French secondary curriculum the reference to modelling is frequent throughout: it is always related to problem solving either inside pure mathematics or in relation to other subjects, particularly sciences and technology. This means that it is not clear if modelling is always referenced to a real world. For example, geometry, where optimisation problems are mentioned, can be considered as a pure mathematical world. That is, modelling in this context could be construed as operating within a mathematical world. We note clearly the reference to a kind of modelling cycle involving the three steps of setting an equation, solving the equation and interpreting the results (BOEN, 2008 p.14). All the mathematical domains are involved in modelling, and specially probability. The competences are mentioned often in a general context because most of these competences are related to general and transversal competences, like to be able to communicate, to be critical, to reason, to argue... For all these reasons **we can conclude that in the curriculum modelling**

is designated to be taught, in all branches of general secondary education. The differences between branches are related to the mathematical level and the nature of the problems (scientific, economic or social) in which it occurs.

In the secondary Spanish curriculum, particularly in the mathematics syllabus of compulsory education we find some reference to modelling, most of which are in the domains of geometry and analysis. We think that the references to modelling that appear in the introduction to the scientific branch can increase inquiry based learning, however it is quite revealing that, in the final year before university we find no mention of modelling. We think the reason may be the following: at this level, the syllabus contains many new topics (matrix algebra, integration and limits) and preparation (often mechanical) for the final examinations for university applications prevents any kind of innovation in the classroom. In the syllabus of Mathematics applied to Social Sciences (literary branch) we find, as expected, several references to the relationship between mathematics and reality. So, we conclude that indeed modelling is**designated to be taught**. However, these official texts give no explicit guidance on how to work mathematics through modelling. They merely recommend that the subject should be worked through problem-based learning (problem must be taken from everyday life). Unlike France, in Spain any teaching resources are published by the Ministry of Education. Possibly further research may well lead us to resources published at the level of the regional government.

Discussion

In contrast with Germany, where modelling is one of the seven core competences of the secondary mathematics curriculum, in neither France nor Spain is modelling so explicitly defined. Official texts discuss modelling both explicitly and implicitly but it is not always clear if students are expected to apply a given model or construct a model in order to solve a problem. However in the French texts is mention of the part of the modelling cycle where the model is built. Indeed, there are several resources from the French Ministry in which can be found classroom tasks where models have to be built, like for example in probability (MEN, 2008). We have also noticedthat it is difficult to compare the French and Spanish syllabi, since these are written in different contexts. For these reasons, we will study now why modelling is designated to be taught.

LEVELS OF DETERMINATION

At the level of civilisation and society, in France, the common base of knowledge and skills (BOEN, 2006) refers explicitly to PISA and to European parliament recommendations.In the text of the latest Spanish education law we find also explicit reference to both the Organization for Economic Co-operation and Development (OECD) and the European Union (BOE, 2006, p. 17160).

At the level of school we remark that in both France and Spain secondary mathematics teachers are subject specialists, which could make it more difficult for them to teach themes linked to other subjects.

In France, over the last few years, new curriculum structures have encouraged schools to integrate different subjects. For example, in grade 10, there is now an optional course on scientific methods and practices (BOEN, 2010c, p.1), which entails one and a half hour per week in students' time-tables. It allows them to explore different areas of mathematics, physics and chemistry, life sciences and earth and engineering sciences. Also, during grade 11, students of the scientific branch have to undertake a supervised project, called TPE (BOEN 2011c). Over eighteen weeks, small groups of students work collectively on a project, using various resources, on a subject chosen by them that connects two topics (as, for example: How can we use satellite images to refine forecasted monsoons? Modification of food is it progress?). TPE bring into play at least two disciplines, including one which is essential to the students' orientation. The realisation of the project is supervised by teachers of the relevant disciplines with two hours per week in the students' timetable. Assessment considers all aspects of the students' contributions, including written and oral presentations, and is part of the final mark for entering university. Clearly, modelling activities with an open building of the model are easier in this kind of structure (themes of convergence, exploration teaching, TPE) than in a one subject lesson. In the Spanish programme, such opportunities are not found. Some regional university institutes (called IREM) take charge of in-service training of mathematics teachers or offer resources, in relation to modelling. Starting in 2012, a network of science houses is developing in France in order to offer in-service training for the teachers and could offer training on modelling or on inquiry based approach.In Spain the problem is that the training of secondary school teachers in didactics is poor (García et al., 2007). In view of these observations, it is understandable that, still, the majority of Spanish secondary school teachers find working in accordance with the official syllabus guidelines difficult. To get a broader view of the resources available for teaching modelling it would be desirable to extend our research to the study of textbooks. In any case it seems clear that, compared to France, Spain is far from incorporating modelling into its mathematic classrooms.

At the level of pedagogy, in France, the *college* syllabus proposes a common introduction for all scientific subjects and defines (BOEN, 2008, p.5) different *themes of convergence* to be worked together by different subjects and supports a common inquiry based approach that fits well with modelling activities. The Spanish education system insists in the connection between all subjects (all the competences have to be developed in all the disciplines). Obviously, modelling activities promote the acquisition of these eight competences. Moreover, the official text of the curriculum claims that "the teaching methodology must be communicative, active and participatory", fostering cooperative work and "highlighting the relationships between subjects and its relationship to reality" (BOE, 2007, p.31682). In the definition of the eight core competences, we realize that these make sense when teachers propose to work on real situations (near to daily reality of students). Indeed, we can conclude that both programmes are based in a pedagogy that could promote the use of modelling as a teaching tool.

At the level of the mathematical domains, in France, resources in numbers, geometry, magnitude, data-organisation and specially in statistics and probability, mention explicitly modelling and propose activities for the class. In Spain there are no national resources like in France. In particular, probability seems a domain underused for modelling in comparison with France.

At the level of mathematical themes and subjects, in France and also in Spain, it is difficult, from official texts, to find a clear link with modelling. We have to investigate resources like textbooks in a next study. Textbooks, in both France and Spain, are non-official resources designating what is to be taught and, when they are used by teachers and students, showing what has been taught.

CONCLUSIONS AND OPEN QUESTIONS

The comparative method shows phenomena in one country (like for example official resources for the teachers in France) absent in the other. The method shows also that the same condition (for example modelling designated to be taught) can produce different consequences because other levels of determination play different roles depending on the country. That is, the comparative method, in fact, helps us to understand more deeply the, not so obvious, conditions of one country in contrast with another. Further research would be able to examine how modelling is taught and learned, and specially the role of non-official resources - like textbooks.

NOTES

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THE VALIDITY-COMPARABILITY COMPROMISE IN CROSS-CULTURAL STUDIES IN MATHEMATICS EDUCATION

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The pursuit of commensurability in international comparative research by imposing general classificatory frameworks can misrepresent valued performances, school knowledge and classroom practice as these are actually conceived by each community and sacrifice validity in the interest of comparability. The "validity-comparability compromise" is proposed as a theoretical concern with significant implications for international cross-cultural research in mathematics education. The paperuses current international research to illustrate various aspects of the issue and its consequences for the manner in which international researchis conducted and its results interpreted. The effects are extensive and constitute essential contingencies on international comparative research in mathematics education.

Key words: International Research, Mathematics Education, Comparability, Validity

INTRODUCTION

This paper identifies key considerations affecting the conduct and utility of international comparative research. Central to the design of such research studies are the dual imperatives of validity and comparability. Unfortunately, as will be illustrated, these imperatives are inevitably in tension. This paper identifies, illustrates and discusses these tensions, utilising very specific examples from current international comparative research. It is argued here that any value that might be derived from international comparisons of curricula or classroom practice is critically contingent on how the research design addresses the dual priorities of validity and comparability. It is further argued that since these priorities act against each other, researchers undertaking international comparative research must find a satisfactory balance between these competing obligations.

Perhaps only the drive to categorise is more fundamental than our inclination to compare (cf. Lakoff, 1987). Indeed, the two activities are intrinsically entwined since the act of comparison involves the recognition of the distinctive attributes of the objects being compared, and these attributes represent a form of categorisation. It is a key premise throughout this paper that comparisons are undertaken between constructed representations, whose structure and attributes are reflective as much of the value system of the researcher as of the objects being represented for the purposes of comparison. That these constructions are encrypted in language and effected socially aligns the paper's theoretical orientation with socio-cultural theorists prioritising the role of language as mediating experience and action (Vygotsky (1978) being the obvious example). In this paper, commensurability is interpreted as the right to compare (cf. Stengers, 2011). And it is our central assertion that this right to compare cannot be assumed, but is contingent on our capacity to legitimise both the act of comparison and the categories through which this act is performed. The need for such legitimisation has been raised for international comparisons of student achievement, but less frequently and less carefully for the cross-cultural comparison of curricula and classrooms. Some examples are cited in this paper of cross-cultural comparisons of limited legitimacy (eg participation or lesson structure) or legitimate comparisons framed at levels of granularity such as to limit or remove explanatory capacity (eg between-desks-instruction or student classroom talk). In each example, I have attempted to suggest how to maximise both the validity and the explanatory power of the categorisation schemes employed for purposes of comparison.

Critical in the legitimisation of these acts of comparison are the validity of the categories we employ and of the act of comparison itself. Much of the focus in this paper is on cultural validity, which is interpreted (with Säljö, 1991) as a key determinant of practice in the international settings we aspire to compare. Research designs, especially data generation and categorisation processes, can misrepresent or conceal cultural idiosyncrasies in the interest of facilitating comparison.

This paper considers this validity-comparability compromise in relation to both curriculum and classroom practice research. Curricular comparisons raise issues related to the structure of school knowledge and the aspirational character of valued performances. Comparisons of classroom practice foreground the performative realisation of school knowledge and introduce the teacher as curricular agent (among other roles), modelling, orchestrating, facilitating and promoting performances aligned with the educational traditions of the enfolding culture. Any cross-cultural comparative analysis faces the challenge of honouring the separate cultural contexts, while employing an analytical frame that affords reasonable comparison.

The paper utilises seven "dilemmas" to reveal some of the contingencies under which international comparative research might be undertaken. The issues raised by each dilemma are not mutually exclusive sets. Specific empirical examples from current international research provide the vehicle by which the entailments of each dilemma can be explored to identify areas of cross-cultural research requiring critical examination.

COMPARABILITY AND VALIDITY IN CROSS-CULTURAL STUDIES

In an international comparative study, any evaluative aspect is reflective of the cultural authorship of the study.

> Culture is thus what allows us to perceive the world as meaningful and coherent and at the same time it operates as a constraint on our understandings and activities.(Säljö, 1991, p. 180).

In seeking to make comparison between the practices of classrooms situated in different cultures, the most obvious comparator constructs become problematic.

Dilemma 1: Cultural-specificity of cross-cultural codes
Use of culturally-specific categories for cross-cultural coding (eg participation, mathematics).

In the Chinese adaptation of the research design for the Middle School Mathematics and Institutional Setting of Teaching (MIST) project, the decision was made not to use the Instructional Quality Assessment (IQA) (Silver & Stein, 1996), but instead to develop a local instrument for the evaluation of mathematics classroom instruction. The reason for the rejection of the IQA instrument for use in Chinese school settings reflected the embeddedness, within the instrument, of particular values characteristic of the cultural setting and educational philosophy of the authoring culture (USA). For example, for the measurement of students' participation in classroom instruction, new criteria are needed that accommodate the larger class size and norms of social interaction of the Chinese mathematics classroom. Figure 1 shows the criteria for evaluating the level of student participation in teacher-facilitated discussion in mathematics classes.

A. Participation

Was there widespread participation in teacher-facilitated discussion?

4	Over 50% of the students participated consistently throughout the discussion.
3	25 to 50% of the students participated consistently in the discussion OR over 50% of the students participated minimally.
2	25 to 50% of the students participated minimally in the discussion (that is, they contributed only once.)
1	Less than 25% of the students participated in the discussion.
N/A	Reason:

Figure 1. Participation criteria from the *Instructional Quality Assessment* (IQA) instrument (Silver & Stein, 2003).

In countries such as China and Korea, teachers in both primary and secondary schools make extensive use of elicited student choral response as a key instructional strategy (Clarke, 2010). In the lessons analysed from one Shanghai classroom, a large number of choral responses (~ 80) were used in each lesson. In the analysis of a classroom in Tokyo, there were a similar number of individual student public statements, but no evidence of choral response. Applying the IQA participation criteria (Figure 1), the regularity and frequency of the use of choral responses would characterise this classroom as participatory at a level comparable with the classroom in Tokyo. Yet the students in the Tokyo classroom participate primarily through individual contributions rather than choral response and the type of teacher-facilitated discussion and the nature of student participation in that discussion in the two classrooms are sufficiently different to make their comparability with respect to participation highly questionable. The cultural authorship of research instruments and their cross-site legitimacy has implications for both data generation and interpretation and must be accommodated carefully through revision or replacement, or through reconception of the nature of the comparison being undertaken.

While Dilemma 1 arises through the misapplication of a culturally-authored categorisation scheme, Dilemma 2 arises from the need to categorise at a level of granularity sufficiently large as to accommodate (mask or gloss over) more fine-grained differences between the objects of comparison. The difficulty here arises not because the category scheme is culturally-specific, but because the objects to be categorised are sufficiently disparate or varied at one level of granularity as to defy comparison, but able to be included in a more inclusive (less discriminating) categorisation scheme that then supports the act of comparison, but disengages the act of comparison from the more fine-grained descriptions that could explain the origins of identified differences.

Dilemma 2: Inclusive vs Distinctive
Use of inclusive categories to maximise applicability across cultures, thereby sacrificing distinctive (and potentially explanatory) detail (eg. mathematical thinking, lesson structure).

In a recent study undertaken by the author and his colleagues, we compared the ways in which mathematics curricula are framed in Australia, China, Finland and Israel. We sought to identify the similarities and differences in the organisation of mathematics curricula in the four countries in terms of their aims, content areas and performance expectations. In particular, we investigated the ways in which "mathematical thinking" was framed through curricular statements.

The key documents analysed in this study were: the Victorian Essential Learning Standards (VELS), the Chinese MathematicsCurriculum Standards (CMCS), the Finnish National Core Curriculum (FNCC) and the Mathematics Curriculum (Israel) (MCI). The four curricula are structurally quite different and prioritise different performance types. The excerpts below capture some of these qualitative differences.

> See mathematical connections and be able to apply mathematical concepts, skills and processes in posing and solving mathematical problems (VELS).

> [Translation] Obtain important mathematics knowledge that is essential for functioning in society and further development (including mathematical facts and experience in participating in mathematics activities) and basic mathematical thinking skills as well as essential skills of application (CMCS).

> The task of instruction in mathematics is to offer opportunities for the development of mathematical thinking, and for the learning of mathematical concepts and the most widely used problem-solving methods (FNCC).
>
> [Translation] Mathematics is not only a collection of calculated algorithmic operations that serve an applied purpose but also a subject with its own structure that includes unique thinking and investigation methods. The goal of the curriculum is to generate a change in the way that students view the subject (MCI).

Any attempt to characterise the relative emphasis given to particular types of valued performance at different grade levels can only be undertaken if a common classificatory framework can be imposed on all curricula. But such a general framework must not be allowed to mask the significant emphasis given to Geometry in grades 7 to 9 in China, or to "Communicating" in grades 3 to 5 in Finland, or the idiosyncratic prioritizing in grades 7 to 9 in Israel of "the evolution of phenomena from the perspective of mathematics." The danger is that the commensurability demands of such comparisons conceal major conceptual differences in the curricular expression of categories of school knowledge. For example, it can be argued that the curricula in Australia, China, Finland and Israel are similar in that they advocate the development of "mathematical thinking" but this conceals important differences in the nature of the mathematical thinking that each curriculum seeks to promote. The act of developing more inclusive categories by combining more fine-grained culturally-specific categories in order to enable cross cultural comparisons runs the risk of distorting the knowledge categories we seek to compare. In cross-cultural research the imposition of an "external" classification scheme for the purposes of achieving comparability can sacrifice validity by concealing diversities reflecting cultural characteristics and by creating artificial distinctions. Comparability is achieved through processes of typification and omission, and each has the potential to misrepresent the setting.

Dilemma 3: Evaluative Criteria
Use of culturally-specific *criteria* for cross-cultural *evaluation* of instructional quality (eg.Student spoken mathematics).

Where research is specifically constructed to be evaluative, the question arises as to the legitimate application of criteria developed in one culture to the practices of another culture. The use of evaluative criteria posits an ideal of effective practice that should be substantiated by reference to research. Problems arise when the research on which a criterion is based is itself culturally-specific.

For example, despite the emphatic advocacy in Western educational literature, classrooms in China and Korea have historically not made use of student-student spoken mathematics as a pedagogical tool. In research undertaken by Clarke, Xu and Wan (2010), classrooms were identified in which student spoken mathematics was purposefully promoted in public but not in private interactions (eg Shanghai classroom 1), in both public and private interactions (eg Melbourne 1) and in neither public nor private interactions (eg Seoul 1). Each of these classrooms models a distinctive pedagogy with respect to student spoken mathematics.

If the occurrence of student-spoken mathematics is identified with quality instruction, then the instructional practice of the classroom in Seoul would be judged to be deficient. The classrooms in Shanghai and Melbourne differed significantly in the extent to which private student-student interactions were encouraged, but the teachers in both classrooms prioritized student facility with spoken mathematics. In the Shanghai classroom, promotion of this capability was developed solely through public discourse, whereas in the Melbourne classroom, private student-student mathematical speech was an essential pedagogical tool. Interestingly, in post-lesson interviews, the students from Melbourne and Shanghai showed comparable fluency in their use of the language of mathematics, while students from the classrooms in Seoul showed little evidence of such a capacity. The comparability issue here is whether it is legitimate to undertake evaluative comparison of frequency and sophistication of student spoken mathematics in Melbourne, Shanghai and Seoul, if the operative pedagogy in the Seoul classrooms does not actually value student facility in spoken mathematics as a learning outcome and therefore cannot be presented legitimately as having "failed" to develop this capacity in the students. Evaluative judgments of instructional quality made in the context of international comparative research must justify the model of accomplished practice implicit in the criteria employed and provide evidence of the cross-cultural legitimacy of these criteria.

Dilemma 4: Form vs Function

Confusion between form and function, where an activity coded on the basis of common form is employed in differently situated classrooms to serve quite different functions (egkikan-shido or between-desks-instruction).

Kikan-shido (a Japanese term meaning "between-desks-instruction") has a *form* that is immediately recognisable in most countries around the world. In kikan-shido the teacher walks around the classroom, while the students work independently, in pairs or in small groups.Although kikan-shido is immediately recognisable to most educators by its form, it is employed in classrooms around the world to realise very different functions. A teacher undertaking kikan-shido in Australia, will do so with very different purposes in mind from those pursued by a teacher in Hong Kong, or, for example, a teacher in Japan. In reporting the frequency of occurrence of an activity such as kikan-shido for the purposes of comparative analysis, the researcher conflates activities that are similar in form but which may be employed in differently-situated classrooms for quite distinct functions. Such conflation can create an impression of similarity although differences in practice are actually quite profound (for more detail, see Clarke, Emanuelsson, Jablonka &Mok, 2006). The distinction between Dilemma 4 and those preceding it, derives from the combination of the legitimate equivalence of form (between-desks-instruction) across cultural sites, and the disparity of function (rather than of yet more fine-grained form) that exists at the same level of granularity between sites (rather than between different levels of granularity across all sites).

Dilemma 5: Linguistic Preclusion
Misrepresentation resulting from cultural or linguistic preclusion (eg Japanese classrooms as underplaying intellectual ownership).

The analysis of social interaction in one culture using expectations encrypted in classificatory schemes that reflect the linguistic norms of another culture can misrepresent the practices being studied. This can occur because characteristics of social interaction privileged in the researcher's analytical frame may not be expressible within the linguistic conventions of the observed setting. For example, the Japanese value implicit communication that requires speaker and listener to supply the context without explicit utterances and cues. This tendency is typically found in leaving sentences unfinished. As a consequence, in Japanese discourse, agency or action are often hidden and left ambiguous. In English, when introducing a definition, the teacher might employ a do-verb: "We define".In a Japanese mathematics classroom, the teacher often introduces a definition in the intransitive sense (*SouNatteIru* = "as it is" or "something manifests itself") as if it is beyond one's concern.Such differences in the location of agency, embedded in language use, pose challenges for interpretive analysis and categorisation of classroom dialogue.

Dilemma 6: Omission
Misrepresentation by omission, where the authoring culture of the researcher lacks an appropriate term or construct for the activity being observed (eg. Pudian).

The Sapir-Whorf hypothesis suggests that our lived experience is mediated significantly by our capacity to name and categorise our world.

> We see and hear . . . very largely as we do because the language habits of our community predispose certain choices of interpretation(Sapir, 1949).

Marton and Tsui (2004) suggest that "the categories . . . not only express the social structure but also create the need for people to conform to the behavior associated with these categories" (p. 28). Our interactions with classroom settings, whether as learner, teacher or researcher, are mediated by our capacity to name what we see and experience. Speakers of one language have access to terms, and therefore perceptive possibilities, that may not be available to speakers of another language. For example, in the Chinese pedagogy "Qifa Shi" (Cao, Clarke, &Xu, 2010), the activity "Pudian" is a key element. Pudian can take various forms: Connection, Transition, Contextualising, but its function is to help students develop a conceptual, associative bridge between their existing knowledge and the new content. There is no simple equivalent to Pudian in English, although teacher education programs delivered in most English-speaking countries would certainly encourage the sort of connections that Pudian is intended to facilitate. Many such pedagogical terms have been collected in a variety of languages (Clarke, 2010), describing classroom activities central to the pedagogy of one community but unnamed and frequently absent from the pedagogies of other communities. It follows that an unnamed activity will be absent from any catalogue of desirable teacher actions and consequently denied specific promotion in any program of mathematics teacher education. It is also likely that such activities will go unrecognised in reports of cross-cultural international research, where the authoring culture of the research report lacks the particular term.

Dilemma 7: Disconnection

Misrepresentation through disconnection, where activities that derive their local meaning from their connectedness are separated for independent study (eg. teaching and learning (cfobuchenie), public and private speech).

Whether we look to the Japanese “gakushu-shido", the Dutch “leren” or the Russian “obuchenie”, we find that some communities have acknowledged the interdependence of instruction and learning by encompassing both activities within the one process and, most significantly, within the one word.In English, we dichotomise classroom practice into Teaching or Learning. One demonstration of the consequences of the inappropriate disconnection of actions that should be seen as fundamentally connected is evident in the comparison of two published translations involving Vygotsky’s use of the term “obuchenie” (discussed in Clarke, 2001).

> From this point of view, *instruction* cannot be identified as development, but properly organized *instruction* will result in the child's intellectual development, will bring into being an entire series of such developmental processes, which were not at all possible without *instruction*(Vygotsky, as quoted in Hedegaard, 1990, p. 350).

> From this point of view, *learning* is not development; however, properly organized *learning* results in mental development and sets in motion a variety of developmental processes that would be impossible apart from *learning*(Vygotsky, 1978, p. 90).

The analogous disconnection of public and private speech in classrooms, and of speaking and listening (Clarke, 2006) has the same effect of misrepresenting activities that may be fundamentally interrelated (not just conceptually but also functionally connected) in their enactment in particular classroom settings.

CONCLUSIONS

The pursuit of commensurabilityin international comparative research by imposing general classificatory frameworks can misrepresent valued performances, school knowledge and classroom practice as these are actually conceived by each community and sacrifice validity in the interest of comparability. In this paper, the"validity-comparability compromise" has been proposed as a theoretical concern that has significant implications for international comparative research. The identified dilemmas offer different perspectives and illustrate some of the consequences of ignoring this central concern. Some dilemmas are not specific to cross-cultural comparative contexts (eg inclusive vs distinctive - Dilemma 2), but should be considered any time difficulties caused by diversity at the intended level of comparison can be avoided by undertaking the comparison at a larger (more inclusive) level of granularity, sacrificing distinctive (and potentially explanatory) detail in the interests of legitimate comparison. Other dilemmas are a direct consequence of the idiosyncrasies of culture and language (eg Linguistic Preclusion or Omission – Dilemmas 5 and 6). The examples also illustrate the steps that can be taken to improve the legitimacy of the intended comparison. Partnerships with those being compared can minimise misrepresentation, but the necessity of the compromise is inescapable. The interpretation and application of international comparative research is critically contingent on researchers' capacity to address those "dilemmas" pertinent to their particular design. As they have been framed in this paper, the dilemmas relate to methodological concerns. However, each dilemma can also serve as an interrogatory instrument: a tool directing the researcher's attention to salient characteristics that, while presenting impediments to comparison, simultaneously provide insight into nuances of meaning and practice. This paperis intended to fuel a wider engagement in the critical interrogation of international comparison as a socio-material knowledge practice.

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MATHEMATICS TEACHERS' BELIEFS IN ESTONIA, LATVIA AND FINLAND

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The article presents results from a cross-cultural comparison of mathematics teachers' beliefs. We report how nationality and language in Estonian, Latvian and Finnish schools shape mathematics teachers' beliefs, the schools' microcultures and the relations between teachers' beliefs and their schools' microcultures. Results indicate cultural differences in school microcultures and teachers' beliefs as well as in how these variables are related to each other.

INTRODUCTION

Research on teacher beliefs has been motivated by the assumption that teacher beliefs largely determine the reality of teaching in classrooms. However, such naïve views have been challenged by case studies indicating inconsistencies between teacher's beliefs and practice suggesting that more emphasis should be paid to contextual factors. In this article we suggest an overall theoretical frame for the role of culture, school microculture, and teacher beliefs in the formation of actual classroom practices. Moreover, we present results of a cross-cultural survey of mathematics teacher belief structures in Estonia, Latvia, and Finland and how these beliefs are influenced by school microculture.

Teacher beliefs

Teaching in schools is orchestrated by teachers. They interpret policies and curricula and implement them in classrooms. Based on this perspective, there has been extensive research on teachers' beliefs, which reflect their priorities for the practices of mathematics classrooms and play a significant role in shaping teachers' instructional behaviour (Thompson, 1992; Schoenfeld, 1998). However, there remains considerable debate about the definition and characteristics of beliefs (see, Furinghetti & Pehkonen, 2002). In this study beliefs are construed broadly as those conceptions, views and personal ideologies that shape teaching practice. Beliefs research in mathematics education has focussed on how teachers view the nature of mathematics, its learning and teaching, and teaching in general (Dionne, 1984; Ernest, 1991; Liljedahl, Rösken, & Rolka, 2007).

Contextual influences on beliefs

The manifestation of teacher's beliefs in practice is influenced by the context: pedagogical traditions in the country, school culture, social background of the students, etc. This makes the relationship between teachers' beliefs and their teaching practice complex; research often reports inconsistencies between teachers' beliefs and their actions (Cooney, 1985; Skott, 2009). There are two levels of contextual factors. The overall cultural milieu includes not only curriculum and other educational policy, but also unofficial aspects of the culture, which impact the values of education and teacher-student relationships. Such influences do not always follow the national borders as religion, and language may be more relevant determinants. An individual teacher must largely take these for granted and just adjust to them.

Another level of context is the local microculture in the school, which is reflected in the school rules and norms and in the way teachers collaborate. The teacher is an important actor of this microculture and may influence its development over time. The importance of school microculture has been found repeatedly in intervention studies. For example, in an evaluation of one large professional development program within mathematics education (Bobis, Clarke, Clarke et al., 2005), the aspects that were considered most effective were the practical resources and activities, the assessment process, the influence of significant people, classroom support, and the opportunity to share ideas. On the other hand, significant barriers to teachers' implementation of the program were time, resources, class management and information overload. Almost all of these can be influenced locally.

So far, few studies have compared teacher beliefs across countries (e.g., Andrews, 2007; Andrews & Hatch, 2000; Felbrich, Kaiser & Schmotz, 2012). The TALIS survey explored conditions of teaching and learning in 24 OECD countries (OECD, 2009). Loogma, Ruus, Talts and Poom-Valickis (2009) constructed two factors for different teacher beliefs: 1) traditional beliefs and 2) constructivist beliefs. Their analysis showed that in some countries teachers tended to choose one view over the other while in other countries there was a strong positive correlation between these two perspectives.

Teacher beliefs and cultural influences in Estonia, Latvia and Finland

Since regaining their independences in 1991, Estonia and Latvia have undergone many changes affecting their educational systems. While natural sciences and mathematics were emphasised in the Soviet curriculum, the focus has shifted towards other topics. Also the attractiveness of the teacher profession has fallen considerably.

In Estonia, there was also a concern among mathematics education researchers that teaching was too much based on drill and practice-methods (Lepik, 2005). Yet, TALIS found Estonia to be one of the countries with strongest support for constructivist teaching beliefs (Loogma et al., 2009). Although Estonian teachers believed more in a constructivist way of teaching they did not directly contrast this view with the direct transmission of knowledge, and could therefore believe in the combination of these two views.

Latvian teachers have been found to be more oriented towards constructivism than teachers in the USA (Ravitz, Becker, & Wong, 2000; Šapkova, 2011). Whilst both primary and secondary teachers supported constructivist ideas, primary teachers reported more implementation of constructivism in their classrooms than secondary teachers (Pipere, 2005). A recent study (Austers, Golubeva, & Strode, 2007) shows that teachers in Latvia, irrespective of the language of instruction, stage of education and school subject, report symptoms of burnout syndrome. Right-wing authoritarianism was found to be above the median, especially in Russian-speaking schools and primary schools, alongside below median levels of social dominance orientation; a positive belief found among teachers from Latvian-speaking schools. In 2006 and 2008 new standards in basic and secondary education were introduced in Latvia. These reforms, as well as the ESF project "Elaboration of the Content of Learning and Teacher Further Education in the Subjects of Natural Sciences, Mathematics and Technologies" (2008-2013), changed the philosophy of the Latvian education towards constructivism. Yet, despite these new standards the mathematics performance of Latvian students in international studies has not improved and is below the OECD average (Shapkova, 2012).

In Finland, the fall of the Soviet Union was one reason for a serious economic crisis. Although this influenced also the educational system, the national policy emphasised mathematics and sciences. A national LUMA-project (1996-2002) was set up to enhance the learning of mathematics and sciences (Ahtee, Lavonen, Parviainen, & Pehkonen, 2007). The national ethos of the time was inspired by the rise of Nokia, generating a vision of Finland as a high-tech economy. As a surprise for Finns, Finland achieved the highest score in PISA 2000 and continued to do well in the following measures. However, Finland was also characterised by less favourable affect (OECD-PISA, 2004).

Research questions

Based on the review of the literature, we consider cultural factors to be relevant for teachers' beliefs. In addition to country (Finland, Estonia and Latvia), we expect the language of instruction (national and Russian) to be relevant in Estonia and Latvia. Moreover, we acknowledge the importance of local school microcultures. Whereas nationality and language are considered to be two independent variables, we see the local micro-culture of the school to be interrelated with teacher's beliefs. Firstly, we acknowledge that the teacher has significant influence on the formation of that microculture. Secondly, our measure of school microculture is based on the self-report of the teacher, thus indicating as much the teacher's personal interpretation of that culture as the local culture per se (Figure 1).

In this paper we explore the following research questions: 1) How do the variables of country and language of instruction mediate school microculture and teacher beliefs?) 2) What kinds of relationships can be found between school microculture and teacher beliefs?

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Figure 1: The theoretical framework for the study

METHODOLOGY

Participants

The data were collected from 815 7-9th grade mathematics teachers in Estonia (N=333), Latvia (N=390), and Finland (N=92). Subsamples of teachers who teach in Russian speaking schools were collected in Estonia (N= 99) and Latvia (N = 96). Data collection was completed in late 2010 and early 2011. The age of the Estonian teachers ranged from 25 to 77 (M=47), the age of the Latvian teachers ranged from 25 to 66 years (M=46), and the age of the Finnish teachers ranged from 25 to 61 years (M=42).

Instrument

A seven-module questionnaire was devised to explore teachers' beliefs. In this paper, we shall analyse and report on three modules: (1) teachers' overall job satisfaction; (2) their general beliefs on teaching and learning; and (3) their beliefs on mathematics teaching and learning. Teachers responded to each item using a 5-point Likert-scale. The first part of the instrument was designed as an indicator of teachers' overall job satisfaction. As many of the items relate to administrative support and teacher collaboration, we consider it a relevant indicator for school microculture. The dimension comprised items for a factor "collaboration and recognition", for example "In our school, staff members are recognized for a job well done". Teachers' general beliefs on teaching and learning were measured using items typical for constructivist (or non-constructivist) philosophies of teaching, for example, "Teachers should direct students in a way that allows them to make their own discoveries" or "Effective/good teachers demonstrate the correct way to solve a problem".

The module measuring teachers' beliefs on mathematics teaching and learning was constructed using 26 Likert statements from Pehkonen and Lepmann (1994). Sample items for the different dimensions are: "Pupils should have an opportunity to independently develop their mathematical understanding and knowledge" (Process), "In a mathematics lesson, there should be more emphasis on the practicing phase than on the introductory and explanatory phase" (Toolbox); "Working with exact proof forms is an essential objective of mathematics teaching" (Proofs). The theoretical background, development and structure of the questionnaire are described more thoroughly in a previous paper (Lepik & Pipere, 2011).

Analysis

In order to reduce data into fewer, but more reliable variables, the different modules of the questionnaire were subjected to principal component analysis with varimax rotation. We removed several variables due to low communality or multiple loadings. In order to determine the number of factors we used different methods, such as scree-test, “eigenvalue greater than 1” rule, and parallel analysis. Several solutions with different numbers of factors were tested. The criteria to select the factors were reliable and allowed easy interpretation of the factors.

Based on the factor analyses, we computed the following sum variables. School micro-culture: Collaboration and recognition (α=.696; 5 items); General teaching beliefs: Constructivist approach (α=.730; 12 items), Traditional approach (α=.577; 4 items); Mathematics teaching beliefs: Process (α=.732; 9 items), Toolbox (α=.677; 6 items), Proofs (α=.592; 4 items).

We examined the possible differences of three countries (Finland, Estonia and Latvia) using Kruskal-Wallis test, and language groups in Latvia and Estonia (national language and Russian) using Mann-Whitney tests for each pairwise comparison. In order to analyse the relationships between teacher beliefs and school microculture, we calculated the Pearson correlations for each subsample.

RESULTS

Cultural differences by countries (Table 1), determined by Kruskal-Wallis tests, were statistically significant for all examined sum variables ($p<.001$). The results show significantly higher scores for teachers from Estonia in all sum variables compared with the teachers from Finland ($p<.001$). With respect to differences between the teachers from Latvia and Finland, with the exception of the Toolbox aspect, teachers from Latvia reported significantly higher scores for all other sum variables (Traditional beliefs $p<0.05$; other variables $p<.001$). The most disparate comparison of scores turned out between Estonia and Latvia, where teachers from Latvia reported stronger emphasis on Collaboration and recognition ($p<.05$), and Constructivist beliefs ($p<.05$), while teachers in Estonia scored significantly higher in Traditional beliefs ($p<.01$), Toolbox aspect ($p<.001$) and Proof aspect ($p<.01$). The Process aspect of mathematics teaching did not show any significant differences between teachers from Estonia and Latvia.

Both in Estonia and Latvia those teaching in Russian reported higher Collaboration and recognition ($p<.005$ in Estonia, $p<.05$ in Latvia) and Proof aspect ($p<.001$ in Estonia and Latvia). Estonian teachers from Russian speaking schools reported higher agreement with Constructivist beliefs ($p<.05$) and Process aspect in mathematics teaching ($p<.01$) than teachers from Estonian speaking schools. Comparing the four language groups from Estonia and Latvia with Finish speaking teachers from Finland, it appears that for all sum variables teachers from Finland scored significantly lower (range of p from $p<.001$ to $p<.05$), except for the non-significant difference between Latvian speaking teachers from Latvia and Finnish teachers for Toolbox aspect.

Group		Collaboration & recognition	Constructivism	Traditionalism	Process aspect	Toolbox aspect	Proof aspect
By country	EST	3.81	4.13	3.39	4.18	3.49	3.43
	LAT	3.92	4.19	3.22	4.17	3.27	3.29
	FIN	3.60	3.96	3.02	3.92	3.16	2.92
By language	ER	3.97	4.20	3.37	4.31	3.51	3.79
	EE	3.75	4.10	3.39	4.13	3.48	3.30
	LR	4.05	4.18	3.26	4.24	3.35	3.53
	LL	3.88	4.20	3.21	4.14	3.25	3.21

Table 1: Means for examined sum variables according to variables of country and language. EST = Estonia, LAT = Latvia, FIN = Finland, LR = Latvia-Russian; LL = Latvia-Latvian; ER = Estonia-Russian; EE = Estonia-Estonian.

The largest similarity was found between Russian speaking teachers from Latvia and Estonia, where Russian speaking teachers from Estonia reported higher scores only for Toolbox ($p<.05$) and Proof aspects ($p<.01$). No difference was found between Latvian speaking teachers and Russian speaking teachers from Estonia for Collaboration and recognition, Constructivist or Traditional beliefs, while the latter was in larger agreement with Process ($p<.01$), Toolbox ($p<.01$) and Proof aspect ($p<.001$) of mathematics teaching.

Significant differences were observed between Latvian and Estonian speaking teachers: Latvian speaking teachers were higher in Collaboration ($p<.05$) and Constructivist beliefs ($p<.01$) while Estonian speaking teachers were higher in Traditional ($p<.01$) beliefs and Toolbox aspect ($p<.001$). Russian speaking Latvian teachers and Estonian speaking teachers showed quite disparate results: the latter scored lower for Collaboration ($p<.001$), Process ($p<.05$) and Proof aspect ($p<.05$), while their scores for the Toolbox aspect ($p<.05$) were higher.

Pearson correlations confirmed some expected results, but also revealed some unexpected results (Table 2). We found a strong correlation between the constructivist view of teaching and process view of mathematics teaching in all groups. A similar strong correlation was found between traditional view of teaching and toolbox view of mathematics teaching.

	Constructivism	Traditionalism	Process Aspects	Toolbox Aspects	Proof Aspects
Collab. LR	.353**	.053	.319**	.038	-.004
Collab. LL	.189**	-.014	.120*	.056	-.048
Collab. ER	.128	-.071	-.056	.048	-.005
Collab. EE	.275**	.037	.207**	.015	.065
Collab. FF	-.027	-.072	.164	.148	-.038
Constr. LR		-.163	.759**	-.006	.286**
Constr. LL		-.130*	.651**	-.005	.239**
Constr. ER		-.052	.522**	.160	.196
Constr. EE		.032	.586**	.116	.160*
Constr. FF		-.177	.695**	.075	.055
Trad. LR			-.088	.523**	-.023
Trad. LL			-.100	.562**	.127*
Trad. ER			.132	.532**	.301**
Trad. EE			.075	.498**	.207**
Trad. FF			-.152	.360**	.224*
Process LR				.007	.223*
Process LL				.124	.227**
Process ER				.197	.246*
Process EE				.136*	.229**
Process FF				.082	.041
Toolbox LR					.318**
Toolbox LL					.252**
Toolbox ER					.351**
Toolbox EE					.240**
Toolbox FF					.061

Table 2: Pearson correlations between collaboration, general teaching beliefs and mathematics teaching beliefs. LR = Latvia-Russian; LL = Latvia-Latvian; ER = Estonia-Russian; EE = Estonia-Estonian; FF = Finland-Finnish.

Moreover, there was interesting variation regarding how collaboration was related to teaching beliefs in different groups. For example, only for Latvian groups and Estonian speaking Estonian teachers' collaboration had a positive correlation with constructivist teaching beliefs and process view of mathematics teaching.

Additionally, only in Finland, the emphasis on proofs had no positive correlation with the process aspect and toolbox aspect of mathematics.

DISCUSSION AND CONCLUSIONS

Teachers' beliefs reflect the way that teaching and learning is conceptualized in different countries. Cross-cultural differences in teachers' beliefs provide important information regarding teachers' inclination to different teaching approaches. The TIMSS and PISA studies have shown that the mathematical attainment of Finnish, Latvian and Estonian pupils are different. Therefore, it would be relevant to assume also that teachers' beliefs would somehow differ in these countries. The country comparison indicates that Latvian teachers emphasize the constructivist teaching beliefs most, while Estonians were the strongest supporters for the traditional beliefs. On the overall level, Finland agreed the least with both of these approaches. We found that all countries and language groups had a similar pattern of higher emphasis for constructivism over traditional general teaching beliefs and for Process approach over Toolbox and Proof approaches for mathematics teaching. We also identified differences within Estonia and Latvia according to the language of teaching, Russian speaking teachers putting more emphasis on proofs.

The results indicate that mathematics teachers' overall teaching beliefs are related to their view of mathematics teaching. Those who believe more strongly in constructivist ideas tend to emphasize the process aspect of mathematics more. On the other hand, those who hold a more traditionalist view of teaching tend to emphasize the toolbox aspect of mathematics in their teaching. Yet, it is important to notice that there was no negative correlation either between constructivism and toolbox-approach or between traditionalism and process-approach. The school microculture – as reflected in teacher perception of collaboration and recognition – seems to have a clear relation with constructivist practices in both Latvian subsamples and in Estonian speaking sample from Estonia but not in the other groups. Such findings are bewildering and ask for further analysis of the data.

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ANALYZING MATHEMATICS CURRICULUM MATERIALS IN SWEDEN AND FINLAND: DEVELOPING AN ANALYTICAL TOOL

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This paper aims to contribute to two interrelated areas. Firstly, it adds new insights into the variation of curriculum materials within and between two countries, Sweden and Finland, with respect to their potential to contribute to various kinds of teacher learning. Secondly, it aims to build on and develop an analytical tool for analyzing curriculum materials. To accomplish these aims we explored two teacher's guides from each country, applying a tool derived from the context of science education. The analysis reveals substantial differences between all four texts concerning the categories in the analytical tool. We suggest how the analytical tool could be developed to more deeply explore its potential for supporting qualitatively varied teacher learning.

Key words: analytical tool, comparative study, curriculum materials, mathematics teacher's guides, teacher learning

INTRODUCTION

Curriculum materials such as commercially produced textbooks and teacher's guides have a strong presence in mathematics education in large parts of the world. These materials are typically a major resource for teachers' planning and practice (Stein et al., 2007; Jablonka & Johansson 2010). One of the tasks of curriculum materials is to bring different discourses together (Jaworski, 2009). On the one hand, there is an academic conceptualization from which the intended curriculum derives and, on the other hand, the socio-cultural settings where teaching and learning occur: the enacted curriculum. Writers of curriculum materials may therefore interpret the intended curriculum and adjust to the socio-cultural settings to function as a bridge between the two different discourses. From this perspective, curricular materials serve as an important tool for teachers in both enabling and constraining their thoughts and actions (Stein et al., 2007). Further, curriculum materials are not only important resources for teachers in designing teaching (Stylianides, 2007), but also for teacher learning (Doerr & Chandler-Olcott, 2009). For instance, Remillard (2000) and Davis and Krajcik (2005) emphasize that curriculum materials could productively contribute to teachers' professional development if they encompass an elaborated attention to the process of enacting the curriculum. Therefore, potentially, well-designed curriculum materials could create opportunities for teacher learning.

There exists no role model for how to design such materials, since teachers' use of, and learning from, curriculum materials are related to their experience, knowledge and the particular classroom situation. This study aims to contribute to the knowledge about teacher's guides and their potential for various kinds of teacher learning in two neighbouring countries with quite similar school systems but different teaching styles: Sweden and Finland. One rationale for undertaking a comparative approach is that through a process of investigating similarities and differences in various countries' curricular materials we reveal some taken-for-granted and hidden aspects (cf., e.g., Andrews, 2010) of teachers' work in classrooms. Such findings, we believe, could contribute to the international research discourse on aspects of curriculum materials and their influence on teaching and teacher learning.

Teachers in Finland use the textbook and teacher's guides extensively (Joutsenlahti & Vainionpää, 2010). There are indications that many Finnish teachers are satisfied with the way their teacher's guides are built, and that they consider them to be very helpful in differentiating their teaching (Heinonen, 2005). Further, Finnish teachers state that the guides provide help and ideas for new ways to teach and simultaneously ensure that the children learn what they are supposed to learn according to the state curriculum (L. Pehkonen, 2004). Swedish teachers also use the textbook to a very large extent, but seldom use teacher's guides (Jablonka & Johansson, 2010). Teachers in Finnish classrooms often lead whole-class instructions whereby all pupils are engaged in the same mathematical area (e.g., E. Pehkonen et al., 2007). In Sweden, on the contrary, it is common to conduct teaching as "speed individualization" and as personalized teaching (e.g., Jablonka & Johansson, 2010) where pupils work, in the same classroom, with different mathematical areas. Due to these differences, it is interesting to compare the curriculum materials used in the two countries. In this paper, we aim to answer to the following questions: *What similarities and differences in curriculum materials exist within and between two countries, Sweden and Finland, with respect to their potential to contribute to various kinds of teacher learning? How is it possible to develop and amend an analytical tool for examining the potential for teacher learning in curriculum materials?*

TEACHER LEARNING AND ARTEFACTS

There are numerous ways of conceptualizing teacher knowledge and teacher learning (e.g., Rowland & Ruthven, 2011). However, instead of digging into these theories and frameworks, we regard teacher learning as a process of social participation in communities of practice according to Wenger (1998), and we understand the artefact of the curriculum material as a resource used in teachers' professional practice. In line with Brown (2009) we emphasize that teachers and curriculum materials participate together in a collaborative relationship, whereby teachers are viewed as active agents in developing and constructing the planned and enacted curriculum. Both teachers and curriculum materials have a role in mediating the relationship, which is shaped by historical, social and cultural factors. This implies that the results of this study could be seen as one piece in the building of an understanding of how curriculum materials, teacher learning, teacher education, culture, etc. are related to, and constitute, each other. Brown (2009) also reflects upon different resources and how they can lead to different opportunities for teachers' and students' learning. We want to stress that a teacher's guide can only hold the potential for teacher learning in practice, and that teachers in the schools form a heterogeneous group with respect to their developing identities, competences and professionalism.

METODHOLOGY

Context

This paper presents one of several studies connected to two larger research programs. The first program is a comparative project examining similarities and differences between mathematics education in Sweden and Finland (e.g., Ryve, Hemmi & Börjesson, 2011). The second concerns a design research project carried out within and together with one municipality in Sweden, within which Finnish curricular materials (translated into Swedish) are tested by some teachers. Both Sweden and Finland have a nine-year comprehensive school that begins at the age of seven, is free of charge, and involves no tracking. Teachers in both countries are free to choose what curricular materials they want to use; since the beginning of the 1980s in Finland and 1991 in Sweden, there is no state control over curricular materials (E. Pehkonen et al., 2007; Jablonka & Johansson, 2010).

The analytical tool

Wenger's (1998) theory of learning is very general; hence we need to ascribe meaning to the artefacts used in the teachers' practice (cf. Hemmi, 2010), in our case teacher's guides in school mathematics teaching practice. For this purpose we apply the analytic tool of Davis & Krajcik (2005), which focuses on opportunities for teacher learning within the practice of science teaching, as a starting point for approaching our data. With regard to conducted research in mathematics education, we modified it to fit our purpose (Table 1).

Categories	Categories for data analysis
1a) General knowledge of students' ideas and strategies	Describes why students might hold particular ideas about mathematical concepts and exemplifies common strategies among students.
1b) Suggestions for how to encounter students' ideas and strategies	Gives suggestions for how to deal with/encounter various ideas and strategies of students and how to enhance their learning and prevent future difficulties.
2) Concepts and facts	Describes concepts and facts within mathematics such as history, field of application, derivations, methods, proofs, correct terminology.
3) Progression and connections	Shows the mathematics progression throughout the school years as well as connections between mathematical topics; for example, explains the future development of methods and concepts.
4) Connecting theory	Supports the teacher's actions in practice beyond the

and practice	curricular materials by connecting theory and practice. Exposes the central ideas in national curriculum and research results for promoting teachers' autonomy.
5) Design of teaching	Supports the teacher's ability to act in practice by suggestions with respect to the design and enactment of lessons, tasks, formative assessment, individualization of teaching, homework, etc.

Table 1: Five categories for data analysis

Data and data analysis

Our selection of teacher's guides was based on two criteria: that they represent curricular materials commonly used in the respective countries and that they represent both older and newer curricular materials. In this paper we focus on four different teacher's guides for first-grade mathematics. As publishers do not give access to sales figures, we based our choice on commonly used curricular materials based on our own experiences as teachers and researchers.

The four teacher's guides we investigate are:

- *FIN 1: Laskutaito* (1999), a Finnish teacher's guide still used in Finnish schools,
- *FIN 2: Min Matematik* (2004), a Swedish translation of a Finnish teacher's guide, *Tuhattaituri*, for use in the Swedish-speaking part of Finland,
- *SWE 1: Matte Direkt Safari* (2011), a Swedish teacher's guide that has been on the market for several years, and
- *SWE 2: Matte Eldorado* (2011), a Swedish teacher's guide that has been on the market only a few years.

Four researchers conducted the analyses together. Two of the researchers are of Finnish ancestry, and examined the teacher's guide that was not translated into Swedish. We first discussed the categories in Table 1 in relation to the empirical data. Then, each member of the research group took special responsibility for one teacher's guide and investigated the extent and qualities of the topics connected to each category. This process was followed by a collective analysis of our findings in relation to the data as a way of checking each other's analysis.

RESULTS

We found substantial variation concerning the presence of the topics connected to the first four categories. Either they occurred regularly in connection to most mathematical areas (++), sporadically with only some sentences (+), or were totally missing (-); see Table 2.

	FIN 1	FIN 2	SWE 1	SWE 2
1a) General knowledge of students' ideas and strategies	++	-	-	+
1b) Suggestions for how to encounter students' ideas and strategies	++	-	-	+
2) Concepts and facts	+	++	-	+
3) Progress and mathematical connections	+	+	-	+
4) Connecting theory and practice	++	-	-	+

Table 2: Occurrence of the topics connected to categories 1-4. ++ occurred regularly, + occurred sporadically, - were absent.

In *Laskutaito* (FIN 1), topics connected to all four categories were identified. This suggests that it supports teachers in acting beyond the curricular materials by connecting theory and research with practice on two pages at the beginning of every chapter (Category 4). These pages present the overall goals as well as how children may think, and what activities a teacher can do to prevent misunderstandings and promote learning (Category 1). For example, it could be a description of prerequisites for learning in geometry according to research, and suggestions for how teachers could work with the ability of spatial perception.

> "**7. Visual memory.** Visual memory can be trained, for example, using the traditional KIM games, whereby 10-20 objects are placed on the table and the children get to see them for a short period of time. When the children then close their eyes, the game leader removes an object and mixes the order of the remaining objects. The children then try to remember which item is missing..." (Laskutaito, 1999, p. 124-125).

Min Matematik (FIN 2) is the only teacher's guide that describes concepts and facts regularly (Category 2). The facts deal with, for example, definitions and correct terminology but also the historical background of, for instance, our number system.

> "**Facts**. 3+1, for example, is an expression. An expression can also consist of a single number or a symbol (for example, a). 4=4 is an equality. An equality that contains one or several unknowns is called an equation." (Min matematik, 2004, p. 27).

Sometimes, in connection to the mathematical facts, there is information about progression (Category 3). For example, when presenting the correct terms involving subtraction it also states that *"in this textbook the concepts of addition and subtraction are used with the pupils in this phase but the terms sum and difference are taught later"*. Hence, under these sub-headings the guide combines facts with progression, but progression appears more sporadically. Topics belonging to the categories 1a, 1b and 4 are lacking. For example, no references to research are made.

In *Safari* (SWE 1), no topics were found that could be connected to these four categories.

Matte Eldorado (SWE 2) deals with all the categories, but not regularly. There is some information at the beginning of the guide about number sense and arithmetic, with descriptions of possible difficulties and ineffective strategies children can use and how to prevent and encounter them (Category 1a & 1b), but this does not appear regularly in connection to new areas. There is a matrix at the beginning of the guide displaying how the textbook and the goals for the students are connected to the goals in the curricula. Every goal is also described in the text (Category 4). Yet, no connections to research are made. The matrix also gives the teacher an overview of the progression from school year 1 to 3 (Category 3). In the ordinary text, topics connected to this category (progression and mathematical connections) occur sporadically. Topics dealing with mathematical knowledge (Category 2) occur only a few times.

All but *Min matematik* (FIN 2) include a general presentation at the beginning of the guide with an explanation of why one should engage pupils with the activities presented in the guide. In *Safari* (SWE 1) this part is very short, and in *Matte Eldorado* (SWE 2) it is extensive. This could possibly be added as an additional category in the analysis tool, as we can see differences in the guides concerning the underlying assumptions about teaching and learning, something to focus on in further studies.

Design of teaching

As to Category 5, dealing with topics more directly connected to the design of the lessons, we found several aspects important to dig more deeply into. It could involve, for example, support for how teachers could differentiate their teaching; what materials they could use to concretize learning, engagement in problem-solving and playing games; and how they could assess students' knowledge of mathematics. We found considerable differences between the guides concerning the structure of these aspects; something that, we hypothesize, may influence teacher learning in practice.

Laskutaito (FIN1) is structured around learning outcomes for pupils. The guide also focuses on lesson plans, but leaves a great deal of room for the teacher to design activities suitable to various students in the classroom as there is no suggestion for a particular lesson plan. The guide is based on the pages of the textbook. Sub-headings *Ideas for how to teach/deal with the current object, Mental arithmetic, Practice and games/plays, including challenges for quick and "talented" pupils,* and *Problem-solving* reappear in the section for each lesson. The guide aims to hold the pupils together but offers them partly different activities, and also suggests a small number of tasks connected to every lesson as homework for the pupils. Concerning assessment, there is a short test after every chapter. Ideas for problem-solving are offered in every lesson.

Min Matematik (FIN2) is structured around learning outcomes for students. The guide focuses on lesson plans, and each session is described on four pages. All the sub-headings (*Discussion about a picture, Mental arithmetic, Suggestion for a lesson* *plan, On the board, A story, Problem-solving, Tips (for example games), Extra, Facts and The following lesson)* reappear in the same order and in the same place on these four pages. The guide suggests a lesson plan with various activities in which the class is held together and where differentiation is organized by extra tasks and problems within the same area. It suggests a small number of tasks connected to every lesson as homework for the students. Concerning assessment, there is a short test after every chapter. Ideas for problem-solving are offered in every lesson.

Matte Direkt Safari (SWE1) is structured around the student textbook, and is not structured according to certain time periods, for example a lesson. Each page in the guide includes information about what *"students learn from the pages"*, sometimes how they should work with the pages in the textbook or with some extra pages from the teacher's guide (for example, what hands-on material they need). On a few occasions there are suggestions for other activities. At the end of each chapter there is a page with ideas for collective activities, like games and outdoor activities. No ideas are offered for differentiation of teaching until the students have finished all the pages in a chapter and received a diagnosis according to which of two paths they can choose. There are suggestions for homework three times in every chapter. No instructions or ideas for problem-solving can be found in the guide.

Matte Eldorado (SWE2) is structured around learning outcomes for students. The guide is not structured in relation to a particular time period, for example a lesson, but is based on the pages of the textbook. The sub-headings *Aim* (with some suggestions for activities supporting learning), *Simplify, Challenge, Observe, Material* and *Go on working* reappear in the same order, but not in all units. Under the sub-headings *Simplify* and *Challenge* are ideas for how to differentiate the teaching regularly. Ideas for collective activities, for example games, are presented at the beginning of the book. Considerations regarding assessment take a dominant place at the beginning of the guide, where different ways of observing pupils' learning are presented. There is one pre-test at the beginning of the school year and one test after each semester. The sub-heading *Observe* aims to help the teacher observe student behaviour during the lessons. There are suggestions for homework on some occasions. General aspects of problem-solving are dealt with at the beginning of the book, while more specific aspects of problem-solving occur at the end of each chapter. The character of the problems seems to be quite different from that in the Finnish guides. For example, they are always placed in an everyday context.

CONLUSION AND DISCUSSION

The analyses revealed significant differences between the four teacher's guides both within and between the countries. Two of the guides (FIN1, SWE2) deal with topics connected to all the five categories, and can hence be regarded as resources for potential teacher learning in practice concerning aspects of encountering pupils' ideas in a productive manner, confronting the teacher with mathematical ideas and concepts connected to the mathematical topics in the classroom, and making visible the demands of the practice concerning the curricular goals. Only one of the guides

(FIN1) offers resources for teachers' access to the practice of mathematics education research. The older Swedish material distinguishes itself from the other three in that topics connected to these categories were not dealt with.

Concerning Category 5 (Design of teaching), we found the following similarities between the Finnish materials that distinguish them from the Swedish ones: Both focus on lesson plans and offer ideas for teaching, mental arithmetic, differentiating, problem-solving, games and homework in connection to *every lesson*. This is not the case with the Swedish materials, which leave more space for the teacher to decide the units they will use in their teaching. This difference could be connected to differences in teachers' work in practice, which in Finland (e.g., E. Pehkonen et al., 2007) often means leading whole-class instructions whereby all pupils are engaged in the same mathematical area or mathematical problem (Franke, Kazemi & Battey, 2007) whereas in Sweden it involves communicating with individual students and pupils working in the same classroom but often with different mathematical areas (e.g., Jablonka & Johansson, 2010). Our analysis shows that three of the four guides emphasize aims and goals in the curricular program, which is interesting in relation to our study of the role of the aims and goals of lessons in school-based teacher education in Sweden and Finland (Bergwall et al., 2012). The older Swedish teacher's guide, *Matte Direkt Safari* (SWE1), represents material that is extremely dependent on the pupils' textbook and, hence, may have some special impact on teacher autonomy. The Finnish guides offer superfluous ideas for various kinds of activities for each lesson, supporting designs of different kinds of resources for students' learning. On the other hand, there is not much room for teachers to work thematically or spontaneously starting from students' ideas if they try to strictly follow the chains in the teacher's guides in their work.

In regard to the first four categories, the tool worked well in its current design. Based on our data analysis, we could add a category containing topics concerning how the textbook authors motivate their choices concerning the progress, structure and different activities. Teacher learning is situated within the practice of studying textbooks and teachers' guides, discussing the materials with other teachers, planning and evaluating the teaching as well as working with the students in the classroom. The professional skill of designing lessons for unpacking mathematical ideas is a kind of participation in practices. The fifth category appeared to embrace a broader qualitative dimension of potential for teacher learning that we should focus on more in further studies. Teachers have various experiences and identities (cf. Wenger, 1998; Brown 2009) and they have also been active agents in developing their own classroom practices. From this perspective, we are eager to extend this study by data analysis of both collaborative discussions between teachers in planning lessons using the teacher's guides as well as classroom teaching in teachers' daily practice.

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BOREDOM IN MATHEMATICS CLASSROOMS FROM GERMANY, HONG KONG AND THE UNITED STATES

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Based on interviews with students from altogether six classrooms in Germany, Hong Kong and the United States, the paper explores students' elaborations of the notion of boredom in relation to their classroom context.

INTRODUCTION

When watching videos from altogether 60 lessons in the countries/regions mentioned in the title, I observed in all classrooms at times some students sprawling on their desks or leaning back with their eyes half-closed, seemingly unlinked to the place and time of their experience, others heavily sighing, drumming their fingers on the desk, or playing around with their pens. Are these cross-cultural expressions of boredom?

BACKGROUND

There seems to be a range of more or less existential experiences in different cultures, summarised in English by the notion of boredom. In the languages relevant for this investigation, these have different connotations. The German *Langeweile* appears first in a dictionary from 1796 (Adelung, 1796), and is described as experiencing with listlessness a long (*lange*) "empty" while (*Weile*) when one is not busy with anything. The Online Etymology Dictionary ("Bore", 2012) finds the use of "bore" in the sense of "be tiresome or dull" in 1768. The word is traced back to variants of the German *bohren* (to drill), the meaning of "bored" being possibly a figurative extension of "to move forward slowly and persistently", as a boring tool does. According to Vadanovich et al. (2011), the Chinese phrases for boredom refer to "having nothing to do" and to "not being of interest".

Taken up by individual psychology, these experiences became classified as an emotion or mood, including elements of discomfort or even resentfulness. Paradoxically, various constructs describe the emotion of boredom as a lack of other emotions, as if emotions would fade away towards zero and then continue on the same dimension with negative values, e.g. as *lack of* interest and satisfaction (e.g. Smith, 1981; Bourgeois, 2001). Further, boredom has been essentialised as a trait called "boredom proneness" or "boredom susceptibility", for which scaled tests have been developed that are used for measuring students' or workers' dispositions for getting bored. Country differences in such measures have been found and elaborated by common sense cultural explanations, for example by Vodanovich et al. (2011). Autochthonous and philosophical terms for experiences summarised in English by the notion of boredom, here intermingle with moral or essentialist interpretations of behaviour. In a psychological study, focussing on an operational definition and its technicality, Sparfeldt et al. (2009) developed a mathematics-related "boredom-scale" and related their results to other mental states (intelligence, self-concept and interest) as well as to achievement, for all of which they found negative correlations with their scale amongst German students in grade four.

"Boredom", or whatever has been translated into this notion, is a category also frequently used by students and researchers in mathematics education internationally. For example, Brown et al. (2008) found that lack of enjoyment and perception of the subject as boring is, amongst higher attaining students in the UK, a common reason for not choosing post-compulsory mathematics at age 16. Amongst Norwegian first year upper secondary students (grade 9), Kislenko (2009) found half of the students claiming that mathematics is boring, but also students stating that mathematics is interesting and boring at the same time. Measures of experiences of students' boredom have also been used in TIMSS. The students' questionnaire includes a question, where they can choose one out of four valences of agreement to the statement "Math(s) [in 1995]/ mathematics [in 1999 and 2007] is boring" (TIMSS context questionnaires, 1995, 1999, 2007). As to cultural differences, Li (2002) derives a model of the ideal Chinese student (a person with *Hao-Xue-Xin*, "heart and mind for wanting to learn") from interview data with high achieving college students, where the strategy to force themselves to persist when faced with boredom is an important feature of the "behavioral ideal". Li (p. 260) explains:

> Like any learners from any cultures, Chinese students too encounter the problem of boredom. Consistent with research findings on achievement motivation in the West … it is also no easy obstacle for Chinese students … a stronger measure is called for to counter the severe demotivating effect of uninteresting content: Force oneself to persist.

By basing this model on empirical data and a Chinese folk term, Li's construction shows how the projected image of a culture itself is a production of that culture.

Instead, the study reported here takes mathematics classrooms as "small cultures" (Holliday, 1999). This is to overcome the assumption of a hierarchical relation between a country's (or a group of countries') culture, and a school classroom's (or any other institutions') culture. Holliday sees the "onion-skin-relationship" model that sees small (sub)cultures as subordinate to and contained within large cultures, as a form of cultural essentialism, as it amounts to explaining differences in smaller units as stemming from the larger culture surrounding the inner layers. Activities in a mathematics classroom could as well amount to the production of cultures that cross the boarders of classrooms and countries. Consequently, it is of interest whether there is cohesion in relation to attributes of classroom practice. Students' motivations for engaging with mathematics for the same six classrooms as in this paper have been reported elsewhere (Jablonka, 2005). Here the focus is on the range of experiences linked to "boredom": *Is there any cohesion visible in the students' engagement and related feelings, expressed as "boredom", that is related to classroom micro-culture?*

Data and method

The investigation draws on student interviews from the Learner's Perspective Study (LPS) (e.g. Clarke et al., 2006), which were conducted in the form of video-stimulated recall interviews. The table below summarises some features of the selected classrooms. Some of the interviews were conducted with groups; hence the discrepancy between the number of interviews and interviewees. The selection of the two (out of three) classrooms videotaped in the LPS is based on maximising differences between classrooms from the same country in terms of student achievement. The rationale for selection of the classrooms in the LPS was based on selecting experienced and competent teachers according to local criteria.

Class	Achievement	Cultural composition	Students interviewed	Number of interviews	Class Size
G1	Average-high	Homogen.	22	11	27
G3	Low-average	Heterogen.	10	8	12
HK1	High- average	Homogen.	19	19	35
HK3	High	Homogen.	18	18	39
US1	Low	Heterogen.	20	20	29
US2	High	Heterogen.	20	20	33
			109	**96**	

Table 1: Features of the classes and number of students interviewed

Although the interviews had as their main goal the elicitation of students' interpretations of classroom events, a number of issues emerged that centred around students' feelings. In addition, all interviews contained an explicit question about whether they liked mathematics (lessons), but the notion of "boredom" was not used by the researchers. In the interview transcripts, all episodes where students talked about feelings and use the notion of "boredom" were identified. Names used in this paper are pseudonyms.

OUTCOMES

In total, 30 students explicitly used the notion of boredom, both, "positively", in saying that mathematics lessons were not boring, or for describing experiences of boredom in the lessons video-taped in the study, or in mathematics lessons in general. The summaries and quotes below display a wide range of experiences.

Germany, G1:

Five students from this class talk about boredom in the interviews.

Gabriel compares mathematics lessons with history lessons and states that mathematics is less boring, which he links to the subject and the teacher. Similarly, Stefan states that there are worse subjects, which are more boring.

Martin in general likes mathematics and links boredom to review, and to a lack of challenge in general; he names some high achieving classmates who are also likely to experience boredom:

Martin: Well for Philip here ...it's also boring sometimes...because he's generally very good in school ...in all the subjects ...yeah and for Lisa ...

Albert likes mathematics, but says that it sometimes becomes boring and then he does not pay attention. He explains:

Albert: Only when the questions you've already done a month earlier and are still being repeated all the time; that's really boring then, when you can already do them perfectly ... and - and it's fun, when there's something new.

Fred is generally not fond of mathematics, stating he likes other school subjects if he gets good marks; he names specific activities that he finds boring:

Fred: Sometimes it is a bit boring a bit ... as a matter of fact, so, with all that numbers, if you then solve only algebraic terms for the whole lesson, well then you are not very keen on it anymore for the next lesson.

As less boring he mentions geometry and contextualised problems.

Germany, G3:

From this class, four students use the notion of boredom in the interviews.

Jeannine says she generally likes mathematics and links her experience of boredom to the specific topic of percentages. She expands:

Jeannine: Erm, yes then there are simply some people for whom the lessons are too boring. Well, I find the lessons quite boring myself sometimes. Well, if you did not at least have something to gabble about, so if I would not gabble now, I believe, I would always fall asleep.

Later in the interview, she also says that gabbling leads to not listening, which in turn leads to not understanding, which leads to boredom. She also says that experiencing boredom is partly related to the teacher.

Mona says she actually likes mathematics when she feels she understands. She gets bored if she feels she does not grasp a method:

Mona: Don't know, somehow I didn't get anything at all, whatever I said was wrong, and then I got bored. Okay, if I understand something then I actually like it. […] I was bored because I didn't get it, and then I automatically switched off.

Later she explains why school in general can be fun:

Mona: Sometimes it is fun, well if there wasn't any school we all wouldn't have met, after all, this is how you should look at it.

Jasara, when asked whether she likes mathematics, says laughing, "it's a bit tricky". She talks about being bored during the lesson and explains how her lack of engagement was linked to an upcoming test in another subject:

Jasara: I was only bored. […] Then I preferred to write and doodle. Actually, I, we had an English test after the lesson well now. Sat an English test and I mean, if Math had been added I think I would have muffed the English test.

Peer does not really dislike mathematics; for him the subject is "mediocre, yes it is not really my favourite subject". He elaborates:

Peer: Yes, that wasn't boring because Selin sat next to me ... this was a little bit funny but now during the lessons mostly because somehow I am on my own now (...) […] And don't get anything Mrs. Md. is doing at the front on the blackboard, then it is going to be boring.

Hong Kong, HK1

Eight students in the interviews refer to what has been translated into the English terms 'boredom', 'boring' or 'bored'.

Polly, like two students from G1, says she likes mathematics and compares mathematics lessons with other subjects:

Polly: No. It is just mathematics lessons uh- no, because in other lessons such as Chinese and Chinese history I do not need to use the brain, no, only listening to Miss talking- so it is boring. […] But I can think in mathematics lessons, so it is not- no, so it is not boring.

Michael talks about the choice of the social base for the communication and the hierarchy between teacher and students, and explains:

Michael: This is because we have to sit in this kind of environment- this particular environment is very boring. Although I am interested in mathematics, I will not like this every- every day.

Int: What do you think it should be?

Michael: I don't know, more freedom. […] If you sit where you like, you will be more interested to have the lesson. […] I think that this is better because there is a gap between the teacher and us. With a gap, we cannot ask so happily- cannot ask so easily. Asking our classmates can be very natural and easy and learn- I think we can learn faster

Nina states she cannot say whether she has positive or negative feelings in relation to mathematics. When comparing the lesson they have just had with others, she "quite liked it" because it was not boring, while there are some other more boring lessons.

Osbert says whether he likes mathematics depends on the topic, indicating that he enjoys things that are "difficult" less. He likes equations, but did not like cosine. He compares mathematics lessons in general with other lessons:

Osbert: Mathematics lessons… It's not as boring as other lessons […] He teaches us something. For the other lessons, the teachers talk about…the texts. But he teaches us, math. […] He teaches us what to do and we're thinking at the same time. We merely listen in other lessons.

Similarly, Paul compares mathematics with Chinese or Chinese history lessons, and mentions humour as a reason for engagement:

Paul: It's more interesting. It's not as boring as Chinese, which requires you to memorize things. Mr. Ng. will say something funny. In Chinese History lessons, the teacher only talks about the facts and asks you to do homework, but Mr. Ng will not. He draws the graphs with us and tells us some jokes.

Even though Patrick is not very fond of the current topic and finds the "graphical method [for solving a system of linear equations] troublesome", he states:

Patrick: Why do I like having math lessons? It's not so boring…you can use your brains, and…talk a little, because he allows you to have discussions at least.

Ruth mentions a lack of variation in lesson structure, and also states that releasing the seriousness could foster engagement. Further, she talks about a "sense of achievement" as important for enjoying a subject.

Ruth: Feeling? Mm… We must listen to the teacher, or we won't understand. And…it was a bit boring. He teaches in the same way every day. I think…we would be happier if he [the teacher] is not so stern.

Asked whether she likes mathematics, she says:

Ruth: Do I like mathematics? Mathematics…I like the arts subjects better, like Chinese. I'm good at arts subjects… when I was small they gave me…a sense of achievement, which couldn't be obtained from mathematics.

Rose generally does not dislike mathematics lessons, but states that they can be boring sometimes:

Rose: Yes. Sometimes, I find it's boring to solve problems again and again.

Further, she states preferences for some topics over others. She likes "drawing graphs", but finds (algebraic) calculations "confusing".

Hong Kong, HK3

Seven students explicitly refer to what has been translated into "boredom".

Janet likes the lessons, and in comparing them with the ones they had earlier she finds them "less boring". She links this experience to the atmosphere and says she likes that it is more "crowded" and also states:

Janet: It seems to have more freedom in this lesson.

Jane also refers to the environment and says:

Jane: Okay, too quiet is not good, it will be boring if it is too quiet.

Further, she appreciates that there is some space for relaxing between phases of work:

Jane: I can daydream when he is teaching us. So I work on the questions when we have to, can't daydream while working on the questions.

Gordon gives as a reason for enjoying the mathematic lesson they just had that he "did not feel bored".

Jessica says she generally does not like mathematics very much and refers to a sense of repetitiveness:

Jessica: A bit boring, just keep on doing exercises. I don't really like it very much, nothing but just keep calculating and calculating, just calculating.

Joyce likes mathematics, as she moved from a feeling of difficulty to a feeling of being at ease with it. She compares the lessons with those she had earlier:

Joyce: The teacher teaches quite well, he is better than the last one, who just did the problems on his own [...] after finishing doing it, was teaching a couple of questions, then asking us to do the exercises, after that he jumped to the next chapter. Very boring.

June says she does not like mathematics very much as she got low scores in primary school, which made her "a bit afraid of it." Yet she "quite" enjoys the lessons, but also says, "It's boring sometimes".

James, when asked whether he likes mathematics, states it is "just average", but there are not any aspects he dislikes. He also compares mathematics with other subjects:

James: Mathematics is just fifty-fifty, but not very boring, because you can have something to do, because in IS [Integrated Science] lessons, we don't even have to make a move, just keep watching the teacher just…

United States, US1

There are two students from this class who elaborate on boredom.

Esperanza likes mathematics, but sometimes, she says, she gets bored, and then she is "day dreaming". This she links to a lack of competitive atmosphere, as it happens "when they don't have a quiz". The quizzes she appreciates because she likes to have good grades. She also compares mathematics with other subjects (science and history), which she does not like, stating that she does not like to "read".

Fred says that mathematics used to be his favourite subject, but, "It's gotten a little bit harder, over- over the years". Now he just wants to learn, while being less excited about it. Seeing himself in the video, he says:

Fred: [laughs] It was too boring. I'm sitting- I was sitting there, trying to keep myself awake, (oh my lord) I'm sitting there [pretends to sleep]/

He links this experience to the fact that the lesson was about reviewing a test. He elaborates on a strategy of reduced attentiveness:

Fred: Because I was paying attention enough to get the facts, but I wasn't exactly paying attention, if you know what I mean.

United States, US2

In the interviews with students from this class, four mention boredom.

Shannon does not explicitly dislike mathematics, but she talks about her preference for the "old" class she has been in, where she never felt "bored". She associates boredom with "having nothing to do". By this she refers to phases after having finished group work tasks earlier than other groups:

Shannon: Alright, now I think it's ... I'm thinking it's kind of this, kind of boring. And so I- I just ... have nothing to do so I'm just kinda sitting there.

Brenda talks about her preference for “hands-on things”, which she enjoys and which help her learning through making her engaged:

> Brenda: It helps me learn a lot. Like, when I actually- like with the lesson when we were measuring things, that really helped me understand because it was hands-on, rather than reading from the textbook, which is really boring.

Amy says she does enjoy mathematics, but not very much and not always. Of the things she finds boring, she does not think she dislikes them at the same time. She does like “graphing things”, but neither fractions nor equations, and adds:

> Amy: I mean I'm- it's not like I don't like it, I just think it's kind of boring. So it's not like, oh I hate math, I just real- kind of boring.

Angie never really liked mathematics because, “it's usually hard for me”. But she enjoys when she gets engaged, which for her is related to having conversations:

> Angie: At the same time while we're doing work, we can actually talk about stuff that happened in the day and stuff like that. […] Um, it just gets me more involved in it. I mean like when I'm bored and I don't even really want to be doing something, then it's like, oh okay, I'm just going to do this.

She links her general engagement to her daily mood, but also compares mathematics with other subjects, which she likes more (history, computer applications):

> Angie: Like if I'm tired and I'm not really in the mood for anything, I don't want- like any classes, but if I'm in a good mood, I like every class.

DISCUSSION

The experiences reported by these students cover a wide range of elaborations on feelings related to classroom practice. A few named specific “troublesome” or “confusing” topics they find boring, (Fred G1, Jeannine G3, Patrick HK1, Rose HK1). In contrast, Osbert (HK1) finds “difficult” topics less boring. Boredom is also elaborated as the feeling of already knowing a topic (Martin G1, Albert G1, Fred US1), but also related to “not getting it” (Jeannine G3, Mona G3, Per G3). Ruth (HK1) elaborates this by saying that she needs to experience “a sense of achievement”. Jasara (G3) explains her deliberate choice for non-attentiveness. A couple of students see their lessons as less boring than in other subjects, as they “use the brain” or “think”, in contrast to just listening (Polly HK1, Osbert HK1, James HK3). Some expand on their lack of feeling to have control over the communication, or on the possibilities for talking to each other (Michael HK1, Joyce HK3, Angie US2). Other atmospheric dimensions that are appreciated as helpful in overcoming boredom, include background noise and a “crowded” environment, as well as the releasing effect of jokes and humour (Paul HK1, Ruth HK1). While Jane (HK3) appreciates having time for “day dreaming”, Shannon (US2) finds idle time boring. A sense of monotony and repetitiveness was also mentioned as causing boredom (Ruth HK1, Rose HK1, Jessica HK3). Brenda (US2) appreciates hands-on activity for overcoming boredom, and Esperanza (US1) competitive situations. Further, tiredness was mentioned by some in their elaborations of boredom.

Altogether, “boredom” was not a very common notion for describing their experiences, much less common indeed than reported in some of the studies cited above. This is perhaps due to the fact that here the students came up with the notion themselves, instead of answering a questionnaire where “boredom” is already included. Some of the experiences from the Hong Kong students can be interpreted as linked to lesson structure and social base of the communication in the classrooms, which appeared more varied in the other cases (as can be seen from the videos). Further, the notion of “thinking” or “using the brain” (as compared to only listening and learning by heart), was only mentioned by students from Hong Kong when they compared mathematics with other subjects. Other experiences might perhaps be related to the lower achievement of the students in Germany in G3 (“not getting it”). For the students from the German and US classrooms, the teaching practice in other subjects tends to be less boring (with the exception of one US1 student). Obviously, “hands-on activities” and other specific features of classroom practice can only be mentioned when they are used.

From these observations it becomes clear that these students’ perceptions of boredom in mathematics do not stand alone, but have to be ‘calibrated’ against their general experiences at school, as they often talk about other subjects. Further, some associate boredom only with specific topics, activities or events in the lessons. This challenges the use of the notion in international quantitative studies. Given the considerable variation of what is perceived as boring (or not) within each class, it seems to be disproportionate to link the students’ experiences to a wider culture of learning habits, as for example claimed by Hess and Azuma (1991), or by Li (2002) cited above. The cultural essentialism implied in the “onion-skin-relationship” model of culture adopted in such explanations, in addition to the questionable labels used for describing the wider culture (such as “East” and “West”), renders such explanations largely unproductive. Studies in other contexts have also attempted to link students’ disengagement to specific features of mathematics classroom practice. For example, Nardi and Steward (2003), in their analysis of interviews and lesson observation data from secondary mathematics classrooms in England, identified issues of pupils’ concern similar to some of those mentioned by the students from this study.

Further, this investigation is a reminder of the importance of including affective outcomes and emotional co-productions of mathematics learning as an important part of studying mathematics learning processes at school. The elaborations of boredom given by the students indeed suggest that framing the interpretation of cognition and emotion in social contexts is a fruitful endeavour and can help to overcome a view of emotions situated in isolated minds, through employing frameworks that recognise that emotional experiences are situated, subconscious and embodied, as for example suggested by Drodge and Reid (2000) or Walshaw and Brown (2012).

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THE PROBLEM OF DETECTING GENUINE PHENOMENA AMID A SEA OF NOISY DATA

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In our paper we consider two principles that are essential for research and, in particular, for qualitative comparative work in mathematics education. First, we consider the explanation of phenomena rather than data as fundamentally important in educational research. Second, we discuss a holistic conception of human knowledge, suggested by Sellars, emphasising the role of common discourse for any theoretical scientific discourse. We relate our discussion to a number of comparative studies in mathematics education, as well as to conceptual modelling, a trans-disciplinary methodological framework that follows and reflects an interactive process in the transition from data to phenomena.

Keywords: Comparative Studies, Philosophy, Methodology, Data, Phenomena

INTRODUCTION

Comparative research in mathematics education, especially qualitative comparative research, is relatively new (see ZDM 2002 Vol. 34,6). One reason for this may be due to the origins of mathematics education in psychology. The first comparative studies were quantitative, as in the First International Mathematics Study (FIMS) (Husén, 1967), and compared results rather than processes. As it has developed, and mathematics education has embraced more sociological and anthropological approaches, more qualitative comparative research has been done, but compared to psychological methods in mathematics education it is often less accepted and understood. Indeed, as Kaiser (2002) observes in her introduction to the special ZDM issue on *Comparative Studies in Mathematics Education*: "it seems that… qualitative strategies are still not well considered" (p. 240).

In this paper we seek to provide a clearer foundation for qualitative comparative research in order to increase its acceptance by the wider mathematics education community. We focus on two principles: the emphasis on phenomena rather than data and the incorporation of stakeholder views. In respect of the first, and acknowledging that qualitative work in the social sciences is typically focused on phenomena, we argue that a more conscious attention to phenomena could increase the recognition of qualitative comparative research methods in mathematics education. The second aspect is the recognition of the perspectives of the stakeholders, e.g., students, future teachers, in-service teachers, and administrators. Qualitative research is more likely than quantitative research to take such perspectives into account, but the relationship between these perspectives and results arrived at by scientific analyses is rarely described clearly. By drawing on Sellars' idea of a "stereoscopic" view we provide a way to describe this relationship, which we feel would make qualitative comparative research methods more compelling.

Compared toqualitative comparative research methods in social science and anthropology, comparative research is less theorised in mathematics education. Anthropology and the social sciences developed a strong theoretical basis for their methodology through their longer history. The challenge is to explain the value of qualitative comparative research methods to mathematics education without having this long history to draw on. The methods can be copied, but the theoretical basis cannot, because its origins are different. Instead of recapitulating the hundred year history of comparative methods in anthropology, we suggest in this paper to go more directly from principles derived from the philosophy of science. We consider especially emphasis on phenomena rather than data, and incorporating stakeholder views into research.Throughout we draw on comparative research studies in mathematics education to provide examples of the two principles in practice. These include Knipping (2003), a qualitative small scale comparison of proof teaching in France and Germany; the Learner's Perspective Study (Clarke, Keitel & Shimizu, 2006; Clarke, Emanuelsson, Jablonka, and Mok, 2006); and the TIMSS Video Study (Stigler & Hiebert, 1999).

THE DATA-PHENOMENA DISTINCTION

The data-phenomena distinction was posed and discussed for the general case of empirical science by Woodward and Bogen in the 1980s. Since then, it has shaped debates in general theory of science. According to Woodward and Bogen, scientific theories are not about data. In particular, it is not data that are explained by mature scientific theories, but phenomena.

> Underlying the distinction between data and phenomenon is the idea that the sophisticated investigator does not proceed by attempting to explain his data, which typically will reflect the presence of a great deal of noise. Rather, the sophisticated investigator first subjects his data to a great deal of analysis and processing, or alters his experimental design or detection technique, all in an effort to separate out the phenomenon of interest from extraneous background factors. [...] Figuring out what one should even try to explain—what the phenomena are in a given domain of inquiry—and what is mere noise is, as we shall see, an important aspect of scientific investigation, especially in relatively immature areas of inquiry like the social sciences. (Woodward, 1989, p. 397)

Data are characterized as the results of idiosyncratic, local measurement processes:

> As a rough approximation, data are what registers on a measurement or recording device in a form which is accessible to the human perceptual system, and to public inspection. […] They typically are of no theoretical interest except insofar as they constitute evidence for the existence of phenomena (Woodward, 1989, p. 393 f.).

In contrast, phenomena are:

> relatively stable and general features of the world which are potential objects of explanation and prediction by general theory (Woodward, 1989, p. 393 f.).

Though the examination of data serves both as a source of discovery and as evidence of the existence of phenomena, phenomena themselves are “not observable in any interesting sense of that term.” (Woodward & Bogen, 1988, p. 305 f.)

Examples for data in physics and empirical psychology are patterns of discharge in electronic particle detectors and records of reaction times and error rates in various psychological experiments.

In Knipping (2003) the dataincludesphotographs and transcripts of audio tapes made in French and German 8th grade classrooms during lessons on the proof of the Pythagorean theorem.

Examples for phenomena in physics, chemistry, empirical psychology and sociology are “gravitational radiation, Brownian motion, capacity limitations and recency effects in short term memory, and the proportionately higher rate of technical innovation among middle-sized firms in moderately concentrated industries” (Woodward, 1989, p. 393 f.).

An example of a phenomenon in Knipping (2003) is that “proofs” in the German classrooms observed follow a leitmotif that can be characterized as “anschauendes Deuten” (a visual contemplative approach), whereas the French classrooms observed follow a discursive leitmotif, where explicit verbal descriptions are essential. These phenomena only became transparent through comparison of the proving processes in the different classrooms and particularly by rigorous argumentation analyses of the proving processes (see Knipping, 2008, for a description of this process). The fine argumentation analyses of the talk in the class and the written texts on the blackboards were necessary to reveal these phenomena,which wereat first,without comparisons, not directly observable in the data.

As a practical and methodological consequence of a clear conceptual distinction between data and phenomena, a large amount of research effort has to be spent to face the resulting problem of detecting a genuine phenomenon rather than some artefact of the experimental setting [1]:

> The problem of detecting a phenomenon is the problem of detecting a signal in this sea of noise [that is: in data], of identifying a relatively stable and invariant pattern of some simplicity and generality with recurrent features—a pattern which is not just an artifact of the particular detection techniques we employ or the local environment in which we operate. (Woodward, 1989, p. 397)

For example, many attempts have been made to explain the phenomenon of the shift in performance of Finnish students on international assessments from average performance prior to 1999 to leading the European countries in recent assessments. However, this phenomenon may simply be an artifact of the methods used to compare mathematics achievement in different countries. Finland’s recent scores are on PISA surveys, while the older scores are on TIMSS. “It may be the case that Finnish educators chose to participate after 1999 in a test oriented to the kind of mathematics curriculum they had been training new teachers to implement” (Stotsky, 2012, p. 299).In other words, the phenomenon of increasing scores is an artifact of the change in testing.

Criteria for evaluating comparative studies in mathematics education are and should be diverse, related to the research questions and aims of these studies, and the theoretical perspectives and paradigms they are committed to. For example Clarke et al. (2012) suggest multi-theoretic approaches "highlighting the dangers of the circular amplification of those constructs predetermined by the choice of theory". We suggest that the philosophical principle presented here may contribute to the selection of appropriate criteria for scientific adequacy in qualitative comparative research methods. To put it in a nutshell, the data-phenomena distinction may form a basis for finding appropriate qualitative counterparts to the criteria of objectivity and reliability used in quantitative approaches. We now turn to Sellar's stereoscopic view, which has potential to serve as a foundation for an adequate specification of the concept of validity in qualitative comparative research methods.

THEORETICAL SCIENTIFIC DISCOURSE AND SOPHISTICATED COMMON SENSE

So far, the data-phenomena-distinction may have helped to highlight that "detecting a signal in this sea of noise [that is: in data], of identifying a relatively stable and invariant pattern" (ibid., p. 397) is essential. But how can this be done? We suggest that a holistic view, as outlined in the following, is particularly promising for comparative approaches in mathematics education.

In the philosophy of language and general epistemology, so-called holistic conceptions of human knowledge (prominently argued by outstanding philosophers such as, W.V.O Quine, Wilfrid Sellars, or Donald Davidson, and taken up and developed further by, e.g., Robert Brandom) have strongly influenced current views of the development of and relation between common, every-day discourse and knowledge on the one hand and theoretical scientific discourse and knowledge on the other.We consider such an approach as important as it can help to identify phenomena by including the perspectives of the involved participants, teachers and students in mathematics education.

As a first step, Sellars calls for scientific theories to connect to pre-scientific and everyday constructions and interpretations of the observable physical world and to take into consideration the complicated inner logic of everyday circumstances.

> I suggested that the most fruitful way of approaching the problem of integrating theoretical science with the framework of sophisticated common sense into one comprehensive synoptic vision is to view it not as a piecemeal task—e.g. first a fitting together of the common sense conception of physical objects with that of theoretical physics, and then, as a separate venture, a fitting together of the common sense conception of man with that of theoretical psychology—but rather a matter of articulating two whole ways of seeing the sum of things, two images of man-in-the-world and attempting to bring them together in a "stereoscopic" view. (Sellars, 1963b, p. 19)

However, the stereoscopic view is itself still part of the scientific and not the common image of the world. A stereoscope allows a viewer to see a three dimensional scene by looking at two dimensional images taken from different perspectives through a suitable frame. Sellars' intent is for research to be done in such a way that the differing perspectives offered by different theoretical ways of seeing be presented in such a way that a researcher sees a single true-to-lived-experience representation of the world.

As a second step, Sellars addresses the question when such a scientific image is "complete". This is not the case when the "common man" has understood and taken over the whole scientific image itself. The Sellarian concept of completing the scientific image is rather functional with regard to the "common man": He shall be able to relate the circumstances and purposes of his actions to the general statements of the scientific theories.

> Thus, to complete the scientific image, we need to enrich it not with more ways of saying what is the case, but with the language of community and individual intentions, so that by construing the actions we intend to do and the circumstances in which we intend to do them in scientific terms, we directly relate the world as conceived by scientific theory to our purposes, and make it our world and no longer an alien appendage to the world in which we are living. (Sellars, 1963b, p. 40)

The Learner's Perspective Study attempts to provide such a stereoscopic view through "complementarity" (Clarke, Keitel, & Shimizu, 2006).

> Complementarity is fundamental to the approach adopted in the Learner's Perspective Study. This applies to complementarity of participants' accounts, where both the students and the teacher are offered the opportunity to provide retrospective reconstructive accounts of classroom events, through video- stimulated post-lesson interviews. It also applies to the complementarity of the accounts provided by members of the research team, where different researchers analyse a common body of data using different theoretical frameworks. (pp. 4-5)

A second element in the Learner's Perspective Study that supports a stereoscopic view is the bringing together of insiders' perspectives with the more typical outsider perspectives. This is especially the case in Clarke, Keitel & Shimizu (2006) where the authors describe their own school systems and cultures, supported by data including the voices of teachers and students in the classrooms studied in their countries. This is done in the framework provided by the overall study, however, so the authors are presenting their perspective with awareness of the differences that exist internationally.

Contrasting approaches to phenomena are evident when the TIMSS Video Study (Stigler & Hiebert,, 1999) is compared with the Learner's Perspective Study. Both sought to compare characteristics of teaching in different countries. In the TIMSS Video study the phenomenon of national teacher scripts for mathematics education was assumed and data was collected in order to study this phenomenon more closely. Because the phenomenon was assumed to occur everywhere, it was not necessary to collect data in multiple locations in each country. An outsider perspective was taken, in order to reveal culturally ingrained characteristics of teaching that are invisible (due to their familiarity) to insiders.

In contrast, in the Learner's Perspective Studythe Hong Kong team (for example) started off by asking whether a Chinese teacher script of mathematics education actually exists (Mok, 2006). Multiple lessons were recorded, in two Chinese cities and in the classrooms of six teachers. Both outsider and insider perspectives were sought and both the teacher and the students' actions in the classroom were recorded (Clarke, Emanuelsson, Jablonka, and Mok, 2006). Data was collected not to study a presupposed phenomenon, but rather to understand teaching "based on the perspective of relevant stakeholders such as teachers and students" (Mok, 2006, p. 133).

In the Learner's Perspective Studyparticipants were asked explicitly for theirviews to bring in the common sense perspective. In contrast, Knipping (2003) observed the participants in their everyday activity of teaching and learning, and their comments on their own activity were later juxtaposed with the results of the scientific analysis to provide a stereoscopic view.

The focus of Knipping's research was classroom proving processes. In order to reconstruct the rationale of mathematical proving practice of teachers and students in the classroom she needed both a view grounded in classroom practices as well as a rigorous analysis of classroom argumentations. Formal mathematical logic cannot capture the rationale of proving practice, asarguments that are produced during proving processes in the mathematics classroomfollow their own peculiar rationale.

As a method Knipping (2008) proposes a three stage process: reconstructing the sequencing and meaning of classroom talk; analyzing arguments and argumentation structures; and finally comparing these argumentation structures and revealing their rationale through an interplay between the structures and the reconstructed classroom talk.

Asa first stage the reconstruction of the common sense meaning of proof in classroom talk is essential. Interpretative methods are used to reveal what teachers and students say and mean, when they produce arguments and proofs in conversation.

Second, the arguments and argumentation structures are analyzed to provide the scientific view. This involves two moves, first analyzing local arguments on the basis of Toulmin's functional model of argumentation, and second analyzing the global argumentation structure of the proving process.

Comparing argumentation analyses of classroom proving processes in different contexts provided a stereoscopic view. Itallowed not only thereconstruction of different leitmotifs of proving processes in classrooms, but also made the perspectives of the participating teachers on proving visible. These two-dimensional images come together into a three dimensional view of the proving processes in the classrooms. In this three dimensional view phenomena stand out and become more accessible to further study.

For example, teachers had communicated to their students during the proving process what they considered to be important, but this was at first not noticed in the data analyses. Only when the phenomena were described in form of leitmotifs, were these statements recognized and their significance acknowledged. One German teachertried to encourage her students during the proving process as follows:

Teacher: Mmh. We don't know yet what exactly to write in the middle. But, you know, what I really like about your answer is that you looked for squares, you could somehow find the area of. But we don't know exactly the lengths of the sides of the inner square. b squared would be a square, that is here somewhere.

Maren: Mmh.

Teacher: … that does not work so well. Maybe you can find something else. Sarah, don't write, don't write, just think, just look. We can write this down later. Jan.

In a French classroom, the teacher guided her students' proving in this way:

Thierry : DCH and BCG.

Teacher: # are ..

Thierry : DCH are complementary.

Teacher: Yes. (9 sec.) And then, are complementary, did you write that? Yes? So?

Stephanie: You write that C, so the angle C is equal to 180 minus

Teacher: You have to say first that HC, why 180?

Stephanie: 180, because it's straight.

Teacher: Well, you have to say it at least, eh? We have not said it yet. We have said it, but we have not written it down. In a proof we have to write everything that we said, so, next line, so HCG equals 180 degrees.

The statements "Sarah, don't write, don't write, just think, just look. We can write this down later" and "We have said it, but we have not written it down. In a proof we have to write everything that we said" were part of the data all along, but once the phenomena of the leitmotifs was recognized in the scientific analysis, such comments could be recognized as also describing the phenomena, providing the "sophisticated common sense" framework for them. Comparing both the argumentation structures and the actual classroom talk provided "two whole ways of seeing the sum of things."

In the Learner's Perspective Study and in Knipping's research we have examplesof comparative mathematics education research in which an attempt was made to investigate both phenomena that were identified by insiders in the community, and also phenomena identified by outside researchers, but which were described in ways that made them visible to the wider community as a whole.

There are other approaches that may address this challenge in comparative empirical research in mathematics education. We conclude with some suggestions in this direction.

THOUGHTS TO THE FUTURE

In this paper we have discussed two principles we consider as essential for research in general, and for comparative work in mathematics education in particular. First, we emphasised the essential distinction between data and phenomena. Second, we discussed the issue and relation of common discourse and scientific discourse and why a 'stereoscopic' view is necessary to overcome unintended shortcomings of educational comparative research. We have referred to ideas from the philosophy of science and discussed how these ideas make a valuable and significant contribution to comparative research in mathematics education. More can be learnt from the works cited and generally from philosophy, and particularly from philosophically motivated socio-empirical studies in the context of mathematics.

Müller-Hill (2011), for example, has studied formalisability in proving practices in mathematical research. She finds that proving practices vary in a significant way in the different subfields of mathematical research, e.g. mathematical logic, algebra, geometry, applied mathematics. In a comparative approach, interviewing mathematicians as experts in these different fields, she investigates to what extent formalisability really plays a fundamental epistemic role, as is often presumed by general epistemology, in these subfields. Her socio-empirical study was based on an approach, called 'conceptual modelling', developed by Löwe & Müller (Löwe & Müller, 2011; see also Löwe, Müller & Müller-Hill, 2010). Conceptual modelling is a trans-disciplinary methodological framework that follows and reflects an interactive process in the transition from data to phenomena. The sophistication in this approach cannot be discussed in detail here. We only want to point out that looking more closely at, and reflecting more carefully on, the conceptualisation and application of the transition from data to phenomena, is vital. It is of particular interest for research in general and for comparative research in mathematics education in particular. This is especially important in respect to differences in quantitative approaches compared to qualitative approaches, and a possible mix of these methods. There is much more potential in the work of philosophy of science and particularly the works we have cited than we can discuss here.

Also the distinction of common discourse and scientific discourse is reflected in Müller-Hill's research and the work of her colleagues. She makes a distinction of 'armchair' epistemology of mathematics, related to normative philosophers' discourse about epistemic aspects of mathematical practice, and the discourse of mathematical practice, related to mathematicians' discourse about their work. Conceptual modelling proved to be particularly helpful to investigate the issue of formalisation not just as a philosophical or logical issue, by not only interviewing practicing mathematicians, but also combining and reflecting on these perspectives to provide a 'stereoscopic' view of the issue.

NOTES

1. In (Woodward, 1989, p. 453), the following example is given:

> The authors of a recent study (Kamien and Schwartz 1982) devote approximately half of their book to arguing that the relationship [a relatively higher rate of technical innovation in moderately concentrated industries] is indeed real—that it has the characteristics of a phenomenon and is not an artifact of various statistical and measurement assumptions they employ. They investigate the relationship by regressing various measures of technical innovation on various measures of firm size and market concentration, the underlying assumptions about functional form being supplied by economic theory. They show that the relationship is relatively robust under different assumptions about how to measure these quantities and that it is fairly constant and stable across different industries. It is only in the second half of their book that the authors turn their attention to what they call 'theoretical explanation'.

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SCHOOL-BASED MATHEMATICS TEACHER EDUCATION IN SWEDEN AND FINLAND: CHARACTERIZING MENTOR-PROSPECTIVE TEACHER DISCOURSE

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Despite many similarities between the neighbouring countries Sweden and Finland, prior studies indicate that conceptualizations and discourses about school-based teacher education are very different. In this paper we add to this picture of differences, and contribute to the research discourse about school-based teacher education, by identifying and characterizing aspects of mathematics teaching made relevant in review meetings between mentors and prospective primary teachers. While the Swedish discourse typically focuses on the students' individual work with textbooks, connections to everyday experiences and teaching as individual supervision, the Finnish discourse emphasizes lesson aims, learning and progression in mathematics through formative assessment and differentiation according to pupils' abilities.

Keywords: Comparative study; Finland, school-based teacher education, Sweden.

INTRODUCTION

To become, and develop as, a mathematics teacher can be seen as a lifelong socialization process, initiated in school and teacher training and continued throughout one's entire professional life. There are many different aspects of this process worth studying, and multiple ways of describing and conceptualizing it (Adler, Ball, Krainer, Lin, & Novotna, 2005). Through this process the teacher should, for instance, develop necessary knowledge in and about mathematics, a number of competencies and teaching skills, as well as a productive teacher identity within a teacher community. A substantial number of studies have engaged in these areas (see, e.g., Jaworski & Gellert, 2003; Rowland & Ruthven, 2011; Rowland, Turner, Thwaites, & Huckstep, 2009; Sowder, 2007).

Whatever perspective one adopts, it seems reasonable to assume that university-based teacher education alone cannot provide teachers with the very specialized and complex competencies needed for productively teaching mathematics and acting as a mathematics teacher. Such competencies must be continually developed through years of critical reflection over one's own teaching practice (e.g., Adler & Davis, 2006). However, research indicates that if deliberately designed, such a critical attitude towards teaching can be fostered in teacher training programs (Rowland & Ruthven, 2011; Rowland, et al., 2009). School-based parts of pre-service teacher education may have a crucial role to play when it comes to creating opportunities for the prospective teachers (PTs) to experience and reflect upon the complexities of mathematics teaching together with mentors (Jaworski & Watson, 1994). Ideally, in such situations, qualified and experienced teachers serve as important guides in socializing the PTs into ways of becoming, and thinking like, a critically reflective teacher (Hegender, 2010; Jaworski & Gellert, 2003). Although school-based teacher education (SBTE) in mathematics has been suggested as an important instance for developing as a teacher (e.g., Frykholm, 1999), the opportunities created for developing competencies in teaching mathematics within this practice cannot be taken for granted. For instance, while several studies (e.g., Ebby, 2000; Krzywacki-Vainio & Hannula, 2008) show that PTs regard the SBTE as the most important element of their education, "too often school experiences, particularly student teaching, are disconnected from university-based components of teacher preparation" (Sowder, 2007, p. 202).

In understanding more about SBTE and the opportunities created for PTs to develop as competent mathematics teachers in such practices, we find the cases of Sweden and Finland of interest to study. Despite many similarities between Sweden and Finland concerning the school system, prior studies indicate that conceptualizations and discourses about SBTE in the countries are very different (Ryve, Hemmi, & Börjesson, 2011) and we are beginning to collect evidence that there are substantial differences between the countries concerning curriculum materials (Hemmi, Koljonen, Hoelgaard, Ahl, & Ryve, 2012) and classroom teaching (Andrews, Ryve, & Hemmi, 2012; Hemmi & Ryve, 2012). In this paper we are particularly interested in characterizing the discourse, where discourse is defined as a recurrent way of communicating (Ryve, 2011), between mentors and prospective primary school teachers (PTs) after classroom lessons in which PTs have taught. We aim at characterizing such meetings by engaging in the research question: *Which kinds of aspects of teaching practice are made relevant in the discourses and how are phenomena related to mathematics teaching construed?*

METHODOLOGY

This paper is situated within the principle that language use can be seen not only as a reflection of reality but also as a tool for constituting reality (e.g., Sfard, 2008; Säljö, 2000). Data were collected in the form of audio-recordings from tenreview meetings between mentors and PTs in Sweden and six in Finland. The meetings took place immediately after a mathematics lesson that a researcher had attended and made field notes of or videotaped. The researcher was also present at the meeting, but took a passive role. Common for all PTs is that they are attending their second (of four) period of SBTE. In our Finnish cases the prospective teachers conduct their SBTE in pairs and hence are two prospective teachers present in the review meetings from Finland. During the Swedish meetings no university lecturer was present, and this relates to fact that during the whole eight-week SBTE period no university lecturer visited the PTs. In contrast, in Finland, university lecturers were present, and took an active role, during two of the analysed meetings. Further, in the Finnish cases the SBTE is conducted at a university practice school where mentors have several hours per week for developing their mentoring. In Sweden the SBTE is organised in ordinary schools where teachers have limited time for mentoring. This means that the review meetings had different conditions and this reflects a structural difference between the two countries.

In this paper we present the initial analysis of four meetings that we have transcribed so far from each country andthis will guide us when analysing the rest of the data.The analysis uses an iterative approach going through several cycles.We used NVivo as a facilitating tool, and all four authors contributed to interpretations of the data. The Swedish transcripts were first analysed by two research team members and the Finnish transcripts were analysed by the Finnish speaking member of the team. We had no explicit categories when we conducted the first analysis but we were searching for aspects made relevant.

We checked each other's analysis and agreed on the following categories that we had derived from the data: *Lesson aims, Learning and progression in mathematics, Differentiation and individualization, Mathematics and its connections to everyday experiences, Assessment, Classroom management, Mathematics textbooks*and *Resource constraints*.

We decided also to attempt to describe howthe role of the teacher, pupil, mentor and PT is conceptualized in the discourses.We then analysed the data once more using all the categories and finally, compared the categories in both countries.When we are referring to the Swedish and Finnish discourses we are only expressing the characteristics of the discussions we have analysed and do not attempt to generalize cross-national differences.

RESULTS

The results section is structured in such a way that all recurrent themes of aspects made relevant within the union of Swedish and Finnish discourse are presented in order tomake the comparison of the discourses of the countries transparent. (SWE M1) refers to a Swedish mentor in Group 1 and (FIN PT12) refers to PT one in the second group in Finland. A university lecturer in mathematics education is represented by U.

Lesson aims

The Finnish discourse is typically anchored in relation to the written lesson plans and the PTs' teaching. For example, the mentors point out the importance of making the aims of the lesson clear to the children by writing a proper headline on the board. Lesson aims also become relevant in relation to issues about pupils' mathematical knowledge and progression as well as the assessment of pupils. Hence, the aims of the lessons are both made relevant and explicit through the lesson plans and stressed as important to make explicit for the pupils in the classroom.

The Swedish discourse contains no explicit statements about formal learning goals for the lessons, or about the outcome in relation to such goals. On a few occasions, these two matters are implicitly included through questions such as "*What was your* *intention that they [the pupils] would get from this lesson?*" (SWE M:2) and "*Did you feel that the children were following you? Did they understand?*" (SWE M:3).

Learning and progression in mathematics

The issue given most relevance in the Finnish discourse is how to develop pupils' knowledge and progress in mathematics. The following conversation, from G1 in Finland, is an illustrative example:

M1:	Why have we done the divisions of numbers so eagerly?
PT11:	Well, if one masters them, they help children even in basic addition, especially when they go to the ten transitions, so they know when they have, say 4+7, how much you need to add to seven to get ten and how much is left.
M1:	Yes […] and now the next work is to teach them how to use them. They're two distinct matters, using them comes when we pass ten.

The connection between formative assessment and ideas for progress is exemplified by a university lecturer, who says "*for example, as a formative test let them write all the divisions of number 9. If one finds this kind of problem-solving strategy, it's easier to remember and check what's missing*" (FIN U2) and then continues "*I wonder if the strategy could be opened, like if we use the nine balls and then move the magnet ruler to make the strategy visible*" (FIN U2).

In the Swedish discourse, issues of learning are typically discussed on a general level, often emphasizing reasoning and interaction: *"[I]t's the journey, they actually learn from the reasoning"* (SWE M4), *"it's very good learning […] when they teach, and learn from, each other"* (SWE M4), *"you could see the difference all the time, that they're advancing conceptually and how much they understand and can begin to reason a little more"* (SWE M4). This last statement is, in principle, the only statement in the Swedish discourse related to the progression of pupils' mathematical learning. There are no statements about pupils' development of mathematical proficiencies other than that of reasoning. When the mentors and PTs discuss forthcoming lessons, there are few statements about how the pupils' knowledge could be deepened. More common are statements such as *"it's important to go back and do more work on liter and deciliter […] they need to repeat, over and over again, in order to cement it"* (SWE M2).

Differentiation and individualization

In the Finnish discourse individualization and differentiation of the teaching was present in all four discussions, with respect to both how to organize the classroom teaching in order to meet the needs of different pupils (*"You've now learnt to handle a class with two grades and think about two groups all the time and you've managed it well. Therefore you've already learnt to differentiate the teaching according to two groups. I haven't demanded that you think so much about the individualization within the groups, only a few aspects of handling pupils working at different speeds"* (FIN

M1)), and how to develop mathematical tasks during the lessons in order to present challenges to all pupils (*"One group of pupils understands immediately, okay this kind of task, and writes it mathematically at once, whereas others need more time, and they should have the right to go on with their thinking process for a longer time"* (FIN PT11)). We conclude that aspects of how to plan and adjust the classroom teaching to ensure that each pupil receives appropriate mathematical challenges is very typical of the Finnish discourse. In the Swedish discourse, the issue of how to arrange the classroom teaching in relation to the pupils' diverse pre-knowledge and interests is not made relevant. However, the importance of helping/teaching the pupils in one-to-one situations is stressed. Supporting pupils who have difficulties was mentioned once, when a mentor referred to special education teachers and their responsibility for this.

Mathematics and its connections to everyday experiences

One of the most specific features of the Swedish discourse is the emphasis on the importance of connecting mathematics to everyday experiences. Such everyday experiences are typically not formalized into mathematics, but are seen as mathematical activities in themselves. For instance, one of the mentors states that *"mathematics […] is loads of everyday experiences"* (SWE M2). Therefore, everyday experiences are typically put forward as relevant in themselves, not for deepening mathematical concepts or ways of reasoning. The situation in the Finnish discourse is different; only once is a connection to everyday life mentioned, in the context of feedback from the mentor to the student: *"And I think that was a really good example you gave [before the lesson] but it didn't appear during the lesson, talking about something like there are no people within a radius of a hundred kilometers or five kilometers, but you forgot it, that would've been a good example about where we can use the concept of radius"* (FIN M4).

Assessment

Assessment is another dominant theme in the Finnish discourse. It mostly concerns formative assessment connected to learning and progression in mathematics, as shown above, but also concerns the principles of giving pupils their grades in mathematics, and the difficulties in working with assessment are acknowledged in the discourse. When discussing pupils with wrong answers in their homework, one mentor states that *"one has to react with some questions and try to sort out whether the pupil is way off track, and this is difficult for the teacher"* (FIN M4). Another typical example regarding formative assessment is the following: "*Yes, the goal of the lesson was to draw a circle according to the given rules and internalizing the concept of a circle. I'm sure they learned to use the compasses by doing it, but I wonder how it is with their ability to draw according to the rules since you gave them a lot of freedom. So we could now think about giving quite strict rules and checking whether they can draw a circle with, for example, a diameter of 6cm or a radius of 3cm, and then it's easy to see who has succeeded and why somebody hasn't succeeded in doing it"* (FIN M4).

The mentors also pose questions about student learning so the prospective teachers have to be aware of how the pupils progress: *"Yes, but otherwise everyone's done the tens table right* (in a test) *and Silja and Aapo are the only ones who aren't ready with the twos table…"* (FIN PT1).

Assessment is only briefly touched upon in the Swedish meetings, when mentors ask the PTs whether they think the pupils understood the content of the lesson, but they do not give explicit suggestions for how one might get this information. The mentors and PTs discuss the importance of meeting the pupils' one to one, or having time to watch them work in smaller groups, in order to get a grip on what parts of the content they understand as well as how they understand them.

Classroom management

The aspect given the most relevance in the Swedish discourse is organizational issues regarding pupils and artefacts. How the pupils should be grouped (*"When I was thinking about it in retrospect, maybe one should've separated Anna and Jennie so that Eric, Maria and John could've been with one of them"* (SWE PT4)), where the books should be stored, the importance of being clear when writing on the board or pointing to exercises, which measuring spoons to use and the length of the teacher's introduction (*"and when they have to sit for so long and listen […] then it'll be tiresome to sit and concentrate"* (SWE M3)) are examples of such issues. This finding of the heavy focus on organizational issues harmonizes with the recent study by Hegender (2010) about mentor-PT discussions during SBTE in Sweden.

Classroom management, like the use of manipulatives and ICT, is also clearly present in the Finnish discussions connected to, for example, formative assessment and disciplinary issues: "*It was good that you solved the problem of who the head of the group is so quickly, so that everybody could start working in time"* (FIN M4). One university lecturer states that if the group is intense and impulsive one could let pupils explain as a way of handling the group: *"When a classmate is saying something the others are trying to listen"* (FIN U3). Another example of managing the class is the following comment about how to start a lesson: "*You wait until you have all the children's attention, all the children are looking at you, they're ready, there's no use saying something before that"* (FIN M4).

The mentors typically also suggest ways to organize the teaching in a way that makes pupils' learning visible and allows them all to be active in engaging with tasks: *"Yes, and again the pupils can have the number cards on their desks as the number area isn't so big when solving equations in their head, so that all the pupils at once can show what number we can put in the square so that you don't need to say 'raise your hands', but here also the teacher gets immediate feedback about who's way off"* (FIN U3).

Mathematics textbooks

Mathematics textbooks are referred to three times in the Finnish discussions, and all these references explicitly deal with the treatment of the specific topics: investigation of divisions (*"Tuhattaituri [the name of the textbook] uses exactly this model where one moves the vertical line when investigating divisions"* (FIN U2)); the notion of a product and the value of a product (*"But the textbooks are like this, it's not only in our textbook but also the others that I've studied, but if we think back to year x, the beginning of my teaching carrier, there they were already separated [product and the value of the product]"* (FIN M1)); and the use of the number line (*"[…] as even some researchers say, this book series uses the number line too much, is it true and whom does it suit, I don't know"* (FIN U2)).

In the Swedish discourse the textbook is more prominent. They do not discuss specific topics in more than one episode; instead, the pupils' use of the mathematics textbook is either mentioned as the main part of the lesson or seen as a goal of the lesson that the pupils can work individually in the textbook. After one lesson when the pupils have worked in the mathematics textbook for practically the whole lesson, one mentor says *"Well it was a very ordinary lesson as many of them are"* (SWE M1). *"[T]he book is something they should be able to work in on their own"* (SWE M3) is a statement from another mentor, who later in the discussion says that *"if one were certain they understood and then started with the book […] one could do a demonstration and then they could work on their own"* (SWE M3).

Resource constraints

Issues about constraints regarding both the time to plan lessons and the length of lessons are brought up several times in all four Swedish meetings. One mentor refers to changes in teachers' working conditions: *"This dilemma that we're struggling with right now, […], when I started working as a teacher, I had plenty of time to, sort of, plan lessons, to really think through what I wanted to do, I made my own material and such. But now that time doesn't exist"* (SWE M1). Especially the PTs express that they do not have time to do all they want: *"I'd planned to do that at the end, but we didn't have the time"* (SWE PT2). The same PT mentions that the lesson in consideration would have been impossible to realize if he had been the only teacher in the classroom: *"I wouldn't have been able to carry this out on my own"* (SWE PT2). The mentor continues: *"No, and that's why we have a special education teacher"* (SWE M2). Hence, an implicit assumption is that productive teaching is more or less impossible to conduct without several teachers present in the classroom.

Resource constraints are made relevant much more seldom, and in a different way, in the Finnish discourse. The mentors and PTs do not focus on time constraints as a problem but instead on how to cope with them. Aspects of the importance of the teacher staying calm are emphasized, as is the importance of not transmitting the feeling of stress to the pupils: *"If the teacher herself has a hurried feeling this is easily transmitted to the children"* (FIN M4). They also discuss issues concerning the teachers' responsibility to adjust the plans for the next lesson by considering issues that could not be handled during the current lesson: "*As a teacher one should hold the threads in one's hands and think that I'm going to sort this out then and how we can repair this next time, it shouldn't be the pupils' responsibility if the use of the time during the lesson hasn't worked as the teacher had planned*" (FIN PT21).

CONCLUSION AND DISCUSSION

The results indicate that there are rather substantial differences between the discourses of the Finnish and Swedish groups. We cannot generalize the results from the present study to the two countries, but they add to the growing body of knowledge obtained from the previous studies.We could further compare the discourses by introducing preliminary results of how the teacher's and students' respective roles are conceptualized as well as discussing aspects of specificity and structure of the discourse. As described above, the teacher's practice within the Swedish discourse is conceptualized in terms of a "supervisor" who should primarily engage in one-on-one discussions with pupils. This finding is in line with findings in Hemmi & Ryve (2012), in which Swedish mathematics teacher educators typically portray classroom teaching as a teacher-talking-with-one-student phenomenon. In contrast, the Finnish discourse puts much more emphasis on whole-class teaching and the teacher acting in whole-class situations.

In the Swedish discourse the pupils are described several times as resources, in teaching and gaining understanding from each other: *"It's very good learning […] when they teach, and learn from, each other"* (SWE M4). One thing that is mentioned as an obstacle to pupils' understanding is their level of maturity, which seems to take precedence over the teaching. An example of this is: *"Well, you also have to realize that they're not that big yet, they don't have it, it's not close to them"* (SWE M4). In the Finnish discourse the teachers' responsibility of students' learning and students' behaviour is stressed in different ways.

Another typical recurrent pattern in the discourses is that Finnish mentors and PTs are rather specific in talking about mathematics, pupils' learning, goals of the lessons, etc. Our analysis of the Swedish discourse suggests that the statements, suggestions and reflections upon the classroom practice are formulated rather generally. This aspect is also related to a fourth characteristic we noticed: the Swedish classroom teaching as well as review meetings are loosely structured, with few explicit frameworks or plans for how to act. For instance, as indicated above, lesson plans are typically not produced or made relevant within the discourse as a way to structure the review meetings. In contrast, the Finnish context is structured through, for instance, lesson plans, visits from university lecturers, and explicit discussions of the lesson goals. Possible explanations for these differences could be the different position of the mentors in the system. Our initial analysis has not revealed differences between the characteristics of the feed-back discussions conducted by the mentors (class teachers) on the one hand and the university lecturers on the other hand in the Finnish context. A deeper analysis of the whole data could inform us whether the presence of the university lecturer seems to have a positive effect on the qualities of the review discussions and in that case how.It would also be interesting to analyze who raises the recurring themes and what this says about the relationship between the participants.

This paper should be seen as a contribution to the research on SBTE and especially on ways of characterizing the discourse of university lecturers, mentors and PTs (cf. Jaworski & Gellert, 2003; Sowder, 2007). The analysis of four meetings from Sweden and Finland shows some similarities, but also many differences. Basically, teaching mathematics seems to be conceptualized in two distinct ways, and in future studies we aim to both complement the study by analysing more data and deepen the analysis by contextualizing the results within a broader frame of similarities and differences between the two countries regarding aspects such as classroom teaching, teacher education, the role of school in society, and cultural patterns. We believe that such an approach could add to the research discourse on the role of SBTE in educating PTs in mathematics, assistance teacher educators in making informed choices about SBTE, as well as deepen the understanding of pupils' learning of mathematics in Sweden and Finland.

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COMPARING MATHEMATICAL WORK AT LOWER AND UPPERSECONDARY SCHOOL FROM THE STUDENTS' PERSPECTIVE

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As part of a comparative study between how students experience and perceive their mathematics education at lower and upper secondary school, a classroom episode is analysed from a theoretical approach combining key concepts from the anthropological theory of didactics and Bernstein's theory of pedagogic discourse. The findings are discussed with reference to the usefulness of the theoreticalapproach.

Key words: mathematical knowledge, transition, praxeology, recognition rules

INTRODUCTION

In addition to the general relevance of increased knowledge about similarities and differences between mathematics education at the compulsory and non-compulsory school levels, low pass rates on the national tests on the first mathematics course during the first year at upper secondary school in Sweden have been reported, especially on vocationally oriented study programmes (Skolverket, 2012). It is therefore of interest to compare students' experiences of their mathematics studies during this transition process. Differences between lower and upper secondary school mathematics that students can meet may be related to curriculum issues, differences in styles and focus of teaching or textbooks, kinds of examination tasks and evaluation criteria, pace and work load. However, this paper sets its main focus on how to identify and describe possible differences in the classroom teaching that may (partly) account for transition problems.

Knowledge in mathematics can refer to different things, such as technical skills in solving tasks by using a certain method, the ability to justify why a method works or to prove mathematical theorems. A common psychological approach for addressing these issues draws on the distinction between procedural and conceptual knowledge. However, to *compare* the character of the mathematical knowledge as it is being practised within the two different institutions (lower and upper secondary school in Sweden), including criteria for what counts as an accepted knowledge production by the institution, an approach that also takes institutional factors into account is needed. One such tool is provided by the anthropological theory of didactics, ATD (see e.g. Bosch & Gascón, 2006), within which the notion*praxeology* (ormathematical organisation) addresses different aspects of the mathematical content in for example a school context activity. More general structural issues regarding the distribution of knowledge in pedagogical contexts are described by Bernstein (2000), where some of the key concepts of his theory such as *classification* and *framing* of knowledge, and *recognition* and *realisation* rules, can be used as analytical tools when researching what possibilities the students have to succeed in the mathematics classroom. In this paper, the use of these theoretical tools for investigating the transition will be discussed and illustrated by preliminary empirical data from grade 9 and first year of upper secondary school. We thus address the question about how and to what extent a combination of these two theoretical approaches will support the comparison of how students might experience differences in their mathematics studies between the end of lower and the beginning of upper secondary school.

BACKGROUND

In Sweden, children begin at the age of seven in the nine years compulsory school. Almost all students proceed to upper secondary school, which is non-compulsory and consists of 18 different national programmes, theoretically as well as vocationally oriented. The students apply for upper secondary school during the last semester of grade 9 in compulsory school. One of the required qualifications for a national programme is to have passed in mathematics in grade 9. In the core school subjects, such as mathematics, national tests are compulsory in order to set a common national standard. Results from May 2011 show that 19 % of the students in grade 9 did not pass the national test in mathematics (Skolverket, 2012). However the national tests are consultative and only 6 % finally did not pass in mathematics in grade 9.

There are three versions of the first mathematics course in upper secondary school. Course 1a is studied mainly in vocationally oriented study programmes, course 1b e.g. in the social science programme and course 1c e.g. in the science programme. To acquire a general eligibility for university studies a student has to follow the study programmes including courses 1b or 1c. Results from May 2012 show that around 35 % of the students did not pass the national test for the first mathematics course. Of these, for course 1a 48 % of around 25 000 students did not pass the national test, for course 1b 30 % of around 30 000 students and for course 1c 9 % of around 7 000 students (Skolverket, 2012). Since all those students passed in mathematics in grade 9 (although not necessarily on the national test), these results point to a concern about the transition from lower to upper secondary school mathematics.

The different levels of the Swedish school system are regulated by national curriculums, including syllabuses describing ‘core content’ and ‘knowledge requirements’ (Skolverket, 2011a, 2011b). While core content is naturally expanded in the first mathematics course in upper secondary school compared to grade 9 in compulsory school, the formulations of the main goals for the subject do not differ considerably between the two school sectors, both emphasising conceptual and procedural proficiency, problem solving and modelling skills, and mathematical reasoning and communication ability. Classroom practice, though, is not regulated in the steering documents and can differ between groups, teachers, schools and school sectors. To understand how the students experience their mathematics studies during the transition stage, it is therefore necessary to investigate potential differences in classroom practices with analytical tools that are flexible enough to make a comparison of these practices possible.

STUDIES OF TRANSITIONS

Most of the existing research on transitions between school-sectors deals with other levels than the transition from lower to upper secondary school. At this level, most studies have focused on less domain specific issues such as identities and motivation (in relation to mathematics, see for example Midgley, Feldlaufer, & Eccles, 1989). However, some studies focus on differences related to the teaching at the two school levels. With a focus on the character of the mathematical work in classrooms, Sdrolias and Triandafillidis (2008) investigate the transition from primary to secondary school geometry in Greece using a semiotic approach, concluding that students' continued construction of mathematically more developed signs may not be supported by the teaching they receive. They observe an increase in logical rigour in secondary school that does not build on children's primary school experiences. In addition, a "rushed move /…/ towards the production of a general law" is seen in teaching at both school levels, with a negative influence both on students' participation and on the "clearness of a mathematical idea" (ibid., p. 167). In Norway, Nilsen's (2012) study employs a semiotic perspective to compare the teaching and learning of linear functions at lower and upper secondary school, concluding that a lack of flexibility in the mathematical teaching, tasks and tests was impeding students' transition towards a more abstract notion of gradient at upper secondary level. The same author also investigated differences and similarities in teachers' beliefs about mathematics teaching at the two school levels (Nilsen, 2009). It was found that teachers at the lower secondary level put more emphasis on "reaching the individual student" (p. 2502), while upper secondary teachers focus more on good explanations and mathematical techniques for students' individual work on tasks.

Research on the transition from upper secondary to university level mathematics has focused on several of its aspects, including differences regarding the content and character of mathematics at the two levels of education. In the Swedish context, Brandell, Hemmi and Thunberg (2008) point to problems created by a gap in topics covered, as well as a change towards a more theoretical mathematical discourse, while Stadler (2009) characterises the transition in terms of how students refer to the different available resources. In an on-going project it has been shown that students' recognition of what counts as the promoted institutionalised mathematics is related to their achievement levels (Jablonka, Ashjari, & Bergsten, 2012). In their study Bernstein's concepts 'classification' and 'recognition rules' have been used to study changes in institutionalised mathematical discourse during students' transition between two levels of education.

THEORETICAL APPROACH

Previous research on transitions has pointed out several features within the receiving institution that seem to be "new" in relation to the students' experiences of mathematics education from the institution they are leaving. These relate to an increase in logical rigour and a more theoretical discourse combined with an increased pace, new mathematical content and work with more advanced mathematical signs, and a teacher focus more on subject matter explanations than "reaching the individual". In order to account for what students experience as different (and possibly problematic) in the passage from lower to upper secondary mathematics education, an analytical tool is needed that makes it possible to analyse the mathematical work at both institutions in a way that makes it open for comparison. Semiotic analyses of the treatment of specific mathematical concepts or ad-hoc comparisons of mathematical content covered will then need to be complemented with a more holistic analysis of the type of mathematical work and knowledge criteria that students experience in the classroom (cf. Nilsen, 2009). This will require a tool that can describe structures in the mathematical work that are *general* across institutions and relevant enough to capture those potential differences that transition research has pointed at, as well as others that have not yet been observed, and *flexible* enough to be applied to both institutions which are steered by different curriculums and different teaching traditions. This would suggest the use of a theory in the sense of Jablonka and Bergsten (2010), with the potential to describe relations between categories and account for aspects not yet observed.

A potential such tool is found within the anthropological theory of didactics, ATD, with its theoretical construct *mathematical organisation* or *praxeology* (e.g. Bosch & Gascón, 2006). By locating the praxeological analysis in relation to different *levels of co-determination* (ibid.), curriculum and teaching traditions can be adhered to. However, while the praxeological analysis accounts for the organisation of the mathematical knowledge in the classroom but not for the overall organisation and structure of the classroom work, we also suggest to employ some notions from Bernstein (2000) to be able to analyse differences and similarities along that dimension at the two school levels, which can be captured by 'classification' and 'framing'. These two theoretical approaches are compatible as they both consider institutional dimensions and share the same intellectual roots (cf. Bergsten, Jablonka, & Klisinska, 2010).

In this paper the theoretical construct 'praxeology', a key concept in ATD, will thus be employed to characterise the mathematical work in an activity such as a mathematics lesson: what types of tasks are given to the students and what techniques are used to solve these tasks (the praxis part of the praxeology, or the 'know-how'), what kind of arguments are used to justify the use of these techniques, and what theoretical background these justifications are based on (the logos part of the praxeology or the 'know-why'). For the study, it is of interest to investigate to which degree there is a difference in the character of the praxeologies, for example in terms of balance and links between the praxis and logos levels, developed in lower and upper secondary school mathematics. To account for constraints on classroom work coming from outside the classroom, such as curriculum, pedagogy and teaching traditions, the ATD employs the theoretical construct 'levels of co-determination'. The ATD is a general theory in the sense of Jablonka and Bergsten (2010) and has been applied in different subjects and school levels (see e.g. Bosch & Gascón, 2006) pointing to inconsistencies and constraints in educational contexts.

Also Bernstein's theory of pedagogic discourse (e.g. Bernstein, 2000) is such a general theory that has been applied in different cultural contexts, and it is here used for comparing the level of explicitness of the knowledge criteria for the students: are there differences in the strength of the *framing of the knowledge criteria* in the two institutions (ibid.)? There may also be differences with respect to the framing of the selection (e.g. who selects the tasks to work on), sequencing and pace, as well as the balance between *instructional* and *regulative* discourse (ibid.), where the balance between these two constituent parts of the pedagogic discourse is expected to change during the transition towards a stronger dominance of instructional discourse. Furthermore, how do students differentiate between what the mathematically relevant aspects of the tasks are and what is less relevant, such as specific context issues? This dimension of comparison may be described in terms of *classification* (ibid.).

Similarities and differences between the two school levels may concern what aspects of mathematical work are emphasised in the mathematics classroom teaching. For example, is solving the task enough or is the student also required to explain why and how a method is working? A comparison of this issue will be possible by way of a praxeological analysis combined with a focus on the framing of the classroom work. Furthermore, knowledge of mathematics includes the ability to communicate about the subject, according to the syllabus. One aspect of communication is writing down the solution of a task. To be able to do this in a way that is legitimate within the institution, the student must know what is expected from him/her, that is, has to be in possession of *recognition* and *realisation rules* (ibid.). The character of the praxeologies, framing of the knowledge criteria and possession of recognition and realisation rules are factors that likely influence students' experiences of mathematics at their respective school level.

AN EMPIRICAL STUDY

The study as a whole will contain analyses of curriculum documents, observations of mathematics lessons at the two school levels, interviews with students and a questionnaire survey. In this paper the main focus will be on the comparison of the classroom work at the two school levels. For this purpose, two mathematics classes were video and audio recorded during three consecutive lessons at two different occasions during the last semester of grade 9. Some of these students volunteered for follow up interviews and most of these students were revisited in their mathematics classes in the first semester of upper secondary school. Thus, three classes in the first year of upper secondary school, following the mathematics courses 1a, 1b, and 1c, respectively, were also video and audio recorded in a similar way as the grade 9 recordings, along with follow up student interviews.

For the purpose of illustrating how the theoretical approach suggested may support the analysis of empirical data that can be used to study the transition process in focus, an episode from a mathematics lesson in one classroom in grade 9 will be discussed and compared to preliminary observations in a mathematics class from upper secondary school. The teacher-student communication selected in the grade 9 classroom took place when the class was repeating the content of the course before the national test. The lesson pattern is typical for Swedish mathematics lessons, at lower as well as upper secondary level: a main part consists of individual work, where the students solve tasks from the textbook; the teacher walks around in the classroom helping the students, who raise their hands to call for help; teaching in front of the class is done mainly at the beginning of the lesson and for a relatively short time (Skolverket, 2003).

A teacher-student dialogue

A student has asked the teacher for help. The task is to calculate the cost of 0,9 kg shrimps, when the price is 95 SEK per kg. The student has written $95 \cdot 0{,}9$ in the notebook but does not know whether she should use the calculator or do the calculation by hand. The dialogue in focus here is about how to calculate $95 \cdot 0{,}9$ without a calculator. The teacher writes a standard algorithm and starts calculating:

1	Teacher:	Nine times five is forty-five and nine times nine is eighty-one, eighty-five [81 + 4], which give eight five five, and then one decimal [writes a decimal sign between the fives]
2	Student:	How do you know that? [that it will be *one* decimal]
3	Teacher:	I was going to make another suggestion, but this might answer your question. If you take nine times ninety-five
4	Student:	Mm [agreement]
5	Teacher:	it will be eight hundred fifty-five.
6	Student:	Mm
7	Teacher:	But you shouldn’t take nine [with emphasis] times ninety-five, but zero point nine [with emphasis].
8	Student:	Yes
9	Teacher:	And because zero point nine is ten times less than nine
10	Student:	Mm
11	Teacher:	it means that if we answer like this [855], the answer will be ten times too large.
12	Student:	Mm
13	Teacher:	Yes

Here the teacher switches between describing techniques and providing justifications. In line 1 the teacher explicitly performs an algorithmic technique for multiplication that the student does not seem to question apart from the last expression, the claim “and one decimal”, for which the student asks for a ground (line 2). As a response, the teacher then in line 3 starts to present an idea of how to explain the position of the decimal sign. The argumentation from line 5 through lines 7, 9 and 11 is used by the teacher to justify the claim made in line 1, where the warrant is implicitly included in a reasoning about powers of ten, again including calculations.

It is the student who raises the question (line 2) about *how* you know that it will be one decimal, though it seems (from line 3) that the teacher had prepared to warrant his claim some way. This question of justification can be either a question about the technique required to find the number of decimals or an explanation about *why* you can come to that conclusion. Due to the other very short contributions from the student, it is likely to conclude that the question *how* was about 'how to do it' rather than to understand. However, the teacher's immediate answer to the question was to provide a rational explanation. In the continued dialogue the same pattern goes on. The teacher provides another justification (warrant) of why the product contains one decimal. When the student (later) raises a question about how to do when the numbers have many digits and decimals, the teacher employs a backing strategy providing a rule that can be used. As this rule is again a technique given without ground, he uses estimation to assure that the answer is correct, employing the qualifier 'reasonable'. This pattern is typical for the dialogues during the three lessons. One of the students asks for help and the teacher gives an explanation including explanations both of how to do it and why it works. The student's responses are mainly short, but sometimes s/he asks another question about *how* to solve the problem. In the excerpt the teacher's explanations are intra-mathematical. However, in other discussions he uses models or metaphors when switching between technique and justification, e.g. 'common economy' when explaining addition or subtraction of positive and negative numbers. He also mentions sign rules as techniques to help doing the calculations.

The excerpts show that both mathematical techniques and reasoning at the level of justification are frequent in the dialogues between students and the teacher. The teacher's explanations often contain different ways of attacking the problem, as well as types of justifications suggested, thus providing a base for a more general praxeology to be developed (cf. Barbé, Bosch, Espinoza, & Gascón, 2005, pp. 237-238). This strategy may also promote the students to get a broader view of the subject mathematics than 'just finding an answer'. However, the students' contributions to the mathematical discussions are mostly short, containing very few questions or conclusions about explanations and do not show signs of engagement in the justification of the mathematical techniques. Hence it is not possible, from these data, to know to what extent the students are aware of the teacher's endeavours of explaining more than just the technique to find the answer to the task. From a more detailed look into what types of tasks and techniques are used, including those appearing in the examination tests and described in the textbook with justifications offered there, it will be possible to describe the character of the praxeology developed in the lessons (cf. Barbé et al., 2005). The mathematical knowledge in focus during the individual discussions between the teacher and the students was strongly classified, though in some explanations metaphors from outside mathematics were used.

Framing of the criteria

Explicit discussions about knowledge criteria, in the sense of what is expected when you write down a solution to a task, are rare during the three lessons. There is one episode from the beginning of the first lesson when the teacher informs that they will get copies of national tests from earlier years, along with solutions, for practicing: "… and then we will go through them with solutions so you will think of that you not just write answers or what is important to include in a solution". That is the single part where this is clearly mentioned. There is also one question from a student about if you just should write down the answer after you have completed a calculation by hand. It is not clear whether this question really is about how to present a solution or if it just aims to check if the task is completed. In summary, explicit information about how appropriate solutions should look like is not frequently offered during the lessons to support students' development of recognition rules for what counts as legitimate knowledge in this mathematics classroom.

The teaching observed in the three lessons does only to a very small extent explicitly discuss the knowledge criteria with the whole class, thus not providing many opportunities for the students to develop appropriate recognition rules. This is then possible only in the individual discussions with the teacher or with other students. The data presented above point to a rather weak framing of the knowledge criteria also in such situations. Otherwise there is neither discussion of what is important in mathematics nor discussions about students' solutions of tasks or standards for written solutions. However, that does not exclude that recognition rules are developed when students work with tasks in relation to solved examples in the textbook, or when they get their result, perhaps with their teachers' comments, on written exams.

Some preliminary observations from the upper secondary classroom

Preliminary analyses from the upper secondary school classroom observations show that there are significant differences between the three classrooms visited. In the class studying course 1a, the praxeology is constituted mainly by the 'know-how'. The teacher's explanations are short and mainly inform the students exactly how to solve a task. In the course 1b class there are more explicit arguments why a certain technique works and the framing of the knowledge criteria is visible. In the third class, studying course 1c, there is more teaching from the front, in some lessons up to 30 minutes. Questions from the students are rare, even when they work individually, but when someone asks a question, the teacher's answer includes both an appropriate technique and its justification. There is generally a strong classification of the mathematical knowledge in these classrooms but answers in student interviews indicate an increasing use of mathematics in other school subjects as compared to grade 9.

In the data from the grade 9 classroom presented, a weak framing of the knowledge criteria was found, indicating that students may develop only weak recognition rules. The preliminary data from the upper secondary school classrooms suggest a more visible pedagogy in the more theoretical courses.

DISCUSSION

Differences between the two school levels that may account for transition problems, or challenges, may be searched at different levels of co-determination. As the general goals and knowledge requirements for the mathematics studies as formulated in the national curriculums are found to be not very different, and the mathematics lessons generally structured by an overall similar pattern, a conceptual framework has been suggested for analysing differences in the organisation of the disseminated mathematical knowledge and the criteria (and their explicitness) for legitimate knowledge productions from the students at the two school levels. By the focus on general characteristics of the organisation of mathematical knowledge in terms of its technical and theoretical levels, and the framing through the teaching practice, it has the potential to detect critical differences, between the two school levels, in the character of the mathematics taught, including the knowledge criteria and their explicitness for the students, as well as differences in pacing and selection and sequencing of the content. The preliminary data indicate such differences as well as differences with respect to the balance between *instructional* and *regulative* discourse as predicted, but this also seems to differentiate *between* the programmes at upper secondary school.

It was also a preliminary assumption behind the study that school mathematics in compulsory school is more "mixed up" with for example everyday contexts than the mathematics in upper secondary school, thus pointing to the relevance of employing Bernstein's notion of classification. The relation between mathematics and other school subjects is also an issue of concern here. Some observations could be done on this dimension already in the preliminary data reported here.

From the rather small empirical basis reported above, it has been possible to characterise the mathematical work in the classroom in terms of the praxeology that seems to be developing, as well as issues related to the classification and framing of the pedagogic discourse, and thus establish links between the empirical data and the key descriptive terms from the theoretical framework suggested. When analysing empirical data from sequences of mathematics lessons at different occasions and in different classrooms at both school levels, the framework has the potential to support the kind of qualitative comparison aimed at here. For the purpose of the overall transition study, students' experiences of the mathematics taught at the two school levels will be investigated, complemented by theory guided analyses of transcripts from student interviews, curriculum documents, textbooks, and examination tests (such as the national tests). By this theory based approach for comparing two different educational contexts, we hope to avoid projecting categories developed within one context on the other which would run the risk of missing out on some critical differences. However, the approach can only work by remaining sensitive to the dialectic between the theoretical and the empirical (cf. Bernstein, 2000, p. 135).

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RE-EXAMINING THE LANGUAGE SUPPORTS FOR CHILDREN'S MATHEMATICAL UNDERSTANDING: A COMPARATIVE STUDY BETWEEN FRENCH AND VIETNAMESE LANGUAGE

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The aim of the present study was to re-examine the influence of language characteristics on mathematical understanding and performance. Third grade French-speaking Belgian and Vietnamese students participated in the study. In Vietnamese, the number-name system has particular characteristics, which indicate explicitly the zero's position (e.g, 2004 is said "two thousand zero hundred remainder four"). Language had an impact only on the number-name related task and did not affect the tasks involving manipulation of symbols. The relative influence of language, the use of tasks in comparatives studies and the different numerical experiences of children are discussed.

Keywords: language support, cross-national comparisons, syntactic zero, transcoding, relativity language

INTRODUCTION

Even though the Arabic number and its mathematical structure are used worldwide, Arabic code(e.g., 45) cannot be used by itself. It is required to be transcoded in a language to be used, to attribute a meaning to those symbols, which are called a verbal-number (e.g., forty five). Therefore, even if the code is used worldwide, this unavoidable transcoding process is going to induce significant differences on its treatment by each individual, depending on their language and cultural context.

Previous cross-national studies showed that due to a more transparent verbal name system, differences inperformances and errors in averbal-Arabic transcoding task were found (Seron&Fayol, 1994; Lochy, Delazer, Domahs, Zoppoth, &Seron, 2004; Nguyen &Grégoire, 2013). The more efficient way of naming numbers in Asian languages is also an important explanation for the differences favoring Asian children in previous studies between Asian and Western children in certain mathematics skills such as abstract counting(Miller, Smith, Zhu & Zhang, 1995; Miller & Stigler, 1987), mental addition (Geary, Bow-Thomas, Fan & Siegler, 1993; Geary et al., 1996), understanding of the canonical base-10 system and place value understanding (Miura, Okamoto,Kim, Steere&Fayol, 1993). The effect of language differences has also supported the Chinese children to surpass their English and American counterpartsin embedded-ten cardinal understanding (Ho & Fuson, 1998)and in the acquisition and use of ordinal numbers corresponding to their ordinal names (Miller, Major, Shu & Zhang, 2000).

Like other verbal systems of numbers in East Asia, the Vietnamese language possesses a transparent name-number system, in particular the name-number for the teens. The number designation from eleven to nineteen isdone with an addition relationship (e.g., eleven is pronounced “ten one”). However, the Vietnamese language has peculiarities when the digit in the tens or hundreds position is a zero. This is not found in other Asian languages like Chinese, Korean or Japanese. Vietnamese uses the word "zero" as a lexical primitive in the number construction. For example, the Arabic number 3024 is named “three thousand zero hundred two ten four”. This zero is not masked in its verbal form like in other languages. There is however an exception when the zero is in the tens place where it is replaced by the word "remainder". For example, the Arabic number 309 is named "three hundred remainder nine". We can understand the word "remainder" as the rest of the division of 309 by 100. Therefore, for the Arabic number 2009, it is said in Vietnamese "two thousand zero hundred remainder nine".

Depending on the position of a zero in a number, two types of zero were distinguished (Granà, Lochy, Girellid, Seron, & Semenza, 2003). The first type is the lexical zero, as in numbers “520”, which is semantically represented. The second type is the syntactic zero as in numbers “508” or “7014”, which is a production of syntax. The syntactic zero is inserted into the number by application of a rule to indicate a missing value in a position.

According to the characteristic of the Vietnamese language and to the linguistic relativity, the syntactic zero may be easier to manipulate in Vietnamese than in other languages. In our previous study (Nguyen &Grégoire, 2013) we examined the impact of language differences related to the syntactic zero on a verbal-Arabic transcoding task in Vietnamese and French. Results from this study also highly showed the advancement of Vietnamese children in comparison with Belgian children (French-speaking Community). However,the mathematical program in Vietnam is more developed on the large numbers topic than in Belgium. As our task comprised many large numbers, the difference between the two countries may be highly due to variation between the two countries’ mathematical programs.

Previous studies emphasized the role of languages in the understanding of mathematics and in the performance on mathematics tasks but notice here that there is also a limitation of this influence. Nunes (1992) had demonstrated that the developed number name system can restructure mental activity and not involve other basic abilities such as memory and logical reasoning. Results from Miller and Stigler’s (1987) work showed that even if Chinese children were better than US children in abstract counting, there is no difference between the two groups in counting objects. In addition, the difference between children in cross-cultural studies really appears in tasks directly related to language. The Chinese children were more advanced than U.S children in abstract counting but they did not differ in object counting and problem solving, which is more symbol related (Miller, Smith, Zhu & Zhang, 1995). The limitation of language influence on the cognitive representation of number is also remarked in the cross-cultural study of Saxton andTowse (1998). In this study the researchers suggested that the difference between performances of English-speaking and Japanese-speaking children was mainly due to the task instructions.

From the perspective of limitation concerns of language influence on mathematical understanding and performance, Vietnamese children may not differ fromothers in the case of a task indirectly involving a name-number. To better understand the influence of language, tasks specifically related to the syntactic zero were built, not only verbal-Arabic transcoding tasks as in the previous study. Four tasks assessing the competence in comparison, understanding of positional digit in the number, verbal-Arabic number transcoding and analog-numerical-representation into Arabic code transcoding were administered in the present study. In the analog-Arabic transcoding task, the children are asked to produce the Arabic number from a number represented with cubes (e.g., a big cube represents a hundred and a small cube represents ten etc.). Therefore, the children in both countries have the same input code and the support from transparent denomination of the syntactic zero is here masked for Vietnamese children. Having the analog-Arabic transcoding task at the same time with a verbal-Arabic transcoding task allowed us to determine more specific effects of language on the interpretation and performance of children. The tasks in the present study comprised only four digits numbers to avoid the variation of the mathematical programs between the two countries Vietnam and Belgium (French-speaking Community).

We investigated whether the numerical language characteristic involving a zero has also an effect on the tasks involving the manipulation of symbols. Children at grade 3 from the two countries Vietnam and Belgium (French-speaking Community) performedfour tasks: understanding of digit value (Comparison), understanding of positional digit in the number (Digit Identification), verbal-Arabic number transcoding and analog-Arabic number transcoding. It was expected that in this examination of transcoding from verbal into Arabic number, Vietnamese children should have higher performances than Belgian children because of the Vietnamese linguistics' support. But this support would disappear on the other tasks such as comparison, number digit identification or analog- Arabic number transcoding.

METHOD

Two groups of children participated in this research. In Belgium, weassessed 56 children of Grade 3 (26 girls and 40 boys; mean age 106 months). In Vietnam, we assessed 92 children of Grade 3 (51 girls and 41 boys; mean age 105 months). The children were randomly selected. The study took place in the second half of the school year.

Four Paper-and-pencil tasks were constructed with twenty four items for each task. The test was performed collectively in the classroom. Children worked individually without time pressure. The average time to achieve each task was between five and fifteen minutes.

The Comparison task consisted of 24 pairs of natural numbers. Half of the items included two digits equal to zero (e.g. 4500 vs.4050) and the other half consisted of items with one digit equal to zero (e.g.6302 vs.6032). Both numbers of each pair had the same digits and the same thousand. They only varied in the position of the digits, which had the form $\overline{a0bc}$ vs. $\overline{ab0c}$ or $\overline{ab0c}$ vs. $\overline{abc0}$.

The Verbal-Arabic Transcoding task asked participants to write the natural numbers from a verbal form into an Arabic number. There were two categories. The Syntactic Zero Verbal Transcoding category was used for numbers containing a syntactic zero (e.g.36**0**4). Numbers without zero, such as 2146, were used in the Non Syntactic Zero Verbal Transcoding category.

The Digit Identification task included none to two digits of zero (e.g. 2146; 3064 and 2006). Children were asked to circle the digit indicating the tens and the hundreds.There are two categories. The category Zero Digit Identification comprised the numbers with the digit needed to be circled equal to zero (e.g. "Circle the digit indicating the tens in the number 4205"). The category Non Zero Digit Identification comprised the numbers with the digit needed to be circled different from zero (e.g. the number 3264).

In the Analog-Arabic number Transcoding task, children were asked to produce the Arabic number from a number represented with cubes (e.g., a big cube represents a thousand and a small square represents a unit etc.).The same 24 items of the Verbal-Arabic Transcoding task were used for the Analog-Arabic Transcoding task. Notice that item order was changed in the two tasks. The Syntactic Zero Analog Transcoding category was used for numbers containing a syntactic zero (e.g.3604). Numbers without zero, such as 2146, were used in the Non Syntactic Zero Analog Transcoding category.

RESULTS

Overall

Two-way repeated measures analysis of variance (ANOVA) was conducted on the correct response rates with the Category (Comparison, Transcoding Verbal-Arabic, Digit Identification and Transcoding Analog-Arabic) as the within subjects factor and the Country (Belgium and Vietnam) as a between subjects factor. The main effect of Country was significant, $F(1,156)=7.15$, $p=0.008$, partial $n = 0.44$, which means that the scores of Belgian children were higher than those of Vietnamese. The main effect of Category was also significant, $F(3,468) = 21.01$, $p<0.001$, partial $n = 0.119$ implies that performances differed across tasks. The interaction between Category and Country was also significant, $F(3,468) = 6.85$, $p<0.001$, partial $n = 0.042$ which means that the achievement across category differed according to the country. To understand this difference more precisely, we made four one-way ANOVA with «Country» as a between-subject variable, separately for each category.

On the category Comparison and Transcoding Analog-Arabic, there are significant differences between Belgian and Vietnamese children, $F(1,156)=9.78, p=0.02$ and $F(1,156)=18.8$, $p<0.001$ respectively for task Comparison and Transcoding Analog-Arabic. The scores of Belgian children were higher than of the Vietnamese in these tasks. Only on the category Transcoding Verbal-Arabic, however, the scores of Vietnamese children were slightly higher than the Belgian but there is no significant difference between them ($p>0.05$). Also, on the task Digit Identification, children in the two countries did not differ ($p>0.05$). The findings are consistent with our hypothesis that the influence of different mathematical languages changed according to the tasks which were associated with language in different ways.

Table 1: Mean rates and standard deviations of correct responses (in percent) in each category by country

	Belgium		Vietnam	
Category	M	SD	M	SD
Comparison	97.0	8.6	89.2	18.9
Digit Identification	77.9	29.8	71.5	41.8
Transcoding Verbal-Arabic	84.5	26.6	90.4	18.6
Transcoding Analog-Arabic	84.9	26.7	65.1	29.5

The impact of zero in the digit identification task

To investigate the impact of zero in the task Digit Identification, the correct response rates in two categories named Zero Digit Identification and Non Zero Digit Identification were compared. Two-way repeated ANOVA was conducted with the Zero Identification (Zero Digit Identification and Non Zero Digit Identification) as a within-subjects factor and the Country (Belgium and Vietnam) as a between subjects factor. The main effect of Country was not significant, $p<0.05$. The main effect of Zero in the Digit Identification task was significant, $F(1,156) = 18.9$, $p<0.001$, which means the children had more difficulties in identifying the digit of a number when this digit was equal to zero. The interaction between Zero Identification and Country was also significant, $F(1,156) = 9.25$, $p<0.05$, which means that the effect of zero differed according to the country.

To understand this interaction more precisely, two-way repeated ANOVA was conducted, separately for each country. The effect of Zero in digit identification was emphasized more for Belgian children than for Vietnamese. In fact, this effect was significant for Belgian children, $F(1,65) = 15.3$, $p<0.001$, meaning that the presence of a syntactic zero leads to lower scores for Belgian children,but it was not significant for Vietnamese children, $F(1,91) = 1.64$, $p>0.05$.

Table 2:Mean rates and standard deviations of correct responses (in percent) in the categoryDigit Identification

	Belgium		Vietnam	
Category	M	SD	M	SD
Zero Digit Identification	69.7	40.5	70.0	43.6
Non Zero Digit Identification	86.1	26.6	72.9	42.7

The impact of zero on the task Transcoding Verbal-Arabic

Table 3: Mean rates of correct responses (in percent) depending on the type of zero in the category Transcoding

	Belgium		Vietnam	
Category	Syntactic Zero	Non Syntactic Zero	Syntactic Zero	Non Syntactic Zero
Verbal-Arabic	85.3	82.0	89.2	93.8
Analog-Arabic	83.2	89.9	57.4	88.2

Two-way repeated ANOVA was conducted with the Category (Syntactic Zero Transcoding Verbal and Non Syntactic Zero Transcoding Verbal) as a within-subjects factor and the Country (Belgium and Vietnam) as a between subjects factor. The main effect of Country was not significant, $F(1,156)=5.3$, $p>0.05$, partial n=0.033. The main effect of Category in the task Transcoding Verbal-Arabic did not differ significantly $F(1,156)=0.25$, $p>0.05$, partial n=0.002. But the interaction between Category and Country was significant, $F(1,156)=7.4$, $p<0.05$, partial n=0.045, which means that the effect of syntactic zero differed according to the country.

To understand the impact of the syntactic zero more precisely, two-way repeated ANOVA was conducted, separately for each country. The results confirmed that the impact of syntactic zero was not significant for Belgian groups,$F(1,65)=2.41$,$p>0.05$, partial n=0.036 but it was significant for Vietnamese, $F(1,91)= 5.7$, $p<0.05$, partial n=0.059 which means the Verbal-Arabic transcoding with a number containing a syntactic zero is more difficult than without a syntactic zero.

The impact of zero on the task Transcoding Analog-Arabic

Two-way repeated ANOVA was conducted with the Category (Syntactic Zero and Non Syntactic Zero) as a within-subjects factor and the Country (Belgium and Vietnam) as a between subjects factor. The main effect of Country differed significantly, $F(1,156)=13.06$, $p<0.001$, partial n=0.077, which showed that theperformances of Belgian children were higher than of the Vietnamese. The main effect of Category in the task Transcoding Analog-Arabic differed significantly $F(1,156)= 44.77$, $p <0.001$, partial n=0.223, indicating that the category Syntactic Zero Transcoding Analog provided less correct responses than the category Non Syntactic Zero. The interaction between Category and Country was also significant, $F(1,156) = 18.65$, $p<0.001$, partial n=0.107.

To understand the impact of the syntactic zero more precisely, two-way repeated ANOVA was conducted, separately for each country. The results confirmed the impact of the syntactic zero was significant for both countries, $F(1, 65) =4.86$, $p<0.05$, partial n = 0.07 and $F(1,91)= 53.38$, $p<0.001$, partial n= 0.37 respectively for Belgian and Vietnamese samples. The impact of the syntactic zero was more highlighted with Vietnamese children than with Belgian.

DISCUSSION

In the current study, the superior performances of Vietnamese children in a verbal-Arabic number transcoding taskover Belgian children, even when this task comprised only numbers of 4 digits, was consistent as reported by our previous study (Nguyen &Grégoire, 2013) related to an effective number-name system of the Vietnamese language compared to the French. In addition, regarding the results of Vietnamese children in the Digit Identification task, we observed a large variation between them (M=71.5, SD=41.8) and the performances across the Zero Digit Identification and Non Zero Digit Identification were similar. If the Vietnamese children had low performances in this task, they failed to grasp the basics of the positional number system. For Belgians, on the one hand the impact of zero was significant on the Digit Identification task and on the other hand the variation between children was smaller than for Vietnamese. In addition, performances of Belgian children were also higher in other tasks involving the Arabic number production such as analog-Arabic number Transcoding. This suggests that Belgian children's difficulty with this task is only slightly due to a general failure of mastery of positional number and rather more to a specific misunderstanding of zero. Even if Belgian children surpassed Vietnamese in this task, they were more affected by the impact of zero than Vietnamese. Vietnamese children may be able to better detect the role of zero, occupied by a position in the number. It can be explained by the support of a greater transparency in Vietnamese number denomination, where the zero's position is explicit. This result also confirmed the previous studies concerning supports of language characteristics on mathematical performances.

The linguisticrelativity hypothesis was well examined by many studies of Muira and colleagues. Looking at the impact of languages closely, Saxton and Towse (1998) replicated a study of Muira with some subtle changes in task instructions. Results suggested that the impact of languages on the cognitive representation of a number was less direct than previously suggested and that the numerical experience played an important role. Recall that in our previous study (Nguyen &Grégoire, 2013), in the verbal-Arabic number transcoding task with numbers from 3 to 6 digits, Vietnamese children at grade 3 were more advanced than Belgians, particularly with large numbers such as 5 and 6 digits. This difference corresponds to the Vietnamese experience of large numbers in school and daily life. Even if only 4 digit numbers were used in this study to avoid the dissimilarity of experience between the two samples, Vietnamese children's performances were always higher than of their counterparts. However, in the Analog-Arabic number Transcoding task, a large gap between Vietnamese and Belgians was observed, which means Belgians surpassed Vietnamese children. One explanation is due to the manipulation experience with analog-representation numbers of Belgian children. In fact, in Vietnam, this number code is rarely used in textbooks and also for mathematical activities in school. These results showed the limitation of language influences on the less related name-number tasks and emphasised the numerical experience in cross-national performance differences.

It can be questioned what kind of task can measure the impact of languages on mathematical understanding and performances. As Saxton and Twose (1998) had pointed out, the task used by Muira and al. (1994) to attest the support of languages didnot measure this impact directly. In fact, the matching numbers task to evaluate the place-value understanding in this study rather involved analog code (the cubes used to represent units and tens) which connect indirectly with language (Dehaene, 1992). Results from work of Miller etal. (1995) also supported the perspective of language influence limitation regarding these kinds of tasks. Chinese children were better than U.S children in abstract counting but they did not differ in object counting or mathematical problem solving, which revolved more on the relation and manipulation of symbols. Therefore, choosing tasks in order to measure correctly the impact of language in cross-national studies must be done carefully.

The results supported our hypothesis. On the one hand, support of language characteristics was also demonstrated with the superior performances of Vietnamese children in the verbal-number transcoding task. On the other hand, this support was not direct and showed its limitation when the introduced task was less related with the number-name. In the task related to Arabic numbers and semantic representations (analog code) of numbers, the effect of language did not appear. Other factors like mathematical experience and differences in approach to mathematics teaching might be a plausible explanation for the differences between the two countries, which favoured Belgian children. The utilisation of varied tasks was useful to underscore the strong effect of different competences related with numerical language. In particular, it also showed how the performances of children in each country changed due to the relation with number-name on the corresponding tasks. It provided a fuller view about the mathematical understanding and performances in cross-cultural studies. It will be worth, in further researches, to re-examine the appropriateness of tasks which were used in previous studies that concern testing the influence of language on mathematical performances.

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COMPARING THE STRUCTURES OF 3RD GRADERS' MATHEMATICS-RELATED AFFECT IN CHILE AND FINLAND

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Affective factors are a significant indicator of the quality of learning. However, cultural differences in affective factors have not been studied comprehensively. In this report we will present Chilean and Finnish 3rd graders' affective structures regarding mathematics. We identified both similarities and differences between the structures in these two countries. The study contributes to our knowledge of cultural comparison of affect research by extending the comparisons to the Latin American – Nordic axis.

INTRODUCTION

In affect research it is widely claimed that affective factors regarding mathematics play a significant role when learning mathematics (for reviews, see Op 't Eynde, De Corte, & Verschaffel, 2002; Leder, 2006; Hannula, 2011). However, international comparative studies have shown that students from one country have different affective relations with mathematics from those elsewhere (Lee, 2009; Pehkonen, 1995). More importantly, the relations between affective variables and achievement also seem to have culturally specific characteristics (Lee, 2009). To date, some cultural features in affective structures have been acknowledged. However, the focus has mostly been on examining the distinction between Western and Eastern cultures, and not Latin American and Nordic cultures.

In this article we study the affective structures of students in Chile and Finland in order to identify both similarities and differences. In so doing, the study contributes comparative to research in comparative education by examining countries representative of both Latin American (Chile) and Nordic (Finland) cultures. In addition to the cultures being different, the countries also differ in how their students have been performed in international assessments of mathematics, such as in PISA and TIMSS (OECD, 2010; Mullis et al., 2000). Our focus is on young pupils (9-year old), so we will also learn about the possibilities of measuring affective aspects by a questionnaire with respect to pupils as young as that. By doing the study, we aim to go deeper in understanding the cultural features that influence the affective side of learning.

THEORETICAL BACKGROUND

Mathematics related affect has been conceptualized in a variety of ways, for example as attitudes, beliefs, emotions, motivation, values and identity. Important dimensions of different theoretical approaches are 1) the distinction between state- and trait-type constructs, 2) distinctions between social (group level), psychological (individual level) or physiological (biological level) theories of affect, and 3) distinctions between cognitive, emotional and motivational aspects of affect (Hannula, 2011; 2012).

In this article, we look at the relatively stable affective traits of individuals. We are interested in the influence of social factors in the formation of affective traits, but we do not theorize affect as a social construct. More specifically, we are interested in the structure that the cognitive, motivational and emotional traits are forming.

The cognitive dimension - beliefs

In this article, we define the cognitive dimension of affective traits as "mental representations to which it makes sense to attribute a truth value" (Hannula, 2011, p. 43). This is very similar to Goldin's (2002) and Op 't Eynde and others' (2002) definition of beliefs. Argued by Op 't Eynde and others (2002), beliefs become from what is "first told". This means that if they perceive no contradiction with given information, (whether true or false), students tend to accept it as true. Only when contradictions appear do students have reason to evaluate their former beliefs, as well as given information in the light of former beliefs.

Students' mathematics-related beliefs can be structured into beliefs about mathematics as a subject, beliefs about mathematics teaching and learning, beliefs about the self, and beliefs about the social context (e.g., Pehkonen, 1995, Op 't Eynde et al, 2002). Regarding beliefs, we will in this study concentrate on the first and the third aspect, beliefs about mathematics as a subject (difficulty of mathematics) and beliefs about the self (self-efficacy). Both aspects, i.e. self-efficacy beliefs and beliefs about difficulty of mathematics can be construed as mental representations that an individual can cognitively evaluate: a truth value or a justification can be attributed at least by the person him/herself.

Emotional dimension - liking

Here, we define the emotional dimension of mathematics-related affective traits, as "typical emotional reactions to typical situations in the mathematics classroom" (Hannula, 2011, p. 45). To be more exact, the definition refers to how the stable emotive trait is revealed during the lessons: the students have their own typical ways of reacting to the situations that emerge based on their long-term emotional traits. Hence, we are not discussing here the fleeting emotional states that occur, for example, during problem solving.

In mathematics education research, mathematics anxiety is a special case of an emotional trait that has been studied extensively (for a meta-analysis, see Hembree, 1990). Another emotional trait that has been studied comprehensively is liking of mathematics (for a meta-analysis, see Ma & Kishor, 1997). In this study, we are interested in the latter, i.e. the enjoyment students derive from doing mathematics. Many studies show that teenagers tend to not find pleasure in doing mathematics (McLeod, 1992; Metsämuuronen, 2010), whereas primary students have more positive emotions (Tuohilampi, Hannula & Varas, accepted).

Motivational dimension

Motivation research has several theoretical approaches and its use of terminology is sometimes confusing (Murphy & Alexander, 2000). Motivation reflects personal preferences and explains choices and, unlike the cognitive dimension, it is not possible to attribute truth to motives, because they are volitional (e.g. Op ‘t Eynde et al., 2002). The trait aspect of motivation is related to the overall values the person attributes to mathematics and to the general motivational orientations for learning. In this study, we are interested in students’ mastery goal orientation, which is one dimension of their achievement goal orientation (Pintrich & Schunk, 2002).

The structure of affect in Chile and Finland

Although it is generally assumed that there is a relationship between mathematics-related motivation, beliefs, and emotions, the theories of their relationships are fairly recent (Op ‘t Eynde, De Corte, & Verschaffel, 2006). In this study, we consider the affective structure to include a cognitive dimension, an emotional dimension, and a motivational dimension, each influencing the affective structure as a whole. In addition, we are interested in students’ effort in mathematics. These dimensions are of interest, because we think that self-efficacy beliefs (cognitive dimension) influence how students attempt to work with mathematics; goal orientations (motivational dimension) imply students’ initiative in mathematics; emotions frame how students experience working with mathematics; and effort shows their resilience in working with mathematics.

Affective structures have been found to be culturally dependent (e.g. Lee, 2009). In Finland, for example, the structures constitute separable dimensions within older (15-year) students (Lee, 2009). Despite being among the most successful on PISA, Finnish students were also characterised by less favourable results on affective measures. Finnish students lack interest and enjoyment in mathematics, they have below average self-efficacy, and low levels of control strategies. On a more positive note, levels of anxiety were low. The study also revealed that gender differences favouring males in affect were larger in Finland than in OECD on average. (OECD, 2004)

With respect to the affect towards mathematics in Chile, positive affective dimensions are connected to good achievement. Also, a study that used TIMSS data and a survey about students’ affect showed that the following dimensions exist among Chilean 8th graders: *liking mathematics*, *importance of mathematics*, *difficulty of doing mathematics* and *importance of luck and talent in doing mathematics*. Also there were found that Chilean students have an inflated self-perception of their mathematical competence and that students who perceived that mathematics is difficult have lower scores in TIMSS. (Ramírez 2005)

METHOD

Affective traits are typically measured by questionnaires. This is an economic, fairly simple method that is familiar to many students, and is particularly appropriate for measuring established, fairly resilient aspect of examinees’ views (trait aspect)

(Leder, 2006). With respect to affective structures, questionnaires, wherein students report their views in relation to different items, enable the collection of data appropriate for statistical analysis in general and correlations in particular.

The data used in this study were gathered during an on-going research study aiming to improve mathematics learning in Chile and Finland. In Chile, the number of participants was 459, and in Finland 466, giving a total of 925 participants. Data were collected during the academic year 2010-2011: February-March 2011 in Chile and September-October 2010 in Finland. The Chilean school year begins half a year later than it does in Finland, so the research phase, despite such apparent difference in date, was undertaken at the same time in the respective school years.

By means of a survey, 3rd grade students were asked their views on effort, competence, enjoyment, difficulty, confidence and mastery goal orientations with respect to mathematics. The instrument has a long history of gradual development. It is a shortened and simplified version of the instrument used in Hannula and Laakso (2011), which was based on a number of earlier instruments: The Patterns of Adaptive Learning Study (PALS) (Midgley, Maehr, Hruda, et al., 2000); The view of mathematics indicator (Rösken, Hannula & Pehkonen, 2011); The Fennema-Sherman mathematics attitude scales (Fennema & Sherman, 1976).

The results in Hannula & Laakso (2011) suggested that the reliability of the instrument might have been compromised in their younger population of 10 year olds because of the relatively demanding language of the questionnaire items. Therefore, we carefully modified the language to make items easier to comprehend. Moreover, we reduced the number of response options from the original 5 to 3. Some of the items were presented as a direct claim (e.g. “I have done well in mathematics”), while some consisted of an indirect claim (e.g.” I am not very good in mathematics”). The items that had an inverse content were recoded to share the same direction with directly stated items.

Analysis

To find out the structure of students’ mathematics-related affect, we did both exploratory and theory-driven factor analyses. The structure, as well as the similarities and differences between the structures in the two countries were inferred comparing different types of factor solutions.

The initial factor analyses were undertaken separately with both data sets (Chilean and Finnish) using the criterion of all the eigenvalues being greater than 1. This is an explorative approach to factor analyses, as the number of the factors is not predetermined. After that, we forced the number of factors according to the theory underlying instrument construction (Hannula & Laakso, 2011).

Statistical criteria were used to support decisions concerning the number of factors, as well as to justify that the explored solutions were appropriate. We tested the following assumptions (see e.g. Leech, Barret & Morgan 2008): The determinant of the correlation matrix should be more than 0,0001: if this value is close to zero, there are considerable amount of collinearity; if zero, the solution is impossible. The

Kaiser-Meyer-Olkin (KMO) measure should be greater than 0.70, and it is inadequate if less than 0.50. The Bartlett test should be significant (p. < 0.05). The analysis itself was made using principal component analysis with Varimax (orthogonal) rotation.

RESULTS

The structure in Chile

Regarding Chilean students, the non-predetermined solution comprised seven dimensions. The dimensions were *negative self-beliefs in mathematics* (inverse items from competence), *easiness and fun* (1 item from competence, 1 item from EoM), *determination* (MGO + confidence), *effort*, *displeasure* (inverse items from EoM + inverse items from DoM), *confidence,* and *enjoying calculations (*1 item from EoM). Within the Chilean data, the statistical criteria were satisfied: det= 0,005< 0,0001; KMO= 0,801>0,70; p= 0.00< 0,05. These five factors accounted for 54% of variance.

The scree plot suggested a 5-factor solution which produced dimensions that could be easily interpreted. These were *determination* (MGO + confidence), *willingness* (effort + 1 item from competence + direct items from EoM + direct item from DoM), *displeasure* (inverse items from EoM + inverse items from DoM), *competence* (inverse items from competence + inverse item from effort), and *confidence*. This structure is presented below (Table 1), so that it can be compared both with the theory-driven solution and with the 3-dimensional solution from Finnish students. In Chile, there are also dimensions concerning negative perspectives related to mathematics. Further, the structure seems to be more unclear within Chilean students, which was also seen in the reliabilities that were lower regarding Chilean data (see Tuohilampi & al, accepted). Within this predetermined solution, the five factors explained 46% of variance.

The structure in Finland

When the number of factors was not predetermined, the solution for Finnish 3rd grade students consisted of five dimensions that complied with instrument construction. The dimensions were *competence*, *enjoyment of mathematics* (EoM), *mastery goal orientations* (MGO), *effort*, and *confidence*. However, the theory-based dimension *difficulty of mathematics* (DoM) was lost: all the items of that scale loaded on the same dimension as the items of competence. Within Finnish data, the statistical criteria were satisfied, though not thoroughly: det= 0,00005< 0,0001 (still, det≠0, see method part for interpretation); KMO= 0,895>0,70; p= 0.00< 0,05. These five factors accounted for 56% of variance.

After the initial solution, we forced a three dimensional solution as suggested by the scree plot. Again, all dimensions were easily labelled. The dimensions were *capability* (competence + DoM + confidence), *enjoyment of mathematics* (EoM + MGO), and *investment* (MGO + effort + confidence). As this solution did not follow the original theorization, it is reported below (Table 1). In this solution, the motivational dimensions effort and MGO (investment) were united. However, all the items of MGO had correlations also to EoM. Confidence did not build up an own dimension. Its items had correlations with competence and with investment. The relation between confidence and competence could be expected as many of these items originate from the same scale (Fennema & Sherman, 1976). Within the predetermined solution, the three factors accounted for 47% of variance.

Comparison

Among Finnish students, enjoyment was connected to mastery goal orientations. Chilean students' emotions were divided: different types of emotions built up separate dimensions. In both countries, mastery goal orientations were connected to confidence. In the Finnish population effort was connected to confidence and mastery goal orientation. In Chile, effort was connected to emotions and easiness. This suggests that in Finland students' effort for learning mathematics is more independent of their feelings than in Chile. Yet, this may not be a positive thing: if Finnish students go on trying and making an effort in spite of feeling no enjoyment, the two belief clusters (emotions and behavior) might become thoroughly distinct. This may in the future turn up to be a barrier for a change, as the change in beliefs needs an open and irritating conflict (Chapman, 2002). The conflict is not likely, if the clusters are too distinct or if the student has constructed explanations for the conflict (e.g. emotions are not important, enjoyment does not belong to school context, making effort does not give satisfaction).

Within Finnish students, the emotions, confidence, and mastery goal orientations were in interaction. Further, in Finnish students' structure, confidence was connected to competence. Thus, poor achievement (low feeling of competence) is likely to decrease confidence and/or vice versa: low confidence is likely to decrease effort, resilience and thus achievement. Whatever the causation, it is possible that once the confidence decreases, emotions become more negative, and this influences mastery goal orientations, effort, and eventually achievement. In Chile, effort connected to enjoyment. What is more, there was also negative dimension of beliefs connected to emotions and perceived difficulty of mathematics. The Chilean students' connection between effort and enjoyment may be protective with respect to that pattern: effort is an attribute to behavior, so if the Chilean students avoid making much effort when not feeling pleasure, they might avoid facing frustration caused by forced attempts with disappointing consequences.

	Finnish students' structure			Chilean students' structure				
Items	Capability	Enjoyment	Investment	Determination	Willingness	Displeasure	Competence	Confidence
Effort: hard working			,454		,604			

Effort: preparing carefully for exams			,591		,693			
Effort: much working			,579		,630			
Effort: working too little (recoded)			,430		,323		,487	
Competence: not that good (recoded)	,738						,743	
Competence: have done it well	,677				,461			
Competence: not the type who can (recoded)	,718						,615	
Competence: weakest subject (recoded)	,601						,657	
Enjoyment of Mathematics (EoM): enjoy pondering		,669			,414			
EoM: pleasant to calculate		,756			,263			
EoM: has been something of a core (recoded)		,542				,628		
EoM: boring to study (recoded)		,762				,543		
EoM: mechanical and boring subject (recoded)		,677				,677		
Difficulty of Mathematics (DoM): easy	,729				,425			,537
DoM: laborious (recoded)	,538					,627		
DoM: difficult (recoded)	,704					,531		
Confidence: can get good grade	,634							,597
Confidence: can succeed	,353		,497					,531
Confidence: would handle more difficult	,533		,324	,536				,407
Confidence: confident that can learn	,365		,498	,409				,479
Mastery Goal Orientation (MGO) item 2		,468	,482	,501				
MGO item 7		,387	,500	,685				
MGO item 10		,471	,571	,493				
MGO item 12		,382	,472	,680				
MGO item 25		,421	,614	,606				

Table 1. Factors in Finland and Chile in the solution suggested by the scree test.

DISCUSSION

This study suggests that 9-year old students' affective structures might consist of different types of connections in the two countries. Further examinations need to be done with respect to verifying the structures, as well as to further elaborate the interaction and hierarchy within the belief structures. More specifically, we noticed that in the Chilean, but not in the Finnish, population the inversely and directly formulated items tended to load on different factors. This suggests that questionnaires may be sensitive to the linguistic or cultural context. In future analyses we intend to explore this phenomenon further.

Chilean pupils had lower reliabilities and are noticed to be lower in their reading skills (OECD, 2010). Thus the quantitative results may be less well justified regarding Chilean students than Finnish. All in all, this study suggests that 9-year old students' belief structures might consist of different types of dimensions and connections in the two countries, and in Chilean culture, the view of affect might be more complex than what has been measured in most studies (see e.g. Chamberlin, 2010).

In an earlier study, Pehkonen (1995) noticed that the cultural differences between the countries in mathematics related beliefs can be larger than the variation within the countries. In our study we observed variation in the structure of affect which, in our opinion, is much more fundamental difference between countries. However, to know about the differences is not enough; we also have to understand the meaning of the beliefs in different countries and cultures.

NOTES

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG12: HISTORY IN MATHEMATICS EDUCATION

Uffe Thomas Jankvist, Kathy Clark, Snezana Lawrence, Jan van Maanen

Keywords: *History of mathematics; Whig history; interdisciplinarity; early childhood; primary, secondary and tertiary education; teacher education; theoretical frameworks and constructs*.

ABOUT THE WG ON HISTORY IN MATHEMATICS EDUCATION

2013 was the third time that the history working group was part of the CERME program. This time the group had about twenty participants, presenting twelve papers and three posters. The educational scope of the contributions ranges from the use of history in kindergarten over primary and secondary school, upper secondary school, tertiary level, and teacher education. In addition to this, the group also has studies on the history of mathematics education as long as they have relevance for mathematical practices of today, as seen from the main themes in the call for papers:

1. Theoretical, conceptual and/or methodological frameworks for including history in mathematics education;
2. Relationships between (frameworks for and empirical studies on) history in mathematics education and theories and frameworks in other parts of mathematics education;
3. The role of history of mathematics at primary, secondary, and tertiary level, both from the cognitive and affective points of view;
4. The role of history of mathematics in pre- and in-service teacher education, from cognitive, pedagogical, and/or affective points of view;
5. Possible parallelism between the historical development and the cognitive development of mathematical ideas;
6. Ways of integrating original sources in classrooms, and their educational effects, preferably with conclusions based on classroom experiments;
7. Surveys on the existing uses of history in curricula, textbooks, and/or classrooms in primary, secondary, and tertiary levels;
8. Design and/or assessment of teaching/learning materials on the history of mathematics;
9. The possible role of history of mathematics/mathematical practices in relation to more general problems and issues in mathematics education and mathematics education research.

THEMES AND QUESTIONS DISCUSSED DURING THE WG SESSIONS

The presentation of papers and following group discussions were ordered according to five general themes deemed important for history *in* and *of* mathematics education:

i. Interdisciplinarity

ii. Theoretical frameworks in history *of* mathematics education

iii. History in pre high school mathematics education

iv. History in high school mathematics education

v. History of mathematics in teacher education and design

In the following, we list the questions which initiated and/or formed the subgroup discussions of the five themes.

Theme I: Interdisciplinarity

- What is true interdisciplinarity? (e.g., the principles, techniques, frameworks, etc. from one discipline that are used to gain new insights within another discipline.)
- How do we 'measure' the level of interdisciplinarity obtained in a given context?
- To what extent does *interdisciplinarity* (need to) go hand in hand with *cooperation between researchers*?
- What is a good example of interdisciplinary research; and what is a non-example?
- Do we consider a study about mathematics education as interdisciplinary (i.e., between mathematics and the social sciences)?

Theme II: Theoretical frameworks in history *of* mathematics education

- What is the difference between *story* and *history*?
- What theoretical frameworks are available already?
- To what extent does history of mathematics education require the study of primary sources?

Theme III: History in pre high school mathematics education

- What are the special challenges when using history in primary school, kindergarten, etc.?
- How do we stay *true* to history, i.e., non-Whig, when applying history of mathematics at pre high school levels? (Briefly, 'Whig' history may be explained as an interpretation of the past through the eyes of the present.)

- How do we determine the effect of history, as opposed to the use of physical materials/resources or other interventions (e.g., drama, poetry, posters, and presentations)?

Theme IV: History in high school mathematics education

- How far can you 'push' the use of primary sources when using history of mathematics at high school level? What are techniques for doing so?
- If one of the aims of using history of mathematics at high school level is to develop students' mathematical awareness (beliefs, images, etc.) about mathematics as a (scientific) discipline, what is then the best way(s) to describe or maybe even 'measure' such development?
- How do we appreciate the principle of 'authentic practice' (i.e. to have the students act as if they were a 17th century surveyor, or a Roman treasurer?)
- What role can history in mathematics education play in building new mathematical concepts with the students? Are there other specific domains in which history in mathematics education was useful, or can be useful?

Theme V: History of mathematics in teacher education

- In the UK there is an increasing public opinion that the universities should get out of teacher training, and that the teachers should be employed by schools where they will train on the job. If this is the case, what role would or could academic research in the history of mathematics have in teacher training?
- What is the role (from a policy/institutional point of view) of history of mathematics in teacher/mathematics teacher education?
- What lessons can we learn about the engagement of teachers with the history of mathematics and their professional progression for the teacher training?
- What part of cultural/historical/heritage implications does the history of mathematics have in teacher training?

SELECTED OUTCOME OF THE GROUP DISCUSSIONS

In the final session, every subgroup gave a report of its discussion of the five themes and the related questions. Providing a full account of all these subgroup discussions is beyond the possible scope of this introductory report, but in order to illustrate what went on in the WG we shall focus on a few of the themes and questions by drawing in viewpoints and arguments on these from all subgroup reports.

The first is theme II. The reason for including this as one of the general themes has to do with our experiences of sometimes receiving manuscripts, e.g., when reviewing for journals, that seem to report more of a story related to mathematics education, than to conduct an actual historical research study. We are delighted to report that this was not the case of the participants of WG12, which was also reflected by the discussions. For example, there was a consensus about *story* being something narrative, whereas *history*, although it may contain narratives (or stories), is structured by theoretical frameworks, the purpose of which includes being able to see benefits or limitations, to communicate results, and to enable the researchers to organize and present findings, assertions, etc. As examples of such frameworks, the participants point to for example constructs from history research, e.g., those of more externalistic historiography of studying factors crucial to the development of institutions, etc. But in the light of bullet 9, frameworks from mathematics education research of course also play an important role in creating a scene for pointing at possible consequences for modern day practice. As to the role of primary sources, all consider these practically a necessity for conducting history of mathematics education. But one important aspect regarding this is that primary sources in this context can be of various different kinds, including written documents, oral records, textbooks, conference proceedings, etc. This is different from when discussing, for example, theme IV, where the reference to primary sources usually refers to original mathematical texts.

The use of history at high school level (theme IV) is something that has been extensively discussed within the context of using history in mathematics education, not least because students at this level to some degree can be successfully exposed to original sources, even if it is still a challenging task for them. But what about using history in pre high school education, such as primary school, kindergarten, and other early childhood education contexts? An actual reading of original texts at this level is often far beyond reach. The participants point to the fact that in practice when using history at younger age levels there is a need for compromise, also in order to make the mathematics itself more accessible to the children. In particular with very young children there may be the need for narratives in the form of telling stories of mathematics, rather than confronting them with the actual history of mathematics. But as one of the subgroups state in their report: “You have to tell stories, but the knowledge of history enables you to tell *true* stories.” To the question of why one would even bother to go to all the efforts of bringing in history of mathematics to younger aged pupils, another subgroup refers to the discussion of providing *context* in the teaching of mathematics stating that lack of context can have a negative influence on learning and that “history provides that context” which is often needed and welcome.

The above naturally links in with theme V, illustrating that sound knowledge of history of mathematics can act as a valuable resource for teacher practice. But equally important is that history of mathematics has a role to play in mathematics teachers’ professional development – something which was illustrated through a few empirical studies already in the late 1970s and early 1980s. Nevertheless, the frequency with which we come across examples from practice of using history of mathematics in mathematics teacher training is still fairly low. Why is this so? It is an open question. But it is clear that it is related to the matter, as one subgroup mentions, of showing teachers, mathematics educators, curriculum designers, and politicians the benefits and potential of using history of mathematics in mathematics education. How to possibly, and partly, do so is addressed next.

A PERMEATING QUESTION OF FRAMEWORKS AND CONSTRUCTS

One topic or question which permeated many of the other discussions and to which we found ourselves returning again and again, is that of which frameworks, theories, or theoretical constructs from mathematics education research may apply best to the various uses of history of mathematics in the teaching and learning of mathematics. The challenge of conducting studies within the scope of WG12 is to find a balance between the three fields: that of the *history of mathematics*, *mathematics*, and *mathematics education* (*research*). This requires knowledge of all three disciplines, often making such studies a relatively demanding task to undertake. For 'outsiders', e.g., math educators who are not as familiar with the history of mathematics, we need to be able to provide convincing arguments for wanting to resort to history in the teaching and learning of mathematics. A sensible way of doing so is to argue by means of theoretical constructs from mathematics education research and to rely on suitable mathematics education frameworks for analyzing data, presenting and discussing results, etc. For 'insiders', who are familiar with history of mathematics, it is important not to be unintentionally anachronistic (or 'Whig') when including history in the teaching and learning of mathematics. From an educational point of view, this is important if having as a goal to foster historical awareness with students. From a research community point of view, it is important if we want to maintain our integrity and strengthen the connections with the research historians of mathematics.

EVALUATION AND ASPECTS TO CONSIDER FOR THE NEXT WG

In accordance with decisions made at CERME-7, more time was allocated to poster presenters during the WG sessions of CERME-8. More precisely poster presenters gave short presentations of their posters in the WG *before* they presented their posters in general. This initiative seemed to function well, and we plan to repeat it again. As always, the history group at CERME works to maintain very close connections to the HPM group, not least within the leading team. As new initiatives for CERME-9, we have in mind to broaden the 'bullets' in the call for papers to also encompass studies related to epistemology of mathematics in relation to mathematics education and the use of philosophy of mathematics in the teaching and learning of mathematics.

PAPERS PRESENTED IN WG12

Alpaslan, M. & Güner, Z.	Teaching modules in history of mathematics to enhance young children's number sense
Bayam, S. B.	Students' views about activities for history of mathematics included in mathematics curriculum
Bjarnadóttir, K.	Arithmetic textbooks and 19th century values
Clark, K. & Phillips, L. G.	"I was amazed at how many refused to give up": Describing one teacher's first experience with including history
Jankvist, U. T.	The use of original sources and its possible relation to the recruitment problem
Kaenders, R., Kvasz, L. & Weiss-Pidstrygach, Y.	History of mathematics as an inspiration for educational design
Kotarinou, P. & Stathopoulou, C.	The history of 5th postulate: Linking mathematics with other disciplines through drama techniques
Krüger, J.	The power of mathematics education in the 18th century
Krüger, J. & van Maanen, J.	Evaluation and design of mathematics curricula: Lessons from three historical cases
Lawrence, S.	Making sense of Newton's mathematics
Mota, C., Ralda, M. E. & Estrada, M. F.	The teaching of the concept of tangent line using original sources
Tsiapou, V. & Nikolantonakis, K.	The development of place value concepts to sixth grade students via the study of the Chinese abacus

POSTERS PRESENTED IN WG12

Moeller, R. D. & Collignon, P.	Calculus and applications – Learning from history in teacher education
Monteiro, T. M.	Ideas about modern mathematics and teacher trainees at Liceu Normal de Pedro Nunes (1957-1971)
Navarro, M. & Puig, L.	Facets of the presentation of the Cartesian coordinate system in Euler's *Introductio in Analysin Infinitorum* and Lacroix's textbooks

TEACHING MODULES IN HISTORY OF MATHEMATICS TO ENHANCE YOUNG CHILDREN'S NUMBER SENSE

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This study aims to investigate potential effects of two teaching modules in history of mathematics in order to develop number sense in young children aged 5. We reanimated two historical artifacts as using tally marks and pebbles for counting activities within symbolic play context. Data were obtained by anecdotal records of the implementation of the modules at the fall of 2012-2013 academic year. Data analysis was made through Dunphy's framework for number sense in young children. Findings revealed that the modules could contribute to various aspects of number sense. We finalized with a discussion about the findings and implications for further research on using history of mathematics with younger children.

INTRODUCTION AND THEORETICAL FRAMEWORK

Number is one of the main concepts for primary mathematical reasoning in young children. They exhibit different kinds of behaviours and skills as a sign of their development in number sense. According to Charlesworth, Lind and Fleege (2003), children are able to relate number of objects by counting, to compare number of objects, and to master benchmarks as five and ten. Children aged between six and eight can make one-to-one correspondence, meaningfully count, and notice cardinal and ordinal numbers in a set of objects (National Council of Teachers of Mathematics [NCTM], 2000). The most basic skill may be rational counting, which requires mastering one-to-one correspondence, arranging names of numbers in exact sequence, being able to continue counting from a predetermined number, and perceiving total amount of objects in a set (Reys, Lindquist, Lambdin, Smith, & Suydam, 2001). Four year olds are initially able to transfer knowledge of rote counting into skills of rational counting (Charlesworth et al., 2003). Another issue is that children should be aware of numbers within verbal and written communication as a part of social life in community (Tolchinsky, 2003). Indeed, it is possible for them to initiate and improve number sense through daily life experiences (Dunphy, 2006b) such as pulling on one glove for each hand before starting to play snowball. Such experiences should be supported by parents and early childhood practitioners since communication with relatively intellectual people builds on children's learning (Vygotsky, 1978).

In Turkey, young children enrol in primary school at six years old. Thus, early childhood education given before this age seems to be crucial for developing number sense so as to meaningfully learn mathematics concepts in primary school. For this purpose, relevant objectives are set in Turkish pre-school curriculum (Ministry of National Education [MoNE], 2012) as to count objects given in a set, to rhythmically count on and back, to determine ordinal numbers in a set, to group objects in benchmarks such as five, and to make one-to-one correspondence between objects. Numbers up to 20 can be used in classroom activities. The curriculum also emphasizes concepts of equity and numerical superiority/inferiority regarding number sense. Though the curriculum sets a variety of objectives, developing number sense in young children does not seem to be an easy and straightforward task. For instance, children are likely to get confused about meaning of cardinal and ordinal numbers of toys in a set (Cooke, 2005). They might have difficulties in one-to-one correspondence such as counting up to seven objects in right order by simultaneously touching some of them more than once (Charlesworth et al., 2003).

It is generally presumed that adventure of mathematics was initiated by different groups of human beings with separating *one* and *many* to get control of their environment for survival and to have a better quality of life. (Boyer & Merzbach, 1991; Burton, 2006). Such needs make them to keep records of concrete objects in daily life. The records connote concepts of number and equity, which was a common abstract property of concrete objects, through one-to-one correspondence (Boyer & Merzbach, 1991). Necessity of numbering plenty of objects makes ancient humans to group these objects in various representations in order to easily count and recognize the product (Boyer & Merzbach, 1991). According to Burton (2006), using *tally marks* and *pebbles* are two of such artifacts around 30.000 B.C. The oldest tally marks are found in today's Czech Republic as 55 parallel line segments drawn on a wolf's bone in groups of five (Cooke, 2005). Pebbles indicate small stones that were round and smooth in shape.

There is an expectation for the re-emergence of certain obstacles and difficulties encountered in the evolution of mathematics while learning mathematics (Jankvist, 2009). We can take advantage of sources in history of mathematics for the aim of preventing problems that might occur in understanding mathematical concepts (Sfard, 1994). In history, human ancestors initiated mathematics by manipulating concrete materials rather than a direct usage of abstract number symbols. It is possible to find an agreement between ancient humans and present-day young children in respect of being mentally ready for learning number concept. It is our opinion that if young children, who are at the beginning of their mathematical account, are instructed through early historical artifacts that used materials as alternative and simple kinds of representation while dealing with numbers, they might overcome possible obstacles/difficulties hindering number sense acquisition. Moreover, history of mathematics can be employed for keeping motivation and engagement high, revealing the human and cultural face of the subject through storytelling and pictures with young children (for details, see Tzanakis & Arcavi, 2000). In order to investigate our claim that historical teaching modules might enhance young children's number sense achievement, we set the following two research questions:

- To what extent do the historical teaching modules have potential effects for developing number sense in young children aged 5?
- In what particular sense do the historical teaching modules have potential effects for developing number sense in young children aged 5?

In an effort to include history of mathematics in mathematics education, there are mainly three different ways: (1) *illumination approaches* in which historical factual information was presented as supplementary to routine mathematics classes, (2) *modules approaches* requiring mathematics instruction through specific cases based on history, and (3) *history-based approaches* meant mathematics courses wholly built on historical perspectives (Jankvist, 2009). In this study, the *modules approaches* kind is adopted since we focus on the process of young children's number sense acquisition through historical materials within two class periods. It may also be characterized as Tzanakis and Arcavi's (2000) *historical package* in which a mathematical topic from the curriculum of interest is taught by means of historical materials in a relatively short period of time.

METHOD

Design of the Study

In this study, we aim at making an in-depth analysis of the learning process in the historical modules. We carefully observed this process occurring within a natural setting, and took anecdotal records. Hence; it can be characterized as an observational case study (Bogdan & Biklen, 1998).

Participants

The study was conducted in a pre-school classroom in Ankara, Turkey. The participants consisted of 12 children (five females and seven males) who were aged 5 years. The children would enrol in primary school in the next academic year.

The children had some background knowledge of numbers which was gained during the pre-school education. They were expected to have the abovementioned knowledge and skills that were defined by MoNE (2012).

The Historical Teaching Modules

The two modules were based on different stories made up by the researchers to enable the children engage in the two historical artifacts including tally marks and pebbles. The modules were implemented in a classroom environment where children could feel comfortable. One of the researchers showed pictures related to the stories and guided the children while working on the tasks. The other fluently read the stories and dramatized certain parts in them. Both of the stories included words and/or phrases such as *centuries ago*, *in very old times*, *times when technology was not invented* to emphasize that the events occurred in past times. As for the teacher, she was in the classroom as a non-participant observer during the implementation of the modules. The modules were conducted in one class hour (30 minutes) within two consecutive weeks. In the following paragraphs, we explained how the modules were conducted.

The first module was named as “Retired Island”. Black and white pictures about the story (referred to old times), wooden sticks, coloured papers and crayons (to draw tally marks) were used as concrete teaching materials. The children were gathered around a table to easily see the pictures and to record the amount of wooden sticks on their papers. Before getting started, transition questions such as “Have you ever gone to sea?” and “Have you ever seen an island?” were asked to prepare the children to the content of the story. After taking the answers, a world map was shown to tell where the story happened. Some other pictures of a ship, storm in sea, a retired island, Captain Jack, a bag on beach, timbers, and a boat were showed hand in hand with the story. After the story was read, it was explained that the children should help Captain Jack so as to count the number of timbers for constructing his boat to escape the island. Additionally, it was stated that the captain determined to record this work in his diary. The children were asked for how to count the timbers (seven wooden sticks). They were invited to draw tally marks on their paper for recording the number of wooden sticks in the meantime. After this task, it was announced that the boat sunk due to the lack of timbers. The number of wooden sticks was increased to 10. A discussion was created about how to notice the higher number of timbers in the record this time, and the children were guided to feel need of grouping the tally marks.

“A Shepherd in Kuka Tribe” was the name of the second module. Materials employed in this module were black and white pictures about the story, sheep pictures and pebbles. The children sit in a half circle position on floor to provide interactivity. At the beginning, they were asked some transition questions such as “Have you ever seen a forest?” After sharing their experiences, a world map was put on floor to show where the story occurred at this time. During the story, some other related pictures as rain forest, a mountain, sheep, a house and a shepherd were also shown. When it finished, the children were invited to help the shepherd so as to count and record his sheep. They were initially asked for how to count and shared their ideas. After that, nine sheep pictures and pebbles were separately placed on floor. The children were expected to count and record the number of the sheep using their own ideas. They also invited to group the sheep and corresponding pebbles in different ways.

Finally, the children were asked some open-ended questions related to the objectives of two modules.

Data Collection and Analysis

In order to collect data, ethical approval document was provided from Research Center for Applied Ethics in Middle East Technical University in Ankara. The preschool administration gave permission for conducting the study with one group of five year olds after being informed about ethical considerations.

Data were collected through anecdotal records of the implementation of two modules at fall semester of 2012-2013 academic year. Anecdotal records were taken for the children's observed behaviour and speech during the implementation.

Data analysis was made considering Dunphy's (2006a) four aspects for young children's number sense, which she introduces in her comprehensive dissertation. She sets a framework regarding number sense in young children who were around five year olds. The framework has four aspects as *pleasure and interest in number*, *understandings of the purposes of number*, *ability to think quantitatively*, and *awareness/understanding of numerals*. Children's *pleasure and interest in number* is their disposition towards practices regarding numbers such as being motivated for games that can be played with numbers. *Understandings of the purposes of number* refers to having some idea about the reasons behind humans' need for numbers, for instance, being interested in why numerals written on home telephone. As for *ability to think quantitatively*, it addresses "to count, subitize, and estimate, and to relate numbers to each other" (Dunphy, 2006b, p. 58) which can be exemplified with proficiently utilizing concrete models for counting. Lastly, *awareness/understanding of numerals* denotes knowing language of numbers (i.e., symbols corresponding to numbers). For example, children should notice numerals written in an elevator for going upstairs home. This framework can be employed for both describing young children's existing number sense and revealing their relevant learning in progress (Dunphy, 2006b).

FINDINGS

From this preliminary study, we explored findings addressing certain aspects of Dunphy's framework to a certain extent. They were delineated in the subsequent subtitles and summarized with respect to aspects, themes, related behaviours and sample quotes (in which R means 'researchers' and C is 'children') in Table 1 as Appendix.

Positive Influence on the Pleasure and Interest in Number

It was remarkable that the children were quite interested in the historical modules. Both of the stories supported by black and white pictures seemed to take their attention and kept them silent. It can be asserted that storytelling made them mentally ready for the actual tasks. The stories, which gained inspiration from needs for counting and keeping records in history, made them engage in the tasks. In addition, concrete materials as pictures, tally marks, wooden sticks, pebbles, and sheep cards also aroused an interest by visualizing the number related tasks.

In the first module, the children loudly counted the wooden sticks one by one and drew corresponding tally marks at the same time. It is possible to state that being actively involved in drawing tally marks by a hands-on experience kept their interest alive in performing the task. At the end, the children were asked for what they liked and/or disliked in the module. It was observed that they were impatient to answer the questions. Eleven children noted that they liked and particularly underlined *drawing tally marks*, *seeing pictures*, *counting*, and *saving Captain Jack*. One of them expressed that she did not agree because of studying on the desk and she added that *play* was the favourable part within this module.

Saying that we had another story in the second module made the children be silent. One of them explicitly stated his wonder. Such behaviours might be a result of their positive experiences in the first module. They were also concerned with pebbles, which were not yet used in the previous activities, and appeared to be impatient about using them. After the story ended, there was a pretty chaos among the children for doing the task with sheep and pebbles. This time all of the children pointed out that they liked the module and then highlighted the following activities and elements: *using stones*, *seeing pictures*, *sheep*, *story*, *putting pebbles for each sheep*, *asking questions*.

Conscious Understandings of the Purposes of Number

In both of the modules, we aimed to contribute the children's understanding related to purposes of numbers through stressing that we as human beings need numbers. The related discussions followed each of the modules. The children concentrated on the need for numbers both in general and in conjunction with the modules' context. They emphasized the underlying reasons for mathematics in daily life activities and technological developments together with those who use mathematics in that manner. In these discussions, it was remarkable that the children included *building ships* to the common answers such as *making inventions*. Among those doing mathematics, they added *students* and *workmen*, which possibly referred *them as helpers* and *Captain Jack and the shepherd* respectively. Herein, we distinguished that the children believed to make a worthwhile task and they felt some self-efficacy in mathematics. The modules may show an idea that mathematics is for everyone and everyone can do/study mathematics.

Potential for Fostering Quantitative Thinking Abilities

We intended to stimulate the children's perceptions about one-to-one correspondence, equity, cardinality, ordering and grouping through using concrete historical materials.

The children successfully matched seven wooden sticks one by one with tally marks in the first module. They drew the tallies in parallel with counting up to seven. When we increased the number of the wooden sticks to 10, we guided the children to group the sticks and asked for the possible reasons behind grouping. They presented the corresponding arguments such as "To make them five." and "To separate them."

They finally decided to group the 10 sticks in three and considered the last one as the rest.

The second module actually proposed to provide an opportunity for articulating the identical mathematical notions of the first. This time it is possible to state that the children did somewhat illuminated mathematics. The mathematical work was more meaningful because they seemed to thoughtfully recall the ideas of one-to-one correspondence, equity, ordering and grouping. To illustrate, when we asked what to do if the shepherd has many sheep (by showing nine), some of them suggested to bring the half to his house. Then they immediately did not find this idea to be feasible and asked as “Shall we divide?” This question brought them to the idea of grouping at the end. We dug this elegant idea and questioned how it was possible to group the nine sheep of the shepherd, they replied as “We drew a line at the centre [addressing the first module]. We may group in five?” After noticing that there was an asymmetry with putting five pebbles in a linear order and four pebbles under that order, they changed their mind as: “In three!” Regarding this new arrangement, they forwarded that the regularity came from the multiplication of three by itself.

DISCUSSION AND IMPLICATIONS

This study indicates that teaching modules in history of mathematics have some potential to assist development of young children’s number sense under the three aspects identified by Dunphy (2006a) as *pleasure and interest in number*, *understandings about the purposes of the number*, and *ability to think quantitatively*. Regarding the first, telling stories, showing old pictures and using concrete materials linked with the two historical artifacts seemed to make their attention alive. Smestad and Clark (2012) also points out that using history of mathematics can arouse interest and mathematical thinking in younger children through storytelling and multiple representations. Herein, we believe that history inspires educational designers to make up meaningful authentic stories that our antecedents has experienced in the paths of mathematical knowledge. What is more, the children may be attracted by simple but original historical materials that have never used before in their classes. As for the second, the children’s explanation for the need of mathematics in daily life might be caused by the content of the two stories stressing the uses of mathematics. In the third, concrete historical materials may make the children’s learning more meaningful in the same manner that children at early ages need to learn by concrete, visual and hands-on experiences (Hohmann & Weikart, 1995). Furthermore, probing questions derived from the necessary skills (e.g., one-to-one correspondence) in the historical artifacts could have catalysed the children’s quantitative thinking based on the idea that reflections of thinking with concrete materials helps quantitative thinking and understanding (NCTM, 1989). Lastly, recording the results through the historical materials appeared to enable them not to get confused during this thinking process. The modules did not seem to influence *awareness/understanding of* *numerals* since they did aim to study mathematics verbally rather than to enhance the children's recognition with abstract number symbols.

The teaching modules could be characterized as practical in terms of the concrete materials used and the time spent. Conducting the modules required materials which were easy to find and cheap. The modules also took reasonable time in terms of implementation which was an issue of concern in the field of using history of mathematics in mathematics education (Tzanakis & Arcavi, 2000). In other words, the stories inspired by the historical artifacts did not appear to steal from the children's learning about numbers. On the contrary, they seemed to mentally warm up the children to learning about numbers and endured their motivation during this process.

According to Carraher and Schliemann (2007), early years are critical and important for young children to gain number sense due to the fact that it took long time. In this sense, early childhood practitioners as implementers of early years' mathematics should be trained on how to use history of mathematics as an alternative way for developing similar mathematical understandings in young children. Accordingly, further research should be made on how the history could be an inspiration for developing similar teaching practices.

NOTES

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Table 1: Summary of the findings with respect to aspects, themes and quotes

Aspects	Themes	Related Behaviours	Sample Quotations
Pleasure and Interest in Number	Interest (I) Pleasure (P)	• Being concentrated on black and white pictures (I) • Keeping silent while listening stories (I) • Active engagement in tasks (I, P) • Loudly counting wooden sticks (I, P) • Giving impatient answers (I, P) • Noting their likes and dislikes (P)	---
Understandings of the Purposes of Number	Need	• Stating that mathematics is necessary • Exemplifying everyday uses of mathematics • Expressing who use mathematics	R: Why do we use mathematics? C: Humans invent something with mathematics. R: What kind of things? C: They use it in electricity, everywhere. They construct ships. R: Who do/use mathematics? C: (1) Students. (2) Workmen.
Quantitative Thinking Abilities	Correspondence (C) Equity (E) Grouping (G) Ordering (O)	• Matching wooden sticks with tally marks / pebbles with sheep (C, E) • Drawing tallies / putting pebbles together with counting in order (C, E, O) • Grouping sticks / pebbles (G)	R: Why did you make in this way? (by showing nine sheep matched with nine pebbles) C: (1) They will be equal. (2) One pebble is put for one sheep. R: Why? C: There will be one pebble on each of the sheep.

STUDENTS' VIEWS ABOUT ACTIVITIES FOR HISTORY OF MATHEMATICS INCLUDED IN MATHEMATICS CURRICULUM

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The mathematics curriculum proposed by The Ministry of National Education, was re-organized within the scope of the study in the experimental group and famous mathematicians and historical anecdotes that played a role in discovery or development of the attainment were given to students (aged 12) as performance projects. The teaching periods of 8 attainments were supported with 24 mathematicians and historical anecdotes during 4 weeks. In the study students' opinions were asked, the qualitative data were obtained via semi-structured interview form and descriptively analysed. Students expressed a positive opinion about the contribution of the method used to affective and cognitive dimensions.

INTRODUCTION

Content and topics in mathematics textbooks are usually without historical background. Through this kind of mathematics education intended for only right answers in the exams, it is imposible to devolop the mathematical thought. In this aspect, my intention is not to teach the history of mathematics, but to study how we can use historical material in the attainments for meaningful mathematics education. There is no use of history of mathematics in attainment of mathematics curricula in Turkey (TTBK, 2009). In this study attainments were supported by history of mathematics.

In addition to enjoyable examples from the history of mathematics and solving problems in several ways suggested by mathematicians in different periods, biographies of ancient and modern mathematicians motivate pupils (Gulikers & Blom, 2001; Furinghetti & Radford, 2008). However, the motivation of teachers is important and a historical approach to mathematics enables the attitude of teachers to develop (Kin Ho, 2008). History is a prerequisite for comprehensive mathematics education (Haverhals & Roscoe, 2010). In this context, I believe that there is a need to raise interest in mathematics and awareness of importance of mathematics for science by using history of mathematics.

MATERIALS AND METHOD

The working group included 24 sixth grade students, (12 female, 12 male), from a rural school in the village of bolu, Turkey in the 2011-2012 academic year. All students were 12 years old. There was one mathematics teacher at the school. The school, where the researcher teaches, was determined as the research school. Thirteen of 24 students were interviewed. The interviews were recorded. The duration of interviews varied from 5 to 8 minutes and they took place over one week.

While preparing interview questions, the factors of attitude test applied for preliminary test and postliminary test were used (Nazlıçiçek & Ertkin, 2002). In line with factors of this test, an interview protocol was designed. The four interview questions were:

- What are the negative things in the class in which the history of mathematics has been used and in the presentation of performance prejects? What are the different and good things in the issues mentioned above?
- Do you think that using the history of mathematics in the class and you presenting this as a performance project had an effect on your learning mathematics or your understanding of mathematics better?
- It is required to think of mathematics class of the last year. Did you like the mathematics class? Why? Have the activities about the history of mathematics been helpful for your liking mathematics? How?
- Do you think mathematics class is difficult or easy? Has the history of mathematics made the class easier or difficult? Why?

Pilot application of interview questions was carried out with four of the seventh grade students who had a similar application about history of mathematics in the past academic year. According to feedback after the pilot application, interview questions were modified.

In Turkey there are five learning areas in primary school mathematics curriculum, which are number, geometry, algebra, probability, and measurement. This study covers eight attainments (i.e. objectives) in the learning areas of algebra, numbers, geometry and probability (TTKB, 2009).

The educational environment based upon these concepts was designed under the leadership of the researcher by giving students of working group performance projects about historical development of thema. In line with the practice, the curriculum suggested by The Ministry of National Education was reorganized with the history of mathematics. Appendix-1 shows a part of the reorganized curriculum.

The performance projects were presented by students as a song and a poem written for a famous mathematician, a drama about discovery or invention of an attainment performed in the classroom, and a poster with information about historical development of the subject or mathematician. The contents of the performance projects were evaluated with a rubric developed by the researcher. Reinforcement practices were made under the guidance of the researcher within the remaining time allocated for the attainment.

The research participant group had four-hour mathematics lessons, which were divided in half on two different days during the week, for four weeks total. In the first hour of the lesson, three students presented their performance projects for an attainment and the researcher evaluated them with the rubric. In the second hour of the lesson, the problematic situation of the attainment was pointed out and the mathematicians' reasons developing or contributing to the subject mentioned in the attainment were discussed. On the second day, the same practice was made and six performance projects were introduced to class during each week and 24 performance projects about history of mathematics were introduced to class during the four-week practice.

The students were encouraged to benefit from different resources appropriate for their level and a mini-library consisting of mathematics and history of mathematics books was built and made availabe. Two of the students' presentations are explained below as a examples.

Firstly, a student, who made a presentation about Ali Kuscu in relation to the attainment of "Makes a prediction based on data.", introduced a model explaining the construction of a simple astrolabe to his/her classmates. An instruction explaining the construction of a simple astrolabe was given out. The predictions and deductions, which Ali Kuscu made about sky by using the astrolabe, were discussed. After the presentations about Ulugh Beg and Omar Khayyam, a drama including the characters of Ali Kuscu, Ulugh Beg and Omar Khayyam was performed by students studying the same attainment.

Secondly, the term "to construct" was used during the introduction of mathematical objects. Before dealing with the attainments in the geometry learning area, it was thought that geometry needed to be constructed and it was decided that one of the performance projects was the birth of geometry. In the drama performed by students, one student represented the Nile River and the Egyptian people, who experienced flooded lands, determined the boundaries by asking priests for help and after ebbing they redetermined the boundaries with measuring instruments. After these experiences, students' views were asked.

Here, a semi-structured interview method was used in order to examine in-depth views of students about using history of mathematics in mathematics education. The purpose of the interview technique is to find out the feelings, views, and beliefs of an individual about the study (Çepni, 2010). Semi-structured interview integrates fixed alternative questioning into looking in depth in the related area (Büyüköztürk et al., 2010).

The data of the study were descriptively analysed. Hence, in the descriptive analysis, the views of individuals interviewed or observed are frequently quoted in order to reflect their views dramatically. The objective of the analysis is to present findings to readers in an organized and interpreted way (Yıldırım & Şimşek, 2001).

The recorded interviews were transcribed. Each student's responses were separately examined. Data were generally categorized according to the responses. After reviewing the categories, sub-categories, which would cover all data, were determined. Some of the categories, which would cover the codes, were combined. Sub-categories covering one another or substituting for an other, were combined by taking the expert opinion. All participant responses, which they gave to the same questions, were included in the analysis. Therefore, there is a difference between frequency and the number of participants (Çepni, 2010).

FINDINGS

Student views about history of mathematics themed lessons in affective dimension

The perceptive dimension of student views was examined according to their positive and negative. Eighty (95%) out of 84 answers in the emotional dimension were positive and four (5%) were negative nature. It can be said that the history of mathematics themed activities effect students positively in an emotional dimension because positive answers were more frequent than negative answers. Detailed themes of positive answers about the method used are given in Table 1.

Table 1: Positive student views about history of mathematics themed lessons

	Frequency	**Percentage**
To like learning about mathematicians' lives	11	14
To be a centre of attraction	1	1
To make models of inventions about attainment	7	9
To like using poem, song and drama	25	31
To learn by having fun	9	11
To think it will benefit future	1	1
To like mathematics more	9	11
To be curious	3	4
To be influenced by a mathematician's life/ to aspire to it	6	8
To participate actively in the learning process	5	6
To overcome the fear of mathematics	3	4
Total	**80**	**100**

Student Medine underlined that her mathematics awareness of the external world developed with the activities.

> Student Medine: Math is also related to play and poems. It spices. Nature is based on math.

Students Eren and Semiha said that they enjoyed at the lessons.

> Student Eren: Lessons are really comical and fun.

> Student Semiha: We performed several plays and at the same time covered several subjects. I had a lot of fun.

When positive student responses are taken into consideration, it can be said that they liked using the forms of poem, song, and drama in history of mathematics-themed lessons. The students had knowledge of previous studies on the attainment and enjoyed the experience in the mathematics lesson.

The other frequency in the emotional dimension is “to like learning about mathematicians’ lives” with 11 responses (14%). If this dimension is thought to be the second important sub-category, it can be said that the deficiency of mathematical acculturation was filled. Student responses support this finding. Students Seda and Emre expressed their views about the subject as follows:

> Student Seda: While doing projects, we are learning mathematicians’ lives. I like this much more. After learning about their lives, I liked math more. I enjoyed learning about their lives. I think we should do many more things about history of mathematics. We should learn about their lives.

> Student Emre: We are learning about a mathematician’s life. We are learning how difficult it was in finding their math theories and we have fun while learning these things.

When student answers and the corresponding their percentages were examined, it can be said that students learning how to discover the subject in the attainment and who contributed to the development of this subject with the play-like activities provided positive student views about mathematics by exceeding the traditional mathematics. These findings are parallel with student views in the study which Lit, Siu and Wong (2001) conducted with a similar experimental subject. Students in that study stated that, they liked the lessons which were entertaining, different, not boring and made understanding easier. In the same direction, students in the study about re-invention of geometry, which Gulikers and Blom (2001) condusted, emphasized that they felt more competent in mathematics and said that they overcame their fear of mathematics and learned while having fun.

When student views and their percentages within the sub-category were taken into consideration, it can be said that students liked mathematics more with the history of mathematics activities and that they overcame their fear of mathematics. The students expressing no negative opinion about this finding reinforced this notion. Similarly, Kin Ho (2008) collected student views about lessons where history of mathematics was covered. Positive feelings were reported, such as belief, interest, trust, and determination in students’ views.

There were four negative student responses about history of mathematics themed activities. Two (50%) out of the four responses said that performance projects of the history of mathematics activities were difficult. However, the students said that they had difficulty while preparing the project but after some time they succeded.

One (25%) of the students' negative responses stated that s/he did not like the presentations of his/her friends.

Student views about history of mathematics themed lessons in the cognitive dimension

The cognitive dimension of student views were analysed under two themes that were the effects of history of mathematics themed lessons on learning and the difference of the method used. The effects of the method used on mathematics learning were analysed in Table 2. Nineteen student responses (38%) showed that the students understood mathematics better and learned it better with the history of mathematics themed activities. Student Semiha expressed that her perceived success for mathematics increased.

Student Semiha: This year my math lessons are better than the ones in the last year and I increased my grades. I tried to practice some stuffs by learning about mathematicians' lives. I started to chew on my lessons more.

Table 2: The effects of history of mathematics themed lessons on mathematics learnings

	Frequency	**Percentage**
To provide foreknowledge	4	8
To learn/understand better	19	38
To learn about mathematicians' lives	9	18
It did not have any effect on my learning	1	2
The lessons got easier	16	32
The lessons got tougher	1	2
Total	**50**	**100**

As seen in Table 2, another important dimension of this sub-category is that the lesson was thought to be easier by the student with the activities including history of mathematics. 16 students (32%) gave this answer. Student Medine and Mahmut expressed their feelings in this regard.

Student Medine: The lesson got easier when historical figures were included.

Student Mahmut: The lesson was not tough because I understood better.

When student answers and their percentages were taken into consideration, it can be said that history of mathematics themed lessons increased the ratio of students' understanding of lessons. Also, students said that history of mathematics made lessons easier by enabling better understanding of mathematics and providing foreknowledge about a lesson can be evaluated as a functionality of the method.

These findings correspond to the study of Haverhals and Roscoe (2010). Who anticipated that the students' academic success would increase because they faced the problem in person by saying that the method they used was the heart of problem-solving.

Another sub-category of the cognitive dimension is the different aspects of history of mathematics themed lessons. While 4 (45%) out of 9 students said that they found the method used different and thought the use of poem, song and drama was different, 3 (33%) out of 9 students underlined that knowing how the attainment was discovered did not look familiar to them.

> Student Hakan: We were learning different things. For example, I masqueraded as sheep. I performed in Tuba's play. A wolf came in. When she came home in the evening, she realized it. In the morning of the next day she understood it by notching and found numbers.

It is clear that a different method used in the mathematics lessons got the attention of students and aroused interest in them. In the same way, Taşkın, Yıldız, and Arslan (2010) took postgraduate students' views on their class entitled "Historical Development of Mathematical Concepts." Participants said that history of mathematics would get the attention of students and enable the subjects to be learned in a meaningful way.

DISCUSSION AND IMPLEMENTATIONS

The students who said that they enjoyed participating actively in the process of learning also mentioned that they overcame their fear of mathematics. Students making active presentations was important for student participation while the history of mathematics was introduced to the class in relation to the attainment.

The biggest reason not to teach or use history of mathematics, in lessons which is often propounded by teachers, is the lack of time to do so. This study is important because it shows that history of mathematics can be introduced to student by being included in major learning areas in primary school mathematics curriculum.

As a result, history of mathematics should be included in mathematics learning environments, especially if it is a goal to develop students' mathematical points of view.

Appendix1: Organized Mathematics Curriculum Based On History of Mathematics

HOURS	LEARNING AREA	SUB-LEARNING AREA	ATTAINMENT	STATEMEANT	ACTIVITY PROJECT PRESENTATION
2 hours	Probability-Statistics	Central Tendency and Spread	1. Makes a prediction based on data.	[!] The student is made to predict present or future situation. [!] It is emphasized that Omar Khayyam, Ali Kucsu and Ulugh Beg have a relation with astronomy and it is discussed how the student would make a prediction in astronomy studies starting from the data s/he has. [!] The students make their presentations.	Uğur – Khayyam Özlem – Ali K. Büşra – Uluğ B.
2 hours	Probability -Statistics	Identifying Probable Situation	2. Compares basic principles of enumeration and uses them in problems	[!] It is emphasized that basic principles of enumeration include the rules for addition and substraction. [!] It is discussed that reasons why Pascal needed to develop probability and Fermat-Pascal correspondence is mentioned. [!] The students make their presentations.	Betül – Pascal Harun – Cauhy İsa – Fermat
2 hours	Geometry	Angles	3. Defines properties of adjacent, complementary, supplementary and opposite angles.	[!] Adjacent complementary and adjacent supplementary angles are defined. [!] It is emphasized that non-shared edges of adjacent angles form another angle. [!] It is emphasized that a "linear pair" is formed when one edge of two adjacent supplementary angles are common and other edges of them are in the same direction but in the opposite ways. [!] The students make their presentations.	Muhm. – Origin of Geo. Ömer F. – Öklit Cihan – Thales
2 hours	Geometry	Measurement of Angles	4. Estimates measures of complementary, supplementary and opposite angles.	[!] One of the letters "s" or "m" is picked as a measure of angle and the other one is mentioned. [!] The student is made to realize complementary and supplementary angles in al Biruni's astronomical observations. [!] It is emphasized that its reading way is important while measuring the angle. [!] The students make their presentations. [!] The contributions of Pythagoras and Descartes to geometry are discussed.	Melike – Biruni Ayşe – Pisagor Mehmet – Descart

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ARITHMETIC TEXTBOOKS AND 19TH CENTURY VALUES

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In this paper two nineteenth century Icelandic arithmetic textbooks are investigated, both written according to the late medieval tradition of libri di abbaco*, practical textbooks. The authors were influenced by the German Protestant tradition that arithmetic was to serve ethics education, as well as by the Enlightenment. Above all, their goal was for the autonomy of Iceland from Denmark. Their textbooks served the purpose of teaching young people wise allocation of their resources with the overall goal of individual as well as national prosperity and financial independence.*

INTRODUCTION

Iceland was a pre-industrial society at the beginning of nineteenth century. At its close it had become modernized in the sense that towns and villages had been formed, a few bridges been built and school legislation was about to be enacted. Two substantial arithmetic textbooks, written in Icelandic, were published during the century. The purpose of this paper is to determine the role of the textbooks in the development of the society, more definitely phrased:

> How did the two textbooks relate to the struggle for Iceland's autonomy from Denmark in the nineteenth century, and which values did the authors deem to serve that purpose?

The research method is historical; exploration of the historical background of the Icelandic society by reference to papers by distinguished scholars, and examination and interpretation of the textbooks themselves. Some research has been done recently, e.g. by Bjarnadóttir (2010; 2012), where the textbooks in concern were situated in the Icelandic heritage of arithmetic textbooks, while in this paper it focussed on the specific aspect of ethics education.

ARITHMETIC TEXTBOOKS

The structure of arithmetic textbooks fell early into a specific pattern, while authors have had some freedom on pedagogical ordering of the material; to present mathematics in context and to develop natural curiosity. Early textbooks were not written for use in institutionalized school systems and the presence of a teacher was not assumed (Howson 1995).

Libri d'abbaco

During the twelfth through the fifteenth centuries, algorithms, based on al-Khwarizmi's arithmetic, appeared in great numbers and in a diversity of languages other than Latin. There are numerous Italian *libri d'abbaco*, arithmetic textbooks written by teachers who spoke the ordinary language of the marketplace. The books typically contained an introduction to the Hindu-Arabic numeration and the accompanying arithmetic operations, in addition to mathematical techniques for business use (Swetz, 1992; Tropfke, 1980).

The protestant reform

The general approach of the Protestant Reform in the early 1500s was to literate the population. Martin Luther's collaborator, Philipp Melanchthon, was particularly keen on nurturing mathematics education. For him, knowledge existed primarily for the service of moral and religious education and he praised mathematics for its ethical role. Most textbooks were written in the vernacular with self instruction in mind while the mercantile community was the main target group (Grosse, 1901).

NINETEENTH CENTURY ICELAND

In the early 19th century, Iceland was essentially a pre-industrial society. The population numbered only 47,000, almost exclusively rural. Reykjavík was becoming the centre of government and a capital town. The number of inhabitants in Reykjavík was 307 in 1801. There were no schools apart from one Latin School which was a theological seminary as well until 1847.

The Icelandic Enlightenment movement, originating in Denmark where in turn it was largely derived from Protestant Germany, had made considerable efforts in publishing educational material by the 1780s. Calamities: volcanic eruptions and earthquakes, followed by bankruptcies of official agencies, caused by Danish participations in the Napoleonic wars, reduced its impact. Two substantial arithmetic textbooks were published in the 1780s, one of them by Stephensen (Stefánsson, 1785), the main Enlightenment champion. Manuscripts were also dispersed from person to person. Reading and knowledge of Christianity were required for confirmation from 1743 upon the responsibilities of the homes, while legislative act, adding writing and arithmetic, was first issued in 1880. Textbooks of the 1800s were therefore not aimed at children, but young people.

By the end of the century, two lower secondary schools offering general education, primary schools in the largest towns, and a handful of schools for prospective farmers, housewives and seamen, had been spontaneously established. Legislation on state support did not exist as yet except for the Latin School and one lower secondary school. Pupils had to have private tuition in Latin prior to admission to the Latin School and both schools required living in student residences, demanding expenses which limited attendance. The Latin School pupils were sons of officials and farmers (Karlsson, 2000), the state lower secondary school pupils were mainly sons of farmers (*Skýrsla um Gagnfræða Skólann á Möðruvöllum*, 1881–1905), but in its parallel private town school, a few girls were registered (*Skýrsla um Alþýðu- og gagnfræðaskólann í Flensborg*, 1883–). The number of pupils in each cohort in the three schools was 40–50 in 1880, or about 3% of the population (*Hagskinna*, 1997).

THE AUTHORS AND DISTRIBUTION OF THEIR TEXTBOOKS

The two nineteenth century arithmetic textbooks describe the current society and propagate ideas of how to make the most of it. Both of their authors later became Members of Parliament; Jón Guðmundsson (1807–1875) during 1859–67 and Eiríkur Briem (1846–1929) in 1881–91 and 1901–1914. They did not serve concurrently at Parliament but both were avid supporters of Jón Sigurðsson (1811-1879), the leader of a movement for autonomy from Denmark, Member of Parliament 1844–1879.

As young men, Guðmundsson was a secretary of Reykjavík Town Magistrate in 1832–1836 while Sigurðsson was the Bishop's Secretary in 1830–1833, a post that Briem served in 1867–73. In the small community of Reykjavík, young men working as assistants to the highest authorities became knowledgeable about matters and men. The single most important matter all through the period from the 1830s was raising Iceland to autonomy. Sigurðsson, the leader, wrote papers about his vision for schools: primary schools, schools for seamen and farmers, and gymnasia by the Prussian model, free trade and financial independence of Iceland. Guðmundsson supported Sigurðsson duly in his newspaper and Briem gave a comprehensive overview of Sigurðsson's work in his obituary of 42 pages (Briem, 1880b). The main issues in the Parliament during Guðmundsson's terms were financial claims related to the autonomy from Denmark (Laxness, 1960). In spite of Sigurðsson's hard work, by 1874, when Icelanders were granted their own Constitution under which the Parliament became a legislative body with own budget, there was hardly any piece of road in the country, no bridged rivers, no transport on sea along the coast and only little transport to other countries (Bárðarson, 1930).

The technological changes that had taken place in the neighbouring countries had only reached Iceland to a limited extent by the turn of the 20th century. Education for the common people was much slower to emerge in Iceland than e.g. in Denmark, and there were relatively fewer towns than in most other European countries. This may be seen as a favourable set of conditions for the reception of 'enlightened' ideas (I. Sigurðsson, 2010). Both Guðmundsson and Briem were influenced by Stephensen's textbook, and his writings about national economy (Bjarnadóttir, 2012).

Jón Guðmundsson

In later life Jón Guðmundsson became best known as newspaper editor supporting J. Sigurðsson. Due to bad health and poverty, Guðmundsson had become 25 year old when he completed the Latin School. Serving as overseer of the King's properties in South-East Iceland, he wrote an arithmetic textbook. *Reikníngslist, einkum handa leikmönnum* / *Arithmetic Art mainly for laymen,* published in 1841, well before he became known for his participation in the struggle for Iceland's autonomy from Denmark. The content of the book is traditional, but its characteristics are his sincere advice to young people on avoiding squandering on luxuries like coffee, sugar and spirits. It was only printed once and used one year only at the Latin School. Other schools had not yet been established. No sources have been found on its use, but as the only available printed textbook for 28 years, it is not unlikely that some number of farmers' sons who had no choice of schools, tried to learn from it.

Eiríkur Briem

The Reverend Eiríkur Briem (1846–1929) became a professor of theology at the Theological Seminary, later University of Iceland, established in 1911. He was also the guardian of the first bank, The National Bank of Iceland, established in 1886, and was entrusted to a number of tasks concerning Iceland's economy in the early 1900 (Bárðarson, 1931). Briem did most of his secondary schooling and the Theological Seminary education staying at home, teaching his siblings. Home education was therefore natural to him. He was only 23 years old and had not yet been ordained when he published his arithmetic textbook, *Reikningsbók*, in 1869 which was expanded into two volumes in 1880. The first volume was republished ten times until 1911 and the second one four times, thus being the main reference book in arithmetic for half a century. It was used for teaching at both lower secondary schools (*Skýrsla um gagnfræðaskólann á Möðruvöllum*, 1880–1905, *Skýrsla um Alþýðu- og gagnfræðaskólann í Flensborg*, 1883–) and the first one at the Latin School (*Skýrsla um hinn lærða skóla í Reykjavík*, 1875–1983). When primary schools became more common from the 1860s, it was recommended as a handbook for teachers.

THE CONTENTS OF THE ARITHMETIC TEXTBOOKS

Table 1 provides a list of the typical content of the *libri d'abbaco* (Van Egmond, 1980) with columns for comparison to the contents of the two arithmetic textbooks: *Reikníngslist* (J. Guðmundsson, 1841), and *Reikningsbók* (E. Briem, 1869; 1880).

	J. G.	E. B.
1. Arithmetic:		
Numeration: the four arithmetic operations, applied successively to whole numbers, fractions and the compound quantities of monies, weights and measures	X	X
Tables: multiplication tables for numbers and monetary units, tables of squares and lists of the parts of monetary units	X	X
2. Business problems:		
Finding the price or amount of goods, usually by the Rule of Three	X	X
Currency exchange, and conversion of units of measurement	X	X
Problems relevant to bartering	X	X
Partnership – dividing profits between members of partnership	X	X

Interest and discount – distinguished between simple interest and compound interest, computed by percentages	X	X
Equation of payments – a series of loans combined for repayment on a single date		
Alligation – metals of varying purities combined in a mixture		X
3. Recreational problems; often including algebraic questions		X
4. Geometrical problems		X
5. Methodological section		
Rule of Three or the “golden rule”	X	X
Rule of the false position, rule of the double false position		
Algebra, solution by an unknowns and equations		X
6. Miscellanious – number theory, tarifs, calendars, etc.		X

Table 1: Comparison of Guðmundsson’s and Briem’s textbooks to the *libri d’abbaco*

The content of Guðmundsson’s *Arithmetic Art* was confined to the above list, while Briem’s *Arithmetic*, in the first edition of its second part, contained more advanced material, such as logarithms for quick backwards computations of compound interests. The fact that the sections of algebra, equations and logarithms were excluded in later editions of the book, witnesses the fact that the general public, who were the readers of the book, did not require it and may not have wanted to spend money on unnecessary information. Books had to be kept at minimal price.

Both authors mention foreign textbooks, e.g. by the Dane Georg F. Ursin, which both may have studied at the Latin School. However, the contexts of their books do in both cases concern the Icelandic farming society and its economic issues.

THE MESSAGES OF THE TEXTBOOKS

The two textbooks describe similar societies, the old farming society of rural inhabitation. The farmers went together with their tenants and servants to the fishing grounds in the spring fishing season. They allocated the revenues of their farming and fishing and dealt with the merchants, mainly by bartering where no money was present, although prices had been calculated in terms of money since 1776 (Karlsson, 2000, p. 244). The farmer deposited his products: wool, meat, fat, fish, knitting and weaving products, and feathers. Against this he drew on his account what he needed. The society was to large extent built up as self-sufficient, and the necessary goods were often confined to grain, such as rye, oats, etc. In addition there were luxurious goods, such as sugar, coffee and spirits. The farmer was also to pay his taxes and various official payments. An important part was the support of paupers and invalids, which expenses had to be computed to a certain degree of exactness.

Each category of arithmetic: the four arithmetic operations by simple numbers, composite numbers (with monetary or measuring units) and fractions; the Rule of Three, simple and inverse; percentages and interests; was followed by six to ten verbal problems by Guðmundsson. The majority concern the above transactions by the merchant, clearing up death estates and heritage, etc. In most categories one meets one or two problems containing advice on sensible allocations of resources.

Briem's textbook contained many more problems and their solutions were generally left to the reader except the very first ones in each section, which served as examples. Problems containing advice and warnings were located in the latter part of the 1869 edition in the sections of percentages. In the 1880 edition the content had been split into two volumes and the ethical advice was exclusively contained in the second volume along with the inverse Rule of Three, percentages and interests. Several recurring themes of the two textbooks are recited below.

Avoid borrowing

Both authors warned seriously against borrowing. An example tells of a man who borrowed money for two years by accepting an obligation of 100 kronas which only gave him 70 kronas. Furthermore, he had to pay 4% rent p.a. In fact he had to pay 27 1/7 % p.a. (Guðmundsson, 1841, p, 241). Usury was clearly not invented in the 21st century. The author added a comment following this and another similar example:

> These 2 examples exhibit how unscrupulous some people are by utilizing others' hardship, and how these in trouble, often out of distress and sometimes out of ignorance, submit to high rents (Guðmundsson, 1841, p. 243).

In Briem's opinion, if was not appropriate to borrow for anything that did not provide revenue (Bárðarson, 1931):

> A farmer borrowed autumn-wool by the end of March against paying it back by the end of June by the same amount of spring-wool; now the price of the autumn-wool was 22 shillings, but of the spring-wool 36 shillings; how much did he have to pay in percentages p.a.? Answer: 254 6/11% p.a. (Briem, 1869, p. 131).

The farmers had to ensure enough wool for their family and servants to spin for knitting and weaving in the late-winter and early spring before the bustle of lamb-births, shearing and hay-making; the old farming cycle in a nutshell. Therefore, the farmer had two options; either to keep his people idle for several months or to borrow wool at high price. The latter was a better option as the home-made products would increase the value of the wool considerably.

Improve farming and avoid spending on luxurious goods

Briem was much occupied with teaching the youth how to make the most of traditional farming and avoid spending. The extended version of 1880 contained:

> A farmer calculated that his household spent 100 kronas on coffee a year. How much is that for twenty years if he could have had 5% revenue of the 100 kronas by allocating

them to improve the land and increase his livestock? Answer: 3306.60 kronas (Briem, 1880, p. 37).

Briem knew how sensitive his countrymen were to taxes and gathered that taxes on luxury spending were the most justifiable. In 1889 when the Budget was unusually difficult after periods of pack-ice, stopping fishing, causing famine and emigration to America, Briem suggested in Parliament an extra tax on coffee and sugar as luxury (Bárðarson, 1931). This suggestion echoes his textbook example nine years earlier.

Guðmundsson (1841, p. 215) told about a coffee-gulper who used to drink five cups of coffee a day. He bought coffee-beans that he thought would last for a year. After twelve weeks he realized it would only last 40 weeks. The task was to find how how long the portion would last if three cups a day were consumed and six cups on Sundays. Briem (1869, p. 107) repeated the task for a whole farm where the master directed how much his servants may drink a day.

Guðmundsson also told a long story of two brothers. Both had been working as free workers for salary, including food and working clothes. This was a novelty. Before, all people had to belong to a farm, either as family or servants. The elder brother had earned more, but spent his salary on spirits, coffee, clothes, knick-knacks, etc. and had less left of his year-salary than his younger brother.

The consumption of coffee seems to have worried many responsible citizens in the 19th century. The import of coffee increased from practically nothing in 1820 to 44 tons in 1840 and to 218 tons in 1866 or fivefold (*Hagskinna*, 1997). The same applied to sugar. A contemporary source (NN1, 1849) expressed concerns that even people in the smallest cottages had to drink strong coffee, up to twice or thrice a day. The farmer could not direct his servants to work, and his servants could not work, until they had had their coffee. Seafarers had to wait for coffee however the weather and other conditions were. In another article in the same issue (NN2, 1849), the writer understood well that many a good farmer was shocked to learn how much of the farm's resources was spent on coffee and its auxiliaries; the grinder, the kettle, the cups, the sweets and the cream, in addition to the firewood- and time-consumption. Coffee consumption was also discussed in the parliament (*Alþingistíðindi*, 1871).

It is by no means unique in history to resist coffee consumption. In the after-WWII period, German economy was ruined. People were urged to drink beer rather than the imported coffee which had high import duties into the 1970s.

Wise allocations and investments

Briem gave an example demonstrating that a tenant who owned 300 kronas was better off by improving and evening out a rented land for the amount in order to be able to feed more cows than else. The other option was to rent the amount to another person by 4% interest p.a. The result was that the tenant would make a profit of nearly 400 kronas exceeding the interest (Briem, 1880, pp. 33–35). This example is not found in the 1869 edition of the textbook where compound interests were also introduced. Briem, as a parish minister during 1873–1880, did similarly himself. He had swamps drained on the farm belonging to his parish church, and set up irrigation on the land on his own cost. However, he was well off after his seven years parish service (Bárðarson, 1931). Depositing money in a bank was still unusual. The first bank was only established in 1886 where Briem himself was involved.

Spirits

Both authors present a number of exercises involving spirits. People mixed the spirit with water, sometimes to lower the price (Briem, 1880, p. 2) but often to gain more profit. Guðmundsson (1841, pp. 228–230) tells a long story of a shady dealer buying spirit in Reykjavík to sell in rural areas and making (too) much profit. Another example concerns a worker in Reykjavík who consumed half a pot of spirit each day from the age of thirty to his death at sixty. The expenditure was gigantic, which must have been the intention to demonstrate (Guðmundsson, 1841, p. 40). Briem was also concerned about alcohol consumption:

> In 1866, 33,029 pots of wine were imported to Iceland which was 2.2% more than the year before; how much was that? (Briem, 1869, p. 125).

In 1866 the number of households in Iceland was about 9,400, and official numbers for import of spirits were 511,700 litres and other alcoholic beverages 88,200 litres (*Hagskinna*, 1997). This means that 54 litres of spirits were consumed on average in each household and 9 litres of other alcoholic beverages. These numbers dropped suddenly in 1873 after an ordinance 1872 regarding a duty on spirits (Árnason, 1988). The duty was still raised in 1875 and 1879, reflected in drops in the import quantities. They rose again, but never to the levels from before 1873. The 1872 ordinance was issued in the period between the Parliament services of Guðmundsson and Briem but was duly supported by Sigurðsson, also for morality reasons (*Alþingistíðindi*, 1871).

Health issues and social problems in connections with alcohol were not mentioned in the arithmetic textbooks, only the waste of money. However, alcohol consumption was considered a growing social problem when a branch of IOGT, the International Organization of Good Templars, was established in 1884 (Karlsson, 2009).

SUMMARY AND CONCLUSIONS

Guðmundsson and Briem both belonged to the typical landless educated elite who had themselves to be economically thinking and thrifty. According to Enlightenment ideas they believed that education would “enlighten” people to allocate their resources wisely. Their textbooks belong to the *abbacus* tradition of practical textbooks as is clearly seen from Table 1. Furthermore, the authors, both theologically educated, were faithful to the protestant tradition in including ethic education in their arithmetic textbooks. They warned people against spending money on luxurious goods, such as coffee, sugar and spirits, for their own financial prosperity but also for the society's. The homes were expected to consume their own products as was the society as a whole. Sources witness that Briem practiced in his own life and in Parliament what he preached in his textbook. Less is known about Guðmundsson. His biography author was more concerned with his activities towards autonomy from Denmark. This was certainly the overall goal of all these men; Sigurðsson, the leader, Guðmundsson and Briem, and for that the population had to be educated and capable of allocating their own personal resources in a wise manner. The conclusion is therefore that the two textbooks by Guðmundsson (1841) and Briem (1869; 1880) were important channels for conveying ideas that the authors gathered would lead Iceland and Icelanders to prosperity and financial independence.

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"I WAS AMAZED AT HOW MANY REFUSED TO GIVE UP": DESCRIBING ONE TEACHER'S FIRST EXPERIENCE WITH INCLUDING HISTORY

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In the following narrative we share the perspectives from our two different roles in thinking about history of mathematics in mathematics education. The first author's perspective provides insight into how a university professor thinks about her graduate students' learning with regard to incorporating history of mathematics in teaching. The second author's perspective presents the reality of the classroom teacher making a first attempt at both studying history of mathematics and incorporating history in teaching mathematics.

INTRODUCTION

In this paper we present simultaneous accounts from two different perspectives about promoting the inclusion of history of mathematics in teaching secondary school mathematics in the United States. The first author's account is from the perspective of university professor, teaching a graduate-level history of mathematics course for teachers and prospective teachers. The second author's account is from the perspective of a secondary mathematics teacher, who took the graduate-level history of mathematics course in summer 2012 and subsequently pursued ways to incorporate history of mathematics in her teaching.

Thus, this paper is less about presenting a formal research study (i.e., with the expected literature review, research question(s), methodology, data collection and analysis, and assertions and conclusions) and more about describing the experience of a classroom teacher's efforts to include history of mathematics in her teaching. In the field of history of mathematics in mathematics education there are several examples of theoretical and empirical contributions, discussing a variety of issues ranging from ideal ways to implement a history of mathematics course for prospective teachers (Clark, 2008; Charalambous, Panaoura, & Phillippou, 2010) to studies on students' experiences with learning mathematics using original source material (Jankvist, 2010). However, there remains a gap in the available literature that provides insight into how and why secondary mathematics teachers include history of mathematics in teaching (or, elementary teachers teaching mathematics, for that matter) [1].

Avital (1997) stated that "[m]athematics is by nature a cumulative subject; most of what was created millennia ago – both content and processes – is still valid today. Exposing students to some of this development has the potential to enliven the subject and to humanize it for them" (p. 3). Moreover, Avital claimed that, "[i]f we want to change the present situation we have to do it through the teachers" (p. 3). Consequently, when working with mathematics teachers with the goal of determining when, how, and why to incorporate history of mathematics in teaching, it is a worthwhile task to examine the different claims for each.

RELEVANT BACKGROUND

In summer 2012 five prospective mathematics teachers [2] and one classroom teacher completed a course on using history in teaching mathematics (or, "Using History" course). The course was taught as a hybrid, in which the course met one evening per week for six weeks and as an asynchronous online course for the remaining six weeks of a 12-week semester. The course was also designed to contain two parts. Each of the six graduate students possessed a different mathematics background and knowing this, the course was designed so that the first eight weeks focused on studying several mathematical topics from an historical perspective. Then, the remainder of the course was devoted to pedagogical concerns and the ways in which history of mathematics can or should be incorporated in teaching mathematics. Table 1 displays the schedule of topics of the course.

Week Number	Topics
1	Overview of research skills and the foundations of early mathematics
2	Content Focus: Cultural and historical development of mathematics; number systems; development of the complex number system
3 – 8	Content Foci: Solving polynomial equations, Geometry, Advanced mathematics topics, Calculus
9 – 12	Pedagogical Foci: Using original sources, Emphasizing historical problems, Cultural- and ethnomathematics

Table 1: "Using History in Teaching Mathematics" course schedule of topics.

The primary intent of the course was to provide future and current teachers with resources, strategies (e.g., aspects of the "hows" per Jankvist, 2009), and reasons for including history of mathematics as part of their practice in teaching mathematics (i.e., the "whys" per Jankvist, 2009). In particular, there were two key course products that prompted students in the course to plan for the role of history in their future teaching. The first, weekly contributions to asynchronous discussion board prompts, allowed the students enrolled in the course to share ideas prompted by weekly assignments, which included mathematical tasks, original source readings, and articles from the field of history in mathematics education.

The second product was the capstone assignment for the course, for which students were required to create a historical lesson for a mathematical concept or topic that they would potentially teach in their future teaching assignment. For the historical lesson, students were directed to include historical information and historical problems or processes. The reason for the requirement to include both anecdotal and mathematical aspects in the lesson was to eliminate the possibility of creating a purely "history as interesting story" (Clark, 2011) lesson, that would most likely not have the potential to engage pupils in learning mathematical content.

THE NARRATIVE

Kathleen's Perspective.

During the course it was clear to me that Lisa was thinking about how history of mathematics could enhance her instruction in ways that the other students could not. This was, of course, primarily due to the fact that Lisa was the only classroom teacher taking the course. In addition to having taught for many years, Lisa was experiencing what many teachers often describe as "hitting a wall" – or becoming somewhat disenchanted with the way they have always taught particular topics or courses. As Lisa explained, she "didn't want to get burned out", and she wanted to include authentic problems to engage students.

In this section, I share two important aspects of my interactions with the students in the course. First, I present examples of Lisa's contributions to the online course discussion board. The sample posts are helpful in revealing Lisa's disposition towards to role of history in teaching. Secondly, I briefly describe the historical lessons that the students produced for their capstone assignment, with particular attention to Lisa's historical lesson.

Discussion Board Contributions.

During the seventh week of the course, students were asked to select one of the topics of focus [3] and to share on the discussion board why they would be most excited to teach the topic using history. Furthermore, students were asked to consider the explicit use of historical problems (or, applications). That is, the students were challenged to consider using history for more than just the exciting stories, anecdotes, or biographies. In response to the discussion board assignment, Lisa stated:

> Trigonometry is usually introduced after the lessons on right triangles. After working on the historical aspect of trigonometry, I think it would have more meaning if the lesson could be taught after the circle concepts. With the circle concepts (especially chord relations) fresh in the students' minds and providing real-world problems from the ancient world, I think trigonometry will make more sense to the students. They may actually have fun with the concept. I do need to do more investigating as to the types of astronomy-based problems to give the class to solve. I am looking forward to building on this lesson and to compare the type of learning from previous years of introducing trigonometry as just ratios in a right triangle. (Discussion Board, Week 7)

Later in the course, Lisa responded to a discussion board prompt that asked for students to express how some of the course readings helped them to think about the possibilities, benefits, and obstacles to using history in teaching:

> Avital's article contained many wonderful ideas that I have been looking for; that these ideas will help me use history in the classroom is a bonus. These days the focus [in teaching] is on reading and rightly so. We also need the same push to work on students'

> ability to think. If the answer does not come quickly, so many students shut down. I have even heard of instances where teachers become frustrated and make the work easier, but making it harder for the student to develop their thinking skills. In the article, Avital mentioned that we should educate students to conjecture “It cannot be done” and then try to prove this conjecture. The problems he presented as examples make this “doable”. An example was asking eleven-year-olds to obtain the sum 45 by adding 8 numbers taken from the set 1, 3, 4, 7, and 11. (This is very doable and the students would love the challenge.) (Discussion Board, Week 9)

Finally, Lisa contemplated a specific way to incorporate history of mathematics in teaching (e.g., Jankvist’s “hows”) toward the end of the course. In her reflection about Jankvist and Kjeldsen’s article in *Science & Education* (2011), she observed:

> What drew me to the article was [Jankvist’s] teaching module. We have been talking about bringing history to the classroom as more than an additive. We need to give it substance and [for it to] have meaning. Many times it has been suggested that if the students ‘witnessed’ the struggles of the men and women as they toiled through the beginnings of mathematical concepts, then the students would find more meaning in their learning of the same concepts. I have been wrestling with the idea of how to incorporate history of mathematics in the classroom in order to share these struggles with the students. I feel Jankvist’s module is a good start. I like the questionnaire that was given to the students before the study. I always wonder what students really thought of math and their ideas of where the subject came from. How did math come about or did it just appear? Many of the historical developments seem to have had some controversy associated with the people involved. The discussions seem to bring the students into the controversy with the debates with one another. All students have an opinion and I feel even the quiet, struggling student will gain insight to the human side of the mathematics. (Discussion Board, Week 11)

Lisa shared a valuable perspective for research in the field of history of mathematics in mathematics education. In particular, Lisa sought out the “Using History” course because she was interested in finding and implementing new strategies to meet the needs of her students, as well as to prevent her own burn-out from teaching mathematics as she always had. Secondly, she was able to read, analyze, and apply ideas that she read about during the course to her particular experience as a veteran classroom teacher. Finally, Lisa used her new learning about history of mathematics, different approaches for incorporating history in teaching mathematics, and reasons why history may contribute to student learning to create her historical lesson in the course.

Historical Lessons.

The six "Using History" students each produced an extensive historical lesson as the capstone assignment for the course. A brief description of the lessons is provided in Table 2.

Student	Lesson Topic	Grade Level	Details
Lisa	Pythagorean Theorem	High School	4 activities; different cultures included; base-60 calculations; proofs
Student 2	Probability	Middle School	1 one-hour lesson; original readings; questions on the readings; mathematical problems
Student 3	Multiplication Strategies	Elementary	3 multiplication methods from three cultures; open-ended questions; mathematical problems
Student 4	Volume of a Sphere	High School	3 methods from three cultures
Student 5	Method of False Position	Middle School	Primarily a description of Fibonacci's application of the method; only two historical problems (out of seven questions)
Student 6	Conic Sections	High School	No history included

Table 2: Brief description of historical lessons.

Four of the six "Using History" students designed lessons that met the required elements of the capstone assignment (Lisa and Students 2, 3, and 4). It was apparent from Lisa's historical lesson that she was an experienced teacher; her lesson was organized with four delineated activities. Each activity represented part of an overall progression to engage pupils with the mathematics of several ancient cultures, ranging from different numeration systems to an example of measurement (e.g., the Pythagorean Theorem). The other students' lessons differed from Lisa's in this key regard. In particular, the other historical lessons contained one main activity, with several aspects (cultural background, historical readings, mathematical tasks) included in the activity.

Lisa's Perspective.

In this section Lisa shares her perspective, both as a classroom teacher and as a graduate student enrolled in the course on using history in teaching mathematics. In the following, then, the first person "I" is used by Lisa.

I have been teaching mathematics at the secondary level for the past 18 years. I enjoy mathematics and would always look for real world applications in the form of activities to make mathematics exciting for the students. In my reflections over the past few years, it seems as though the students are becoming more disconnected from mathematics. In this "quick fix" world, students are looking for formulas to solve all the problems. Consequently, I wanted to find something that would broaden the students' understanding of mathematics as well as give them a whole new perspective of the subject.

As part of my search to find content and strategies to include in my teaching, I met with Kathleen to discuss course offerings at the Florida State University. She told me about her "Using History" class and I immediately thought this could be the answer to my dilemma. I always believed my understanding of mathematical concepts would be enhanced if I knew about the origins of the concepts. And, more importantly, I hoped that it would do the same for my students.

The course proved to be fascinating. The courses I teach range from middle school mathematics to honors-level geometry. As I studied the required course readings, my thoughts were always focused on how I could use the material in the classroom. I found many ideas in the course text (*Math Through the Ages: A Gentle History for Teachers and Others*), as well as the assigned journal articles. For example, Avital (1997) suggested introducing open problems to develop critical thinking and problem solving. Meavilla and Flores (2007) presented an interesting activity of analyzing problems as they were given in their original language. One of my concerns has been students' poor problem-solving skills. Thus, the ideas suggested by Avital and Meavilla and Flores caught my attention as a potential strategy to help students develop this important life skill.

I also found the lives of mathematicians fascinating and feel students would enjoy learning the stories behind the individuals responsible for mathematical ideas. Additionally, I feel that my students could possibly relate to the struggles these great thinkers faced. It would be interesting to hear my students' opinions of what particular mathematicians dealt with in their lives. My thoughts during the "Using History" course were to make plans for when to introduce these interesting activities throughout the year. During the summer, I had other (though unexpected) encounters with the history of mathematics. At one conference I attended one of the lesson plans presented involved Eratosthenes' measurement of the Earth. A brief history was given and an outdoor activity for participants followed. This lesson provided proof that introducing history into lesson plans does not take the huge commitment of time that many teachers fear.

Since completing the "Using History" course, I must admit that I have been slow to introduce history into the classroom. My lessons for the academic year began with integers and measurement and this presented the perfect opportunity to introduce the many number systems of the ancient world. I began my 7th grade classes with the different numeration systems of the ancient Egyptians, Babylonians, and Romans. I expanded on the Roman Numerals by mentioning that it is believed that it was motivated by the human hand. The numbers 1 through 4 were the fingers and the numeral for 5, "V", was the hand, which showed how the thumb formed a "V". In class, we discussed what it would have been like working with other groups from different parts of the world, with each having their own number system.

Later in the academic year I spent a class period in my two Geometry Honors classes working on proofs of the Pythagorean Theorem. After distributing the handout with various proofs, I let the students be to see if they could come up with the proofs. As we worked together on deciphering the directions, the students became excited about setting up the problems. One student even jumped up in the air when he figured out a proof, and he quickly asked for another. In fact, other students asked to try another. I was amazed at how many refused to give up and say the usual 'I don't get it'.

The greatest challenge in planning lessons informed by history of mathematics, however, deals with the resources available. Thus far, as I develop lessons and activities, I started small and as I continue to work on the plans, the ideas continue to evolve into many sub-phases. My slow start with introducing history as part of my lessons is due in large part to my lack of confidence with the many sources that available. For example, I read several interesting articles and books to learn about a number of mathematicians and found enough discrepancies to make me hesitant about the reliability of the information that I would use with students.

This, however, will not deter me from developing more lesson plans that draw upon the history of mathematics. For example, as a result of the class discussion on ancient number systems, I believe it will be fun for them to perform operations within each of the different number systems and I will design a lesson on this topic. In the lesson, I want students to compare the different number systems to each other, as well as to the way in which we perform operations in our base-ten numeration system. Although my efforts to include history in my lessons may appear to be a slow start, I do not plan on letting history go by the wayside.

Kathleen's Additional Response.

As a final orientation for the reader, the narrative returns to Kathleen's first person perspective.

Lisa's reflections brought three important issues to my attention. When Lisa first shared with me the lesson on Eratosthenes' measurement of the Earth that was presented at a professional in-service meeting for classroom teachers, I missed a valuable opportunity to share published examples of this activity with Lisa. Indeed, seeing a lesson plan that used history of mathematics – in the context of the in-service meeting – was an important cue for Lisa regarding the viability of incorporating history in her mathematics teaching. However, taking the time to share further examples with Lisa, particularly from resources that she would have valued (especially given her desire for more scholarly resources resulting from her experience during the "Using History" course), would have contributed to the ongoing discussion in which we were (are) engaged. For example, sharing lessons on the topic found in the *Historical Modules for the Teaching and Learning of Mathematics* (Katz & Michalowicz, 2005) or the excerpt on the story of the ancient history of Earth measurement (Führer, 1991) would have, taken together, emphasized Lisa's idea that there are important stories to be shared with students and the mathematical content.

Reflecting on the story presented by Führer (1991) prompts another reaction to Lisa's perspective. As Lisa shared in her narrative – as well as many times during our meetings together – she felt overwhelmed by the number of resources that she perceived as providing different historical information. Führer stated the exact situation that Lisa found herself in: "[Most mathematics teachers] would be quite ready to include historical aspects in their lessons, provided someone would help them to overcome their feelings of precariousness concerning the uncertainty of history and the methodological problems of teaching it seriously" (p. 24). Though I tried to alleviate Lisa's "feelings of precariousness", her perceptions remained. Quite possibly, however, her strong feelings were tied to the need to please me, as her university course instructor. As a result, in my future work with Lisa I will be sure to emphasize Führer's exact sentiment that, "It cannot be the job of the mathematics teacher to idolize the standards of historical science. The teacher must not lie, but should free herself from the heavy burden of exactness" (p. 24).

Finally, Lisa's revelation that she discussed the origin of the Roman numeral "V" as it related to the angle formed by the thumb and index finger of a human hand reminded me that the work to incorporate history in teaching mathematics must – at least from its initial stages – be a true collaboration among classroom teachers who want to provide an historical perspective for their students and mentors (e.g., math historians, mathematics educators, history of mathematics course instructors). With respect to the example about the origin of the Roman numeral "V", Lisa and I could have worked together while she planned for including this story (and subsequent discussion with her students). Then, when confronted with a possibly too convenient explanation about the possible origin of "V", we could have searched sources together to investigate alternative explanations. For example, Menninger (1969) discussed the development of the Roman numerals I, V, and X from notches cut into tally marks: "At the very beginning, the Roman numerals observed the basic and simple laws of ordering and grouping, like the notches in tally sticks" (p. 241). Furthermore, V may have resulted from a half-symbol (the "V" formed from half of the "X" symbol). [4]

NEXT STEPS

I argue that the inclusion of history of mathematics in meaningful ways (i.e., addressing both content and affective dimensions for learning mathematics) may most successfully come to fruition in classrooms when sought after by teachers who are "looking for another approach" or are fearful of becoming "burned out". In our joint narrative Lisa and I described first, her initial journey to investigate how to incorporate history in her teaching – and her first steps to do so. Lisa's desire to seek formal instruction on how to do so marked the beginning of her journey. From my perspective – that of a mentor of sorts and fellow traveller – introducing teachers to the tools that they can use to plan for incorporating history in teaching mathematics is essential. In the case of Lisa, studying and working with a variety of resources presented tangible ways for her to begin to change her practice, but this also presented a challenge for her. Our continued collaboration will include discussing Lisa's lessons as she develops them, with attention to appropriate resources.

Our initial collaboration prompts important considerations for future research. Indeed, I have not had access to classroom teachers interested in using history of mathematics in their teaching in a long while. I have renewed interest to investigate the following with Lisa – and hopefully others:

(1) How do mathematics teachers select historical stories to use with their students?

(2) In what ways do teachers envision that historical stories will promote or support the learning of mathematics content?

(3) What are the most effective resources available for teachers to use to incorporate history of mathematics in teaching, in their current classroom context?

(4) What resources are needed to promote or support teachers' further use of history of mathematics in teaching, and how can academic historians of mathematics contribute to the construction of these resources?

NOTES

1. In this paper I use "elementary" to refer to pupils aged 5 to 11 years and "secondary" to refer to pupils aged 12 to 18 years.

2. These five students were enrolled in a master's (graduate) program designed to gain certification to teach secondary mathematics.

3. Topics for week 7 included early number theory, trigonometry, probability, the concept of infinity, or set theory.

4. The first author is grateful to Jan van Maanen for his suggestions on how to improve the original draft of this paper.

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THE USE OF ORIGINAL SOURCES AND ITS POSSIBLE RELATION TO THE RECRUITMENT PROBLEM

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Based on a study about using original texts with Danish upper secondary students, the paper addresses the possible outcome of such an approach in regard to the so-called recruitment problem to the mathematical sciences. 24 students were exposed to questionnaire questions and 16 of these to follow-up interviews, which form the basis for both a small quantitative analysis and a qualitative elaboration of this.

Keywords: Original sources; recruitment problem; view of mathematics; HAPh.

INTRODUCTION

Usually when discussion falls on the use of original sources in mathematics education focus is on the teaching and learning of mathematical in-issues such as abstract mathematical concepts, mathematical ideas and notions, theorems, proofs, etc. (e.g. Jahnke et al., 2000). And it is true that original sources indeed have a lot to offer in this respect – for recent empirical examples, see Clark (2012); Glaubitz (2011); Kjeldsen & Blomhøj (2012). But as occasionally suggested, a use of original sources may have *more* and quite *different* things to offer our educational systems as well. In fact, a use of original sources may be a way of dealing with more general problems such as: *recruitment* of students to the mathematical sciences; *transition* of students between educational levels, in particular between upper secondary level and university; *retention* of students once they have entered the mathematical sciences at university level; and the dimension of *interdisciplinarity* often appearing somewhat artificial to students when implemented in a classroom situation. In this paper, I shall address the first of these, i.e. the recruitment problem – for an elaboration of the three other possible roles of original sources, see Jankvist (preprint) – and I shall do so by relating to empirically collected data from upper secondary students, in order to both strengthen and unfold the claimed role of original sources in relation to this educational problem.

HAPH-MODULES AND ORIGINAL SOURCES

From February 2010 to May 2012, I followed a Danish upper secondary class of 26 students who through two teaching modules were exposed to extensive readings of original sources. An overall purpose of the modules were to introduce the students to aspects of history, application, and philosophy – abbreviated *HAPh* – of mathematics and to do this simultaneously in one module relying on original sources (for more detailed description, see Jankvist, In press). In the first HAPh-module, implemented in April-May 2010, the students read Danish translations of the following there texts [2]:

- LEONHARD EULER, 1736: *Solutio problematis ad geometriam situs pertinentis*
- EDSGER W. DIJKSTRA, 1959: *A Note on Two Problems in Connexion with Graphs*
- DAVID HILBERT, 1900: *Mathematische Probleme – Vortrag, gehalten auf dem internationalen Mathematiker-Kongreß zu Paris 1900* (the introduction)

The overall theme of this module was mathematical problems, as addressed by Hilbert in his introduction to the 1900 lecture. To make Hilbert's quite general observations a bit more concrete, the students first were to read the two other texts, each of which addresses a mathematical problem. Euler's paper from 1736 addresses the Königsberg bridge problem: how to take a stroll through Königsberg crossing each of its 7 bridges once and only once – today this paper is considered the beginning of mathematical graph theory. With the dawn of the computer era two centuries later, graph theory (and discrete mathematics in general) found new applications. Dijkstra's algorithm from 1959 solves the problem of finding shortest path in a connected and weighted graph – today it finds its use in almost every Internet application that has to do with shortest distance, fastest distance or lowest cost. Furthermore, Dijkstra also discusses a method for finding minimum spanning trees, a problem relevant for the building of computers at the time, and since then used in telephone wiring, etc. (see also Jankvist, 2011).

The second HAPh-module was implemented in September-October 2012, and here the students read Danish translations of the following three texts:

- GEORGE BOOLE, 1854: *An Investigation of the Laws of Thought on which are founded the Mathematical Theories of Logic and Probabilities* (chapters II and III)
- CLAUDE E. SHANNON, 1938: *A Symbolic Analysis of Relay and Switching Circuits* (first parts)
- RICHARD W. HAMMING, 1980: *The Unreasonable Effectiveness of Mathematics*

The title of Hamming's paper made up the theme for this module. Hamming discusses 'the unreasonable effectiveness of mathematics' from the viewpoint of engineering (and computer science) asking why it may be that so comparatively simple mathematics suffices to predict so much, this making up the 'unreasonable' aspect. To provide the students with a possible concrete example of this, they were first introduced to Boole's idea of a two-value algebra – and the context in which this was conceived by Boole in 1854, namely that of trying to describe language (and thought) from a logical point of view. Next, the students were to study a later – to some degree contemporary – application of Boolean algebra by Shannon from 1938. By relying on a set of postulates from the now further developed Boolean algebra ($0 \cdot 0 = 0$; $1+1 = 1$; $1+0 = 0+1 = 1$; $0 \cdot 1 = 1 \cdot 0 = 0$; $0+0 = 0$; and $1 \cdot 1 = 1$) and by interpreting these in terms of circuits, Shannon is able to deduce a number of theorems which can be used to simplify electric circuits and thus the building of such considerably (see also Jankvist, 2012a).

EMPIRICAL SETUP

As part of the empirical setup the students were exposed to a series of questionnaires and follow-up interviews during the 2-year period in which I followed them (for specific details, see Jankvist, 2012b). In order to get an indication of the possible effect of the modules and the students' readings of original sources in relation to the recruitment problem, their final questionnaire of March 2012 included the following set of questions:

1. Have the two modules provided you with a different view of what mathematics is; how it comes into being; and what it is used for? If yes, explain how and in what sense. If no, then why not?
2. Did the two modules encourage you to study or in any way concern yourself with mathematics (and/or natural science) after upper secondary school? If yes, how and why? If no, why not?
3. Whether you answered 'yes' or 'no' to the above question (2), do you then consider the two modules to have provided you with a more *enlightened basis* on which to either select or deselect mathematics (and/or natural science) to be part of your future education? If yes, how? If no, why not?

Question 1 above of course concerns the students' beliefs about mathematics as a scientific discipline, which was one of the main objectives of the overall study (Jankvist, 2012b), but which is also relevant for the following reason: Often when students either select or deselect mathematical sciences as part of their higher education, they may in fact be basing their choices on 'incorrect' assumptions. In the panel on empirical research at HPM2012, David Pengelley referred to this problématique as "reality vs. fantasy" [3]. This phrase has to do with students – including upper secondary ones – not having an (accurate) idea of what mathematics is about when practiced as a scientific discipline at the tertiary levels, e.g. by pure and applied mathematicians at universities. As found in Jankvist (2009), upper secondary students' answers to the question of what professional mathematicians do typically range from having no clue at all to believing that they perform some kind of 'clean-up job' consisting in finding 'errors' in already existing formulas and proofs, more efficient ways of calculating already known quantities, etc. Often such views has to do with their belief of mathematics as something a priori given; static and rigid – a belief of course not unrelated to textbooks' usual presentation of mathematical topics. Only very few students seem to believe it possible that mathematicians can come up with actual *new* mathematics. Therefore the students know neither what they accept to study if they choose to engage with the mathematical sciences, nor what they reject to study if they do not. The claim, which I am of course trying to make, is that a study of original sources may provide students with a truer image of mathematics as a scientific discipline, both pure and applied, because history in general and original sources in particular show mathematics-in-the-making as opposed to mathematics-as-an-end-product (Siu & Siu, 1979), i.e. the usual textbook presentation. Question 3 addresses this aspect by asking the students if they think that the modules enabled them to make their choice on a more enlightened basis. Question 2 is a more straight forward question asking if the modules in any way encouraged the students to pursue a study of the mathematical sciences – or to put it on the edge; if the study of original sources 'attracts' or 'rejects' in terms of recruitment.

I shall split my analysis of the students' answers into two parts: one in which I perform a small scale quantitative analysis of the students' questionnaire answers; and another in which I try to elaborate and deepen this by drawing on the follow-up interviews with the students.

QUANTITATIVE DATA AND RESULTS

Out of the 27 students, 24 answered the final questionnaire. Of these 24 students, 16 were exposed to follow-up interviews. The possible answer combinations of the students are given in table 1. The reason for distinguishing between students who were exposed to interviews and students who were not is that sometimes students would change their answer during interviews. In particular to question 3 (Q3) some students would alter their original answer, since apparently the phrasing in the questionnaire was not entirely clear to all of them. Any such changes are taken into account in the column 'Quest.+Int.'.

Combination			**Quest. + Int.**	**Quest. only**	**Total**
Q1	*Q2*	*Q3*	*16 students:*	*8 students:*	*24 students:*
Yes	Yes	Yes	6	0	6
Yes	No	Yes	7	1	8
No	No	Yes	2	0	2
Yes	Yes	No	0	1	1
Yes	No	No	1	5	6
No	No	No	0	1	1

Table 1: Number of student answers according to occurring combinations [4].

A first observation based on table 1 is that a total of 21 students (88%) agreed to the HAPh-modules having provided them with a different view of what mathematics is, how it has come into being, and what it is used for (Q1). 7 students (29%) agree to the

HAPh-modules having encouraged them to study mathematics or natural science as part of their future studies. Out of these 7 students, 4 already had decided to pursue a higher education related to the mathematical sciences, but the remaining 3 may be characterized as possible ‘win over’ students. Regarding question 3 (Q3), a total of 16 students (67%) agree to the HAPh-modules having provided them a better foundation to either select or deselect mathematics and/or natural science to be part of their future education. Taking into account that the 15 of these were the ones exposed to follow-up interviews and that 4 of these actually altered their answer during the interview after having been explained the meaning of question 3 more clearly, it is reasonable to assume that the total number could have been considerably higher had all 24 students been exposed to follow-up interviews [5]. The observation that the majority of these students actually answers ‘Yes’ to question 1 (5+1 students) only supports this further, since the majority of the interviewed students who answered ‘Yes’ to question 3 also answered ‘Yes’ to question 1. I shall return to this observation in the final discussion.

SELECTED QUALITATIVE DATA

In order to deepen some of the reported findings, I shall display a selection of students’ answers from questionnaire and follow-up interviews in order to illustrate students’ rationale behind their answers as well as possible changes in these [6].

Christopher, who represents the ‘Yes-Yes-Yes’ combination in questionnaire as well as interview, replied to question 1 that the modules provided him with “a different way of seeing things, that it [mathematics] isn’t only calculations with numbers”. When asked to elaborate in the interview, he replied:

> Well, you can say that what gave me some [insight] was all this philosophy, which lies behind, but also the way in which it has evolved... that it has evolved in order to describe a certain thing; for example that Boole used it to describe one thing, and then Shannon saw, okay, I apply it for this other thing and then develop it according to that. This connection; that it is two completely different things they are working with and they then can use the same [mathematics]... that this mathematics can be applied in so many different contexts. (Christopher, March 29th, 2012)

Regarding question 2, Christopher was already set on studying something related to the mathematical sciences, but he stated in his questionnaire answer that “the modules definitely did not reduce this desire”. Also Christopher stated that he simply had been “thinking more” while working with the original texts during the modules, than he usually did when working with the textbook in his regular classes.

Another example of a ‘Yes-Yes-Yes’ student is Katharine, who provides the following answers. Question 1: “Yes, since here connections are made between problem, solution, and practice, so that we in the end can exploit it. Then you get the ‘whole ride’ which makes it easier to understand.” Question 2: “Yes, because it wakes you up! And you then want to find out how other things were created also.” Question 3:

"Yes, because they [the modules] have provided me with an insight in what mathematics at a higher level can do." When asked, immediately after the second HAPh-module, about reading original sources, Katharine said:

> I like the original [texts] better. You kind of get inside the head of those people and think, well that's how they... Because, if there is another one [a secondary source] trying to interpret it, then I feel that they can't really figure out the original, so they take it to a lower level. Whereas I feel that you are challenged more when reading the original [text]. You get to sense how he [the author] has structured it, how he has thought, and so. Of course, it is okay to have the small explanations afterwards on what is meant with this and that... So you see, okay, that was a quick summery, and I understood that. Then you feel that you have won something, because you understand his [the author's] intention, how he carried it through, and so on. That was incredibly exciting. You felt that you got to know them a little more personally and how they expressed themselves using mathematics, explained [things], and so. Also, it gave you ideas on how to express yourself mathematically, in your hand-in tasks etc. I found that very exciting. (Katharine, November 3rd, 2011)

To illustrate the 'Yes-Yes-No' combination and the change of this into 'Yes-Yes-Yes', we shall look at an extract from the interview the student Tobey:

> Tobey: Yes, they [the original texts] gave me an understanding of how you need to think completely different. [...] It has been quite an instructive experience in that regard; kind of an *aha*-experience once you began thinking about it in relation to all the [questionnaire] questions afterwards.
>
> Interviewer: Besides you being surprised due the two modules, did they have any other impact on you?
>
> Tobey: What they impacted is that I now may consider, well not to study mathematics directly, but to study something where you use mathematics to more than what you use it for in physics... Because it is a deeper discipline than what you usually think it to be, with just formulas, plus, and minus. [...] There's more to it. It can be applied to several things, at least in relation to these... It would be cool to look at those problems which have been posed, but which have no solution yet... It would be cool to be involved in finding the solution to just a single one of them. It would be completely awesome.
>
> Interviewer: But then you answer ['No' in question 3] here... What I might have hoped is that regardless of you wanting to study math or not, then the modules might have provided you guys with a more... well, done that you could either select or deselect on a more enlightened basis?
>
> Tobey: But it has! I mean, after these [questionnaire] questions my answer has definitely changed, because yes, they can do that. (Tobey, March 29th, 2012)

As Tobey, the student Nikita also expressed being more open to possibly pursue a study related to mathematical sciences after the modules than she had been prior to the HAPh-modules:

> I do think that I have got more of a reason to select it, than I had in the beginning [of upper secondary school], because we've seen several different aspects of it [mathematics] due to these modules. If I had only been working with the textbook and so, my answer would definitely have been 'No', I believe... because it's very monotonous and much of the same, whereas with the two modules we've had the opportunity to think differently and view things through different lenses and... Yeah, see the interrelations in a more comprehensive way than we usually get things presented. So, personally I've discovered that there is much more to mathematics than what it says in the textbook. [...] I think it surprised me that someone actually has been sitting and working with these things, and then arrived at this. Because before I've only thought about mathematics as something just being there, and us as just having these and these things which we could make use of. I've never given it a thought that someone had sat down and worked on it and arrived at something to be used in certain contexts... I've never thought about it like that, only in the way that it's in the textbooks and that's just the way it's given. (Nikita, March 29th, 2012)

In both questionnaire and interview, Nikita is quite clear on answering 'Yes' to question 3, almost as if this is implicit to her due to her answer to question 1. Due to Nikita's positive change from questionnaire to interview regarding question 2, she is counted as a 'Yes-Yes-Yes' student in table 1. Regarding her encounter with original sources, Nikita had on a previous occasion expressed herself positive regarding this:

> ... not only did you have to understand what it was about, you also had like the language of it, and it has been a different way of thinking compared to the mathematics we are usually taught, where we have this formula and it works like this, this, and this. Here you got all the background knowledge, and how he arrived at it, etc. For me, I personally think that I get much more interested, when I see it all, than if I'm only told that now we are studying vectors and we must learn how to dot these vectors and then we must be able to calculate a length, right. That's all very good, but what am I to use it for? Whereas, when you know about the background, the development up till today, that I think was exciting. (Nikita, November 3rd, 2011)

As a representative of the 'Yes-No-Yes' combination we shall take a look at an extract from the interview with the student Liza, who is also quite settled on her positive answer to question 3:

> Interviewer: Let us take *you*: You are not interested in studying mathematics?
>
> Liza: No.
>
> Interviewer: No. And after you've followed these modules, do you then feel that you know what you say 'No' to, to a higher degree than if you had not been through the modules?
>
> Liza: For sure, I do.

Interviewer: You do?

Liza: Definitely, yes. I would never have thought mathematics at the university to be about things such as Euler...

Interviewer: As [Euler's] graph theory?

Liza: Graph theory! Or I might have thought, okay, there is something related to graphs. But graph theory, no. So yes, it certainly did provide me with a more enlightened basis. (Liza, May 22nd, 2012)

Another student representing the 'Yes-No-Yes' combination is Salma, who give the following three questionnaire answers. Question 1: "Yes. It has shown me how mathematics develops, and at the same time how mathematicians work with mathematics. And that mathematics is its own language." Question 2: "No, I must admit that it hasn't. I do find it [mathematics] quite interesting, but there are things which excite me more." Question 3: "I have never considered studying mathematics. But if I had, then it would have been nice with these modules, since I feel that you will know much better what you agree to study."

A third representative of the 'Yes-No-Yes' combination is Sophia, who explains her encounter with original sources in mathematics as follows:

> Regarding the modules, even though it has been a little dry from time to time, I do think that it has been nice to get the historical [dimension], to read the original texts, and do it the way they did, the people who developed things. [...] ... in order to get it at a slower pace... to try and figure out 'what the fuck is going on here?' That is, instead of just sitting and doing exercises, which you do in school. To try something completely different, something which might be more similar to what they [the originators] actually did. (Sophia, March 27th, 2012)

DISCUSSION AND RESULTS

Both Tobey and Nikita are representatives of the previously mentioned potential 'win over' students to the mathematical sciences as a consequence of their studies with the original texts in the HAPh-modules. For both of them it seems quite clear that this 'encouragement' to possibly pursue mathematics further is due to the effect of the original texts on their view of mathematics. Nikita gives as reasons the interrelations between different parts of mathematics which the texts reveal, the fact that mathematics has come into being by the hands of human beings, and not least the different way of working when studying an original source as opposed to the regular textbook. Tobey stresses the dimension of creativity in research mathematics and refers enthusiastically to the posed but yet unsolved problems in mathematics (with an implicit reference to the text by Hilbert). Also, Katharine was an example of a 'win over' student, since she prior to the second HAPh-module had no intentions of possibly pursuing the mathematical sciences. Regarding the four other students who gave positive answers to question 2 (table 1), the thing to notice is that the modules –

and the reading of original sources – did not diminish their desire to pursue the mathematical sciences. As illustrated by the quotes of Christopher and Katharine, it may have even enhanced it.

Although the above suggests that a use of original sources may have as an outcome that some students can be encouraged to study the mathematical sciences, the more important finding of the study is that in relation to question 3. Namely, that 2/3 of the students agree to the modules having enabled them to either select or deselect future studies involving mathematics and/or natural science on a more enlightened basis. For the majority of these students this appears to be directly related to the original texts having provided them with a different view of mathematics as a discipline (Jankvist, 2012a). Of course, that a use of original sources can change students' view of mathematics is not a new finding. Still, the present study confirms it once again. What is new in this respect, however, is that this study, although small in scale, suggests a direct connection between a students' positive answer to question 1 (Q1) and a positive answer to question 3 (Q3) [7]. Hence, the more important finding of this study is not necessarily that a use of original sources may 'win' some students over to the mathematical sciences, but that the students who are 'won over' are done so on a more enlightened basis. (Equally important is of course that the students who deselect the mathematical sciences also do so on a more enlightened basis.)

This again has a direct relation to the problem of retention, as mentioned in the introduction. Because if the students who enter the mathematical sciences at tertiary level have a more realistic image of the discipline which they are about to study, then surely one would expect a higher degree of retention among such students.

NOTES

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2. The precise references to the original sources may be found in the teaching modules, which are available as texts no. 486 and no. 487 at: http://milne.ruc.dk/ImfufaTekster/

3. The panel on "Empirical research on history in mathematics education: current and future challenges for our field" at HPM2012 in Deajeon organized by Uffe Thomas Jankvist along with panellists David Pengelley, Yi-Wen Su, and Masami Isoda.

4. No answers of the combinations 'No-Yes-Yes' and 'No-Yes-No' occurred.

5. As in any other interview situation, there is always the possibility that the interviewee is trying to please the interviewer by answering what (s)he thinks the interviewer wants. The way of trying to avoid this here was to invite the students to elaborate on their answer to question 3, and when doing so some students would realize more clearly the meaning of the question and change their answer from 'No' to 'Yes'.

6. All student quotes have been translated from Danish.

7. If we follow through with this, we may assume that the 6 'Quest. only' students who answered 'Yes' to Q1 but 'No' to Q3, might have altered their answers had they been explained the question more thoroughly in an interview session. Taking the 2 'Quest. + Int.' students who answered 'No' to Q1 but 'Yes' to Q3 as a source of error, we would get 20 out of 24 (83%) instead of 16 out of 24 (67%). But this is of course to some degree speculation.

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HISTORY OF MATHEMATICS AS AN INSPIRATION FOR EDUCATIONAL DESIGN

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Use of historical and cultural perspectives in education supports the development of mathematical concepts which are thus not as usually based on logical relations. They allow embracing new individual contexts of experience as well as methods of science and humanities. The use of models of historical development and the need of understanding of phenomena that foster or block learning challenges teachers as well as learners. It will be shown at the hand of concrete and relevant examples how historically, culturally and socially inspired problems can encourage an alternative approach to well-known mathematical concepts and deepen understanding.

INTRODUCTION

The employment of historical examples can enrich mathematical teaching in various ways. From the viewpoint of a historian, problems of inadequate methodology and a lack of contextual understanding might easily occur when historical material is used in the classroom. In this paper we do not put stress on the professional handling of historical material. We would like rather to consider the use of historical material as a source of inspiration in educational design and as a diagnostic mean.

In the first part we discuss different approaches to the employment of historical content at the hand of existing forms of presentation of historical material in teaching. Hereby we pay attention to the representation of the historical material and its potential to initiate in the class room a discussion of the socio-cultural aspects connected to mathematical topics. The second part will exemplify the relations of various models of historical development and will make related positions in learning theories explicit. This constitutes the basis for the conceptual understanding of historical aspects by means of mathematical awareness in the third part. Here we are guided by examples from mathematics textbooks and examples from the education of pre-service mathematics teachers for grammar school.

FORMS OF EMPLOYMENT OF HISTORICAL CONTENTS IN TEACHING

The use of historical material in educational design has a long tradition. For instance Walther Lietzmann gave in 1921 a lecture course on recreational mathematics at the University in Göttingen and emphasized the important role of social and cultural aspects as well as the potential of historical contexts for the design of mathematics instruction. He encouraged the publication of corresponding materials with direct use for teaching. Although Lietzmann gave in (Lietzmann 1922) extensive references, the discussion of the authenticity of the presentation of historical content (Fried 2001) was not a predominant issue. In Lietzmann's book impulses to reflect about mathematics and incentives to deal with mathematical problems are given by means of varied, witty and curious comments. In contrast to the approach of a historian, where the existence and under-standing of later mathematics is assumed, and is even part of the background, Lietzmann's approach does not involve an understanding of modern mathematics. Historical contents and examples are used to make other points of view accessible to the reader, to free her from thinking routines by the use of metaphors, unexpected representations, and confrontations in order to enhance a deeper understanding and ludic handling of mathematical objects.

There are various models of historically oriented mathematics teaching developed by historically interested mathematics educators (see Fauvel & Van Maanen 2000, Jankvist 2009), however, for Lietzmann history of mathematics is a source of inspiration for storytelling. In most current mathematics textbooks we can find some historical content. The presentation forms range from historical anecdotes to excerpts from historical documents (Kronfellner in (Kronfellner 1998) gives an overview of commonly used forms of historical content in the classroom). These materials are usually designed as insets or appendices that fit in the textbook design. They are additions to the canonical representations in the given textbook. The related omissions and circumlocutions thus easily lead to erroneous ideas about the historical development of mathematical ideas. In developing a socio-cultural context of the historical material the teacher should be aware of adjustments that have possibly been made.

Many textbook examples of historical inserts in mathematics textbooks show the effort to bring authenticity with learning and reading habits in line. The language and symbolism used correspond only to a small extent with those in the original historical sources. The presentations often build on the student's presumed skills and routines. The student is more likely to reproduce logical steps in the modern style than that she is stimulated to think in the framework of the historical context.

In the design of learning environments based on historical sources the deliberate inclusion of national diversity in language and culture that can be found in the classroom can often be rewarding. An example would be the inclusion of original texts. Another practical link between the mathematical content and Islamic culture is shown in (Moyon 2011), by attributing geometric problems of area decomposition to inheritance rules in the form of rules for splitting acreage.

Admittedly, a playful handling of historical events, oriented at imagination, transfer and variation is in danger of losing its historical authenticity but it can stimulate a much broader mathematical activity. Everyday knowledge and intuitive ideas can contribute amenable to different perspectives. When you want to understand historical hypotheses related to the constellations of celestial bodies (Jahnke and Wambach 2011), unwanted references to reality and modern general knowledge can interfere with the unbiased view. The latter interference can be avoided by an initial settlement of the problem in a fantasy world. The necessary alienation of everyday experience of space and time can be generated playfully by the transition into computer games, science fiction and fantasy-inspired worlds. The resulting relationships between socio-cultural and extracurricular mathematical approaches allow diverse developments. Versions of the transmission of mathematical problems in other worlds (in the literature there are many examples, e.g. Jules Verne, Kurd Laßwitz, Ian Stewart, Stanisław Lem, Terry Pratchett), and social relations in the world of mathematical objects (Isaac Asimov, Edwin A. Abbott) allow direct involvement of dramaturgic tools to deal with interaction of cultural and historical mathematical and scientific phenomena. Taking together these and many other examples, we can see a wide variety of possible inspiration from historical material in the mathematics classroom.

HISTORICAL PERSPECTIVES ON MATHEMATICAL DEVELOPMENT AND CONCEPT DEVELOPMENT IN MATHEMATICS EDUCATION

The references and school examples listed in the first section show that historical materials in textbooks usually occur as a reference to historical sources or as an illustration by means of a historical record of the mathematical concepts, objects, methods, or mathematicians just treated in the class. History here is strongly identified with the existence of historical sources, usually without paying attention to the historical source itself. This may partly be due to the fact that the use of historical sources for general goals of mathematics education, such as to experience mathematics as a living, evolving science or to create access to the cultural heritage, are difficult to interpret in such a general formulation. Thus the tool is turned easily into the goal and the historical work is reduced to mere reference to the existence of historical sources. The indeterminacy inherent in the historical approach can be ignored because there is the simple control feature *availability of the historical source*. The exciting and very challenging task of designing a learning environment based on the historical source cited in the textbook by introducing historical contexts rich with potential for socio-cultural, mathematical and scientific development lies in the hands of the teacher. How can, however, a short section of a historical text be turned into a mathematical development and what can actually be understood under mathematical development?

In designing such learning environment the question of the underlying concept of a scientific theory, the choice of the general philosophy and of the appropriate developmental model plays a crucial role: is the historical progress based on endogenous and exogenous factors; is it based on the activities of some eminent personalities, or are there rather economic, political and social factors that contribute to the development of this subject? Is it necessary, for an appropriate understanding of the development of mathematics, to turn to methods such as source-analysis and source-interpretation, ethno-mathematical approaches or other perspectives?

For a historian in the choice of the developmental model the scientific criteria are in the foreground. When contextualizing a historical fragment, the distinction between perspectives on history as a tool and as a goal and further differentiation of these categories as introduced by Jankvist (2009) seems natural and fruitful. In the development of learning environments by means of historical sources the entertaining potential can also play a role. Whiggish or present-centered representations (for this distinction and for a discussion of their unscientific nature see (Wilson et al 1988)) can still inspire genetic instruction or, as the first part exemplified, by means of relocating the actions, personalizing circumstances, and introducing other ways of alienation we can trigger a change of perspective leading to deeper understanding. The mathematics educator, not bound by the norms of scientific rigor of the historian, is open to other developmental models and many more possibilities of the use of historical sources in developing teaching environments.

In a class, the concern is less on the development of mathematics as such as on the development of specific mathematical notions and concepts.

For the development of mathematical concepts in the classroom an important role is played by the implicit phase in which a mathematical concept is not systematized, defined or referred to, which phase is for the historian hard to grasp and not often studied. As appropriate structures or regularities at this stage, however, often appear in the form of a problem solving method or a representation not really anchored within the context of the language of mathematics, the implicit phase can be helpful in instructional design in motivating a definition or in helping to formulate a problem aimed at introducing a concept.

From a historical perspective, there are helpful approaches to extend intension-extension of the conceptual dynamics, such as in Hans Wußings model by *ostension* (Scholz 2010). The presentation of the development of Euler's formula, as Imre Lakatos in (Lakatos 1976) has reconstructed it, is oriented before all on the development of mathematics as a language. This developmental model may be whiggish for a historian. However, it makes important issues of epistemological debates of the last century accessible to a class discussion and it makes tangible the otherwise hardly conveyable idea that even the meanings of mathematical concepts are negotiated. For the long-term conceptual development in the classroom it is worth to think about what historical or other socio-cultural developmental models reflect the dynamics described in Figure 1. Even for local ordering and linking of concepts in school mathematics it is useful to deal with regularities in the development of mathematical language and with regularities in the development and formalization of mathematical methods from a historical perspective (Kvasz 2008).

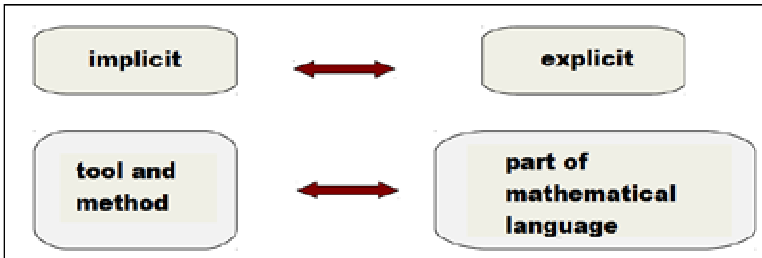


Figure 1: Dynamics of the development of mathematical concepts.

DESIGNING LEARNING ENVIRONMENTS BY THE AID OF HISTORICAL PERSPECTIVES AND MATHEMATICAL AWARENESS

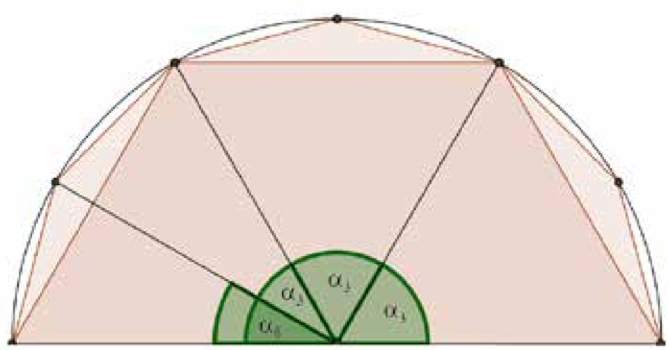
The development of mathematical concepts in the classroom is based on omissions, substitutions, rearrangement distortions, misrepresentations and other customizations transformations of relevant historical processes. On the other hand, there is not the one historical process of a concepts development but many aspects and perspectives under which one can see development as already discussed in the previous chapter.

The following example (Figure 2) shows that historical jackets of mathematical problems do not necessarily initiate a change in routine approaches or standard solutions. In teacher training seminars the solutions students had for this text book problem were hardly related to elementary or historical approaches.

These solutions, with only minor variations occurred in multiple (parallel) seminars on mathematics education. The tasks were part of a seminar presentation prepared by students on real numbers. The student's answer was to express half the length of the edges of half a regular n-gon by the sine of the angle π/n. They expressed their calculation using the calculator and multiplied this length by n/2 also with a calculator. The presentation of the solution consisted of giving symbolic expressions. The irrationality of the terms of the sequence, which approximate π was not noticed because the computer automatically rounded. Reflections on the convergence of the sequence and the transition from geometric objects (lines) to arithmetic sequences, i.e. sequences of numbers were not made.

Approximation of π

The determination of the number π was in the history of mathematics an important task. Various processes have been developed. One method is to approximate a semicircular arc of radius 1 by a line of equal chords.



a) Calculate π for approximately three equal sections. Evaluate the results.

b) Develop a process that allows you to find a better approximation of π.

Figure 2: Example of an exercise in historical jacket.

Despite the historical jacket and the instruction for a direct calculation, the students calculated the solution with calculator using the sine function even for half a hexagon – although the three equilateral triangles give the approximation of π by 3 accurately. If you want to combine the cognitive process with the historical one, the reason for the superficial solution can be seen as a lack of *experimental* and *intuitive* awareness (see Kaenders and Kvasz 2011). The easy change between the geometric and arithmetic representation was based less likely to automation or a deeper understanding of the limit, but on the lack of experience with pathologies. The latter historically led to the necessity of the precise formulation of the limit concept. The two examples in Figure 3 (Lietzmann 1949) above do not represent this historical development of a precise formulation of a geometric limit. However, they show that they are necessary and can therefore lead to doubts about the solution and give rise to rethink the solution.

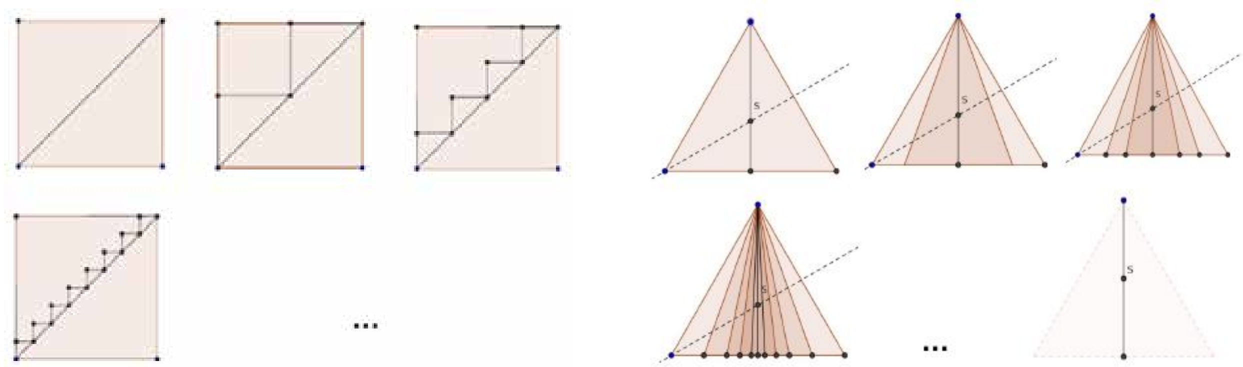


Figure 3: A mistaken approximation of the length of the diagonal of a square. (left) A mistaken determination of the center of gravity of a segment as the limit of the centers of gravity of a family of triangles. (right)

Simplifications of the long historical development of the concept of limit in the classroom are of course essential. Omission of pathologies in this example could be seen in relation to the historical development as an aggravating, trivializing simplification rather than a support to conceptual understanding.

The next example relates to a change in the usual representation of the solution of systems of equations. Given a system of linear equations with two equations and two unknowns, such as:

$$2x + 2y = 3$$
$$-5x + 2y = 2.$$

The approach proposed by the math textbook begins with the algebraic solution of the system: by equivalent transformations, the two equations can be converted into two equations, from which the solution x = 1/7, y = 19/14 can directly be read of. The corresponding geometrical solution process is started with the visualization of a given system of equations. Here, the linear equations can be brought into a form from which the transition *equation* ↔ *geometric object* is routine operation.

Since this has been trained in the context of drawing graphs of linear functions, it is a common textbook exercise to find the equation of the form y = mx + b, when the line in a coordinate system is graphically given. Just examples of the form x = b cannot be selected. In the next step, the transition between the expression y = mx + b, and the function graph is automatized. The visualization of the algebraic method or the "geometric solving" of the system of equations is now to determine the two function expressions corresponding to the initial equations, to draw the graph and read of the coordinates of their intersection.

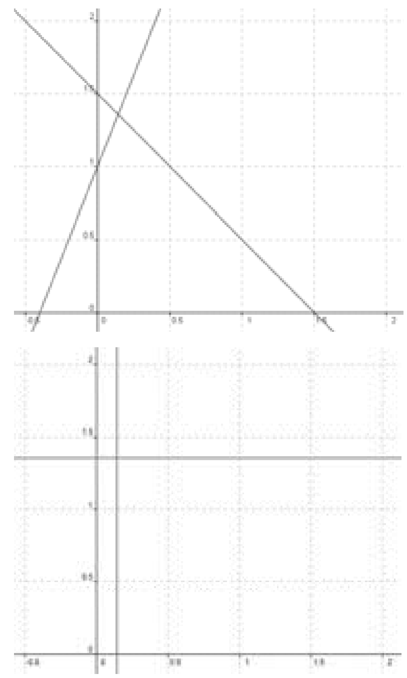


Figure 4: Visualizing of equivalent systems of equations:

$$\left|\begin{array}{l} 2x+2y=3 \\ -5x+2y=2 \end{array}\right| \Leftrightarrow \left|\begin{array}{l} x=1/7 \\ y=19/14 \end{array}\right|$$

The lower situation does not exist in the language of school books.

The use of the concepts linear function, with slope and intercept, the ability to draw straight lines and the associated determination of the intersection of two graphs of functions by reading of the value of a function at a point are required as appropriate skills. The concept of equivalence transformations – in this case to determine the nicest representatives $x = x_0$, $y = y_0$ from the set of all pairs of linear equations with the solution set $\{(x_0, y_0)\}$ is not evident in the described approach: in the geometric representation of the equivalent transformations in each step the two lines preserve the intersection of the lines. Since the straight line $x = x_0$ is not common in the understanding of function graphs, the geometric representation of the algebraic solution method is reduced to the visualization of the given lines, and reading of the intersection. The described way of visualizing the equations, graphs is probably chosen also because the understanding of analytical geometry of curves in the plane, especially of lines is not developed.

The concept illustrated in Figure 4 of equivalent transformations and conservation laws, which is essential for the solving of systems of algebraic equation of higher order, and appropriate geometric representations by hyperplanes cannot be generalized by the transition to function graphs and values of a function at a point.

The language used in the example shows above all a lack of contextual awareness. Instead of a change of representation a visualization of an equation is performed. From this we can conclude a lack of logical awareness, too, i.e. the role of systems of equations in an adequate development of theory is ignored. When we orient ourselves in the historical process, then the later and in a different context developed representation of functions as graphs of functions in Cartesian coordinate systems is rather misleading here – the Cartesian coordinates go back to René Descartes (1637) and the concept of a coordinate change to Christiaan Huygens (1656). One could denote the misplaced use and the transfer of concepts that are developed in other contexts as *ahistorical implant*.

Another example deals with the currently conventional introduction of integral calculus, where three different aspects of the integral are introduced simultaneously:

- Integral as oriented area,
- Integration as anti-derivative,
- Integration as a way to determine a function from a given function of change.

One aim of this introduction is the motivation and direct introduction of the fundamental theorem of differential and integral calculus. To allow a direct connection between calculation of area and the determination of a function from the function of change, the change function is replaced by a step function, and thus the oriented area under the curve is replaced by the area of rectangles. The original functions are monotone, the determination of the step function is carried out graphically by estimating the areas of respective triangles. The replacement of the function by a step function aims at simplicity of calculation, but does not fit into the framework of the developed mathematical language of calculus: The geometric transition from secant to tangent with slope or the notion of actual speed would suggest an approximation with piecewise linear (rather than piecewise constant) functions. A historical insert of the parabolic segment method of Archimedes would show that it is a technically challenging problem and so, it could motivate the need for the later introduced Riemann sums.

The language used shows *experimental* and *intuitive* awareness. Upon further conceptual and technical development of the concept of the integral taken simplifications can nevertheless lead to motivation and understanding of issues and the development of logical and theoretical awareness counteract. The summary described the technically difficult problem of the definition of the Riemann integral, and of conceptual understanding of the relationship between differential and integral calculus will not do justice to the complexity of the task, so you might call the representation as a caring shortening.

A well-known transformation of historical contexts is what Freudenthal formulated as a criticism to the New Math movement *(anti-didactic) inversion*, which even today often determines the representation of higher mathematics at the University:

> "… the final result of the developmental process is chosen as the starting point for the logical structure in order to finish deductively at the start of the development. This genetic-logical inversion expresses itself as a didactical – or rather antididactical – inversion." Hans Freudenthal, 1991, Chapter 11, S. 305.

In this kind of concept development the focus is on the long-term development of *logical* and *theoretical* awareness. The inclusion of an implicit history of the development of a concept and the *ostension* of a notion would lead to a more balanced mathematical awareness that also includes *intuitive* awareness.

And finally, we introduce the well-known parable of Achilles and the tortoise by Zeno of Elea (495 - 430 BC.) Still today it is treated in many math textbooks. Unfortunately the Zenon paradox causes rarely the expected confusion or amazement. That may be partly due to the role of the problem as an application to the convergence of geometric series, which might inhibit a direct confrontation with the verbal formulation of the paradox.

It was interesting to observe that the arguments in the teaching seminar consisted in the verbal recapitulation of the steps. However, to find own formulations and phrases related to the initial formulation of the Zenon paradox were a problem to our teacher-students. Paradoxes force a change of perspective, re-orientation and provoke cognitive conflicts. One way to achieve this would be by the following possible question of Thomson (1954): Suppose a referee actuated whenever Achilles catches up to the new position of the turtle, the switch on a lamp. If the lamp is switched on or off when Achilles overtakes the tortoise? Also indirectly present ideas, such as the overall presence and familiarity with real numbers and their completeness, separate the students from the original and the paradoxical phenomena. The neglect of such indirectly involved factors when considering historical episodes can be called *cultural alienation*.

For the analysis of the concept development of the textbook examples, it was helpful to further investigate, identify and name the changes made to the historical development of concepts (e.g., reductions, simplifications, omissions, ...). To draw attention to several emerging problems in the integration of historical content, we have introduced these following terms: *(ahistorical) implant*, *(caring) abridging*, *(trivializing) simplification*, *(anti-didactic) inversion* and *(cultural) alienation*. For the previous investigation of handling and development of concepts in teaching at the hand of concrete textbook examples and the description of possible problems of precise formulation, of conceptualization and transfer the perspective of mathematical awareness was useful. Based on an analysis of the mathematical language, we examined two aspects: First, what qualities of mathematical awareness do not occur in the language and symbolism and second, by which evolution of the subject could these qualities of mathematical awareness be developed.

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THE HISTORY OF 5TH POSTULATE: LINKING MATHEMATICS WITH OTHER DISCIPLINES THROUGH DRAMA TECHNIQUES

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We present the design and implementation of a cross-curriculum project concerning the history of 5th postulate that was carried out among 11th grade students in a public school in Athens. We used History of mathematics as a unifying framework for an interdisciplinary curriculum through 'Drama in Education' techniques. Drama in education contributed to a creative way for the students to reflect on broader learning issues (emotional, cognitive, social and political) mostly because of Drama experiential dimension. Students' active involvement through this framework helped the students to conceive both mathematics and other disciplines knowledge in a meaningful context and made them realize that mathematics is a cultural and social construction; a component of our world.

INTRODUCTION: HISTORY OF MATHEMATICS AND INTERDISCIPLINARITY

Specialization in education is a modern phenomenon, which results in viewing present day school mathematics as completely separate from other subjects of the curriculum. Meanwhile, school administration and timetabling of classes also often work against efforts to make links between subjects (Grugnetti and Rogers, 2000: pp.52) making mathematics teaching ineffective (Anice, 2005).

This finding leads to the reassessment of the content and way of teaching mathematics. Interdisciplinarity is suggested as a way of "injecting" meaning to mathematics. This rationalization process is associated with personal experiences and the collective engagement in an activity. Interdisciplinarity is perceived as a complex process which, in the context of teaching design, requires the detection of each child's personal rational experience and the cooperation in groups based on common interests in complex matters (Kalavasis, 2008: pp. 24).

Interdisciplinary methods strive to create connections between traditionally discrete disciplines such as mathematics, the sciences, social studies or history, and English language arts (Coffey, 2011). During interdisciplinary teaching, educators apply methods and the language of more than one academic discipline to examine a theme, issue, question, problem, topic, or experience, enabling students to see the links between various subjects and consequently to see the knowledge in a unified way, making their school experience more compatible with their real life. The objective of interdisciplinary teaching can be orientated towards content, method, competence and ways of thinking, as well as to the characteristic forms of subject- parallel and comprehensive-orientation (Bechmann, 2009: pp.32).

Throughout the literature we encounter different kinds of interdisciplinarity. In Jantsch's (1972) five level taxonomy of disciplinarity the three middle ones that may be considered as 'true' types of interdisciplinarity are: *pluridisciplinarity,* when various disciplines, usually at the same hierarchical level, cooperate without coordination about a common theme, *crossdisciplinarity,* when the axiomatics of one discipline are imposed for support upon other disciplines *and interdisciplinarity (proper)* when there is coordination in all involved disciplines by a higher level concept (Jankvist, 2011).

Research has shown that interdisciplinary and cross-curricular teaching can increase students' motivation for learning as well as their level of engagement and thus provide conditions for effective learning (Thaiss, 1986). Barton and Smith (2000) explain that interdisciplinary units enable teachers to use classroom time more efficiently and address content in depth, while giving students the opportunity to see the relationship between content areas and engage in authentic tasks. '*Interdisciplinarity is increasingly viewed as a necessary ingredient in the training of future oriented 21st century disciplines that rely on both analytic and synthetic abilities across disciplines'* (Shriraman, 2009: v).

The History of mathematics provides a suitable framework both for integrating varied disciplinary areas of curriculum, as well as for helping students with their understanding both of mathematics and other subjects. Keeping in mind that the history of mathematics as the history of ideas is strictly linked to the history of human beings, we have to analyse the cultural, political, social, economic contexts in which ideas arose (Grugneti and Rogers, 2000). Interdisciplinarity through history of mathematics can reveal this wider aspect of mathematics as a cultural activity; as a human activity both done within individual cultures and also standing outside any particular one.

This integration of history is not confined to traditional teaching delivery methods, but can often be achieved through a variety of media such as doing projects, watching films, constructing models, researching history in libraries, devising dramatic presentations, surfing the www, which add to the resources available for learner and teacher (Nagaoka, 2000).

In this paper, based on our research results, we claim that the use of 'Drama in Education' in Mathematics teaching motivate students to reflect on broader learning issues (emotional, cognitive, social, and political) mostly because of Drama experiential dimension.

HISTORY OF MATHEMATICS AND 'DRAMA IN EDUCATION'

Drama in Education is (DIE) is a form of theatrical art in which the child while creating and playing roles, projects himself into fictional characters and situations, exploring and expressing his ideas with his body and his voice (Alkistis 2000: pp.78). It is a highly structured pedagogical procedure utilizing specific rituals and techniques of dramatic art which grants us, through the creation of an imaginary world, the context for teaching a notion, an idea, a fact, a solution of a problem as well as the potential of cultivating personal and social skills. It constitutes a dynamic and creative tool for the teaching of different subjects of the syllabus through collective and experiential activities (Andersen, 2004), putting children in the position of the actor (experience), audience (critical ability), author (meaning) and director (form) (Neelands 2008). DIE combines a) form and content b) action and reflection c) logic and imagination d) thought and emotion and e) body and soul.

Regarding the teaching and learning of mathematics, the research of the influence of introducing drama in the teaching process, limited in primary and junior high school, is very encouraging in students' understanding and retention of mathematical notions (Saab, 1987, Omniewski, 1999, Fleming et al., 2004, Duatepe, 2004) and in creating positive impact in their attitudes towards mathematics (Duatepe, 2004). Specific techniques such as the 'as-if' approach can create the context for teaching a concept, an idea or an event and offer opportunities for exploring mathematics in a variety of historical, social, political and cultural contexts (Kotarinou et al., 2010, Ponza 2000b, Pennington and Faux, 1999).

THE RESEARCH METHODS AND CONTEXT

Empirical data for this paper arose from our research on exploring the dynamics of Drama in Education Techniques in teaching Geometry in high upper school. This paper discusses the following research question: does and in which way interdisciplinarity through history of mathematics in a DIE setting affect students' learning as well as their epistemological beliefs about mathematics?

The setting: the research was carried out in a group of 26 (11^{th}-grade students) from different directions of studies in the 2^{nd} High School of Ilion (Athens, 2010-11) during four months.

The method: we designed and implemented an interdisciplinary didactical intervention, a teaching experiment (Chronaki, 2008). The teaching experiment concerned the teaching, through Drama in Education techniques, the axiomatization of Euclidean and Non-Euclidean Geometries as well as the history of Euclides' fifth postulate. The question of judging the effectiveness of integrating historical resources into mathematics teaching may not be susceptible to the research techniques of the quantitative experimental scientist. It is better handled through qualitative research paradigms such as those developed by anthropologists (Fauvel & van Maanen, 2000, pp. xvii).

Exploitation of ethnographic research techniques (observation-interviews) helped us gathering research data. All students' presentations were videotaped and analyzed regarding the proper use of mathematical notions in their dialogues, while some episodes of students' group work were audio recorded and analyzed regarding the role of Drama as a mediating tool for the negotiation of meaning and the development of understanding.

The interdisciplinary project 'The history of 5th postulate. From Euclid till non-Euclidean geometries': Within this paper we present the unit of the teaching experiment which concerned the history of the fifth postulate as one of the five set by Euclid and it being challenged as an independent one by mathematicians of classical Islam and by the West mathematicians of the 19th century, untill its replacement following its refusal by Bolyai, Lobatscevski and Riemann.
The researcher in teaching role carried out the project in 18 teaching periods, in Geometry, History, Literature, Greek Language, and Ancient Greek Language class. The three main points of the interdisciplinary project were: the mathematics per se, the Mathematicians and the historical, cultural political, social, economic contexts in which these mathematical notions arose. Our teaching aims were: a) the students acquainting with the axiomatization of Euclidean Geometry b) the students perceiving the role of the postulates in the axiomatic foundation of a science c) through the errors of different proofs of the fifth postulate, the students to acquaint with the various equivalent with the 5th request propositions that characterize Euclidean Geometry d) students perceiving that Mathematics is constructed through its development located in various specific historical, geographical, cultural and social contexts.

The project consisted of the following units-steps:

1. *Euclid's Elements and the axiomatization of Euclidean Geometry* (6h): The relation between Euclid's axiomatic foundation of Geometry and Aristotle's 'Logic' was studied as well as the definitions, the Postulates and Common Notions from the Book I of Euclid's 'Elements'. Knowing from Thomaidis and Tzanakis, (2010) research, that the study of original texts in the classroom creates a new didactical environment in which students actively participate in the classroom discourse and exhibit a positive attitude towards the subject, which never happens in conventional geometry teaching, we gave students, in Ancient Greek language class, excerpts from Euclid's *s* original texts and requested them to read and translate the text. In Geometry class, students in groups were asked to answer questions concerning the mathematics of the text, after having studied some relevant excerpts of the book 'The Historical Roots of Elementary Mathematics' by Bunt Jones and Bedient (1976). Then students prepared and had performaces concerning the postulates, the common notions and some definitons of Euclid axiomatic foundation of geometry, using drama techniques as 'role-playing', 'reportage', 'alter-

ego', 'interview'. We must notice here that in this unit students for the first time came into contact with Euclid's own form of the fifth postulate.

2. *Euclid and the historical, cultural and political frame of his era* (4h): The purpose of including elements of the history of mathematics has to do with showing the students that mathematics is dependent on time and space, culture and society, that mathematics is not 'God given' and that humans play an essential role in the development of it (Jankvist, 2009:2734)**.** For this reason, in order the students to know the historical context in which Euclids' Elements were written, a digital presentation was held by the researcher in History class, concerning Alexandria in the Hellenistic period, while excerpts from the book "The Parrot's Theorem" were read concerning the history of this era, as well as the reasons for the blossoming of Mathematics in this historic period and area. Knowing that Dramatization is an important tool in the repertoire of a teacher for humanising and contextualising the development of mathematical concepts (Hitchcock 1996a; 1996b; Ponza, 2000a, 200b), the chapter 'Euclid's conceit' from J. P. Luminet book 'Euclid's bar', presenting the mathematician Euclid and his era, was read expressively by some students in Literature class, while some scenes of the same chapter concerning differences in thought between Pythagorians and Euclid, as well as historical anecdotes about Euclid were dramatized.

3. *'History in shadow': the controversy of Euclid's Fifth postulate till 18th century (5h)*: There is a common belief held by many, teachers and students alike, about the static nature of mathematical concepts (Jahnke, 2000). The history of Euclid's Fifth postulate provides the potential to undermine this entrenched perception. In mathematics class the unsuccessful efforts of Arabs mathematicians Thabit ibn Qurrah, Al-Haytham, Omar Khayyam and Nasir al-Din al-Tusi as well as Saccheri and Lambert to prove the famous Euclid's fifth postulate were presented, while equivalent to the 5th postulate propositions were presented to interpret the mistakes made in these efforts. In history class the development and the reasons of development of Mathematics in the Islamic world, from the 8th to the 13th century AD, were discussed. For this reason, a short extract from the book *''The Parrot's Theorem''* of Denis Guedj (pp. 269-272), which refers to the ''House of Wisdom'' in Baghdad and its role in the collection and translation of the work of the ancient Greeks, was read. A combination of 'Shadow Theatre' and 'role playing' was used for presenting all these unsuccessful efforts as well as the mistakes made at the proofs.

4. *János Bolyai, Lobachevsky, Riemann, the founders-creators of non Euclidean geometries (3h*): As the biographical allusions serve the purpose of humanizing concepts (Ponza, 2000a) students, in Greek language class, studied the biographies of the two latter mathematicians from Bell's book '*Men of Mathematic's''* as well as quotes from letters from Gauss to F. Bolyai and berween father and son Bolyai. Different Drama

techniques as ‘Letters’, ‘Portrait’, ‘Role on the wall’, ‘Conscience alley’, ‘Conflicting advice’ were used for presenting their work and some snaps shots of their lives.

Integrating history of mathematics invites us to place the development of mathematics in the scientific and technological context of a particular time and in the history of ideas and societies (Jahnke, 2000). Hence, we emphasized that many times dominant philosophical stances during a historical era act as obstacles to the challenge and the overcome of assumptions in mathematics, as well as to the publication and widespread acceptance of new mathematical theories. An example lies in our case of concern, the controversy of the uniqueness of Euclidean Geometry. Within this context we mentioned the example of Immanuel Kant, the dominant philosophical physiognomy of the century that preceded the discovery of non-Euclidean geometry, whose ideas about space and geometry acted as an obstacle on the acceptance of non-Euclidean geometry. Some attribute the non-publication of Gauss's ideas on non-Euclidean geometry just in his unwillingness to conflict with Kant (Davis, 2007: pp. 133). In this unit

RESULTS AND DISCUSSION

Analysis of the dialogues in students’ Drama performances suggests that students conceived the mathematical notions that they had been assigned to present, integrating them correctly in their dialogues. As Ponza (2000a) refers: ‘*This method of teaching is not just intuitive. When students write or dance or perform mathematics they work out, they analyze, organize and solve’.*

The students’ answers in the interviews, two months after the end of teaching experiment, indicate that students themselves believed that through Drama based instruction they learned better and easier.

-(A)The way we presented, using theater, it isn’t that we just sat up and said it, we had to prepare it, and this required to have understood it.

-(I) in the way we did all these, I think that one learns more easily. What we learned (through this procedure) wasn’t designed for learning by heart or to show to the teacher that we really learned that. I think it was easier to learn in this way and to retain this knowledge in our mind.

- (G.) Well. Until now, geometry was completely indifferent to me, and still is, if they teach it the way they do. Because to learn something by heart it is not nice, while if there is a story it is a little more interesting, while if you do a sketch it is nicer and it becomes a more interesting lesson. Because you learn more this way.

What was the main effect of Drama in students’ learning? In this teaching experiment, due to students’ responses, it was Drama that motivated all students’ active participation in the teaching experiment.

-(S.) If we have done all the tasks except the presentations, we would have been bored to death.

- (N.) Without drama, I don't think that anyone would be interested (in the project).
-(V.) It was completely different than just writing stuff, from comprehending something yourself and trying to pass it to others. It was very nice.

How did interdicsiplinarity affect students?

•***In unifying knowledge:*** Interdisciplinarity enabled students to see the links between various subjects and consequently to see the knowledge in a unified way.

-(St.) It was something very different, something not expected and we had to wait till this grade of high school to be able to do something similar, a similar activity that is related with something different from the syllabus. Indeed the specific issue that connects Mathematics with different disciplines as Ancient Greek, History, was very interesting. It was interesting because we talked about Euclid but not only about the postulates and the various theorems, but we talked about the historical and cultural context that prevailed in Alexandria at that time.
-(J.) This was interesting and we had not done something like this before. I think this did History, which I do not like, more interesting because I learned about mathematicians and I like this more.

•***In students' beliefs about Mathematics' historical and cultural context:*** Students experienced a diachronic evolution of geometry and its interaction with the historical, the scientific, and the cultural evolution, as well as with social, economic and political conditions of each era. The introduction of mathematics history into the course curriculum attracted the attention of pupils, since it was the first time that they had such an experience. Students welcomed the activities related to the history of geometry, which linked the science back to the specific historical, political and economic context within it developed

-(Char.) It was nice that we learned about this era, while in the ordinary lesson we didn't learn anything about this, it was just triangles, squares.

From students interviews it appeared that there were students who linked the development of mathematics with the economic development of the region in which it was developed.

-(St.) We realised that in order for the science to be evolved, there needs to be evolution in all other areas first, starting with financial areas and then cultural ones'.

•***In students' beliefs about Mathematics as a human creation:*** From the follow-up answers in the interviews concerning students' beliefs about the evolution and development of mathematics, it seems that students believe that mathematics is timeless and superhuman and ahistorical.

-(A) The historical connection of mathematics is very important. Mathematics seemed like having come from heaven. We had just been told the 1 and 2 and the numbers and the calculations, all ended there. The historical connection of mathematics is very important, as

it is its historical evolution as well as that of Geometry.
- (V.) Previously, I thought that the knowledge of mathematics came from God, from heaven. I couldn't be in the process of thinking that someone thought of it.

Through the efforts of proving Euclid's fifth postulate and through the mathematicians' biographies, students perceived mathematics as a historically evolving human creation and also realized that other cultures also, have contributed to what is defined as Western mathematics.

- (Tz.) It was nice, because if you only see the result it is boring. You must learn a little bit of what has happened in the past; who created it, so it was nicer, I learned more things.
- (Th.)...We saw how theorems and postulates have been created and it was interesting, because it is difficult to understand them if you see them in their final form.

Students met for the first time the struggle of many mathematicians that has not been crowned with success. They emphasized the continuous and strenuous effort that was made by the mathematicians of the wider Islamic world and later by Renaissance mathematicians, to prove the fifth postulate.

-(Tz) In the beginning I said, well, as they don't find it, why don't they stop? Yet they have been trying and trying. They liked all these too much.

We asked students to tell us their thoughts from their acquaintance with the biographies of Bolyai, Lobachevsky and Riemann. Students in their responses highlight the dedication of all these people in mathematics and the perseverance in their objectives.

- (Mar.) ...I saw that these people in a great part of their lives were dealing with mathematics and they had set a target, trying to reach it. It was very interesting, all this effort to prove something, really interesting.

CONCLUSIONS

Our research offers considerable evidence of the effectiveness of History of Mathematics in linking mathematics with other school disciplines. History of Mathematics became an open door towards interdisciplinary work and consequently towards a wide range of possibilities for all subjects. Students had the opportunity in language class to write texts, which included mathematical discourse, while in History class for the first time they experienced mathematics as a cultural treat. This indisciplinary approach was the key role for the historical and social context in which mathematical concepts were created to be illustrated and provided students the opportunity to modify their epistemological conceptions about mathematics. Likewise, cultural construction and the contribution of different cultures in the creation of mathematics were showcased. Drama provided students the incentive to work while through drama techniques students were able to experience all these dimensions of mathematics, not only mentally but emotionally and physically.

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THE POWER OF MATHEMATICS EDUCATION IN THE 18TH CENTURY

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In the Dutch Republic in the 18th century mathematics was considered very important for many professions. However there were hardly any national or regional educational institutes which provided mathematics education. Three orphanages in different towns received a large inheritance under condition that they provided education for technical and artistic professions. The Foundations which were established had a curriculum in which mathematics was the main subject. The influence of several curriculum components and some external curriculum factors are recognisable in documents and archival data. This is illustrated by the history of one student.

THE *FUNDATIE VAN RENSWOUDE*

During the 18th century in the Dutch Republic mathematics was considered very important for professions such as navigation, architecture, the technical weapons, shipbuilding, water management and also fine arts (Van Maanen, 2006). However there was a serious shortage of mathematically trained professionals. The main reason for this was the absence of national or regional institutions which offered mathematics education (Beckers, 2005; Boekholt & Booy, 1987). The example of the Dutch Engineering School at Leiden University in the 17th century (Krüger, 2010) had resulted in lectures in Dutch language on fortification, navigation and architecture at a few institutes for higher education. But they occurred only intermittently and were not for those on a low income. Professional education was mainly left to individuals, the quality varied and the fees could be high. The Dutch adopted a religious form of Enlightenment; they believed in progress through mathematics and science, which also revealed God's work. Many saw the value of education, also for poor people; initiatives of wealthy individuals, of learned societies and of local government became more common in the Dutch Republic during the 18th century (Dodde, 1991; Roberts, 2010, Smid, 1997). An early example of such an initiative is the legacy of Maria Duyst van Voorhout, baroness of Renswoude: in 1754 three orphanages, in Utrecht, Delft and The Hague, each inherited about 500 000 Dutch guilders (HUA 771, inv. 3). The main condition was that the money should be used to select talented orphan boys, at least 15 years old and teach them separately from the other orphans in:

> Mathematics, Drawing or Painting Art, Sculpture or Stone Cutting, practices in building dykes to protect our Country against floods or similar Liberal Arts….

It seems that Mathematics was meant as a profession, but also as the main subject taught: all the professions mentioned used mathematics in some way. One of the other conditions was that the capital should be administered by an independent administrator and that there should be some alignment between the three Foundations (HUA 771, inv. 1). As a result, in 1756 three Foundations were established, each called *Fundatie van Renswoude* (in full: Foundation of the Baroness of Renswoude), with communal rules, the General Regulations (HUA 771, inv. 5). The archives of the Foundation in Utrecht are well preserved, they provide much information about the intended and implemented curriculum. A question one may ask is:

Which factors and which actors influenced the curriculum of this Foundation?

The main subject in all three Foundations was mathematics or rather mathematical sciences. When one analyses the available information on the curriculum many curriculum components, which are distinguished in present day research, are recognisable. In this paper I will discuss some of these components, but also some other factors which were of notable influence on the curriculum of the Foundation in Utrecht. This will be illustrated by the history of one of their students, Dirk Kuijper.

CURRICULUM RESEARCH

Fig. 1: Spiderweb model

A useful model to describe curriculum and its components is the spiderweb model used by Van den Akker (2003). Characteristics of the spiderweb model, which has ten components, are the visibility and influence of these components in the intended and/or implemented curriculum and the interconnections between the various components. If one component changes this effects some or all of the others and each component may be influenced by others (fig. 1). In this paper I add the component Alignment, as this is mentioned explicitly in all documents of the intended curriculum. For instance, the regents of the three Foundations were to discuss with each other the state of affairs and progress in the three institutes at least once a year. These meetings promoted the wish to find a mathematics teacher of similar quality as the one in Delft. Also within the Foundation teachers and housefather had to inform each other of the students requirements and behaviour (HUA 771, inv. 5, inv.8).

There are other factors which are influential, but a major difference with the curriculum components mentioned is the mainly one-way direction of this influence. They could be named 'external factors'. Finance and transition are the examples mentioned here. Transition refers to the preparation of students for the next stage of their life, after completing the curriculum. See table 1 for an overview of curriculum components and external factors.

Rationale	Why are they learning?
Aims & Objectives	Toward which goals are they learning?

Content	What are they learning?
Learning activities	Through which activities are they learning?
Teacher role	In which way does the teacher facilitate the learning?
Learning materials	With what materials are they learning?
Grouping	With which students are they learning?
Location	Where are they learning?
Time	When and for how long are they learning?
Assessment	How far has their learning progressed?
Alignment	Is the learning coherent with the larger curriculum?
External factors	
Finance	Which financial means are available to facilitate learning?
Transition	In which way is the start of students in the next stage facilitated?

Table 1: Components and external factors of the curriculum, based on Van den Akker (2003)

The influence of these curriculum components) and external factors are to some extent illustrated by the history of student Dirk Kuijper.

DIRK KUIJPER (1766 – 1830) AND THE *FUNDATIE VAN RENSWOUDE* [1]

Fig. 2 Dirk Kuijper

Dirk came into the care of the town orphanage on 3 May 1769, when he was about three years old. Thirty years earlier, his prospects would have been rather bleak. However, since 1756 the Foundation, of which the big house was situated next to the orphanage, offered a better opportunity to talented boys. In 1756, only some of the boys of the right age (15 – 18 years old) could read a little and none could do arithmetic. So the regents had appointed a carefully chosen teacher to teach the eligible boys reading, writing and arithmetic. This was the preparatory school and in 1773 Dirk was one of the 22 boys in the preparatory school, with ages ranging from 5 – 15 years old (HUA 771, inv. 37). Of this group 10 boys would in the following years be admitted into the Foundation.

The mathematics instructor (mathematics teacher) of the Foundation, Laurens Praalder, was the main teacher and also the supervisor of the Foundation school. He visited the preparatory school at least once a week to watch progress and to discuss with the teacher the potential of each boy. Once a year the regents asked his advice on which boys should do an admission exam to assess if they qualified for entrance into the Foundation. On 28 April 1779 Dirk was one of the boys who was allowed to take the exam, which consisted mainly of arithmetic. On 11 June 1779 the board of regents admitted Dirk, 13 years old and one other boy, who was 12 at the time. The average age on admission was 15 years.

Mathematical education

The education process in the Foundation was structured into three phases. During the first phase, which lasted about two years, teaching would be in small groups and the program was the same for all the students. After about two years, depending on the progress of the individual student, a profession would be chosen. In most cases the student would start in a part time apprenticeship with a work master, while continuing theory lessons in the Foundation in the evenings and during some afternoons. The third and last phase would usually consist of a fulltime apprenticeship, preferably in some other part of the country or abroad. During the first two years Dirk would have about 32 hours of lessons per week, of which at least 20 were taught by the mathematics teacher. Drawing lessons, by the drawing master, took eight hours and the remaining hours were spent on writing, French language and religion. Outside teaching hours there was homework to do, mathematical exercises, drawings to finish, etc. Lessons took place in the large lecture room (fig. 3), homework was made in the dining room, under supervision. In both rooms books and writing materials were available; the mathematics teacher also liked to do practical exercises with the students, preferably based on measurements in the environment (Krüger, 2012). In spring 1781 Dirk had been nearly two years in the Foundation; he had made sufficient progress to choose a profession and start an apprenticeship. His first choice was instrument maker, however as at that time there was no suitable instrument maker in Utrecht willing to take on an apprentice, Dirk became a student of the mathematics instructor. During the following years Laurens Praalder taught him several mathematical subjects as a general preparation for technical professions.

Fig. 3 Large lecture room

In February 1785, Dirk had studied the following mathematical subjects: arithmetic (whole, rational, irrational and decimal numbers, calculating roots and powers of numbers), geometry (Euclid I – VI, XI and XII), algebra (including linear and second degree equations and series), the theory and practice of surveying (including trigonometry and the use of logarithms), civil and military architecture, mechanics and geography.

The importance of social and cultural education

Mathematical subjects were very important, but not sufficient to succeed. Dirk also learnt to draw, both technical and artistic drawing, French language and how to write properly. Students would need these skills in most of the professions for which the Foundation provided education. They also needed the skills to maintain a position on a middle class level of society. So like all students Dirk learnt proper table manners, how to read a newspaper, how to behave in company and similar skills.

Specialisation

In the course of 1784 Laurens Praalder suggested that Dirk might become an army engineer, a very unusual choice for the Foundation in Utrecht at that time. Perhaps the fact that the Director-General of the Fortifications, Carel Diederik Du Moulin, some of the regents and the secretary-administrator of the Foundation all were members of the Provincial Utrecht Society (PUG), had something to do with this new possibility of career choice for the students. PUG, a scientific Society, was established in 1773 by Laurens Praalder and J.van Haeften and attracted a good number of members (Van Haeften, 1781). Du Moulin was very much in favour of mathematical instruction for officers in the army. On 1 December 1784 the regents decided that Praalder should contact Du Moulin to enquire if he would be willing to take on a student of the Foundation. Du Moulin quite liked the idea and after an exchange of some letters on what was expected of Dirk he was sent on his way to the Grebbe on 12 February 1785. He had with him with a warm coat, field maps, a suitcase with the necessary clothes and linen, a good quality surveyor's chain, a surveyor's level and an astrolabe. Dirk was to write every eight days to report on the spending of the extra pocket money he received and on his whereabouts.

Apprenticeship and politics

During 1985 Dirk worked as an apprentice-engineer for the army. Some of his letters are kept in the archives; he wrote on the work he was doing and where he was staying. He wrote to the regents in general, to regent Nes, and later on also to mr. Van Voorst, the secretary-administrator from 1788 – 1802 (HUA 771, inv. 38; van Lier, 1954).

Towards the end of Dirk's first year, in September 1785, his supervisor, a major Kupfer, wrote a letter to the regents in which he was very positive; he mentioned Dirk's diligence, good behaviour and skills. The major proposed to employ Dirk in the army, in that way he could earn some income as well. The regents decided to decline this offer. The Foundation was responsible for her students until they could earn an income which allowed them to be independent and they considered Dirk too young to be left on his own. So on 28 December 1785 Dirk was back in the Foundation home, the army had its winter stop. That winter Dirk continued to study theory of military engineering. In the spring of 1786 Dirk worked under guidance of captain Ulrich Huguenin, who also appreciated Dirk very much and who in the years to follow would be important for his career. Huguenin had studied mathematics while in the army and had also taught mathematics at a private school for artillery officers. Like Du Moulin he considered knowledge of mathematical sciences very important. Dirk started to learn German during that year, on his own request. Possibly Huguenin had advised him on this to enable him to study the German military literature. Huguenin was in the artillery, so this meant for Dirk a change from military engineering to artillery. He also proposed to let Dirk enrol in the army. Unfortunately for Dirk, the political situation had changed; in October 1786 the patriots had come to power in Utrecht. As a consequence the regents of the Foundation were either patriots or neutral, as most Orangist regents were replaced by patriots. The army in principle supported the Orangists, so the regents could not allow a student of the Foundation to fight in the army against their patriotic comrades. Dirk was not allowed to continue with his work for the artillery, he remained in the Foundation; during 1787 he supervised defence works and enlarged a map of Utrecht, on order of regent Eyck, an active and popular patriot. In September 1787 the Prussian troops intervened, the Patriots lost power and a number of them, like regent Eyck, went into exile for some years. Most patriotic regents were replaced by Orangists, so a military career became again a possibility for Dirk. In December 1787 regent Kien told the meeting that he recently had met captain Huguenin who had expressed his regrets that Dirk Kuijper had not returned into the army. The regents decided at once to invite Huguenin for a dinner and to tell Dirk about the good opinion captain Huguenin had expressed. Soon after it was decided that Dirk should train as an artillerist, but in accordance with the regulations he remained a student of the Foundation for the time being. That meant that the Foundation paid for most of his cost of living and study.

The start of a career

On 27 February 1788 Dirk left for The Hague. As was common in the army he moved around frequently and was often short of money, as life in these army towns was expensive. Early in March he wrote, that he hoped to be accepted as a cadet-bombardier on 25 March, and that he had bought a German textbook, which had been advised by captain Huguenin (fig. 4). Being a cadet-bombardier meant that he earned a little bit of money, about 40% of the amount he needed for board and living. From then on his career went well. In 1788, possibly under pressure of the political situation, the stadtholder finally agreed to establish three Artillery Schools. In May 1789 when Dirk was in Breda, with Huguenin, he was promoted to lieutenant-to-be. Soon afterwards Dirk also became a junior teacher at the new Artillery School in Breda, with Huguenin as Director. He still had to study hard to be able to fulfil the demands of his teaching position. The Foundation received another bill, of a German bookseller. In August 1790 he became lieutenant and was appointed tutor at the Artillery School, which enabled him to become independent. He was dismissed with honour from the Foundation and in December 1790 married a wealthy orphaned young lady, ms Stuerman, from Baarle Nassau. In 1795 he became director of the Artillery School in Groningen. In that position he admitted and taught some students of the Foundation in his school.

Fig. 4 Letter by Dirk Kuijper, 14-3-1788

Learning materials

Learning materials were considered very important by teachers and regents. They consisted of student's notes, many books, tools and instruments, such as surveying tools, compasses, drawing utensils and gauging rods. Laurens Praalder wrote his own teaching texts to use during the first and part of the second phase (Krüger, 2012). The students made extensive notes, during the first phase they may have used the available books during homework time. In phase two and three they needed more specialized books for their future profession. On average ca. 3,6% of the yearly expenditure was spent on learning materials (Table 4).

Information about the books Dirk possessed come from an inventory Dirk made in January 1787, from his letters and from the invoices of the booksellers (HUA 771, inv. 38, inv. 90, inv, 103, inv. 129, inv. 130).

On 24 January 1787 Dirk possessed some mathematics books all students used, on algebra and *Euclid* in the edition of P. Warius. For surveying he had *Werkdadige Meetkonst* by J. Morgenster & J. Knoop, a leading book on trigonometry and surveying (Van Maanen, 2006), which was used by many, but not all of the students. Also on his list were five books on military architecture; one in the style of the Dutch military architect Menno van Coehoorn, which used mainly geometry; one in the style of the French military architect Sebastian le Pretre de Vauban, published in 1784, in which book algebra was combined with geometry. These books represented different points of view on military architecture and also different mathematical approaches. He possessed a French grammar and a German-Dutch dictionary. During 1786 and 1787 the regents bought three books of B.F. de Belidor (*Sciences des ingenieur, Le bombardier françois* and *Nouveau cours de mathématique à l'usage de l'artillerie*). This suggests that Dirk's command of French was at least sufficient to study from these more specialised books. In 1788 Dirk bought *Anfangsgründe der Artillerie*, by K.A. Struensee (fig. 4). In 1789 in Breda he purchased *Magazin für Ingenieur und Artilleristen*, 11 volumes, by A. Böhm; *Lehrbegriff der gesamten Mathematik*, 8 volumes, by W.J.G. Karsten and *Mémoires d'artillerie : contenant L'artillerie nouvelle* by H.O. Scheel.

So from 1787 on he used more specialised military literature, which was written in French or German. But he also bought the eight volumes of a general mathematics publication in German. This extensive work by Karsten is still available in print.

Finance [2]

Finance is rarely a topic in research on mathematics curricula. However, the financial means available were very important in the 18th century, as they are today. The amount of money at the start of the Foundation was a factor which permitted the building of a home suitable for physical care and for education. The financial situation also made possible to engage good quality teachers who were well-paid. At least as important was financial management, which in Utrecht became more professional from about 1771. From 1772 – 1809 on average the yearly income and expenditure of the Foundation were in balance, both slightly more than *f* 15000 (HUA 771, inv. 89 – 92). In 1810 the income was severely reduced, due to French regulations (Table 3).

	1772 – 1809, average	1810
Income	*f* 15909	*f* 6938
Expenditure	*f* 15378	*f* 11660

Table 3 Average income and expenditure of the *Fundatie van Renswoude* in Utrecht.

Between 1772 and 1787 there are sufficient data to estimate the relative expenditure of some entries and their range (HUA 771, inv. 89, 90). See table 4.

	Average of total	minimum	maximum
Household and clothing	30,6 %	*f* 4011	*f* 5506
Salaries	24,9 %	*f* 3686	*f* 4202
Work masters	6,5 %	*f* 250	*f* 1705
Learning materials	3,6 %	*f* 226	*f* 1076
Pocket money etc.	2,7 %	*f* 188	*f* 806
Gift at dismissal	4,7 %	*f* 54	*f* 2695
Students abroad	4,1 %	*f* 0	*f* 2494

Table 4 Yearly expenditure 1772 - 1787

During this period costs of household and clothing were on average 30,6% of the total expenditure; salaries of teachers and personnel were on average nearly 25% of total expenditure. The salary of the mathematics teacher was by far the highest, *f*1500. The costs of work masters fluctuated, but over the years the relative cost increased, from about 1,5% to about 10% of the total expenditure. Learning materials included paper, writing utensils, books and some instruments, the cost fluctuated, but there is no trend visible. A student, who became independent and thus was dismissed, usually received equipment and money to assist him in the start of his career. The remainder of the expenditure, ca. 22% of total cost, was for maintenance of the building, interest, administration, costs of meetings, etc.

Dirk Kuijper on average paid *f* 5 per week for board and lodging. He received some income from the army: about *f* 2:10 per week. The additional funding by the Foundation also was *f* 2:10 each week. The Foundation also paid for travelling costs, books and clothes. In January 1789, when he got promoted and had to make extra costs, he received an additional *f* 250. On his promotion to lieutenant and his appointment as a tutor at the Artillery School, in August 1790, his salary became *f* 49 every six weeks, which was sufficient to become independent. He was dismissed from the Foundation on his own request and received *f* 350 as the remainder of his dismissal gift.

DISCUSSION

In the curriculum of the *Fundatie van Renswoude* in Utrecht the components and external factors, mentioned in table 1, are clearly recognisable, perhaps with the exception of grouping. In Utrecht, but also in Delft, the selected boys were cared for and educated in a location close by but separate from the orphanage. As a result it became possible to give these students not only a sound mathematical education, but also a social and cultural education, thus facilitating the transition to their future profession and position in society. This is illustrated by the history of Dirk Kuijper. He would very probably have remained a labourer all his life if the Foundation had not only offered him the opportunity to learn mathematics quite well, but also to learn languages and social skills. While he was a student of Laurens Praalder he was taught surveying theory and practice, military architecture and mechanics. This made a career as an engineer in the army a possibility, but water management, surveyor or mathematics teacher would have been equally possible (Krüger, 2012). Very important was the financial support of Dirk by the Foundation. So it was not only the mathematics which was fundamental for Dirk's career, the financial support was essential as well, as was to some extent the broader curriculum.

Important actors in the implementation of this curriculum were clearly the mathematics teacher, the regents and in this particular case also C.D. Huguenin. Not all regents had the same importance, i.e. regent Eyck, regent Kien and regent Nes seem to have been more involved than others, as was mr. Van Voorst, the administrator. Though it must be kept in mind that in archives only traces of what really happened, are to be found. In a way the students could be seen as actors influencing the implementation of the curriculum as well. Dirk himself asked permission to study German and so later on he could ask for and study from the relevant specialised publications.

In the 17th century the Dutch Engineering School at Leiden University had similar, but more restricted aims as the *Fundatie van Renswoude* (Krüger 2010). In 1863 a new type of secondary school was introduced, with mathematics and science as important subjects. Its aims were general education as preparation for the higher positions in industry and commerce and preparation for the Polytechnic School and the Military Academy. Most of the curriculum components and external factors are recognisable in these curricula as well. In all three cases mathematics is important and mathematics has to be applicable, but from the only subject in the 17th century it became one of 16 subjects in the 19th century. Comparison of the three curricula will show which factors are consistently important and in which way actors influence the success of a curriculum.

NOTES

1. Sources: Hua 771, inv. 11, inv. 12 (resolutions of meetings of regents), Langenbach (1991), unless otherwise stated.

2. A guilder (*f*) was 20 pennies, so *f* 2:10 is two guilders and 10 pennies, or two and a half guilders.

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EVALUATION AND DESIGN OF MATHEMATICS CURRICULA: LESSONS FROM THREE HISTORICAL CASES

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Our central question is: Which factors and actors influence the content of mathematics curricula? In answering this historical question we also hope to provide inspiration and references for those involved in curriculum design and evaluation. We will base our answer on historical data from three mathematics curricula from the 17th, 18th and 19th century:

1. The curriculum of the Dutch Engineering School established in Leiden in 1600.

2. The 1756 curriculum for Utrecht orphans, funded from a huge legacy.

3. The curriculum of the HBS schools, established by the Dutch government in 1863.

We test whether the theoretical framework of current curriculum research also applies to these historical case, and sketch a comparative perspective.

HISTORICAL RESEARCH AS A BACKGROUND FOR CURRICULUM DESIGN

We became interested in historical decisions to define or renew the mathematical curriculum because of current and ongoing discussions about this issue in the Netherlands. In discussions and plans the arguments are always shortsighted. They refer to the actual situation and to the needs for the future. Never, or at least not often, crucial episodes in the past enter the scene. Yet, historical research has a valuable set of characteristics which makes it interesting for the curriculum evaluator and designer: the historical process is completed, and therefore it is possible to study the process and at the same time its outcome; also the current observer is not involved herself or himself in the discussions of the past, which makes an unbiased approach easier; and finally the current curriculum in any country in the world is a historical construct. Therefore we expect that historical research will produce results that are important not only for historians but also for curriculum specialists.

A first and superficial analysis reveals that personal preferences of influential individuals play an important role in curriculum development, next to arguments about societal needs. This first analysis brought us to our central research question

Which are the factors and actors that influence to a high degree the content of mathematics curricula?

and in order to structure the research we subdivide this question into the following three:

- Which motivations and whose ideals are influential on the content of the formal curriculum?
- Which factors and which people determine the interpretation of the formal curriculum and its implementation?
- Which factors and which people are important for successful implementation of a curriculum?

Method

First we select three cases of new mathematics curricula in the Netherlands, which were designed in three subsequent centuries (17th-19th). For each case the listed questions will be investigated. We will analyse the data and also make a comparison between the cases. The projected outcome is a historical description, and also a list of conditions, which are important for successful curriculum design and development. Comparison of these conditions with recent curriculum design theory and with developments in mathematics education should result in criteria which are decisive for successful mathematics curricula today. We refer to curriculum studies such as (Goodlad, Klein & Tye, 1979) and (Van den Akker, 2003), and 'borrow' their terminology, especially the following division in stages (or 'domains'):

- the intended curriculum, the ideal of persons who initiated the curriculum and the translation of the ideal into the formal curriculum;
- the implemented curriculum, interpretation and implementation of the formal curriculum, by teachers and through teaching materials;
- the attained curriculum, success (or lack of success) in relation to the students.

The selected historical cases are:

1. The Leiden Engineering School, established in 1600 and affiliated to Leiden University, also known as Duytsche Mathematique, shows the first known example in the Netherlands of a formal curriculum. The school flourished until ca. 1665.

2. The three Foundations of Renswoude, founded in 1756, offered professional education to talented orphans. Mathematics was the main subject, at least during the first two years of their education. The focus is on the Foundation in Utrecht, operational since 1761 and still existing today, although with different goals.

3. In 1863, the Dutch government established the **H**ogere **B**urger**s**chool (HBS), a secondary school for children of citizens who would not enter university, but who were to take up higher technical or administrative positions in society or enter the Polytechnic School in Delft, the present Technical University of Delft.

For each period the data will be presented and analysed separately. They will be combined and compared in the final section, which also presents our conclusions.

THREE HISTORICAL CASES OF CURRICULUM DEVELOPMENT

The curriculum of the Leiden Engineering School (Stevin and Van Schooten)

Leiden had a university since 1575, the first one in what in 1588 would be the Republic of the Seven United Netherlands, a federal state in which the House of Orange played a crucial role. Since 1568 the Dutch were at war with the Spanish king Philip II. The war, combined with strong economic development, led to the need for a type of professional that was not yet existing, the military and civil engineer. Therefore, in 1600, the Dutch leader Prince Maurits of Orange decided to connect to Leiden University an Engineering School called *Duytsche Mathematique*. Its name points at two central characteristics of the curriculum: it was taught in vernacular, i.e. Dutch, and mathematics was the main subject.

The formal description of the curriculum, written by Simon Stevin at the request of Prince Maurits, is still available. The elaboration and implementation by one of the first teachers, Frans van Schooten Sr, is amply documented, as well as the interpretation of Stevin himself in in part 1 and 2 of his *Mathematical Memoirs* (Wisconstighe Ghedachtnissen, 1605-1608).

Prince Maurits asked his former fellow student and later private mathematics tutor Simon Stevin to write an 'instruction' for the teaching at the Duytsche Mathematique. This 3 page manuscript, dated 9 January 1600 and signed by Maurits, is preserved in the archives of Leiden University library. It was published in (Molhuysen 1913, annex 338, 389*-391*) and in a five page pamphlet (published by Jan Paedts Jacobsz, Leiden 1600). Stevin's instruction is truly the intended curriculum, as the excerpt in Figure 1 shows:

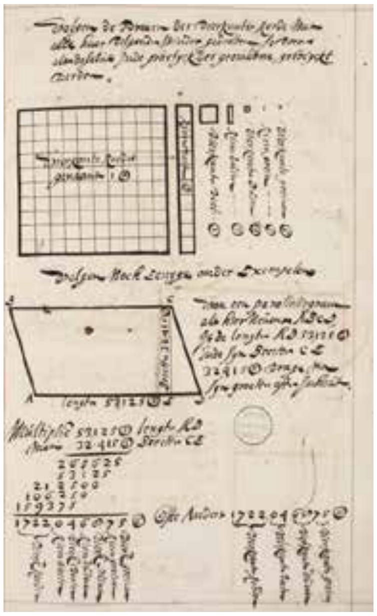
> Arithmetic: the four operation in whole numbers, rational numbers and decimal numbers, also the rule of three in those three types of numbers
>
> Surveying on paper, that is calculating area with the use of decimal numbers
>
> Measuring a circle, parts of a circle and area, ... learning to subdivide rectilinear figures and curvilinear figures into several parts, such as triangles or other figures, to check calculations
>
> Measurements on paper of dykes and works to learn how many … feet the works contain.
>
> Fieldwork, learning how to use tools properly.
>
> Mapping on paper what is measured in the field and the reverse, from a map setting out stakes in the field
>
> Fortification, learning the names of the parts from wooden or earth models. They will learn to make maps of towns. They will draw on paper the perimeter of fort or towns with four, five or more bastions and stake them out in the field.

Figure 1: excerpt from Stevin's 'Instruction'

The emphasis on practical geometry is striking. Practical geometry also involves the calculation of perimeter, area and volume, so arithmetic is part of the programme, but subjects as the area of circle sectors and measurements on conic sections are not "since engineers will only rarely hit upon these". Note that in the field of arithmetic Stevin prescribes "decimal numbers", which he had introduced himself in his *Thiende* (1585). The teaching of Van Schooten, who was one of the first to use decimal numbers, contributed to their introduction. Stevin also indicates that topics are to be taught both theoretically ("on paper") and in practice ("actual surveying in the field"), which is important since ruler, compass and right angle, useful for work on paper, are to be replaced by the typical tools for fieldwork. Students were expected to spend the summer months to service the army in helping with the fortification works. There is even advice how the theoretical course of arithmetic and geometry should be taught: every lesson to be split in half an hour group instruction and half an hour tutorial in which questions of individual students are answered.

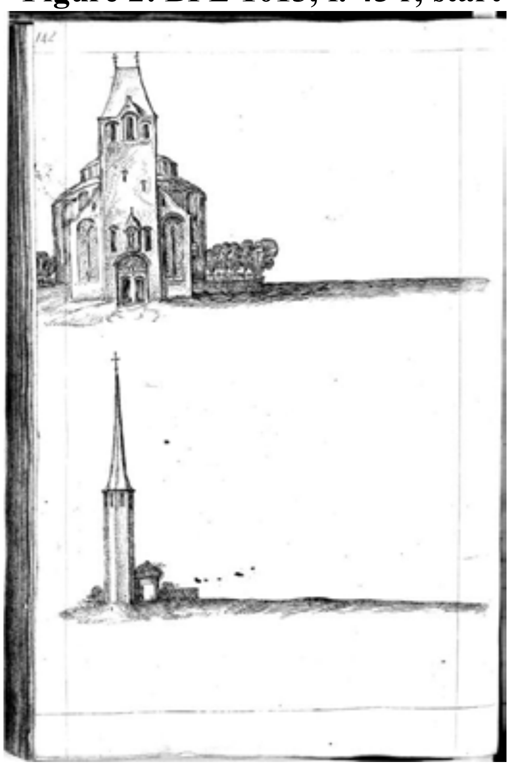
An implemented curriculum soon followed. Stevin's own *Mathematical Memoirs* from the years 1605 to 1608 (Stevin, 1608) can be considered as such. Yet, it is quite unlikely that this expensive folio-size book was used directly in teaching. Moreover Stevin's instruction of 1600 is well recognizable in the texts that Van Schooten used in his teaching at the Duytsche Mathematique, and therefore we take these documents, which are mainly in manuscript form, as an account of the implementations in actual teaching. Van Schooten succeeded the two first teachers of the school, Van Merwen and Van Ceulen. Both were appointed at the 10 January 1600 meeting of the Board of Leiden University in which the school was erected and Stevin's instruction was registered. They both served until 1610, the year of their death. By then Van Schooten had already assisted Van Ceulen in the fieldwork.

It took many deliberations and even some petitions by students until in 1615 Frans van Schooten (1581/2-1645) was appointed successor to Van Merwen and Van Ceulen. A series of manuscripts, kept in the university libraries of Leiden and Groningen, indicate how Van Schooten interpreted Stevin's instruction. A comprehensive description with detailed sources, is to be found in (Krüger, 2010). The central source that informs us about Van Schooten's teaching is the Leiden manuscript *BPL* 1013, 256 leaves in folio, which bears the title "Mathematische Wercken door F. van Schooten" (… works by …). The manuscript can be dated c.1622; for a discussion of the evidence and further physical properties of the manuscript, see (Van Maanen, 1987). *BPL* 1013 discusses arithmetic (extraction of roots, decimal numbers, calculation of area), geometry (definitions and axioms, propositions (Euclid), constructions), surveying (measuring distances in accessible lands and calculations), use of trigonometric tables (measuring in inaccessible lands, making maps, measuring heights (or depths), also measurements without use of tables, c.f. Figure 4), solids (calculations on all kind of shapes and materials, calculating content) and fortification (definitions, plans and calculations).



The structure of *BPL* 1013 is clear: each topic starts with an introduction. Van Schooten defines the new concepts (cf. Figure 2, the start of the practice of surveying; also interesting on this page is the use of decimal numbers; it shows the decimal comma as well as circled integer, Stevin's 1585 notation). The introduction is followed by a carefully chosen set of worked problems, in rising order of difficulty. Van Schooten designed these problems in an almost industrial manner. At least the section where he shows how to measure distances and heights contains drawings of landscapes prepared by an artist, before he himself drew the specific configuration of the problem. Some of the landscapes remained empty because Van Schooten had run out of problems (cf. Figures 3 and 4).

Figure 2: *BPL* 1013, f. 45 r, start of section on surveying



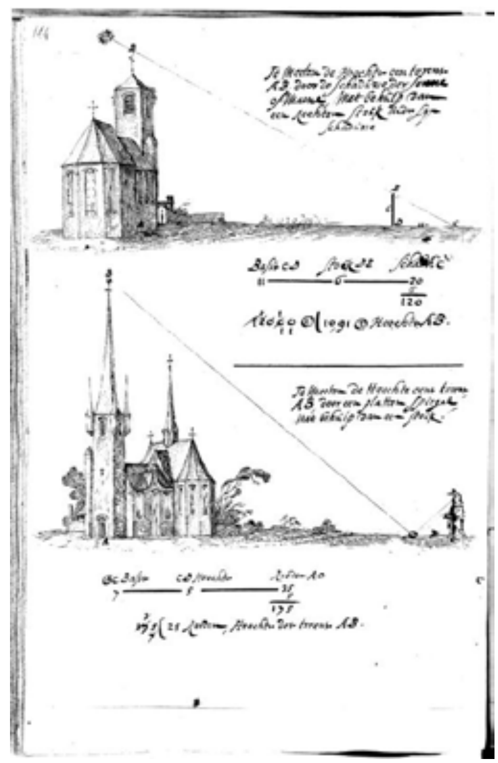


Figure 3: *BPL* 1013, f. 69v, unused diagrams **Figure 4: *BPL* 1013, f. 70v, heights**

The curriculum of the Utrecht branch of the Foundation of Renswoude

The case that comes up now is a curriculum by private initiative, although eventually three institutes in three different Dutch cities were involved. The wish to establish a mathematical curriculum was a last wish: it was expressed in the last will of a very wealthy woman, Maria Duyst van Voorhout, baroness of Renswoude (1662–1754). The will, signed in 1749 and executed from 1756 onwards, allotted to orphanages in

Delft, The Hague and Utrecht each half a million guilders. These huge sums should afford the orphanages to

> … select some of the most talented and suitable boys, at least 15 years old, to set them apart from the orphanage in order to teach them Mathematics, Drawing or Painting Art, Sculpture or Stone Cutting, practices in building dykes to protect our Country against floods or similar Liberal Arts….” [Utrecht Municipal Archive: *HUA* 771, inv. 1)]

And so it happened, with some variation between the three independent but cooperating institutes, which all three were called after their founding mother “Fundatie van Renswoude” (foundation being the common name for an institution, funded by a legacy). The focus in this article will be on the branch in Utrecht. Its mathematics curriculum was described in (Krüger, 2012). In a remarkable way the structure and progress through time of the Utrecht institution parallels the curriculum models of Goodlad et al. and Van den Akker, summarized above.

First, the last will of the Baroness van Renswoude made a concise and precise statement about her aims: promising orphan boys should receive an education in technical or artistic professions, which would respond to the needs of the nation. Mathematics was the first “practice” that the boys should learn. This concise visionary statement can be considered as the intended curriculum. It was elaborated into a formal curriculum by the executors of the testament in cooperation with the regents of the three orphanages. Together they composed an on 17 May 1756 signed “General Regulations”, a document that specified how the visionary statement would materialize. The Regulations describe a variety of aspects: the learning environment and structure of the course, the professions for which it should cater, the subjects to be taught (reading, writing, arithmetic, drawing, principles of mathematics and if necessary French and English language), the accountability of the professionals involved (governors, teachers and Foundation personnel), assessment of the students and also a consultation procedure between the three branches of the Foundation.

The implementation in the Utrecht branch required some strong measures. Up to 1756 few boys mastered reading, writing and arithmetic, so a schoolmaster was appointed to raise the entrance level of the boys. The Utrecht orphanage was so crowded that it had no room for further teaching activities, so a new spacious Foundation house was constructed. In 1761, an “Instruction” for the mathematics teacher was agreed upon. As to the mathematical topics the Instruction was brief: “General mathematics, Military and Civilian Architecture, Surveying, Etc.” The interpretation was left to the mathematics teacher (the “mathematician”). More detail was attached to the procedures of admission and examination (students were to be examined twice a year in the presence of the regents). The mathematician also had to advise on the future profession of a student. In this sense, the intended curriculum focused more on organizational aspects of learning than on its cognitive aspects.

The Utrecht regents made a lucky choice with their appointment of Laurens Praalder. Praalder (1711-1793) was a mathematical practitioner, versatile and widely respected.

He had no academic background, started his teaching career in a region where navigation and building dykes were central topics. In 1751 he was appointed at the Rotterdam Naval School, and in 1761 called to the Foundation. Praalder brought the implemented curriculum, which was still a draft, into practice. He taught at Utrecht until 1792, so for more than 30 years. He had a broad and practical view on the subjects and teaching methods, integrating mathematical theory with physical experiments and practical work with his students, especially in surveying. The duty in finding proper professions for his students was with Praalder in the good hands; he was linked in an extensive and varied network, where he established many useful apprenticeships for his students.

The Instruction had a rather general statement about the topics to be taught. So Praalder faced both the duty and the freedom of choice. He adopted the drafted structure of a course in three phases. Phase 1, which lasted two years and had two examinations per year, was devoted to theory and practical exercises in mathematics, drawing, French language and religion. In Phase 2 Praalder continued the theory but also shifted towards the preparation of the apprenticeship. Phase 3 was the actual internship, often out of town; some students even went abroad. The Foundation paid attention (and money) to providing the students with proper textbooks, which Praalder used next to his own extensive collection of notes; when he retired in 1792, 19 notebooks on a variety of subjects were copied for the sake of his successor.

From 1761 - 1810 the average number of students was 11.5, with a maximum of 15. Their average age was 15. They entered at an age between 12 and 18 and studied and lived within the Foundation for 8.2 years (an average again). A total of 71 boys were admitted, 52 of them completed their studies and started a professional career. Some succeeded well, e.g. as governmental officials, and in general the conclusion is that the Foundation provided education which made a seamless connection with a professional career. The programme was designed as such, and it was effective.

Secondary education for children of citizens, the HBS in the 19th century

In the Netherlands during the first half of the 19th century, secondary education continued to be a matter of private initiative. However booming industry and commerce required a nationally regulated educational system which would cater for subjects such as mathematics and science, languages and economics. Both the Royal Academy for Civil Engineers in Delft and the Royal Military Academy in Breda struggled with students who entered with insufficient knowledge of mathematics and science. The growing middle-class repeatedly proclaimed that it wanted better education possibilities for their children. After several unsuccessful attempts, the liberal statesman J.R. Thorbecke (1798 –1872), who also promoted and formulated the 1848 revision of the constitution, managed in 1863 to introduce the law on secondary education (WMO, 1863). The WMO 1863 distinguished several types of school. A really new school type was the Hogere Burgerschool (higher secondary school for citizens) or HBS. It was modelled after the Prussian Realschule. As a consequence of WMO 1863, for the first time in Dutch history the government established and paid a limited number of exemplary national HBS and partly financed a number of HBS established by local councils.

The WMO 1863 defined two types of HBS, a three year school and a five year school. We focus here on the HBS with five year course. Its objectives were to provide general education with science and mathematics as important subjects for sons of the middle class and to provide admission to Delft Polytechnic School. Students with an HBS certificate were exempted of the first exam of the Polytechnic School. The school programme was ambitious, with 18 subjects: mathematics, mechanics and technology, physics, chemistry, natural history, cosmography, Dutch, English, French and German, geography, history, political science, two economic subjects, technical and artistic drawing, writing, and physical exercise. All subjects apart from writing and physical exercise were assessed in a regional final exam. As Latin was not a subject taught in the HBS, the HBS certificate did not give admission to university. Knowledge of Latin remained a condition for university admission.

Only for mathematics the syllabus was prescribed. In his "Explanatory Memorandum", Thorbecke mentioned the topics he deemed suitable for the HBS: (1) arithmetic (continuing where primary school had finished), (2) algebra, including quadratic equations, arithmetic and geometric series and Newton's binomial theorem, (3) plane and solid geometry, (4) trigonometry and goniometry and (5) descriptive geometry up to curved surfaces. This rather full programme was indeed assessed in the final exams, as is shown in the report on examinations of 1887. The mathematics programme of the Polytechnic School consisted of: higher algebra, spherical trigonometry, analytic geometry, descriptive geometry and applications, differential and integral calculus, surveying and geodesy. Thus the mathematics programmes of HBS and of the Polytechnic School fitted together, at least on paper.

At least two other people were influential in the preparation of the legislation. P.L. Rijke (1812–1899), physics professor at Leiden University, advised on the programme for the HBS and wrote a first draft of the law of 1863. He was in favour of experimental physics; Thorbecke, however, favoured a more mathematical physics programme. D.J. Steyn Parvé (1825–1883) taught mathematics at the Athenaeum in Maastricht and published his ideas on mathematics education in the Netherlands in 1850. Some of his ideas on the role of mathematics were very similar to those Thorbecke expressed in his Explanatory Memorandum. From 1858 until 1863 Steyn Parvé worked at the Ministry, where he was a close co-operator of Thorbecke. In June 1863 he was appointed Inspector of secondary education; the Inspectors were very active in supervising and guiding the schools for secondary education.

The HBS with five year course soon became very popular. Quite a number of students who passed their final exams took a course in Latin and gained entrance to university. Four of the five Dutch Nobel laureates received their education at a HBS. The fifth, Van der Waals, who was born in 1837, taught physics at a HBS. The WMO 1863 stated that teachers at the HBS needed a university degree or equivalent. Although there were worries about a possible lack of qualified teachers, the existence of these schools offered employment opportunities for graduates and so encouraged enrolment in a university programme for mathematics and science.

Our collection of data on the design and implementation of the HBS curriculum continues. The influence of supervision (national inspectors and local committees), of regional certificate exams, of the Polytechnic School and possibly the Royal Military Academy and of the financial aspects will receive further research.

A COMBINED AND COMPARATIVE PERSPECTIVE AND CONCLUSIONS

When we compare our findings for the three cases, we see them in this perspective:

- current curriculum theory, as described in the **Method** section, applies well to all three cases. In all cases there was a single person (Prince Maurits, the Baroness of Renswoude, Thorbecke), who translated a societal need into a visionary intended curriculum and who took care of funding. Co-workers took care of the further elaboration of the curriculum. Maurits' vision was materialized by Stevin and Van Schooten, the Renswoude testament turned into a regulation thanks to the regents of the Foundation, and Rijke and Steyn Parvé assisted Thorbecke in specifying and introducing his ideas about mathematics under the new WMO law.
- current emphasis in practical curriculum development differs considerably from the historic cases. In the 19th century the interest in detailed mathematical content is observed already, but in the two earlier curricula there is more emphasis on the societal context of the students. The Renswoude testament went very far in this.
- the attained curriculum is strongly depending on the quality and energy of the teacher. For the HBS case our data should first be analyzed, but the result in the two earlier cases is the merit of inspired teachers (Van Schooten, Praalder)
- in all cases assessment is a crucial part of the whole set-up. For the Duytsche Mathematique we have not paid much attention to the surveyors examinations, but these played an important role in the admission to official functions. In the case of the Foundation the examination system was strict. The regents attended the examinations, which implied that it was at the same time a quality check for the teacher. The HBS directly started with common certificate examinations for each region; here the inspection enters the scene, acting on behalf of the government.

There is clear evidence that vision towards society prevailed in the stage of the intended curriculum, that funding, a good team and trust was crucial in the stage of the implementation, and that the curricula attained results thanks to sound teaching, care for examination and coherence. We intend to be more specific and more operational at a later stage of our research, so that we can turn the above comparative observations into recommendations for the curriculum designer and evaluator.

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The next three documents are available via SGD (= Staten Generaal Digitaal), i.e. the Digital Archive of the Dutch Parliament:

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http://resourcessgd.kb.nl/SGD/18651866/PDF/SGD_18651866_0000766.pdf (Curriculum of HBS in 1865-66)

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MAKING SENSE OF NEWTON'S MATHEMATICS

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This paper describes a project conducted by the author, developed at the Bath Spa University in England, and which included teachers in training and their pupils, working alongside each other in order to make sense of Newton's mathematics. It drew on the original sources: "Newton's Mathematical Wastebook", albeit in its electronic format. The two main aims of the project were to engage teachers and pupils in a joint research enterprise, and improve on teachers' subject knowledge by asking them to prepare resources based on Newton's original work for their sessions with secondary pupils. The project described here was part of the educational remit of the ***Newton Project*** *website aiming to put all Newton's work (and interpretations related to it) on-line.*

BACKGROUND TO THE PROJECT

The outreach for the *Newton Project*[i] consisted of engaging thirty one student teachers, attending the PGCE Secondary Mathematics course at the Bath Spa University, and thirty five pupils with whom the teachers worked (of varying ages, 13 to 18 year olds) from two mathematics departments in two Bath Schools.

The benefits of the work with the original sources from the history of mathematics have been explained at length in various publications in the past twenty years, most notably in Laubenbacher & Pengelley (1996). We particularly mention this publication, as it was the description of the excitement of learning that we wanted to initiate in our teacher students:

> As with any unmediated learning experience, a special excitement comes from reading a first-hand account of a new discovery. Original texts can also enrich understanding of the roles played by cultural and mathematical surroundings in the invention of new mathematics. Through an appropriate selection and ordering of sources, students can appreciate immediate and long-term advances in the clarity, elegance, and sophistication of concepts, techniques, and notation, seeing progress impeded by fettered thinking or old paradigms until a major breakthrough helps usher in a new era. No other method shows so clearly the evolution of mathematical rigor and abstraction.[ii]

As one of the issues in teacher education, identified by the author in previous studies,[iii] is the lack of subject knowledge and interest in learning more about the mathematical content, this approach seemed worthy of at least an experiment, and certainly worthy as an attempt to bring Newton's mathematics closer to school teachers and pupils.

The main aim of the project was therefore to engage both teachers and pupils to read original sources in the search for deeper understanding of mathematical concepts taught at secondary level.[iv] Whilst the mathematics which is being taught and practiced now is quite different to that which Newton knew and developed, a correlation was established between the two through attempts to read diagrams, equations and Newton's explanations of his thought processes in the development of calculus. As we shall see at the end of this paper, teachers reported that the understanding of some of the most fundamental concepts of A-level mathematics became deeper and meaningful to both pupils and teachers as a consequence of their engagement in the project, and the teachers were able to prepare and extract enriched pedagogical material as examples for their pupils.

Teachers were divided into five working groups, investigating the:

1. General history of calculus
2. Introduction to the study of mechanical curves and the tangent problems
3. Development of Binomial Theorem
4. The development of the Fundamental Theorem of calculus
5. The spread of Newtonian science in Europe.

The teachers worked on 'deciphering' Newton's original manuscripts (given in electronic format) and preparing extracts of these to present to the secondary pupils. They all looked at the possible links between their topic and the topics from the A-level syllabus.

EXPECTATIONS IN THE PROJECT AND TEACHER TRAINING

'Making sense of Newton's Mathematics' was an experimental project – it tried to establish how and if, the novice teachers and their students make more sense of mathematical concepts through the learning process by having access to original works of mathematics. The expectation was not for all teachers to make use of all the resources available to them, nor to be able to explain each and every concept mentioned in Newton's *Mathematical Wastebook*, a notebook Newton kept whilst studying mathematics from the late 1664 to 1665.[1] We wanted, and asked teacher students, to make a correlation between what is being taught at A level (pre-undergraduate mathematics course in UK, 16-18 year olds) and what we expect our pupils to understand, with what Newton and his contemporaries really discovered and worked on.

Teachers were therefore asked to manage their workload in such a way that they familiarised themselves with all available resources, and produced one good explanation and/or resource for one concept they wanted to understand better themselves, for teaching during their practice in schools.

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[1] For further details about this notebook and its electronic copy see the webpage *Wastebook* at the University of Cambridge http://cudl.lib.cam.ac.uk/view/MS-ADD-04004/.

Mathematics teachers sometimes expect their pupils to take things for granted as not all mathematics can be explained by what is already known to them.[v] There is also a noted disparity between what actually students understand and what they think they understand (and the same goes for teachers).[vi]

Even if the question of the subject knowledge (including the historical account of the development of a concept) was not an issue with teacher trainees, there is also the issue of the lack of time to explain the whole development of a concept, and there is not often enough time to explain why exactly something works as it does in mathematics as the lesson time is limited, and the syllabus to be covered is usually substantial.

In the last several paragraphs therefore, we have introduced many issues relating to the subject knowledge (or the lack of it) in teacher students. In order to avoid vagueness, let us summarise some of the findings from the studies described at length elsewhere[vii]:

a) Teachers generally have poor level of subject knowledge (of those mathematical topics that are prescribed by the National Curriculum in Britain) at the beginning of their Teacher Training course,

b) Teachers have very poor, if non-existent, knowledge of the historical mathematics (apart from the incidental and the anecdotal)

c) Teachers have no, or very vague, knowledge of the benefits of understanding mathematical content in the context in which the concept taught was invented

Another recent study by the author[viii] also concluded that teachers are, at this time of their professional development, exploring the ways which support their exploration and discovery of mathematical concepts in a way that offers new insights, allowing them deeper understanding of the mathematics. Teachers also seem very keen at the beginning of their teaching career to undertake intense study in this area, i.e. the history of mathematics.[ix]

Having these findings in mind, and the principle of 'reorientation' as described by Furinghetti (Furinghetti, 2007), the project aimed to link student teachers' learning of mathematics from the original sources and compare it with mathematics of the modern/current syllabus. The reason was to get teachers to both pay attention to the development of subject knowledge, and re-position themselves in terms of the pedagogy related to it. The latter (the repositioning) would be achieved through greater understanding of both the concept in question, its historical development, and in trying to overcome the difficulty in trying to explain the concept from two different perspectives (the concept as originally developed, and the concept as presented in the modern-day syllabus).

HOW NEWTON'S APPLE HIT THE GROUND

Majority of the pre-university syllabus in mathematics in England is based on the study of calculus (and most of other topics appearing can also be linked to Newton's work).[x] The study of curves however is devoted, almost entirely, to the study of quadratics and cubics, albeit without any attempt to consider them in the wider scheme of things either in terms of their classification or as part of wider classification of curves.[xi] For example the study of conics is now redundant in the English pre-university mathematics curriculum, so the study of parabola is entirely based on its algebraic analysis and does not mention that this too is a conic section curve.

It was considered that some historical insight into the study of curves by Newton, his persistent attempts to 'resolve problems by motion'[xii], and his study of the curves as described in the writings of Descartes and van Schooten (1615-1660) as well as their description of dynamic generation of curves, was crucial to his later work on fluxions and subsequently his formulation of calculus. To this end, an introduction to the work of van Schooten and Descartes was deemed useful. Newton built his work on, as we already said, Descartes among others. In his study of tangents, Newton looked at finding not only the one circle which touches a given curve at a given point, but to finding the 'best fit' curve – the one which would most closely approximate the curve at the given point. His work on this concept is described in *Epistola Posterior* 1676 and can be seen on the diagram in Fig. 1. The three circles there all touch the curve at *A*, but only *c1* is the one which fits the curvature of the curve at point *A*.

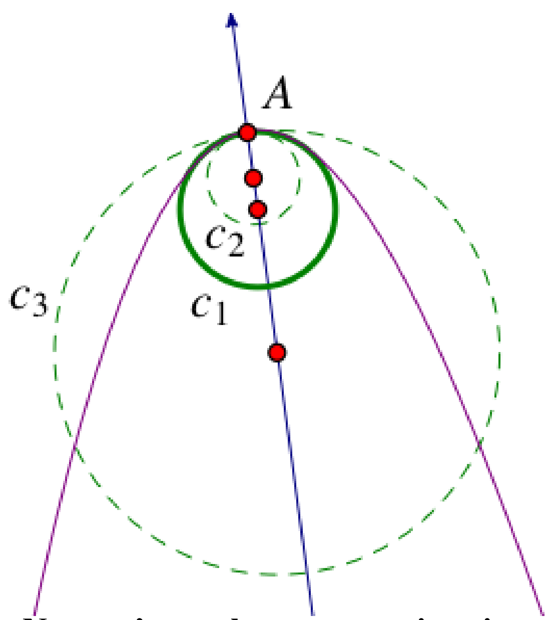

Figure 1: Newton's work on approximating the curve at the given point; see *Epistola Posterior*, 1676.

Figure 2: Newton manuscript 4004: Mathematical Wastebook - p.14. Cambridge University Library.

Newton uses the infinitesimal quantities to find the slope of a tangent to a curve at a given point, and calls these *fluxions*. At this point the link between the study of the curves and in particular their 'dynamic qualities' or description became apparent and the teacher students were encouraged to consider the following:

- What is the difference between geometry of Euclid and the mechanical or dynamic study of curves?
- How do we find the gradient of a straight line? How do we apply this to differentiation?
- What are Newton's fluxions?

Further, inspired by looking at Newton's study of dynamic description of curves which he learnt of by studying van Schooten's work (as per the illustrations in Fig. 2 & 3) the student teachers created a number 'mechanical devices' both from cardboard and in the virtual world, by using dynamic geometry software.

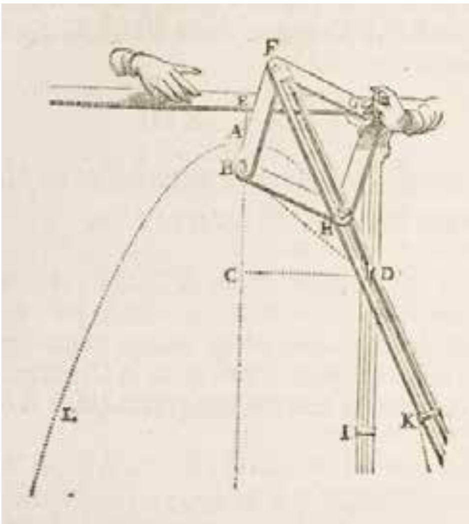

Figure 3: van Schooten's construction of the parabola, 1646.

METHODOLOGY – GATHERING DATA

All the teachers were aware that the project would assess the suitability of the approach to both learning mathematics and teacher training. All teachers were part of the collaborative teaching cycle: they did research, lesson planning, teaching and evaluation in groups of five and six, and were then given a task to complete a reflection on the way they worked in researching the orignal sources, trying to make sense of it and structure it for the discussions and lessons with their pupils, and the teaching sessions. They evaluated how their pupils learnt mathematics by doing informal interviews with pupils and incorporated these findings into their reflections. The author of the paper was present during the discussions and guided teacher students in their work, but there was no detailed analysis done of the lesson plans (apart from group discussions) or of research materials during the project.

Whilst small in scope, the project was used as a basis to evaluate possible outcomes and applications of the *Newton Project* in education, in particular in teacher training and the use of original sources that *Newton Project* offers in electronic format on its website, in the classroom.

Teachers were asked to reflect upon the project immediately after the completion of the teaching sequence, i.e. within a week.

TEACHER STUDENTS' ASSESSMENT OF THEIR DEVELOPMENT

Whilst the teachers originally commented on the amount of time they were going to spend learning what they thought they already knew, in their reflections (anonymous) they commented on the usefulness of the exercise. Some of their conclusions are given below:

1. After the Newton Project I feel I understand the area well and can now explain it to others. The link between Newton and Binomial theorem was interesting and I did not know such a link existed before this project… I found that this helped put things into context and improve my understanding.
2. In the whole I believe this project has been a fantastic learning curve, as it consolidated skills I was strong at and strengthened those I was weak at. My research skills, for one, are an area that I lack in. My groups topic was, *The History of Calculus*, having such a topic forced me to do research allowing me to develop a skill I otherwise wouldn't have.
3. This was not only a learning curve for the pupils, but also for me.
4. The 'deconstruction' process was absolutely pivotal in creating process and breaking down the 'lesson' into manageable chunks for the students… So the 'deconstruction' process' was important for both students and teachers.

Some of the benefits the students therefore themselves identified:

- Working from original sources may lead to learning about the links between the topics teachers did not know of before, therefore enlarging their understanding of both the topics they teach, and the interrelationship between mathematical discoveries
- The research skills were improved – a necessary skill for a teacher, but one which is rarely put to the test in teacher training
- The deconstruction process on the topic is integral to the teaching, and the opportunities were all the more available when the teachers had to engage themselves with the learning process.

The assessment of the views of students was overwhelmingly positive; it remains to be seen in further analysis, and in a more detailed study, how this type of work can be modelled as a common practice in teacher education.

CONCLUSION

Whilst the evaluations of the project are purely qualitative, out of thirty one teacher students only one had a negative comment in their evaluation. This related to the fact that the student teachers are already under a lot of pressure to complete various tasks; learning about the origin of calculus this student did not deem necessary when preparing to teach it. Because the evaluations were anonymous, it is impossible to say anything about the progress of this student in the latter part of the course.

The structure of the project allowed for not only pupils in schools and teacher students to learn some new facts about mathematics, but also teacher mentors and the author of the project and the paper, learnt about some new connections between the topics, as well as the resources which they will be able to use in the future. The learning therefore occurred at all levels, promoting the model of learning in mathematics whereby the history of mathematics offers a field through which the professional learning landscape is being developed for teachers at the same time as the mathematical knowledge is being developed for their pupils.[xiii]

Out of all the outcomes, the most unexpected one was perhaps an occasion during which a student was able to help the teachers by translating the original text from Latin, thereby contributing directly to the teachers' understanding of the mathematical content. In this context, 'making sense' meant learning of teachers and students side by side, in a very literal way.

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NOTES

i Newton Project is an online resource developed by the team of historians of science, aiming to put all Newton's writings online, regardless of their 'discipline' classification. See http://www.newtonproject.sussex.ac.uk/prism.php?id=1.

ii Laubenbacher & Pengelley 1996, p. 257.

iii Lawrence and Ransom, 2011.

iv See in particular Furinghetti, 2007 and Lawrence, 2008.

v Nathan and Kenneth R. Koedinger, 2002.

vi See Knuth, 2002.

vii As described by Lawrence and Ransom, 2011.

viii See Lawrence, 2012.

ix As described by Lawrence, 2012.

x For example, and to mention only a few: number series, binomial theorem, calculus.

xi Whilst it is accepted that Newton's classification of quadratics and cubics is beyond the remit of this level of study, it is nevertheless the considered that the study of motion in curves – for example Newton's work on 'to resolve problems by motion' – MS. Add. 3958, fols 49-63 – were part of his view of curves, and therefore integral part of the principles of calculus.

xii See Ms. Add. 3958, fols. 49-63, and another version, *De Solutione Problematum per Motum*. There is evidence on this work in his *College Notebook*, MS. Add. 4000, see http://www.newtonproject.sussex.ac.uk/view/texts/diplomatic/NATP00128 (accessed 1st September 2012). See also the transcribed paper version in Hall & Hall, 2009, p. 15.

xiii See Lawrence, 2008.

THE TEACHING OF THE CONCEPT OF TANGENT LINE USING ORIGINAL SOURCES

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This paper reports a mathematics experiment conducted in a Portuguese secondary school, with 11th and 12th grades pupils (pre-university). We have used original historical sources for teaching the mathematical concept of tangent line to a curve as well as its connection with the concept of derivative. Emphasis will be given to the conception of the designed module, to the analysis of the pupils' answers and to the pupils' final outcomes. We will infer both benefits and disadvantages of using such activities into the teaching of mathematics.

Keywords: Tangent line; derivative; historical sources; history of mathematics; mathematics education.

INTRODUCTION

The use of history of mathematics in mathematics teaching is increasingly advocated by experts in mathematics, history of mathematics and mathematics education. Many reasons have been presented to justify why it is advantageous to use history of mathematics in mathematics teaching, within those we emphasise the humanization of mathematics, the motivational tool, the dynamical vision of mathematics and its evolution and/or the support for development of an internal image of mathematical concepts (Fauvel&Maanen, 2000; Jankvist, 2009).

Nevertheless some disadvantages have also been identified (Siu, 2006, p. 268-269) and on these we may highlight the time consuming problem, the pupils' concerns about the benefits of this use and the lack of teacher training.

In Portugal we may speak, at least theoretically, of a tradition in the use of history of mathematics in mathematics teaching, which dates back to 1772 when a Faculty of Mathematics was created at the newly reformed University of Coimbra (Mota & Ralha, 2011).

According to this long lasting tradition we have developed an experiment, as part of our PhD research on the historical evolution of the concept of tangent line and its teaching, which aimed at examining the use of original historical sources in teaching mathematical concepts.

THE TEACHING OF THE CONCEPT OF TANGENT LINE

The concept of tangent line to a curve is a concept that becomes familiar to most pupils during their compulsory education but, as Vinner showed (1991, p. 65-81), the pupils' concept definition and concept image of tangent line are not usually the same and this does not allow learners to correctly work with the concept. To avoid this situation, Tall (1990, p. 58) defended that students should face enriching learning experiences to help them construct concept images that are as close as possible to the concept definition.

As Tzanakis & Thomaidis (2011, p.1654-1655) showed history of mathematics can be used in mathematics teaching to, among others, uncover/unveil concepts, as a bridge between mathematics and other disciplines, to get insights into concepts by looking from a different point of view and to compare old and modern. Within our experiment we have tried to achieve such goals by using original sources. Our guidelines were: the works of Euclid, Archimedes and Apollonius, the work of Roberval, Descartes, Fermat, Newton and Leibniz, as well as the notion presented by the Portuguese mathematician José Anastácio da Cunha in the 18th century.

The concept of tangent line in the Portuguese curriculum and textbooks

In Portugal, on one hand, the curriculum for mathematics advocates explicitly (in every grade) the use of the history of mathematics. On the other hand, for our concept of tangent line to a curve we find it approached in the relationship between the derivative of a function at a given point and the slope of the tangent line to the graph of the function at that point. We may, for example, read [2]:

> The use of historical examples or references to the evolution of mathematical concepts will help pupils to appreciate the contribution of mathematics to Humanity problem understanding and problem solving. Some suggested situations: polynomials according to Pedro Nunes, history of differential calculus, history of complex numbers. (Departamento Ensino Secundário, 2001, p. 20)

Curriculum directives are, as expected, vague but, in Portugal, most textbooks and teachers follow only the so called “illumination approaches” (Jankvist, 2009, p. 245-246). In our opinion, Portuguese textbooks [3] aim at informative purposes more than formative ones, illustrated with the use of mathematicians pictures and very scarce references (sometimes even misleading) to the work of these mathematicians. Therefore such historical facts may often become more distracting than informative.

Methodology and data collection

Our experiment took place in two different moments with different approaches and was conditioned by the school executive board directives: we were asked to use the fewest possible classes and we were not given permission to the collection of images or for pupils’ interviews. Thus, data collection was carried out exclusively from the pupils’ written answers as well as our field notes but we believe that our results are still valid. In fact, having developed the experience in usual classroom environment we believe that we have gotten more genuine reviews and reactions from pupils.

In a first phase, involving an 11th grade class of twenty-one pupils, we have used “the modules approach” (Jankvist, 2009, p. 246) with a “historical package” that had the duration of three-class periods of ninety minutes each. This choice was due to the fact that we wanted to focus on a small topic, with strong ties to the curriculum, and we had limited class periods available.

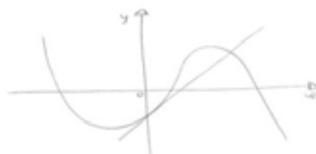
Table 1: Structure of the historical package.

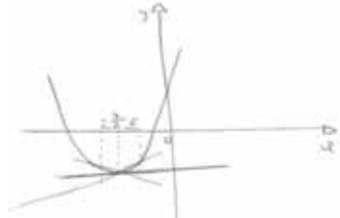
Author	Original Source used [4]	Question/task presented to pupils
Euclid	*Elements III*, Definition 2 (Heath, 1956, p. 1) and *Elements III*, 16 (corollary) (Heath, 1956, p. 39)	**1.** Is this definition suitable to all curves? Justify. **2.** Present a method to draw the tangent line to a circle at a given point.
Archimedes	*On Spirals*, Definition 1 (Heath, 2002, p. 165)	Is Euclid's definition suitable for Archimedes' spiral? In case of a negative answer present an alternative definition for a tangent line to a curve.
Roberval	*Observationes sur la composition des mouvements et sur le moyen de trouve les touschants de lignes curves* (Roberval, 1736, p. 22-23) *Traité des indivisibles* (Roberval, 1736, p. 209-212)	**1.** Use Roberval's method to draw the tangent line to a cycloid at a given point. Make a small report about the taken steps and justify it with transcriptions from the original text. **2.** Using the concepts presented by Roberval and your knowledge in the theory of movement establish connections between the concepts of movement, velocity and tangent line.
Fermat	*Methodus ad Disquirendam Maximam et Minimam* (Fermat, 1891, p. 121)	**1.** Using Fermat's method, determine the extremes of a 2nd degree polynomial function. **2.** What is, geometrically, the interpretation of Fermat's method?
Leibniz	*Nova Methodus pro Maximis et Minimis, Itemque Tangentibus, qua nec Fractas nec Irrationales Quantitates Moratur, et Singulare pro illi Calculi Genus* (Leibniz, 1684/1983, p. 5-7)	**1.** Using an image of the differential triangle, establish a relationship between the slope of a tangent line to a curve at a given point and the function derivative at that point. **2.** Establish a relationship between function monotony and the sign of the derivative. **3.** Using the established relationship present a method to determine the extreme of a function.

Both authors and tasks were chosen to represent different approaches to the concept of tangent line as well as its relation with other concepts, namely the Greek "touch without cut" approach, the relationship with physics and, in particular, with movement, the limit position of the secant approach and the relationship with the concept of derivative.

Our pupils worked in five small heterogeneous groups but their answers, in the different groups, were quite similar. Hereby we present the most relevant data:

Table 2: Data collection of the module-based approaches.

Question/task	Collected Data
Euclid 1	All groups answered "no" and presented a graphic of a curve where the tangent line at a point cuts the curve at another point to justify their answer (below we present two examples of pupils' answers).  
Euclid 2	All groups presented a correct method to draw a tangent line to a circle, for example: - *Trace the radius through the given point;* - *Construct the perpendicular line to the radius through the given point.*
Archimedes	All groups answered "no" and presented an alternative definition. Bellow we present two examples of pupils' answers: ***Example 1:*** *Tangent line to a curve is a straight line that cuts the curve in only one point of the curve near the considered point.* ***Example 2:*** *A straight line is tangent to a curve if it touches the curve without cutting it at that point.* In fact, all the presented definitions had as common idea the fact that the tangent line touches the curve at the contact point and no other in a neighbourhood.
Roberval	The pupils presented many difficulties in the implementation of the proposed tasks, despite the examples presented in the 'historical package'. They just managed to perform the task after a few explanations of the teacher, mainly one related to the description of curves through movement.
Fermat 1	All groups, after some discussion and despite presenting some doubts about the method, were able to follow the method's directives and correctly determine the minimum of the function $f(x) = x^2 + 3x + 2$.
Fermat 2	Only one group was able to present the geometrical interpretation of Fermat's method, which was the following:

	 *The tangent line is horizontal.* To allow all pupils to fully understand the method, the teacher had to thoroughly explain it.
Leibniz	All groups recognized, by using similarity of triangles, that the slope of the tangent line at a given point is the derivative of the function at that point. All groups establish the correct relationship between the function monotony and the sign of the derivative and used that relationship to present a method that allowed the determination of extremes of a function.

The second phase of the experiment took place almost one year later and involved pupils of the 12th grade, ten of them (group A) that had participated in the first part of the experiment and twelve (group B) that were participating for the first time in the experiment. In this phase, it was used "the history-based approaches" (Jankvist, 2009, p. 246-247) by confronting the pupils with different definitions of the concept of tangent line and propositions about this concept, presented in a historical order, that were discussed and analysed with the pupils aiming at achieving a clearer and more complete concept image and concept definition of tangent line. We have chosen to work with the same mathematicians used in the first phase of the experiment, adding a new one: the Portuguese mathematician José Anastácio da Cunha (1744-1787) aiming at dealing with yet a different point of view on the concept alongside with a national/cultural component. The definitions/propositions analyzed are the following:

Table 3: Structure of the history-based approaches.

Author	Definition / Proposition
Euclid	*Elements III*, Definition 2: A straight line is said to touch a circle which, meeting the circle and being produced, does not cut the circle. (Heath, 1956, p. 1)
Archimedes	*On spirals*, 13: If a straight line touch the spiral, it will touch it in one point only. (Heath, 2002, p. 167)
Roberval	*Observationes sur la composition des mouvements et sur le moyen de trouve les touschants de lignes curves*, Axiom or Invention Principle: The direction of the movement of one point which describes a curve is the tangent line of the curve at the position that the point occupies. (Roberval, 1736, p. 22)

Fermat [5]	Tangent line is the limit position of the secant when the two points of the curve tend to meet. (Eves, 1992, p. 391)
Leibniz	*Nova Methodus pro Maximis et Minimis, Itemque Tangentibus, qua nec Fractas nec Irrationales Quantitates Moratur, et Singulare pro illi Calculi Genus*: drawing the tangent line is to draw the straight line that joins two points on the curve that are distant from each other an infinitely small distance, or the side of a polygon of infinite angles that we consider equivalent to the curve. (Leibniz, 1684/1983, p. 7)
José Anastácio da Cunha	*Princípios Matemáticos II*, Definition III: If the sides of an angle meet at the vertex such as it is not possible to draw two straight lines between them it is said that the sides of the angle are tangent to each other. (Cunha, 1790, p. 13)

First the pupils were asked to write their comments and opinions on the definition validity, evolution and/or applicability and afterwards to debate their writings in the class. The most receptive and participative pupils were the ones who took part in the first phase of the experiment and almost all comments, in the class debate, were lead by those pupils. Hereby we present the most relevant data:

Table 4: Data collection of the history-based approaches.

Author	Comments (group A)	Comments (group B)	Comments (class debate)
Euclid	This definition is not valid for all curves.	This definition is correct and is the one generally used.	Group A tried to convince group B that this definition shouldn't be used today since it is not valid for all curves by using examples of curves where the tangent line cuts the curve.
Archimedes	Reference to the uniqueness of the tangency point that is not valid if the curve is a straight line.	Accepted that this property is valid for all curves.	Group A presented their example to group B.
Roberval	Pupils were able to establish, even though with some difficulties, the	Pupils weren't able to understand the relationship between the presented	An explanation on the part of the teacher was necessary in order to make the relationship between mathematics and

	connection with physics.	definition and their concept image of tangent line.	physics evident to all pupils.
Fermat	Recognized this as the generally used definition of tangent line.	Established a relationship with the concept of derivate.	The two groups' approaches were considered complementary.
Leibniz	Pupils referred to the last part of the definition as having no connection with the rest and so it is unnecessary.	Established a relationship with the concept of derivate.	Recognized the similarity with the treatment given by Fermat and discarded the last part of the definition.
José Anastácio da Cunha	Stated the difficulty of the definition and weren't able to understand the connection of the definition with the geometrical treatment of the functions they are used to do.		The teacher explained the presented definition, presenting Anastácio da Cunha's use of the definition.

In the end pupils were asked to present their own definition of a tangent line to a curve and they have chosen the definition referred as Fermat's.

CONCLUSIONS

The use of original sources to teach the concept of tangent line has proved to be not only important for us to realize many positive aspects, but also to identify some obstacles to overcome.

The first major difficulty was faced when constructing the historical package. The choice of the sources to be used, the translation of the original sources into Portuguese that have raised problems of terminology and notation, and the choice of the questions/tasks made the construction of the module to be time-consuming and scientifically intense. It is only possible to develop this type of work if the teacher has some teaching experience and a profound knowledge of the concept/subject to teach together with historical interest and practice. We are sure that we might never develop such competence if it was not for our own process of finding, disclosing and intense debate on many details on the studied sources. This calls our attention to the need of introducing/consolidating history of mathematics in teachers' training as well as the need of a real collaborative work among colleagues that are mathematics teachers, to share materials, experiences, ideas that will make less time-consuming the preparation of this sort of activities.

During the three-class periods where the history package was used, it was hard to get the commitment and stimulate pupils' interest in this type of activity. The first arguments used by the pupils were that mathematics is not history, they do not attend history classes anymore and that they do not like history. Pupils were not, at all, aware of the importance of the historical development of mathematics for understanding mathematics itself, so they discard this sort of activities even before they begin working with them. To overcome this pupils' arguments/attitude history of mathematics might be introduced into mathematics teaching early and often and history classes and mathematics classes might even become, in some aspects, coordinated disciplines.

After the first refusal and the beginning of the task, pupils faced the complexity of the presented texts. The texts had been previously prepared by ourselves but, even so, since Portuguese pupils are not used to read mathematical texts, it was difficult for them to understand the presented historical material. This barrier might only be overcome if pupils are more often faced with reading and analysing mathematical texts. In fact, if we decide to use this historical package again, it might be necessary to prepare its application, by presenting earlier to pupils small excerpts of original sources so that students gradually become familiar with that sort of texts.

Pupils also presented difficulties with the establishment of connections between mathematics and physics. Although Portuguese mathematics curriculum explicitly defends the interdisciplinarity of mathematics and physics, that is rarely a fact in teaching practices. This experiment shows us that a true collaborative work between the teachers of the two disciplines is necessary, for example, for pupils to realise the importance of mathematics to explain physic phenomena.

Another difficulty was related to the purpose of the task. In Portugal, history of mathematics is not a topic evaluated in final examinations, so pupils of these levels (prior to being selected for universities) tend to discard anything that will not be part of national examinations and to concentrate only in "model questions" which might prepare them to the final exam. Pupils showed only interest in the questions concerning to Leibniz's work, since it seemed to them that this might be useful for their exam preparation and they were not able to see any more immediate advantages in any other part of the work.

After using the historical package with the pupils we realised that some changes are necessary. The first one is the awareness that it is necessary more time for pupils to fulfil the tasks presented. We also noticed that it is necessary to introduce more information concerning Roberval or Fermat's method for pupils can perform the tasks and understand it. The use of secondary sources (such as the work of Eves concerning Fermat's definition of a tangent line or the differential triangle, in the Leibniz method) may seem awkward but justified by the need of additional information that would help pupils fully understand the original sources.

The real advantages for pupils with the use of original sources came to be fully acknowledged by them in the second year of the experiment. When asked to comment the definitions/propositions presented, the pupils who took part in the first phase of the experiment were the most receptive and participative ones. They also presented less doubts and less difficulties. This shows that, even though they had some initial concerns, the work made in the previous year developed the inwardness of concepts that gave them the knowledge to perform easily the 2nd phase tasks. At the end it was clear to all participants that the pupils involved in the two years of the experiment had a more solid formation than the others, showing the utility of this sort of tasks to develop a profound and more complete knowledge of the concept. When faced with counterintuitive examples such as the tangent line to a straight line or the tangent line to a curve at an inflection point, pupils showed little difficulty and did not present the problems that are often reported (Vinner, 1991; Tall, 1990) concerning to the concept of tangent line.

It is our conviction that the contact with the historical evolution of the concept allowed our pupils to fully understand it: they internalized the concept, they constructed clearer concept images, they achieved a correct concept definition in their minds. They even performed better in their final examinations.

NOTES

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2. All translations into English were made by the authors.
3. We have analyzed the textbooks Matemática A 12º, Novo Espaço 12 and Xeqmat 12 which are the most used in the country.
4. All texts were translated into Portuguese by the authors.
5. Although Fermat does not present a definition of tangent line in his work he has presented a method from which Eves has composed the definition used in our experiment.

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THE DEVELOPMENT OF PLACE VALUE CONCEPTS TO SIXTH GRADE STUDENTS VIA THE STUDY OF THE CHINESE ABACUS

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The paper presents part of the findings of a study that intended to use the history of mathematics for the development of place value concepts with 18 sixth grade Greek students. In the given pre-tests students faced difficulties in solving place value tasks, such as regrouping quantities and multi-digit subtractions. Also, they vaguely explained the carried number, a notion which is structurally associated with calculations. We held an instructive intervention via a historical calculating tool, the Chinese abacus. In the post-tests students improved their scores and they often put forward expressions influenced by the abacus investigation. To a smaller extent we attempted to highlight the historical dimension of the subject.

INTRODUCTION

Studies have shown that many students are not aware of the structure of the numbers. A great difficulty is in developing an understanding of multi digit numbers. Students need to understand not only how numbers are partitioned according to the base-10 structure, but also how these values interrelate (Fuson, 1990). Resnick (1983b) used the term 'multiple partitioning' to describe the ability to partition numbers in non-standard ways, e.g., 34 can be decomposed into 2 Tens and 14 Units. This ability is essential for competence in calculations and many types of errors that studies have been observed in subtraction (Fuson, 1990; Lemonides, 1994) are due to the students' difficulty to acquire this competence. As a consequence, they cannot interpret the carried number, a concept structurally associated with calculations (Poisard, 2005b).

In this paper we focus on the difficulties that the students of the present study faced in the above concepts and the way that we tried to address these difficulties with the use of the history of mathematics. Initially, we present the reasons that historical instruments may positively contribute to mathematics education. Then we describe the didactical use of the historical instrument that we used in the intervention, the Chinese abacus. Afterwards we present an overview of the intervention: the objectives, the design with the use of history, and an example of a didactical session. Then, a brief quantitative and a more detailed qualitative analysis of the results follow. Finally, we discuss the findings and further research issues.

THE ROLE OF THE HISTORY OF MATHEMATICS IN THE CLASSROOM

Researchers have long thought about whether mathematical education can be improved through incorporating ideas and elements from the history of mathematics. Tzanakis and Arcavi (2000) offered a list of arguments and Jankvist (2009) distinguished these arguments between using "history-as-a-goal" (learning of the development and evolution of mathematics) and using "history-as-a-tool" (learning mathematical concepts). Jankvist also classified the approaches in which history can be used. One of these is the 'modules approach'. Modules are instructional units suitable for the use of history as a cognitive tool, since extra time is required to study more in-depth mathematical concepts, and as a goal. (Jankvist, 2009). Among the possible ways that modules can be implemented using history as a 'tool' as well as a 'goal', is through the use of historical instruments since they can illustrate mathematical concepts in an empirical basis. They are considered as non-standard media, unlike blackboards and books, that can also affect students emotionally (van Maanen, 2000). Students explore them as historical sources for arithmetic, algebra, or geometry and they may also enable students to acquire awareness of the cultural dimension of mathematics (Bartolini Bussi, 2000).

Chinese abacus: A historical calculating instrument

The Chinese abacus is comprised of vertical rods with same sized beads sliding on them. The beads are separated by a horizontal bar into a set of two beads (value 5) above and a set of five beads (value 1) below. The rate of the unit from right to left is in base ten. To represent a number e.g. 5,031,902 (figure 1) beads of the upper or/and the lower group are pushed towards the bar, otherwise zero is represented.

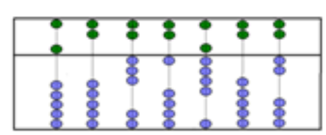

Figure 1: Representation of numbers on Chinese abacus

> Many characteristics of our number system are illustrated by the abacus (Spitzer, 1942). Unlike Dienes' blocks the semi-abstract structure of the abacus becomes apparent as the same sized beads and their position-dependent value has direct reference to digit numbers. The function of zero is represented, as a place-holder. Furthermore, it may illustrate the idea of collection, since amounts become evident in terms of place value. Finally, the notion of carried number emerges. Poisard (2005b) argued the fact that we can write up to fifteen units in each column and make exchanges with the hand reinforces the understanding of the carried number in operations. From the definition of the carried number (Poisard, 2005b) highlighted its relation to the functionality of the decimal system to allow quick calculations: "the carried number allows managing the change of the place value; it carries out a transfer of the numbers between the ranks" (p. 78).

Based on the studies about students' difficulties and the possible positive contribution of the history of mathematics via the Chinese abacus in place value understanding, the present study set the following objectives:

1. To study whether sixth grade students recognize the structure of our number system when handling numbers.

2. To study how they verbally explain the carried number and how they use it in written calculations.
3. To study to what extent an instructive intervention with the Chinese abacus would help students handle possible difficulties and misconceptions.
4. To highlight the historical context of the abacus and enrich teaching with a variety of approaches where students are actively involved.

In the present study we adopted Poisard's (2005b) proposal for the didactical use of the Chinese abacus; we used all the beads in order to record up to 15 units, unlike the standard technique where one of the upper beads (value five) is not used at all. This allowed us to add new elements in the present study, such as the use of regrouping activities as essential knowledge (Resnick, 1983b) before implementing the written algorithms of addition and subtraction.

RESEARCH METHODS

The study took place in an elementary school in Thessaloniki during the school year 2010-2011. The participants were 18 twelve-year-old students (9 girls and 9 boys). The criterion was that the students would be able to participate once a week during the hours when their school program was to work on a two-hour project. Four students had a very weak cognitive background and eight students often relied on procedural rules due to partial conceptual understanding.

For the first two objectives two questionnaires (pre-tests) were administered in November. **Questionnaire A** consisted of six closed-type questions and one that required a written explanation (Appendix II, p.10). After the intervention similar questions were administered as post-test. The questions were created with the following in mind: (a) the literature about students' difficulties (b) the Greek mathematics curriculum so as to ascertain that they constitute important and prerequisite knowledge in the beginning of grade 6, and (c) the feasibility of teaching via the abacus. For integers the questions concerned: named place value, expanded form, regrouping, rounding, subtraction, and multiplication. For decimals: transforming from verbal to digit form, number pattern, addition, and subtraction. Two of the questions that are subjected in the present analysis concern exchanges between classes: sub question 3b, which concerned regrouping and comparing quantities, and sub question 7a, which dealt with subtraction with carried number. In order to study how students perceive the concept of carried number used in the subtraction tasks, we administered **Questionnaire B**. It consisted of Poisard's (2005b, p.101) four open questions. The same questions were given as post-test. Here we present student responses to the question: *what is a carried number?*

The design of the intervention with the use of the History of Mathematics

For the other three objectives we implemented a five-month instructive intervention. It was inspired by modules approach (Jankvist, 2009). We designed a didactical sequence for the teaching of mathematical concepts that was allocated in sections (integers, decimals, and operations). For every session a teaching plan was elaborated including procedure, forms of work, media and material. The outcomes were recorded and several sessions were videotaped as feedback for the researchers. The introductory and closing activities aimed at using history mainly as a goal.

Initially, the arguments mentioned below are aimed at exploring why history would support the learning and raise the cultural dimension of mathematics. They were based on Tzanakis's & Arcavi's (2000) arguments and were grouped under Jankvist's (2009) categorization. We have included a third category placing pedagogical arguments in an attempt to emotionally motivate as well as develop critical thinking. Thus, students are expected to:

A. History as tool

1. develop understanding by exploring mathematical concepts empirically
2. recognise the validity of non-formal approaches of the past

B. History as a goal

1. become aware that different people or in different periods developed various forms of representations
2. perceive that mathematics were influenced by social and cultural factors

C. Pedagogical arguments: motivate emotionally, develop critical thinking and/or metacognitive abilities

Some examples of the interrelation between the activities chosen and the arguments for such a choice are presented below. (The arguments are in parentheses). Introductory and closing activities: Presentations about number systems of the antiquity: Roman, Babylonian, Greek, Mayan (B1, C); students create numbers and discuss the effectiveness of the systems (A2, B1, C). Presentation about the ancestor of the abacus, the counting rods (B1); form rod numerals and compare with the modern representation (A1, A2, C). Information about the abacus (B2); compare the two forms (abacus and rods): advantages/disadvantages similarities/differences (B1). After the intervention students presented their work to an audience in the role of the teacher (C); they elaborate on information about the cultural context of the abacus that led to prevail over the counting rods (B2, C) for a multicultural event. Main part: Students investigated place value with handmade abaci, web applications (A1, A2, C; Appendix I, p.10) and worksheets designed by the researchers (A2, C); they analysed the abacus's representations/procedures and corresponded with the formal one (A1, A2); contests between groups (A1, C).

The implementation of the intervention through an example of a session

The example is based on Poisard's (2005b) proposal for subtracting on an abacus with carried number. The method mainly taught to Greek schools and other European educational systems is the 'parallel additions', which uses the relation $a-b= (a+10^x)-(b+10^x)$. The other method, the 'internal transfers', is taught in second grade as an introductory method so it is rarely used over the years. It allows exchanges between classes and is the only that can be implemented on abacus when using all beads.

Previous knowledge on abacus: decompose quantities; perform subtractions without trading. Procedure: The teacher forms the minuend of the subtraction 933-51 on the abacus. The number 1 can be subtracted immediately by removing one unit bead (figure 2, step1) but in the tens column the regrouping process must be put forward. A student removes a one-bead from the hundreds and replaces it with two five-beads in the tens (figure 2, step 2). Having a total 13 on the tens he/she removes one five-bead and gets the result (figure 3, step 3). The student is encouraged to explain in terms of place value: *I decompose 1 hundred to 10 tens and then subtract 5 tens.*

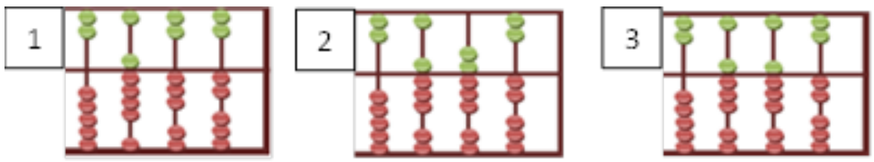

Figure 2: Example of the subtraction method 'internal transfers' on abacus

Observation from the teaching: A student solved the subtraction 4,005-8 initially on the blackboard. She transferred a 1 thousands' unit directly to the units' position; she subtracted and found 3,007. We also observed this error (Fuson, 1990; Lemonides, 1999) in some answers of the pre-test. When prompted to use the abacus, the student correctly implemented the decomposition process and explained it in terms of place value. Our discussion then revolved around the two results, so that the student reflected on her incorrect thought when she solved on the blackboard. One of the reasons that she did not make a mistake on the abacus – apart from the intervention's influence – is possibly the visual-kinetic advantage of the tool; the space that occupies the intermediate columns may act as a deterrent for the eye to arbitrarily surpass them. Also, since we use the hand to remove one upper class unit bead, the fingers are merely guided to the next column in order to replace it with 10 equivalent lower units. The role of the teacher was crucial at this point to link the semi-abstract with the abstract technique, and at the same time to emphasize the common underlined mathematical theory.

DATA ANALYSIS AND RESULTS

Questionnaire A: The total score of Questionnaire A was 100. The t-tests showed a statistically significant difference between the two measurements of students' scores ($t= 5.243$, $df = 17$, $p <0.001$), with a pre-test mean of 50.5 and a post-test mean of 78.2. For the questions 3 and 7, the t-tests showed a statistically significant difference between the means of the two measurements: Question 3 ($t=6.172$, $df=17$, $p<0.001$) pre-test: mean 4.67, standard deviation 5; post-test: mean 12.9, standard deviation 3.5. Question 7 ($t= 2.807$, $df = 17$, $p < 0.05$) pre-test: mean 17.1, standard deviation 11.2; post-test: mean 22.6, standard deviation 9.0. The qualitative analysis that follows concerns the sub questions 3b and 7a. It aims to find if the scores' improvement is connected with better understanding through the investigation of the abacus. For question 3b we studied students' written explanations.

Sub question 3b, Pre-test: 'Compare *8 hundreds 2 tens 1unit* __ *7 hundreds 11 tens 16 units* using the sign of inequality/equality. Explain the way you have thought'. **Post-test**: 'Compare *6 hundreds 3 tens 3 units* __ *6 hundreds 14 tens 13 units* using the sign of inequality/equality. Explain the way you have thought'.

Types of reasoning	Pre-test	Post-test
correct	4	16
incorrect	8	
insufficient/no explain	6	2

Table 1: Reasoning analysis for answers to sub question 3b

Four students on the pre-test (table 1) gave correct justifications, while on the post-test the majority of them were correct. Examples of students' written explanations on the pre- and post-tests follow. The abbreviations used are H=Hundreds, T=Tens, and U=Units.

Correct reasoning: *1T=1H and 1T. Also 16U=1T and 6U. So we have 700+110+16=826.* Incorrect reasoning: They saw individual numbers on both sides: *The second is bigger than the first in two numbers.* They isolated digits and arbitrarily formed a number: *826 less than 71,116.* They compared the hundred's class, possibly recalling a vague knowledge of upper classes: *The first number has 1H more so it is bigger because hundreds matter.* Insufficient reasoning: *Because 7hundreds 11tens 16units is bigger.*

Post-test: A figurative explanation appears (figure 5). By circling and using arrows, students were depicting the abacus process of composing ten units to a higher class.

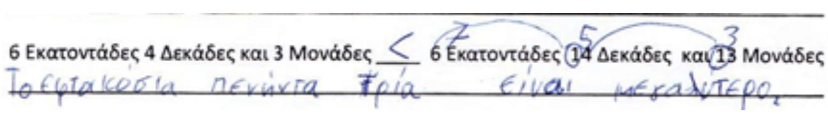

Figure 3: Sub question 3b – Example of regrouping at the post-test

Translation: *Seven hundred and fifty three is bigger*

A more detailed response: *I get 10 from 14 T and make 1 H. The H now are 7. Then we have 13 U. I take 10 U and do another 1 T. The number is 753 greater than 643.*

Sub question 7a, Pre-test: Solve the subtraction 70,005-9 in vertical form. **Post-test**: Solve the subtraction 40,006-9 in vertical form.

	carried number not noted	parallel additions	Totals
Answers	10	6	**16**
Success	4	5	**9**

Table 2: Management of the carried number on the pre-test (sub question 7a)

Two students did not answer this question. From table 2 we observe that half students succeeded. The visible method was ‘parallel additions’, since the rest of the students did not note the carried number. The types of errors are categorised in table 3.

Question: 70,005-7	carried number not noted		use of carried number	
Types of errors	N	Examples	N	Examples
Carried number	**5**	60,008 70,010 81,098,		
Copying numbers	**1**	7,005-7		
Number facts			**1**	69,997

Table 3: Types of errors on the pre-test (sub question 7a)

The main type of errors (table 3) seemed to be the management of the carried number. For example, in the result ‘60,008’, though the carried number is not noted, the error is the transfer of 1 thousand to the units’ position.

Sub question 7a, Post-test: Almost all students succeeded and the number of students who did not use the carried number decreased because of the use of the new method that requires the notation of the carried number (table 4).

	carried number not noted	parallel additions	internal transfers	**Totals**
Answers	4	7	7	**18**
Success	3	6	7	**16**

Table 4: Management of the carried number on the post-test (sub question 7a)

The method ‘internal transfers’ appears and along with ‘parallel additions’ was applied successfully (table 4). The method ‘parallel additions’ was applied mainly by students who had successfully applied it during the pre-test, while the method ‘internal transfers’ was given by those who had not be able to handle the carried number correctly.

Figure 4: The method ‘internal transfers’ as implemented on the post-test

***Questionnaire* B: ‘What is a carried number?’**

Explanations	**N**
find/use/ something in calculations	9
Example with addition	5
I don't know/remember; I cannot describe it	4

Explanations	N
When the number exceeds 10	1

Table 5: The interpretation of the carried number (Pre-test)

Explanations with the use of an example	N	Verbal explanations	N
Composing e.g.10hundreds=1 thousand	**6**	Ten units of a position move to the next position as one unit	**3**
Decomposing e.g.1hundred=10 tens	**1**	Number we keep aside/use in operations for transfer	**2**
Composing/decomposing	**1**	Borrowing from a number	**1**
		A format of tens, hundreds, etc., for transfer	**2**
		Convert a number of ten and over to another format	**1**

Table 6: The interpretation of the carried number (Post-test)

The explanations with the use of an example differ between the two tests (table 5 & 6). At the pre-test students just performed an addition while in the post-test they put forward composing and decomposing examples. Verbal explanations at the pre-test seemed meaningless. Only in one answer we detected an attempt of mathematical explanation; *when the number exceeds 10*. At the post-test we can still observe a difficulty to explain but most students used the idea of exchanging (e.g., 'transfer', 'convert the format'). One is specific: *10 units move to the next class as 1 unit;* others mix the knowledge before and after the intervention: a *number we keep for transfer.*

DISCUSSION

The results of the pre-tests showed that most students did not have a profound understanding of the numbers' structure; almost all could not recognise the numbers behind a non-standard partitioning (Fuson, 1990; Resnick, 1983) and half failed to solve a four-digit subtraction across zeros, a task that other studies have shown is difficult (Fuson, 1990). In addition, they could not interpret the notion of carried number (Poisard, 2005b) considering it as an aid in operations but more of a vague nature. At the post-test, almost all displayed a better conceptual understanding. Using schematic representations and place value explanations influenced by the abacus' activities, they successfully regrouped non-standard representations to standard numbers. As for the subtraction task, the students that had unsuccessfully managed the carried number in the pre-test, they implemented successfully the abacus's method 'internal transfers', which requires the reverse process of decomposing numbers. In agreement with Poisard (2005b) the method has the advantage of illustrating the properties of our number system when they have not been adequately understood. The regrouping activities on the abacus and their connection to the algorithms of addition and subtraction changed students' perspective about the concept of the carried number. They explained it as an exchange between classes, either verbally denoted or through an example.

Despite the limitations of the study, such as the small sample and the lack of relevant experiential studies about the Chinese abacus, except Poisard's (2005b), the authors believe that the reasons for using the history of mathematics were accomplished in a quite satisfactory way. By elaborating on place value concepts via the abacus, students developed understanding on an empirical basis (literally with their hands). By analysing processes with the historical tool, students appreciated that mathematics of the past also lead to results that have logical completeness. In general, Bartolini Bussi's (2000) argument was verified that in the tactile experience offered by the ancient instruments one may find the foundations of mathematical activity.

During the intervention we recognised the crucial role of the teacher in the teaching/learning process. Students may learn to calculate correctly with the tool, but without conceptual understanding. Also, as the example from the didactical session showed, they may achieve understanding place value concepts when calculating with the tool but they continue to misapply the written calculations because they do not connect the two processes. That is why teachers should encourage students to gain insight into the relation between the tool and the concept that it represents (Uttal, Scudder, & Deloache, 1999), otherwise its semiotic function will not be transparent.

As further research we suggest the study of the Chinese abacus with younger students for the teaching of simpler concepts (Zhou & Peverly, 2005). In addition, it would be innovative the production of material involving the history of mathematics in the primary education with the cooperation of university departments and classroom teachers.

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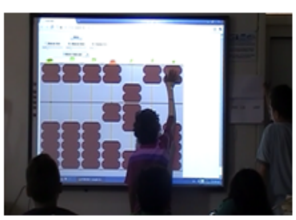
APPENDICES

APPENDIX I

Abacus for demonstration

Student's abacus



Web application

APPENDIX II

Questionnaire A-Post-Test

1 Write the place value of the digit "2" of the number 27,257,275

_______________ _______________ _______________

2. Write the number 4,018,379 in the expanded form as in the colored example

(4x____________)+**(0x**____________)+**(1x**__________)+**(8x**____________)+**(3x**__________)+**(7x**___10_____)+**(9x**____________)

3. Compare the numbers using the signs >, <, =. Explain written the way you have thought

A. 4 Tens and 3 Units ____ 3 Tens and 13 Units

B. 6 Hundreds 4 Tens and 3 Units ____ 6 Hundreds 14 Tens and 13 Units

4. Round the number 5,283

a.	To the nearest ten	
b.	To the nearest hundred	
c.	To the nearest thousand	

5. Write the following decimal numbers with digits

a.	Eight tenths	
b.	Forty six hundredths	
c.	Seven thousandths	
d.	Twelve tenths	

6. Fill in the gaps using the sequence of numbers

0.2	0.5	0.8		

7. Perform the operations in vertical form:

40,006 - 9   5.4 + 3.46   21 - 3.56   623 x 82

CALCULUS AND APPLICATIONS – LEARNING FROM HISTORY IN TEACHER EDUCATION

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The current scientific body of calculus may appear as if its development were carried out on a linear path without many optional possibilities which appeared along the historical processes. Under a historical perspective stressing the conditions of its genesis many decisions were made along the way. This contribution emphasizes the importance of generating an awareness of the unfolding character of the history of calculus when addressing student teachers. The aim is an enhanced teaching process of the body of analysis with its pure and applied parts.

TOWARDS AN AWARENESS OF HISTORICAL DEVELOPMENT

Modern calculus textbooks are characterized by an axiomatic representation and often offer physical applications. The main concepts are the real numbers, the notion of a function and the notion of a limit. All are presented by a certain way of formalization: definitions, theorems, proofs, added by a few examples illustrating the propositions. This representation of the body of calculus was developed during the first half of the last century and is still regarded as "modern" and constitutes the result of a long historical non-linear development with discontinuations.

From a "postmodern" point of view questions arise and the necessity for a broader perspective: Are there alternative ways that calculus could have taken? Why did the idea of Archimedes concerning "exhaustions" not lead to a first draft of the Riemannian integral? Why were the steps which Newton, Euler and Weierstrass took, in fact, a dynamical idea of the limit, the notion of a function, the epsilon-delta-concept, widely accepted? Under what circumstances the infinitesimal numbers of Leibniz or the non-standard-analysis would have become more important? In how far do computers influence the way we practice and interpret calculus and its applications? Already the process of answering these questions helps deepening the understanding of the development of calculus.

EDUCATIONAL ASPECTS FOR STUDENT TEACHERS

Within the context of teacher training students should receive a deepened insight in the process of mathematics and develop an appropriate concept of science by considering historical circumstances. Infinitesimal calculus is both – a branch of pure mathematics as well as a tool in applied mathematics, and it often happens that its use lacks critical reflection. This situation provides a favorable opportunity to discuss the diversity of doing research – within and through mathematics.

Taking account of the different aspects of pure and applied mathematics, calculus shows an odd ambivalence. On the university level lectures of calculus are usually considered to be a part of pure mathematics. On the other hand it embodies strong connections to numerical mathematics. Additionally, the methods of infinitesimal calculus play more and more a decisive role in mathematical modelling. Besides well-known modelling procedures in "exact" sciences such as physics, contemporary modelling is increasingly concerned with other initially non-mathematical scientific issues such as economy or social sciences. These subjects deal with quantities that are not originally discrete in every case. This raises the question of whether infinitesimal concepts are a suitable tool for describing the "reality" in those subject areas. There are different possible answers that depend on the perspective and the targets under investigation and the perception of science. Future teachers need to develop an awareness of the genesis of calculus since it will enable them to teach the sense of it in class.

WAY OF REPRESENTING – THE POSTER

In the center of the poster two timelines represent the history of calculus and the actual representation in textbooks. It also displays the important correlations of the topic predominantly in the form of a concept map and includes several text passages giving guidelines and showing some of student's elaborations and (mis-)conceptions. Based on the well-known didactical triangle an extended scheme shows the connections to teaching-relevant aspects of history. Proposals concerning teacher education are included.

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IDEAS ABOUT MODERN MATHEMATICS AND TEACHER TRAINEES AT LICEU NORMAL DE PEDRO NUNES (1957-1971)

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This poster presents a longitudinal analysis of the texts published in Palestra magazine by mathematics teachers, not teacher trainees[1]*. In our study we divided in four categories the ideas that were advocated by this magazine: Mathematics and programs; Geometry teaching and axiomatics; Modern algebra and algebraic structures; Teacher training and math didactics. We see a convergence of views on the need for the introduction of modern algebra in programs and math classes; the moderate interest in the geometry axiomatization among pupils; and a paradigm shift in teaching methods, which call for student participation in the learning process and also for the teacher continuing education.* ***Key-words****: Portuguese mathematics education history, Teacher trainees, Modern mathematics.*

It is important to investigate the history of school subjects (Chervel, 1990). The international movement of Modern Mathematics was felt in Portugal in 1950s and 60s and introduced a change in contents and methods in the teaching of this discipline (Matos, 2006), to which the teacher training had to respond.

We are interested knowing a little better what ideas about modern mathematics and mathematics education circulated at Pedro Nunes Normal Secondary School (*Liceu*). With this objective, we focused on *Palestra* magazine, which was founded by Dias Agudo, methodologist and dean of the *Liceu*. The magazine was subsidized by the Portuguese State. The first number was published in January 1958 and the last one was number 42, published in 1973, which excluded mathematics sections. The last one with a mathematics section was number 41, published in 1972.

The chronological scope of this study is limited by the reopening of the training programs in this *Liceu* in October 1956 (which had been suspended in 1947) and the beginning of José Veiga Simão's mandate, who sets in motion a new reform of the Portuguese educational system. This was preceded by a new training model in 1969, which led to the teacher training reduction from 2 to 1 year.

In our study we divided in four categories the ideas that were advocated by the magazine[2]: Mathematics and programs (Calado, 1958; Dantas, 1958; Leote, 1958 and 1964; Silva, 1959); Geometry teaching and axiomatics (Leote, 1958 and 1964; Paulo, 1959 and 1962); Modern algebra and algebraic structures (Calado, 1958; Leote, 1958; Paulo, 1963); Teacher training and mathematic didactics (Calado, 1958; Dantas, 1958; Leote, 1958 and 1964). Leote (1958) argued that teachers should "enjoy and encourage" (p. 37) the creative activity that students possess. He further sustains that the teacher must be an investigator and shouldn't think that concepts that he himself took years to learn are obvious to pupils. Leote and Dantas argued that the heuristic method, although desirable, does not respond to all educational needs. A real difficulty that those teachers were faced with was the overcrowding in classes, which had 42 students. For Dantas (1958) a problem resided in the "limited time to teach [...]. The pace of discovery is very slow" (p. 99). Dantas cites the conclusions of the Congress of Mathematics Education held in Salvador da Bahia from 4 to 7 September 1955: "All [the methods] are good as long as the teacher leads the student to participate rather than to observe." (p. 101). Calado (1958) warned about the need to "review the scientific recruitment and preparation of teachers of secondary education" (p. 91) in the light of new concepts and language which are intrinsic to modern mathematics. Addressing the Minister of National Education, Pinto Leite, who was present at the *Liceu*, Calado requested that in the Secondary Schools with teacher training "there should be beginners' courses or seminars on Algebra of the Logic, on the Foundations of mathematics and on Modern Algebra carried out by renowned teachers "(p. 102). Calado argued that these courses should be compulsory for the trainees of the mathematics group and disseminated to all teachers of Mathematics and Physics of secondary education. A response to this desire came to pass with the lessons of Silva (1959).

In our study, we came to the conclusion that there was an upgrading of knowledge and of the presented proposals, just like in the rest of Europe and the United States. With respect to teacher training, though not discordant, the authors refer to different facets of this issue: Gonçalves Calado, Silva Paulo and Sebastião e Silva contribute to scientific education; Furtado Leote and Martha Dantas reflect about the teaching of mathematics. In the *Liceu*, the presence of leading figures of the ministry of education and of the Portuguese State was frequent, including the Republic's President Almirante Américo Tomás, who visited the *Liceu* in 1959 and 1966. Between 1957 and 1971 we found 50 mathematics teacher trainees in this school and, possibly, they heard these words.

NOTES

1. For teacher trainees you have another work (Matos & Monteiro, 2012).

2. Calado: 1958, *Palestra*, 1, 89-105; Dantas: 1958, *Palestra*, 3, 97-103; Leote: 1958, *Palestra*, 1, 37-48; Leote: 1964, *Palestra*, 21, 110-123; Paulo: 1959, *Palestra*, 5, 97-106; Paulo: 1962, *Palestra*, 13, 78-88; Paulo: 1963, *Palestra*, 17, 29-44; Silva: 1959, *Palestra*, 6, Separata 1-66.

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FACETS OF THE PRESENTATION OF THE CARTESIAN COORDINATE SYSTEM IN EULER'S *INTRODUCTIO IN ANALYSIN INFINITORUM* AND LACROIX'S TEXTBOOKS[1]

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This paper studies the presentation of the Cartesian coordinate system in Euler's Introductio in Analysin Infinitorum *and in Lacroix's* Traité du calcul différentiel et du calcul intégral *and* Traité Élémentaire de Trigonométrie Rectiligne et Sphérique, et d'Application de l'Algèbre a la Géométrie*, searching for what components made possible its systematization, and bearing in mind students' difficulties*.

INTRODUCTION

It is a well known fact that students have difficulties with understanding and dealing with the representation of functions in the Cartesian coordinate system (CCS). This didactic problematics has led us to determine "which texts must be sought out in history and what questions we should address to them" (Puig, 2011, p. 29). The texts chosen are Euler's *Introductio in Analysin Infinitorum* (1748), and its French translation from 1796-1797, and Lacroix's *Traité du calcul différentiel et du calcul intégral* (1797) and *Traité Élémentaire de Trigonométrie Rectiligne et Sphérique, et d'application de l'Algèbre a la Géométrie* (1797).

REASONS TO CHOOSE THE TEXTS

The reasons to choose Euler's *Introductio* and Lacroix's textbooks are, first, that we wanted to study texts from the moment in history when the present way of representing functions in Cartesian coordinates was being constituted, and from the moment when it was being incorporated as a teaching topic in textbooks. Next, the main reason to choose Euler's *Introductio* is that it is one of the first books to deal with Cartesian coordinates in a systematic way. Lacroix's textbooks have been chosen because 1) they elementarize mathematics in order to teach it (Schubring, 1987), 2) they deal specifically with Cartesian coordinates in a progressive way, and 3) they had a big impact in the teaching of mathematics, not only in France, but also in Spain. Lacroix's *Traité Élémentaire de Trigonométrie Rectiligne et Sphérique, et d'application de l'Algèbre a la Géométrie* was translated into Spanish, as part of the *Curso completo elemental de matemáticas*, a Spanish translation of Lacroix's textbooks. This translation was widely used because it was established as an official textbook by King Fernando VII's law-ranking decree of 1824, whose Article 42 states "in all these

[1] This is a result of a research program funded by the Ministerio de Ciencia e Innovación (EDU2009-10599) and Ministerio de Economía y Competitividad (EDU2012-35638) of Spain.

Chairs [referring to the Chairs of Mathematics and Sciences in Universities], lessons will last one hour and a half in the morning, and one hour in the evening; being used as textbook the Pure Mathematics by Mr. Lacroix translated by Rebollo". As far as we have found, the Spanish translation *Tratado elemental de trigonometría rectilínea y de la aplicación del álgebra a la geometría* was published eight times, the 8th edition being from 1846. We have used the Spanish 6th edition of 1820 (Lacroix, 1820), and the original French 4th edition (Lacroix, 1807). No Spanish translation of Lacroix's *Traité de calcul* was published. We have used the original French first edition (Lacroix, 1797).

COMPONENTS OF THE CONSTITUTION OF CCS

As a result of our study, we state that the main components that make possible the systematization of Cartesian coordinates as presented in these texts are:

1. The endowment of negative quantities with meaning both in algebra and geometry, and the setting of a fixed origin (of coordinates).
2. The constitution of the concept of abscissa.
3. The movement from the notion of applicata (a segment raised at the end of the abscissa) to the ordinate concept (a distance measured in the ordinate axis).
4. The movement from coordinates as segments to coordinates as distances, and the consequent movement to coordinates as numbers.
5 The establishment of absolute coordinate axes, i. e., axes not specific to the curve.

A study of student difficulties in the light of these components is in progress.

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG13: EARLY YEARS MATHEMATICS

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***Keywords**: Children play; counting; curriculum; digital tools; family; geometrical shapes; kindergarten and primary school; measurement; relational thinking; teachers' role*

The working group on Early Years Mathematics was established at CERME 6 in Lyon, in 2009. The aim of this working group was, and continues to be, the sharing of scholarly research related to mathematics education concerning children aged 3-8. This age group spans preschool through the early grades of primary school, and takes into consideration that in different countries children begin primary school at different ages.

Since that first meeting in 2009, there has been increased appreciation for the need to provide young children with a rich environment where they can explore mathematical ideas and relationships and increased recognition of the need to explicitly promote young children's mathematical knowledge. There has also been an increased appreciation for the need to explore, investigate, and research different aspects of early childhood mathematics. This was reflected in the working group of 2013 where we saw an increase in the number of presentations (19 papers and one poster were presented) as well as an increase in the number of participants (34 participants from 10 different countries). Table 1 presents an overview of the presentations and participants according to nationality.

Country	Number of papers	Number of posters	Number of Participants
Canada	1		1
China	1		1
France	1		2
Germany	6		12
Greece	1		1
Israel	2		1
Italy	4	1	7
Norway	1		3
Sweden	2		5
Turkey	-		1
TOTAL	19	1	34

During the current conference, there was a growing interest in exploring cultural differences among the various cultures and more of an emphasis on researching mathematics education before the beginning of primary school. In the following sections we offer a brief review of the papers presented at the conference.

The interest in cultural differences was sparked by two papers, one from China (Sun) and one from Italy (Ramploud & Di Paola). Within the mathematical context of addition and subtraction, these papers explored task design in the different countries, different epistemologies, pedagogies, and beliefs related to cognitive processes. Different cultures also have different approaches towards early childhood education. This was apparent when discussing two papers related to the concept of measurement. The paper from Germany (Zöllner & Benz) explored children’s ways of comparing lengths indirectly when given a variety of tools and explicit directions to compare the lengths. On the other hand, the paper from Sweden (Lembrer) explored how different concepts related to measurement arise as children engage in a map drawing activity.

Many of the papers focused on specific mathematical content. As mentioned above, addition and subtraction and measurement were the focus of four papers Sun; Ramploud & Di Paola; Zöllner & Benz; Lembrer). Four papers focused on geometrical shapes (Acar-Bayraktar; Bezold & Ladel; Maier & Benz; Tabach, Tirosh, Tasmir, Levenson & Barkai). The geometry studies were based on various theoretical frameworks, and explored learning and teaching geometry in different situations such classifying, defining, reasoning, describing, explaining, and working on properties. The studies also utilized different materials such as drawn representations of figures as well as concrete objects. Four papers related to counting and enumeration (Bartolini-Bussi; Guidoni, Mellone & Minichini; Ouvrier-Buffet; Sinclair & Sedaghatjou). These papers described very different ways of promoting children’s knowledge of these skills, including the use of narratives and storytelling (Guidoni, Mellone & Minichini), the use of concrete materials and lists (Ouvrier-Buffet), and the use of digital technology (Sinclair & Sedaghatiou). Another paper described promoting prospective primary teaches’ knowledge of counting and enumeration through semiotic mediation and the use of a giant abacus (Bartolini-Bussi).

In addition to specific mathematical content, mathematical processes such as reasoning, generalization, and explanations may also to be encouraged during the early years. This was the focus of two papers (Levenson & Barkai; Vighi). The paper from Israel (Levenson & Barkai) theorized about the functions of explanations during mathematical activities while the paper and poster from Italy (Vighi; Vighi & Aschieri) showed how even young children in kindergarten are capable of reaching general conclusions when engaging in a mathematical puzzle.

Finally, one paper (Jung & Vogel) addressed the issue of coding children's mathematical activities while they are involved in situations of play and exploration, revealing the existence of preferred mathematical domains for particular children during the process of problem solving.

One of the new areas of research which arose during the CERME 8 conference was that of Information and Communication Technology (ICT) and how it can be integrated into early years mathematics. Once again, cultural differences were noted. In Sweden (Lange & Meaney), a study was conducted which explored the mathematics which may be hidden in popular games played on the iPad. A paper from Canada (Sinclair & Sedaghatjou) presented a study related to specifically designing a game for promoting young children's knowledge of number concepts. A third paper (Hundeland, Erfjord & Carlsen) explored the roles of the teacher when employing ICT related mathematics activities. A fourth paper (Bezold & Ladel) analyses how ICT can support the mathematical learning processes, focusing on exploration and discovery of mathematical relations. This new area of research led to additional research questions, such as the difference between engaging children with concrete versus virtual manipulatives.

Besides the paper described above (Hundeland, Erfjord & Carlsen) which investigated the teacher's roles when integrating ICT with mathematics learning, two other papers focused on early childhood teachers. One paper (Bartolini-Bussi) described a program for developing preschool teachers' mathematics knowledge while a second paper (Kröger, Schuler, Kramer & Wittmann) explored the beliefs of kindergarten and primary school teachers related to mathematics teaching and learning. However, teachers are not the only ones which may influence children's mathematical learning. Two studies (Acar-Bayraktar; Tiedemann) focused on the possible roles of the family in supporting mathematics learning during the early years.

The obvious common thread which runs throughout all the papers described above is the desire to improve mathematics learning for young children. That being said, as can be seen from the diversity of papers described above, there are many aspects to consider and many avenues yet to explore. At the last session of the conference, a questionnaire developed by the group leaders, was passed out to participants. The questionnaire included the following four questions: (1) Have you participated in the past in this working group? (2) If so, have you noticed a change? (3) Do you have specific questions? (4) What have you learned? In general, 20 participants had participated in this working group in the past and some of the changes they noticed were in the way the group was organized, pairing papers together for a fruitful discussion, having reactors to each paper, and allowing time for small group discussions. Other issues which arose either as questions or as new ideas learned during the conference, related to the different cultural perspectives, to the practice and acceptance of different research models, and the issue of ICT. For several participants, this was their first time taking part in this group. We hope to see additional participants join this working group at the next CERME in 2015.

THE SECOND DISCERNMENT INTO THE INTERACTIONAL NICHE IN THE DEVELOPMENT OF MATHEMATICAL THINKING (NMT) IN THE FAMILIAL CONTEXT

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As a reflection of mathematics education on developmental psychology and cognitive development, Götz Krummheuer created the concept of the "interactional niche in the development of mathematical thinking (NMT)" as a new theoretical framework in the mathematics education. This theoretical framework has been adapted as a developmental niche in the familial context (NMT-Family). Through an empirical study in familial play situations more details of the NMT are investigated. In this paper it will be stepped into the second discernment into the NMT-Family through the study erStMaL-FaSt.

1. INTRODUCTION

The IDeA (Center for Research on Individual Development and Adaptive Education of Children at Risk) is a research centre, which investigates extensively the development of children at risk and the processes of individual learning. This research centre is constituted by the German Institute for International Educational Research (DIPF) and Goethe Universität Frankfurt.The financial support provided by the Ministry of Higher Education, Research and the Arts from the state of Hessen. [1]

One of research project of IDeA Center is a Project erStMaL (early Steps in Mathematics Learning), which investigates the mathematical development of children with regard to their migration background. It is designed as a longitudinal study to follow children from the age of three, until the third year of primary school from a socio-constructivist perspective. While the first survey period contains only kindergarten children, the second survey period contains the same children in primary school ages (see also Acar Bayraktar et al. 2011).

In the scope of the project erStMaL, a family study is performed, which is designed as a longitudinal study and named as erStMaL- FaSt (early Steps in Mathematics Learning-Family Study). The study deals with the impact of the familial socialization on the mathematics learning and due to the following three criteria, 8 participants are chosen from the project erStMaL. The criteria are the ethnic background (German or German/Turkish), the duration of the formal education of the parents and the sibling situation within the families (see Acar Bayraktar and Krummheuer 2011, Acar Bayraktar 2012). Data collection comprises of recorded videos and their transcripts. Once in a year, an appointment is arranged with each family. This leads step by step to a collection of data from each child. In each appointment the erStMaL child is video-recorded together with members of the family while they are playing in different settings.

For the family study two mathematical domains are chosen: Geometry and Measurement. Four play situations are conceived, due to these two mathematical domains. The members of the family are supposed to choose at least 2 games out of 4 and to perform them. For participation of all families, instruction manuals of each play are made both in German and Turkish, which can be spoken freely by families during play situations. The game materials are provided and put at the disposal of the family in the recording room. Currently, the new play situations are set up for the third observation phase in September.

In this paper it will be stepped into the second discernment into the NMT-Family.

2. THEORETICAL BASIS

> "The play, for the child and for the adult alike, is a way of using mind, or better yet, an attitude toward the use of mind"(Bruner 1983, p. 69).

erStMaL-FaSt enables families freely to play with their children. During each play situation with maintenance of father/mother/sibling the child explores something about the issued mathematical domain. This accompaniment of family provides to the child some "learning offerings" and interactive negotiation about the mathematical play. During the interaction of such various mathematical learning situations, there occur different emerging forms of participation and support.

For the comparison among the various mathematical learning situations and for the longitudinal analyses, the concept of the "interactional niche in the development of mathematical thinking" (NMT) will be used, which has been constituted by Krummheuer (2011a, 2011b). He explains NMT as in follow:

> The concept of the "interactional niche in the development of mathematical thinking" (NMT), consists of the provided "learning offerings" of a group or society, which are specific to their culture and will be categorized as aspects of "allocation", and of the situationally emerging performance occurring in the process of meaning negotiation, which will be subsumed under the aspect of the "situation" (Krummheuer 2012).

NMT- Family is a *subconcept* of NMT and offers the advantage of more close analyzes and comparisons between familial mathematical learning occasions in early childhood and primary school ages.

In view of the design of FaSt, three components of NMT-Family are shown and then their details given below:

NMT-Family	**component: content**	**component: cooperation**	**component: pedagogy and education**
aspect of	mathematical	Play as a	developmental theories od

allocation	domains: Geometry and Measurement	familial arrangements for cooperation	mathematics education and proposals of activeness for parents on this theoretical basis
aspect of situation	interactive negotiation of the rules of play and the content	leeway of participation	folk theories of mathematics education, everyday routines in mathematics education; MLSS

Content: In the practice of erStMaL-FaSt, children and their families are confronted with mathematical play situations, which are – as mentioned - either in mathematical domain “Geometry” or in mathematical domain “Measurement”. The play situations in erStMaL-FaSt are designed to offer the families opportunities for interactive negotiations. From the situational perspective, in these play situations, processes of negotiation emerge, in which the rules of play and/or mathematical topics might be chosen as themes.

Cooperation: The process of cooperation between the adult and child provides the opportunity to refine their thinking and to make their performance more effective. Depending on this cooperation, a different leeway of participation comes forward. Krummheuer meant Leeway as a colloquial meaning of “room for freedom of action” (Krummheuer 2012).

“Leeway of participation” („Partizipationsspielraum“, Brandt 2004) is one of the interactionistic approaches, by which a child explores his/her cultural environment while co-constructing it. “Leeway” is taken here in the colloquial meaning of “room for freedom of action”. So, this is a concept belonging to the situational aspect. Brandt (2004) explains that the participants interactively accomplish different margins of leeways of participation that are conducive or restrictive to the mathematical development of a child. (see also Krummheuer 2011c; 2012). Alongside of contents, the children are involved in the social settings in the play situations, which are variously structured as in child-parents interaction and/or child-sibling interaction. These social settings need to be accomplished in the process of interaction.

Pedagogy and Education: Developmental theories and theories of mathematics education describe and delineate learning paths for the children’s mathematical growth from which point of view. With the respect to the folk pedagogy, the participating adults and children become situationally active and operant in the concrete interaction. The cognitive development of each individual is constitutively bound to the participation of these individuals in a variety of social interactions. During these interactions and participations in the mathematical discourses, there occurs a “support system”, which is proposed as a concept for the learning of mathematics and called “Mathematics Learning Support System” (MLSS) , with respect of Bruner’s concept of a Language Acquisition Support System (LASS)

(Bruner 1986, p. 77; see also Acar Bayraktar and Krummheuer 2011, Krummheuer 2011b, Tiedemann 2010).

In the patterns and routines of interactions between child and families, MLSS occurs in different ways. In the mathematical competitions of play situations in FaSt, adults maintain the play, in which, possibly, emerges a support system. During these play situations they impart *their knowledge* by giving for example explanation to the statements during the negotiation of meaning. Not only through "the right given instructions" but also through "the wrong given instructions" by families, there occur some different types of support. Through the negotiation of the given instructions, children and parents lay out new interpretations, which support the development of the child either in a positive or a negative way (see Acar Bayraktar and Krummheuer 2011, Acar Bayraktar 2012).

With the respect to all these three components, it will be introduced one chosen scene as an example to show how the spatial abilities (spatial thinking) in the interactional niche in a familial context emerge.

3. A PLAY: BUILDING 02

The mathematical play "Building 02" refers to geometry and spatial thinking. The family is supposed to build three-dimensional version of the picture with wooden bricks, which all are in the same size and weight. Supposedly, they perform the relations between two- and three-dimensional representations. The player chooses one card from the deck and builds a wooden corpus from the image on the card. In the play, cards are placed on the table face down. Each card has a difficulty level ranging from 1 to 4. The cards with the number 1 are the easiest. The cards with the number 4 are the hardest, whose transitions between the various blocks are fluid and are purposed more complicated.

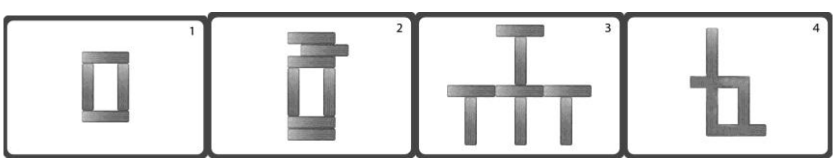


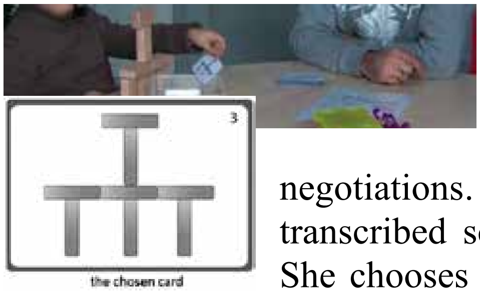
Fig 1. The game cards in different levels

An Example: Family Ak

The required information about Family Ak is in the following table:

Aleyna	6;9 years old	Single Child	German, Turkish rudimental
Father	Studied 15 years	Higher Education	German, Turkish rudimental
Mother	Studied 12 years	Higher Education	German, Turkish rudimental

Fig 2. Information about Family Ak



the chosen card

In the chosen game, Aleynas' game partner is only her father. Her mother is bystander and accompanies them behind the cameras by watching and making interpretations during their negotiations. In total, they play 5 rounds by turns. The chosen and transcribed scene is the first round, which begins with Aleyna. She chooses a card, which is shown on the left side. She builds the figure up while her father reads the instruction manual of play. Until she builds the figure up, in each step she asks her father and mother, if she does it right. They both don't give explanatory answers but the mother motivates her by saying "slowly" and "be concentrated". She accomplishes the corpus vertically on the table as shown on the right side. After she finished this construction, a conversation between father and daughter emerges, whose transcript is shown in the following.

the built figure

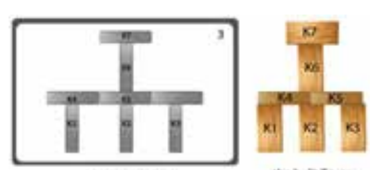
After her corpus is done, her father poses a question:

Transcript [2]

09	Father:	is it correct? No.
10	Mother:	just look accurately at it, Aleyna. there are two blocks, on it- or? there comes one more block up on it.
12	Father:	<u>O.K. be quiet. don't interfere.</u> *has a look at mother*
13	Aleyna:	*grimaces* ..nooooooooo! it is correct.
15	Mother:	just look
16 #	Father:	*shows with his right indexfinger on the card* just look. there are three parts. one, two, three.
18 #	Mother:	it is not true like that. honey?
19 <	Aleyna:	I've- *opens her mouth, looks grimly, handles K8 with her right hand*
21 <	Father:	yes. you've lost. *takes the card away* now it is father's turn-
23 >	Aleyna:	*takes the card furiously with her left hand from her father* nooo! *sets K8 on the Z Side near K3*
26 >	Father:	<u>but it can't be played like that</u>
27	Aleyna:	*puts K9 diagonally on K8 and K5*
28	Father:	no <u>not like that.. not like that</u> *smiles*
29	Aleyna:	*lays K9 under* ıııııhhhhhh!
30	Father:	<u>O.K.</u> now it is dady's turn. *removes K6 and K7, puts them into the box*

the built figure

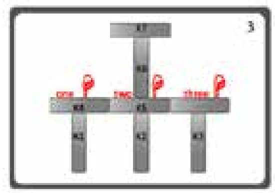
The father asks if aleyna built up the figure correct. He rephrases her question and answers, that the figure is not built correct. Hereby he expresses no further reason. By this he obviously deprives Aleyna of becoming informed about "the right figure" and also he limits her leeway of participation. Beside Aleyna's father, her mother tells her that she should look at the built corpus carefully. This

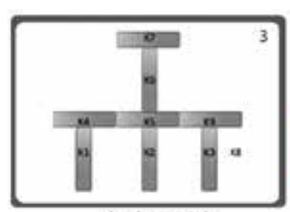


the chosen card the built figure

reaction can be interpreted, that she calls her attention to the block, which actually has to be laid next to the K5 on the right side on the card, in the built figure and hereby she gives her a chance to think about her oversight. Then she tells her daughter, that there are only two blocks on “it”. Probably the pronoun “it” refers to the structure, which consists of K1, K2, K3. Hereby she might try to open a discussion that could foster her daughter to think about her oversight. Likely to make easier Aleyna’s task, her mother gives her a clue, that how many blocks on which structure should be put. But then her father punctuates and warns her mother about, to be quiet and not to interfere herself to their play. After the both commentaries made by her parents, Aleyna gives a response, which is not clear enough to determine if she gives it to her mother’s commentary or her father’s. But as a consequence, she does not accept the critics and tells that the built corpus is correct.

After Aleyna’s denial, her mother proposes her to look the built figure carefully. Then her father tries to call her attention to her oversight (?) on the card. By showing with his index finger, he counts the interjacent blocks on the card. This reaction of him could be interpreted as he is influenced by his wife’s struggle to show Aleyna her oversight. Hence, he may try to show and make it clear how the structure actually has to be built. As a confirmation of the father’s explanation, the mother tells her that the built figure is not correct without the third block, which has to be laid next to K5. The mother calls her, probably to get a response from Aleyna. Aleyna tends to respond her mother and probably tries to rectify her own oversight. But the father cut her off and says that she lost the turn. Hence, he takes the card away and tells that his turn begins. Presumably, he does not have enough perseverance to explain to Aleyna what her oversight is and how she can rectify it. Against her father’s reaction, she takes the card furiously from him and puts K8 vertically near K3. Possibly she does not loose the game. It also could be that she very much involved in the correction of her initial conctruction and would like to continue regardless what game thy are playing. Then the father reacts as if she is in the contravention of the play rules by working over the built figure. Without saying anything, Aleyna goes on constructing the figure and puts K9 diagonally on K8 and K5 (see picture in below).





the chosen card

the built figure

As response, the father gainsays her action and tells her smilingly that the built corpus is not the same with the figure on the card. This could be as a interpreted as the smirk of the winner of the game or as a cordial attempt to mitigate the tension in the dispute. Aleyna lays K9 back on the table as if she abandons insisting, that the built figure is correct and she did not loose her turn. At the end of Aleyna’s turn, the father says that his turn is up and removes blocks of the built corpus in the box. Obviously this is the end of Aleyna’s turn.

4. CONCLUSION

As a summary, , a developmental niche for Aleyna emerges slightly in the chosen play situation, although there are an oversight of Aleyna and deficient information by father. While her father cuts out that she did wrong and lose the turn, her mother tries to give some hints and to call her attention to her oversight. With both emotional motivations, encouragement by her mother and declining reaction by the father, Aleyna realizes her oversight and tries to work on it again. In spite of her endeavour her father insists, that she doesn't build the corpus correctly and thus loses her turn. However this reaction of her father works on Aleyna and urges her to continue improving her construction. Actually and finally she does not come up with the correct solution but her activities give reason to assume that her competencies in spatial structuring and visual discrimination are enhanced.

She sets the different faces of blocks in the figure, and also she cannot coconstruct the built figure. According to the National Research Council Committee on Early Childhood (National Research Council 2009, pp. 186), the children at age 4 can identify the faces of 3-D shapes to 2-D shapes, can match faces of congruent 2-D shapes and can represent 2-D and 3-D relationships with objects. Furthermore at 5 years old they can build complex structures from pictured models. Considering these statements, it could be assumed that Aleyna's spatial thinking is not developed enough to think about parts and to relate them to the whole. But after constructing the figure with oversight, realizing and trying it to coconstruct, are the indicators to perceive Aleyna's developmental niche. Both together, encouragement by her mother and disapprovement by herfather, directs and provides her to see "mistakes" and to struggle getting at the "truth".

According to this analysis the three components of an interactional developmental niche in familial context can be structured as in follow:

Component "Content": Block Building provides a view of children's initial abilities to compose 3-D objects. In the chosen play, four goals are pursued: Spatial structuring, operating shapes and figures, static balance between blocks, identifying the faces of 3-D shapes to 2-D shapes. By National Research Council is also reported that 5-years-old children can understand and can replicate the perspectives of different viewers. These competencies reflect an initial development of thinking at the relating parts and wholes level (National Research Council 2009, p.191). Aleyna realizes the spatial relations between 2-D and 3-D objects. She can relate some parts with the whole. So, as an allocation aspect occurs the spatial structuring and operating with shapes during the play. As a situational aspect, Aleyna negotiates with her father and mother. In this trial structure, there emerges a negotiation about the built figure. But neither her mother or her father assists her to explore how the figure actually has to be and what she overlooks on the built figure, there occurs a *consensus,* that Aleyna built the figure incorrectly.

Component "cooperation": The play situation is changeable, but directed by father. While he plays with his daughter, the mother does not refrain herself from interfering in their play and giving help to her daughter. Thus she plays such a role as a contributor in their play.

While her mother encourages Aleyna seeing her oversight, she also gives her a chance to an exploration and discussion on it. This means that she opens up a leeway of participation for Aleyna. While her father disapproves her, he also motivates her to struggle getting her due. So, the parents together in their seemingly uncoordinated moves open up a leeway of participation for Aleyna. On the other side, because the father always tries to end up her turn and brings no more discussion about the built figure, he limits the leeway of participation for Aleyna. Thus in all, father and mother together co-construct a leeway of participation in this play situation.

Component "pedagogy and education": The chosen play situation refers to the spatial structuring in geometry. In the chosen scene, by the organizing and setting objects, the graphical- and spatial-development of Aleyna are slightly assisted. While she is called attention to her oversight and is motivated co-constructing the built figure, she is not directly assisted, how this figure actually has to be. One can conjecture that the parents follow a rather constructivistic idea of helping and educating.

NMT **Family Ak** **Building 02**	**component:** **content**	**component:** **cooperation**	**component:** **pedagogy and education**
aspect of allocation	Geometry, Spatial structuring, operating shapes and figures, static balance between blocks, identifying the faces of 3-D shapes to 2-D shapes.	Playing with father, and Mother as a contributor	Theory of the development of spatial skills and spatial structuring
aspect of situation	negotiation between father, mother and Aleyna about the built figure; *consensus*	Different leeways of participation offered by the parents	Emotional motivation from parents provides Aleyna to explore her oversights and to endeavour coconstructing the built figure.

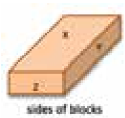
"The play under the control of the player gives to the child his first and most crucial opportunity to have the courage to think, to talk, and perhaps even to be himself " (Bruner 1983, p. 69).

In the chosen example, with mother's encouragement and father's disapprovement, Aleyna realize her oversight and tries to resolve it. Although there occurs an "antagonisms" between Aleyna and her father, she can interpret her deficiency. This is an over careful learning progress, in which the interactional developmental niche slightly occurs for Aleyna. Clements and Samara reports that spatial processing in young children is not qualitatively different from that of older children or adults, but children with the age produce progressively more elaborate constructions (2007, p. 512). Hence, It will be really exciting to go on augmenting the examples of Aleyna, and to find out how NMT-Family functions work on Aleyna's spatial development in her familial context.

5. NOTES

1. http://www.idea-frankfurt.eu/homepage/about-idea
2. Rules of Transcription

Column 1	Serially numbered lines.
Column 2	Speech timing
Column 3	Abbreviations for the names of the interacting people.
Column 4	Verbal (regular font) and non-verbal (italic font) actions.
<u>underlined</u>	Speech is in Turkish
bold	Accentuated word.
<	Indicates where people are talking at the same time.
>	The next block of simultaneous speech is indicated by a change in arrow direction.
#	There is no break, the second speaker follows immediately from the first.



The sides of block are defined as X Side, Y Side, Z Side in transcript.

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BAMBINI CHE CONTANO: A LONG TERM PROGRAM FOR PRESCHOOL TEACHERS DEVELOPMENT

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This paper aims at reporting about a five years program for preschool teachers development, with the author as a teacher educator, on appointment of the city council of Modena. The project has involved every year about 25 teachers. In this paper the political background and the structure of the program are reported together with some outcomes concerning the materials for teacher development, the activities realized in schools by the teachers and the construction of a community of inquiry involving teachers, teacher educators, pedagogists and policy makers.

INTRODUCTION

Modena is a city of about 190.000 inhabitants in the region Emilia-Romagna. In Italy compulsory education starts at 6 (first grade of primary school) but there is an ancient sensitivity to the organization of public school institutions for children aged 0 – 6, especially in some regions, like Emilia-Romagna. After the Second World War, parents, educators and politicians of this region united their efforts to provide child care for young children. Originally inspired by the need of women to return to the work force, this education system has been developed in different places (e.g. Reggio Emilia, Modena, Bologna) with programs that have caught the attention of early childhood educators worldwide. In order to fill the social gap between peasant, working and middle classes (*cognitive democracy*) the civil society agreed to start high quality schooling as early as possible, as 6 years old children might have already experienced the devastating effects of social inequalities. In Reggio Emilia, Loris Malaguzzi developed the so-called Reggio approach, well known all over the world (http://www.reggiochildren.it/?lang=en), while in Modena, Sergio Neri developed a parallel approach, with similar formats for teacher development and similar involvement of parents and the civil society. The differences between the two approaches may be very roughly outlined focusing on different curricular choices for children aged 0-6, with a major focus on *enculturation* in Modena and on the so called *"hundred languages of children"* in Reggio Emilia.

I have worked in both programs, for a short time on appointment of Reggio Emilia city council and for a longer time on appointment of Modena city council. The latter collaboration was started in the eighties, when Sergio Neri was in charge of the pedagogical coordination of the municipal preschools, and was resumed in 2007 after a ten years interruption. An outcome of the first phase collaboration was the preparation of the field of experience *Lo spazio, l'ordine, la misura*, i. e. early year mathematics, of the *Orientamenti* (1991, the national suggested curriculum for preschools: http://www.edscuola.it/archivio/norme/programmi/materna.html). This contribution concerns the second phase of collaboration, started in 2007.

THE PROGRAM FOR TEACHER DEVELOPMENT

As preschool is not compulsory in Italy, in the case of this program, the educational policy has to be distinguished into two parts: the national one; the local one.

The political choices at the national level

In 2007 a major effort was started by the Ministry of Education to design anew the standards (*Indicazioni*) for students aged 3-14, thus involving also the level of preschools (covered by *Orientamenti* from 1991). General educators called up the complete responsibility for the new document for the age 3-6, without consulting experts of specific subjects (contrasting the choices made in the early 90s), and decreased the reference to the adult culture as a curriculum organizer and increased the freedom and responsibility of children themselves in defining the direction of their learning. The six fields of experience of the *Orientamenti* were reduced to five, with the merging of mathematics and science in only one field of experience (*la conoscenza del mondo*, i. e. world knowledge). In the fifth field of experience words like numbers and counting totally disappeared, as considered too much oriented to mathematics in the adult sense. The draft document of *Indicazioni* was officially issued in July 2007 to raise comments from schools. The part of the *Indicazioni* for children aged 3-6 was quickly put in the shade, as comments from schools were not positive at all. In 2012 a new version of *Indicazioni* has been issuedfor students aged 3-14. In this case, although the number of fields of experience for preschools has not been restored to six, the weight of early years mathematics has been increased distinguishing in the fifth field of experience two subfields, concerning mathematical and scientific experience. http://hubmiur.pubblica.istruzione.it/web/istruzione/prot7734_12)

The political choices at the local level (Modena)

In 2007, when I started the second phase of collaboration with the Modena city council, preschool (and especially the presence, if any, of early years mathematics) was in the heart of the debate at the national level, as learned societies had strongly criticized the choices made by general educators. To design anew a program for preschool teacher development it was necessary, first, to agree on the intended curriculum for schools. Officially, the 1991 *Orientamenti* have not been abrogated as the new *Indicazioni* were still only a draft to be discussed by schools. The local authority for instruction, Adriana Querzè, a very intelligent and open-minded policy maker, agreed that, for mathematics education, *Orientamenti* (1991) still represented the framework of the municipal preschools in Modena as a model of enculturation, updating the original text to meet the needs of schools in the new millennium. Hence the political choice was to focus on early years mathematics (especially on numbers and space) to be tested in the large laboratory of the 22 municipal preschools.

The local organization

Every year, eight different projects are offered for teacher development by Modena city council: Italian language, Mathematics, Science, Art, Music, English, Orienteering and Philosophy for children. Enrolment in one project is compulsory for each teacher of the 22 municipal preschools and included in their working hours. Local teachers from governmental, private or religious preschools are welcome to (yet not forced to attend) the meetings. Also prospective preschool teachers from the Faculty of Education are welcome, as a part of their compulsory schedule for practicum. The system of Modena municipal preschools and the programs for teacher development are coordinated by a team of pedagogists (education committee). For the Mathematics project, two pedagogists have been involved in sequence: Maria Teresa Corradini (2007/2011) and Mariavittoria Vecchi (2011/now). I am the only permanent mathematics teacher educator with some other invited teacher educators for specific meetings. In 2010 a new member was appointed by the city council to join the small steering committee: Susanna Stanzani, a former teacher, expert in multimedia documentation; for nearly two years, she has part time taken care of the collection of materials from schools and authored the multimedia report on behalf of the city council.

For each project, every year the pedagogist specifies, in cooperation with teacher educators, the focus of the yearly project, the timetable and the contents of each meeting and the way of documenting school activities. She is in charge also of special meetings at schools to discuss with small groups of teachers problems found during school activities. In this sense, she represents a bridge between teacher educators and teachers. Her feeling of teachers' attitude is precious to detect whether there are problems or needs in the implementation of activities.

At the beginning of every school year a group of teachers is appointed (about one per school) for each project. Teachers are encouraged to be permanent members of the projects, but, for Mathematics project, in the period 2007-2012 there was a large fluctuation, because of different reasons (teachers' retirement; teachers' moving to other schools; new teachers entering municipal preschools and so on). This fluctuation represents a problem as every year it is necessary to welcome newcomers and to organize meetings so that they feel comfortable and ready to work. Now (2012) expert teachers are expected to take part as tutors in the small group sessions (third meeting, see below).

The general structure of five half day meetings with the group of about 25 teachers is shared by all projects. All the meetings but the third involve teachers, teacher educators and pedagogists.

First meeting (October): welcome to newcomers, summary of the past experiences and launch of the year activities (adult education and examples from classrooms)

Second meeting (November): continuation; design of some activities for different students' age (3, 4 or 5 years old);

Third meeting (January: small groups of teachers with pedagogist and expert teachers, in schools): 2-3 sessions for discussion on the ongoing activity (problems, needs, criticisms, new ideas);

Fourth meeting (February): discussion about the issues of the third meeting;

Fifth meeting (May): presentation and discussion of some activities realized in the schools; small exhibition of materials from schools.

If necessary, additional meetings with the pedagogist or the teacher educators are agreed with groups of teachers to meet specific needs.

Occasionally each project presents to the large group of teachers (about 160) of municipal preschools the outcomes of the project. The Mathematics project, called Counting and Measuring at the beginning, and later *Bambini che contano* (Counting Children), presented the partial outcomes in September 2010, after a three years activity, with speeches by the teacher educator, the pedagogist and a group of teachers. The multimedia report of the project will be published on-line at the beginning of 2013 at the website of the documentation centre for education of the town council of Modena (http://istruzione.comune.modena.it/memo/)

The outcomes of a complex long term program of teacher development concerns different issues. The general effectiveness is measured by changes in teachers' beliefs and school activities. They both have been focused in the program, the former with teachers' interviews and the latter with the documentation of school activities.

THE REPORT *BAMBINI CHE CONTANO* (COUNTING CHILDREN)

The multimedia report is divided into several parts and organized into three frames: http://memoesperienze.comune.modena.it/bambini/index.htm. The first part (introduction) is accessible from the left frame and concerns the intended program from a political perspective (video interview with Adriana Querzè), from a pedagogical perspective (interview with Mariavittoria Vecchi) and from the mathematics educator perspective (interview with Maria G. Bartolini Bussi). In the same introduction video interviews with teachers focus the links between theory (as perceived in the meetings with teacher educators) and practice (the school realization) and the changes, if any, in their system of beliefs about early years mathematics.

The main frame contains a rich repertoire of activities realized in the schools, with teachers' design and report. For each activity, there is a collection of commented children's protocols (drawings, transcripts of individual interviews, transcripts of either small group or large group discussion, video clips, and so on). School activities are divided into four parts, each structured in different chapters: Hands and finger counting; Slavonic abacus; Everyday mathematics; Numbers in space. The main frame has been designed to be independent from the other parts of the multimedia report. It is targeted to laymen (as long as they are interested in education) and may be used, for instance, to present school activities to parents and members of the civil society.

The right frame links to a rich collection of materials for teacher development, that put at disposal of teachers printable texts, videos, annotated and translated references usually available only to researchers in specialized literature, a glossary and a short presentation of each quoted author.

The reference to scientific literature in teacher development has been discussed and agreed by the teacher educator, the pedagogist and the policy maker. This choice cannot be taken for granted: in most cases in Italy teacher development is likened to teacher training with exemplary activities to be repeated in schools. The long term feature of this program allowed maintaining a high quality offer, to work at a slow pace, involving teachers as protagonists of the innovation and treating them like professionals of culture and education rather than like imitators of recipes.

THEORETICAL FRAMEWORK

To approach the issue of counting and measuring some foci were agreed with the pedagogists and teachers at the beginning and reconsidered every year (in the first meeting). The approach drew on the firm belief that mankind came to construct mathematics as a cultural object, producing *artefacts* which embody mathematical meanings and processes, although the emergence of meanings for users cannot be realized without specific activities (Meira, 1998). Hence we focused:

- The function of some selected cultural artefacts, developed by mankind.
- The teacher's role as cultural mediator in the enculturation process.

The theoretical framework for analyzing and designing the teacher's role in the classroom process was slowly reconstructed with teachers after the framework of semiotic mediation (Bartolini Bussi & Mariotti, 2008), that had already been validated from primary school on. In this paper I only report the main ideas from the perspective of the teacher's role.

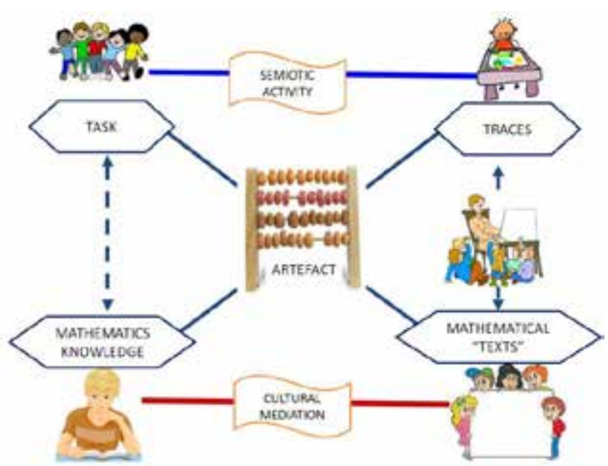

Figure 1: Semiotic mediation

The teacher is in charge of two main processes: the design of activities; the functioning of activities. In the former the teacher makes sound choices about the artefacts to be used, the tasks to be proposed, the pieces of mathematics knowledge at stake, according to the curricular choices. In the latter, the teacher exploits, monitors and manages the children's observable processes (semiotic traces), to decide how to interact with children and what and how to fix in the individual and group memory.

Some cultural artefacts were considered by the group as paradigmatic at the beginning (others were added later taking into account teachers' suggestions). A giant

Slavonic abacus (especially for children aged 4 and 5, see fig. 2) was chosen at the beginning as paradigmatic by the whole group (see below).

Mathematics knowledge at stake was shared exploiting the research studies on counting and measuring processes. This choice was strongly biased by the initial choice to give value to numbers, counting and measuring; later a focus on space too was introduced.

Exemplary tasks were collaboratively designed by the whole group during the meetings. Others were creatively designed by teachers during school activities. Tasks were intended for individuals, for small groups or for large groups.

The design process is encapsulated by the left triangle of the fig. 1. The other parts of the scheme concern the functioning in the classroom. When children are given a task they start a rich and complex semiotic activity, producing traces (gestures, drawings, oral descriptions and so on). The teacher's job is first to collect all these traces (observing and listening to children), to analyze them and to organize a path for their development towards mathematical "texts" that can be put in relationship with the pieces of mathematics knowledge into play.

EXAMPLE: THE GIANT SLAVONIC ABACUS

The artefact

At the beginning of the second year of activity (2008/09) all the municipal preschools were given a giant Slavonic abacus. Teachers themselves had designed it with forty beads because this number meets the most common needs of school activity (e.g. counting children in the roll, counting the days per month in the calendar). The large size fosters large body gestures (even steps) to move the beads. Intentionally, schools received a dismantled abacus, as most teachers agreed that the very assembling could have been an important part of the exploration of the artefact.

Figure 2: The giant Slavonic abacus

The mathematics knowledge at stake

The "embodied" mathematical meanings were analyzed in the meetings:

Partition, to separate counted beads from beads to be counted;

One-to-one correspondence, between beads and numerals;

Cardinality, given by the last pronounced numeral;

Sequence of early numerals, to be practiced in counting.

Place value (early approach), as beads are divided in tens.

THE SYSTEM OF TASKS FOR THE SLAVONIC ABACUS

A system of suitable tasks was collaboratively produced in the meetings, drawing on the same methodology of projects for primary and secondary school (Bartolini Bussi & Mariotti, 2008; Bartolini Bussi, 2009; Bartolini Bussi et al., 2011). The aim is to foster children's productions of different voices to allow the teacher to organize a *polyphony of voices*, which, according to the theoretical framework of semiotic mediation, nurtures the individual construction of mathematical meanings during classroom discussions. In this way the same artefact is looked at from different perspective. Some tasks tested with 4 or 5 years old children follow. Teachers have arranged these tasks (or similar ones) according to their educational planning.

Task 1: The first impact

Tasks are different if the abacus is assembled (A) or if it is dismantled (B): *A) What is it? Have you seen it before? What's its name? B) What have I carried today? What do you see? Do you know objects with many beads?* Such tasks, to be used in either small or large group discussion, aim at evoking earlier experiences and involving children. At the end the name may be introduced (in Italian *pallottoliere* – abacus has the same root as *pallina* – bead). This tasks foster the emergence of a *narrator* voice. An individual drawing of the materials is in the fig. 3 (A).

Task 2: The structure of the artefact

How is it made? What do we need to build another one? How to give instruction to build another one? Such task, to be used in either small or large group discussion, aims at identifying the components and naming them in a correct way and at describing the spatial relationships between them. They foster the emergence of a *constructor* voice. After discussion, individual drawing tasks are given: *draw our abacus*. The previous verbal analysis of the structure of the artefact fosters the production of very detailed drawings, with, for instance, the right number of beads and the realistic representation of legs and other parts (fig. 3, B and C).

Task 3: The use of the artefact

The task is functional to the context where the artefact is used. For instance it may be used to keep the score in skittles or to count the present children during the call. It may be given in small or large groups. *How do you use it to keep the score? How do you use it during the call?* This task fosters the emergence of the *user* voice.

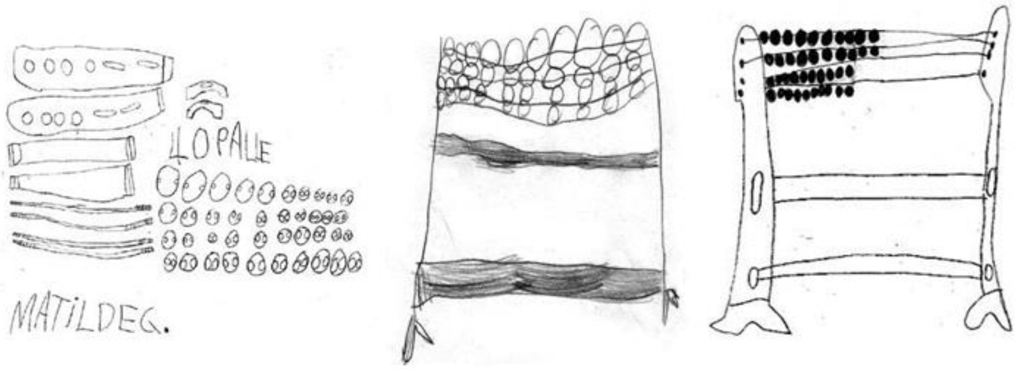

Figure 3: Drawings of materials (A) and of assembled abaci (B,C)

Task 4: The justification for use

In this case too the task is connected with the context. Children are asked to explain *Why does it work to keep the score?* and similar. This is a very difficult task, that fosters the emergence of the *theoretician* voice, to explain what mathematical meaning or process makes us sure that the functioning is effective. This task may be given indirectly, showing a puppet that makes mistakes and encouraging children to comment and to correct it, if they do not agree, explaining why.

Task 5: New problems

These last tasks were not designed in advance but emerged together with creative solutions in classroom activities. For instance, in a classroom, children proposed to use the Slavonic abacus to plan the preparation of tables for lunch. They suggested registering on the abacus the number of children for each table on a different line. When they realized that the tables to be set were 5, they told that there were not lines enough and decided to create a new line on the floor, lining up 10 small cubes and moving them accordingly. These self posed tasks foster the emergence of *problem poser and solver* voice.

THEORY AND PRACTICE

The system of tasks for the giant Slavonic abacus may be usefully referred to literature on mathematics education.

The emergence of the *narrator* voice in the task 1 is related to *devolution* (Brousseau, 1998) as it fosters the personal involvement of children in the tasks. The emergence of the *constructor* voice in task 2 is related to Rabardel *instrumentation* (1995) as it concerns the component of the artefact as an object. The emergence of the *user* voice in task 3 is related to Rabardel *instrumentalization* (1995), as it fosters the emergence of individual *utilization schemes.* The emergence of the *theoretician* voice in task 4 is related to mathematical meanings, hence consistent with the Vygotskian approach through semiotic mediation (Bartolini Bussi & Mariotti, 2008). The emergence of *problem poser and solver* voice in task 5 shows that, in spite of the teacher's cultural guidance, children creative ideas have space to be developed. However, it is better to say that, in this program, those tasks were not exercises of application of that literature, but rather *experiments developed, interpreted, analyzed and generalized, in collaboration between teacher educators, teachers and pedagogists in a dialogic way, exploiting the literature.*

In each task, *explanation* is in the background. There are, however, different kinds of explanation s(Levenson & Barkai, 2013), which are related to the different voices: a narrative explanation (task 1); a communicative explanation (task 2); an argumentative explanation (task 3) that develops into a proving explanation (task4).

CLOSING REMARKS

The protagonists in the recorded interviews show that an inquiry community (Jaworski, 2005) has been constructed with shared values, beliefs and competences.

> Adriana Querzè (policy maker): "We have played with the title "Counting children" as our children count with hands, with artefacts, with abaci, with situations, but, above all, count as they can exercise their rights of citizenship. Knowledge makes up citizenship as it allows reading and understanding reality. In the case of math and science, it allows constructing skills in analyzing and arguing, hence, in a broad sense, in communicating one's own point of view. We believe that it can be done also with young children, who are "citizens in the age of development", yet "citizens" to all purposes.[…] Parents can understand from this documentation what their children do in school, also for learning, strictly speaking. […] We believe that teacher education and development must be high quality […] and we want to dispel the myth that teaching young children is easier.[…] The "good" mathematics questions, the tasks, are not very different in preschools and university. Yet to look for answers to good questions with 3 years old children is much more difficult, hence requires a deep knowledge of mathematics and of methodology".

> Mariavittoria Vecchi (pedagogist): "It is necessary to join disciplinary, methodological and didactical competences together with relational competences. On the one hand teacher education and development cannot be reduced to practice and, on the other hand, a purely theoretical knowledge, with no elaboration on the job, is fruitless. Teachers' practices need to be deeply analyzed together with the theoretical models. It is necessary to give teachers high quality cultural tools, to enhance their planning autonomy, to make them keen on what they are doing and to construct a community of inquiry where the voices of mathematicians, teachers and pedagogists dialogue with each other".

> Marina (teacher): "I feel more competent in mathematics and more expert in the organization of school activities".

> Laura (teacher): "The child is the centre of this mathematical experience. The teacher has to come along with child in this experience".

> Rosa (teacher): "Meetings are very important for our professional development as we bring there our experience with children and discuss with each other".

Cinzia (teacher): "Children do mathematics in a spontaneous way. The most important thing is to raise that at the level of consciousness [..]. This program is very demanding, but I am enthusiastic. It's up to us to transpose knowledge to practice".

The above short excerpts from the multimedia report show that the program *Bambini che contano* is the offshoot of a synergetic effort of teacher educators, pedagogists, policy makers and teachers. Each of them has played a specific role and collaborated to construct an inquiry community where to develop their existing knowledge and beliefs system (Kreiner, 2011).

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NARRATIVE CONTEXT AND PARADIGMATIC TOOLS: A TALE FOR COUNTING

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We present a design study about the use of narration to frame a work on counting in a first grade class. We propose to use the ideas of 'narrative' and 'paradigmatic' ways of thinking (Bruner, 1986) in order to design, manage and analyse the development of children's experience. In this specific experience the role of logical tools is played by the criteria used to carry on the count process, conceived as coordination of two different semiotic activities.

INTRODUCTION

Stories and storytelling continue to arise interest of practitioners in different research fields, from psychologists and anthropologists that characterize this kind of activities as crucial in human cognitive and cultural development, to educators recognizing their value in order to pursue educational goals for specific disciplines: "*The value of the story to teaching is precisely its power to engage the students' emotions and also, connectedly, their imaginations in the material of the curriculum*" (Liljedhal and Zazkis, 2009, p. 3). Actually, stories can provide controlled contexts for experimenting parts of the complexity of life and for creating meaning in our lives and environments (Green, 2004). In this sense telling a story is a way of creating meaning also for mathematical structures: in a story context we can offer pupils an authentic experience of modelling by leading and supporting them in recognizing the mathematical structures as embedded in facts and embodied in human ways of managing them. Our research group has been working on this idea for several years, with the aim of recognizing some features of narratives – and of the way of using storytelling – which can enable mathematical meanings to emerge among the several ones intertwined in a narrative context.

In this paper we present a particular design research (Cobb et al., 2003) built upon the theoretical perspective of the paradigmatic and narrative ways of thinking pointed out by Bruner (1986). We used this frame in order to actually design a teaching-learning trajectory using storytelling with first grade children, with the aim to explore children's counting strategies and suitable didactical mediations to make their mathematical skills develop. In particular we aim at exploring opportunities (in terms of capacities of designing settings and interpreting the correlated dynamics) provided by a driving-conjecture about counting as *focusing of attention*.

THEORETICAL FRAMEWORK

As many scholars have been pointing out in the last years, a mathematics classroom practice exclusively built on standardized word problems and routine exercises does not educate students to a genuine mathematical modelling disposition (see for example Verschaffel et al., 2000). One of the hypotheses put forward claims that the verisimilitude and likelihood that something told could be true and real in same coherent “world” (even thought it could be not true and real at all) have been very often neglected in the stories used to do mathematics at school (see for example the narrative ruptures in the word problem texts detected by Zan (2011)). On the contrary storytelling can – by means of its verisimilitude – represent a special experience of teaching and learning also in a mathematics classroom (see for example Liliedhal and Zazkis (2009)), precisely for its power to evoke situations (even tough completely fantastic) in which children can project themselves according to their previous experience and own sensitiveness. Indeed telling a story, as well as proposing a “story problem”, entails a triggering process of the children’s natural interest for human actions and interactions (Donaldson, 1978).

According to Bruner what makes a narrated tale really interesting for people is its “violation of canonicity”, that is a situation that a listener feels as unusual with respect to the expectations determined by the contextual background (Bruner, 1990). The problem of restoration of canonicity is crucial in the development of the tale and we think that it can be used in an education perspective as an occasion to introduce or to build new cultural tools. In particular, Bruner (1986) points out two complementary ways of thinking: the narrative one arranged as an open space of possibilities that frames the temporal sequence of events and the choices taken time after time; and the paradigmatic one linked with the formal constraints (logical, algebraic, physical and so on), which rules the space of possibilities opened by the tale. On the mathematics education level we assume that within a story the violation of canonicity can be used to introduce or build paradigmatic mathematical tools in order to establish again the scripts of the narrative space. For this goal we also recognize the need to let children directly experience the actions described in the narration. In this view dramatization – as acted narrative – represents a natural step to be explored, along with the development of the story (for another example at kindergarten level see Mellone et al., 2012). Outside of the dramatization – and while it develops – the children, supported by adults, can explore how to overcome the breaches in canonicity building the paradigmatic tools that are useful to solve the tricky situations.

The design study we are going to outline here is placed in the trend focused in the use of stories in order to build new mathematical tools and meanings: in particular, it concerns a work around the counting process. Regarding the paradigmatic aspects of counting we refer first to Gelman & Gallistel (1978) who describe the counting process through five principles: the One-one Principle, the Stable-order Principle, the Cardinal Principle, the Abstraction Principle, the Order-irrelevance Principle. Such principles have the merit of carefully describing the counting process giving, at the same time, instruments to detect the difficulties that children meet during this activity. But, nowadays the new insights coming from neurosciences (see for example

Piazza (2010)) show that the counting process is, beside being an external phenomenon, an active mental process of either recognizing a naturally discretized aspect of reality, or imposing a more or less rhythmical discretization of continuous magnitudes. In both cases the core of such a process is a (perceptual and) internal gesture of *focusing of attention*, in the sense that what we count is always a “closed” internal action-of-individuation, variously correlated to others (from utterance to external doing). This conjecture becomes useful mainly when we want to study the counting of events developing along time. In this situation, one of the problems that needs to be firstly faced is that of deciding what are the rules to proceed into the count, that means understanding “what we count when we count”. This metacognitive activity can be supported by a semiotic activity (in a vygotskian sense) by expressing, and in this way stabilizing, the external events in correspondence of which we can add one to the count: in other words, using paradigmatic tools to lead the focusing of attention.

Our hypothesis is that this aspect of the semiotic activity linked with counting is different from – and needs to be coordinated with – the ones described by Gelman and Gallistel, which mainly deal with the use of labels (“numerons” for them). In other words, in counting events children need at first to recognize the discrete events corresponding to “add one to the count” and then to coordinate this recognition with the use of labels (as described in Gelman and Gallistel’s principles).

THE DESIGN STUDY

The research has been carried out in two primary school classes. The research team is composed by the class teachers and by some researchers. Two researchers play an active role in the ordinary educational work, spending about two hours a week in each class throughout the whole school year. The research is still going on, following the class through years. In this work we only account a part of the path developed in one of the involved classes, when it was at first grade. This specific part lasted two months at the beginning of the school year (that is seven working sessions of two hours each). Our research work can be defined as a design experiment (Cobb et al., 2003) for its ecologic and engineering features in exploring the opportunities that the conjecture of counting as focusing of attention can offer in planning and managing long term educational paths framed in a narration context. The classes also dealt with number issues in several other ways, working in the daily activities with the teachers. Nevertheless the teachers’ work and our design research were closely intertwined and reciprocally supported.

We believe the goals of mathematics education should include letting children experience rich activities that, in the case of counting, mean also involving them in purposeful activities of counting events. Having this in mind we built an ad hoc narrative frame, encompassing an element bringing about a breach in canonicity and letting children give themselves up the story development and directly act (as characters) in dramatized forms of the tale. We aim at leading children in building their own (geometrical and numerical) semiotic tools and use them in order to enable a restoration of canonicity and a consequent development of the story. In particular, in order to support the children into the coordination of the two semiotic activities linked with counting, we propose to lead them to move to a representation level, where *both* the events and the objects to be counted are represented with certain mobile marker objects (in the case of counting objects the markers should be different from the objects). In the analysis we will develop in the following paragraph, we underline the moment when we "went out" of the tale and the way we worked on the coordination of the two semiotic activities described in the previous paragraph. Our data consist in researchers' and teachers' observations, in excerpts from classroom discussions and children's representations.

Lastly, we would like to highlight the choice we made to put problems that were difficult enough in order to be hardly handled without the use of suitable tools (as marks or signs) and, at the same time, accessible and comprehensible enough in order to let the children recognize them as manageable. In our case, a difficulty was related to the presence of several rhythmical processes to be simultaneously controlled; another difficulty concerned the fact that the number of times (or the number of things each time) to be counted, though small enough, exceeded the limits related to the *subitizing* (for a review on this issues, see (Dehaene, 1997)).

THE DEVELOPMENT OF THE TALE

Temujin has been taken captive and is compelled to serve a sentence: he has been tied to a grindstone and forced to go round and round, in order to hull rice. Every morning Temujin is woken up by two beats of a gong, struck by a guard, and begins its job. He has to stop at the evening, when the guard beats the gong one time. Each time Temujin completes a turn around the grindstone, four rice grains are hulled and cumulatively collected in a holder. Several 'accountants' have the task of counting the number of turns accomplished by Temujin and the number of days of work, in order to report them to an inspector.

In this phase, as in the following ones, the action presented in a narrative form is also contextually dramatized by the children, so that they have to deal with the necessity of orienting their actions, leading to a certain development of the story. A child played the role of Temujin (fig.1); two children played the role of the grindstone, having autonomously to decide when the rice grains should be dropped in the holder (fig.3 and fig.4); a researcher played the role of the guard (actually signalling the end of a day every time Temujin completes two turns, but this was not made explicit with the children); some children had to count the number of turns completed by Temujin (fig.5); other blindfolded children had to count the number of the days spent (fig.2). The dramatization has been repeated many times and each child has played several roles in order to let children became familiar with the canonicity of the situation. Each dramatized sequence lasted about six "story days" (twelve child's turns) and, at the end of each sequence, a sharing was held in order to check if the counting worked.


fig. 1 fig. 2 fig. 3 fig. 4 fig. 5

The design of the story has been driven by the conjecture according to which counting is basically an oscillation between recognizing and imposing a rhythm, meant as complementary aspects of an intentional and aware addressing of attention and action. The considered situation is characterized by different rhythms: Temujin has to comply with the rhythm marked by the gong; the rice dispensers have to decide the right moment when they have to drop the rice grains and – at the same time – they have to warrant the right number (four) of dropped grains at each turn; the accountants have to establish how to record number of turns or days they counted. We intended to support children to try and orient themselves among these different rhythmical structures, by mean of discussions devoted to collectively treat each issue.

Initially we observed the counting strategies children adopted. In this phase we distinguished three kinds of attitudes:

1) Some children count properly. They recognize the occurrence of a repeated structure, distinguishing the base elements: two beats plus one beat of gong; four rice grains. These children are able to express the operative procedure they follow in order to rhythmically modulate the action

Elena: Maya shouldn't add four grains if Alyce has already dropped them [Maya and Alyce played the role of the grindstone], because Temujin didn't do two turns! […]

Ilario [a counter of the days]: I add one after two beats, he after one beat [turned to another counter of the days]. It's the same! We always count the days, only I count the mornings and he counts the evenings.

Here Ilario evidently recognizes there are alternative criteria for deciding when one has to proceed in the counting.

2) Other children tendentially singsong, in the sense that they count throughout the whole development of an action (i.e. during a whole day or a whole turn), even if they understand there are moments when the counting has to be stopped, for example during the night, when one sleeps. The conduct of these children is coherently commented by the first ones:

Stefano: Gianluca [a counter of the days who singsongs] is counting the minutes or the seconds, not the days.

Alyce: The seconds, not the days! […]

Camilla: Ornella [a counter of the turns who singsongs] isn't counting the turns! She's counting the steps!

3) In an intermediate position there are children that struggle to keep counting properly. These children understand the constraints to be complied with in order to proceed with counting, but they apparently make a great effort to avoid the singsong, or to avoid counting two times the same base elements. It is interesting to highlight Lucio's conduct. As a blindfolded counter of days, he counted "one" in

correspondence of morning beats and “two” in the correspondence of the evening one, while he tried to avoid this mistake:

Lucio: One… two… oh no, wait!

With the exclamation “wait!” aimed at himself, Lucio brings out the fact that counting is actually an oriented activity, which implies effort. An analogous situation concerns those children who play the role of parts of the grindstone. Maya’s case seems to be emblematic. At the beginning Maya had difficulties in playing alone the role of grain dispenser; then she completely relied on Alyce, who helped Maya and suggested her when dropping the grains. In the meantime Maya tried to autonomously move, oscillating among three facts: the peremptory indications by Alyce, the continuous wheeling around of Antonio (who played the role of Temujin) and her attempt to coordinate herself with them and to autonomously decide when dropping the rice grains.

In a second phase, we provided the counters with some objects, as bottle tops and seashells, to be used as markers during the counting. Nevertheless, the introduction of these supports did not change the scenario. In fact, the use of markers seems to be not particularly useful – at least for little quantities – for those that are already able to count properly, as highlighted by these comments

Eleonora: I count in my head!

Lars: I make it like this [counting on his fingers] and I make it well!

At the same time those who had difficulties in controlling the several rhythmical scans appear to be even more confused by the invitation to use the markers. At this point we considered opportune for the group to move outside from the temporally ordered narration and from its heuristic dramatization, in order to build the atemporal paradigmatic prostheses, the explicative tools aimed at establishing rules and controlling procedures in counting. This is a crucial point in the development of our intervention. Our tale comprises several activities, which can be effectively managed through some kind of (mathematical) formalisation. Now, the activity aimed at formalising does not belong to the narrative. It rather needs for another setting, other actions and other discourses, in order to build the tools used for explaining and controlling what happens at the narrative level. Therefore children are continuously invited to go back and forth between the dramatized narration and the paradigmatic level, in alternating and complementary phases. So we transferred into the gymnasium to work on structuring the space, in order to be able to determine when a child completes a turn. The children formed a circle and were invited to discuss about the movement of some classmates, who had the task of completing a turn around that same circle. The discussion was aimed at settling criteria for understanding when a child completes a turn. At the end, the group agreed on the advisability to mark out a line on the floor, in order to recognize the end of a complete turn. We made the same for what concerned the counting of the days spent. In this case the group should have dealt with a structuring of the temporal events: the discussion focussed on the distribution of beats of gong in a day and on the different ways to recognize the passing of the time, following some suggestions given by those who count properly.

After this activity, we came back to the narrative context, providing children with markers that were used by most of them. Therefore objects were used in order to record the number of turns and days, as well as signs were used to decide 'what' had to counted: the crossing of a line, the sound of two close beats, the sound of a separate beat, a special sequence of sounds (two close beats plus a separate one). It seems interesting to recognize the usefulness of support given from the practice of proper counting together: being supported by others, children that have difficulties can find time and ways of coordinating words and gestures with what they perceive, distinguishing the events to be counted and being able to record all of them, without getting confused in the complexity of those operations.

Subsequently, a second part of the story was told.

Suddenly an inspector arrives, with the task of verifying whereas Temujin has served the sentence. The inspector declares Temujin will be released when he will have worked for no less than 7 days (and no one more) and will have completed 16 turns. The accountants are so consulted in order to decide about the Temujin's destiny. But different groups of accountants express contrasting opinions about the possibility of setting Temujin free. Therefore the inspector gets initially angry. At the end he will accept other conditions, but he will ask for a written document, which certifies the actual Temujin's work.

The new situation is clearly designed so that the condition put by the inspector is incoherent with the constraints of Temujin's activity: for those who count days, he may be free; for those who count turns, he may not. This is the point where we introduce in the tale a studied violation of canonicity. A character with a normative role (the inspector) provides a criterion to decide if the prisoner can be set free or not. According to canonical expectations, the given condition should be granted or not, so that one should be able to decide about the release of Temujin. In our case, the contradictory judgements of the accountants, and the consequent impossibility of deciding, put the group at an impasse. This doubtful situation needs explanations and strategies to allow a plausible development of the tale. At this point the paradigmatic mode, the arithmetic in our case, is committed to explain the rifts in the narration (Bruner, 1990). Through the necessity of settling this matter, we aimed at making explicit (at least at a certain level) the multiplicative structure, which is always intertwined to the counting operation (characterized by the ratio of two turns per day, which was left hidden during the whole development of the activity). At the same time, we aimed at raising the necessity of producing a representation of what had happened in order to encompass the several rhythms that had characterized the considered processes.

Meanwhile, one looks for plausible solutions to the missing satisfaction of the conditions necessary to set the prisoner free. Anyway, no child observed that the inspector's conditions couldn't be fulfilled together: this suggests that the ratio, which correlates number of turns and number of days, is not well understood by pupils even if it is clearly “smelled”: here we move in a zone of proximal development (Vygotsky, 1934). In fact, when we ask to dramatize again the situation, children begin to identify the correlations among the several rhythms as a crux, even if they do not have a symbolic system which enables them to highlight the inner contradiction of the inspector’s request

Elena: You finish the day too soon! You should let him do more turns! [talking with the researcher that beats the gong]

Researcher: I only beat when the night comes.

Elena: We can change both the things. Day and night longer and Temujin faster.

Lucio: Or we could say to the inspector Temujin worked for seven days and did sixteen turns!

Elena: We can’t say he did sixteen turns! We can’t cheat! The inspector will realize it!

Lucio: How does he realize it?

Elena: If he counts the grains, he realizes it!

We are at a moment when the children try to identify the crucial ratio between turns and days, but they are not able to make it explicit. We decide to intervene, simplifying the situation and addressing the work toward a clarifying semiotic activity: the inspector negotiates his request, asking Temujin to work for 8 days doing 16 turns. In return for the prisoner’s freedom, the inspector asks for a written document, which certifies the exact number of days of work, completed turns and hulled grains. Children work in little groups on drafting several documents using mobile markers. This activity is carried out under adult’s supervision, gradually helping to organize the groups work. The adults support and partially address the activities, fostering the circulation of ideas among groups, asking for explanations, giving suggestions. At the first, we attempted to share ways of representing days, turns and grains:

Eleonora: We can make two circles, which are the two beats.

Stefano: But I counted a day each three beats.

Eleonora: But I don’t find it good, so.

Researcher: What do you mean about making a white circle, as a sun, in order to represent a day?

All children agree.

This short excerpt represents an attempt of coordinating the two semiotic activities of focussing of attention and labelling. In fact during the first phase of the path the adult mediation leaded the semiotic activity related to identifying events that corresponds to add one to the counting. Now one has to coordinate this semiotic activity with that described by Gelman and Gallistel (1978). By means of the use of shared markers (white circle for the days, macaroni for the turns, pumpkin seeds for the rice grains) the groups build several documents, actually several ways of giving form to the rhythmical structures and to their mutual correlations (see for example fig. 6, fig. 7, fig. 8, fig. 9). The produced representations are shared and read together (fig.9), in order to test their comprehensibility and coherence (fig.11). In some cases the particular rhythmical disposition of the markers on the sheet, as well as their spatial relationships, enable to highlight the more "difficult" relations which were not yet made explicit, as the number of hulled grains per day, or the number of turns per day. A final discussion (fig.9) aims at selecting the "best" document for the inspector (or at producing a shared new one).

> Camilla: According to me, one better understands our one [fig.8], because it really says what happened. [She points out the first column of the array] There is the day, then he has made a turn, so the grindstone has dropped four grains, then another turn and other grains.
>
> Elena: Oh yes, but our one [fig.10] is more orderly. The sun, then two turns, then the grains.

We would like to emphasize the different features of the strategies adopted in these two representations, according to their authors' comments that had the opportunity to emerge due to the use of mobile markers. On the one hand, Camilla is interested in producing a representation faithfully keeping the temporal development of the narrated and dramatized action. On the other hand, the introduction of markers as meaningful signs leaded Elena's group to reconsider and reorganise signs according to their meaning, in a semiotic activity that actually presents a fruitful ground for the further dealing, in a significant way, with the roots of the multiplicative structure and its dimensional issues.

fig. 6 | fig. 7 | fig. 8 | fig. 9 | fig. 10 | fig. 11

CONCLUSION

Our design study showed the relevance of encompassing the possibility of getting out from the tale, in order to build mathematical tools. This became a methodological assumption during our work and could be suggested as a central element in designing activities where math education is carried out through a storytelling. For example children that presented difficulties in counting the events, such as turns and days, didn't seem to take advantage of using mobile markers during the dramatization. Rather children benefited by discussions aimed at establishing semiotic rules, which helped them to focus their attention, such as marking out a line on the floor in order to recognize the end of a complete turn, or focusing on the sequence of beats in order to recognize the end of a day. This supports our assumption that the *focusing of attention* plays a central cognitive role in the counting practice and suggests introducing quantity markers only after activities devoted to clarify what one has to control in counting. After the second part of the tale, the use of mobile objects and the work with their arrangements actually supported the children in the difficult task of catching and precisely expressing the several simultaneous rhythmical processes in the story, enabling children to progressively appropriate of role and meaning of conventional representations. In fact the following step was dealing –within the frame of the same story – with the base-ten representation (Guidoni et al., 2011).

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USE OF DIGITAL TOOLS IN MATHEMATICAL LEARNING ACTIVITIES IN THE KINDERGARTEN: TEACHERS' APPROACHES

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In this paper we study how kindergarten teachers used digital tools in mathematical learning activities in the kindergarten context. With digital tools we mean mathematical software for computers and ICT applications for digital white boards, mathematical games and memory games. Within this context we study the various approaches the kindergarten teachers took when orchestrating mathematical learning activities. Through analyses of these mathematical learning activities, we find that the kindergarten teachers took three different approaches in their orchestration, the assistant approach, the mediator approach, and the teacher approach. These approaches all carry qualities and potentials for children's learning of mathematics, but in different ways.

INTRODUCTION

The Norwegian kindergarten is regarded by OECD (2006) as situated within a social pedagogy tradition, i.e. an educational institution where core enterprises are upbringing, care, play, and learning. A new curriculum for kindergarten was launched in 2006 in which mathematics was comprised for the first time as a separate subject area (Ministry of Education and Research, 2006b). Additionally, policy documents as regards implementation of information and communication technology (ICT) in the kindergarten were made (Ministry of Education and Research, 2006a). However, the combination of mathematics and the use of ICT has been more or less left for the kindergarten teachers to elaborate and implement in learning activities.

With this as a background we initiated a project called ICT Supported Learning of Mathematics in Kindergarten. In this project we aimed at collaborating with kindergarten teachers to gain insights into children's mathematical learning processes when interacting with digital tools. We argue in accordance with Plowman and Stephen (2003) and Sarama and Clements (2004) that research is needed to identify the mathematics learning potentials of digital tools. Hennessy (2011) explored interaction possibilities when using interactive whiteboards (IWB) in teaching. She came up with affordances such as the multimodal nature of interaction possible and the opportunities for direct manipulation of objects.

In this study we are particularly interested in analysing the ways the kindergarten teachers use these digital tools and how they interact with the children in their orchestration of mathematical learning activities. For the present study we have formulated the following research question: *What characterises kindergarten* *teachers' approaches when using digital tools in mathematical learning activities?* This question is interesting to pursue due to the fact that there is an insufficient amount of literature in this area of research.

THEORETICAL FRAMEWORK

In this study we adopt a sociocultural perspective on learning and development, a theoretical stance originating in the work of Vygotsky (1986) and further developed by socioculturalists such as Rogoff (1990) and Wertsch (1998). Two main concepts within this perspective are indubitably the notions *mediation* and *tool*.

Mediated action through the use of tools

In mediated action humans use several tools. Language, both oral, written, and body language, plays a fundamental role in mathematical learning activities (Goldin-Meadow, 2009; Roth, 2001). In order to communicate and interact, both adults and children use various kinds of language to collaborate, discuss, and make sense.

Relevant for the study reported here is the use of digital tools, i.e. in our case mathematical software engaged with by the children through their use of computers and IWBs. Following a sociocultural perspective, use of digital tools mediates mathematical ideas and concepts. Thus, learning becomes a process of mastering these tools and performing in appropriate ways when engaging with the applications, since "our mastery of such tools is a critical element of what we know" (Säljö, 2010, p. 62). By using digital tools in their orchestration of mathematical learning activities, the kindergarten teachers seek to mediate mathematical ideas and concepts for the children to make sense of. The kindergarten teachers' use of digital tools in mathematical learning activities thus establishes opportunities for the children to appropriate mathematical concepts, ideas, and actions.

Orchestration of mathematical activities

As already mentioned several times we view the kindergarten teachers' actions and communication through the metaphor of *orchestration*. Following Kennewell (2001), we see orchestration as managing "the visual cues, the prompts, the questions, the instructions, the demonstrations, the collaborations, the tools, the information sources available, and so forth… (p. 106). The notion of orchestration is thus used to describe what kindergarten teachers do when hosting mathematical learning activities using digital tools.

When considering teachers´ orchestration of digital tools for the benefit of learning mathematics, researchers have pinpointed many aspects. Goos, Galbraith, Renshaw and Geiger (2003) studied teachers´ orchestration of calculator use in mathematics teaching. They observed clear differences in the orchestration and argue for the importance of directing students to explore the tasks, use the digital tools to discuss the solutions of tasks, teachers´ ability to hold back information and stimulate collaboration and discussion among students.

Mathematics teachers' activities when using technological tools in the classroom is discussed by Monaghan (2004). He argues that to integrate technology in mathematics teaching is a complex undertaking, and teachers experience multifaceted processes when integrating the use of digital tools in their teaching. Moreover, Monaghan argues that the use of technology, computers and calculators, plays a vital role with respect to the social interactions between teacher(s) and student(s). In referring to the research literature, Monaghan argues that there has been a change of emphasis in describing teachers using technology and their role. The term 'facilitator' is used as a metaphor for teachers' roles within a radical constructivist paradigm. Contrastingly, the term 'mediator' is used as a metaphor for teachers' roles within a social constructivist paradigm. This term is used about teachers who play "an active part in the students' learning" (p. 329) through social interactions.

Similar arguments have also been advocated by Zbiek, Heid, Blume, and Dick (2007). These authors coin two roles mathematics teacher take when implementing technology in teaching, called Technical Assistant (The teacher assists the students with software and hardware difficulties) and Counselor (The teacher is familiar with the mathematical ideas and concept addressed in applications and supports the students upon request). The former role is described in a similar way as Monaghan's (2004) term facilitator, while the role labeled Counselor might share similarities with Monaghan's term mediator.

The study of Zbiek et al (2007) identifies teachers' roles when interacting with students at school. As will be seen later in the paper our identified approaches taken by kindergarten teachers differ from the labels coined by Zbiek et al. From the analysed data we needed to divide the Counselor role of these authors in two, the Mediator approach and the Teacher approach, since these approaches more distinctively address the studied kindergarten teachers' interaction with the digital tools and the children.

METHODS

In our project we collaborated with three kindergartens called Bee Pre-school centre, Swan Pre-school centre, and Frog Pre-school centre. As methods of data collections in the project we used video data of 12 sessions where kindergarten teachers orchestrated mathematical learning activities by the use of digital tools. Video recordings of the 12 sessions, each of approximately 30 minutes of duration, where complemented by observations and field notes. From initial analysis of these sessions and our general discussions regarding our experience within the project, a hypothesis emerged concerning the kindergarten teachers' approaches to the use of digital tools. Secondly, based on our reasoning, we returned to the video data and looked at some sessions more in depth. Guiding our analysis was the communication between adult and children and to what extent they engaged with mathematics. From this process, we were able to characterise three approaches the kindergarten teachers used. Thirdly, we returned to the data in order to identify episodes which illustrate these three approaches. The three excerpts chosen are meant to illustrate these three approaches. Conversations with the kindergarten teachers, meetings in kindergartens and a workshop at the University were also arranged. However, data from these events are not particularly used in the study presented here.

ANALYSIS AND RESULTS

The data we present comprises video data from three kindergarten teachers, one from each of the three kindergartens. In the excerpts below the kindergarten teachers used digital tools covering applications with mathematical elements such as counting, comparing sets, measurements, and shapes. We have identified three different approaches the kindergarten teachers took, the Assistant approach, the Mediator approach, and the Teacher approach. After describing these approaches, examples of adult-child interaction will be analysed to justify the identification of the three approaches as well as point to what these approaches encompass.

The Assistant approach (AST)

When the kindergarten teachers take an Assistant approach to their orchestration we characterise what they do as assisting the children with minor issues such as starting and running the software; they organise the activity so that one or two children interact with the digital tools at a time, they point at places where to touch the screen or keyboard, i.e. keystrokes, to answer software inherent questions and tasks. In one of the sessions we videotaped, the kindergarten teacher made small remarks regarding where to press the various buttons at the IWB to navigate and choose different games to play. The kindergarten teacher led the session, by pointing at what children should do to engage with the ICT application and by asking the children whether they wanted to play another game.

The Mediator approach (MED)

The second approach that we have identified is the approach called Mediator. When the kindergarten teachers took the Mediator approach they orchestrated the mathematical activity by being more active in interpreting the digital tools the children engaged with. The Mediator approach is further characterised by the kindergarten teachers reading text within the applications, and they supported the children in interpreting the screen. The teachers helped the children to become aware of crucial elements and aware of parts of the screen. Taking this approach it is the applications which determine the interaction.

The Teacher approach (TEA)

The third approach the kindergarten teachers took when orchestrating mathematical learning activities involving interaction with digital tools, we have called the Teacher approach. This approach is characterised by the kindergarten teachers' use of questions and comments with respect to the children's interaction with the applications. When taking the teacher approach, the kindergarten teacher actively chooses what applications for the children to engage with and monitors the pace of children's interaction with the digital tools. Taking this approach it is adults who determine the interaction.

The case of Ann

In this excerpt one child is engaged with an IWB in a mathematical learning activity with the use of a digital tool involving numerals and counting (implicitly). The application was designed for the kindergarten age group and it was called "Labbe Langøre" [1]. Three children were present. Sound is included in the digital tool, giving the child instructions of what to do at various places and stages within this short excerpt. In this application numerals are shown on a poster. Four trees are also shown, each of them having a numeral posted on them. The child is supposed to identify similar numerals.

Ann (AST): Your turn [Judy walks towards the IWB and up on a bench] Do you want me to move that bench?

Judy: Yes

Ann (AST): Then you can press the START button [Judy chooses an application called 'Memory']

Ann (TEA): Can you choose 'the Mathematical Cat'? [Judy chooses that application, and she presses the correct numeral. Two snakes emerge on the screen]

Ann (MED): Press the cat and see what happens [When pressing the cat, the software confirms whether the answer is correct or incorrect]

[Judy does that and the cat confirms that she is right. Then a new task shows up]

Ann (TEA): Can you count them [the animals] when they appear?

[The numeral 5 is shown on the poster, and Judy presses the numeral 5 on one of the trees. Five frogs emerge and Judy points at one after another while Ann counts loudly]

Ann (MED): One, two, three, four, five

The orchestration could be described by the kindergarten teacher having responsibility for hosting the session. From the excerpt it is evident that Ann is shifting between all three approaches. Ann serves the children by managing the computer and the software, assisting Judy in her interactions with the applications. The digital tool took care of much of the necessary information the child was in need of since oral sound was included in the applications. The Mediator approach was thus not so often needed in this case. Ann seems to adapt her interventions, comments and questions with respect to both the child in action and the actual application the child interacts with. Ann also took the Teacher approach in her goal-directed instruction of Judy to choose the mathematical application and Ann addressed the mathematics implicitly present in the application.

The case of Egil

In the following the kindergarten teacher ran an application on an IWB from the software named “Salaby” [2]. The application concerned the clock – the attention was on the relationship between written time slots such as 11:00, the hour hand and the minute hand. In this particular episode, all the minute hands were fixed at 12. The application presented twelve different clocks which represented twelve different time slots. The written time slots should be dragged and dropped to the corresponding clock symbol. Egil starts the application in front of 15 children and one other adult. Egil shifts between different approaches during the excerpt.

Egil (MED): Herman, try one o’clock. [After some hesitation, the child points at the clock symbol that represents one o’clock.]

Egil (TEA): How did you see that this was one o’clock?

First, Egil supported Herman in interpreting the application and Egil continued with a question which requested an argument. This may indicate that Egil wanted to promote a learning goal beyond just interacting with the application. Additionally, an important part of the mediation is to describe what is visible at the screen:

Egil (MED): The hour hand points at one and the big minute hand points straight up. It points straight up in every clock here: Twelve, twelve, …., twelve [He points at every clock on the screen, one after another until he has pointed at all twelve clocks]

Egil (MED): But, on which of this clocks does the hour hand point at eleven? Raise your hand if you believe that you know the answer.

Apparently, the mediation approach changes suddenly when Egil asked the question that give us association to a normal classroom conversation. However, the content of the question draws attention to the meaning of the application. In order to respond correctly, the children have to identify the correct clock symbol. We claim that this question is part of Egil’s mediation approach.

There is a blurred border between the Mediation approach and the Assistant approach. The Mediation approach is linked to the message the software brings and the Assistant approach is linked to the physical and organising part of the software. For example, Egil selected what child who was supposed to use the software, and he adjusted the children’s behaviour and made them focus at the screen. Use of IWB gave the children challenges related to the touch technology within the screen. A typical Assistant approach that we observed was when the adults helped the children with the drag-and-drop functionality.

The case of Unni

In the observed lessons at Bee Pre-school centre, the children were organised in groups of two children and one adult sharing a portable computer. Below two episodes are presented with one adult, Unni, and one child, Trine, aged 5 years. The software used is again “Salaby” [2]. In the application at stake here, cars are supposed to be elevated into different numbered garages (numbered 2, 4, 6, 8, 10 and 1, 3, 5, 7, 9 respectively on each side from bottom to top of the garage). In order to drive a car into correct garage, first an addition task such as 2 + 3 is supposed to be solved. Then the user needs to correspond that sum with the correct symbol, 5.

In the episode below, Trine started by choosing the car to the right where the task is 5 + 5. She was able to move the car into the elevator and then the episode below appears. Unni showed the numbers with her fingers (5 + 5) and Trine was able to respond orally with correct answer and to recognize how ten is written. This we consider as evidence that K3 after a brief mediator comment adopted a Teacher approach helping the child to understand why the car should be placed in garage number 10.

Unni (MED): Now you have to wait a little bit. Where is it supposed to be parked?

Trine: There (Trine points to the car with her finger, appears to be in doubt).

Unni (TEA): Yes, where will you park the car? How many fingers do I have on my hand (Unni shows her left hand to Trine)?

Trine: Five

Unni (TEA): Yes, and then you add 5. How many fingers will that be?

Trine: Ten

Unni (TEA): How does number ten look?

Trine: (Trine points to the correct numbered garage 10).

An indicator of Unni’s concern and judgment for the mathematics at stake is also visible in an episode occurring immediately before the one displayed above. Unni seemed to be reluctant to Trine’s choice to park the car on the right and suggests she chooses the car on the left. We believe this suggestion from Unni came based on a judgment that Trine rather should start with an easier task to solve. However, Trine decided to keep her choice and Unni then accepts this and adapted her support.

Unni (MED): Which car do you want to park?

Trine: (Trine points to the car on the right hand side where the task is 5+5)

Unni (TEA): Maybe you rather take the other one (Unni appears to think that the other car with the task 2 + 3 might be a more suitable task for Trine).

Trine: (Trine still points to 5+5)

Unni (TEA): You want that one. Which number will that be? 5+5, how much is that?

Trine: (Trine drives the car into the elevator)

These two episodes were brief and the time spent from Unni was to engage in mathematics with the children. We argue that episodes presented illustrate a typical pattern in Unni's approach to orchestration of the sessions. Unni spent most of her time taking a Teacher approach. Unni uses many questions indicating that she adapts what mathematical challenges the children should deal with. We also found a few utterances where Unni explains the technical actions with the tool taking a Mediator approach.

DISCUSSION

In this study we have seen that the kindergarten teachers orchestrate children's engagement with the digital tools differently. We agree with Monaghan (2004) and Zbiek et al. (2007), that it is a complex endeavor to implement the use of digital tools in mathematical learning activities. We have identified three different approaches the kindergarten teachers took when orchestrating the activities and interacting with the children. This is not to say that other approaches were not taken, but the three identified approaches were the most dominant ones. We have identified these approaches and called them an Assistant approach, a Mediator approach, and a Teacher approach. The approach called Mediator has several similarities with what Zbiek et al. label Counselor. Our description of the Assistant approach shares similarities with what Zbiek et al. call 'Technical Assistant' and what Monaghan labels 'facilitator'. However, what we call the Mediator approach differs slightly from what in the research literature has been labelled mediator role taken by teachers (Monaghan, 2004). We interpret that mediator role as encompassing a broader perspective than how we use the label mediator approach in our study. The way Monaghan (2004) refers the use of the term we interpret as comprising both what we call Mediator approach and Teacher approach. Our use of the Mediator approach is more in line with what Zbiek et al. (2007) call Counselor role. These discrepancies might be due to the fact that Zbiek et al. studied mathematics teachers at school level. We have, on the contrary, studied teachers at kindergarten level. By taking the Mediator approach, the kindergarten teachers become a bridge between the digital tool on the one side and the children on the other side. The kindergarten teachers support the children in order for them to make sense of the digital tools and for them to know what to do at various places when interacting with the applications. Thus, the kindergarten teachers mediate what the applications are about and come up with questions that make the children pay attention to relevant elements within the tools.

When taking the teacher approach, the kindergarten teachers purposefully decide for the children what digital tools to engage with. This approach initiates mathematical reasoning amongst the children, since the comments and the questions request predictions and justifications on behalf of the children. Thus, the kindergarten teachers utilise the digital tools to mediate mathematical ideas and address mathematical learning goals.

We argue that all three approaches to orchestrating mathematical learning activities in the kindergarten carry qualities and potentials when it comes to the children's opportunities for appropriating mathematical tools and actions. By taking these three different approaches the kindergarten teachers adapt to the situations and contribute with their support as the children and the situations request. With respect to the sequentiality of the interaction, the different approaches are needed to a differing degree. Typically, within the initial phase of interacting with the digital tools the Assistant approach is often needed in order to keep the activity going. However, when children are engaging actively with the mathematics within the digital tools, the Mediator approach is needed to support the children's sense making of the digital tools. Eventually, the Teacher approach carries affordances as regards the children's mathematical learning opportunities. The applications are used to serve mathematical learning goals formulated by the adults.

The main difference between the Mediator approach and the Teacher approach is that when using the Teacher approach, the adult focuses particularly on the mathematics implicitly present within the applications to serve pre-formulated mathematical learning goals. Thus, when taking the Teacher approach the kindergarten teachers are seen to orchestrate the children's mathematical learning process.

From our analyses it is also relevant to discuss the quality of the digital applications the kindergarten teachers used in terms of the mathematics learning opportunities created and whether the applications became utilised as tools, in a sociocultural parlance, for mathematics learning. The quality of the applications used, we argue, cannot exclusively be judged from the outset, since we believe their quality heavily depends on the competent utilisation of the applications by mathematically and didactically competent kindergarten teachers.

NOTES

1. The software "Labbe Langøre" is a DVD manufactured by http://www.kallekunskap.se/

2. The software "Salaby" is manufactured by Gyldendal, http://www.gyldendal.no

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KINDERGARTEN CHILDREN'S REASONING ABOUT BASIC GEOMETRIC SHAPES

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In our study we attempted to investigate the criteria used by preschool children in distinguishing basic geometric shapes, namely circles, rhombus, squares, triangles and rectangles. Besides the detection of the syncretic level as it was described by Clements et al. (1999), analysis of our observations revealed seeds of relational thinking in very early age, which via suitable collaborative tasks designed by teachers may advance preschoolers geometric thinking.

INTRODUCTION

Researchers' interest in preschool mathematics has considerably grown during the last ten years. Many research papers have been published and lately a book by Levenson, Tirosh and Tsamir (2011) entitled Preschool Geometry: Theory, Research and Practical Perspectives has appeared, while during the 6th Conference of European Research in Mathematics Education (2009) a new working group in Early Years Mathematics was established. Yet, geometry and spatial thinking are often ignored or minimized in early education (Sarama & Clements, 2009).

During preschool education children's concept images are based mainly on perceptual similarities of objects in their immediate surroundings and personal experiences. Gradually, through instruction, children pay particular attention to certain common attributes of shapes, and begin to develop informal definitions of shapes; they usually "define" a shape detailing his properties. Understanding of geometric concepts involves an invocation of a set of mental images along with a corresponding set of properties for that class of objects (Fischbein, 1993). Vinner and Dreyfus (1989) pointed out that students might not understand a concept in depth, if they do not bond the "concept image" and "concept definition" appropriately.

In the research, students' understandings of geometric figures are mainly analyzed taking as framework the van Hiele's levels of geometric thinking (van Hiele, 1986) and the hypothesized geometrical shape learning trajectory developed by Clements and Sarama (2009). Clements and Battista (1992) enriched van Hiele's theory, suggesting a sixth level of thinking more primitive to the "visual level". At this *pre-recognition* level children perceive geometric shapes by attending to only a part of the shape's characteristics: "[…](*Children) may distinguish between figures that are curvilinear and those that are rectilinear but not among figures in the same class; that is, they may differentiate between a square and a circle, but not between a square and a triangle.*" (Clements & Battista, 1992, p. 429). Gradually, experiencing multiple examples of a shape, children mentally construct increasingly elaborated shape-schemas, which engender the formation of mental prototypes (ibid., p. 208). If children's schemes remain chained to a prototype this prototype may be the source of incorrect responses, especially in the transition phase to the next description-level. (Clements & Sarama, 2007, p. 505). Furthermore, Clements and Battista suggested that geometric levels of thinking coexist. They used the term *'*syncretic level' in order to indicate *"…a combination of verbal and imagistic knowledge without an analysis of the specific components and properties of figures. At the syncretic level, children more easily use declarative knowledge to explain why a particular figure is not a member of a class*" (Clements & Battista, 1992, p. 211).

Our research tries to shed new light on the ways pre-school children construct geometric concepts through an investigational context that promotes communication and language exchanges. Basic assumption of this work was that a structured teaching setting with emphasis on language exchanges between participants, on the design of meaningful for children activities and on literacy practices, can promote the construction of school knowledge categories such as geometric shapes as well as children's reasoning development. In fact, the data of this paper derive from a more extensive linguistically oriented research grounded on the sociosemiotic perspective of Systemic Functional Linguistics[1] (SFL) and its model of context analysis, which interrelates the socially constructed knowledge (semiotic representation of reality) with the realization of language in context-situation (Giannisi & Kondyli, 2011; 2012). Data analysis focused on: a) *relational clauses,* which encode meanings of "being" and of "having" (e.g. *All these are circles / The square has 4 angles*) b) *Taxonomic relations*, which refer to *class/subclass* (e.g. *The triangles and some slanting triangles* [classifying triangles]) or *part/whole* relations (*These [rhombi] are two triangles stuck together, not stuck together... it is a rhombus)* c) *Reference* types realizing degrees of abstraction from closely contextualized - e.g. *This/these is a../are* (use of deictics or/and deixis – *phoric reference*)- to more decontextualized speech - *e.g. Rhombi don't go with triangles* (generic, *non phoric* reference) d) use of *causality* and *conditional* structures -e.g. *If we turn it* [square], *it becomes another shape*.(Unsworth, 2001)

The results here, concise the basic conclusions concerning the identification and classification of all basic shapes used in that research, as these are elaborated in order to address to a more specialized public. From this point of view, our venture gets the meaning of a cross examining revision.

DESCRIPTION AND METHODOLOGY OF RESEARCH

A total of 18 students from a sub-urban public kindergarten school participated in this study. The older students of the class (12 girls and 6 boys, aged 5:6 – 6:3) worked together with their teacher-researcher in small groups of 3 (2 girls and 1 boy) during the period between April and May of the school year 2008-2009. The corpus comprises of 20 video and tape-recorded meetings (about 13 hours in total).

Children's involvement in geometric concepts was set within a scenario frame of action: the production of books, which were to be sent to schools that couldn't afford to make such a purchase (action proposed to students on the occasion of the International Children's Book Day). Each group undertook the task of creating a book about geometric shapes as well as an insert card game of geometric shapes. The book to be produced was described from the beginning as a "learning book" (i.e. an informative leaflet/dictionary of "basic terms") for children. In this case, the knowledge of geometric shapes was considered to be a precondition for the receivers to play the card game found in the book.

The whole project was carried out during the successive meetings of each group, and was divided into activity modules, coherent and consistent with each other, according to the original context of the project (the production of the book), and, partly, to the level of their difficulty. The general objectives of each meeting's tasks were clearly explained at the beginning of the session. The teacher participated in the discussion as a coordinator, asking open, general questions, related to the aim of each activity. Special attention was given so that the teacher would not direct children's answers but encourage the discussion between the members of each group.

In brief, the tasks of all groups included the following activities:

A] Classification of flat geometric shapes (circle, ellipse, triangle, rhombus, square, rectangle) which would be pasted on the pages of the book after discussion and agreement between the members of the group (36 non identical paper geometric shapes of the same color and texture, with variations within each category regarding the shape qualities and size of the shape).

B] Addition of texts: Giving "titles" to every shape category (*categorization by naming*) as well as producing informative texts for the reader (*definitions, descriptions of their categorizations*), which were dictated to the teacher.

C] Creation of book's card game (a deck of 27 cards of 4 basic shapes: circle, triangle, square, parallelogram) -after the completion of phase B- and a sample performance of the game in a whole class meeting according to instructions read by the teacher. In order to design the game, each group was asked to collect objects found in the classroom and to classify them in different baskets by shape, according to the selected facet (designed game would preserve only the shape categories and the total number of cards of the initial game). Children were given samples of empty cards and asked to find different objects (at least 6) fitting to the size of the card in order to draw (outlining object) the 4 geometric shapes. Suitability and classification of the objects were then discussed within the group and each child undertook the drawing of 8 cards (picking 2 objects from each basket-shape category).

D] Presentation of all groups' books with the participation of the students of another kindergarten class and performance of a description game (a child picks a card from the deck and describes the shape depicted on it to the rest of the students, who try to identify it).

The whole project presupposed a) collaboration so that a common target (the production of the book) could be achieved b) linguistic interaction and negotiation of meaning in order to agree and compose book's content.

For the purposes of the research, each set of activities could function as a complementary source of information and as a means of checking coherence and progress of children's response.

RESULTS

Two sets of data are analyzed in this paper. The first set of data includes children's attempts to classify and define shapes according to the demands of the interrelated/ closely connected activities of Task A and B. A second set of data includes the subsequent descriptions of shapes during the 'description game' (task D). The results presented below concerning identification and classifications of shapes focus on the decisions taken by groups of children. The accompanying excerpts of dialogues, although necessarily restricted here, are indicative of the procedures across tasks' performance, as well as of the semantic choices and of classification criteria constructed within groups.

Circles-ellipses

Based on data from the collected material, both circles and ellipses were treated as closely related categories by all groups of our sample. During their final categorization, 3 out of 6 groups classified the items of the specific categories in a unified category under the name "circle". Moreover, in all groups there are statements which correlate circles and ellipses as semantic categories of a lower semantic organization level, within a general category called "round" or "cycline/ circular".

At the following excerpts, the group of children (G1, G2 etc) and each task of the project (A, B, C, D) are included in a parenthesis before the enumeration of each excerpt (e.g. G1A/). In dialogues the children are referred to with their initials and the teacher with the abbreviation (Tch).

(G1A/2) T.: *Madam, I gather all the round ones* (ellipses and circles).

(G3B/68) K: *And if we cut them like this (a virtual cut on an ellipsis), they become curves like circles.*

(G4A/84) Z: *This is the 'cycline' group* (shows only the circles).

However, in the discussion that followed about the items of the 'cycline' group, Z and A support the common inclusion of circles and ellipses in a group named "circles" (G4A/86), while N disagrees, supporting that ellipses are not circles. In that discussion Z. refers to a "circular group" including ellipses as well.

(G4A/86)

Tch: *Which ones are the circles?*

Z: *These and these* (circles and ellipses).

Tch: *Take a look you too N and A. I want you to participate. Are these all circles?*

Z: *This is the circular group, family (...) All these are circles, I think.*

Although circles and ellipses seem to be closely connected categories clearly distinguished by the others in task A, in the cases of groups of children who decided to make separate categories the distinction is accurate in 2 groups and almost accurate in one group, which added one ambiguous (circular) ellipse in cycles. The distinction between circles and ellipses is achieved mainly by reference to the circle prototype. Thus, children characterized ellipse as a "(a bit) stretched" circle, or "like a circle", "so and so", "(circles that) look like eggs".

(G1A/3)

F: *Well, normally* (indirect comparison to the circle prototype), *this one,* (showing an ellipse) *is a little more stretched, it's an egg.*

(G1A/4)

F: *It's a circle, but it's a bit stretched.*

(G1B/6)

Tch: *Yes, and you said that they are circles. Are they circles?*

F: *Like circles.*

K: *Like circles, but "so and so" she said.*

(G4A/86)

Z: *This is the circular group, family.*

Tch: *N, take a look. Do you agree that all these are circles? It's you that will decide.*

Z: *I think that they are all circles.*

N: *These are not circles* (he points at ellipses). *Those are* (he points at circles).

Z: *They are. But they look like an egg, that's why.*

It is noteworthy that the characterization "stretched"/ "slanting", as a critical distinctive quality between circles and ellipses, is also used with small variations in the correlations of square-rhombus, as well as for triangles in relation to the inclination of their sides ((G3A/75) K: *The triangles and some "slanting triangles"*). In all cases, children have a prototype in mind: circle, acute isosceles triangle. Besides, representations of such prototypes existed as a classroom shape chart. The wall-pictures function in an assistant way as a means of cross-examination, though children seem to use them also to enforce their arguments.

Triangles

In general the "triangle" appears to constitute a generally "obvious" and recognizable category of shapes, with which children are familiarized. Thus, there was no overlapping with the rest of the categories in the material, with the exception in some cases with the rhombus. The rhombus constitutes the shape most commonly associated with the triangle for reasons more thoroughly examined later.

In addition, some triangles are treated as more typical or representative, and some others appear to move along the edges, constituting a subject of discussion or disagreement within the groups, arousing the production of argumentation, which in

turn brings into light the more typical qualities of the category (G3A/74).
In some cases, typicality does not stem only from comparisons between members of the category, but also as a function of their relative spatial placement, as implied by the usage of the word "upside-down" or "crooked/ stretched triangle" in the following excerpts.

(G2A/191)

G: *Oh, I stuck it upside-down!*

(G3B/210)

M: *I mean... Let's say this is a pair of scissors* (she holds a pencil) *and I cut it a little like this* (she points to a diagonal on a square). *How will it become?*

Tch: *What will it become?*

K: *A crooked/bent/stretched triangle.*

Concerning the hypothesis of the prototypicality of the shapes, in relation to their spatial arrangement, more research is needed. Nevertheless, from children's actions on the material, as well as from their corresponding comments and discussions arises a clear ranking at the representativeness of the members of the category "triangles". The shapes that children initially recognize as triangles are the two acute isosceles triangles and the acute scalene triangle (e.g. G3A/74). It appears that the main criterion for characterizing a shape as a triangle is the existence of three acute angles.

(G3A/74)

[Takes in hand first the 2 acute isosceles triangles and then the right-angled isosceles. Then grabbing the right-angled scalene triangle, asks]

M: *Should we make this a triangle too?*

Five out of six groups agreed on a common classification of all triangles in task A (except of a case G2A/41, where a child refuses to see the right-angled triangle as a "whole" triangle) while the sixth group (G5) had difficulty to decide between several alternative classifications for right-angled triangles focusing on their distinctive attributes (the right-angled isosceles triangle, which was correlated with pyramid "like a pyramid", was considered either as a unique representative of a possible subcategory or as the basis for the common classification of all right-angled triangles –the right-angled scalene triangles were also suggested as a separate category; see indicatively excerpt G5A/118). However, during task B, group 5 also suggested their common classification with the rest of the triangles on the basis of the number of sides.

(G5A/118)

X: *Madam, what shall we do with these?* [2 right-angled triangles left apart] *If we make it like this, it looks like..* (Touches the tip corner of a right-angled scalene triangle)

Tch: *Like what?*

X: *Like a triangle I think, but (...) like a fin.*

Kind: *Are we going to leave them separately, is it another group?*

E: <u>*Let's make it to be two, a bit separate*</u> (she removes the two triangles from the main triangle group) *but be alike, <u>not be with another group</u>, so they are alike.*

Rhombus

Despite the suggestion of some children that rhombi could be included in the triangle category, finally a separate category is formed where a rhombus is described:

1) As "two triangles stuck together -G1A/13 (and vice versa, "if we cut the rhombus in half, we get two *pointy* triangles").

(G1A/13)

F: *Separate, separate, I say. Rhombi don't go with triangles.*

K: *They do, they do, because we have... it picks here, it picks here too... and here it's the same* (he points at the vertexes and at the acute angles in triangles and rhombuses).

F: *Yes, but you can't* (you shouldn't), *yes, but they are not the same…normally* (...) *These here, these are two triangles stuck together, not stuck together… it is a rhombus...This is a triangle, right?* (she raises a rhombus and shows the 2 tangent sides to the other children; then she turns it upside down and shows its other 2 sides)*….and another triangle under it... and so it becomes a rhombus. This is only one triangle* (she shows a triangle from K's groups).

(G3B/76)

M: *when we cut this here* (shows with a motion a diagonal cut on a rhombus on the page), *it will become a pointy triangle.*

2) By enunciation of its typical characteristics: number of angles and sides.

(G2A/163)

M: *I say it's not for a group* (meaning triangles and rhombuses do not belong in the same group), (…) *this one has 3 angles...this one has one, two, three, four...*

The "theorems in action" (Vergnaud, 1996) the children apply in creating the rhombuses' categories are:

1) The rhombus is a «crooked/stretched square».

(G1A/194)

T: *If we turn it like that...*(turns a rhombus sideways so that the two parallel sides be horizontal.

F: *Well, it's a crooked square.*

2) The rhombus is an «inclined» square.

(G2B/47)

M: *The square... it has 4 angles and if we turn it, it becomes another shape(…) It becomes the rhombus. Now, that is easy for me.*

Squares-rectangles

Five groups classify squares and rectangles in separate groups named correspondingly "Square/-es" and "Rectangle /-s" or "Parallelogram" (2 groups of children), while one group (G5) decides to unify rhombi and squares in a common category named "Rhombus" due to similarity with rhombi in attempts of square rotation (while handling shapes they used both terms "square" and "rhombus"). Regarding squares and rectangles, shape categories often correlated focusing on their contrasting characteristics, the base of children's classification seems to be the square, as in 3 groups of children the most ambiguous rectangle is added to the group of squares (possibly as a marginal representative at the edges of the category). We should also mention that children use the words 'parallelogram' and 'rectangle' as identical. In Clements et al.'s (1999) study children also tended to accept "long" parallelograms as rectangles.

The inequality of the rectangle's sides is recognised as its basic distinctive characteristic implied or explicitly notified as in excerpt G1B/172.

(G1B/172)

Tch: *Yes. How do we distinguish these by those?* (i.e. squares and rectangles)

F: *I know, because this is a bit longer and the two lines are bigger and the other two smaller* (points at a narrow parallelogram on the page consecutively the horizontal and vertical sides) *and because this one* (square) *has the lines that are equal.*

DISCUSSION - CONCLUSIONS

Though the findings of this research are suggestive and could not be generalized, a general remark is that the children were able reliably to identify shapes. All children used formal names for shape categories (circles, triangles, rhombuses, squares, rectangles /parallelograms) except for the ellipse (described as a mirror or an egg). Children distinguished shapes into two large categories: those without angles (circles and ellipses) and those with angles (triangles, squares and rectangles). They used different criterion (visual and property) for examples and non-examples of shapes negotiating within their group the meaning of shapes' category. Their positions do not concern stable, irreversible or of the same kind criteria, but issues introduced for further discussion in correlation with the tasks they had been assigned.

The criterion of *angle* and *line* (side), which was more or less directly employed and mentioned by all groups of children in the previous tasks, prevails at the game performance of task D, where shape should be described with no reference to its name. Adjusting to the demands of the task (more typical, school-type descriptions

/competitive game), children chose more typical-decontextualized terms like "angles" and "lines", which they considered more appropriate in the certain context. On the contrary, plenty of "(looks) like..." type metaphors were used in shape descriptions of the more "open" texts of task A and B (e.g "circle looks like o (letter)", ellipsis is "like an egg", "like zero", "like a mirror" etc). This kind of metaphors, which are also found in task A, but prevail at task B, allow for the essential qualities of the shapes, as well as the general communicational-educational goal of the project. We could say that such metaphors constitute an alternative strategy for shape identification, a kind of descriptive definition related to definitions through examples.

The procedure of shape classification children were involved, highlighted two main elements about their knowledge on geometric shapes:

1. Despite the existence of "prototypes" (in the case of triangles), their prototypical visual image was challenged by the presence of non- prototypical triangles and the communicational context. This means that if the context of the activity affects to such an extent children's way of thought, more emphasis must be placed on the kind of activities suggested at this level of education.

2. There is evidence from our study that most children were able to formulate correlations between circle-ellipse, rhombus-square and square-rectangles, a fact, which allows us to advance the hypothesis about the existence of a transitional stage between descriptive and relational level of thinking. If confirmed with further research, this finding could have significant impact for the instruction of geometry in kindergarten.

Apart from confirming the existence of the syncretic level, our study offers plenty of indices favouring our fundamental hypothesis that when preschool children collaborate in the context of a meaningful activity (van Oers, 1998) they can construct meanings that characterize levels of geometric thinking attributed by research to elder children.

NOTES

[1] Giannisi, P. (2010). *Definitions and classifications by kindergarten children in the context of school literacy practices: the case of geometrical shapes.* MA thesis (supervised by Kondyli, M.), University of Patras [in Greek].

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BELIEFS OF KINDERGARTEN AND PRIMARY SCHOOL TEACHERS TOWARDS MATHEMATICS TEACHING AND LEARNING

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Kindergarten and primary school teachers' professional actions and beliefs are assumed as crucial factors for a successful transition. Individual interviews with kindergarten and primary school teachers, video observations in both institutions and focus groups were analysed to describe these beliefs. A spectrum of different beliefs was outlined. Kindergarten and primary school teachers have different views on mathematics teaching and learning due to different curricula and different trainings. The research results are used to support the process of professionalization.

INTRODUCTION

In Germany, there exist differences between kindergarten and primary school concerning curricula (1), teachers' qualification (2) and the organisation of (mathematical) learning (3).

(1) In the *Orientierungsplan* for kindergarten (Ministerium für Kultus, Jugend und Sport Baden-Württemberg, 2011) no standards but goals are outlined. Everyday situations and their learning potential are described. In the *Bildungsplan* – the curriculum for primary school (Ministerium für Kultus, Jugend und Sport Baden-Württemberg, 2004) – specific content and process ideas are formulated as well as standards and competencies children should gain in different subjects.

(2) Kindergarten teachers are mainly trained in vocational schools. Since 2004 students have the possibility to gain a bachelor degree in early childhood education. Primary school teachers study at university and have an academic degree.

(3) In Germany, children attend kindergarten from the age of 3 to the age of 6 in heterogeneous age groups. Daily routine in kindergarten are for example free play or open assignments. Primary school usually starts at the age of 6, depending on the federal state of Germany. Classes are mostly organised in homogenous groups. The morning is structured around lessons in different subjects.

In this study, the alignment of kindergarten and elementary school teachers' beliefs of mathematics instruction and teaching practices is seen as a basic condition for the interconnection between early childhood and primary school education. However, it is assumed that due to the different conditions differences and inconsistencies exist. Predominant beliefs in both institutions will be described.

THEORETICAL FRAMEWORK

Two aspects are important for the theoretical framework: Beliefs and aspects concerning transition from kindergarten to primary school.

There is no universal definition of ‚beliefs‘ (Pajares, 1992). Furthermore, there are different almost synonymous terms like ‚concepts‘, ‚subjective/implicit theories‘, ‚attitudes‘ or ‚values‘. The following findings on beliefs shall give an overview on what beliefs are (Pajares, 1992, p. 324f.):

- „The belief system has an adaptive function in helping individuals define and understand the world and themselves.“
- „By their very nature and origin, some beliefs are more incontrovertible than others.“
- „The earlier a belief is incorporated into the belief structure, the more difficult it is to alter. Newly acquired beliefs are more vulnerable to change.“
- „Individuals’ beliefs strongly affect their behaviour.“

Green (1971) characterised beliefs as followed: Beliefs are quasi-logical, they have a psychological centrality and are organised in clusters.

In spite of the various meanings of beliefs, there is a broad acceptance that teachers’ instructional practices are influenced by their belief systems (Calderhead, 1996). These practices impact on students’ learning and beliefs concerning mathematics (Hiebert & Grouws, 2007). Therefore, much effort was made to investigate mathematical beliefs of teachers in the recent two decades all over the world (Philipp, 2007).

In Germany, there are several studies concerning mathematical beliefs of high school teachers (COACTIV, Kunter et al., 2011), a few concerning the beliefs of kindergarten teachers (Thiel, 2010; Benz, 2012) as well as primary school teachers (Schuler, 2008; Bräunling et al., 2011).

Transition from kindergarten to primary school is marked by continuity and discontinuity. Therefore, two ideal-typical positions exist (Roßbach, 2006): (1) Differences should be reduced to increase continuity and to allow a gradual and smooth transition. (2) On the other hand, discontinuities can be seen as challenges beneficial to (personal) development.

According to Hacker (2004), there are several reasons for a gap in transition from kindergarten to primary school:

- different understanding of education of kindergarten and primary school teachers
- different training of the two professions

- different establishments who are responsible for the institutions (state, private establishments etc.)

To allow every individual a continuous educational biography (Heinze & Grüßing, 2009), cooperation between both institutions – kindergarten and primary school – is indispensable (Hacker, 2004). Goals and contents of both institutions should be made transparent (Griebel & Niesel, 2009).

RESEARCH OBJECTIVES

The goal of this interdisciplinary project is to design a well-grounded structural model of epistemological beliefs on teaching mathematics.

The data will be gathered with a questionnaire survey in two federal states of Germany – Bremen and Baden-Württemberg.

In a qualitative study data was collected for designing the questionnaire in a bottom-up process. The data was also preliminary used for analysing and interpreting the beliefs. In this paper, we will present only the qualitative study of Baden-Württemberg.

Various research questions, outlined below, guided the data collection and the data analysis. The following ones are important for the qualitative study.

1. What kind of hands-on materials are used in kindergarten and primary school?
2. How do kindergarten and primary school teachers support the transition from kindergarten to primary school in regard to mathematics?
3. What do kindergarten and primary school teachers consider relevant for the mathematical development of children?
4. What kind of transition practices can be found?

METHODOLOGY

The data was collected by using guided interviews subsequent to video-observations (Dinkelaker & Herrle, 2009) and with focus groups (Morgan, 1997).

The guided interviews were conducted with five kindergarten and five primary school teachers. All interviewees were female. They differed from age and teaching experience.

The interview manual contained the following topics: Planning and arrangements for teaching, interaction between teacher and child, materials used for learning mathematics, goals of teaching mathematics, cooperation between kindergarten and primary school. After each interview notes were taken in a postscript (Witzel, 1982): Comments on situational and non-verbal aspects, conversational contents and the explanation of priorities of the interviewees.

The professionals were asked to show a typical mathematical kindergarten situation or a typical mathematics lesson for first-graders. All situations and lessons were videotaped.

In two focus groups (Morgan, 1997), 35 kindergarten and primary school teachers exchanged their views on early mathematics education, the transition from kindergarten to primary school and the cooperation between both institutions. We invited highly experienced experts of whom some work in teacher training. We asked them to bring along typical hands-on materials, textbooks, etc.

Focus groups were conducted because they are an economic alternative to conventional interviews. More individual opinions can be gathered in one run. In addition, the atmosphere in focus groups is more relaxed than in interviews. Not everyone is needed all the time thus there are possibilities to withdraw. As well, thoughts and utterances can be stimulated by listening to other participants. Therefore, there are often more ideas than in a guided interview with only one person (Bortz & Döring, 2006).

The interviews and the audio data from the focus groups were transcribed verbatim. The data was analysed by coding according to the qualitative content analysis (Mayring, 2010). One part of this analysis, the paraphrase, was used to create items for the questionnaire.

ANALYSIS AND PRELIMINARY INTERPRETATION

The following analysis refers to the outlined research questions (1–4); ordered chronologically.

(1) Almost all shown materials and activities showed and used in both institutions refer to the content area ‚number and operations‘ (NCTM, 2000). The kindergarten teachers participating in the qualitative study used everyday objects (cf. figure 2-3), special mathematical programmes and trainings (cf. figure 1) or both.

According to kindergarten teachers everyday objects comprise objects from nature, e.g. nutshells, stones or chestnuts, as well as objects that surround children in their everyday life, e.g. Schleich figurines (authentic figures made from rubber), games, dice, etc. Kindergarten teachers use these materials because mathematics can be found everywhere and is a part of everyday life. Therefore – according to their point of view – no specific materials are needed. This way, children can understand the omnipresence of mathematics. The use of every day materials refers to the belief that learning mathematics in kindergarten can and should take place in day-to-day life.

Special mathematical programmes and trainings are inserted to prepare all children of one age group or disadvantaged children for school (*Zahlenland* Preiß, 2007; Friedrich & de Galgóczy, 2008; *Mengen, zählen, Zahlen* Krajewski et al., 2007). The key contents in the programme ‚Zahlenland‘ (English equivalent: land of numbers) (Preiß, 2007 or Friedrich & Galgóczy, 2008; cf. Figure 1) are knowledge of numerals and counting abilities. Each numeral is presented in a separate story using special characters. Matching songs and various activities complete the examination of one numeral. Some kindergarten teachers mentioned that they work with such programmes to prepare all children ideally for school and to make the transition in mathematics easier. This reasoning refers to the belief that the preparation for school is more successful while using programmes than teaching and learning mathematics in everyday situations.

Kindergarten teachers who do not work with special programmes do so for various reasons. Either, they did not attend further education programmes for mathematical training or they do not see an importance in teaching some mathematical aspects before school as transition was no problem until now. Furthermore, some kindergarten teachers consider teaching and learning mathematics explicitly as inappropriate for this age group.

On the other hand, primary school teachers prefer didactical hands-on materials for the arithmetic instruction (cf. figure 4-6). These hands-on materials (cf. figure 3) shall help children to subitize and to develop the part-part-whole-scheme. The use of these materials reflects the primary school teachers' emphasis on the number concept.

Figure 1-3: Hands-on materials used in kindergarten

Figure 4-6: Hands-on materials for arithmetic instruction in primary school

(2) In the interviews and in the group discussions the primary school teachers stress the heterogeneity of children at the beginning of their school career which influences and sometimes hinders their work. In the teachers' view the reason for the children's heterogeneity is based on the fact that the children attend different kindergartens. Because of this, they gained different mathematical experiences. Teachers try to meet the heterogeneity with group and individual work as well as action-oriented teaching. The goal is to facilitate the beginning of school for all children. Also, during the visits of the cooperation teachers in the last year of kindergarten, mutual areas and actions for a successful transition in mathematics are named in cooperation with the kindergarten teachers. The children should complete these areas and activities in kindergarten in order to obtain optimum conditions to start school. This reasoning of the primary school teachers refers to the belief that mathematical education in kindergarten should be mostly compensatory in the sense to achieve more homogeneity.

Kindergarten teachers emphasis the preparation of all children for school too (cf. (1)), but heterogeneity is often described as normal and unproblematic. Mathematical education in kindergarten is seen less as compensatory but as individualized.

(3) In connection with the question which relevant skills children should acquire before entering school, implicit demands are made towards the other institution. This is especially true for primary school teachers. In some cases the interviewed primary school teachers expressed different abilities and skills the children should master: On the one hand, these are domain specific skills like counting, subitizing up to five and on the other hand daily living skills such as to tie shoes, to hold a pencil correctly, etc. Writing numerals in kindergarten is seen critically, because the primary school teachers fear that children in kindergarten could learn something wrong. This reasoning refers to the belief that (mathematical) education in kindergarten should merely provide the basis for learning mathematics in school and should not anticipate school contents.

Some kindergarten teachers expressed several times that they consider counting as very relevant and important for the development of the children and for a successful start of their school career. Other kindergarten teachers, however, see their role primarily in mediating and arising joy and curiosity for mathematics. The primary school teachers interviewed think primarily of special skills that are important for a successful start in school. The kindergarten teachers, however, do consider some skills as relevant too but put their main emphasis on the teaching of certain attitudes. These differences are probably related to the aims of the institution. Schools have clearly defined standards to achieve but kindergartens do not. This can already be found in the underlying curricula.

(4) In Baden-Württemberg, the cooperation between kindergarten and primary school is recorded in the curricula. The way and extend of the cooperation is not described. Nevertheless, different forms of cooperation were established and used in

the last few years. Often, one teacher per school is responsible for the cooperation between kindergarten and school. This teacher is called *cooperation-teacher*. The cooperation-teacher is not always the teacher of the first-graders. He/She cooperates with those kindergartens from which children will enter their school. The cooperation-teachers visit the children in their last year of kindergarten on a regular basis, normally every second month. In this time they play with them, look for things kindergarten teachers could do with the children till entering school so that they are well prepared, etc. Part of the cooperation often is that the kindergarten children take part in a lesson in school once.

Both kindergarten and primary school teachers mentioned that there is not enough time to have an intensive cooperation. They agree that there should be much more capacities for having an in- and extensive cooperation. Nevertheless, both think that cooperation between the institutions is a key component of the transition process. The children should already get to know their future teacher and future classmates before their first day of school. They criticised the fact, that the cooperation-teacher and not the teacher of the new first grade is responsible for the cooperation. ‚Round tables‘, at which kindergarten teachers and primary school teachers meet on a regular basis, reciprocal visits, the broadening of cooperation over a longer period of kindergarten time and the integration of the entire team into the alliance in order to abolish the restriction of responsibility to only one cooperative force, were wishes expressed by the respondents.

SUMMARY AND DISCUSSION

The transcripts of individual interviews and focus groups with kindergarten and primary school teachers were preliminary analysed to describe the beliefs of those professionals.

In kindergarten, everyday objects and programmes are used primarily whereas in primary school didactical hands-on materials are used. The use of everyday objects focuses on a broad view of mathematics – mathematics is not only reduced to numbers, counting and calculations – which is at the same time non-specific. Some interviewees described how they work with these materials. Those reports suspect a limited view on mathematics as the materials are primarily used for counting. To what extend the views of the teachers are related to their choice of material needs to be proven.

The interviewed primary school teachers name different mathematical skills as helpful to be taught in kindergarten. These are mainly counting, knowledge of numbers, subizing and the assignment of number images and numerals. The interviewed kindergarten teachers name domain-specific skills less often. They strongly emphasize that they see their task in imparting fun with numbers and to arouse curiosity about mathematics, which should be continued by the teachers in school. Here, the influences of the tradition of the institutions are visible. In primary schools standards have to be achieved. In kindergarten, a socio-pedagogical view is dominant which influences the educators' work as well as their point of view (OECD, 2006).

Between the previously surveyed teachers a consensus on the relevance of cooperation exists. Often, the children get to know the school in their last year of kindergarten. The cooperation-teacher visits the children in kindergarten and they are allowed to attend a lesson in school. Two views on the cooperation can be distinguished: (1) A child-centered/pedagogical view: Kindergarten and primary school cooperate to allow the children to get to know their possible future teacher, the new environment at school, etc. and to facilitate their first day at school. Primary school teachers and kindergarten teachers with this point of view maintain a strong cooperation. Projects are initiated together, they are organised over a longer period of time and do not only focus on future 1st-graders. (2) Cooperation and kindergarten is only seen as a preparation for school. The aim of kindergarten is to prepare the children appropriately for school to allow primary school teachers a direct start with ‚proper lessons'. In this case, cooperation is limited. The cooperation-teacher visits the children in their last year of kindergarten approximately three times to diagnose the children and to give hints on individual training programmes.

The curricula of both institutions have an influence on the perspectives of the professionals as well as on their practices. Also, it can be interpreted that the different traditions of kindergarten and primary school are reflected in their beliefs. One example for the inconsistencies of beliefs is that some interviewees work with programmes in kindergarten and emphasize the omnipresence of mathematics at the same time.

The influences of beliefs as well as the quantitative distribution are still not clear. The questionnaire based study in Bremen and Baden-Württemberg will provide information about this issue.

Findings from the project will serve the process of professionalization of both professions in training as well as in in-service trainings.

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IPADS AND MATHEMATICAL PLAY: A NEW KIND OF SANDPIT FOR YOUNG CHILDREN?

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In Swedish preschools, it is the responsibility of teachers to provide learning opportunities through children's play. Although anecdotal evidence suggests that ICT is becoming more common in preschools and that young children actively engage with ICT in play situations at home, there is little research on how ICT can contribute to playful mathematics activities. Data from a study of young children using iPads in home situations indicates that Bishop's six categories of mathematical activities can be seen in free, downloadable apps. It was also clear that features of some apps made the children respond to the apps in more playful ways.

ICT, MATHEMATICS AND YOUNG CHILDREN

Although there are not requirements for what children should learn in preschools, in the Swedish curriculum it is clear that preschools and teachers have a responsibility to provide learning opportunities based on play, including mathematical ones (Skolverket, 2010). Therefore, if information communication technology (ICT) is to be used to support learning then it must be through play and this means finding out from children what playful learning with ICT could be. To do this, we take the advice that Prensky (2006) gave to parents about not buying "educational" computer games for their children–"a far better strategy in my view, is to take the games your kids already play, and look inside them for what is educational" (p. 184). Consequently in this research, we identify the mathematics in the apps that one child played and begin an investigation into why they were engaging.

Almost all previous research, which has looked at how ICT supported mathematical learning, has been from the perspective of investigating what young children learnt from engaging with specifically-designed educational software in preschool settings (see for example Highfield & Mulligan, 2007). After reviewing the literature on children using ICT, Sarama and Clements (2009) suggested that the affordances of computers made them more advantageous for developing mathematical thinking than physical objects because, "compared with their physical counterparts, computer representations may be more manageable, flexible, extensible, and 'clean' (i.e., free of potentially distracting features)" (p. 147).

An emphasis on school mathematics in these specially designed programs is problematic in situations where learning is supposed to occur through play. This is because play has certain features, as summarised by Docket and Perry (2010):

> The process of play is characterised by a non-literal 'what if' approach to thinking, where multiple end points or outcomes are possible. In other words, play generates situations where there is no one 'right' answer. … Essential characteristics of play then, include the exercise of choice, non-literal approaches, multiple possible outcomes and

> acknowledgement of the competence of players. These characteristics apply to the processes of play, regardless of the content. (Dockett & Perry, 2010, p. 175)

The use of play as the basis for learning activities affects the roles available to the teacher and children. From examining an activity where preschool children explored glass jars, we found that although the teacher could offer suggestions about activities, the children did not have to adopt them and could suggest alternatives (Lange, Meaney, Riesbeck, & Wernberg, 2012).

However, from related research (Johansson, Lange, Meaney, Riesbeck, & Wernberg, 2012), it was clear that mathematical variety was not lacking in these play situations. We found that all of Bishop's (1988) six mathematical activities, which underlie the Swedish preschool mathematics curriculum (Utbildningsdepartementet, 2010), were present in play situations that children engaged with. Bishop's activities have a different background and focus than the topics often connected with school mathematics, although there is some overlap. The activities were:

> Counting. The use of a systematic way to compare and order discrete phenomena. It may involve tallying, or using objects or string to record, or special number words or names.
>
> Locating. Exploring one's spatial environment and conceptualising and symbolising that environment, with models, diagrams, drawings, words or other means.
>
> Measuring. Quantifying qualities for the purposes of comparison and ordering, using objects or tokens as measuring devices with associated units or 'measure-words'.
>
> Designing. Creating a shape or design for an object or for any part of one's spatial environment. It may involve making the object, as a 'mental template', or symbolising it in some conventionalised way.
>
> Playing. Devising, and engaging in, games and pastimes, with more or less formalised rules that all players must abide by.
>
> Explaining. Finding ways to account for the existence of phenomena, be they religious, animistic or scientific. (Bishop, 1988)

Of the 12 situations illustrating the 6 activities that we considered in Johansson et al. (2012), 3 came from a sand pit. It was clear from what the children were doing–filling buckets, making pretend gardens, driving toy cars–that they were involved in play because, for example, the children exercised considerable choice, adopted non-literal approaches, produced multiple outcomes and were acknowledged as competent in their actions. Our thinking about how a sandpit facilitated children's play and supported their engagement with mathematical ideas made us want to explore whether ICT, iPads in particular, could provide similar mathematical learning opportunities. Anecdotal information suggests that many young children play with ICT at home and we considered that information from home situations could be informative for preschools. Our research questions for the study are:

1) What mathematics can be seen in apps for the iPads that young children play?

2) What features of these apps support children's play?

METHODOLOGY

Data was collected from children, aged between four and six, engaging with apps on iPads. In this paper, we use video data of a six-year-old boy, Miguel, who currently lives in England. Although he had played computer games at home, he had not used an iPad before and so was not familiar with the apps. Over four days, he played games for 1-1½ hours each day. The data discussed in this paper is from the final session. By this time, Miguel was familiar with the apps and made choices of what to play based on his interests. Being videoed was also no longer a novelty. Although he had already spent time in an English primary school, he would not have started school until the month when the data was collected if he had been living in Sweden.

Miguel was recruited as a co-researcher to evaluate the suitability of the apps for a four-year old child. We explained that he was better at making these judgements as we were too old. One of the authors, Tamsin who was his aunt, videoed Miguel's activities with a small video camera. At times, there was discussion between them, sometimes initiated by Miguel and at other times by Tamsin. The discussions were not planned but occurred as a natural part of Miguel's engagement with the apps.

Before meeting with Miguel, 33 different apps, which were recommended by one of three sources, were loaded onto the iPad. The apps were all free, although some operated only for a short period before payment was needed to move to a higher level. Fifteen apps were recommendations from *Pappas appar* (Pappa Daniel, 2011), a blog where a Swedish father had placed links to free apps for iPhones and iPads for young children, based on his own children's interests. Four apps came from Apple's store (Apple, 2012), under the category of being education and mathematics. The remaining apps came from a US mother's blog (PragmaticMom, 2012). On this site were recommendations from the mother as well as from other people. Once the apps were downloaded, Tamsin played them all. Some apps were about basic number facts. However, in most apps the mathematical ideas were less visible.

Initially the data were analysed to identify which of Bishop's six activities occurred in the apps that Miguel played. This information was recorded in a table. During this analysis, notes were made about the features of the app that seemed to support Miguel's engagement with it. Once the analysis of the mathematical categories was completed, the notes about the features were considered and general themes/points identified. One of the themes, which was connected to Miguel's willingness to use an app, appeared to relate to how visible the mathematics was.

We could distinguish 3 levels of visibility of the mathematical ideas. To clarify how this visibility was related to the definition of play we used Basil Bernstein's (1971) ideas about the degree of classification and/or framing of content. In our context, framing is about whether it is the app or the player who has control:

> over the selection, organization and pacing of the knowledge transmitted and received in

> the pedagogical relationship. … Strong frames reduce the power of the pupil over what, when and how he receives knowledge and increases the teacher's power in the pedagogical relationship. However, strong *classification* reduces the power of the *teacher* over what he transmits as he may not over-step the boundary between contents, *and* strong classification reduces the power of the teacher *vis-á-vis* the boundary maintainers. (Bernstein, 1971, pp. 51-52 italics in the original).

In our case, the app and by implication its designer, replaces the teacher in the pedagogical relationship. Wang, Berson, Jaruszewicz, Hartle and Rosen (2010) discussed the importance of "the virtual world product developers who incorporate decision making options that the users can manipulate" (p. 36). As exercise of choice was one of the key features of play (Dockett & Perry, 2010), consideration of who controlled what content was used and how it could be used were of interest. Although Tamsin took part in discussions with Miguel, she made no suggestions about "the selection, organization and pacing" (Bernstein, 1971, p. 51) of what he did. The exception was after Miguel had spent considerable time having a dog in the *Talking Tom 2* app make farting noises, Tamsin suggested that he found another app to play.

In Miguel's engagement with the apps, it was possible for us, as researchers, to identify the mathematical content, such as ideas about numbers, but it often would have been invisible to others, especially young children. At the most visible level, a player could not engage with the app without being aware that they were using mathematics as was the case with *Factor Ninja*, described in the next section. In this case, the mathematics could be considered as strongly classified and well insulated from other content (Bernstein, 1971).

At the next level, the mathematics was semi-visible, lying somewhere in between a strong and weak classification. Often apps had several parts and one or more of them required Miguel to use mathematical concepts. However, the focus of the app was not on the mathematics. Instead the mathematics was used to achieve another purpose. In *Tavern Quest*, the player earned money when a meal was sold. This could be used to quicken up meal production or to buy necessary skills for a quest. In this case, the content was not strongly classified because it was integrated with other content, but it was strongly framed because there was no choice in the selection of the content that had to be used.

At the third level, the mathematics was invisible. For example, in *Toca Hair Salon - Christmas gift*, measurement ideas about the length of the Christmas Tree's hair were part of considerations of how the hair should be styled. In these cases, the designer did not insulate the mathematics from other contents and the content selection depended upon the purpose of the app. Classification and framing were weak (Bernstein, 1971). Emilson and Folkesson's (2006) research in a Swedish preschool, suggested "a weak classification and framing can promote the possibility for children to participate on their own terms" (p. 235). Therefore, classification and framing in an app is likely to affect whether it is considered as play by the children.

IPADS AND BISHOP'S 6 MATHEMATICS ACTIVITIES

Over the hour, Miguel played eleven different apps, sometimes for a very short time, such as *Tavern Quest*, and others for much longer periods, such as *Meeblings* and *Toca Hair Salon – Christmas Gift*. Only in *Talking Tom 2* was it not possible to see any of Bishop's six mathematics activities.

Counting

Factor Ninja had mathematics as its sole content and required composite numbers to be sliced into their prime factors. Therefore, the contents of this app were strongly classified (Bernstein, 1971). Miguel did not know what prime or composite numbers were but was aware that there was no choice in the selection of knowledge he needed to use. This was the only app where he went to the explanation to find out what he had to do. However, the explanation required too much reading for a six year old. He also "died" too quickly to learn the rules from using the app. Consequently, he did not engage for long. Nevertheless, he named the numbers as he was slicing.

Knattematte appeared to be very similar to a school-like number activity. In one part of the app, a simple addition was given in numerals and then with apples. Then a choice of four numerals appeared on the screen. The correct numeral had to be tapped, but when Miguel tapped a wrong numeral, it just shook. As well, he found he could eat the apples by tapping on them. Although the mathematics in the app appeared to be strongly classified and framed, the way Miguel engaged with it suggested that he was able to blur the boundaries between contents and determine what content should be used.

Figure 1: Duplo Jams–train activity

In other apps such as *Duplo Jams*, the counting was less visible. In one activity a train ran along a track. Duplo bricks fell from the sky and if the train moved slowly, they landed on it (see Figure 1). After a discussion about the relationship between the train's speed and catching the bricks, Miguel said "I've just got, now I've got 3". Then another brick landed on the train and Tamsin said "There you go, how many now?" As two more bricks fell, one after the other, Miguel stated "I've still got lots." Tamsin said, "How many's lots?". After some distractions, Miguel counted them and said "Eight, eight". Knowing how many bricks were on the train was not essential for the game, but the way the bricks fell, one at a time, facilitated a systematic noting of how many there were. Thus in this app, counting could be considered both weakly classified–there were no distinct boundaries between it and other content, such as the train's speed–and weakly framed as Miguel could control what content was his focus.

Locating

Locating is about exploring one's spatial environment. In some apps, location skills were needed and this supported them being discussed. In one part of *Bamba Ice*, Miguel had to tap a numeral between 1 and 9 on a cash register to indicate the price he should pay for an ice cream he had designed. A purse then opened to reveal an equal number of coins to the price chosen, although this match was not clear from how the coins were shown in the purse. Miguel had to drag the coins one at a time to the cash register drawer to pay for his ice cream. Sometimes his dragging meant the coin was just outside the drawer, so it was returned to the purse and he had to start again. Although in this last session, we did not discuss why it took him a long time to get all the coins from the purse to the drawer, in a previous game, Tamsin had mentioned how important it was to make the coin go wholly into the drawer. The choice of what Miguel had to do in order to move to the next stage was controlled by the app, making it strongly framed. However, the lack of visibility about what content he was expected to use–location skills–suggested the activity was weakly classified.

Measuring

Measurement knowledge was needed only for some parts of the different apps, affecting the degree of visibility. In the app *Order Up*, Miguel was expected to judge when different items were "cooked" by reading a scale. However, as was the case with several apps, the mathematical content was not always Miguel's focus.

Miguel: Cook eggs, cooking eggs this time, I've never cooked eggs before.

Tamsin: No, it looks like you've got to cook both sides as well. Do you remember how to flip them?"

When the meter moved to perfect for one side, he flipped the egg

Tamsin: Yep, flip it, that's it.

Then he moved to another part of the app and the "cooking" was not completed and the cooking of the egg was labelled "poor". Then as Miguel refocused on the "cooking", Tamsin intervened so the scale reading became part of the discussion.

Tamsin: Argh, you didn't wait till it was all done.

Miguel: Of what?

Tamsin: Hang on, wait till that one gets up there (pointing to where the meter has to get to for it to be perfect). Remember, it's got to be put in here (pointing out the dinner plate which must be pressed in order to save the meal).

He tapped on the dinner plate when the meter reached “perfect” on the second side.

Tamsin: That’s it go, go!

Miguel: Perfect!

In home situations, the boundaries between school and everyday knowledge, such as between measurement and cooking, are likely to be blurred, suggesting that classification is weak. However, in a virtual environment of an iPad app, the blurring did not seem so clear to Miguel. Although he had seen eggs being cooked, it was unlikely he had cooked any himself. With Tamsin’s intervention, the need to read the scale was reinforced making the measurement knowledge strongly classified. The app design also meant that Miguel had little choice about what to do, suggesting that framing was also strong.

Designing

Designing was a large part of several apps that Miguel played over the four days. The one that he repeatedly returned to was *Toca Hair Salon*. In this app, Father Christmas or a Christmas Tree could get a new hair style, by using various equipment and accessories. In the final session, Miguel placed Christmas decorations of the tree and coloured its needles different colours. He turned his finger into a hair dryer by clicking on an icon and rearranged the needles (see Figure 2).

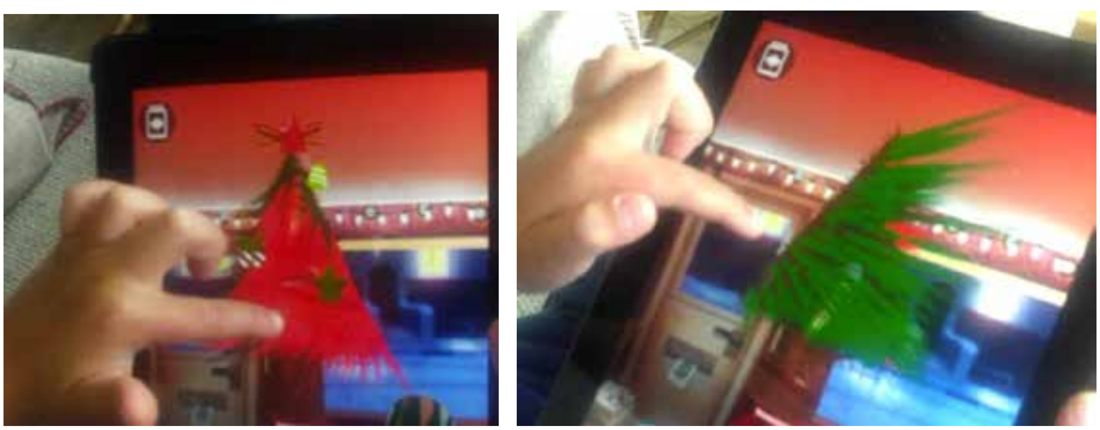

Figure 2: Christmas Tree gets a new hair style

Both characters in this app changed facial expressions as Miguel gave them different styles. The tree’s face showed surprise when the hair was cut and then became sad when the needles were made so long that they covered its face. Miguel said “Why is he feeling so sad? I’m putting his hair back.” Tamsin answered “Maybe because it’s a little too long for him?” Miguel said “Because he doesn’t like it on his head.” Tamsin replied “Yeah, it’s in his face. I don’t like having hair in my face either.” Although the tree’s face seemed to register happiness, sadness and surprise, it was not always clear what the connection was to Miguel’s actions. Nevertheless, the feedback that Miguel gained from the characters encouraged his continued engagement.

The designing knowledge that Miguel used was not isolated from other contents about hair, hair styling and Christmas trees, suggesting that classification was weak. Framing also appeared weak because Miguel had significant control over what he chose to do and the order in which he chose to do it.

Playing

Apps designed for young children require them to conform to certain rules but these rules were usually embedded. Given that children have limited if any reading skills, apps need to be operated intuitively. For example, in the *Meeblings* app, different coloured Meeblings had to be tapped in a specific order to rescue other Meeblings. After several unsuccessful attempts at a level, Miguel checked the solution, by tapping on this word and watching what he had to do. This suggests that what is considered to be the content of rules is neither weakly or strongly classified. However, the constraints they placed on players' choices about what they could do indicated they were strongly framed. In the train game in the *Duplo Jams* app, Miguel had to know from other experiences, that raising railway barriers might move the train forward. When he first came to them, Tamsin had to tell him what to do. In a later attempt, Miguel did it himself and said "It's easy, see. It's easy, easy this time."

Explaining

When Miguel first engaged with the apps, he rarely talked about what he was doing even when questioned. His attention seemed focused on determining what he had to do. Only after he was familiar with the apps did he seem able to discuss them. However, when he became comfortable with the app, it still had to be challenging for him for him to continue to play it. Therefore, opportunities for discussions, including explaining what was happening, only arose on a few occasions. Of Bishop's (1988) six activities, this was the one which was the least represented.

The embeddedness of the rules also meant that models of explanations were rarely provided, although sometimes, Tamsin provided them. In *Play Lab* a shape was traced over and then appeared, as a solid shape on the screen. In one game, the shapes were combined to form a car which drove along a road. Miguel's tracing was not always recognised by the app and twice when he tried to trace a circle, a square appeared. His car ended up with two squares on its roof.

Miguel: What is that?

Tamsin: I think it's a car, isn't it? … A car with parcels on the top.

Miguel found that by tapping on the car, which was now driving, he could make the squares wobble and eventually fall off.

Miguel: See, I'm trying to put the parcels down. See I put them down.

Miguel's explanation of how he made the squares jump off the car involved focusing Tamsin's attention on his actions. Miguel's explanations were weakly classified because what they was not clearly insulated from other content, such as making statements about what was occurring. As well, framing was weak because the control of what was and what was not transmitted was largely up to Miguel who used his everyday knowledge to form the explanations.

CLASSIFICATION, FRAMING AND SANDPIT PLAY

Almost all of the apps that Miguel chose to use on the final day of his "research" using the iPad contained one or more of Bishop's mathematical activities, although opportunities to provide explanations were limited. Nevertheless, as Prensky (2006) suggested, the apps that Miguel engaged in did contain educational material.

At the same time, it was clear that six-year olds, who are at the end of preschool in Sweden, may not recognise that they were doing mathematics while engaging with the apps. This was because the focus was on something else. In apps, such as *Knattematte*, which most adults would consider to be about number knowledge and thus strongly classified, Miguel found other aspects of the app to focus on. When the mathematics was invisible, there was a blurring of boundaries between mathematical and other contents as Miguel responded to the different challenges in the apps, making is weakly classified. A similar case could be made for the mathematics that children engage with when playing in a sandpit.

On the other hand, the apps that seemed to be most play-like in the ways that Miguel engaged with them were those that were weakly framed. In these cases, Miguel choose what he did and how and this facilitated him to try out different approaches and explore different outcomes, such as was the case with *Toca Hair Salon*.

It cannot be assumed that because a researcher considers an app to use mathematical concepts that children can build on the concepts or make connections to other concepts. However, it cannot be assumed that children do not do this either and so longitudinal research is needed in this area. Regardless of whether an app was considered to be weakly classified, framed or both, describing what he was doing, especially when he used explanations, seemed to provide Miguel with opportunities for his mathematical thinking to become more visible. It also seemed to allow him to be challenged in other ways than those required by the apps. Thus, apps have the potential to provide opportunities for playful mathematical learning but interaction with an adult would contribute a significant amount to realising this potential.

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YOUNG CHILDREN´S USE OF MEASUREMENT CONCEPTS

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This paper describes preschool children´s measurement representations as they engaged in drawing a map. The use of small cars, boats, trains and aeroplanes helped the children to make the connections between two and three dimensional space. They also made connections to their awareness of length. The children's own experiences provided the motivation and stimulus to provoke their mathematical thinking about quantifying different attributes of objects.

INTRODUCTION

In this paper, I focus on children's experiences of area and different forms of length, such as breadth and height, by analysing the children's descriptions given while they were drawing a map. The children's own thoughts were used as a starting point to discuss the mathematics by myself, as the teacher. This case study is a part of a larger study which arose from a desire to understand how children's outside experiences can promote mathematical learning.

Doverborg and Samuelsson (2011) highlighted the need for children to learn from their own experiences in a way that made sense to them. Their research indicated how children perceived mathematics to be useful both at the current time and in the future. However, Uttal (2000) found that children's developing conceptions of maps were affected by their understanding of the surrounding world. Uttal suggested that younger children perhaps have not yet developed the ability to understand and connect their outside experiences to activities inside preschool. Uttal's study showed children's difficulties in encoding, remembering or understanding information. He suggested that to capture and comprehend all aspects of a three dimensional world on to a two dimensional sheet of paper is impossible for young children.

According to the Swedish preschool curriculum, early childhood settings should facilitate mathematical learning through play (Skolverket, 2011). Play has a leading role in developing children's knowledge from an early age (Vygotsky, 1933/1966). Activities can be based on a creative form of play, with opportunity for a variety of expressions. Children´s everyday experiences can be represented in their visual creativity and provide opportunities for conversations. Nevertheless, it can be difficult to see children´s illustrations with anything other than the adult eyes, but when we listen to children, as they draw, we can understand their thoughts, thus providing insights into their interests and background (Coates & Coates, 2006).

Social interaction in play can promote the construction of mathematical knowledge. Edo, Planas and Badillo (2009) stressed that teachers and children interactively construct shared meanings when engaging in activities, such as play. Consequently, the preschool teacher has to find meaningful situations and ways to communicate that challenge the children to ask questions, reflect and discuss (Clarke, Clarke, & Cheeseman, 2006). The conversation between adults and children can be a part of the learning process in mathematics, where the teacher has a supporting role to help the children build an understanding of measurement.

When children attend preschool they bring with them experiences from outside preschool which can be the basis for developing children´s mathematical thinking (Clarke & Robbins, 2004). However, teachers' perception that they must follow the curriculum can result in them providing activities suggested by the curriculum but which do not build on children's own understandings (Doverborg & Pramling Samuelsson, 2011).

MEASURING CONCEPTS USING CHILREN'S OUTSIDE PRESCHOOL EXPERIENCES

Concepts of measurement are described in relationship to the concepts of attribute, unit and scale (New Zealand, Ministry of Education, 2007). The attribute that is compared can be area, length, volume and time, etcetera. Unit and scale measurement concepts can be applied to most attributes but initially it is very important for children to be able to identify what attribute is to be measured (New Zealand, Ministry of Education, 2007). Once the attribute is identified, children are then able to do direct comparisons by placing two objects next to each other. They also develop an understanding of transitivity, in which a third object is used to compare two other objects. If the first object is smaller than the third object but bigger than the second object, it is possible to say that the second object is smaller than the third object.

In measurement, units are used to measure an attribute and to quantify the amount of an object. Bush (2009) described children´s understanding of measurement, with focus on usage of identical units and iteration. For an accurate measure, the units must be identical. Iteration is the repetition of a unit when measuring involves detailing an amount and is one of the underlying concepts connected to unit. Connected to the need to understand iteration, children also have to understand the idea of tiling, which is when units are placed repeatedly, with no spaces between. These are counted in order to find the measurement amount (Bush, 2009). Relativity involves understanding how units compare in size to other known objects (New Zealand, Ministry of Education, 2007). McDonough and Sullivan's (2011) research suggested that children also need to understand that a larger unit can be subdivided into repeated parts which can be counted, to produce a measurement of the object. This concept leads to the use of standard units, such as metres and centimetres.

When using a scale to measure, any point can act as the start or end point. However, without an awareness of the concept of unit, incorrect measuring can occur (McDonough & Sullivan, 2011). The concept of scale also includes an understanding that marks on a scale represent the end point of each of the units. Therefore, the end point when something is placed on a scale, indicates the amount in the same way that counting individual units does (New Zealand, Ministry of Education, 2007).

The Swedish preschool curriculum suggests that the preschool should engage children in activities that develop their ideas about measurement and space, as well as other mathematical concepts (Skolverket, 2011, p. 10). This means that teachers are responsible for building on children's understandings of attribute, unit and scale as they engage in an activity.

Children's everyday experiences outside preschool can be a starting point for building measurement strategies in preschools (Castle & Needham, 2007). Clarke and Robbins (2004) collected data that illustrated children´s experiences at home and in their neighbourhood and showed a variety of mathematical contexts. For example, there were sequences of children measuring ingredients and cooking at home. These provided evidence of mathematics in everyday experiences, although they were not recognized by parents or teachers. Meaney (2011) also found that a six-year old girl engaged in a number of measurement activities at home, often associated with the child's physical engagement in a task. In particularly, she suggested that measurement of time, often considered hard because of its abstract nature, was the focus of many discussions between the child and her mother. This contradicts suggestions that length is the easiest attribute to measure.

Fleer (2010) suggested that younger children in preschool probably are unaware of the value of their own experiences and the teacher has to encourage this awareness. Within preschools, there are possibilities for knowledge creation, nevertheless children should have the opportunity to form their own experiences and make choices in the light of these.

There appears to be little research which shows how children's share their previous measurement experiences and then teachers make use of them to develop their understandings. For example, Castle and Needham (2007) investigated younger children's understanding of measurement concepts, but not their thoughts about them. In McDonough and Sullivan's (2011) research, children were assessed on their preconceived understandings about how children learn length measurements.

The aim of my research is to understand how children's outside experiences can promote mathematical learning. Teachers' ability to recognize and work with children's outside preschool experiences can affect the mathematical activities that they offer to children. The research question is:

How do children use measurement concepts in an interaction that draws on their outside preschool experiences?

METHODOLOGY

This paper presents a case study which is a part from a larger study (Bryman, 2012). Over recent years, researching early childhood education by listening to and observing children has become common (Dockett, Einarsdottir, & Perry, 2009). A case study approach recognises that within social and cultural settings, children as competent participants have a right to have their voices heard and to be taken seriously. The larger project investigates the relationship between children´s outside preschool experiences and their mathematical learning in the preschool. In this paper, I present one episode in which I was involved as the teacher where children made connections to their outside preschool experiences whilst drawing a map. Their involvement showed use of many of the measurement concepts described earlier.

In order not to lose the spontaneous aspect of their play, field notes, first by myself, but later, after the children invited me into the play, by a colleague were made instead of, for example, video recordings. The latter would have provided the possibility to analysis the data several times. However, given that the wider project was about documenting naturally occurring incidences in a preschool setting, it was decided to use field notes instead.

Rather than being set by me, the activity began as a play session with a group of five children aged between two and six years before breakfast. It was the children, one boy in particular, who suggested drawing a map, which became the focus. From being an observer I became an active participant in the activity. I am aware that several of the questions that I asked had an impact on the dialogue sequences during the activity (Hasselgren & Beach, 1997). On the other hand, I was one of these children's preschool teachers and we interacted in ways that seemed typical of our normal forms of interaction.

Analysis of the interactions was done by looking for examples of the measurement concepts of attribute, unit and scale. Examples of the children's use of these concepts are provided in the next section. The exchanges were originally in Swedish but are provided in English. It is not always easy to translate young children's Swedish as their language is developing, so it has been tidied up in places to make it more understandable. This has changed the form but not the content.

CHILDREN'S STRATEGIES IN MEASURING WHILE DRAWING A MAP

The group of five children consisted of three boys and two girls. Child 1 is six years old, child 2 is two, child 3 and child 4 are both five years old and child 5 is four.

Child 1 handed out toy vehicles to the other children, at the beginning of this activity. During the activity, the children shared and swapped toy vehicles between themselves. They all had experiences about travelling and used their knowledge to draw the map. The dialogues show how the children used the toy vehicles with the measurement concepts of attribute, unit and scale.

The activity began with a boy picking up paper and pens. The following exchange then transpired.

Dorota: Why did you take out the paper?

Child 1: We must make space

Dorota: Make space? What do you mean?

Child 1: Space for boats, trains, cars, airport, roads, you can take busses.

Dorota: Are you thinking about a map?

Child 1: We will have roads, airport, harbour, train station. We will find it, we will draw on the paper.

By saying "we must make space", Child 1 appeared unclear about which attribute he was talking about. However, he clarified this by saying that space was needed for the toy vehicles, suggesting that it was area. After that he said "we will find it", which was followed by looking at the toy vehicles and the sheet of paper. This suggests that the child was making a visual estimation of the different amounts of area that would be needed for roads, airport, harbour, and etcetera. However, there is no explicit comparison mentioned either between the different vehicles or between the vehicles and the space on the paper. If there is a comparison, it is implicit. This is similar to what Meaney (2011) found in her study of a six-year old child's use of measurement concepts. In this study, many of the comparisons were to an unidentified other, making them also implicit.

Later Child 4 helped a younger boy, Child 2, to count busses and draw train stations. Then Child 4 had an idea about drawing a railway and two train stations. She described this to Child 2. Child 4 had experiences of travelling by train and, therefore, may have known that trains travelled from one station to another, although she did not explain why she needed to have two train stations.

Child 4: We are drawing roads and two stations.

Dorota: Why two stations?

Child 4: I do not know, but the train has to go somewhere (Looks at child 2. He had trains, which he gave to her)

The child used physical objects to visualise her thoughts. By saying that the train has to go somewhere, Child 4 implies a comparison between the area taken up by a train, and more implicitly its journeys, and the area on the piece of paper. The presumption seemed to be that the paper had a large enough area to cater for the railway line so the train could "go somewhere". Child 4 placed the two trains side by side and used them to draw lines on either side of this pair. Then, she moved the trains forward and drew new lines, again either side of the pair of trains and repeated this three times. After that she drew the rest of the railway without using the two trains. She had designed a railway across the paper. Each train had the role of an identical unit, as each train was the same width. In placing the trains side by side, there was no gap between them, suggesting that this child understood the unit concept of tiling. In this way, Child 4 determined the width of the railway from using the toy trains. After that, she drew a station at each end. This may indicate that she was using them as end points for the length of the railway line, which is related to concept of scale.

The next example shows Child 5's explorations about width, from putting two vehicles side by side and using his experiences and knowledge about traffic and directions. Child 5 started by drawing a road, which he linked to the train station.

Child 5: How much space do I need? I want to have a two-lane street, so my car can drive in both directions.

Dorota: What do you think? How much space does your car need, how wide is the car?

He looked at the car and drew a straight line beside it. He moved the car sideways, and drew another line. The street was compared to how wide the car was. Child 5 used two identical units, the cars, and put them side by side on the road, to see if they fitted into the space. By doing so, he subdivided the width of the road in order that the cars could drive in both directions. The cars were placed side by side with no gaps between, indicating tiling. He estimated the width of the road by placing these cars together in a similar manner to what Child 4 had done with the trains.

The next exchange shows again the importance of the toy vehicles in supporting the children's measurement representation so that they could draw the map as they wanted it to look. The toy cars were not only used by Child 5 to draw roads of an appropriate width, but also to draw a line in the middle of the road.

Dorota: What are you doing?

Child 5: Dividing the road so the cars know on which side of the road they should drive. You know when you drive you should have this line there (pointing at the line, he drew in the middle of the road)

Dorota: Here! On the left side of the car (pointing on the line in the middle)

Child 5's experiences outside of the preschool made him aware the road should be designed so the cars knew on which side of the road they should drive. He placed two cars side by side. Then he took away one and drew a line. These two cars represented the width of the road, which can be considered as a single unit in its own right. In this case, the cars could be seen as supporting the partitioning of this large unit of a road into smaller units, the width of one car. Being able to move backwards and forwards between seeing the car or the road as the unit provides a way of seeing the complex relationship between them.

The follow exchange illustrates how child 4, the girl who drew the railway, began to draw a harbour. The harbour was needed because, as the children discussed, it was possible to travel by boat. Child 4 took a pen and drew a line in front of a boat, then she put another boat behind the first and repeated this until she has five boats, lined up one behind the other, like cars parking in a street, and drew two lines. She said:

Child 4: I'm drawing a harbour, I place my boats behind each other and I have five boats. I have to draw all five to get space

Dorota: Do you make space for your boats?

Child 4: Yes, I know how large a harbour should be now

Dorota: How do you know?

Child 4: My first boat is behind this line (she points) I have drawn two lines now, you see (she takes away the boats and points on two lines)

Dorota: Okay, a line in front of the first boat, and a line behind the fifth boat

Child 4 uses the boats, as physical objects, to find out and measure the area needed for the harbour. To do this, she builds of the attribute idea of comparison, by using length as a default for area measurement. The iteration is of five boat lengths, which forms the area of the harbour, when boats are placed one after the other. Similar to when she was drawing the railway line, child 4 used five identical units and filled a space without gaps, suggesting the unit concept of tiling. By drawing a line at the end of the last unit, she identified the end point for her measurement, which is for a component of the concept of scale. As the teacher, I took the opportunity to use ordinal terms, “a line in front of the first boat, and a line behind the fifth boat” to highlight these endpoints.

A discussion with children in relation to measurement occurred again when the children tried to draw streets and a runway for an aeroplane. The children used each other’s ideas to work out how they could make enough space on the paper for vehicles. Child 1 and Child 3 noticed what Child 5 did when he drew roads and did the same with the runway for aeroplanes.

Dorota: How is it going? Do you have space for all the planes?

Child 1: We have five aeroplanes and only two can be in air.

Dorota: Be in the air?

Child 1: One lands, and one lifts off (he points to the map), you see, we have drawn a take-off and landing runway. Other planes are here.

Dorota: Okay, what were you thinking when you drew the runway?

Child 1: The aeroplane takes a lot of space, we have tried.

The children needed the support of physical objects when, for example, they discussed the width of a runway to make sure it would be possible to fit an airport on the map. Child 1 took two aeroplanes and placed them on the runway, side by side. In this way, he compared the width of an aeroplane to the width of the runway. By placing two aeroplanes on the runway side by side, Child 1 and Child 3 used the concept of iteration, to measure the width of the runway–it was two aeroplanes wide. They used aeroplanes to determine the size of the area that they had to draw, by lifting one aeroplane and landing the other.

Sometimes physical objects were not sufficient for developing some ideas and it was myself, as the teacher, who provided the stimulus. When it was time to draw a bridge, I challenged them to think more about height and width. To begin with, I took a piece of paper and said “how long should the bridge be?” A girl replied, “as long as a car”. Then I cut a piece of paper, so that it was as long as the car the girl gave to me.

Dorota: Is it a bridge? (I looked at Child 4. who put the piece of paper on the map), is it a bridge?

Child 4: No, how should we make one? What should we do?

Dorota: (took a larger piece of paper, gave it to child 5) Can you cut out a strip, which has the same width as this piece (the piece child 4 cut, which was too short). It is as wide as two cars. This has sufficient width to be a road in two directions.

Child 4: We take two cars, put them on the piece of paper, one after the other. Is it enough?

Dorota: How do we know that the bridge has enough width and height to allow a train to drive under?

Child 5: A train must be able to drive under the bridge, we try (teacher takes a train, holds up the piece of paper and pushes it upwards until there is space enough to drive the train under it.)

Child 4 together with child 5 wanted to build a bridge for cars to drive over and trains to go under. The width of the paper was compared to the width of two cars. Child 4 said "we take two cars, put them on the piece of paper, side by side and cut". In placing two cars side by side, they showed a concept of iteration. The cars were identical units and these units filled the space without gaps, thus tiling was used. Relativity in measurement takes place, when they needed to cut a piece to fit two kinds of units, cars and trains. The piece of paper, the cars and trains, are compared directly.

DISCUSSION AND CONCLUSION

This activity was initialized by the children and the map was a product of their engagement. To produce the map, the children used several measurement concepts to solve problems. I consider the children´s creativeness, in map making, to be the key for making connections between their ideas about how to measure the spaces they wanted on their map and the measurement concepts described in the literature (Bush, 2009; New Zealand Ministry of Education, 2007; McDonough & Sullivan, 2011). The children used informal units, such as the toy cars, boats and places, to measure attributes of objects, such as the area for a harbor, the width of a road, height of a bridge and length of the airport runway.

The results of this study indicate that children's own experiences were the background for the activity and could be drawn upon whilst they were playing. Using their own experiences allowed them to link the knowledge they possessed with knowledge about measurement concepts. The activity allowed them to be creative. At times, they were not able to express their thoughts verbally but did so through gestures, when they were using the physical objects.

As a teacher I could recognise the mathematics in children's actions and drew their attention to concepts of measurement, especially in the bridge episode. I could help children to address challenges they had when building the bridge. The knowledge that the children had about the need for the bridge to be tall enough for a train to go under it and wide enough for two cars to travel on it provided them with background to what the problem was that they had to solve. Their understanding of what the problem was meant that my questions prompted them to think again about how long the paper for the bridge needed to be.

As Doverborg and Pramling Samuelsson (2011) stressed there is a need to use children´s own experiences as a basis for their mathematical activity. In this case study, children's outside preschool knowledge about travelling and their experiences with cars, trains, aeroplanes and boats allowed them to use and develop understanding about measurement concepts. This illustrated how Doverborg and Pramling's ideas could become a reality when children are supported to discuss their ideas. As the teacher, by engaging in their play, I confirmed the value in these experiences through the social interaction and promoted the construction of mathematical knowledge. In many ways this was similar to what was documented in Edo, Planas and Badillo's (2009) research. Listening to what the children had to say contributed to finding a meaningful situation, in which it was possible to challenge the children to ask questions, reflect and discuss (Clarke, Clarke, & Cheeseman, 2006).

Further research is needed to understand how children´s background knowledge can be used by preschool teachers in activities and discussions. The research described in this paper has shown how concepts of measurement can be used but further research is needed about how other mathematical concepts can be developed by preschool teachers drawing on children's outside preschool experiences.

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EXPLORING THE FUNCTIONS OF EXPLANATIONS IN MATHEMATICAL ACTIVITIES FOR CHILDREN AGES 3-8 YEAR OLD: THE CASE OF THE ISRAELI CURRICULUM

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Explanations are an inherent part of mathematical activity. They serve various purposes and may take on different roles. This paper focuses on the functions explanations may serve in the preschool as well as in the beginning of elementary school. It investigates the use of explanation-related terms in the preschool and in the first and second grade mathematics curriculum in Israel.

Key words: Preschool; first and second grades; explanations; curriculum documents

INTRODUCTION

The mathematics education community has placed great emphasis on the giving and evaluating of explanations by students of all ages (Mueller, 2009; Yackel & Cobb, 1996). They are part of the reasoning processes we wish to encourage among students, "Students need to explain and justify their thinking and learn how to detect fallacies and critique others' thinking" (NCTM, 2000, p. 188). They are also part of communication processes which include "sharing thinking, asking questions, and explaining and justifying ideas" (NCTM, 2000, p. 194). From what age should we expect students to offer explanations when engaging in mathematical activity? Do the types of explanations and the purposes of giving explanations differ at various ages? This study examines the roles of explanations in official Israeli mathematics curriculum documents, focusing on the transition between preschool and primary school. We chose to examine official curriculum documents for several reasons. First, the national curriculum often sets the standards for what is learned and how that learning is assessed. Second, textbooks in Israel take their cue from national standards and must be approved by the Ministry of Education. In addition, prospective and practicing teachers are explicitly exposed to the curriculum during professional development and often use the guidelines to construct lesson plans. The curriculum is especially important for preschool teachers who often design their own activities based on curriculum suggestions as textbooks and other curricula materials are less available for this age group.

In Israel, there are separate curriculum documents for students of different ages. At the preschool level, the Israel Mathematics Preschool Curriculum (IMPC) (Ministry of Education, 2008) covers concepts and competencies that children should reach by the time they enter first grade. It also lists explicitly and separately which of those concepts may be promoted and which skills should be enhanced for children ages 3-4, 4-5, and 5-6 years old. For example, by the time children enter first grade they should be able to count backwards from ten but the curriculum suggests that this skill should be fostered from the age of 4 and not from the age of 3. The preschool curriculum also includes examples of activities that can be used to promote and assess the required skills. The Israel Mathematics Curriculum (IMC) for elementary school (Ministry of Education, 2006) covers concepts and competencies which should be fostered among students in grades one through six. Each grade is dealt with separately. In addition, an official supplementary document to the elementary mathematics curriculum was published by the Ministry of Education entitled "Milestones" (2009). This document includes the curriculum as well as specific standards for each content topic and explicit examples of activities that can be implemented in classes. Thus, in this study we examine the preschool curriculum document and the "Milestones" document. We focus on preschool children ages 3-6 years old and first and second grade students in elementary school (ages 6-8). The main questions of this study are: (1) Are young children (ages 4-8 years old) expected to give explanations when engaging in mathematical activities? (2) What are the functions of explanations at these ages?

THEORETICAL FRAMEWORK

How may explanations be characterized? Philosophers of science tend to view the concept of explanation as a logical relationship between questions and answers. According to van Fraassen (1980) an explanation must answer a why-question. However, Achinstein (1983) takes the broader view that many different kinds of questions may be asked when attempting to gain understanding and it follows that the act of answering any of these should be regarded as an act of explanation. Within the field of mathematics, the notion of mathematical explanation is closely related to other notions such as ‘generality’, ‘visualizability’, ‘mathematical understanding’, ‘purity of methods’, and ‘conceptual fruitfulness’ (Mancosu, 2008).

In mathematics education, explanations may be characterized by referring to their functions and forms. Yackel (2001) views explanations given in the classroom as a social construct whereby their functions and forms are interactively constituted by the teacher and students. Thus, an explanation is considered to be an aspect of discourse and its first function is communicative, “Students and the teacher give mathematical explanations to clarify aspects of their mathematical thinking that they think might not be readily apparent to others” (Yackel, 2001, p. 13). In a traditional mathematics classroom, an explanation may describe the steps of a procedure used. In the inquiry-based mathematics tradition, explanations communicate interpretations and mathematical activity to others in order to convince others that solutions are legitimate. Krummheuer (2000) found that when learning mathematics, students “’tell’ or ‘narrate’ how they came to their solution, or better put how one can come to a solution” (p. 24).

An additional function of explanations is to rationalize actions, both for the giver of the explanation as well as for the receiver. In that sense, explanations may also have the function of convincing oneself or another person of some assertion. In the field of argumentation (Krummheuer, 2000), an explanation may take on the role of data which supports some assertion, or a warrant which legitimizes a previous explanation, or a backing for the warrant. Thus, one may view explanations as the building blocks of argumentation. Convincing and explaining are also related to proving. When discussing proofs, de Villiers (1990) characterized an explanation as providing insight into why a statement is true as opposed to verifying the truth of the statement. An explanatory proof (Hanna, 2000) may help students see why a theorem is true; it is both convincing as well as illuminating. Nunokawa (2010) referred to proofs as full explanations which often contain critical ideas. He further claimed that explanations not only communicate the student's existing thoughts but may also generate new objects of thought by directing new explorations which may then deepen the student's understanding of the problem at hand. Thus, an underlying function of explanations is to expand students' mathematics learning.

METHOD

The first stage of this study was simply to gather each instance of when the curricula use explanation-related terms. In Hebrew, the term 'explanation' may be translated as 'hesber' (הסבר). This is the most common translation and when translating back the term 'hesber' to English one always gets 'explanation'. There is also a second word in Hebrew which is sometimes translated to explanation and that word is 'nimuk' (נימוק). However, 'nimuk' may also be translated as justification. On the other hand, a specific word for justification, other than 'nimuk', exists. This word is 'hatzdaka' (הצדקה). Finally, as mentioned above, explanations are sometimes related to proofs; thus the term for proof, 'hochacha' (הוכחה) was included in this study. To summarize, when examining the curriculum documents, four terms were taken into consideration: 'hesber', 'nimuk', 'hatzdaka', and 'hochacha'.

The second stage consisted of an inductive process, whereby each instance of the four explanation-related terms was analysed with the help of guiding questions. This led to the development of categories. A separate analysis was conducted for the introduction sections of the curricula and the other sections. This separation was due to the more general format of the introductions as opposed to the specific examples of activities given in the other sections. In the introduction sections, the following questions guided our analysis: How do explanations fit in with the general aims of teaching mathematics at each level? To what purpose do we use explanations in a mathematical activity? In the other sections of the curricula we asked ourselves: In the given context, would an explanation be an answer to a "how" question or to a "why" question? Would the explanation be used to evaluate procedural or conceptual knowledge? Does it seem that the explanation is the culmination of some mathematical activity or reasoning process or might the explanation be a stimulus for further mathematical activity? Each of the authors of this paper categorized the instances on their own and then compared the analyses. A third researcher validated the final categories.

RESULTS

As mentioned in the previous section, we began by counting the number of instances each term appeared in the different curriculum documents. As can be seen from Table 1, the term 'proof' does not appear in any of the surveyed documents and the term 'justification' appears once. Recall that the term 'nimuk' does not have a clear translation and seems to fall somewhere between explanation and justification. To sum up, there are 15 instances of explanation-related terms in the preschool curriculum and 11 instances in the elementary curriculum.

	Preschool (ages 3-6)	Elementary school (ages 6-8)
Explanation	13	8
'Nimuk'	1	3
Justification	1	-
Proof	-	-

Table 1: Frequency of explanation-related terms found in different curricula

We now focus on how the different terms were used, beginning with the introductions to the two curricula and moving on to specific examples of suggested activities.

'Explanations' in the curricula introductions

The introductions relay the general aims and reasons for teaching mathematics at these ages. In the preschool curriculum there is one instance of the term justification, one instance of the term 'nimuk' and three instances of the term 'explanation':

> Participating in mathematical activities develops mathematical skills such as the ability to count and enumerate, add different amounts together, and identify shapes and solids, as well as thinking skills such as such as the ability to make comparisons, the ability to sort, and the ability to **justify** oneself and it is important to develop both of these aspects when working on mathematical activities. In order to develop mathematical understanding as well as a child's (general) thinking skills, one should request the child to **explain** his actions. The **explanations** allow the child to justify (**nimuk**) his actions allowing the teacher to better understand what the child meant. (Emphasis not in the original, p. 12)

According to the introduction, there are two separate aims of engaging children in mathematical activities. The first is to promote their knowledge of mathematical concepts and skills and the second is to promote general thinking skills and abilities. How do explanations fit in with those aims? Regarding the first aim, having children explain their actions can promote their mathematical understanding. Thus, giving explanations is a means to achieving a goal. Regarding the second aim, giving explanations is the goal. We note that it does not mention that giving mathematical explanations is the goal or that explanations should be based on mathematical ideas and principles. Instead, we desire in general to promote children's ability to justify their actions and this ability can be promoted while engaging children in mathematical activities. Another general ability, being able to verbalize one's ideas, is mentioned later on in the introduction, "mathematical discourse … develops a child's verbal abilities, that is, his (or her) ability to formulate and **explain** in words what he (or she) is doing" (INMPC, 2008, p. 14). Finally, a child's explanations can be used by the teacher to assess mathematical understanding.

In the introduction of the elementary school curriculum the terms explanation, justification and proof do not appear but there are two instances of the term 'nimuk'. One of them is related to developing children's mathematical reasoning skills such as, "deductive reasoning, …raising conjectures, generalization and **justification** (nimuk), …etc." (p. 8). The second instance is related to evaluating students' thinking, "When the teacher is assessing the students, she should follow students' engagement with task… and listen to their **explanations** (nimuk) as they implement the mathematical activity" (p. 14). In the first instance, being able to offer justifications is seen as a mathematical skill which we wish to develop. The second instance reminds us of the introduction to the preschool curriculum which suggests listening to children's explanations in order to assess their mathematical knowledge.

To summarize, from the introductions we learn that explanations are both a means and an end – the means to achieve mathematical understanding but also a skill, onto itself, to be developed. In the preschool curriculum there is no direct mention that explanations should be mathematical in nature or based on mathematical principles. In the elementary curriculum they are connected to specific mathematical reasoning skills such as making generalizations. In the next section we gain additional insight into how explanations may be used in mathematical activities.

'Explanations' in suggested activities

Following the introductions, each curriculum suggests specific examples of activities which may promote different competencies. While in the introductions we find several reasons for promoting the use of explanations, we are left unsure of what is meant by the term 'explanation'. In addition, the introductions aim at general abilities and general mathematics reasoning skills, not only those related to explanations. In this section, we review the examples of activities that include explanations and analyze what the specific function of an explanation might be in a mathematical context. These functions were not explicitly written but arose from the contexts. Below, we list the functions we found along with examples of activities from the preschool and elementary school curricula which highlight these functions.

Function 1: Explanation as a description of one's thinking process or way of solving a problem (i.e. How did you solve the problem? Explain.)

<u>Preschool curriculum</u>	<u>Elementary school curriculum</u>
Mathematical topic: Comparing sets by counting	Mathematical topic: Operations with natural numbers
Suggested activity: In competitive games	Suggested activity: You can buy the

where the winner is the one with more or less cards, tokens, etc. the teacher can see the way children use counting in order to compare two sets and the way children **explain** how they acted.

following toys in the store. (A picture of several toys and their corresponding prices is given.) Tamar has 15 NIS. She wants to buy two toys. Which toys can Tamar buy? **Explain**.

In the preschool activity, the teacher is encouraged to request from the children an explanation of how they acted, what they did in order to compare the sets. In the elementary school activity, we infer from the problem situation that students are requested to explain how they arrived at an answer. The explanation can be in the form of an arithmetic sentence or a verbal description. In both the preschool and elementary cases, the explanation allows the teacher to evaluate procedural knowledge. In essence, the explanation tells us what the child did but not why he did what he did. This brings us to the second category.

Function 2: Explanation as justifying the reasonableness or plausibility of a strategy or solution (i.e. Why did I choose to solve the problem in this way?)

Preschool curriculum

Mathematical topic: Measurement

Suggested activity: When comparing the measurements of two items with the help of a mediating tool, does the child use a convenient and appropriate mediator? Can the child **explain** why he chose that specific mediating tool?

In the above activity, the child is asked to solve a problem, in this case a measurement problem. He is not asked to explain what he did but instead to explain why he chose to solve the problem in such a way. For example, when comparing the lengths of two tables, the child may use his foot to measure the length of each table. However, if asked to compare the lengths of two papers, he might use paper clips as a measurement tool. The child is then asked to explain why he chose to use his foot in the first case and paper clips in the second case. This type of explanation is thus different from the first type. Yet, in this second category, explanations do not necessarily draw on mathematical properties nor are they necessarily related to general properties. In the third category, explanations are also given as an answer to a "why" question, but tend toward more general mathematical properties. In the elementary school curriculum, no activity was found for this category.

Function 3: Explanation as an answer to a "why" question where the underlying assumption is that the explanation should rely on mathematical properties and generalizations (i.e. Why is this statement true/false? Explain.)

Preschool curriculum

Mathematical topic: Shapes

Suggested activity: The teacher will draw a picture for the child made up

Elementary school curriculum

Mathematical topic: Operations with natural numbers

Suggested activity: (a) For each of the

of different shapes in various sizes and orientation. The teacher then asks the child to color, for example, all of the triangles and asks the child to **explain** why he colored or not colored a certain shape.

numbers below, try to write an addition sentence using two consecutive numbers.

____+___ =27 ;____ + ___ = 13
____+____ =30 ; ____+___=81

(b) Which (kinds of) numbers could be the sum of two consecutive numbers? **Explain**.

As opposed to the first category of explanations, this category focuses on conceptual, rather than procedural, knowledge. In the preschool activity, when asking the child to explain why he did or did not color a certain shape, we are essentially asking the child to explain why that shape is or is not a triangle. This type of explanation allows the teacher to evaluate the child's conceptual knowledge of triangles as well as their preconceptions of triangles (Vighi, 2003). The elementary school activity begins with specific arithmetic examples but then asks a general question. By requesting the child to explain her answer to this general question, we are encouraging her to think about the properties of natural numbers, more specifically, the properties of even and odd numbers. In this case, the explanation allows the teacher to evaluate children's conceptual knowledge of even and odd number and of consecutive natural numbers. In addition, the one instance when the term ‘nimuk’ was used in an elementary school activity, it was used in this sense.

Function 4: Explanations as a step in directing new explorations leading to generalizations (i.e. Find all possible solutions and explain)

<u>Elementary school curriculum</u>

Mathematical topic: Geometry

Suggested activity: Cut a rectangle along a straight line generating two polygons. Which (kinds of) polygons can be the result of this action? Can you get two squares? A triangle and a pentagon? **Explain**.

We found it quite difficult to interpret the term ‘explain’ in the above activity. It was clear that the underlying purpose of the request was to have the child verbalize his thoughts. This, of course, may be said of each of the examples given above. But, what might we learn from the child's explanation in this case? In order to gain a better understanding of the activity, we note that it was labelled by the curriculum as an inquiry-based activity. Children are requested to investigate what might result from cutting the rectangle along a straight line. While two possible results are suggested, the underlying aim is for children to try cutting the rectangle in different ways. The function of the explanation here could be viewed as a combination of functions mentioned previously. On the one hand, children can explain the situation by saying what they did – I cut the rectangle this way and got two triangles and then I cut the rectangle this way and got two rectangles. On the other hand, an explanation might rely on mathematical properties, such as explaining under what conditions two squares will result from the cutting. In our opinion, the ultimate purpose of the explanation in this activity is to encourage children to think of additional possibilities. If children explain what or why they did some action, it might lead them to think of other possible ways to cut the rectangle, which may possibly lead to a general conclusion covering all possibilities. In the preschool school curriculum, no activity was found for this category.

To summarize, following the introductions, there were 10 instances of explanation-related terms in the preschool examples of activities and 9 instances in the elementary curriculum. Table 2 summarizes the number of instances (%) related to each function.

Function	Preschool	1st - 2nd grade
1 – Explanation as a description of one's thinking process or way of solving a problem	5 (50)	3 (33)
2 – Explanation as justifying the reasonableness or plausibility of a strategy or solution	2 (20)	-
3 – Explanation as an answer to a "why" question relying on mathematical properties and generalizations	3 (30)	5 (56)
4 – Explanations as a step in directing new explorations leading to generalizations	-	1 (11)

Table 2: Frequency of explanation functions at different ages

Note that the most frequent function of explanations in the preschool curriculum was simply to have children describe what they did. These explanations are not necessarily mathematically-based. This seems to be in line with the preschool curriculum introduction which clearly stated that engaging children in mathematical activities can also promote general thinking skills. Moving on to elementary school, the functions of explanations become more mathematical in nature and less descriptive.

DISCUSSION

To begin with, we see that having children offer explanations while engaging in mathematical activities is encouraged from an early age. It is not something left to the later years. We also see that, as mentioned by the NCTM (2000), explanations are part of both communication processes and reasoning processes we wish to promote. We do see, however, a subtle shift from the preschool to the elementary school. In the preschool, it seems that more emphasis is placed on the communication aspect of giving explanations and less on the reasoning aspect while the opposite seems to be true in the first and second grade curriculum. During the early elementary years, it seems that explanations become more mathematical in nature, relying on mathematical properties and supporting mathematical explorations.

Looking back at the specific functions of explanations cited in this paper, we see much in common with the literature background. For example, both Yackel (2001) and Krummheuer (2000) claim that explanations are often given in a narrative format, conveying what was done in order to solve a problem. This is, in essence, the first function mentioned above. However, an explanation may also clarify the rationality of an action (Krummheuer, 2000), which is the basis for the second function mentioned above. The rationality of an action may or may not be based on mathematical properties. Thus, the third function of an explanation might be to specifically ground an activity in mathematics. Finally, in line with Nunokawa (2010), the fourth function of an explanation could be to lead students to new understandings and knowledge. However, despite the relationship between explanations and proofs found in the literature (e.g. Hanna, 2000), at this age there is no mention of proofs in either document and the term justification is hardly used. The absence of these terms is notable when considering studies which have shown that young children are capable of proving or refuting conjectures raised by themselves and others (e.g., Stylianides & Ball, 2008).

We do not believe that our way of categorizing the above examples is the only way. In fact, we are in the middle of an international comparative study investigating the uses of explanation-related terms in mathematics curricula in four different countries. Initial results indicate that explanations may also serve other purposes such interpreting day-to-day occurrences in a mathematical way (e.g., explain what it means when the carton of milk says it contains 3% fat) and clarifying personal view points (e.g., explain why statistics is important). We also do not believe that each of the examples we presented in this paper necessarily falls into exactly one category and not another. Much is dependent on the classroom context. For example, the elementary activity presented under the first function could lead to explanations of the third type based on children's knowledge of numbers. This might be encouraged if the teacher were to ask for a general statement concerning the combination of prices leading to 15 NIS, perhaps an explanation that if one toy costs more than 7.5 NIS, then the second toy must cost less. Likewise, the preschool activity presented for the second function of an explanation may lead to additional exploration and comparison of lengths and measurement, thus qualifying it for the fourth function of an explanation.

Taking into consideration that our interpretations are just that – our interpretations – one might ask, why bother analysing how explanation-related terms are used in curriculum documents. To begin with, we wanted to raise the issue that the different explanation-related terms are open to interpretation and that even among mathematics education researchers, the same word may be used but with different meanings. In addition, as mentioned in the introduction, these documents are used by teachers and others when planning lessons and activities. Our analysis can be used as a preliminary investigation into how the curriculum may be interpreted by others. Our analysis may also serve as a guide to others in understanding how explanations can serve different purposes. Finally, we hope that our study will lead others to investigate the use of the term 'explanation' in additional contexts, such as text books and curricula materials and perhaps become more sensitive to the different purposes and functions that explanations, and perhaps justifications and proofs, may have when teaching mathematics at all ages.

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SELECTING SHAPES – HOW CHILDREN IDENTIFY FAMILIAR SHAPES IN TWO DIFFERENT EDUCATIONAL SETTINGS

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To investigate the geometric competencies of children from 4 to 6 years old in two different educational settings – England and Germany – 80 children were given geometric tasks via clinical interviews. The children were interviewed at the beginning and at the end of one school year. In this paper, the results of one task – the selecting shapes task – are illustrated with the focus on children's conceptualisation of geometric shapes as well as their reasoning why certain figures were chosen as representatives of a geometric shape and why others were not.

Keywords: concept formation, geometric shapes, preschool, clinical interviews

INTRODUCTION

In the last couple of years, the importance of early learning has been widely discussed. One of the remaining questions is how the education in the early years should look like. The study at hand investigates the geometric competencies, in particular the concept formation, of children from two countries with two different concepts of elementary education: Germany (Baden-Württemberg), where learning through play and with this a constructivist view of learning is at present the main concept for kindergarten education (Schäfer, 2011; Rigall & Sharpe, 2008) and England, where the elementary education is rather systematic, curriculum based and rather instructive, and where the competencies of the children are tested via "stepping stones" which they should have acquired. There, the children enter school in the year when they have their fifth birthday, but many children go to a reception class before that. So the entering school age is about two years earlier than for children in Germany. The focus of this research is on describing the competencies the children of each educational setting acquire. As a consequence of the study, hypotheses will be formulated to what kind of competencies each way of education might lead. In this paper, one task is chosen to support some findings leading to such hypotheses.

THEORETICAL BACKGROUND

For the presented shape selection tasks, the competencies that are needed are (1) being aware how a certain shape in several varieties looks like, (2) being familiar with the properties of the single shapes, (3) being able to verbally express these properties, and (4) being able to distinguish examples of shapes from non-examples of shapes. In the following, it first will be illustrated what constitutes a concept in general, followed by two theoretical models of concept development, before some empirical results of previous shape selection studies are presented.

According to Vollrath (1984) a comprehensive conception of geometric shapes is shown through being able to (1) name the shapes, (2) give a definition of the shapes, (3) show further examples of this category and (4) name all properties. However, this description was given for secondary school children. Concerning the development of such concepts, there are different suggestions as for example the two theoretical approaches proposed by Szagun (2008): First, the "semantic feature hypothesis", where the general features are learned before specific features and where the features are either present or not and apply for every member of the class, e.g. "all kinds of dogs belonging to the category "dog" are four-legged and bark". And second, the "prototype theory", which is the generally more accepted theory, some members of a category are categorised as more typical than others (Szagun, 2008). However, in order to give a complete picture of what we know of the geometric concept formation, how a concept develops has to be complemented by research findings on geometric concepts.

With the observations of Piaget & Inhelder (1975), research focusing on children's concepts of space and geometric shapes began. His topological primary thesis stated that children first realize topological features, such as "open", "closed", "interior" and "exterior". According to Piaget, the children are not able to name and to distinguish between geometric shapes before the age of six.

Another body of research has focused on children's reasoning about geometric concepts that they have formed (van Hiele & van Hiele, 1986). The van Hieles, who also created a hierarchical developmental description, constitute that children realise shapes as whole entities from the age of four onwards and are not able to distinguish shapes by their properties before primary school (from 6/7 up to 9/10). Several studies (e.g. Battista, 2007; Burger & Shaughnessy, 1986; Clements & Battista, 1992; Lehrer et al., 1998) concluded that such a hierarchic developmental description is not discrete or independent and that students also preferred different levels for different tasks. Some research also proposes that the characteristics of the single levels develop at the same time but in diverse intensity (Clements & Battista, 1992; Lehrer, 1998).

The shape selection task that will be shown in the following was already conducted in several studies (Burger & Shaughnessy, 1986; Clements et al., 1999; Razel & Eylon, 1991). To summarise the main results of these studies, most children identified circles accurately, only a few of the younger children chose an ellipse and another curved shape as circles (Sarama & Clements, 2009). The squares were also identified fairly well in these studies, between 80% and 90% of the children identified them correctly. Clements et al. (1999) found that children had some difficulties in selecting squares, for they were less accurate in classifying squares without horizontal sides (Clements, 2004). There are no circles deviating from the prototype and square prototypes only occur concerning position. Consequently, the children had more difficulties in recognising triangles which were identified correctly by about 60% of the children. Some studies (e.g. Burger & Shaughnessy, 1986 or Clements et al., 1999) revealed that children's prototype of a triangle seemed to be an isosceles triangle. The majority of children did not identify a long and narrow, scalene triangle as a triangle, although they often admitted that it has three lines and three corners.

Another research also concerning the shape selections of children was conducted by Tsamir, Tirosh and Levenson (2008) in order to examine whether there are prototypical non-examples, when children are asked to determine whether a figure is a triangle or not. They found that some figures were intuitively identified as non-examples for triangles and that more children correctly identified non-triangles as such. Another study (Levenson et al., 2011) dealt with the question what it means for preschool children to know that a shape is a triangle. They investigated whether all examples and non-examples are created equal. They found that over 90% of the reasons given by the children were based on the essential attributes of a triangle.

The study at hand complements these previous studies by investigating the competencies of children in two different educational settings and by illustrating children's understanding of geometric shapes in the light of these different settings.

EMPIRICAL STUDY

Research Questions

The underlying research questions are:

(1) Concerning the choice:
What kind of representatives do children select for a certain shape?
How do they explain their selection and how do they justify the attributes that make a figure a representative of a certain shape?

(2) Concerning the two points of investigation:
What differences can be described after a year?

(3) Concerning the two different educational settings:
What differences between the results of the children from different educational settings can be observed?

Subjects

The research gathered 81 four to six year old children, of which 34 are of English nationality and were attending a local primary school, near Winchester. The age of the children at this primary school ranges from four to eleven years. The other 47 children were from Germany and attending a kindergarten in Karlsruhe, where children from the age of three to six, up to primary school, can go.

Method

The study was conducted in the form of qualitative interviews, in order to get as much information about children's knowledge about geometric shapes as possible, although it might illustrate an "artificial" situation. The order of the tasks as well as the material was predetermined but in accordance with the nature of qualitative interviews this order could be altered or complemented if some of the child's answers happened to be interesting or leading into another direction worth being examined. There were altogether two points of investigation, without a special intervention, one at the beginning of the school year in October 2008 and one at the end of the school year in July 2009. Thus, it must be taken into consideration that the English children were instructed in geometry during the year. The German children also had some kind of "instruction", because a lot of material with geometric shapes was available in the German kindergartens and the children could choose to play with these materials.

Tasks

In order to investigate children's knowledge of shapes and to illustrate the concept formation of the children, five different tasks were conducted in the interview of which the shape selection task – identifying and discerning shapes – will be explicitly presented in this paper. Here, the children were asked to "put a mark on each of the shapes that is a circle" on a page of separate geometric figures. After several shapes were marked, the interviewer asked questions such as: "Why did you choose this one?", "How did you know that one was a circle?", "Why did you not choose that one?". A similar procedure was conducted for squares and triangles.

RESULTS

In the following, the main results are presented by illustrating both measuring times, distinct by country. The answers of the children were thoroughly examined and then categories were generated to which the single answers of the children could be grouped to. Sometimes, the answers of the children could be grouped to more than one category, therefore the added percentages could be more than 100%. In other cases, it is not illustrated if the children didn't mark anything, so the added percentages might be less than 100%.

Identifying circles

The children were shown a picture with nine correct circles and six other shapes, like an ellipse, two twisting shapes, a semi-circle, a triangle and a square (cf. fig. 1). At both points of investigation, the majority of the children (81% of the German and 82% of the English children at the beginning and 84% of the German and 76% of the English children at the end of the school year) could distinguish between circles and non-circles correctly. In this case, "to distinguish correctly" basically means that they marked only circles as such and that they didn't select non-circles as circles. Children who could not distinguish between circles and non-circles correctly, either marked additionally other shapes or left some of the correct ones out. At the beginning of the school year, the English children either marked the ellipse (9%) and the semi-circle (3%) additionally as circles whereas the German children, although not many, chose one of the two twisting shapes (no. 8 (2%) or no. 10 (2%)). At the end of the school year, more German (14%) than English children (6%) additionally chose the ellipse as circle. As before, the semi-circle was only chosen by English children as circle but by none of the German children.

The reasoning of the children could be grouped into the following categories (cf. tab. 1): (1) no justification, (2) visual (referring only to the appearance of a circle), e.g. "it looks like a circle" or "has (hasn't) the shape of a circle" or (3) expressed by gestures "is not a circle, for it looks like this" (then they were for example drawing the shape with their fingers into the air)", (4) using comparisons to other shapes or objects, e.g. "is not a circle because it looks like an egg", (5) naming properties, e.g. "this is a circle, because it is (isn't) round" or "is not a circle for it has corners" or (6) using the proper terms for the shapes, where the children just named the shapes: "this is a circle" or "this is a semi-circle and that is a triangle". Whatever justification was chosen, it was always correct, meaning that if a child used for example gestures to justify a circle it was drawing a circle with the fingers but not drawing another shape.

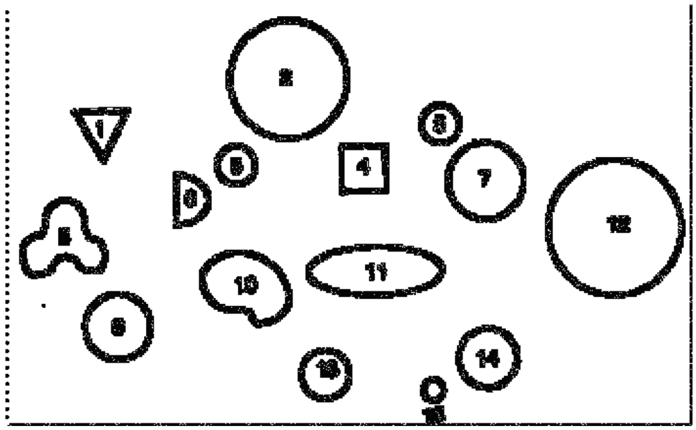

fig. 1: Identifying circles (Razel & Eylon, 1991, in Sarama & Clements, 2009).

	(1)		(2)		(3)		(4)		(5)		(6)	
	G	E	G	E	G	E	G	E	G	E	G	E
2008	9%	12%	0%	9%	19%	18%	23%	15%	21%	26%	5%	21%
2009	0%	3%	5%	6%	2%	9%	47%	15%	60%	18%	12%	38%

tab. 1: Justifications for circles

Altogether, the German children used more comparisons than the English, especially at the end of the school year. At the first point of investigation, about the same amount of children from both countries used gestures and slightly more English children were mentioning properties to explain their choices. At the second point of investigation, more English than German children used gestures and far more German than English children used properties in order to justify their choices.

Identifying squares

At this task, the children were asked to put a counter on all the squares they see. Nearly half of the children in both countries (47% German children and 44% English children) marked only all correct squares at the first investigation. At the second point of investigation, more than twice as many German (44%) than English children (21%) marked all correct squares and nothing else (cf. tab. 2).

At both investigation, clearly more English than German children (more than twice as many at the end of the school year) marked only squares but not all of them. They mainly marked either only horizontal lying squares leaving out squares in other

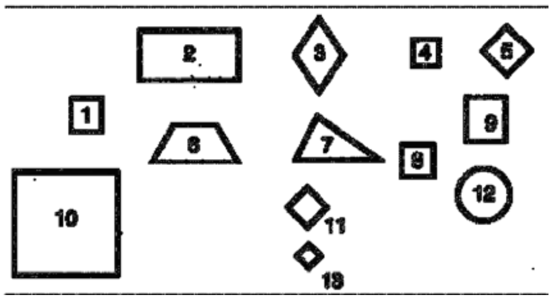

fig. 2: Identifying squares (Razel & Eylon, 1991, in Sarama & Clements, 2009).

positions (no. 5, 11 and 13) or marked only all the horizontal lying square plus one square in another position (no.5). They justified this choice by either saying that "this is too aslope to be a square" or "if you turn a square it becomes a diamond" and that "only in this (horizontal) position it is a square". The choice of the additional in another position lying square no. 5 was justified by for example highlighting that it (no. 5) "is more alike" the horizontal lying squares and "looks different than the other two shapes (no. 11 and 13)".

	Only all the squares		Only squares but not all of them		All the squares and other shapes		Not all squares and other shapes	
	G	E	G	E	G	E	G	E
2008	47%	44%	28%	53%	16%	3%	9%	0%
2009	44%	21%	28%	68%	16%	9%	9%	3%

tab. 2: Selecting squares

Clearly more German than English children at both investigations marked all the squares and other shapes. The other additional shapes that were marked by the German children were mainly rectangles (16% at the first investigation and 21% at the second investigation) but also the diamond (no.3), which was selected by 14% at the beginning and by 12% at the end of the school year. The trapezoid was selected by only one German child at the first and by two children at the second investigation but by none of the English children. In order to briefly summarise the justification of the children, it can be stated that they either reasoned by mentioning (1) the visual appearance or through (2) gestures, (3) comparisons to other shapes or objects, (4) describing their properties (informally or formally) or (5) just naming them with the geometric term. One result is for example that far more German (16% at both investigations) than English children (3% at both investigations) used comparisons in order to justify their choice of examples and non-examples of squares.

Identifying triangles

Here, the children were asked to put a counter on all the triangles they see in the picture (cf. fig. 3). Altogether, there were six triangles (no. 1, 4, 6, 8, 11, 13), two shapes consisting of triangles (no. 10, 12) and six non-triangles (no. 2, 3, 5, 7, 9, 14). The triangle selection task was more demanding than the circle or square selection task, because there were less "intuitive non-examples" (e.g. Tsamir et al., 2008), meaning shapes that are clearly no triangles, but more "non-intuitive non-examples" or "nearly triangles", lacking one attribute as for example straight sides (no. 3, 5, 7 and 14) or three corners (no. 9). So, only a few children marked all the correct

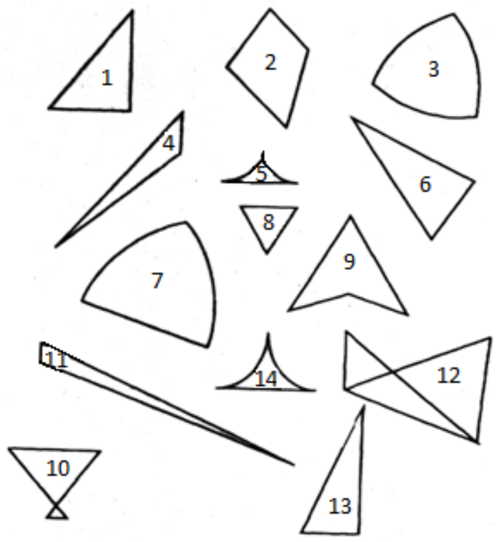

fig. 3: Identifying triangles (Razel & Eylon, 1991, in Sarama & Clements, 2009).

triangles and only these (cf. tab. 3). About a quarter of the children in both countries selected at the first investigation all triangles and other shapes. At the second investigation, there were clearly less German and about the same amount of English children selecting all triangles and other shapes. The majority of the English as well as the German children marked not all the triangles and other shapes.

Examining which shapes were selected most often, it becomes obvious that most of the children marked the equilateral triangle (no. 8) as such, although it's representation was upside down, immediately followed by the right-angled triangle (no. 1). The "nearly triangles" no. 14 and no. 9 were chosen more often as triangles as the two scalene triangles no. 11 and no. 4.

	Only all the triangles		Only triangles but not all of them		All the triangles and other shapes		Not all triangles and other shapes	
	G	E	G	E	G	E	G	E
2008	5%	6%	9%	26%	28%	24%	56%	44%
2009	12%	9%	26%	29%	14%	26%	44%	35%

tab. 3: Selecting triangles

Altogether, there were more children marking triangles as triangles than children marking other shapes as such. At the beginning of the school year, more German (40%) than English children (15%) marked convex shapes (no. 3 and 7) or the four-sided shape (no. 9) (65% of the German and 47% of the English) as triangles and about half of the children of both countries marked concave shapes (no. 5 and 14) as triangles. At the end of the school year the English children tended to mark slightly more frequently the concave shapes and to the same amount the four-sided shape (no. 9). The children often explained the acute triangles (no. 4 and 11) as being "too pointy", "too thin", "too long" or "too aslope" for a triangle or as in the case of the right-angled triangles as "too straight at the side", "only one long side" or "so high at one side". Altogether, the children often explained in an informal way why certain shapes – actually correct triangles – were not triangles, as for example: "too aslope for a triangle" or "not equilateral". The non-triangles were often justified as triangles by using parts of a correct definition such as: "three sides and three corners" without regarding that the sides were not straight or There also were children, especially English children, who could give a perfect definition of a triangle at a previous task, e.g.: "a triangle has three corners and three straight sides" but were still choosing at the shape selection tasks concave or convex shapes, obviously not having straight sides or selected no. 9 as triangle, which nearly resembles an equilateral triangle.

DISCUSSION

Altogether, the German children, although not formally instructed, more often selected correct circles and correct squares than the English children and about the same amount of children from both countries selected all the correct triangles. However, the English children chose less often wrong shapes as triangles, which shows that they are more familiar with what does not constitute a triangle, presumably due to the instruction in school. The reason why the German children identified circles more correctly is that the English children sometimes simply didn't mark one or two of the circles, but even the explanations of the children didn't reveal why they left some correct circles out. Still, what became obvious in the reasoning of the children is that the German children used far more comparisons than the English in order to explain their choices (cf. Maier & Benz, 2012). The English children rather used the names of the shape in order to explain why a shape is or isn't a circle, e.g.: "this is not a circle because it is an oval" or "that is not a circle, it's a semi-circle". Such terms were not familiar to the German children who, for this reason, rather chose comparisons, such as: "this is not a circle, for it looks like an egg". Why there were far more German than English children at the end of the school year using the properties of a shape to justify their choice does not become obvious through the interviews. Still, other research suggests (e.g. Levenson et al., 2011) that "young children, even those who do not attend a preschool with an especially enriched geometrical environment, (still) employ reasoning with attributes" (ibid. p. 28). This could be the case, because there were now more English children using the correct geometric term and the German children were rather familiar with some properties such as "round" or "acute" than with the geometric term. Another reason for the sometimes different choices of the children could be the material that is used in the single institutions. Consequently, one reason why the English children might only describe a horizontal lying square as a square and one that stands on one of its corners as "a diamond" and not a square anymore, could be the illustrations in the classroom or school books, only showing squares in horizontal position. The reason why they preferred marking isosceles or equilateral triangles as triangles, might be because the material they use for exercising only shows isosceles or equilateral triangles. If the children do not have a checklist of attributes in their minds, they are likely to just choose figures they know, predominantly prototypes. Still, the English children selected less often wrong shapes as triangles than the German children, revealing that the Germans were not introduced at this stage to aspects of definitions of a triangle with how a triangle has to look like. Therefore, the fact that a triangle should have *straight* sides was often not considered by German children.

CONCLUSION

The children of the research seem to conform to a large extent the first van Hiele level (the visual level) or somewhere between the first and the second (descriptive) van Hiele level. Still, the children could be on different levels for different tasks, so the levels seem not to be discrete as previous studies already revealed (e.g. Burger & Shaughnessy, 1986; Clements & Battista, 1992; Lehrer et al, 1998). Furthermore, it can be concluded that the way of teaching as well as the used material are influencing the concept formation of the children. Therefore, it should be discussed *how* to introduce shapes and how to actively support the children's concept formation to develop a comprehensive knowledge about shapes, rather than first clarifying *when* would be the best time to introduce geometric concepts to the children. In order "to know" a shape, being able to identify a wide variety of examples and non-examples is essential (c.f. Levenson et al., 2011). A careful introduction of shapes is important, because research indicates that a lot of educational materials introduce children "to triangles, rectangles and squares overwhelmingly in limited, rigid ways" (Sarama & Clements, 2009, p. 216) as was assumed in this study as well, and moreover that such rigid visual prototypes can rule children's thinking throughout their lives. The visualisation and usage of material in class rooms might be too limited as for often only prototypes are shown. However, the usage of prototypes can be very helpful but teachers as well as kindergarten educators should be aware of the variety of representatives of a certain shape and let them explain what properties a shape needs to have in order to be called "a triangle" for example. An isolated memorising of definitions is to be seen critically and more emphasis should be placed on being able to connect a concept with many representatives as examples. Still, even without being formally instructed but instead informally, children are able to acquire a comprehensive knowledge of concepts.

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A FOCUS ON HIDDEN KNOWLEDGE IN MATHEMATICS THE CASE OF "ENUMERATION" AND THE EXAMPLE OF THE IMPORTANCE OF DEALING WITH LISTS

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The aim of this paper is to bring some elements (based on French research works) for the discussion. The curriculum of French preschool is briefly presented as well as the context of a French research which underlines that some kinds of knowledge are not identified as such but are required for the learning of counting in preschool (and also in other levels of mathematics learning). The knowledge "enumeration" is brought to light and defined (Briand, 1993). Another kind of situation is also developed regarding the construction of lists and their reading (Loubet & Salin, 1999-2000). The conclusion opens on the international perspectives of these kinds of results.

THE CURRENT FRENCH CURRICULUM OF PRESCHOOL

This curriculum (BO - 2008, June, p. 12) applies to children from 3 to 5 years old. It is split into six parts: discovering language; discovering writing; becoming a pupil; physical development; knowledge and understanding of the world; creative development. Mathematics are integrated in the wide field "Knowledge and understanding of the world" which contains the following components: discovering the objects, discovering the matter, discovering the living, discovering the shapes and the magnitudes, approaching quantities and numbers, finding one's bearing in time and space. The main advantage of such an approach is to promote an interdisciplinarity that avoids the compartmentalization of disciplines at school. The inconvenience lies in the fact that there are no clear directions for the teachers, who have to construct an organized teaching of mathematics. In this curriculum, there is no prescription for the didactical approaches that should be used in the classrooms. Moreover, some mathematical terms are vague. That is the case of the question of "classification", for instance. This knowledge is dispersed in several paragraphs of the curriculum (of preschool and primary school): the mathematical words (to classify, to sort, to order, to list, to define, etc.) are often neither used nor linked in a correct manner (but that would require another article focused on the topic).

CONTEXT OF A FRENCH RESEARCH

Sometimes, children require some specific hidden knowledge (i.e. not taught and not identified as such in the curricula) in order to learn an identified knowledge of the curriculum (BO - 2008, June). The teacher cannot take it into account in his teaching either because this knowledge is diffuse or because it constitutes a kind of metaknowledge. It is very difficult for a teacher to grasp a knowledge which is not an identified knowledge in mathematics, and to connect it to some other taught kinds of knowledge; that raises the question of pre-service and in-service teacher training and the question of the components of the curricula.

The main question for a didactical research is twofold: what is precisely this hidden knowledge? How can one design didactical situations to develop this knowledge? Researchers have dealt with these questions in France for a long time (in the eighties, their results were published in the nineties and are now revisited), regarding the field of reasoning, of geometry (Berthelot & Salin, 1992), of logic (Loubet & Salin, 1999-2000), of the construction of the first numbers as well as for arithmetical operations or combinatorics (Briand, 1999 for instance; Margolinas, 2012). The main theoretical background of this research is the Theory of Didactical Situations (Brousseau, 1997), which is made up of two main steps: finding problems where the knowledge aimed at is the best tool for the resolution of the problems, and, starting from such problems, designing (a)didactical situations [1] suitable for students of a given level. Such a design uses several theoretical tools, mainly the notions of devolution of the problem [2], didactical variables [3], and institutionalization of knowledge and processes [4]. The Theory of Didactical Situations identify four stages in a learning (devolution of the problem; action of the students; formulation; and validation, mainly without the teacher, in the situation). The question of designing adidactical situations works with the question of the validation.

Brousseau (1984) and Briand (1993) have underscored several preliminary steps in the learning of counting. One of them is the notion of “collection”. Indeed, in order to count, a pupil should perceive the object “collection” in order to measure it. To determine a collection brings a measurable space from a mathematical point of view; it requires several skills for a pupil including among others “pointing” (i.e. the act by which one gives a sign to an object but without structuring the set of objects; it can be the finger-pointing) and “denominating” (i.e. the act by which one gives a name to each object; it implies a structure of the collection). In the various situations designed by French researchers involving collections and actions on them, the first ones can deal with lists (to point a collection for instance, but also to match a picture and an object and then to elaborate a code to point several objects, and to represent the structure of a collection). Then, several knowledge and skills are prior to counting, such as: Constituting a collection, Dealing with lists (when memory is not enough to remember the collection), Drawing collections and each element of them, Coding the elements, Decoding a list, Pointing, Denominating, Representing the structure of a collection. Such skills are now revisited and experiments are conducted in the current school context.

This paper will deal with two kinds of fundamental knowledge prior to counting: to enumerate and to deal with lists. What enumeration is will be explained first and will be followed by starting situations dealing with lists (a wider set of didactical situations is available in Loubet & Salin, 1999-2000 and will be discussed during the Congress).

ENUMERATION: A CRUCIAL KNOWLEDGE

A hidden knowledge which is required for counting

Several research works deal with the construction of the concept of number, exploring the manipulation of collections in particular. I will not develop here the results of Fuson et al. (1982) regarding children's elaboration of number word sequences and the five counting principles of Gelman & Gallistel (1978) (the one-one principle; the stable-order principle; the cardinal principle; the abstraction principle; the order-irrelevance principle). I will highlight a hidden knowledge which is required for counting.

The following situation comes from a wider study (50 pupils, 6 years old – see Briand, 1993; this kind of study is now revisited by Margolinas (2012) who underscores the importance of the results of Briand and proves that these results also overlap the didactic of French). The pupils have a sheet where trees are drawn (Figure 1). The teacher asks the number of trees. The pupils can draw on the sheet, as they want.

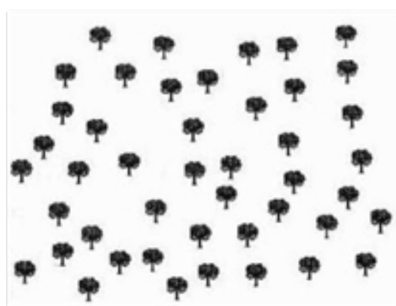

Figure 1 – Trees (in this case, the pupils have to structure the collection in order to count the trees)

In order to succeed, the pupils have to take the following steps:

1) *To be able to distinguish two different elements of a given set.*
2) *To choose an element of a collection.*
3) To say a number word (which is "one" or the successor of the previous number word in the continuation of "words-number").
4) *To keep in mind the set of the chosen elements.*
5) *To identify the set of elements which are not yet chosen.*
6) *To start again (with the collection of the elements that have not been chosen yet) 2-3-4-5, as long as the collection of the objects which are to be chosen is not empty.*

7) *To know when one has chosen the last element.*

8) To say the last number word.

In this series of actions required for counting, only the steps in straight (steps 3 and 8) refer to the numerical sequence. What Briand (1993) calls "inventory" is a task represented by the steps in italics: the pupils have to look over all the elements of a finite collection one time and only one. This task characterizes a non-taught knowledge, called by Briand "enumeration". Enumeration is clearly linked to the one-one principle (Gelman & Gallistel, 1978). The number word is a tag (and such a tag is a convention) but a tag is not necessarily a number word. Learning enumeration is not an explicit part of teaching. It is a paradox that "enumeration" is not considered as a fundamental component of counting up activities or measurement of discrete quantities (for the construction of first numbers, as well as for arithmetical operations or combinatorics). The research of Briand (1993) (following the research of Brousseau, 1984) proves the necessity of activities involving "enumeration" in the pre-numerical domain. The following excerpts underline its existence.

Analysis of some strategies: the lack of the "enumeration" knowledge

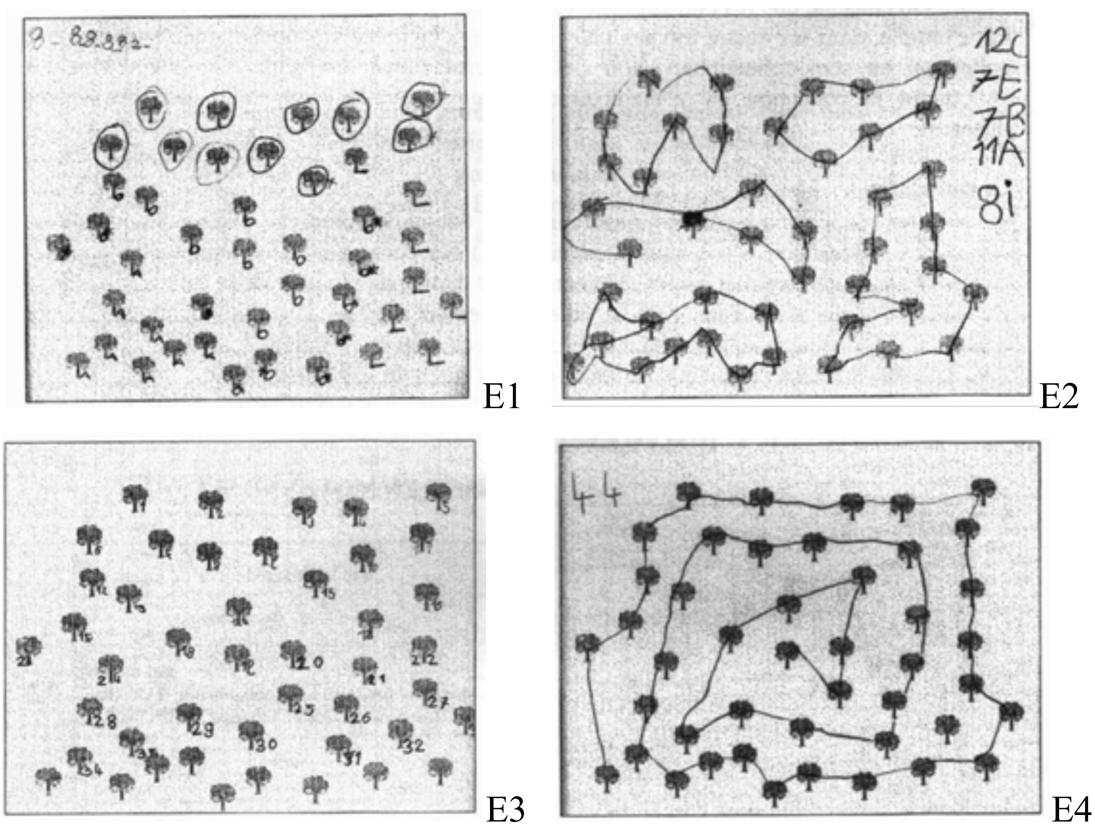

Figure 2 – Four processes (pupils are 6 years old)

Several strategies can emerge (see Figure 2). All of them need an organization of the space in order to take into consideration all the elements of the collection, without forgetting one of them. The lack of such a spatial organization implies difficulties in counting.

The first two pupils structure the collection into subcollections.

> E1: this pupil builds subsets and counts trees of each subset with a specific marking. A partition of the whole set of the trees is done. He builds the writing '8 8 8 8 8 2'. He misses the tree with a circle and a cross.
>
> E2: some trees are linked and then constitute a subset. A partition of the whole set of the trees is done. A letter represents each subset.

The other two pupils use an order to act on the collection.

> E3: this pupil explores the collection by line. The numbers are written but the pupil stops her/his counting (at 35) because there is no more line and the pupil "does not know" where 36 is or where she/he can mark 36 (although this pupil can count way after 36).
>
> E4: this pupil draws a path like a snail that will facilitate the counting. She/he finds 44 because she/he has counted the number of jumps and not the number of trees.

These excerpts show that the "enumeration" appears as a crucial knowledge. One can define the verb "to enumerate" with the following elements (see Briand, 1993, § 1.4 to 1.7 of chapter 1):

To enumerate: this is an action that involves taking into consideration each element of a collection, one and only one time. This verb refers to numbers from an etymological point of view, even if this action does not require the knowledge of the numbers. But the action "to enumerate" is a necessary action to count a collection. The enumeration implies the mobilization of several kinds of mathematical knowledge (from the exploration of space to combinatorics): it depends on the nature of the collection (if the objects are visible or if the objects are defined by properties).

A fundamental situation for enumeration

Briand (1993) has designed a fundamental situation for enumeration, for pupils between 4 and 5 years old. A didactical analysis of this situation will be developed during the working group with the help of videos.

Figure 3 – The matchbox situation (pupils 4-5 years old)

A pupil has, just in front of her/him, several opaque matchboxes with a small hole (in order to put a match in the matchbox) and a lot of sticks (which are "safe matches") in a plastic box. The task is the following: each pupil should put one and only one match in each matchbox and then declare that she/he has finished. When she/he thinks that she/he has successfully accomplished the task, another pupil (or two) comes to verify with her/him, by opening the matchboxes. The pupil has succeeded if there is one and only one match in each matchbox. It is a recurrent task that the teacher can organize for the pupils to work on their own, for instance from November to February.

Several didactical variables can be listed: the nature of the space (here, it is the table, a micro-espace); the number of boxes; the fact that the matchboxes can be moved or not.

An ergonomic fact which both imply a non-built collection and an inferred enumeration

The way matchboxes are placed on the table can impact the learning (as well as the place of the box of matchsticks).

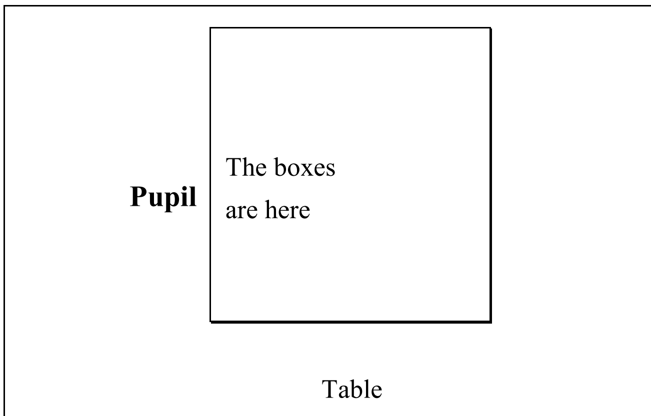

Fig. 4a - Classroom A

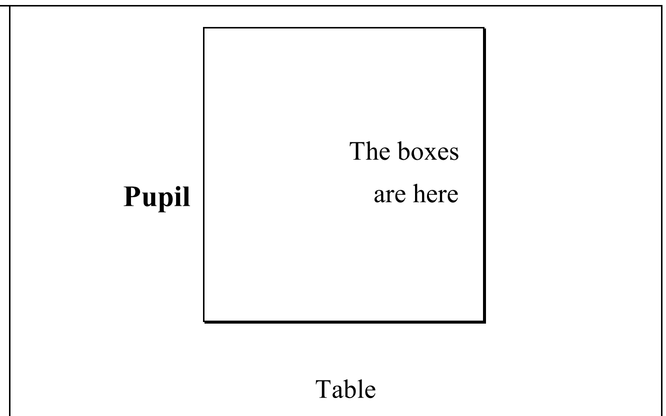

Fig. 4b - Classroom B

In order to put a matchstick in a matchbox, the pupil has to:

1. take a matchbox;
2. take a matchstick;
3. put the matchstick in the matchbox;
4. physically separate the filled matchbox and the empty matchboxes;
5. restart.

To succeed, the pupil has to build the collection of filled matchboxes, from the second matchbox on. In the case of Classroom B, the first stage requires the pupil to take a few steps: the pupil has no reason to mix filled boxes with empty matchboxes. The enumeration is not learnt in such a case. In the case of Classroom A, the pupil consciously has to organize her/his collection of filled matchboxes. The difference in the place of the matchboxes can explain some differences in the results between these two classes.

DEVELOPPING REASONING AND LOGICAL SKILLS WITH THE DESIGNING AND READING OF LISTS

Several activities based on "lists" take part in the development of the logical thought in preschool: building a collection starting from a list, inventing the list as a tool in order to remember a collection or to communicate its content, building lists, building symbols in order to designate objects and then building a list etc. A list represents the easier way to designate non-structured collections of objects. Loubet & Salin (1999-2000) develop a set of situations (experimented with the COREM – Centre d'Observation et de Recherche sur l'Enseignement des Mathématiques, with several classes; all the experiments were analyzed with the tools of the Theory of Didactical Situations).

Introducing the idea of "list" for 3 year old pupils: a challenge

The following steps come from the research work of Loubet & Salin (1999-2000). It will be discussed during the working group.

1) To elaborate a collection: Doggy's suitcase

The main goal of the teacher here is to elaborate a common background for further situations in the classroom. The recurrent story is the following: Doggy is a puppet who brings a suitcase with various objects inside (3 the first day, up to 20 – it requires around 20 sessions and 2 months). The pupils look at the objects and they learn to name them. When all the pupils have touched and named the objects, the teacher puts these objects in the suitcase which is closed and Doggie explains that he will come the next day and bring a new object if and only if the pupils remember all the previous objects.

2) The game of the pictures: the pictures will now take the place of the objects, but the objects are still here. Each pupil gets a picture (at first 1, then 4, and finally 7) and has to take the matching object.

3) The game of the lists: the need to make up a list and to build the elements for its construction

This game takes two days and requires a collection of objects, and two boxes. The first day, the teacher takes some objects from the first box and puts them into the second box. Every pupil can look at these objects and touch them. The box is then closed, nobody can touch it. The second day, each pupil is asked by the teacher to play: if she/he names all the objects of the box, she/he has won. Otherwise she/he has failed. Each time the pupil names an object, the teacher shows her/him this object. If there are still objects in the box and if the pupil thinks that she/he has enumerated all the objects, the teacher shows her/him the objects too.

There are several didactical variables:

- the size of the hidden collections: when the number of hidden objects is small, the pupils can succeed using their memory. Memory is not useful when the number is big, hence the advantage of a list (to play with this didactical variable leads pupils to think that they have to find another idea to succeed);
- the construction of a list (invention of some pupils, diffusion in the classroom and role of the teacher): when pupils fail and think that their memory is inefficient, the role of the teacher is fundamental in order to encourage them to find another process. The knowledge now involved concerns the one-one link between the objects and their designation. It also depends on the following didactical variable;
- the composition of the collection (when the pupils have to construct the designation of the objects): a fundamental step here concerns the moment when pupils have to “decode” the written work of their friends.

CONCLUSION – CONSEQUENCES IN EDUCATION

Teaching requires knowledge that it does not support. Education is so far unable to make a didactic transposition of the enumeration leading to this knowledge. Therefore it is under the responsibility of the pupils. This generates difficulties for pupils and teachers. Current research (Margolinas, 2012 and in press) underscores that some kinds of knowledge such as enumeration are still “transparent”, even if didactical works and official documents (Emprin & Emprin, 2010, pp. 27-29) point them out. And yet teachers have knowledge about enumeration and know about the difficulty involved by enumeration.

The role of the individualization of the processes of the pupils is also important (i.e. the teacher takes care of every process in her/his classroom). The devolution of the adidactical situations to the pupils is fundamental and requires a specific didactical contract. The validation of the processes of the pupils comes from the situation. What kind of posture the teacher should have? How can teachers design sequences integrating such (partially hidden) knowledge? The place and the role of oral and written formulations should be analyzed more precisely.

The role of the teacher is then twofold: he has to present the problem to the pupils, to make sure that its devolution is made, and to convince the pupils that they can succeed in performing the task and that the solution of the problem is up to them. Besides, the institutionalization can both concern knowledge (when it is identified, such as enumeration) and processes (lists). Moreover, recent research (Margolinas & Laparra, 2011) points out that some kinds of knowledge such as enumeration are also involved in other disciplines (French). Here is an example (Margolinas, in press): the pupils (5 years old) have to fill out the paper of Figure 4, using the tools of Figure 5 (scissors, a box, a pencil, and an alphabet).

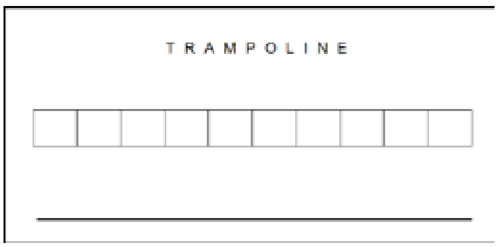

Figure 4

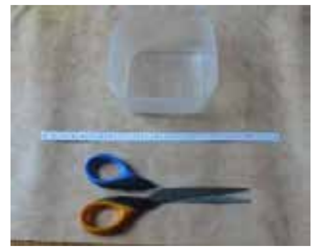

Figure 5

In order to stick the letters in the right position, pupils should make two enumerations (one with the first word "TRAMPOLINE" and one with the configuration of the individual letters-label). That underscores how enumeration is a transversal knowledge and not only a mathematical content. Further research should question more deeply this kind of knowledge and what it implies as far as pre-service and in-service teacher training are concerned. The case of the preschool and primary school with one teacher for several disciplines is a complex configuration, but it reveals transversal knowledge, when researchers coming from several disciplines work together.

NOTES

1. An *adidactical situation* is one in which the students is enabled to use some knowings to solve a problem "without appealing to didactical reasoning [and] in the absence of any intentional direction [from the teacher]" (Brousseau, 1997, p. 30).

2. The notion of *devolution* is "the act by which the teacher makes the student accept the responsibility for an (adidactical) learning situation or for a problem, and accepts the consequences of this transfer of this responsibility" (Brousseau, 1997, p. 230)

3. A *didactical variable* correspond to a potential (yet often implicit) choice for the teacher that modifies the accessibility of different strategies for solving the problem.

4. The teacher gives a cultural (mathematical) status to the new knowledge and he/she requests memorization of current conventions. He/she structures the definitions, theorems, proofs, pointing out what is fundamental and what is secondary.

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数学 [SHUXUE: MATHEMATICS] TAKE A LOOK AT CHINA. A DIALOGUE BETWEEN CULTURES TO APPROACH ARITHMETIC AT FIRST AND SECOND ITALIAN PRIMARY CLASSES

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** University of Palermo (Italy)

This is not about comparative philosophy, about paralleling different conceptions, but about a philosophical dialogue in which every thought, when coming towards the other, questions itself about its own unthought

F. Jullien

ABSTRACT

Aim of this paper is to exploit intercultural dialogue to analyze two cases of task design about straws and word problems in different cultural traditions (the Eastern and Western one). By means of two paradigmatic examples developed in Italy, we aim at showing, on the one hand, the effects and advantages of intercultural dialogue and, on the other hand, the need to take into account and to respect culturally rooted pedagogies, avoiding uncritical transfer from one culture to another.

INTRODUCTION

Why introduce a multicultural perspective in the didactics of mathematics at the beginning of primary school? Why choose specifically China? The positive international results achieved by China, and more in general by the countries of the Confucian area (China, Korea, Japan, Singapore), in the OCSE PISA 2009 judgments concerning mathematics, cannot but raise questions in those dealing with education and particularly with the didactics of mathematics. In particular, it seems necessary to study how and whether these abilities are developed from the first grades of primary school and whether there are paradigmatic elements to be considered in advance. Beside this element there is an important context fact: in Reggio Emilia there is a large Chinese community, which causes the need to constantly create new occasions for dialogue among schools, teachers and social context. For some years, the Universities of Modena, Reggio Emilia and Palermo have started a number of research projects (Spagnolo 1986, 2002; Bartolini Bussi et alii 2009, 2011b), in cooperation with teachers of primary (UNIMORE) and secondary school (UNIPA). These projects aim to question ourselves about “our” (Italian) and “their” (Chinese) teaching methodologies. We interpret this collaboration between school and university in the attitude expressed by G. Prodi: “It is important to maintain contact with school: the 'University - School mixed groups' can play a very important role both in the spread of didactic innovation and in the didactic research.

These groups require the adoption of a really joint relationship: pre-university school teachers invited to participate in a didactic research have to do it fully, and not only as auxiliaries responsible to collect data and protocols." (Prodi, 1991).

In this work we will deal with some aspects of the first arithmetic learning in the early grades of primary school (1st and 2nd grades).

COUNTING RODS: A PARADIGMATIC EXAMPLE FROM THE TEXTBOOKS OF CHINESE PRIMARY SCHOOL

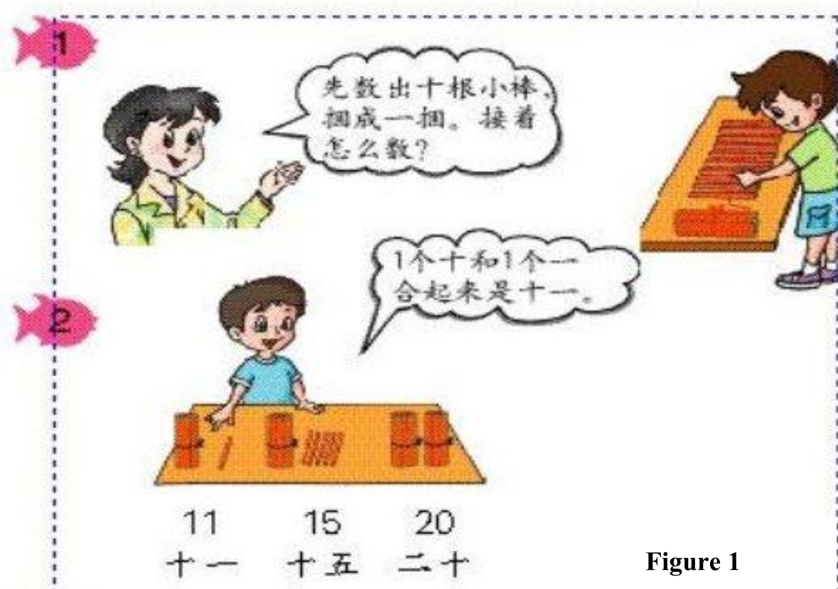

Figure 1

Our analysis focused firstly on textbooks of Chinese primary school as rigid vehicles of 2001 school curricula, reapproved in 2010. In them it seemed to grasp the strategic importance of two artifacts which are typical of mathematics in China: the *counting rods* (for which we used common straws) and the *problems with variation*.

The page below is taken from textbooks for the 1st semester of the 1st grade of primary school. First of all, we would like to draw attention to the use of the counting rods and the practicality the child shows counting and binding them, in order to obtain numerical correspondences.

However, attention should be paid above all to the language. In the second comic strip from above it is possible to note the use of both Arabic figures and their Chinese equivalent.

This seems to lead the pupil, more or less explicitly, to conceptualize the calculation through three different "translations". In fact, let us try to read this page from the bottom up. In the lowest part, correspondence between the two codes can be found (Chinese writing - Arabic writing). This is then exemplified with the image of the counting rods the child is observing and handling.

The pupil's balloon refers to a multiple reading of the used codes, which are useful to the learning child to consolidate positionality. This step has to be attributed to the predominance of methodology on content, which Chinese culture brings along. Indeed, the continuous codes *contamination* is found also in other passages of the textbooks and tends to privilege the use rather than the creation of stable concepts.

To substantiate this statement, it seems interesting to show below two other images, always taken from the same book and following one another (pages 17 - 18):

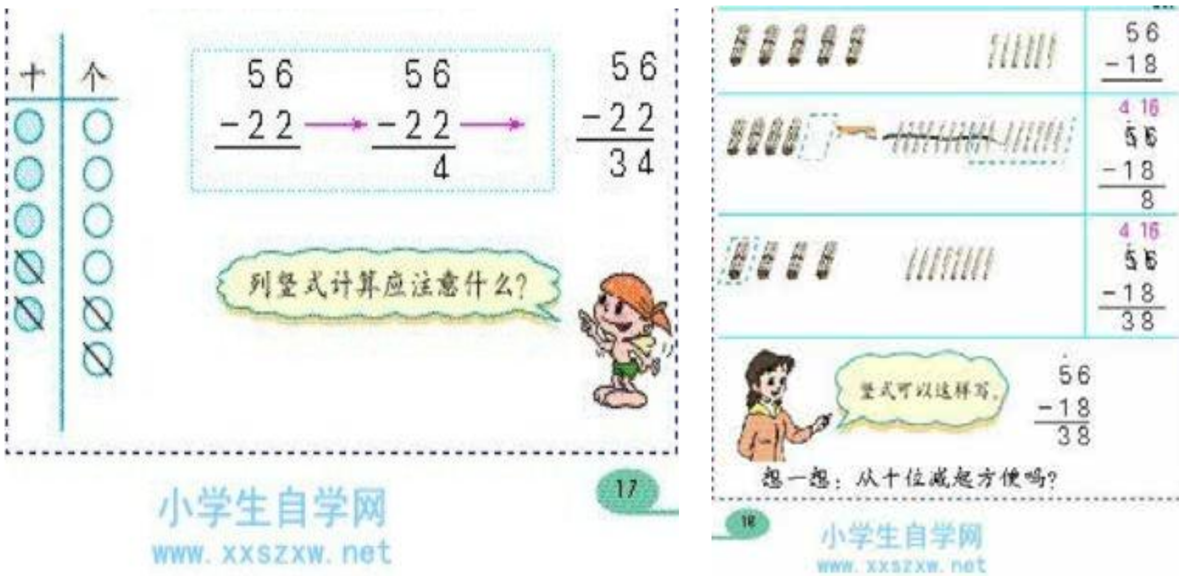

Figure 2

Here it is easy to note how the use of schemes is totally linked to a didactics focused on the research of solving *methods*, rather than on the presentation of isolated contents. This means that children in the early grades of primary school are faced with a text which, on two consecutive pages, modifies the iconic representation, appropriately replacing the column scheme on page 17 with the use of the artifact of the rods on page 18. The former, in fact, would be unsuitable and harbinger of possible cognitive hitches to perform a subtraction with the decomposition of the tens. The solving methods are so perfectly interchangeable because the reason why they are introduced is their way of using them.

PROBLEMS BY VARIATION: TWO PARADIGMATIC EXAMPLES FROM THE TEXTBOOKS OF CHINESE PRIMARY SCHOOL

If we analyse the 1st grade text carefully, we find several examples of formulation of this kind of problems, which, from a given situation, proceed by variation, integrating addition and subtraction. Let us look at picture 3 to the right.

Figure 3

First of all, the book shows a school where the teacher addresses the issue of the problems in a totally decontextualized way. This is clear when observing the blackboard. The represented dinosaurs are divided into a group of 6 and one of 2. What can be noted on the pieces of paper the children hold in their hands is the use of a mathematical writing code to translate the situation the teacher shows them on the blackboard. Referring in particular to the codes on the pieces of papers, it is possible to note how this figure introduces a first seed of problems with variation in which the child handles with a problematic situation that can be solved both with addition and subtraction. What do Chinese children do already from the 1st grade of primary school? They DEAL contemporaneously with what in the West is usually decomposed into four subsequent conceptualizations: addition, subtraction, the strong interdependence between them and the commutative property of addition.

The interdependence between the arithmetic operations is an extremely interesting factor for us, because it refers to acquisitions of western formal mathematics, in which the algorithms are not four, but only two, that is addition and multiplication. The examples pertaining problems with variation which we are going to analyse below will better explain these reflections.

OUR TRANSPOSITIONS: PARADIGMS FOR A DIDACTIC PLANNING

The intention here could be mistaken for an attempt to *translate*, or even worse *import* a Chinese text into the Italian culture and more in general in the Western. On the contrary, the educational paths we are going to analyze, developed by the group of research professors with whom we have collaborated for some years, aim to create real transpositions of the Chinese model into the Italian one. The goal of this research activity, as we said, was to try to understand if it is possible to transpose in the Italian context some of the fundamental elements which characterize Chinese mathematic didactics in the first years of primary school. The first activity was developed by Mr. A. Ramploud in his first class of primary school. He introduced the straws at the beginning of the school year. The didactic activities planning focused on this tool to convey a series of mathematical meanings.

Already from the analysis of the Chinese textbooks, it is easy to discover how, using such a tool, it is possible to try to convey to the children the structure/scheme of positionality. Directly dependent on this aspect is the importance given to the "ten-bundle" and therefore to the decimal system. Another significant element, in this initial analysis of the potentialities of the counting straws, is the fact that they allow also the manipulation of very big numbers (hundreds) already after a few days from the beginning of school.

This perspective seems even more significant if we think that in the first class of the Italian primary school, they approach numbers only up to 20. Finally, by building manipulative calculation paths, it is possible to develop the abilities of one-to-one correspondence object-straw and accompany the path towards the abstraction of numerical concepts. From these analysis, which provides some possible potentialities of the straws-tool, specific tasks for the pupils have been structured. First of all, since we work with first class-children of primary school, we asked them to freely examine the straws-tool.

The children, during this activity, handle the straws and describe their appearance and possible uses. In a second stage, the teacher asks them: "What can the straws be useful for?" The records show different possibilities, going from *playing* "They can be used to play", to *drinking* "I use them to drink" and to *counting* "You know, we can use them to count", "When you don't know how many they are, you can use them to count". On the basis of these indications, the teacher readjusts the didactic activity, proposing further reflections and choosing the most appropriate perspective to convey the desired mathematical learning.

At this point the teacher replies, having laid several hundreds of straws on the table: "You said they are used to count, but how many are these straws?". This situation generates a series of hypothesis made by the children, who give back a ingenuous mathematical learning and start different explorations in order to answer the question. It is possible to appreciate solutions linked to estimates like the following one: "I think they should be 62.99 billions..." or solutions to calculate the amount using distribution: "We can give 10 to everyone". It is clear that this second modality is adopted by the teacher, who provides the class with a new work possibility, introducing the "ten-bundle". In that, a link with the analyzed Chinese text can be seen. Through this procedure of continuous readjustment of the activities, the children get to discover the ten and to count all the straws, finding out that they are 512. These activities have shown an appreciable improvement and strengthening of the skills related to positionality, as observed in subsequent tests.

This path, based on Chinese mathematic didactics, through the mediation of Chinese textbooks, looks also consistent with the theoretical reference frame illustrated by A. Mariotti at the Taiwan PME 2012 (Mariotti, 2012). These elements drive to speak about a real transposition, i.e. the appropriation of methodologies, in order to reconsider one's own way of doing school.

The other example to which we want to pay attention is the one developed by Mrs. L. Maffoni in her first class. Always starting from the introduction of the straws, she followed the path to identify the "ten-bundle". However, here it is essential to focus on a key-element: in the definition of the "ten-bundle" the children created a very interesting neologism, which, used by the teacher, is able to convey a series of mathematical concepts. In fact, when she asked the pupils what they were doing, they answered: "we are '*elasticando*' (The meaning is *binding the straws together using an elastic band*) the straws".

Figure 4

This expression was the occasion to focus on children's informal learning, which can be used to develop out-and-out mathematical concepts, such as ten, composition and decomposition. Apart from this, the teacher decided to further develop the activity with the straws. She chose to incline the children towards additive mathematical problems, submitting them some problematic situations starting from problems with variation. Analyzing the nine problems-scheme (Bartolini Bussi et alii, 2011) during the planning stage, she chose the first row, which provides the children with a static situation with a very short text.

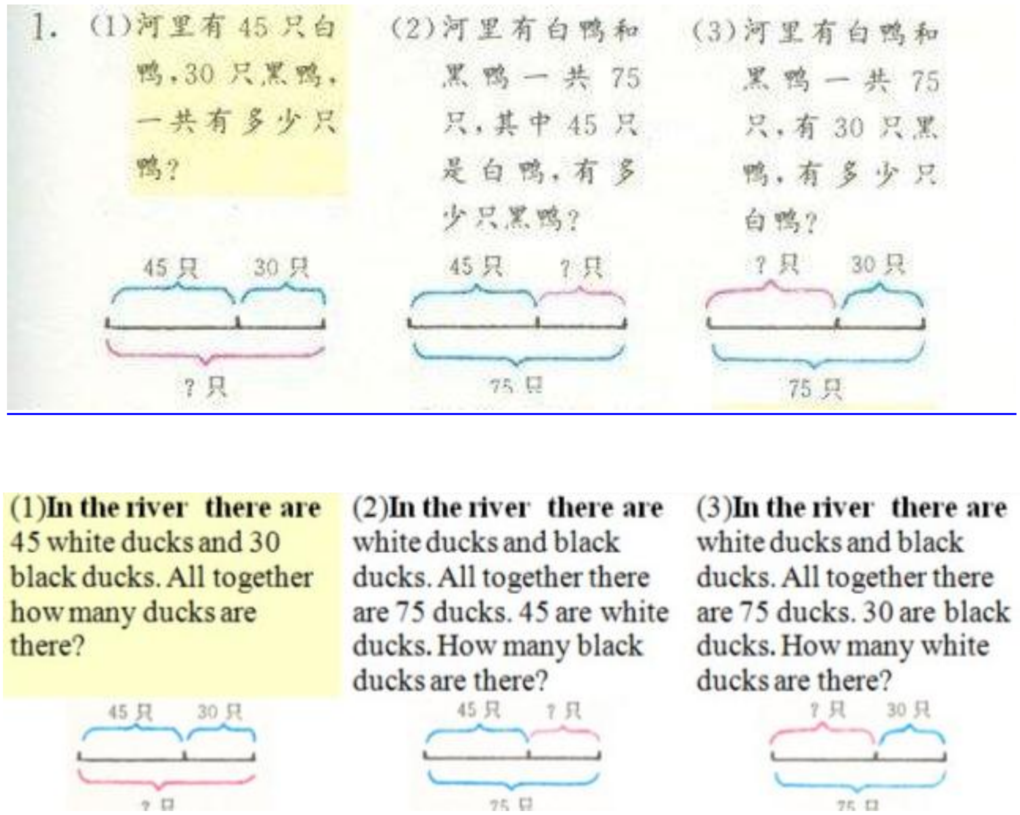

Figure 5

These aspects, of course, are essential for pupils of the first class with initial reading - writing skills. However, the major *gap* in comparison with Italian didactical practice is the simultaneous presence of addition and subtraction, which is hardly found in the Western didactics. Again, we are in front of a necessary process of transposition. Indeed, simple translation and importation of a scheme like the Chinese one in the Western didactics can be unsuitable for the activated paths. In this case, the teacher introduces a further element in Chinese problems with variation: the dramatization of the situation. She proposes different situations to the children, which have been documented on different videos.

Nevertheless, referring to what we consider to be the paradigmatic element, we will concentrate on the following proposal. The teacher sellotapes the straws on the blackboard and asks them to read the consequent situation: "On the blackboard there are 9 yellow and 6 green straws. How many yellow and green straws are there?". The children, invited by the teacher, stand up and near the blackboard to count the straws, or, in some cases, they do the same operation from their desks.

The additive situation is clear: all answer that the straws are 15. At this point, the teacher *varies* the situation (with reference to Chinese problems with variation) and asks: “So, on the blackboard there are 15 yellow and green straws. 9 are yellow, how many green straws are there?”.

In front of this new situation, the pupils seem disoriented and some of them propose the additive situation “they are 15” again, but the teacher points out again: “I asked how many green, not yellow and green, straws are there”. Leading the discussion with the children, she comes to introduce subtraction, which allows them to count the straws and answer correctly: “the green are 6”. The teacher, always referring to the structure of Chinese problems with variation, does not stop and asks: “Well, on the blackboard there are 15 yellow and green straws. 6 are green, how many yellow straws are there?”. The disorientation of some children shows the difficulty of this age group to deal with these variations, but in the discussion a proposal emerges: “All you have to do is to remember the question you asked us before!”. The child, here, starts to reason not about the element and the operation, but about a schematic solving process, which finds the core of this activity in the identification of the relations among the operations underlying the teacher’s three requests.

These examples seem paradigmatic to underline once again how these activities, transposed through the analysis of Chinese books, spur the child to look for solving schemes, which lead him to a methodological learning, perhaps able to create links between different contents, often too atomised in our schools.

To better understand the implications that can be caused by this kind of proposal, we believe it is important to refer to the activity conducted by F. Ferri and documented during the 2011 SEMT (Bartolini, Canalini, Ferri, 2011). It is clear that the blend of the didactic methodologies here discussed and presented to the teacher in training aims to begin the construction of mathematic meanings, which teachers can develop in their classes.

COMMENT ON THE THEORETICAL CONTEXT: FRANÇOIS JULLIEN’S REFLECTION. POSSIBLE TRANSPOSITIONS

Allen Leung, Hong Kong researcher on the concept of variation, but above all friend of ours and privileged speaker, while e-mailing on the problem of transposition, spoke about *learning across cultures*. For us, it means that learning, including that of mathematics, cannot but prescind from a dialogue, a *crossing* between cultures. Let us develop this dialogue, this *crossing* further on. François Jullien (sinologist and philosopher), in an interview in 2005 at the Modena Philosophy Festival, says: “It is not China in itself that makes the thought feel lost with its language, culture and way of thinking, but the fact that, dealing with Chinese thought, a extraneousness is reintroduced, which is not ‘China’, but the sidelong glance on thought that arises reflexively from the extraneousness in which China is”. (Jullien, 2005)

The studies of F. Spagnolo and M. G. Bartolini Bussi present this approach of contamination, continuous dialogue from *distances*, and cultures *crossing* in the didactics of mathematics. Starting from these theoretical points of reference, we will try to reread this passage like this: in order to redraw attention to the Western and specifically the Italian didactics of mathematics, it can be useful to observe it from a perspective of absolute distance: that of Chinese didactics of mathematics, that is, in a school system with cultural and organisational characteristics which are totally *different*, totally *extraneous* and *foreign*. Therefore, F. Jullien becomes the privileged interpreter to try to describe this approach, using the *obliquity* of thought he speaks about (Jullien, 2006). To better explain this concept, we think it is appropriate to evoke an image: let us imagine to be on a becalmed sea, on a ship. Our eyes can wander in any direction and what stands out in front of them, at an always unreachable distance, is the line of the horizon. Similarly, putting ourselves *in the logos*, we place ourselves in the centre of a specific cultural horizon, whose boundaries can really seem unreachable.

However, this interpretation has to induce us also another consideration: this position, placing us in a "centre", seems to allow us to draw a universalistic perspective, which extends the power of the *logos* to infinity. If we do not put ourselves in the logos anymore, but we *go towards the other*, that is, we adopt also the passivity dimension which allows the differentiation process to make other perspectives emerge, than the image changes: we are no more on a ship offshore, where our eyes lost to infinity towards the horizon, *in* the horizon. But, we may imagine to find ourselves in an Iceland "moonscape" , where our feet can really walk on the fault, that is Earth's fissure, the place of the boundary, which is the differentiation process. Why looking from the fault, from the fissure, puts ourselves in front of something different? Because, suddenly, logic hypertrophy shows itself in front of me and you.

CONCLUSIONS

This brief contribution opened with some questions, which are important to resume. Why introduce a multicultural perspective in the didactics of mathematics at the beginning of primary school? Why choose China? As said before, teenagers' positive results in China and in the wider Confucian area cannot but make us think about that specific cultural tradition. Still, this would mislead our intervention, if we would not consider these fifteen-year-olds as the result of a long educational process, whose bases are in pre- and primary school. From this perspective, to understand how Chinese didactic paths are structured, starting from school curricula translated into textbooks, gives rise to a series of reflections, which show how cultural differences can turn into an asset for interpretation and dialogue, providing us new interpretative keys for our cultural and didactic reference points (in this we see F. Jullien's *unthought*).

In confirmation of this path effectiveness, from the didactic point of view, we believe it is sufficient to think over the activities we proposed as transposition examples starting from Chinese textbooks. In them, it is appreciable the fact that *didactic blend* gave positive results right in the very important initial stage of mathematic meanings building, corresponding to the age of 6 to 8 years old. In fact, both positionality and the approach to decimal system have been favoured by this perspective, this introduction of these elements, which are distant but never completely extraneous.

Not only that, but a path for the strengthening of the use of even very big number already in the early stages of primary school has been started. Moreover, the introduction of problems with variation into Italian didactic practice provided very interesting information, which should be examined more in depth.

Among them, we mention the shift of child's reflection from an arithmetic to a pre-algebraic way of thinking. This is proved by the attention of older children for schemes (Bartolini, Canalini, Ferri, 2011). Instead, younger children (6 - 7 years old) focus more on the quest for the unknown element through variation. From this reflection we have thought to create the summer school for teachers, aimed to develop educational paths capable of tackling *difference*.

This research in progress aroused big interest and participation, though only the training part with the teachers has been executed. Indeed, although we forecasted an attendance of 20 – 30 teachers, the registrations have been 87. Besides that, the participation in all three days of training has been the evidence of a deep interest in this kind of job prospect. This educational activity will be developed in the classes of the teachers who participated and expressed their will to continue on this research path.

This will allow us to carry out new broader-spectrum tests, in order to get, by September 2013, a series of data confirming the indications obtained from the mentioned experiments. In conclusion, our intention was to create a course allowing teachers to reconsider the educational perspective from a continuity point of view, where the age group of 3 or 6 to 8-year-olds, has a central value on which the whole children educational path is founded.

Moreover, the didactics they carry out every day in their schools can be contaminated by "more distant" approaches, in order to create transposition occasions, deconstructing pedagogic - didactic categories often too eurocentric.

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FINGER COUNTING AND ADDING WITH TOUCH COUNTS

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This paper describes the design of a digital technology focussed on early number sense (especially counting and adding). This "TouchCounts" application (designed for the iPad) takes advantage of the direct mediation through fingers and gesture of the touch screen interface. Through an a priori *analysis, we show how the perceptual and motor aspects of the application can support the development of cardinality. Using a theoretical perspective informed both by tool use and by embodiment in mathematics thinking and learning, we provide a case study analysis of how a 5 year-old child gains emergent expertise in producing and transforming numbers.*

INTRODUCTION

Current mathematics education software has been developed for the desktop/laptop paradigm of technology use where the mouse and keyboard are essential interfaces. Even software for interactive whiteboards (IWBs) does not take full advantage of touch-screen capacities because the mouse/keyboard interface is the default interaction mode. In addition, IWBs, while providing a social space for interaction, do not allow individual students, or small groups of students, to each interact directly with the software. In contrast, iPad devices permit both whole-class and individual interactions. Also, iPads enable collaborative interaction between two or three students on a single device (as it recognizes multiple inputs from different individuals simultaneously), something that computers, with a single mouse, have not been able to offer. Their small size (book-sized at ~24 x 19 x 1 cm) overcomes obstacles faced by teachers using school computer labs (e.g., awkwardness of obtrusive monitors).

The touch-screen iPads also enable direct mediation, allowing children to produce and transform objects with fingers and gestures, instead of through a keyboard or a mouse. Recent neuroscience research has shown that there is a neurofunctional link between fingers and number processing, and that finger-based counting may facilitate the establishment of number practices (Andres, Seron, and Olivier 2007; Kaufmann et al. 2008; Sato et al. 2007; Thompson et al. 2004). Research has already shown that consistent use of fingers positively affects the formation of number sense and thus also the development of calculation skills (Gracia-Baffaluy and Noel 2008). This suggests that using the fingers to create numbers in a correctly ordered way, with both visual and auditory feedback can support the development of number sense and provide the foundation for successful arithmetic achievement.

The large majority of existing iPad applications are designed for game-like interactions in which learners practice arithmetic operations. Lange and Meany (in these proceedings) described several mathematics-related applications and used Bernstein's (1971) notions of classification and framing to characterise them. They found that none of the applications they used in their study were strongly classified (explicit in the content focus) and weakly framed (open to student agency). As an application specifically focused on number but open to student expressiveness, *TouchCounts* illustrates well this missing category. Indeed, our interest was in developing an expressive technology that supports the development of meanings related to numbers and operations. A similar project has been undertaken by Ladel and Kortenkamp (2011), who have developed a multi-touch-table environment.

DESIGN OF THE *TOUCH COUNTS* INSTRUMENT

Currently, there are two sub-applications, one for Counting (1, 2, 3, …) and the other for Adding (1+2+3+…).

Counting World (1, 2, 3, …).

In this world, learners tap their fingers on the screen to create small numbered circles that are also represented through both symbol (written numeral) and sound (spoken word) as fingers are placed onscreen. In the default mode, gravity makes these circles fall off the screen, unless they are placed on the horizontal bar (see Figure 1). Adding more fingers continues the counting. Fingers can be placed onscreen all at once to create a group of numbers. So, for example, placing five fingers on the screen creates five numbered circles but produces only the word 'five' orally. Every finger touch produces a number; it is thus not possible to move existing objects on the screen.

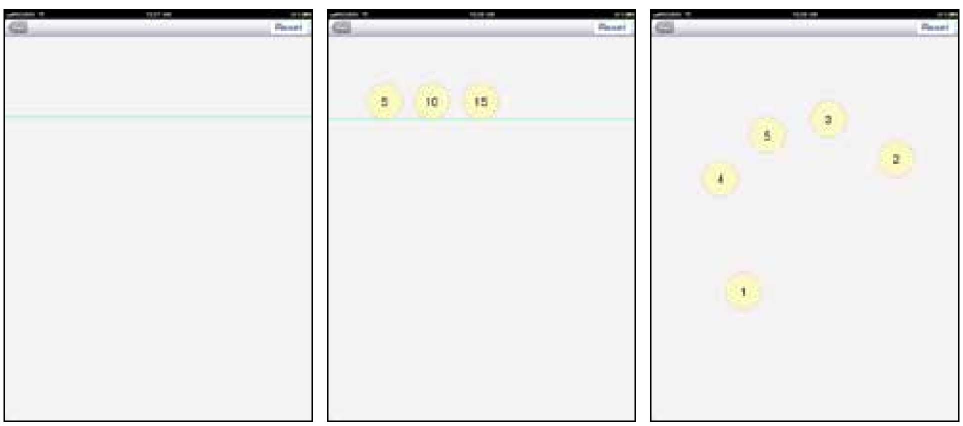

Figure 1: (a) Default Counting world; (b) Default Counting world with numbers on the horizontal bar; (c) No gravity Counting world

The goal of this simple application is to assist young children in developing an understanding of the one-to-one relationship between their fingers and numbers. Children at this age, when asked to count, do not necessarily associate the words for each number with the objects being pointed to (this is often called "rote counting"). They tend to recite the numbers as if it were a song and point at the same time, but not always coordinating the two actions (Fuson, 1988). The Counting world should directly supports two of the five aspects of counting identified by Gelman & Meck (1983): (1) when counting, every object gets counted once and only once (one-to-one correspondence principle); (2) the number words should be provides in a constant order when counting. Further, when the gravity option is turned off, it becomes evident that the last number that is counted is the number of items on the screen, which is a third principle of counting.

Being able to select specific numbers to "pull out" is evidence of having objectified number, that is, of being able to think about a particular number as being more than just an element in the process of counting. More specifically, in order to place a given number on the bar, one must know what the previous number will be. This objectification of number enables the move from ordinality to cardinality, the latter being necessary for answering the "how many?" question.

Nunes and Bryant (2010) have argued that children need to make three types of connections between number words and quantities: "they need to understand cardinality; they need to understand ordinal numbers, and they need to understand the relation between cardinality and addition and subtraction." While the Counting world focuses on ordinality and the objectification of number, the Adding world specifically targets the idea of cardinality.

Adding World (1+2+3, …).

While tapping on the screen in the Counting world creates numbered objects, tapping in the Adding world creates a group of discs labelled by the cardinality of the group (see Figure 2a). Placing five fingers on the screen will create a group of five discs arranged around the circumferences of a circle as well as the numeral 5 in the middle of the circle. This focusses attention on the cardinality of five. As Vergnaud (2008) has argued, understanding cardinality involves more than knowing that the last number in the sequence of counting objects in a set is the number of objects in the set. It involves being able to use numbers in operations and, more specifically, being able to count on. Once two or more sets have been created, they can be added by using a pinch gesture (see Figure 2b), which provides a fundamental metaphor for addition (see Lakoff & Núñez, 2000).

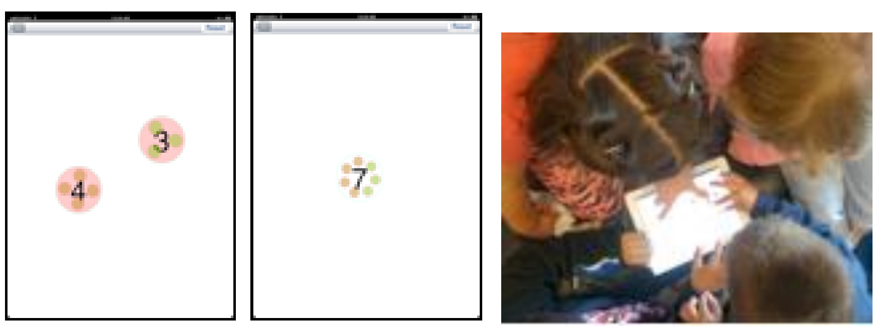

Figure 2: (a) Two groups in the Adding world; (b) resulting sum with colour-based record of the addends; (c) Children using the pinching gesture to add two groups

Thus cardinal numbers are ones that can be acted on (in this case, added). The explicit use of gesture in this application is not only based on the affordances of the iPad device, but also draws on recent research highlighting the important relation between gestures and learning (Goldin-Meadows, 2004), and the recommendation that children be exposed to and encouraged to use more gestures (Cook & Goldin-

Meadow, 2006, Singer & Goldin-Meadow, 2005). When two or more groups are added, the discs in them retain the colour of the original sets so that the sum retains a trace of its construction (see Figure 2b). When two or more sets are added, the value of the sum is given orally. Children need not know how to add before using this environment. And while a teacher might introduce the word ‘adding’ to the task, it does not appear on the screen. As such, words such as “making,” “putting together,” “joining” can all be used to describe the action of pinching sets together. Note that the pinching gesture is symmetric, which means that there is not order implied to the sum of A and B.

For Vergnaud, the student who can answer how many objects are in a set if you add some objects to the set that they have just counted has a sense of cardinality. By having the Adding world groups labelled with their cardinality, the action of counting on is facilitated since the child will focus more on the cardinal number displayed in the group than on the objects in it.

Given the presentations of Sun and Ramploud and Di Paola, we are interested in developing tasks that can help promote a more unitary approach to addition and subtraction. This might involve, for example, asking children to create a group of 7 using an existing group of 3. Current attempts to design “take away” gestures have been discarded either for physical reasons and pedagogical ones.

THEORETICAL PERSPECTIVE

Broadly speaking, we situate our work within the area of mathematics education research that examines the role of technology, tools and artefacts in mathematics thinking and learning (see Hoyles & Laborde, 2010). Within this area, given the tangible nature of the interface, we are particularly interest in the embodied practices that allow learners to interact with digital technologies. While other theoretical approaches such as instrumentalism, sociocultural theory and semiotic mediation do not discount the role of the body in mathematical practice, Nermirovsky’s (Nemirovsky, Kelton, Rhodehamel, 2013) perceptuomotor integration approach focuses specifically on the way that mathematical expertise develops through a “systematic interpenetration of perceptual and motor aspects of playing *mathematical instruments*” (in press, *emphasis in original*). This approach shares many similarities with the emerging body of work in mathematics education that moves away from a mentalist focus on structures and schemas toward a description of lived experiences in which learners’ activities are at once bodily, emotional and interpersonal (Radford, 2009; Roth, 2011).

The perceptuomotor integration approach assumes that mathematical thinking is centrally constituted by bodily activity, which may be more or less overt, and that mathematical learning occurs through a transformation in the lived bodily engagement of a learner in a particular mathematical practice. This approach takes a strong stance toward embodiment, seeing it not just as a precursor or underpinning of mathematical thinking, thereby further promoting a mind/body dualism. Instead, mathematics learning entails transformations in the lived body experience, not just at the primary school age when children interact with physical manipulatives, but for learners of all ages. Thus, taking *TouchCounts* as a mathematical instrument, we will be interested in learners' developing fluency and the concomitant changes in the way they touch, move, talk, gesture, etc.

METHODOLOGY

The interview took place near the end of the kindergarten school year in the resource room of an elementary school located in northern British Columbia. Several children were interviewed, all between the ages of 5 and 6. The interviews protocol was intentionally open-ended since we wanted to see what children would be able to do without specific instruction and what kinds of questions/investigations they would initiate on their own. If the child did not notice particular features/techniques (that the bar keeps numbers from falling down, that the Reset button restarts the counting, that the pinching gesture assembles groups) the interviewer provided an explanation. In addition, the interviewer asked each child, after a period of play, to place a certain number on the bar (usually 5) in the Counting World and to make a group of 7 in the Adding World. Other tasks were given when the child seemed to have exhausted a certain investigation. Our hypothesis in terms of instrumentation was that the children would discover the main features of Touch Counts on their own. In terms of number sense, we hypothesised that the children would like to create big numbers and that both tasks (placing just 5 on the bar and making groups of 7) would be challenging. In this paper, we have chosen to focus on a girl named Katy, who has just recently turned 5. Her teacher described her as one of the weaker students in the class.

RESULTS AND ANALYSIS: ONE-FINGER INSTRUMENT PLAY

We divide the results into two sections, one focussed on the Counting World and the other on the Adding World. For readability, we analyse each section in turn.

Counting World: What kind of number is going to come after?

The session began with the interviewer saying, "Let's start with number." Without any instruction, Katy started by placing her right index finger on the screen and swiping it downward (Figure 3a). She did this slowly, repeating the numbers as she goes (saying some of them out loud, like 2, 3 and mouthing the others). After 9, she put her head down, created a number then repeated 10 out loud (Figure 3b). She lifted her head up at 14 and kept making numbers. At 17, she put several fingers on the screen at once and the iPad said 21. She paused and smiled. She then continued, with her index fingers, to make numbers up to 27, saying the numbers at the same time as the iPad. She looked up, no longer watching the screen and continued swiping and saying numbers (Figure 3c). She had automated her number-making, swiping the screen in a rhythmic way without having to look.

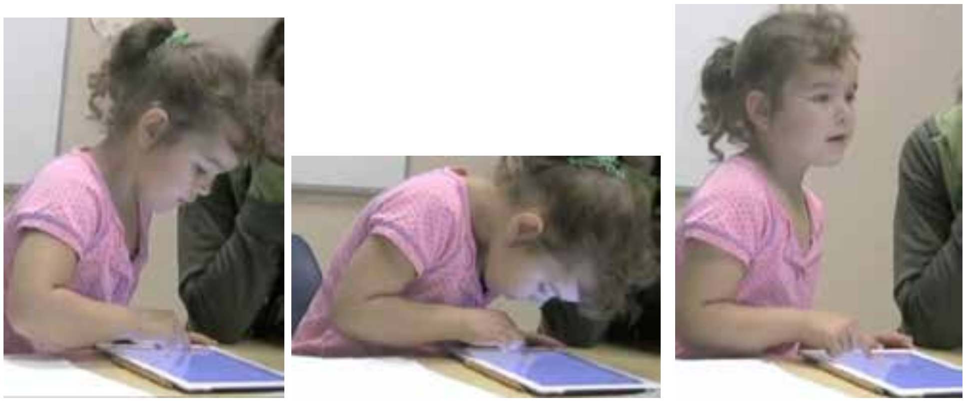

Figure 3: (a) first interaction; (b) bending over around 10; (c) counting beyond 27

After accidentally pressing the Reset button, Katy began counting again. The interviewer then invited her to put numbers above the bar. When she put her next number above the bar, she said, "It stops the number. Why?" She then tapped her index finger more quickly and said, "pop." She continued putting numbers above the line, saying "pop" each time, and lining them up in a row moving from right to left. She stopped at 47, sat back and smiled. The interviewer asked Katy to press the Reset button. The interviewer then said, "I want to see just five up here." Katy tapped with her index finger above the line. The interviewer says, "I don't want to see one." Katy resets (without being asked). She then put 1 above and 2, 3, and 4 below the bar, and then 5 above. She sat back and smiled. When the interviewer prompts her to put just 5 on the bar, Katy resets, put 1, 2, 3, 4 below the bar, saying four out loud. She paused, then put 5 above the bar, saying five out loud.

During this whole interaction, except for a brief multiple-finger tap (perhaps accidental), Katy used her right index finger. However, the way she touched the screen changed from a slow swipe when she first creating numbers, to a quick tap ("pop") when she was putting numbers above the line. As she tried to put 5 above the bar, her tapping got slower. When she succeeded in getting only 5 above the line, her tapping became quicker and rhythmic, suggesting that she could anticipate when 5 would come.

The interviewer then asked whether Katy could put 5 and 10 above the line.

40	I:	Imagine five and ten are your best friends and they are the only ones you want to have come over to your house.
41	Katy:	[*Smiles, looks up*]. Okay. [*Sits up in her seat. Taps 1, 2, 3, 4 down and then 5 up. Smiles. Taps 6 down*.] What kind of number is going to come after?
42	I:	After 6? What do you think?
43	Katy:	Don't know. One, two, three, four, five, six, seven! [*Taps 7 down and then looks up*]. Eight. Does he go there?
44	I:	He's not your friend, just ten.
45	Katy:	[*Taps 8 down*.] Nine. [*Looks at interviewer*.]
46	I:	Not your friend.

47 Katy:		Is nine going to come after?
48 I:		You just did eight. What do you think?
49 Katy:		One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen. No. [Closes eyes, laughs and tilts her head up. Puts 9 down. Then puts 10 down.]
50 I:		Oh. Ten was our friend.
51 Katy:		[*Resets. Puts 1 up. Resets. Puts 1, 2, 3, down. Pauses. Puts 4 down. Puts 5 down*.]
52 I:		Oops. We lost five.
53 Katy:		[*Resets. Puts 1, 2, 3, 4 down tapping quickly right below the line, up to 5 up. Taps 6, 7, 8 in the same place below the line. Looks up at the interviewer.*] What's next?
54 I:		After eight?
55 Katy:		Yeah. [*Taps 9 down. Puts 10 up. Smiles and claps*].

Katy tapped the numbers 1 through 4 quickly in a specific area of the screen. In [41], she was pleased to have correctly put five above the bar and then immediately put 6 below. Katy then counted up from 1 to figure out what would follow 6. She used the same strategy after 8, but this time overshooting 9. After mistakenly putting 10 below the bar, Katy reset three times before [53], when she successfully got to 8. She then succeeded in [55] at putting 9 below and 10 above the bar. In [53] her tapping from 1 to 8 had become even quicker, with no pausing before 5.

Hoping to encourage Katy to use more than one finger at a time, the interviewer asked whether she could make many friends at a time. But Katy responded by tapping rhythmically (with one index finger) below then above the bar, up to 98, saying "I'm doing a pattern." The interviewer repeated the invitation to use more than one finger. Katy put her right hand down (touching the screen also with her palm) and smiled. She created numbers up to 205, then sat up and made more numbers with the left hand, this time with only the fingers (not the whole hand). She switched back to her right hand, pressed Reset with her index finger and said, "I don't want no friends."

As Katy engaged with *Touch Counts* through exploration and with the interviewer, she developed several dimensions of tool fluency. Without explicit prompting, she could make numbers on the screen. This was an activity she seemed to enjoy, as she patiently counted higher and higher, repeating the numbers with the iPad. She quickly became adept at pressing the Reset button and placing numbers above the bar. She also became proficient at being able to place a given number *n* (a best friend) on the bar by tapping below the bar *n-1* times and then tapping above the bar. Katy was able to articulate a strategy for deciding where to tap her finger, as evidenced in [41] "What kind of number is come after?" Instead of saying that she has objectified number, we see her actions more in terms of developing local fluency around 5 in that she forges relationship (what's before 5, what's after 5) in the neighbourhood of 5. However, it is evident in her work on trying to get just 5 and 10, that she uses the routine for placing 5 on the bar in order to predict when 10 will come.

The episode shows her ability to pick out a given number developed through perceptuomotor integration. At first, her tapping was slow and irregular, and was often accompanied by her own oral counting, and sometimes by repeating the iPad. But as she made mistakes, hearing and seeing, for example, the number 5 fall off the screen, she would reset on her own and tap anew. The tapping became quicker and more rhythmic until eventually putting 5 on the bar involved tapped her finger four times on a spot below the line and then moving to tap the 5 above the line. We notice too that Katy, despite several prompts from the interviewer, strongly preferred one-finger playing. So, despite the fact that she can "objectify" 5 and 10, we see her as an ordinal Touch Counts player—she creates numbers one at a time.

Adding world: 7 involves making more ones

The interviewer switched to the Adding world. Katy immediately put her left hand on the screen, creating a group of 4, then a group of 5, then 4, then 2. When the interviewer asked her to bring two groups together, Katy uses her right index and middle finger to gather groups of 4 and 2. But when she tried to bring other groups together, Katy inadvertently created new groups. The interviewer showed her how to gather groups with two index fingers. Katy tried this, making a group of 5. She then unintentionally creates a group of 7. The interviewer asked, "Can you make a group of seven for me?" Katy tapped with her index finger several times, then made a group of 4. She then tried gathering the groups together, but ended up creating several more groups of 1 and 2. Seeing that she was having trouble making groups, the interviewer told Katy that she could use two hands. Katy eventually makes a group of 4 (using one hand only). The interviewer asked her again "How could you make seven?"

71	Katy:	Four and one [*pinches groups of 4 and 1 with her right hand thumb and index fingers, making a group of 5*].
72	I:	You have 5 now and how many more do you need to put in there [*pointing to the group of 5*] to get seven?
73	Katy:	One? [*Katy struggles to put to groups together. Gathers the group of 6 and a group of 1*].
74	I:	Six! Oh, you are almost there, you got six.
75	Katy:	[*Makes groups of 1 and a group of 3*.] Not three. No. You go. [*Drags 3 to the corner of the screen.*] I need you one. One? [*Holds 1 with her right hand index finger and gathers to the group of 6 using her thumb*.] Again.

When asked whether she wants to make another group of 7, Katy used her middle and index fingers and she continued to gather more 1s together to make another group of 7. The interviewer asked her to gather a group of 2 and 4 (already on the screen), but she said, "I'm going to make more ones." She continued to work diligently until the interviewer showed her again how to use her two index fingers to gather groups. She did it herself, made several groups, and then returned to using just one hand.

In moving to the Adding world, Katy experienced difficulty in gathering groups together since her fingers would land on a blank part of the screen instead of on an existing group. At first, her gathering was haphazard, sometimes resulting in a combination of groups but most often in the creation of a new group. She more or less refused to use two hands to make the gathering easier. Nonetheless, she became fluent in gathering two groups, especially when one of the groups was a group of one. Using the strategy of successively adding groups of one, she was able to create several groups of seven. In Vergnaud's sense, Katy was evincing a sense of cardinality as she was operating with the groups, adding 1 to 1, 2, 3, 4, 5 and 6. However, at no time did Katy intentionally add anything other than a 1 to an existing group in order to produce a group of 7. Indeed, when invited to gather a group of 2 and a group of 4, Katy was insistent on "making more ones." We hypothesise that adding-on one, in *Touch Counts* at least, offers an intermediary kind of expertise for children like Katy, who are deeply oriented toward ordinality but are using an instrument that expresses cardinality.

CONCLUDING REMARKS

As we saw in the Counting World, there seems to be a close connection between Katy's preference for one-finger (or one hand) actions and her almost exclusive use of adding-on one. We are interesting in examining the correlation between this kind of one-finger interaction and students' number sense. In addition, we would like to study whether explicit instrumentation of two or more finger touching (in both the Adding and Counting Worlds) might help develop children's number sense.

Our goal in this paper has been to describe *TouchCounts* and provide a rationale for its design. In our exploratory study, we have shown that children easily and sometimes spontaneously learn to play this instrument. Further, and this is central to the motivational context of mathematical learning, as Bartolini-Bussi noted in the working group, students enjoy playing with *TouchCounts*, spending a great deal of time creating numbers, looking at them and listening to them. We further noticed that the design decision to not let users move numbers that had already been created in the Counting World occasioned didactical opportunities in that they had to start the task over again, thus gaining further number fluency.

We have also shown that there may be a strong relationship between the way children actually use their fingers in playing this instrument and the way they think about numbers—a finding that is consonant with the dialogical nature of our theoretical framework. Finally, we have shown that the tasks offered in the two worlds enabled Katy to develop a certain kind of expertise in working with cardinal numbers, which suggests ways of thinking about ordinality and cardinality that are specific to this instrument rather than technology-independent, as suggested in the literature.

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THE STRUCTURES, GOALS AND PEDAGOGIES OF "VARIATION PROBLEMS" IN THE TOPIC OF ADDITION AND SUBTRACTION OF 0-9 IN CHINESE TEXTBOOK AND ITS REFERENCE BOOKS

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Different cultures with their own advantages and disadvantages are, rather than oppositional, complementary. In this paper, we attempted to articulate a Chinese "indigenous" pedagogical practice. This "indigenous" practice, "variation problems", in the topic of addition and subtraction of 0-9 in Chinese textbook examples and their goals and pedagogies in their textbook reference books are presented. To grasp its distinctiveness, a comparison between Chinese and American textbooks is carried out. It might enable us to see which parts of the different educational systems can learn from each other.

INTRODUCTION

Different curriculum traditions are developed in different cultural communities (for a general discussion, see Xie & Carspecken, 2008). Priority of "contextualization problems" in the interest of facilitating connection situation is generally regarded as the common curricular trend in the West (Clarke, 2006). In contrast with "contextualization problems", variation problems in the interest of facilitating connection concepts and methods play important roles in the eastern curriculum. It is generally perceived as one of the most valuable experiences within Chinese mathematics education community (e.g. Sun, 2011). From outside perspective, Marton (2011) argue that variation practice stems from the Chinese language expression. This study tries to indicate how this practice reflected Chinese own advantages and disadvantages of curriculum. It is interesting to note there is robust literature in textbook comparison in the field of math education. The main direction of textbook comparison focuses on contents and problems. For example, Fuson, Stigler & Bartsch (1988) concentrated on the grade placement of topics, content topic covered and page space of each topic. Li (2000) stressed the "problem" perspective (1) number of steps required; (2) context; (3) response type; and (4) cognitive expectation. However, there are little textbook studies reveal the textbook difference from variation perspective.

Problems with variation

There are two mainstreams of variation practice studies: one developed from the Chinese tradition as local Curriculum design model (e.g. Gu, Huang & Marton, 2004), another evolved from the European tradition, mainly from learning perspective (e.g. Marton and Booth, 1997). Following the European tradition, Watson and Mason (2005) pointed out two important parameters of mathematics structure: the dimensions of possible variation and the associated ranges of permissible change should be stressed in the use of examples. In this study, we aim to introduce European readers to variation practice from the Chinese tradition. A number of studies (e.g. Gu, Huang, & Marton, 2004; Sun, 2007) consistently claimed that variation practice offers some advantages in Chinese mathematics education. For example, Gu ,Huang, & Marton (2004) argued that, by adopting teaching with variation, even with large classes, students still could actively involve themselves in the process of learning. The "paradox of Chinese learners" might originally be a misperception by Western scholars due to the limitation of their theories. Sun (2007, 2011) presented a Chinese pedagogical phenomenon in organizing a curriculum with an emphasis on discerning relationships through variation approach and argued that there exists an "indigenous" variation practice uniquely rooted in cultural backgrounds. It has been used broadly in example or exercise design to extend the original examples, known widely in a certain way as "one problem multiple solution" (OPMS, 一題多解, varying solutions), "one problem multiple changes" (OPMC, 一題多變, varying conditions and conclusions), and "multiple problems one solution" (MPOS, 多題一解, varying presentations) .This practice is typically regarded as a natural strategy to deepen the understanding in local curriculum as a daily routine, which perhaps makes the "indigenous" practice distinctive. Sun (2011) described these practices and their roles in the topic of fraction division. Reader could further wonder why Chinese textbook authors design them in this way and how to use them. In this study, we explore structures, goals, and pedagogies of problem with variation in the topics of addition and subtraction of 0-9, the most vital and central concept for later mathematic learning.

Addition and subtraction of 0-9

Addition and subtraction of 0-9 is vital and central concepts for later mathematic learning, which would influence numeracy, algorithm understanding of multi-digit addition and subtraction, of multiplication and division, of decimals, of fractions. It is central to developing number sense and is also the basis for the four fundamental operations on numbers and concepts that comprise elementary school mathematics. Not only does it connect to all important concepts, it is also a prerequisite for any real understanding of whole and rational number system". For example, US national performance of subtracting with regrouping of 2-digit is only 28% correct in Grade 2 and their error remains extremely common until high school (Fuson & Li, 1990). Cross-cultural comparison indicated that Chinese teachers have a deeper conceptual understanding of subtraction with regrouping, a solider knowledge of abundant connections and much more flexible way to explain problems than their American colleagues (Ma, 1999). Is it related to their learning resources?

As mentioned before, it seemed that Chinese arithmetic development, textbooks, their textbook reference books, and particular variation practices, might be a good clue for understanding Chinese mathematics education system rarely known outside of Chinese community. To enable us to see which parts of the different educational systems can learn from each other, in this study, we would like to go further to explore structures, goals and pedagogies of “variation problems” in the topic of addition and subtraction of 0-9. For comparison, a USA textbook and their according for teaching guide was chosen as a “mirror”, which reflected the curriculum constructed upon different philosophical traditions: Dewey’s instrumental pragmatism in the case of USA compared with dialectical materialism in the case of the Chinese mathematics curriculum (Xie & Carspecken , 2008) . The research question of this study is restated as follows: *What are structures, goals, and pedagogies of variation problems in the topics of addition and subtraction of 0-9 in Chinese textbook and reference book?*

A Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005) used for over 30 years by the majority of students composed of diverse backgrounds was chosen, a representation of the Chinese national curriculum by most scholars of textbook comparative study. This textbook with its textbook reference book (Elementary Mathematic Department, 2005) is an authoritative guide for all teachers on what to teach/learn and how to teach/learn as national examination problems are required to be “from textbooks, but above textbooks”. An American textbook Teacher edition (similar to Chinese guide book) (Gonsalves, Grace , Altieri, Balka, Day, 2009) identified as a widely used mathematics textbook was chosen to act as a “mirror”. Here we translated “indigenous” variation practices into codes by examining “problem set with /without concept connection” or “problem set with /without solution connection” (Sun, 2011). Note that this textbook is not claimed to be representative of all the textbooks used in USA, but we consider that reflect the typical practices in USA.

Why do we choose Chinese textbook? It is deserved to note textbooks play different roles in their system. Generally, teachers play much more central role than textbook in an education system. However, since Chinese teachers generally have limited space to re-design task due to the fact that Chinese curriculum evaluation (exam) is unified by government and curriculum content is required to follow strictly the unified standards and the unified textbooks, textbooks play much more central role than teachers in Chinese education system. Therefore, Chinese textbooks have been regarded as the most authoritative books in local culture. They have been both driven and governed by government system since Tang dynasty, different from those driven by markets in western culture. Chinese textbooks as textual art of pedagogy are required to rigorously present what teacher should teach and what a student should learn than those in other places. Therefore, they are generally designed by local experts collected in the entire county. They are expected to play multiple functions in Chinese mathematical education system, such as main media for teaching and learning in the classroom, self-learned instrument for out-of-school learners, tools of teachers` professional development by intensively studying textbooks and its reference series (教學參考書) (e.g. Ma, 1999) as one of the important professional development notions in Chinese system. These textbooks have been played much more important roles in Chinese system than those in the west.

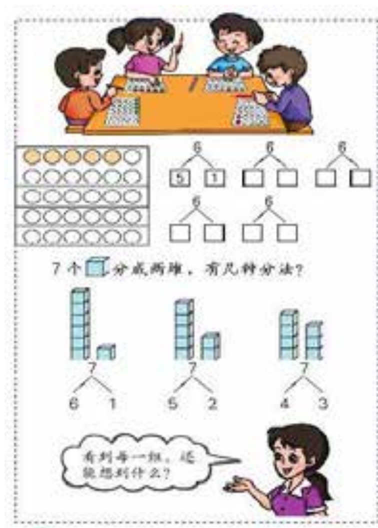
THE STRUCTURES, GOALS, AND PEDAGOGIES OF VARIATION PROBLEMS

The Chinese textbook includes 13 examples with problem sets of concept connection (OPMC), in total, accounting for about 87% of all 15 examples. Furthermore, the Chinese textbook includes 2 examples with problem sets of solution connection (OPMS), in total, accounting for about 13.4% of all 15 examples. None of these examples appear in the American textbook. The invariant mathematical meaning of addition and subtraction, that is, part-part-whole concept is the "core" idea highlighted in Chinese textbooks, which are not pointed out in American textbook.

The design of addition and subtraction content of 0-9 are typical, which could reflect the consistent features in other 4 chapters too. In the following, we present the structures, goals, and pedagogies of variation problem in the topics of addition and subtraction of 0-9 in Chinese textbook /reference books and American textbook of student/ teacher edition. We will begin with the structures, goals, and pedagogies of OMPC.

Structures, goals, and pedagogies of OPMC

OPMC in the topics plays two roles: providing foundation and making concept connections. In what follows, these are introduced with illustrative examples of providing foundation. It is interesting to note that addition or subtraction is not introduced directly, but its knowledge foundation as knowing number by OPMC is systematically provided. Fig. 1, 2 shows two examples introducing the quantity concept of 4, 6, 7, called cardinal number, by the problem variations with composition

	
Fig. 1. The example introducing the cardinal number concept of 4 using the problem variation with composition and decomposition concept connection in the Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005, Vol. 1, p.19)	**Fig. 2. The example introducing the cardinal number concept of 6,7 using the problem variation with composition and decomposition concept connection in the Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005, Vol. 1, p.44)**

and decomposition concept connection in the Chinese textbook.

It is noteworthy that the design is unique that knowing number, concept of addition, and concept of subtraction are united together in all 6 chapters and gradually expand from 0-5, 6-10, 11-20, two-digit, three-digit, above four-digit in the Chinese textbook, which is separated into 20 chapters with tittles of pattern and number sense, that of addition strategy, that of subtraction strategy in the American textbook. Their design goals and pedagogies of figure 1& 2 are explained in the following in its reference book.

> Knowing numbers is the premise of calculation. Conversely, calculation will help to deepen understanding of numbers. For young children, the strategy combining knowing number with basic calculations would be, not only easy for learning number concept, but also conductive to consolidate basic calculations learned inversely. (Elementary Mathematic Department, 2005, P.34)

The goal and pedagogy of figure 2 is explained in the following in Chinese reference book.

> The teaching should follows the following procedure: counting → understanding of the order of number → comparison of two adjacent numbers → writing digit →order of number →composition and decomposition of number. The composition and decomposition of number is the focal point. This arrangement, on one hand, reflects the rich meaning of number concept, on the other hand, also reflects logical order of knowing number as foundation of basic calculations (Elementary Mathematic Department, 2005, 67).

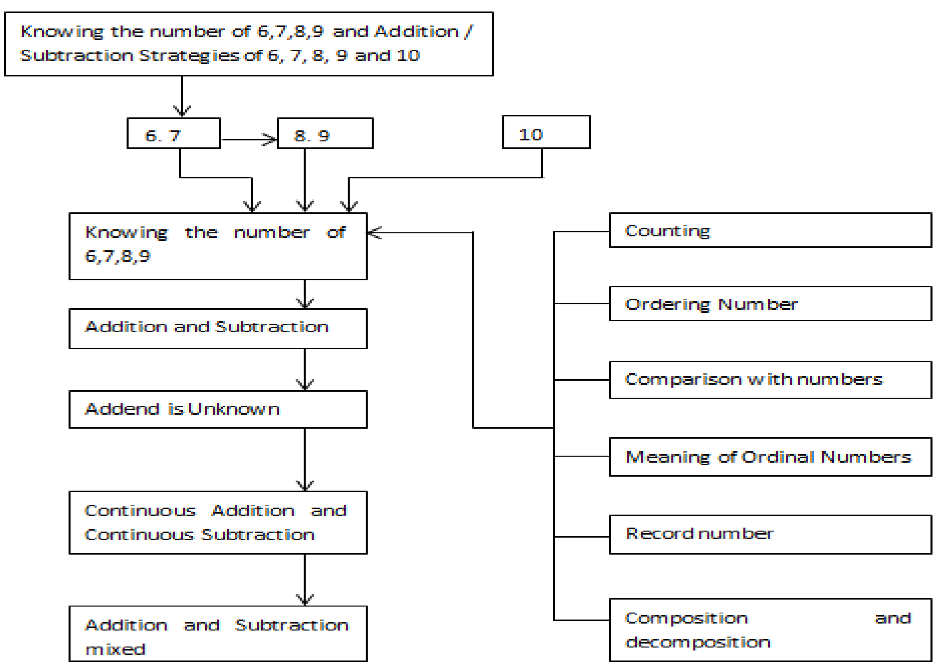


Figure 3.The concept structure of knowing number 6-10 in Chinese teaching reference book (Elementary Mathematic Department, 2005, P.35)

The design above mainly reflect Chinese curriculum tradition with focus on goals – “two bases”, namely, the curriculum foundation of addition and subtraction is “part-part-whole” (pre-algebra thinking foundation) relationship. In fact, “two bases” is regarded as the most valuable tradition in the history of Chinese curriculum reform by local experts, different from those in other counties, such as problem solving, communication, reasoning in USA. It is impressive that the concrete foundations, similar to knowledge package (Ma, 1999), in every unit clearly are presented in Chinese teacher guide book. Fig.3 is the concept structure of knowing number 6-10, the concrete curriculum foundation in a unit, in Chinese teaching reference book (Elementary Mathematic Department, 2005, P.35).

It is impressive that two concepts of addition and subtraction are always almost elicited together in term of examples of OPMC in Chinese textbook, rather than separated in the American textbook. In what follows, these are introduced with illustrative OPMC examples of making connections.

Fig. 4. The example introducing the subtraction concept using the problem variation with concept connection in the Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005, Vol. 1, p.20-21)

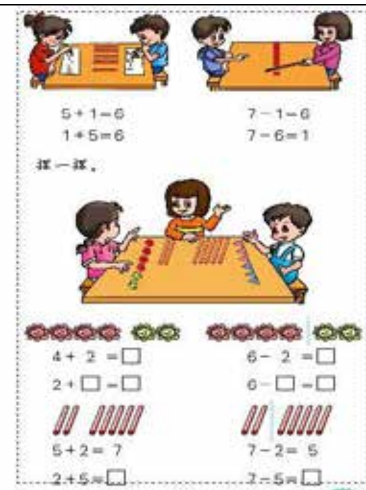


Fig. 5. The example introducing the subtraction concept using the problem variation with concept connection in the Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2005, Vol. 1, p.45)

Figure 4 shows a paradigmatic example of introducing the subtraction concept by OPMC: 1+2=3, 3-1=2.The problem set intends to help learners to recapitulate the relationship of addition and subtraction, and the meaning of “equal” by three figures, 1, 2, and 3, among the two problems, which may help students to focus on concept variation, rather than digital variation (general feature in the US counterpart). Figure 5 shows a typical example of introducing addition , subtraction , and exchange law of number 6 by three groups of OPMC: 5+1=6, 1+5=6 ; 4+2=6, 2+4=6, 6-2=4,6-4=2 ; 5+2=7, 2+5=7, 7-2=5,7-5=2.The three group of problem set intend to help learners to recapitulate the relationship of addition and subtraction, and the meaning of “equal”, which stress the invariant concept of part-part-whole .

The goal and pedagogy of this design is explained below in its reference book.

> The teaching idea of meaning of subtraction is same as that of addition. Textbook use the same situation to elicit subtraction which indicates the relationship that subtraction is the inverse of addition. Therefore, appropriately combining subtraction with addition in teaching will be helpful for students to grasp the relationship and difference of addition and subtraction, which will deepen the understanding of the meaning of addition and subtraction too (Elementary Mathematic Department, 2005, P.39).

Compared with Chinese design, every example in American textbook naturally introduces the concept of addition and subtraction isolated or with limited concept connection as “basic arithmetic facts” such as, “5+7=12” or “12–7=5”) for students simply to memorize (Ma, 1999). In the American textbook, each addition example uses multiple, different, inconsistent concepts, such as “counting”, “combining”, and “adding”. Each subtraction example uses multiple, different, inconsistent concepts, such as “taking away”, “comparing”, “cross out”, and “identifying inverse operation of addition”. The comparisons are weaker than those in Chinese textbook in each circle. The concept of part-part-whole is addressed as modelling subtraction, different from the most central status as the knowledge foundation mentioned above in Chinese textbook.

Structures, goals, and pedagogies of OPMS

It is impressive that multiple-solutions are always almost elicited together in term of examples of OPMS in Chinese textbook, rather than single solution in each example in American textbook. Figure 6 is a typical “prototype” example of OPMS in the Chinese

Fig 6, The example introducing three additions solutions using the problem variation whit solutions connecting (Mathematics Textbook Deveiopers Group for elementary School, 2005, vol.1, p.16)

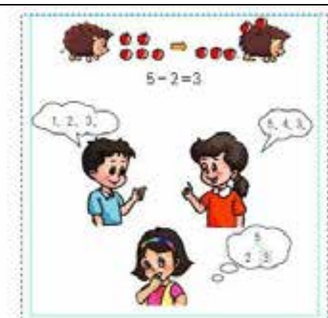


Figure 7, The example introducing three additions solutions using the problem variation whit solutions connecting (Mathematics Textbook Developers Group for elementary School, 2005, vol.1, p.25)

textbook. In the problem variation above, 4+1=5 is designed to naturally introduce a solution system of addition. Within the problem set in the example, there are three solutions given. The first one is that of addition by counting from 1 to 5. The second solution is that of counting from the addend 4 to 5. The third is that of addition by regrouping 5 with 4 and 1. Within the problem set, three addition solutions are presented. Fig. 6 is the problem variation of OPMS above, 5-2=3: The first solution is that of counting what is left from 1 to 3; the second one is counting down 2 from 5(5, 4, 3). The third is that of "separating 5 into 2 and an addend 3 as the unknown. The two group OPMS above intend to help learners to recapitulate the relationship of three solutions of addition / subtraction, and the result of "same", which stress the invariant connection of multiple solutions .

The design goal is explained in the following in its reference book.

> Algorithm diversification is one of the basic philosophies of the "new curriculum standard". It states that: "it is natural students use divertive methods because of different living backgrounds and from different perspectives; teachers should respect their thoughts, to encourage them to think independently, to advocate the diversification. (Elementary Mathematic Department, 2005, P.34)

The design pedagogy is explained as follows in Chinese reference book.

> After students` presentation of multiple solutions, teachers may prompt a discussion on which solution is the simplest one, which help them realize the decomposing-solution is simpler than others. Teacher should guide student from the solution of low level to that of high level. (Elementary Mathematic Department, 2005, P.44)

Compared with Chinese multiple-solution feature, examples in American textbook always elicit the single solution with limited solution connection. It is deserved to note each example in American textbook use many "inconsistent" solutions, such as *"use number line to add"; "doubles"(3 +3 = 6, 5+5=10), "doubles plus 1"(8+9= 8+8+1=17), "compensation*"(*6+8=7+7=14*), and "*reference number*" (*6+7=5+1+5+2=10+3=13, 5 as reference number*) for addition strategy , such as "*counting back"*, " *use of fact families for addition and subtraction*", and *"doubles", "use number line to subtract"* for subtraction strategy emphasizing the importance of applying influenced by Dewey's instrumental pragmatism philosophy (Xie & Carspecken, 2008). Although Chinese textbook authors use multiple solutions, double-number –solution, using-number line-to add, and count-back-solution are not introduced. Only one invariant, "consistent" solution of "decomposing/composing–number-solution" (developed making-a-ten-solution latter) is addressed by OPMS in all the addition /subtraction examples in the chapter, also other chapters with a focus emphasizing the importance of analytical and neopragmatism (Xie & Carspecken, 2008).

DISCUSSION

Many readers may argue that the variation approach may be confusing and that a sequential organization with time gaps ("one-thing-at-the-time") should be preferred. In fact, variation approach might come from different kinds of pedagogical traditions and philosophies developed for centuries. The issue of variations in problem sets directly reflects the old Chinese proverb, "no clarification , no comparison" (沒有比較就沒有鑒別), rather than "to consolidate one topic, or skill, before moving on to another," a notion broadly used in most textbook development (Rowland, 2008) in Europe and throughout the world . It coincidently emanates from the work of Marton, “variation is a necessary condition for effective discernment” as the soul of variation theory. In contrast, this “one-thing-at-the-time” design would clearly provide fewer opportunities for “making connections” compared to those of contemporaneous variation approaches. The “one-thing-at-the-time” design might possibly reflect a hidden conception, making a connection could naturally happen. In this context, the curriculum role of making connections could either be relatively neglected or taken for granted. It is deserved to note, OPMC and OPMS aim to provide opportunities for making connections and further discern, compare the invariant feature of the relationship among concepts and solutions, since comparison is considered the pre-condition to perceive the structures and relationships that may lead to mathematical abstraction. The invariant is repeatedly highlighted by the design as central idea to design Chinese textbook. Conversely, the invariant concept /solution are not stressed after multiple concepts/solutions are presented in USA textbook, which possibly link to fragmentation understanding pointed out by (Ma, 1999).

This study provides the structures, goals, and pedagogies of variation problems, compared with the American system, in the topics of addition and subtraction of 0-9 in Chinese textbook and its reference book, which is consistent with the findings on fragmentation in US textbooks and connectedness in Chinese textbooks (Ma, 1999). The comparisons above inspire us to develop much more coherence curriculum by addressing knowledge foundation, concept connections, highlight the invariant concepts and solutions in Chinese mathematics education system. Variation approaches may be critical in developing concept-connection curriculum and instruction rarely figured out before. The priority of “contextualization” in the interest of facilitating engagement, motivation, and meaningfulness is regarded as the common curricular trend in the West (Clarke, 2006). In this light, variation problems suggest a way in which way western counterparts could learn from content-orientation curricula in China. Clearly, variation problems are two-edged sword which could lead to more learning challenges compared to “contextualization” problems because they require the use of multiple concepts and solutions targeted. This study inspires us that textbook comparison with their goals (why textbooks are designed in this way) and pedagogies (how textbooks are used) in their textbook reference books would provide a much more *integrated* window for understand curriculum than textbook series alone in the field of textbook comparison (e.g. Fuson & Li, 2009).

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HOW FAMILIES SUPPORT THE LEARNING OF EARLY YEARS MATHEMATICS

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This paper is based on the assumption that parents are of prime importance for their children's mathematical development. It gives account of a study that focuses on the question of how families support the learning of early years mathematics. Thereby, everyday playing and reading sessions with pre-schoolers and their mothers are the context of investigation. By means of three very short trancripts from these sessions, different kinds of support systems will be reconstructed. In particular, the research question is about the aim of the support system: What is the support system focused on? The related analyses show the diversity in young children's mathematical experiences.

INTRODUCTION

Although the discussion about early years mathematics is often confined to institutional contexts a holistic perspective challenges us to keep not-institutional contexts in mind, too. When thinking about early learning processes, the family is one of the most important places to be recognized. For the young children spend much of their time with their parents, observe them and become step by step a member of their culture (Warren & Young 2002). As a consequence, the young learners become amongst other things a part of their parents' mathematical culture as well. For this reason, the family cannot be ignored as a context of particular importance for questions about early years mathematics.

This opinion is confirmed again by Street, Baker and Tomlin (2005) who conducted a large study in Great Britain about home and school numeracy practices. One of their central results is that learning difficulties might arise from differences between home and school discourses. The researchers explain that for some children there is a gulf between these contexts: "The school replicates the Primary Discourse of middle class homes whilst it presents children from other backgrounds with a Secondary Discourse." (Street et al. 2005, p. 7) According to the authors, many children are restricted in their prospects to succeed in mathematics education because they are confronted with a problem of discourse: The switch between home and school discourses can be difficult because of different values, rules and patterns (ibid., p. 44ff. and 70ff.). This result exemplifies why research about families is also important with regard to institutional contexts. Everyone who wants to teach children mathematics has to know about their earlier ways of learning mathematics to avoid difficulties:

> "If children are frequently presented with tasks which are unconnected with their earlier ways of knowing about mathematics, they may come to reject it and to begin to feel that they are failing." (Pound 2006, p. 20)

In summary, it can be said that the families' support of learning (early years) mathematics is of prime importance for the children's development. While we already know something about *what* children learn in the familial context (Blevins-Knabe 2008, S. 2; Blevins-Knabe et al. 2000, S. 50; Anderson 1997, S. 492), we know only a little about *how* they learn it.

> "Studies of the processes by which parents encourage early numerical development in the context of parent-child interactions during routine, culturally relevant activities at home are scarce." (Vandermaas-Peeler, Nelson, Bumpass & Sassine 2009, P. 67)

For this reason, in my research, I focus on the question of how families support the learning of early years mathematics. With this question in mind, I observed mother-child-dyads in game playing and reading sessions at their homes for a number of times in the course of a year.

THEORETICAL FRAMEWORK

In studying early years mathematics, we necessarily do it with a certain conception of what learning mathematics is all about. In my opinion, children do not encounter mathematics itself, but a cultural practice that is recognised as mathematical by capable members of the belonging culture (see Sfard 2006). In other words, I regard mathematics as a social construction and learning mathematics as a social construction too. This idea of learning is explicitly described in Sfard's theoretical work. She defines learning mathematics as "individualizing mathematical discourse, that is, as the process of becoming able to have mathematical communication not only with others, but also with onself." (Sfard 2006, p. 162) Against this background, supporting the learning of early years mathematics means to help young children becoming fluent in a mathematical discourse.

Concerning the support, I draw on the notion of support as it was developed by Bruner (1983) and Rogoff (1990). Bruner worked on the question of language acquisition in the very early years of a child's life which was for him mainly a question of culture acquisition. Assuming that a child primarily has a need for social contact, Bruner supposed that he has to learn his mother tongue in order to become a part of the given culture and, in this way, to relate to people around, first of all to his mother. In this context, Bruner postulates the existence of a so-called *Language Acquisition Support System* (LASS). This support system is established by a child and his mother and is realised in the form of 'formats'. Here, a format is "a standardized, initially microcosmic pattern between an adult and an infant that contains demarcated roles that eventually becomes reversible" (Bruner 1983, p.120). The adult-child-dyad creates "a predictable format of interaction that can serve as a microcosm for communicating and for constituting a shared reality" (ibid., p. 14).

Bruner sees this kind of interactional and recurrent support as a condition for the child's learning. Being part of the LASS, the child becomes part of the adult's culture and, in doing it, he learns his mother tongue. According to Bruner, an increasing autonomy within the support system can be understood as an indicator for learning progress. From my point of view, the important aspect of Bruner's concept is to understand support as a support system, that means as something that is established by at least two persons. Therefore, support is not any longer an individual achievement of the mother but a certain kind of format that is established by the interlocutors. The child reacts on what the mother does and vice versa. In this way, a mother-child-dyad creates a support system that helps the child to become part of the given culture. Transferring this notion of a support system to the field of mathematics education, I assume that families establish a support system for mathematical learning processes by means of their everyday discourses. In dependence on the LASS-concept, I refer to it as a *Mathematics Acquisition Support System* (MASS).

It was Rogoff (1990, 1989) who pushed the interactional equality of adults and children even more to the spotlight than Bruner did:

> "The mutual roles played by children and their caregivers rely both upon the interest of the caregivers in fostering mature roles and skills and on children's own eagerness to participate in adult activities and to push their development." (Rogoff 1989, p. 209)

Rogoff (1990) calls the process of becoming a competent participant in a specific type of discourse 'appropriation'. In that way, she emphasizes that learning and support of learning take place within social activities and are something different than a cognitive individual performance. In the process of appropriation, children "can carry over to future occasions their earlier participation in social activity." (Rogoff 1989, p. 213). In regard to a MASS, this means that the child supports his learning as much as the mother does. Support systems are projects of cooperation. By the way, this is the reason why the title of this paper is "How families support…" and not "How parents support…" The children themselves should be included in the group of persons who support the learning of early years mathematics. Against this background, I understand a MASS as a condition for processes of appropriation.

One could think of many investigations concerning this MASS. In this paper, the function of the MASS is investigated as it is realised in mother-child-discourses in everyday playing and reading sessions: What is ensured by the MASS? What is the MASS focused on? In order to clarify the theoretical perspective for this specific research question, I refer to a study conducted by two German linguists: Hausendorf and Quasthoff (2005) reconstruct how children's competences in telling a story as part of a conversation develop in the course of time. At the same time, the researchers investigate how this development is influenced by an emerging support system. In terms of this object of research, the authors follow Bruner (1983) and his idea of a LASS. For this reason, they describe the support processes as a *Discourse*

Acquisition Support System (DASS). Although I neglect their specific linguistic findings, in my opinion, one special theoretical element that they developed is really helpful to my concern. Hausendorf and Quasthoff talk about so-called ‘jobs’ in order to describe empirical phenomena. Here, ‘job’ is a structural task that has to be mastered when a story should be told as part of a conversation (ibid., p. 124). ‘Picking something out as central theme’, ‘closing’ and ‘leading back to the conversation’ exemplify jobs that the linguists have found in their data. These jobs are established interactionally and are mastered by the discourse partners jointly (ibid., p.122f.). What the child is not yet able to do, is done by its adult interlocutor. So these jobs are determined by their function (ibid., p.125). They are characterized by their contribution to the story as a whole. This functional orientation makes the idea of a job suitable for my aim. But when I intend to explore the function of a MASS I cannot really talk about structural jobs as Hausdendorf and Quasthoff do. For that reason, I use their further theoretical results, too. The linguists connect the jobs that are structural in nature to a higher aim. Namely, they state that the aim of all jobs is to complete the story and that is what is ensured by the DASS (ibid., p. 294). That means the support system guarantees nothing but the completion of the story. On this level, I can refer to that linguistic work. So one can say that there is a kind of higher job by which all the structural jobs are directed. And this higher job is what I intend to reconstruct as the focus of support systems in mathematical discourses. I call it a *support job*. This is the task that is mastered by means of the support system. Just like the structural jobs, these support jobs should be determined by their function: What is ensured by working on a certain support job? This approach to mathematical discourses allows for a functional investigation of the MASS in families.

Against this background, it is important to notice that Hausendorf and Quasthoff could find only one support job in their data. Namely, they reconstruct that the DASS is always focused on the completion of the story (ibid., p. 294). To be a little bit more abstract, one can say that the support job they found is to ensure the child’s participation. The child should tell a story and master the jobs as good as possible. Only when the child is not yet able to cope with a certain job the adult performs it instead.

METHODOLOGY AND RESEARCH METHODS

The data

Over one year, I met ten German families at their homes on a regular basis. In all cases, it was a mother-child-dyad that took part in the project. The year of the regular meetings was in each case the child’s last year before entering primary school. When meeting the families, I invited them to engage in games and picture books that I had brought with me. The families received no further instruction for dealing with the material or for their discourses in general. All the playing and reading sessions were filmed so that a considerable data corpus came out of that. But only those scenes that showed a somehow mathematical topic came into consideration as a possible object of analysis.

The analysis

On the basis of my theoretical framework, I consider a MASS – and with it the support job – as a phenomenon that emerges in the interaction between a child and its mother. For this reason, my analyses are of a reconstructive manner. They are analyses of interaction (see Cobb & Bauersfeld 1995). This method refers to the interactional theory of learning, is based on the ethnomethodological conversation analysis (see Sacks 1998) and was devised by a working group directed by Bauersfeld. In contrast to the conversation analysis, it is aimed at the reconstruction of thematic developments in face-to-face interactions. For this reason, it is especially suitable for the field of mathematics education where the researcher is always interested in the content of a discourse, too. In a second step, the results of the analyses of interaction can easily be interpreted with a special focus on support systems and their inherent support jobs.

INSIGHT INTO THE RESULTS

In the following, I deliver insight into the results of my work by means of very short examples. They give an impression of the data and exemplify the three support jobs that I could reconstruct in mathematical mother-child-discourses. The scenes were chosen with the objective of showing substantially different support jobs. Later on, I will assume that the three support-jobs that are illustrated in the following are the only ones to be found. If this assumption is correct, there will be no support job that cannot be described with the given terms.

The support job "participation"

The first scene is with Paco (5 years 4 months) and his mother, Mrs. Czipin. The two are playing a game called "Max Mümmelmann" which is about rabbit families. The belonging board has the form of a regular octagon with one point on each side. Besides each of these points, one puts a deck of hidden cards. When it is your turn you roll the dice, move the counter which is a wooden rabbit and the counter for all players and draw a card from the hidden deck beside the counter. The cards show rabbits that are characterized by numbers from one up to six. On each card, the number is represented in two ways: First, you can see the digit on the card; second, the rabbit has as many dots on its coat as the number indicates. The six different rabbits are introduced as family members. So, the rabbit children have the numbers from one to four, number five is the rabbit mother and number six, finally, is the rabbit father. Below, each card is named by means of the number on it: A '2-card' is a card that has the digit '2' on it and shows a rabbit with two dots on its coat. To win the game, you have to be the first who completes a whole rabbit family: 1, 2, 3, 4, 5, 6. In the following scene, Paco has already three rabbit cards: 3, 6, 2. They are laying in front of him in exactly this order. Now, it is Paco's turn. He rolls the dice and moves the counter [1].

Paco: (*draws a 2-card, moves his 3-card a little bit to the left and puts the drawn 2-card right beside it:) 3,2,6,2*

Mother: Do you have that already or not?

Paco: No. […]

Mother: Not the colour, you have to watch the numbers.

Paco: (*looking at his mother:)* I have the five. (*pointing at the drawn 2-card:)* But this is the two, you see.

Mother: (*pointing at Paco's 2-card:)* You already have a two.

Paco: (*first looking at his cards, then at his mother:)* Which one do I have to put back?

Mother: One of the two. Each number only once.

In this short scene, Paco performs his fourth move. He rolls the dice, moves the counter and then draws a card from the right deck. As he draws another 2-card, he has to put it back. Nevertheless, Paco integrates the drawn card into his row of cards without being irritated. In a first step, his mother advises him of the rule: For deciding whether one still needs a drawn card or not, only the number is relevant, but not the colour of the rabbit. By giving him this hint, the mother supports Paco in performing his move. The only important question seems to be how Paco can participate successfully in the game. But Paco insists upon his opinion: He is convinced that he still needs the drawn 2-card. In answer to that, the mother refers directly to Paco's fault. She points at his first 2-card and, thereby, she makes clear that he has not noticed the first 2-card in his row, which is not under the rules. Thereupon, Paco obviously realizes that he has to put one of the 2-cards back. So one can say that the support job that Paco and his mother work on is focused on the game itself. The established support system ensures that Paco can participate in the game as a player who is as independent as possible. That way, the game can go on smoothly. As Paco's participation is the central purpose of the MASS, I named this specific support job as "participation".

The support job "improvement"

The second example, which should be discussed in this context, is a short scene with Alina (6 years 1 month) and her mother, Mrs. Gerlach. It is interesting because their MASS is determined by a substantially different support job. Just like Paco and his mother in the example above, the two are playing the game "Max Mümmelmann". It is the mother's second move. She rolls the dice and moves the counter.

Mother: (*drawing a 2-card:)* Now, I may draw a card. Look here. (*showing the drawn card to Alina:)* I have **a**…

Alina:	A mother?
Mother:	No. (*pointing at the digit on the card:)* Do you know this one?
Alina:	It's a…
Mother:	**A**?
Alina:	**Two.**
Mother:	A two. Right. (*still showing her card to Alina:)* And how many dots does my rabbit have?

In this second scene, the mother performs her second move. She rolls the dice, moves the counter and, finally, draws a card that she does not have yet. In this special move, Alina is integrated. She is asked to name the digit on the mother's card. This claim persists until Alina can meet it. Although she suggests a familial framework of reference ("A mother?"), which is actually suitable as well, the mother obviously insists on a mathematical one. Not until Alina refers to the relevant card as a two, her mother agrees. With regard to the topic one can say that knowing the number words might be helpful to talk about specific cards or moves, but it is not essential. It is even isolated from the concrete issue of the mother's move. When comparing this short scene with the one before, we can see that the established MASS is no longer focused on the child's participation. Instead, Alina is part of a move that is not hers. She has to use number words in order to name a given digit. More generally, we can say that Alina is asked to practice on her mathematical skills. So the support job that the two work on is to improve Alina's mathematical skills. Therefore, I named the support job exemplified by this second scene as "improvement". As the mother's final question indicates, the support system will continue to be focused on Alina's improvement. Next, she is asked to determine the number of dots on the rabbit's coat.

The support job "exploration"

The last example is a short scene with Tonio (5 years 7 months) and his mother, Mrs. Liermann. Just like the two mother-child-dyads before this dyad is playing the game "Max Mümmelmann". In particular, they are talking about one special card in the game. It is the card with the rabbit called Max Mümmelmann on it. When you draw this "Max Mümmelmann"-card (shortcut: MM-card), you may take a card of your choice from somebody else. In the following scene, which shows a third support job, Tonio creates an imaginary example in which the drawing of the MM-card leads to the victory. It is Mrs. Liermann's turn. She has rolled the dice, moved the counter and drawn a card. As she has rolled a six, she may now perform another move.

Tonio:	Look here. (*showing one finger for each number word:)* If you have a one, a two, a three… and a six…
Mother:	(*nodding:)* Yes.

Tonio:	Then, you can… And if the other one has the five, you can simply take his five.
Mother:	Right.
Tonio:	Then, you have already won.
Mother:	But, in this case, you still need another number.
Tonio:	(*shaking his head:)* Which one?
Mother:	The four. Or did you mention it?

In this scene, Tonio develops an example of a situation in which the drawing of the MM-card means winning the game. He thinks of someone who has already the cards with the numbers one, two, three and six. According to Tonio, this person is the winner of the game when he draws the MM-card and can take the five from another player. The mother considers that this devised person might still need the four, too. However, she thinks it is possible that her son already mentioned the four. After this scene, the two will work on that example so that Tonio is finally able to display it in a correct and complete version. The short transcript shows that the emerging support system is not really connected to someone's move. Instead, Tonio starts exploring the situation and his mother takes part in that exploration. They come up with a situation that helps understanding the concrete meaning of the MM-card. In that process, they talk about ideas, suggestions and better possibilities. Thereby, Tonio is a kind of director. He decides on the topic, on the development of the discourse and on its end. So the MASS that this mother-child-dyad establishes ensures primarily that Tonio has enough room for his free exploration. His mother helps him when he asks for help but she allows him to decide on the course of the situation. Mathematical aspects get only relevant when they are helpful for Tonio to make progress with his exploration. In view of this central focus, I named the third support job as "exploration".

SUMMARY AND CONCLUSIONS

First of all, the examples give an impression of the fundamental differences in familial Mathematics Acquisition Support Systems (MASS). In this context, the function of a specific MASS was the matter of interest. For that reason, I reconstructed so-called support-jobs. To distinguish the three support jobs that I could find it might be helpful to see them in connection with their situational context. In the first scene, the support system is established while the child is performing its move. Thereby, the MASS enables the child to overcome some difficulties in its participation. So, the support job the mother-child-dyad works on is the child's **participation** in a smoothly running situation. In the second scene, however, the support system is established while the mother is performing her move. In this context, the demand on the child does not really follow from the game itself.

It is rather initiated by the mother. She uses the playing situation as an opportunity to work on the child's mathematical skills. In this regard, she sets a problem and evaluates the child's solution. So the support job is focused on the child's **improvement**. It is important to note that this does not mean that the child's skills really do improve. But you can see that the support system is focused on it. In the third scene, the support system is not linked to someone's move at all. The child engages in an exploration of the game which is rarely restricted in terms of time or method. The child is free to explore its ideas, questions and interests and it is the support system that ensures that there is enough room for that purpose and help if necessary. So, the support job is focused on the child's **exploration**.

When comparing the three support jobs, one can notice the following: The support job "improvement" serves the mother's idea of the situation. She interprets the playing situation as suitable for working on her daughter's mathematical skills and insists on this view. By contrast, the support job "exploration" serves the child's idea of the situation. In the context of playing, he develops a plan of a free exploration and carries it out. And, finally, the support job "participation" points to the idea of the material. Both the mother and her child are working on realising a smoothly on-going playing situation. The mother, the child and the material – one can hardly think of another factor that the support job could refer to. For that reason, one might at least be doubtful of further support jobs to be found.

By means of further analyses, I found out that the families can be characterized by a certain support job. Comparisons over time and also over different materials showed that each family steadily establishes a certain kind of support job. In other words, the support job does neither depend on the point of time nor on the material (games and picture books). When relating these results to those from Hausendorf and Quasthoff (2005), one can see that *Mathematical* Acquisition Support Systems are not fixed to a certain support job. Instead, they vary in terms of their focus.

This variety of MASS in the familial context is a challenge for every institutional context in which children should learn mathematics. The young learners have different experiences in terms of content, discourse practices and support systems. When Paco, Alina and Tonio attend elementary or primary school they need a teacher who is able to cope with their diversity.

1. Transcription rules: (1) **Bold** text marks stressed utterances. (2) (*Text in parentheses*) refers to non-verbal actions.

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TWO CHILDREN, THREE TASKS, ONE SET OF FIGURES: HIGHLIGHTING DIFFERENT ELEMENTS OF CHILDREN'S GEOMETRIC KNOWLEDGE[1]

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This paper presents three different geometrical tasks which involved the same set of geometrical figures. An analysis of the affordances and constraints of each task is discussed along with the results of two children's engagement with these tasks. Results indicated that not all children take advantage of the opportunities afforded by a given task and thus a combination of tasks is necessary in order to assess both strengths and weaknesses of children's geometric knowledge.

INTRODUCTION

During the preschool years, children are developing and refining their spatial and geometric thinking (Clements, Swaminathan, Hannibal, & Sarama, 1999). Promoting geometric concepts and reasoning is also considered an important aim of several preschool programs (e.g. NCTM, 2006). Whether engaging in free play in a geometrically enriched environment or whether engaging in teacher-directed tasks, how can we know if and what children have learned from these activities? How can we know if we have achieved our goals? Ginsburg and Golbeck (2004) claimed that employing standardized procedures for measuring young children's learning is not appropriate. They note that many children are uncomfortable in or may be unfamiliar with the testing situation, and may display variable interest in the task. Instead, they suggested testing methods which might include clinical interviews and observations, methods that are "designed to be sensitive to the needs and peculiarities of young children" (p. 194). Yet, even when employing interviews and observations, children's competence may be linked to the specific nature of the tasks. What knowledge comes to the fore as children engage in different tasks?

In this article, we describe a study which aimed to investigate how different aspects of kindergarten children's geometric knowledge may become evident as they engage in different geometrical tasks. Using the context of two-dimensional figures, kindergarten children were presented with three different tasks. Two of the tasks employed the same set of geometrical figures, which was a subset of the figures employed in the third task. Keeping this in mind, we ask the following questions: (1) What elements of geometric knowledge are revealed by each task? Will the same elements come to fore in each task or will different elements be revealed in different tasks? (2) Will children display the same level of geometric reasoning (i.e. according to van Hiele) in different tasks or will different levels of reasoning be employed in

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different tasks? This paper presents in depth results of two children who took part in this study.

YOUNG CHILDREN'S GEOMETRICAL REASONING

Much of young children's knowledge, including their geometrical knowledge, is based on their perceptions of their surroundings. Later on, examples serve as a basis for both perceptible and non-perceptible attributes, ultimately leading to a concept based on its defining features. Such a process was described by Vinner and Hershkowitz (1980) who introduced the terms concept image and concept definition in reference to geometrical concepts. Visual representations, impressions, and experiences make up the initial concept image. Formal mathematical definitions are usually added at a later stage. Fischbein (1993) considered the figural concepts an especially interesting situation where intuitive and formal aspects interact. The image of the figure may promote an immediate intuitive response not necessarily based on logical and deductive reasoning. "Sometimes, the intuitive background manipulates and hinders the formal interpretation" (Fischbein, 1993, p. 14).

With regard to geometrical reasoning, van Hiele (1958) theorized that students' geometrical thinking progresses through a hierarchy of five levels, eventually leading up to formal deductive reasoning. At the most basic level, students' use visual reasoning, taking in the whole shape without considering that the shape is made up of separate components. Students at this level can name shapes and distinguish between similar looking shapes. Regarding naming, Markman (1989) proposed that when children hear a new name for an object, they assume it refers to the object in its entirety and not to its parts. In addition, children assume a given object will have one and only one name.

At the second van Hiele level students begin to notice the different attributes of different shapes but the attributes are not perceived as being related. Attributes may be critical or non-critical (Hershkowitz, 1989). In mathematics, critical attributes stem from the concept definition. Definitions are apt to contain only necessary and sufficient conditions required to identify an example of the concept. Other properties may be reasoned out from the definition. Burger and Shaughnessy (1986) claimed that an individual's reference to non-critical attributes has an element of visual reasoning. Thus, they further claimed that a child using this reasoning may either be at van Hiele level one or at van Hiele level two, as he is pointing to a specific attribute, and not judging the figure as a whole.

At the third van Hiele level, relationships between attributes are perceived and definitions are meaningful. If the student points out that a figure is a quadrilateral because it has four sides and therefore it also has four angles and vertices, then that child may be operating at the third van Hiele level. Finally, we note that research has suggested that the van Hiele levels may not be discrete and that a child may display different levels of thinking for different contexts or different tasks (Burger & Shaughnessy, 1986).

METHODOLOGY

Participants

The two children reported on in this paper were both scheduled to enter first grade during the following school year. They learned in two different classes, but both classes participated in our program *Starting Right: Mathematics in Preschools*. This program integrated professional development for teachers with onsite guidance by a program staff member (Tsamir, Tirosh, Tabach, & Levenson, 2010). Throughout the year, teachers engaged students with various geometric activities to promote their knowledge of triangles, non-rectangular quadrilaterals, pentagons, hexagons, and circles. At the time of the study, children were expected to be familiar with the names of these shapes as well as with the mathematical language used to describe these shapes (e.g., vertices, straight and curved lines, open and closed shapes).

The set of figures

The set of two-dimensional figures which served as the context for this investigation included intuitive and non-intuitive examples and nonexamples of each of the shapes mentioned above. Figure 1 illustrates with triangles how figures may be grouped along two dimensions: a mathematical dimension and a psycho-didactical dimension.

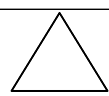
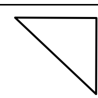
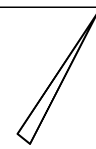
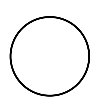
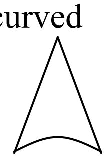
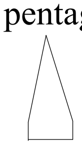
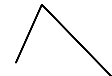
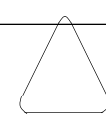
Dimensions	Psycho-didactical				
Mathematical	Intuitive	Non-intuitive			
Examples	 Prototypical triangle	 Right triangle		 Narrow scalene triangle	
Non-examples	Circle 	“Triangle” with curved side 	Elongated pentagon 	Open "triangle" 	 Rounded "triangle"

Figure 1: Intuitive and non-intuitive triangles and non-triangles

Regarding the examples, the equilateral triangle, and possibly also the isosceles triangle, are considered prototypical of all triangles (Hershkowitz, 1989). Shaughnessy and Burger (1985) showed that triangles which are rotated, triangles without one side horizontal to the page, and very narrow scalene triangles are often not identified as triangles. These may be considered non-intuitive examples.

Regarding nonexamples, because the circle is intuitively recognized as such by even young children (Clements, Swaminathan, Hannibal, & Sarama, 1999) it may be considered an intuitive nonexample for a triangle. This reasoning holds true for other geometrical figures which the child can identify and name. Other easily identifiable nonexamples are those which are visually far removed from the prototypical triangle. On the other hand, nonexamples which are visually similar to the prototypical triangle may be considered non-intuitive nonexamples of that shape (Tsamir, Tirosh, & Levenson, 2008). Among this group of nonexamples, we specifically chose figures such that each nonexample would violate a different critical attribute. This allowed us to investigate the child's knowledge of each critical attribute separately. For example, in Figure 1, the rounded "triangle" is missing vertices, the "clown hat" has a curved side, the open "triangle" is not closed, and the stretched pentagon has five, instead of three, sides and vertices.

Examples and nonexamples for each of the other shapes were chosen in a similar manner, taking into consideration the necessity to limit the amount of figures presented at once to the children. The entire set of figures is presented in Figure 2 in the exact manner in which they were presented to the children for two of the tasks. The regular octagon, a shape that was not specifically taught to the children previously, was chosen as a non-intuitive nonexample for a hexagon, in order to investigate if the children would discern between shapes that had many sides, if they would actually count the sides, and not just group them together indiscriminately. It was also thought to possibly be a non-intuitive nonexample for a circle, especially the smaller octagon which was visually similar to a circle. In addition, we included a concave quadrilateral, pentagon, and hexagon as non-intuitive examples of each respective shape. The concave quadrilateral was drawn visually similar to a triangle and thus was considered to be a non-intuitive nonexample for a triangle.

Figure 2: The entire set of figures.

The tasks

The tasks were presented to the children in the order presented below. Task two was implemented immediately following task one on the same day. Due to the age of the children, their ability to sit and concentrate, and other time constraints, the last activity was implemented a week later.

Task one: Free-sort. All 22 cards were placed on the table in front of the child as in Figure 2. The interviewer said, "There are lots of shapes on the table. I would like you to sort them. However you like. You decide which cards go together. You can put as many cards as you want together in the same group." As the child began working on the task, moving the cards around and grouping different cards together, the interviewer asked, "Why did you put those cards together?" The interviewer also reminded the child that he or she could make changes along the way, take away a card from one group and place it with another group and that a group may contain just one card if needed. When the child seemed to be finished, the interviewer asked, "Are you satisfied? Would you like to change anything?" When the child indicated that the sorting was done, the task was considered completed. This task was completely open-ended and had not been implemented with the children in their kindergarten class.

Task two: All-at-once. All 22 cards were placed on the table in front of the child as in Figure 2. The interviewer asked, "Is there a triangle here?" If the child answered yes, then the interviewer asked the child to point to the appropriate card without moving it. The interviewer then asked, "Is there another triangle here?" And again, the child was asked to point to it. This continued until the child indicated that there were no more triangles. The interviewer then asked the same set of questions with the same procedure for the quadrilateral, pentagon, hexagon, and circle in that order. This task was a completely closed task that was somewhat familiar to the children in the sense that they had practice identifying figures but had no experience dealing with 22 figures at once that could not be manipulated.

Task three: One-shape-at-a-time. This task differed from the previous two tasks in that the child considered one shape at a time. The interview began by considering only triangles. The interviewer held in her hand the cards with the figures shown above in Figure 1. The cards were presented one at a time and each child was asked: Is this a triangle? Why? The child was allowed to take hold of the card, rotate it, and take his time to consider the one figure. After the child answered, the interviewer took back the card and placed another on the table. The same questions were repeated for each card.

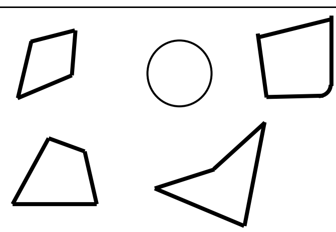
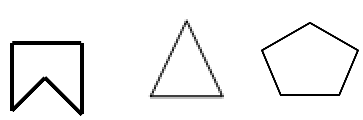
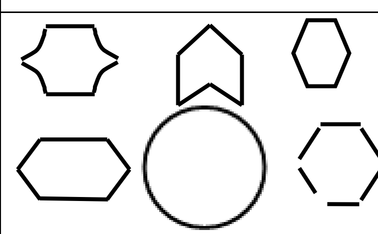
Is this a quadrilateral?	Is this a pentagon?	Is this a hexagon?
		

Figure 3: Is this a…?

When this set of cards was completed, the interviewer went on to quadrilaterals, pentagons, and hexagons using the figures in Figure 3. The same set of questions was repeated each time. This task was familiar to the children in that they had much practice in identifying one figure at a time and explaining their reasoning.

RESULTS

Johnny

On the *Free-sort task*, Johnny built the groups shown in Figure 4. In his words, there was a group of circles, triangles, quadrilaterals, pentagons, hexagons, not triangles (there are two groups of not triangles), not quadrilaterals, not pentagons, not hexagons, and one group for which he has no name. In general, it seems that Johnny sorts the figures according to geometrical shapes naming the figures as he goes along. At no time during this activity did he mention explicitly any critical or non-critical attributes of the figures presented.

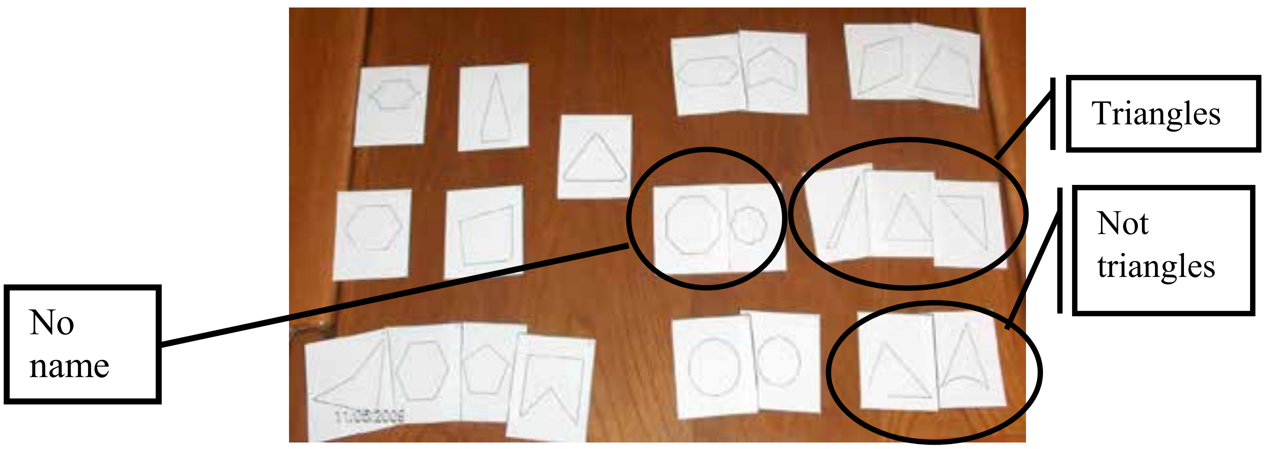


Figure 4: Johnny's sorting of the 22 figures

Within the group of "pentagons" we note that Johnny included two figures which were not pentagons, the regular hexagon and the concave quadrilateral. Yet, Johnny does not include any open figures or figures with curved lines in his groupings of the different polygons. In other words, he does not include non-polygon figures with polygons.

The first group of cards which does not consist of what Johnny terms as examples of geometric shapes includes the triangle-like shape that is not closed and the triangle-like figure with a curved side. Johnny groups together these shapes and says "not triangles". What does he mean here? There are many other figures which are also not triangles. The group of circles and the group of hexagons are also not triangles. In addition, the "not-triangle" figures are also not pentagons and not hexagons. Yet, he chooses to relate to them as "not triangles". What's more, he later relates to the quadrilateral-like figure with a rounded corner, as a "not quadrilateral". Perhaps Johnny is trying to tell us that these figures look like triangles or quadrilaterals but that he knows that they are not triangles and quadrilaterals. Perhaps Johnny is naming these figures "not triangles" and "not quadrilaterals". If this is the case, we may surmise that the name of the figure is the criterion Johnny used in his sorting and that Johnny has consistently used one criterion throughout this sorting. It is also possible that Johnny is sorting the figures according to geometrical shapes and then subdividing them into examples and nonexamples. In other words, open and curved figures are identified in relation to the polygon they most closely resemble.

On the *All-at-once task*, For the most part, Johnny identified correctly all of the shapes. There were three exceptions. When asked to identify pentagons, he failed to point to the regular pentagon and to the concave pentagon. In addition, he pointed twice to the concave hexagon – as a pentagon and as a hexagon.

On the *One-shape-at-a-time task*, Johnny correctly identified all of the figures on each sub-task. For each figure presented to Johnny, he consistently noted both the number of vertices and the number of sides. When the figure was open or had curved lines, he correctly identified the figure as a nonexample of the requested shape, explicitly referring to the critical attribute which was violated. From this task it seems that Johnny has a wide concept image of the figures presented.

To summarize, Johnny's ability to name the figures became known only from the *Free-sort* task. This is important because some children are able to identify a shape when given the name but may not be able to name the shape on their own. From the same task, we get a sense that for Johnny, nonexamples are connected to examples. In other words, a nonexample is not generic but related to some specific figure which it is not an example of. This too, is information about Johnny that we did not learn from the other tasks. Both from the *Free-sort* task and the *All-at-once* task, it seems that Johnny's knowledge of pentagons is less stable than that of other figures. Yet, on the pentagon sub-task of the *One-shape-at-a-time* task, he made no errors. Perhaps handling 22 figures at once raised the level of difficulty for Johnny, or perhaps, he merely has a narrow concept image of pentagons. Finally, if we had only engaged Johnny in the *Free-sort* task, we might have surmised that he reasons visually with geometrical figures, actions indicative of reasoning at the first van Hiele level. Only from the *One-shape-at-a-time* task were we able to learn that Johnny is capable of using mathematical language and critical attribute reasoning, indicative of the second level of geometrical reasoning according to van Hiele.

Randy

Randy's final groupings on the *Free-sort* task is presented in Figure 5. In her words, there is a group of figures with eight vertices, six vertices, five vertices, four vertices, three vertices, two vertices, and a group without any vertices. Throughout, the only shape Randy explicitly names is the circle. She placed the rounded corner triangles with the circles because "it too does not have any vertices." She does not say that this shape is a circle nor does she say, for example, that the open "hexagon" is a pentagon because it has five vertices. Rather, she consistently counts points, or what she calls vertices, and groups together figures according to their number. Her concept image of a vertex is somewhat blurry. She claims that both the regular hexagon and the curved "hexagon" have six vertices. However, the point connecting curved lines is not called a vertex. When discussing the "triangle" with a curved side, she points to the endpoints of the curved side and asks the interviewer, “Are these vertices?" When the interviewer does not respond, Randy goes on, “I think not. Because this is a curved line. But, let's say that there are three [vertices]. In other words, Randy notes that the figure has a curved line and questions the legitimacy of calling the points, vertices. Yet, she decides to act as if they are vertices and continues sorting along this line of reasoning. In the end, Randy sorted all the figures by one critical attribute – the number of vertices. (The only time she mentions the sides is when grouping the triangles.) In doing so, the world of geometrical figures, with its examples and nonexamples, becomes somewhat blurred.

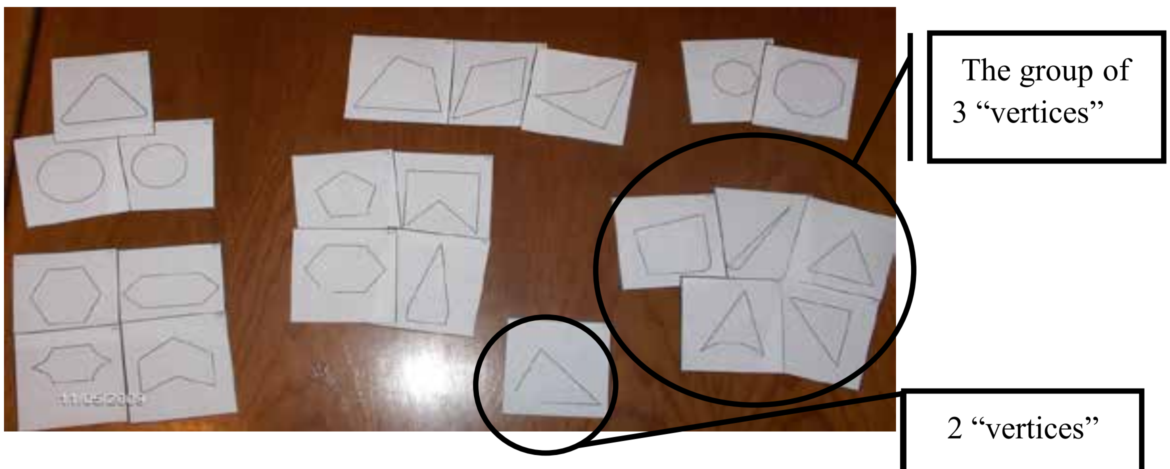


Figure 5: Randy’s sorting of the figures

On the *All-at-once task*, Randy correctly identified all figures except for the concave quadrilateral, which she missed pointing to. On the *One shape at a time* task, she correctly identified all of the figures. She not only related to sides and vertices but also to the figures being closed. For example, when identifying the prototypical triangle, Randy comments, "It has 3 vertices, 3 sides, and it is closed." Reference to closure only took place on the triangle sub-task. When reasoning about quadrilaterals, Randy referred only to the sides and vertices while for the pentagons and hexagons she referred only to the vertices. When presented with a nonexample, she explicitly referred to the critical attribute which was violated using correct mathematical language to express her reasoning. Interestingly, when identifying the open "hexagon", Randy claimed "it is open and it has 5 vertices." In other words, although the figure is open, Randy still counted the points. For the curved “hexagon” Randy said that it is not a hexagon “because it has six vertices but only two sides."

To summarize, from all three tasks, we see that vertices play a dominant role in Randy’s geometric reasoning. This was first evident from the *Free-sort* task and was backed up by the *One-shape-at-a-time* task, where slowly but surely, vertices were the only remaining critical attribute mentioned. If we had only implemented the sorting activity with Randy, we may not have learned that she is aware of other critical attributes. We may also not have learned that Randy is capable of identifying figures or we may have thought that handling 22 figures at once was too much. Yet, on the *All-at-once* task, she identified all but one figure. On the other hand, if we had only implemented the *One-shape-at-time* task, we may not have learned of Randy's dilemma regarding points versus vertices. We may also have missed that for Randy the number of vertices is perhaps more critical than the other critical attributes.

DISCUSSION

As mentioned in the background section, assessment of children's geometric knowledge may vary greatly depending on the set of examples and nonexamples used in the study. However, in this study, the set of figures was constant. What varied, were the tasks themselves. Thus, as we look back on the results of this study, we focus on the tasks themselves, their similarities and differences, their affordances and constraints, and the knowledge which each task brought to light.

The *Free-sort* task was both open-ended and unfamiliar. As an open-ended task, there was no correct or incorrect way for the children to sort the figures. Yet, different aspects of children's geometric knowledge could still be assessed. Ability to name figures was one such aspect. This task also allowed us to investigate the relationships children might perceive between figures and the generalizations children might have constructed along the way. This is in accordance with Lane (1993) who claimed that tasks of this nature may shed light on the cognitive processes that underlie performance, such as discerning mathematical relations, organizing information, evaluating the reasonableness of answers, generalizing results, and justifying an answer or procedure. Johnny's groupings of "not triangles" and "not pentagons" may reflect that Johnny relates nonexamples to examples. That is, a figure is not merely a nonexample. It is a nonexample related to some specific shape.

The *All-at-once* task was a closed task. Children could either correctly or incorrectly identify each figure. The obvious constraint of this task was the set of figures. While children were familiar with the task of identifying various figures they had no experience dealing with 22 figures at once that could not be manipulated. This was a new challenge. Thus, the task afforded us a glimpse into which figures may be confused with other figures (such as the confusion between pentagons and hexagons evident by Johnny). This task was also the only task where the child's response was indirectly challenged. Recall that even after the child had pointed to all of the triangles (or quadrilaterals or pentagons, etc.) he was asked yet again if he could identify another triangle. Thus, this task also afforded us a glimpse into the child's confidence in his ability to identify figures.

The *One-at-a-time* task was both closed and open in that it began with an identification which was either correct or incorrect but continued with an explanation which was more open in nature. The closed nature of both this task and the *All-at-once* task and the focus on identifying figures on both tasks allowed us to assess which figures children may find more difficult to identify. Having children explain their reasoning on the *One-at-a-time* task afforded us the opportunity to gain insight into the van Hiele levels at which they might be operating. Explanations also brought to light children's knowledge of appropriate mathematical language.

As noted above, different tasks afford children different opportunities to use and display geometric knowledge. But, not all children take advantage of the opportunities afforded by a given task. If a child only employs visual reasoning on one task, can we conclude that he or she is operating only at the first van Hiele level of reasoning? Previous research suggested that the van Hiele levels may not be discrete and that a child may display different levels of thinking for different contexts or different tasks (Burger & Shaughnessy, 1986). This study supports this suggestion as knowledge and reasoning which did not necessarily come to the fore on one task, sometimes appeared on another task. Thus, we conclude, a combination of tasks is advantageous when assessing both strengths and weaknesses of children's geometric knowledge. Our challenge as mathematics educators and researchers is to continue analysing the affordances and constraints of different tasks in order to optimize learning experiences.

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GAME PROMOTING EARLY GENERALIZATION AND ABSTRACTION[1]

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The paper presents and studies some cases of mathematical reasoning observed during a game, proposed to pupils 5-6 years old. The analysis of pupil's behaviours in front of the task furnishes some examples that prove the possibility of early and significant mathematical activities of generalization and abstraction.

THEORETICAL FRAMEWORK

Relations between play and learning in early mathematics are object of investigation and study. In particular, Bartolini Bussi (2008, p. 262-263) emphasises that:

> "The collective play is important both as socializing element, and as "instrument of production" of problematic situations. Fundamental is the encounter with the play with rules; it requires the overtaking and the solution of problems that we can relate to three areas."

These three areas are: language (understanding, presentation and invention of new rules), socialization (respect of rules) and abilities of mathematical kind (order and behaviour organization).

Schuler (2011), about the relations between play and learning, observes that theoretical models and empirical researches confirm the evidence of learning while playing, since plays offer situational conditions for learning. In particular, she highlights three main blocks: affordance, liability and conversational management:

> "[…] rules can offer mathematical activities beyond a material's intuitive affordance and thus create liability. Intuitive affordance of materials is replaced in games by (*the affordance of*) keeping the rules and winning *the game*" (Schuler, 2011, p. 1919).

Moreover she emphases "the central role of the educator". In fact, sometimes it is difficult to adopt a good equilibrium between a free and spontaneous play and a guided play. In other words, "Play is not enough. […] children need adult guidance to reach their full potential" (Balfanz, Ginsburg, & Greenes, 2003), but when the teacher proposes a play finalised to promote particular abilities, he risks to force in some way the child and to impose directions of work connected with the finality of the play.

We chose to report here some definitions of 'generalization' and 'abstraction' since they are concepts very difficult and their meaning depends from the theories. In the

1. Work done in the sphere of Italian National Research Project Prin 2008PBBWNT at the Local Research Unit into Mathematics Education, Parma University, Italy.

following part of the paper we will refer to these quotations, with the aim to explain our ideas and results.

Generalization is cited as typical form of mathematical thinking. Very often it is related to algebraic reasoning, but it is possible to observe the use of generalizations also in other mathematical activities. Generalization is often associated with abstraction, but, Mac Lane (1986) writes:

> "Generalization and abstraction, though closely related, are best distinguished. A 'generalization' is intended to subsume all the prior instances under some common view which includes the major properties of all those instances. An 'abstraction' is intended to pick out certain central aspects of the prior instances, and to free them from aspects extraneous to the purpose at hand. Thus abstraction is likely to lead to the description and analysis of new and more "abstract" mathematical concepts." (p. 435-436).

Malara (2012, p. 57-58) describes the 'generalization process' as:

> "… a sequence of acts of thinking which lead a subject to *recognize*, by analyzing individual cases, the occurrence of common peculiar elements; to *shift attention* from individual cases to the totally of possible cases and *extend* to that totality the common features previously identified".

Also Mac Lane (1986) uses the locution "shift of attention", but referring it to abstraction:

> "*Abstraction by shift of attention*. Some abstractions arise when the study of a Mathematical situation gradually makes it clear that certain features of the situation – perhaps features which were originally obscure – are really the main carriers of the structure. These features, with their properties, are then suitably abstracted" (p. 437).

In his theory of 'universal model' Hejny (2004) distinguishes six different stages: motivation, isolated (mental) models, generalisation, universal (mental) model(s), abstraction, abstract knowledge. In particular, concerning the 'Stage of generalisation' he writes:

> "The obtained isolated models are mutually compared, organised, and put into hierarchies to create a structure. A possibility of a transfer between the models appears and a scheme generalising all these models is discovered. The stage of generalisation does not change the level of the abstraction of thinking" (Hejny, 2004, p.2).

He describes the 'Stage of abstraction', as a stage in which a new concept, process or scheme is constructed, bringing a new piece of knowledge.

Dörfler (1991) differentiates two forms of generalization: empirical and theoretical.

> "The basic process in **empirical generalization** is to find a common quality or property among several or many objects or situations and to notice and record

> these qualities as being common and general to these objects or situations. The common quality is found by comparing the objects or situations, with regards to their outward appearance, isolated mentally, and detached from the objects and situations.
>
> In contradistinction to this form Dörfler introduces another one - called **theoretical generalization** – and describes it with the help of a theoretical model for processes of abstraction and generalization which can often lead to the genuinely mathematical concepts (propositions, proofs, etc.)" (Ciosek, 2012, p. 38).

Dörfler criticizes the first, since it realizes only a recognition process, while the second can contribute to the concepts construction. In its "model of the processes of abstraction and generalization" he emphasises the role of "invariants of actions" that emerge as a consequence of the repetition of actions and define a "schema".

Finally, we want to speak about row-column arrangements and related difficulties:

> "Children have a different awareness of the rows and columns arrangement. Some of them prefer rows, some of them columns. It appears that it was difficult to see both rows and columns, especially for young children" (Rożek & Urbanska, 1998, p. 304).

RESEARCH QUESTION

The present research is placed in the theoretical framework of early mathematical education by play. In particular, it deals with development of reasoning promoted in children from a game (play with rules). The initial hypothesis is that a suitable play can produce an early and spontaneous use of generalization. Our aim is to give answers to the following question:

> "Is it possible to promote in children the recourse to generalization in a game context?".

METHODOLOGY

The participants

The experiment took place in the last year of kindergarten (pupils 5-6 years old), in school years 2010/2011 (15 children) and 2011/12 (13 children), working weekly. They worked sometimes in groups (7-8 pupils), under the guide of the teacher[2], sometimes individually. The researcher was present for the whole activity.

The game of coloured houses

We describe only a part of a wider research based on a game (without winner), named "The game of coloured houses', invented from the author of the present paper[3] (Vighi, 2010). The game is based on a disposition of houses with three different

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2. I wish to thank the teacher Palma Rosa Micheli (Scuola dell'Infanzia Statale "Lodesana", Fidenza (PR), Italy), for her collaboration and helpfulness.

3. The idea of this game bore during a conversation with prof. E. Swoboda (University of Rzeszów, Poland).

colours (green, red, yellow) in a grid 3×3, respecting this fundamental rule (as an 'axiom'): in each row and in each column must be present three different colours. The possible different villages constructed following this rule are twelve. We report here some examples:

G	R	Y
Y	G	R
R	Y	G

a

Y	R	G
G	Y	R
R	G	Y

b

Y	R	G
R	G	Y
G	Y	R

c

Figure 1: Examples of villages

We observe that *a* and *b* present an exchange of colours (Green with Yellow), *c* is obtained from *b* exchanging the second row with the third row, *a* and *c* present an exchange of columns (they are 'symmetric').

The game can be seen as a simplified version of Sudoku, with a grid 3×3 (instead of 9×9) and only three 'symbols' (it is possible to use also digits 1, 2, 3 or letters in place of colours). From the mathematical point of view, it is a 'Latin Square', an $n \times n$ array filled with n different symbols, each occurring exactly once in each row and exactly once in each column.

The game can be executed with orderliness or by means of 'method of trials and errors'. It requires the contemporaneous management of rows, columns and colours. It is difficult, but important since it prepares to the concept of array that has a fundamental role in different significant mathematical concepts as multiplication, area of rectangle, Cartesian plane etc. Observing and working with two-dimensional dispositions of objects, pupils can make argumentations, discover theorems and use hypothetic-deductive reasoning. The game involves also local and global observation: during the construction of the village child applies rules working 'step by step', afterwards it needs to control if the whole village respects the task. So, she/he passes from an activity of construction to the 'lecture' of an array.

The procedure

The activities took place in the every day context. Teacher presented the game and she conducted it, promoting and fostering the viewpoints of children, without forcing their thinking, but waiting to listen their suggestions. Researcher observed, recorded on video, later she analyzed and transcribed dialogues, making comments.

Firstly teacher worked on the linguistic level: to understand the rules of the game it needs to clarify the meaning of the words "row" and "column" and their use. Afterwards she proposed the game in a context of motor activity (Fig. 2a), with the aims to involve pupils, to promote understanding, application of rules and socialisation also. Subsequently pupils played with coloured tiles and appropriate supports: small cardboards to fit in appropriate grooves (Fig. 2b) or to glue in a sheet of paper opportunely arranged (Fig.2c). At last teacher proposed a general conversation and a discussion about the constructed villages and their features.

ANALYSIS OF CHILDREN BEHAVIOURS

We present here only a brief qualitative analysis of the activity concerning village's construction with tiles.

a b c

Figure 2: different ways to play with coloured houses

A first important observation is about different behaviours, emerged from the analysis of videotapes: in some cases the child brings a tile and after he puts it in a box, in others cases the 'anticipatory thinking' leads to observe before the map of the village and after to choose an useful tile.

The village constructions were very different. Some pupils worked realizing systematically rows, starting from the top and proceeding without hesitation from the left to the right, constructing before the first row and afterwards the second and the third; in this way, the game became a simple exercise. But, as we forecast, most children had difficulties to coordinate rows and columns: some respected rules only in rows, others only in columns, others in part in row and in part in columns, jumping from the one to the other. Some pupils putted tiles on the grid without any organization. Sometimes rules were respected only locally, so, when the village was finished, its global construction was wrong. Sometimes child arrived in front of an impossible situation, for instance he cannot finish her/his village since no colour can be used without make a mistake. So, the concept of 'impossible' appears, which is important in mathematics, since it forces to cogitate and to formulate hypothesis of change and promoting hypothetical-deductive reasoning. For example, we report here the sequence followed by Mattia:

Y	R	G	Y	R	G	Y	R	G	Y	R	G	Y	R	G	Y	R	G
					Y	R		Y	R		Y	R		Y	R		Y
										G			G	R	Y	G	R

Figure 3: Mattia's construction

Mattia stops and, observing the first column, he says: "There are two yellow". So, he uses the number two to explain why the column is incorrect. Finally he decides to change some tiles and his village is correctly finished.

During the game pupils make argumentations as “It is impossible to have a red house here”, or “Here it must be a yellow house”. They use also hypothetic-deductive reasoning: “If I put here a green house, then …” and so on. In this way, they use theorems that are consequences of rules.

The final activity of control was often suggested from the teacher. In many cases we observe that if the first construction of the village is wrong, for children it is difficult to come back and to modify their own previous execution. Child, invited to control her/his village and eventually to modify it, showed different reactions: refusal or inability to continue, procedures based on trials and errors, fast and opportune individuation of modification. In fact, as Vygotskij and Luria (1987) observed, to show the contradictions present in their conclusions creates a big difficulty in pupils, but obviously it depends on didactical contract.

THE ‘LITTLE LADDER THEOREM’

After the village’s realization, teacher putted the sheets of paper with villages on classroom walls and she suggested pupils to compare their productions. It is well known that comparison is fundamental in mathematics, to construct concepts: to think about analogies and differences could promote it. A lot of suggestions emerged during this comparison: a local way of seeing leads to remark only some couples of tiles with the same colours, placed in the same places (e.g. “In the first village there is a green house here, in the second also”), while a global way to see promotes the individuation of “equal villages” or “symmetric villages” etc.

In particular, as Brousseau (1983) highlighted, the passage from the work on the desk (micro-space) to the observation of the classroom walls (meso-space), along with the visual perception stimulated by colours, promoted an important ‘shift of attention’ (Malara, 2012) from rows and columns, explicitly mentioned from the game rules, to the monochromatic diagonal present in each village. Some pupils said that “Villages have three houses disposed as a ladder, with the same colour”, others observed villages and confirmed this conjecture. To indicate the monochromatic diagonal discovered in each village some pupils used the locution “little ladder” since the disposition of tiles suggested the steps of a small ladder, others “bandy row”, some others ‘arrow’.

It is an “empirical generalization” (Dörfler, 1991), but also a generalisation in sense of Hejny (2004): the structure appears as generalizing isolated models, the connection between isolated models, created from a particular disposition of tiles, produces generalization.

In this way, pupils found and formulated a theorem: “In each village, there is a diagonal with houses of the same colour”. We named it the ‘Theorem of little ladder’. It is an example of “theorem in action” (Vergnaud, 1983), a theorem which children use when they deal with problems, obviously without a mathematical formulation, but as relational invariant.

Afterwards some of them observed the presence of two different kinds of diagonals: “from down to up or vice-versa”. It promotes the observations of both the diagonals and the formulation of another theorem: “In the other diagonal there are three different colours”, but this was perceived as no relevant aspect, since it seems to respect the game rule.

FROM GENERALIZATION TO ABSTRACTION

In the following activities we decide to investigate if children use ‘diagonal rule’. We observed different behaviours, as example we report here Chiara’s strategy (Fig. 4).

Figure 4: Chiara’s strategy

Chiara, using the ‘little ladder theorem’, starts with a yellow diagonal, she continues with two correct passages, after she makes a mistake that leads to have at the end a ‘wrong village’. When the teacher asks to control the village, Chiara realizes that in the third row there are two green tiles near, but she don’t perceive the error present in the second row, since the red tiles not are next to each other. Chiara remember and use ‘small ladder theorem’, but her execution shows mistakes and a difficult management of game rules. In fact, for Chiara the presence of a monochromatic ladder is only a new rule of the game, a starting point for her activity.

In the school year 2011/2012, we decided to submit to pupils of preschool (5-6 years old) a new game in 3D, consisting in the construction of a ‘palace’ (a cube), of three floors using 27 cubes, with similar rules: “In each wall face it needs to have three different colours in each row and in each column” [4]. In particular, the previous activity on village construction was realized putting coloured cubes (instead of tiles) one near to the other in a grid three times three (Fig. 5). Really cubes are more similar to houses and, as research documents, to work in 3D dimension is suitable for young

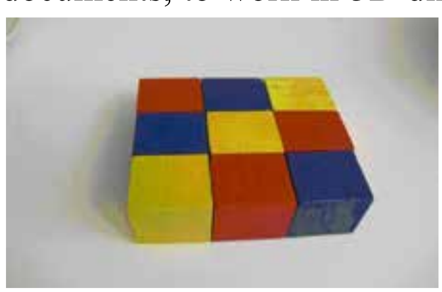

Figure 5: Village of coloured

4 The complete experience is described in (Vighi, 2012). See Poster Presentation in CERME 8.

pupils.

This change of didactical variables (from tiles to cubes) promoted an important breakthrough, since, as Rożek (1997) writes, in a row-column arrangement of figures the distance between objects influences in depth the observation. After the village construction with cubes, it happened that a child observed the presence of a yellow line (Fig.5), suggested from the colour and also from the idea of "straight line" and he incorrectly said that "there was a mistake". Teacher suggested that in fact all rows and columns respected rules and the child replies that "Yellow cubes are as an arrow" ('little ladder' in the previous experience).

Another breakthrough happened when a child observed that in the other diagonal there were three different colours: he indicates it with his hand, accompanying with gesture and sound: "here, blue, yellow, red, pum, pum, pum" and he added: "A 'point' entirely yellow, another of three colours, it is an 'X'!". In other words, the disposition of diagonals suggested the mental image of letter X. Its presence in all villages produced a passage from isolated models to a general model in the meaning of Hejny (2004), but … not only this.

Of consequence of "X discovery" some children changed their way of village construction, also in the following activities. They started putting cubes in 'X disposition' and completing the remaining parts. Obviously, in this way the game becomes easier: the construction of a coloured village changes a lot, since starting from diagonals, the placement of the other houses is obliged. We describe here, for instance, Nicholas strategy (Fig. 6):

Y			Y		B	Y	R	B
	Y			Y		B	Y	R
		Y	R		Y	R	B	Y

Figure 6: Nicholas strategy

There is a fundament difference between the use of one diagonal or two diagonals as starting point: in the first case (Chiara example) it is only an use of a theorem obtaining by generalization, in the second (Nicholas example) it appears another important and very strong 'shift of attention', but related to abstraction, in the sense of Mac Lane (1986): 'letter X' is understood as one of the "main carriers of the structure". It is an example of "theoretical generalization" in the meaning of Dörfler (1991): the 'letter X rule' becomes an "invariant of action" and it creates a "schema".

This finding of X was unexpected for us. As we said before, when we planned the activity our interest was on row-column arrangements, early reasoning etc. It is very surprising that, after its discovery, some pupils chose to adopt and to use the method of 'letter X' in the following activities. In particular, some children (38%) always used the 'rule of two diagonals', 32% the 'rule of a diagonal', and the remaining 30% other various methodologies.

The main result of the second experiment, maybe promoted from the use of wooden cubes, is the recourse to generalization and abstraction, promoted from the “X discovery”.

CONCLUSIONS

The “Game of coloured houses” is surely motivating for pupils: speaking with the teacher, I knew that children often proposed the game at home to their parents or brothers. It is a first goal; in Hejny’s theory the “motivation” is the first stage of development and structuring of knowledge.

The experimentation furnished positive answers to our research question. In particular, the mathematical learning is related to understand and to use mathematical methods. We expected only generalization, but also abstraction appeared. So, learning is realized on meta-mathematical level, since it is related to generalization (Malara, 2012) and abstraction processes (Mac Lane, 1986).

When we planned this game and its implementation in preschool, our aims were different, but generalization appeared along with the discovery of a monochromatic diagonal in each village. The use of X-strategy, we think, promoted the passage from generalization to abstraction. The recognition of an “invariant” or “schema” (letter X) to describe it, is a “symbolic description” in the Dörfler’s theory meaning.

So, for the young pupils involved in our experimentation, generalization and abstraction become an instrument of work. Also ‘theory of semiotic mediation’ (Bartolini Bussi & Mariotti, 2008) can furnish an interesting interpretation of the present research: an artifact (cubes) and a task (game rules) promoted the use of a sign (letter X) that express relation between artifact and knowledge. As we said before, in our case it is a metacognitive knowledge and it appears unexpectedly. We are determined to analyse our experience having the mentioned theory as theoretical framework.

About methodology, we agree with Schuler (2011): “Potentially suitable materials and games need a competent educator with regard to didactical and conversational aspects”. In our experience the role of the teacher and the conversational management appeared determinant. We want to add another important conclusion about “conversational management”: as we wrote before, the game was entirely conducted from the teacher with the presence of researcher as observer. The latter had the possibility to “peek and catch” some observations made from children, while the teacher was involved in the action. After an exchange of opinions with the observer, teacher took advantage from these suggestions and she used them in the following activities and conversational managements, improving in this way the activity in classroom.

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VIDEOCODING – A METHODOLOGICAL RESEARCH APPROACH TO MATHEMATICAL ACTIVITIES OF KINDERGARTEN CHILDREN

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In this paper a methodological research perspective is taken on the mathematical activities of Kindergarten children in mathematical situations of play and exploration. Videotaped verbal and gestural expressions of children as well as activities with the material are encoded with the help of a coding guideline, which is developed for mathematical activities. So courses of mathematical activities can be traced during a mathematical situation of play and exploration and even over several data collection points. At the same time connections of using different mathematical activities between the participants become clear. The introduced methodology of video coding connects qualitative with quantitative analysis methods.

INTRODUCTION

Starting point for the development of an instrument for video coding are videotaped mathematical situations of play and exploration, which occur within the framework of the project ‘erStMaL’ [1]. One overarching aim of ‘erStMaL’ is to trace the development of mathematical thinking of children aged between four and nine years from a mathematics-educational perspective (cf. Acar Bayraktar, Hümmer, Huth, Münz & Reimann, 2011; Krummheuer, 2011). The mathematical domains numbers & operations, geometry & spatial thinking, measurement, patterns & algebraic thinking and data & probability serve as reference points for the conception and the development of the mathematical situations of play and exploration (cf. Clements & Sarama, 2007). Within the ‘erStMaL’-project the mathematical situations of play and exploration [2] serve as an empirical research instrument (Vogel, in preparation). The developed situations provide a situational framework within the children work in tandems, together with a guiding adult, on mathematical tasks. The guiding adult has an expert status concerning these mathematical tasks. These situations are designed dialogically. The children’s mathematical activities, which are usually tied to the selected materials (artefacts) of the particular situation, are the centre of consideration (cf. Wells, 1999; van Oers, 2004). The materials (artefacts) are selected in a way that they, on the one hand show a narrative character and on the other hand they provide connection points for mathematical activities. The materials can therefore be seen as ‘culture tools’ (Bodrova & Leong, 2001, p. 9). Overall, an ‘overlap situation’ between the world of mathematics and the world of experience is generated by the designed mathematical situations of play and exploration (cf. Prediger 2001; Vogel, in preparation).

THEORETICAL BACKROUND

In terms of a social-constructivist perspective on learning mathematics, the situations of play and exploration activate the involved person’s ability to negotiate the objects` mathematical meanings. Dealing with the objects and how the objects can be used for the solution process of the mathematical task has to be clarified in a negotiation process by the children (cf. Brandt & Höck, 2011). This theoretical learning approach can be combined with the theoretical development approach of the co-construction, which is discussed primarily for early education processes (cf. Brandt & Höck, 2011). In the context of these interactionist approaches, the area of tension “between interaction processes between partners with equal rights” and “interaction processes with a rather disparate role allocation and clear differences“ (Brandt & Höck, 2011, p. 249, translated by R. Vogel) must be taken into account amongst the participants. At the same time, the mathematical situations of play and exploration create an area in which knowledge can be expressed situationally (cf. Vosniadou, 2007). According to conceptual changes, which do not exclusively exchange concepts but also change perspectives, a further development of mathematical concepts is integrated (cf. Vogel & Huth, 2010; Vosniadou, 2007).

RESEARCH PERSPECTIVE

Against this theoretical background, it is necessary to develop an analytical tool that enables the reconstruction of the childrens’ mathematical activities in the situational processes of negotiation. Through this reconstruction, on the one hand it is possible to track the development of mathematical concepts of children over the course of time. On the other hand, video sequences can be identified, which can be analyzed with other qualitative analysis tools on a micro level, in terms of the mathematical concept development. Particularly, the significance of the interaction of different semiotic resources for the development of mathematical thinking can be worked out (cf. Vogel & Huth, 2010; Givry & Roth, 2006).

VIDEO CODING – THEORETICAL FRAMEWORK AND APPLICATION

The presented method of video analysis combines qualitative and quantitative analysis steps. By using a coding guideline and by determining frequencies of the emerging categories, quantitative analysis of the videotaped mathematical situations of play and exploration are carried out. Ginsburg and Seo (2004) investigated children’s everyday mathematical activities and developed inductively mathematical content codes. For the development of our coding guideline a qualitative content analysis has been fulfilled. In this process an offset of deductive and inductive developed categories has been created (cf. Mayring, 2000). Figure 1 shows an overview of the process of category determination. Several relevant mathematical concepts from the five mathematical main domains, which are the basis of the different situations of play and exploration, formed the starting point (cf. Clements & Sarama, 2007). Considering the video data, this first category-system was supplemented and extended inductively through the mathematical activities that were expressed by the participating children. Therefore we looked at various videotaped situations of each domain considering the first four data collection points. After determining a constant category system the coding accordance of different persons was checked and the categories were proved and verified again.

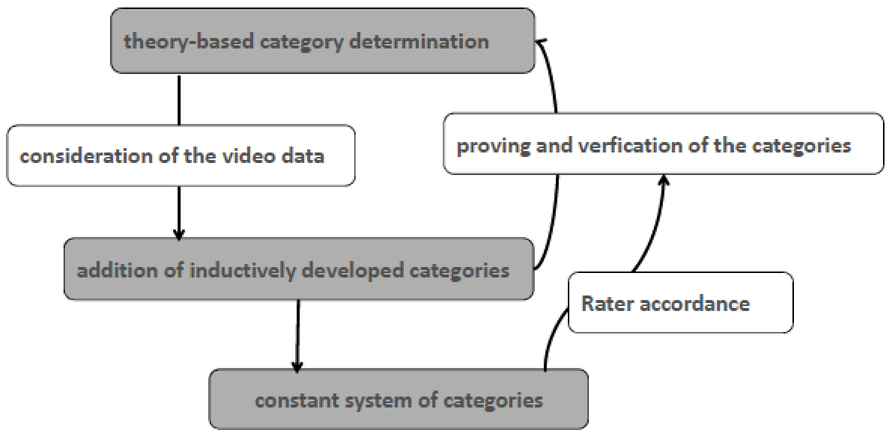

Figure 1 Process of developing the coding guideline

The coding guideline consists of eleven main categories, which are divided into several subcategories. The main categories refer to mathematical activities, which are attributed to the five mathematical domains already mentioned above. In the following paragraphs the main categories will be presented briefly.

The main category ‘determination of quantities – operation (QO)’, which is attributed to the domain of ‘numbers & operations’, will be presented in detail because of its importance for the selected example. A reference to empirical research context is given. It is a domain that has been elaborately researched for mathematical early education. A distinction is made between counting as a series of numbers (right or wrong) and one to one assignment of numbers and objectives (cf. Fuson, 1988). Additionally, the field of subitizing is mentioned. This kind of entry of quantities is often described as a preliminary stage of counting. ”Results suggested that spontaneous focus builds subitizing ability, which in turn supported the development of counting and arithmetic skills.” (Clements & Sarama, 2007, p. 473). Another subcategory serves to register children’s activities and statements in the fields of seriation (cf. Clements & Sarama, 2007). Further subcategories serve to register simple operative activities in the field of addition and subtraction. Subcategories are indicated with numbers after the short cut of the main category (e.g. QO1 for ‘counting’, QO2 for ‘determination of quantity without recognizable counting processes’).

The main categories ‘mathematical structures (MS)’ and ‘patterns (pattern units, band ornaments, parquets) (PA)’ are attributed to the domain of ‘patterns & structures’: The central focus of the main category MS is on the construction of structures in a set of objects as well as between several sets (cf. example of the video coding process). It becomes apparent that the children integrate structures of their every day life into the situations and partially interpret these structures mathematically (MS1). The category PA includes geometrical activities with regard to the work with patterns, e.g. band ornaments should be added, continued or should be reinvented by the children. Here, it is taken into consideration that children might create patterns, which cannot be interpreted as patterns from a mathematical perspective, but are described as patterns by the children. As a result, sequences from the video data can be identified where several patterns of interpretation from within the children's world and the world of mathematic become apparent. Comparable subcategories have been developed for dealing with parquets.

Within the coding guideline, activities in the domain of geometry and spatial thinking are determined through three main categories: 'topological fundamentals and activities (TP)', 'components of spatial thinking (ST)' and 'geometric shapes and 3-D figures - transformation between plane and space (GE)'. There is a distinction between dealing with closed and open lines and finding ways in narrative contexts e.g. within 'railway-situations' and activities of Euclidian geometry. Components of spatial thinking should also be taken in account in a separated category. The domain of 'measurement & size' is represented by the main category 'measurement (ME)'. It is proposed that within the subcategories, activities of direct or indirect comparisons are coded.

The mathematical domain of 'data and probability' is included in three main categories: 'data (DA)', 'chance (probability) (CH)' and 'combinatorics (CB)'. In view of the example, the domain data must be emphasized. Focus is on elementary and complex processes of sorting as well as on adequate representations for comparison of quantities. The category 'sundries (SU)' subsumes all activities (expressions) that can not be interpreted mathematically.

The process of coding involves mathematical interpretations of verbal and gestural based statements of the children as well as actions with the material according to the coding guideline. The multimodal statements of several persons are coded separately. Therefore, a coding unit of 30 seconds is allocated to a maximum of two subcategories. The following example of transcription [3] reproduces most of the coding units 16 and 17 (cf. Figure 2) and it also shows which subcategories are allocated to the involved persons' statements.

Coding unit 16 (extract from the 30 seconds, start):

<	René	look. one two three four *pointing at the dots of a 'ladybug-card' which is in front of him* one two three four *pointing at the dots of the 'ladybug-card' in his hand*
<	Marie	**two** *taking a 'ladybug-card' with two dots out of the middle and dropping it in front of him on the floor together with the other 'two-dots ladybug-cards'*
	B	yes-

Marie and three\ do I have a three/ *observing a 'ladybug-card' with three dots, which is lying in the middle and then looking at a 'ladybug-card' which is in a row in front of her on the floor*

René *taking the 'three-dot ladybug-card' out of the middle*
he/ . I have a three\ *dropping the card to the other 'three-dot ladybug-card' in front of him on the floor*

Coding (Subcategories):
René: counting (QO1) & elementary sorting operations according to one category (DA1)
Marie: recognizing structures within sets (MS1) & elementary sorting operations according to one category (DA1)
B (guiding adult): mathematical structures (stimulus) (MS (x))

Coding unit 17 (extract from the 30 seconds, start):

< René Mum and Dad\ *pointing at two red 'four-dots ladybug-cards'*
Brother and Sister\ *taking two red 'three-dots ladybug-cards' in his hand*
Mum and Dad\ Brother and Sister\

< Marie *pushing all 'ladybug-cards' lying in front of her on one pillar*
mine is a whole kindergarten\...should be a whole kin-

Marie that is a whole Kindergarten\

B *laughing* a whole Kindergarten/

Coding (Subcategories):
René: recognizing structures within sets (MS1)
Marie: recognizing structures within sets (MS1) & determination of quantity without recognizable counting processes (QO2)
B: mathematical structures (MS)

EXAMPLE OF AN ANALYSIS

For this paper the mathematical situation of play and exploration 'Ladybugs' is selected for an exemplary analysis. In the settings the participating children are observed in constant tandem (pairs). In these pair settings the children always attend the same mathematical situation however for each data collection point the mathematical tasks and materials are adapted to the children's age if necessary. For the current analysis the selected tandem consists of René and Marie. They deal with this situation at three different data collection points. The data collection point T2 is missing due to the research design (situation with the kindergarten teacher). At the data collection point T1 René and Marie were 4;9 years old.

In the 'Ladybugs'-situation the children can differentiate between similar objects, which differ in several attributes. The children work with 'ladybug-cards' which differ in colour (red, green, yellow), in number of spots on them (one, two, three, four) and in the shape of the spots (circles, triangles, squares). No 'ladybug-card' appears twice. In the first working phase the children are encouraged to sort the ‚ladybug-cards' according to different criteria and to establish different 'ladybug-groups'. In the second working phase the member of the research team presents a triplet of big 'ladybug-cards'. The big 'ladybug-cards' also differ in colour, shapes, size and number of spots. The children should decide which of the three cards does not fit.

The results of the analysis are visualised in two different forms. Below, a time course that shows the emerging mathematical activities, its transitions and connections at one data collection point in one situation is presented. Here, it is possible to follow the interaction between the participating persons regarding its influences on the mathematical activities carried out by one person. The second form of representation shows the quantities of the mathematical activities over time. The percentage frequency of main categories or subcategories at different data collection points can be compared and a possible transformation may become apparent. Children's preferred mathematical domains can be described through comparing and contrasting their percentages.

Time course

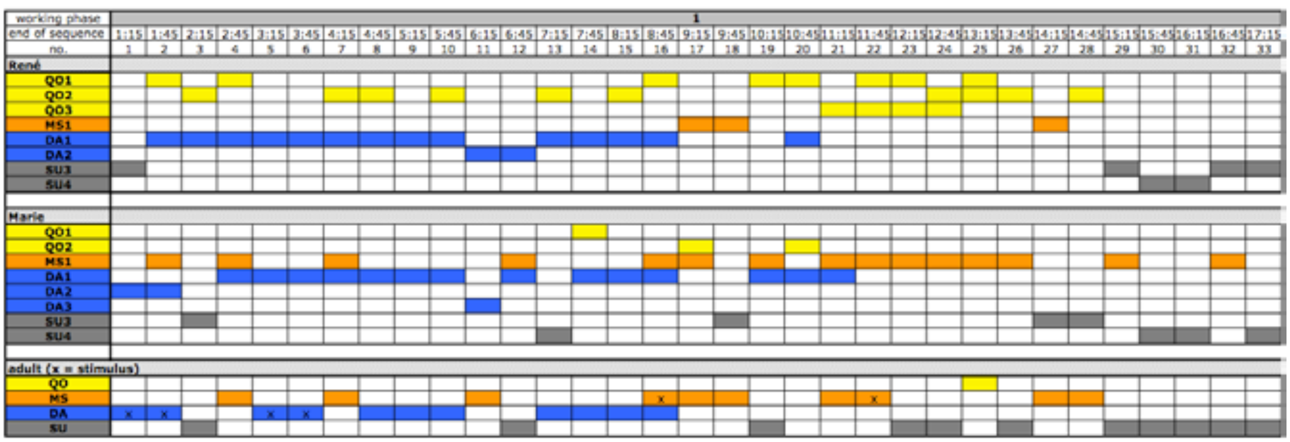

Figure 2 Time course of the first working phase of the 'Ladybugs'-situation carried out at the data collection point T3

Figure 2 shows the time course with the coding of the mathematical activity of the two children and the guiding adult in the first working phase of the 'Ladybugs'-situation at the data collection point T3 (Marie and René: 5;10 years). The guiding adult starts with impulses from the domain data (DA), which are picked up by the children. In the sequences from 1-16 the 'ladybug-cards' primarily are sorted by one specific criteria (colour, shape, number) (DA1). Marie and René combine the sorting process with various mathematical activities. René has a particular interest in determining the numbers of spots on the 'ladybugs-cards'. Therefore he uses different strategies: 'counting (QO1)' and 'determination of quantity without recognizable counting processes (QO2)'. Marie is specifically focused on the structural relationships of the objects of different sets (MS1). She uses the context of family and kindergarten. There is an intensive exchange between the main categories QO, MS and DA to be seen during the coding units 16 and 17. This indicates a so-called ‘dense’ sequence, in which different mathematical concepts within the activities become apparent and are placed in relation to one another (cf. example of the video coding process). After those coding units Marie remains in the domain of algebraic structures until the end of the working phase, deepening her idea of structuring the sets. Rene’s mathematical activities can mainly be attributed to the domain ‘determination of quantities – operations’. Once he has sorted the ‘ladybugs-cards’ according to colours he starts to bring the cards in an order according to the number of spots upon the back and determines which ‘ladybug-group’ is the biggest one. Over time it becomes evident that during the span of a mathematical situation of play and exploration sequences can be observed in which the activities of the involved persons cannot be interpreted as mathematical activities. Nevertheless, these activities can also be important for the children and the course of the situation. From coding unit 29 until 33 Marie begins with a detailed description of her ‘ladybug-kindergarten-groups’ and starts talking about a fictional excursion for the groups. René is sitting next to her, not entirely detached and listening partially. Overall, it becomes obvious that the children combine activities from within different mathematical domains and they switch from one domain to another.

Quantities of the mathematical activities

If you compare the percentage frequencies of the emerging major activities of René to the ones of Marie you will get to the point that they remain true to their favourite mathematical activity (Figure 3). The fact that through all the data collection points the domain data remains constant is a result of the situation’s and the material’s design. Compared to Marie René’s obviously preferred activity is out of the domain of determination of quantity - operations. The courses over time allow the following interpretation: At point T1 from the data collection the children bring up a diversity of mathematical domains. This diversity of different mathematical activities decreases over time. It is possible that this is a result of the fact that the children already know the task and the material and it does not require much testing anymore.

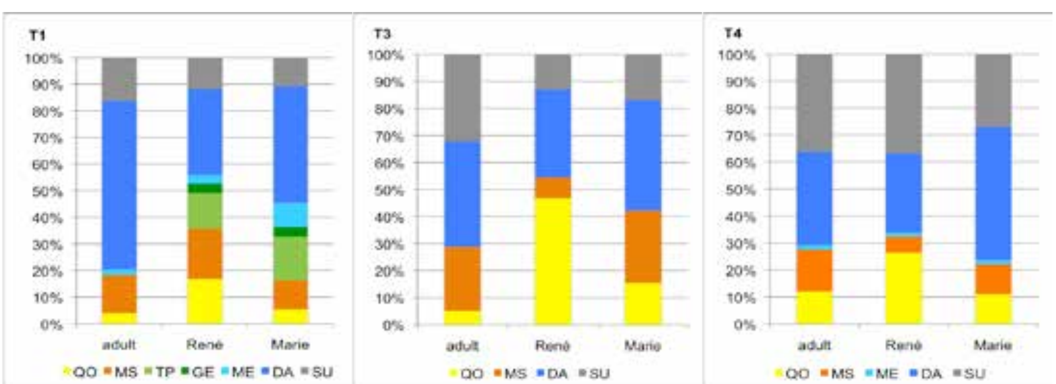

Figure 3 Percentage frequencies of main categories at different data collection points

If you focus solely on the main category there does not appear to be a major change between the data collection points. But when taking the subcategories into account one notices that there are changes in the mathematical activity over time. Figure 4 shows the subcategories of the main category 'determination of quantities – operations (QO)'. The subcategories are counting (QO1), determination of quantity without recognizable counting processes (QO2), ordering (QO3), operations – addition and subtraction (QO4) and operations – multiplication and division (QO5).

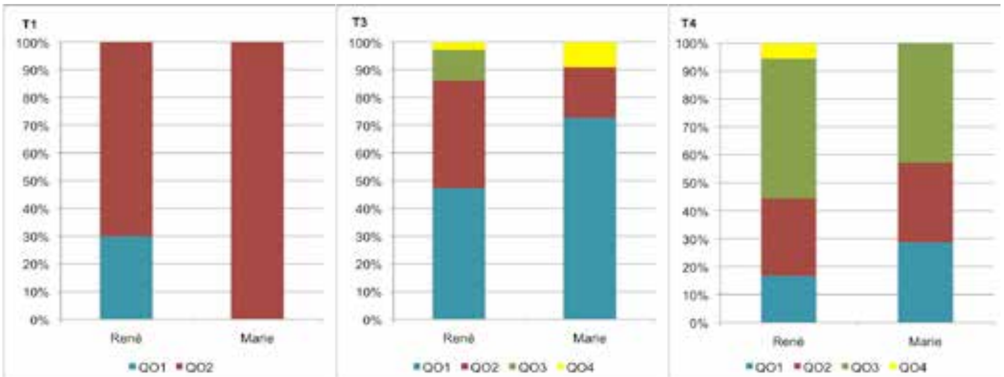

Figure 4 Percentage frequency of the subcategory 'Determination of quantities – operations' at different data collection points

At the first data collection point Marie's activities out of the domain 'determination of quantities – operations' can be exclusively assigned to the subcategory determination of quantity without recognizable counting processes (QO2). She distinguishes between less and many 'ladybug–cards' however without counting them, whereas René tries to confirm these assumptions with noticeable counting processes. Nearly one year later at the third data collection point the children focus on counting processes. At the data collection point T4 (Marie and René: 6;5 years) the focus of the children is on sorting the 'ladybug-cards' in series according to the number of spots and bare counting processes are less frequently realized.

CONCLUSIONS AND FUTURE PROSPECTS

The example already shows the potential of this type of video analysis. The results show that the mathematical main categories remain constant over time, while the subcategories change. Therefore the results of the video coding process offer reference points for the reconstruction of the development of mathematical thinking in different mathematical domains. The results of analysis of various mathematical situations of play and exploration and different tandems of children suggest the existence of preferred mathematical domains for particular children during the process of problem solving. This result could be used to examine specific groups of children in their mathematical preferences. In addition, the results show that the procedure of video coding enables an identification of sequences with reference points for further analysis regarding mathematical concepts. The coding can also be used as an evaluation of the situations of play and exploration which is necessary for a possible advancement to a diagnostic tool.

NOTES

1. The project 'erStMaL' (early Steps in Mathematics Learning) is a longitudinal study, which accompanied the children during their time in kindergarten and primary school. 178 children in 12 day-care centers participate in the study. Data collection takes place twice a year and is carried out in the familiar environment of the day-care-centers. The project is established at the IDeA centre (Individual Development and Adaptive Education of Children at Risk) in the context of the LOEWE-Initiative.
2. All the situations of play and exploration were developed by the research team of the project 'erStMaL'.
3. Explanation for understanding the transcript: / … lifting the voice, \ … lower the voice, - voice in abeyance, pause in speech: one point according one second, action are in italics, < … happened at the same time.

ACKNOWLEDGEMENT

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HOW FOUR TO SIX YEAR OLD CHILDREN COMPARE LENGTHS INDIRECTLY

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This article will introduce one part of a research studying how four to six year old children develop an idea about indirect comparison of length, with the aid of diverse tools. In a qualitative survey the solution processes of altogether 40 children have been observed. Hence, this paper will show the system of categories and will also present initial results.

INTRODUCTION

"We are convinced that dealing with measures is the most suitable way to introduce five-year-old children to the world of figures and mathematical terms" (Castagnetti & Vecchi, 2002, p. 14). In spite of this statement, Copley notices that "measurement is an often neglected mathematics topic for young children“ (2006, p. 17). Furthermore, Benz states that, during mathematical activities, German kindergarten educators mainly focus on arithmetic (2012, p. 22). Only 17% of the kindergarten educators who took part in that survey (n=589) answer that measurement is part of their mathematical activities in kindergarten. A fact which is "astonishing, because the reference to everyday activities is very obvious concerning measurement“ (ibid. 2012, p. 22). This is also true for the area of length "although of the […] physical quantities, length is undoubtedly the most primary“ (Buys & Moor, 2005, p. 18). It can be stated that in descriptions of the development of a concept of length, comparing objects indirectly constitutes an important aspect (e.g. Battista, 2006; Clements & Sarama, 2009; Piaget, Inhelder & Szeminska, 1975). Therefore, this article focuses on comparing items indirectly.

Looking on empirical studies it can be stated that there is a lack of studies investigating especially younger children`s ideas about length and also about children's competences of comparing indirectly (Gasteiger, 2010). Still, these findings are indispensable in order to have the possibility in kindergarten or in school to tie in with the abilities and perceptions of the children.

Nevertheless, there are some studies which investigated children`s concept of length focusing especially on competences of comparing indirectly. But these studies took place with children starting school or attending school (e.g. Hiebert, 1981; Nunes, Light & Mason, 2003; Schmidt & Weiser, 1986). The focus of this paper will be on competences concerning comparing items indirectly of four to six year old children.

THEORETICAL BACKGROUND

The following section deals with a normative aspect of what kind of competences are identified for comparing items indirectly. Empirical results are included as well while describing these competences. One must have in mind that all research results are referring to children in primary school. Only the study by Schmidt and Weiser (1986) children were asked who were just starting school.

Concepts of length and its linear character

In order to be able to compare lengths indirectly it is necessary to develop an idea about lengths and its understanding of the attribute (Nührenbörger 2002, p.100). This means that children have to learn to focus their attention only on the length of an object while ignoring its other attributes, for example like colour or taste.

Comparing objects directly

In order to compare objects directly understanding additivity (Clements & Sarama, 2009), decomposing (Battista, 2006) and part-whole relationships are essential. Referring to children just starting school, Hiebert (1984) showed with his survey that the idea that the length of an object consists of the sum of its different parts is not new to these children. Griesel (1996) and Nührenbörger (2002) are convinced that children who are capable of comparing objects directly have learned that it is possible to divide the longer object in two parts. Whereas one of these two parts is as long as the shorter unit that has to be compared and the second part is as long as the overhanging piece. Consequently, children who can compare things directly would also understand the different concepts of additivity, decomposing and part-whole relationships.

Conservation and transitivity

Regarding the importance of conservation there are different positions. Piaget et al. (1975) emphasized the importance of understanding conservation and transitivity. They are convinced that children are only able to measure when they have learned to understand transitivity and conservation. In contrast to that, Hiebert (1981, p. 208) observed that first-grade children who did not yet have developed an understanding of conservation and transitivity were able to implement different measurement strategies successfully (cf. also Schmidt & Weiser 1986, p.150f.).

Unit iteration

If a shorter medium is used to compare an object indirectly the comparison will only be successful if the children can control the unit iteration and if they are able to count these units. Together with the transitivity many authors highlight the key role of this expertise (Battista, 2006; Clements & Sarama, 2009; Piaget et al., 1975; Schmidt & Weiser, 1986). While repeatedly conducting the unit iteration, it is important to focus on the fact that there are no gaps in between the units. Schmidt and Weiser (1986) observed in their survey that even those children starting school who implemented the idea of the unit iteration did not attach value to an entire measurement without gaps.

Application of suitable adjectives

In order to be able to successfully compare objects indirectly children have to be able to describe the results with the aid of suitable adjectives. Relational terms such as *longer, wider, lower...* are prerequisites for being able to communicate about lengths (Gasteiger, 2006, p.11). Nührenbörger (2002) showed that faulty comparisons were not wrong because the comparison itself was wrong but because the comparison and its result were not described correctly with the second graders. Whereas Schmidt and Weiser (1986) could prove that children starting school use relational terms properly in connection with measures even if they do not have an idea about measurement. For example, when being asked if three meters or five meters are longer the children answered correctly. It seems as if children who have an understanding of figures a "transfer within" (Schmidt & Weiser, 1986) takes place. The relational term 'is longer than' can – at least verbally – be referred to relations between figures in a correct way ("is a bigger number than", "is bigger than", "is more than") (Schmidt & Weiser 1986, p. 145).

Regarding these aspects it becomes obvious that comparing length indirectly requires many different competences. Is dealing with measurement, especially length, (Castagnetti & Vecchi, 2002) to demanding for young children particularly an indirect comparison? In order to answer this question the following research question has been addressed.

RESEARCH QUESTIONS

Which individual solution processes can be observed for the different aspects of comparing two different lines indirectly?

STUDY DESIGN

In order to get first insights into children's solution processes at the age of four to six for comparing indirectly a qualitative method was chosen: Qualitative interviews were conducted. All children had to solve the same task, however, the following strategy was oriented on the children's expressions and strategies. Consequently, the following study is a cross-sectional study with „investigating" interviews (Lamnek, 2005) with the aim to investigate the strategies of children when doing an indirect comparison.

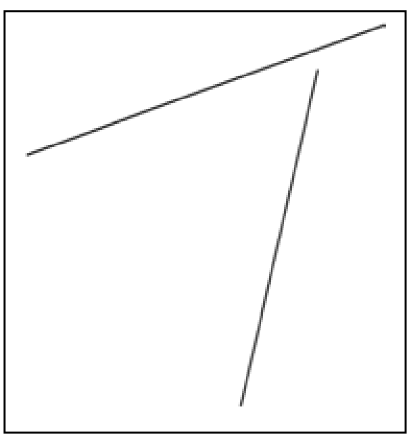
The interview is an artificial situation for the children where they are confronted with tasks that they would not have asked in such a way or would have answered in another way in a different situation. A participant observation would picture a „natural" situation but with the help of the interview it is ensured that the child passes through the steps of the solution of the task. Moreover, the child uses the material of which the interviewer is convinced to get the most information of the strategies children are using to do an indirect comparison (cf. Marotzki, 2011, p. 114). The artificial situation can lead to the fact that a child tries to meet the expectations of the interviewer and answers respectively (cf. Beck & Maier 1993, p.

162). The children might possibly not show their original concept concerning the indirect comparison but show the concept, which is in their opinion the “correct“ concept or the concept, which they think the adults expect them to use. Through the possibility to ask questions in the interview and to demand explanations concerning the strategy that is used this differentiation between the original strategy of the child and the strategy that is shown isn’t a problem. In contrary, it establishes more possibilities to unscramble the actual often complex view of the children in how to manage an indirect comparison.

The qualitative interviews were videotaped. In the analysis of the interviews categories were generated to describe the children’s processes.

The research was conducted with 40 children as a cross-sectional study with four to six year old children. The children were selected randomly.

On the basis of one task, which was used in the interviews, the generating of the categories will be described and first results will be illustrated. Following with Oswald there will be also some quantitative data given, due to the fact that quantitative data can be one aspect of qualitative reality (Oswald 2010, 186). Moreover, through quantitative data expressions like typically, generally, frequently, rare, can be avoided. These expressions are not clearly defined and might complicate the comprehension. Besides the interpretation of the qualitative data there will be quantified details (e.g. number of children using the same arguments).



In the beginning, two stripes of sellotape (1,20m and 1,30m long) were taped on the floor (see figure). Initially, the children were asked to state which line is the longer one. After that, they not only had to give a reason for their idea but also had to give an idea about how to proof their answer. The second step included giving the children several tools which helped them to proof their first statement: different wooden sticks and cords, a 30cm long ruler, a set square, a measuring tape, a surveyors tape and a folding yardstick. Use of the terms “longer” and “shorter” the clarification what property it is on, if one talking about “length”, was previously audited by other tasks.

RESULTS AND INTERPRETATION

While the children were trying to solve that task the following steps could be identified: at first length comparison without any tools, selection of a tool, the application of the tool and the reasoning.

In the following, some parts of the classificatory scheme that was generated to analyse the children`s actions will be explained and interpreted.

First length comparison without any tools

All children followed the request to compare the two lines with each other. It was especially interesting how the children reasoned the results of their comparison. Thereupon, the following four categories could be generated:

- The endpoints were related to each other as if compared directly (n=10).
- “I have seen it” was explained as a substantiation (n=8).
- Imaginary units were counted (n=4).
- 18 children did not give any explanation.

Luka (female; 5;4 years old) can be included into the category *Imaginary units were counted*. She tried to implement the task in the following way:

Interviewer: What do you think, which of these stripes is longer?

Luka: (points her finger in the air) This one!

Interviewer: Why do you think so?

Luka: Because I have counted it.

Interviewer: Can you show me again how you did that exactly?

Luka: (taps her finger along the line and counts imaginary units) 1,2,3,4 .... 40 (doing this, she not only skips some numbers but also not every tap with her finger corresponds to one number).

Luka knows that if two lengths are measured and the figures are contrasted it is possible to compare them with each other. Thus, she equalizes the idea of measuring with the idea of counting. In the course of this, she does not consider equal units.

Selection of the tool

Referring to this solution step it becomes apparent that in this survey[1] fourteen children took a measuring tape, sixteen a folding yardstick, five a surveyors tape and five took a ruler. Solely four children used a stick or a lace as a non-standardized tool. Finally, one child did not use any tool[2] at all.

Nunes et al. (1993) could observe that most six to eight years old children – provided that there is one at hand – prefer a standardized tool to compare things indirectly. Consequently, the study underlying this article approves Nunes et al. observation for younger children. It can be assumed that children equate the process of comparing things indirectly with a measurement process and thus, while measuring, try to imitate adults. Likewise, it could be possible that the children taking part in the survey did not take sticks or laces because these are ordinary toys, whereas standardized measuring tools could be much more attractive in the children`s eyes.

[1] In Zöllner (2012) there are slightly different numbers of children, because there vague situations were not included. In terms of intersubjectivity. These ambiguous cases were discussed later with a team of researchers and, consequently, they can be included here.

[2] Some children used different tools which have been counted.

Usage of the tool

While analysing the usage of the tool the focus will be on how children apply it, whether they have taken into consideration to use the tool straight and on how the children deal with the endpoint. Here, not only the approaches of those children who used a standardized measuring tape have been regarded but also of those children who used a stick or a lace.

Applying the tool[3]: It is salient that all children applied the tool in a way that the stripe`s and the tool`s starting point meet each other (n=31). Except for the ruler, the starting point of each tool equals the scale`s zero point. Four children tried to apply the tool so that one of its endpoint coincides with the endpoint of the stripe and five children applied it without any visible relation to either the one or the other endpoint.

Process: To compare the lines successfully the children have to put the tool straight on the line. Due to the fact that, when using an inflexible tool, such as a ruler, one automatically uses it in an even way. It is not possible to come up with a statement explaining whether the children have consciously tried to apply the tool in an even way and, thus, those cases will not be regarded. Six of those children who have used a measuring tape tried to use it in an even way, meaning that they tried to use it straight.

Reading of measured values: Thirteen children either tried to read a number from the scale of the tool on their own or asked the interviewer to read it to them. In other words, those children paid attention to the scale`s figures and use them as an aid. Whereas others used a certain point as a marker (without paying attention to the figures) or they used their finger. On the other hand, eight children opened the tool so far that it equalled the length of the line. Eleven children completely ignored the endpoint of the line in relation to the endpoint of the tool.

Usage of the tool by the second line: Measuring the second line, ten children did not use any tool (neither the one used for measuring the first line nor another one). Therefore, one could conclude that they observed adults using the meter and, thus, try to imitate these movements without having developed an idea of comparing indirectly.

In observing of how the children applied the tool on the second line it was examined whether they did it in a similar way as they had tried to measure the first line or whether they chose another approach.

It could be observed that twenty-four children used the same tool in the same way to compare with the first as well as with the second line, whereas six children used a different approach for using the tool with the second line. . To determine the result

[3] Regarding those children who used several tools, in the analysis the focus was on that solution process which the children favored most. Two children have processed several approaches with the same tool and, thus, both of these processes have been acquired.

twenty children used the same approach as before. Four children did not do that. Six children completely ignored the endpoint of the second line in relation to the endpoint of the tool.

Generally, it can be stated that those children who used a standardized tool were more successful in comparing the lines indirectly than those who used an arbitrary tool. Only one of these children that had used an arbitrary tool gave an explanation. Indeed, this explanation does not refer to the process of measuring itself but the child tried to correlate the endpoints as if it is a direct comparison.

Again, this result equals the ones that Nunes et al. (1993) observed in their survey with older children. This investigation showed that children are more successful with a standardized tool even though – accordingly to its construction – they did not use it correctly. This approach becomes visible when regarding Elsa`s (female; 6;3 years old) approach:

Elsa:	(opens the folding yardstick and tries to apply it in a way that its starting point coincides with the line`s starting point (130cm). Then she uses the edge of her hand to mark the line`s endpoint) So far, that much.
Elsa:	(still holding her hand on the folding yardstick, she walks towards the other line (120 cm) and applies it so that the "handmarked" point equals the starting point of the line. The yardstick`s endpoint is above the line`s endpoint) Well no, they are not of equal length!
Interviewer:	And which one is the longer one?
Elsa:	This one! (points to the line which is 130 cm long)

Elsa chose a folding yardstick to compare the stripes indirectly. She used it very proficient but, nevertheless, as if it was a non-standardized tool. This means that, considering the fact that she used her hand as a marker, the scale on the yardstick is not relevant to her at all.

Conclusion

Finally, after measuring the children have not only been asked again to compare the length relation of the two line, but they also have been asked to reason their thoughts. Some children (n=9) came up with their first explanation again – which shows that using a tool did not make them choose another reason to describe their result. Ten children did not reason their result at all. Twelve children reasoned their result with the aid of markers and a direct comparison. With the help of a marker, a figure or by adjusting the tool the children tried to depict the first line on the tool. After that, they compared their result with the second line and deduced their outcome. Two children gave up this task before they came to an end. Seven children reasoned because of the larger/ smaller relations between numbers, meaning that those children used a standardized tool and, if necessary, they asked someone to read the figure to them or they read it themselves. After that, they compared those figures to be able to conclude the stripes` lengths.

Luka can also be included into this category. Due to the fact that both numbers (130 and 120) exceed her active range of numbers, she always asked, after having measured both lines with the measuring tape, which of the two numbers is the bigger one. Nevertheless, she correctly correlates the bigger number with the longer line. As this example shows, it is noticeable that Luka accomplishes a “transfer within” (Schmidt & Weiser 1986) from the numbers to lengths.

Summary

Summarizing the results and referring them to the aspects that have been emphasized before the following results become apparent: The participating four to six years old children had no difficulties to clearly identify the different lengths and their linear character at this task. This is not surprising because concerning lines, the linear character and the usage of the concept “length” is clear. All children of that survey could easily compare objects directly. Some children had the idea: Measuring equals counting.

As it was described above, some researchers claim that developing an idea about transitivity is the most important aspect in order to be able to compare things indirectly (Piaget et al, 1975). Indeed, observing the children`s actions it becomes obvious that none of them employs transitivity in its mathematical sense (if A > B and B>C then A>C). Moreover, the tool has the function of representing the length of the first line. In the following, the children compare the second line directly with the aid of their chosen tool. In the case that the children consider the figures on their standardized tool, they use them as a marker. Then, they either follow the same approach as described above or, as Luka did, they compare the two figures and thus, identify the longer line. Consequently, in these situations those children do not think deductively and transitively. Concerning the conservation the following two aspects could be seen:

Shifting a rigid tool (a stick, a ruler or a yardstick), none of the children wondered about the fact that its length always stayed the same.

Nine children did not pay attention to the course (lace and measuring tape). Regarding those children it can be assumed that they either have no idea about conservation at all or that they cannot recognize the conservation in connection with this situation.

In order to compare an object indirectly, these young children prefer standardized measuring instruments and are more successful when using standardized measuring instruments. They use seldom shorter tools in order to compare indirectly, regardless of whether it is standardized or not. The unit iteration of a shorter tool was not observed.

In order to describe their result the children were able to choose correct adjectives. Due to the fact that the interviewer could ask the children in case of vagueness it can be assumed that the children understood the relational terms correctly.

DISCUSSION AND CONCLUSION

In conclusion it can be stated that many of the competences considering normative aspects for an indirect comparison can be observed by four to six year old children.

All children had an idea on how to solve the task and it became obvious that concerning comparing items indirectly children already have acquired some competences and thus do not enter school as tabula rasa. Many children already have experiences with comparing lengths. Dealing with length (Castagnetti & Vecchi, 2002), especially indirect comparison, can therefore constitute a challenging task for young children.

Looking on instruction about length this study reveals that emphasizing the unit iteration before using a standardized tool does not correspond the children`s natural approach. These insights can provide a basis for an instruction approach which is orientated on children's own constructions.

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CUBES OF CUBES

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We present the three dimensional version of a play that is the topic of a paper sent for submission to Working Group 13. Its aim is to complete the presentation made in WG13, showing the potentiality of the play when we pass from 2D to 3D dimensions. In Cerme 8 session poster, the poster will be presented accompanying it with the same equipment utilised during the experimentation. The study of pupils' behaviours in front of the task furnishes some examples that prove the possibility of an early mathematical activity of generalization.

Keywords: kindergarten, play, generalization.

THEORETICAL FRAMEWORK

Generalization is often cited as typical form of mathematical thinking, but without using a definition or specify its meaning.

In her analysis of the act of understanding, Sierpinska (1994) considers four basic mental operations: identification, discrimination, generalization and synthesis. Her definition of generalization is the following:

> "Generalization is understood here as that operation of the mind in which a given situation (which is the object of understanding) is thought as a particular case of another situation. The term 'situation' is used here in a broad sense, from a class of objects (material or mental) to a class of events (phenomena) to problems, theorems or statements and theories" (p. 58).

Hejny (2004) write also:

> "The generalisation of isolated models (experiences and pieces of knowledge) is determined by finding connections between some of isolated models. This web is the most important product of the stage of the isolated models" (p.5).

The authors present and study an example of generalization that appears during a play. It is well known that the play can promote logical and mathematical competences. Schuler (2011) highlights that:

> "[…] *play* and *relationship of playing and learning* have to be explored more closely when talking about mathematics for the early years" (p.1912).

In particular, in the play utilised in this research, an important role is done to row-column arrangements. Rożek & Urbanska (1998) studied in depth this topic:

> "The children have a different awareness of the rows and columns arrangement. Some of them prefer rows, some of them columns. It appears that it was difficult to see both rows and columns, especially for young children" (p. 304).

THE 'CUBE OF CUBES' CONSTRUCTION AND ITS PROPERTIES

The play named "Cube of cubes", proposed to pupils of kindergarten (5-6 years old), is based on the construction of a 'palace' of three floors (a cube 3x3x3), following these rules: "In each wall face it needs to have three different colours in each row and in each column". Pupils in groups following the indications suggested from the teacher. The materials used are: 27 wooden coloured cubes (9 red, 9 yellow, 9 blue) (Fig. 1a), their wooden support, named from children "palace" or "house with a lot of floors", and a wooden rotating disk (Fig. 1b) to facilitate gestures and the observation. In a second time, teacher removed the support and she putted the cubes one near to the other (Fig. 1c).

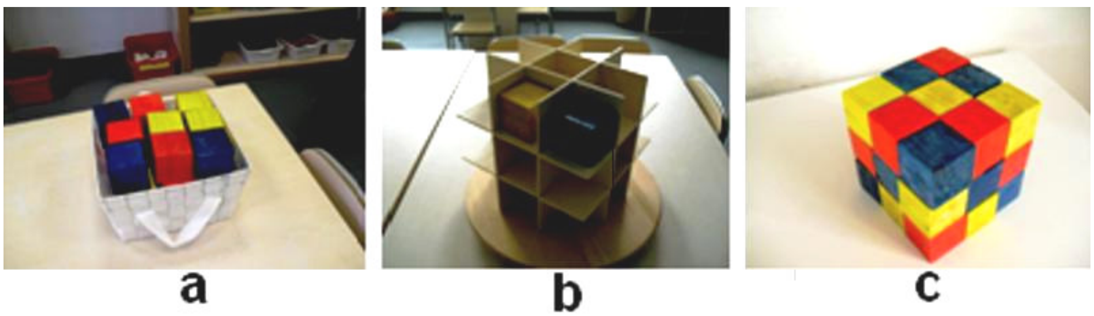

Figure 1: Materials and an example of 'Cube of cubes'.

Observing the final cube, children noticed a lot of properties, in particular they found a theorem: "In the cube there is an "internal monochromatic diagonal" and the other diagonals of cube are of three different colours". It is an example of generalization from 2D to 3D (the 2D version of the theorem is in Vighi paper, WG13, Cerme 8).

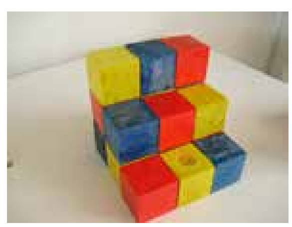

Figure 2: The monochromatic diagonal

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WG14: UNIVERSITY MATHEMATICS EDUCATION

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WG14 (*University Mathematics Education*, hereafter UME) was introduced in CERME7 (Nardi, González-Martín, Gueudet, Iannone & Winsløw, 2011) in recognition of its distinctiveness as an area of mathematics education research: within UME, researchers in mathematics education are often teachers of mathematics and in recent years the number of mathematicians specializing in UME has been growing. Furthermore, as the papers presented in CERME7 and CERME8 demonstrate, theories and methods from research on mathematics teaching at other levels must and do find specific adaptations within UME.

The WG14 Call for Papers invited contributions from a wide range of research topics. These included: mathematical reasoning and proof in university mathematics; challenges for teaching mathematics at university level (including the perspectives of university teachers); the role of ICT tools (e.g. CAS) in the teaching and learning of university mathematics; transition issues "at the entrance" to university studies of mathematics, or beyond; novel approaches to teaching Calculus and Linear Algebra; the teaching and learning of advanced university mathematics topics (beyond Calculus and Linear Algebra); challenges of teaching mathematics to students in non-mathematics degrees; assessing the learning and teaching of mathematics at university level; and, theoretical approaches to the study of teaching and learning mathematics at university.

This report draws on the presentations, reactions to, and discussions of the 21 *Research Papers* that were accepted for presentation at the conference and publication in the proceedings. In the proceedings there are also six *Short Contributions* to WG14, most based on posters presented and discussed at the conference. This report also draws briefly on these. Its structure is based on that used for the WG14 presentation on the final day of the conference. This presentation was structured around four broad themes: *Transitions*; *Affect*; *Teacher practices*; *Mathematical topics*. In what follows we present the papers clustered under each of these four themes. We do so with a focus on *theoretical perspectives* and *research paradigms* underlying the research and on key *results* presented in the papers. We conclude with a few thoughts on issues of rigour and quality that emerged from the discussions of papers across the four themes.

Transitions

A substantial number of WG14 papers raised transition as an important issue in students' learning of mathematics at university level as well as in teaching practices and approaches. In this sense transition, as a key issue, runs across many of the papers. Three of these studies however focused exclusively on transition. Two focused on the transition from secondary to tertiary education and proposed approaches that can enhance students' first year experiences – teaching for students' enculturation in the mathematical thinking and teaching for students' conceptual understanding. The third study discussed the transition at the other end of the students' experience at university, namely the transition from university mathematical studies to secondary education teaching, and analysed the nature of the mathematical knowledge in these two institutions. All three papers deployed qualitative approaches (interviews and observations). Specifically:

Hoffkamp, Schnieder & Paravicini highlight the importance of students' enculturation in the foundations of mathematics and propose a teaching approach that is based on the philosophical aspects of mathematical epistemology and introduces first year university students to mathematical reasoning, argumentation and proof. In the paper the design and the rationale of a math-bridging course based on the above assumptions are presented with clear recommendations for further research.

Breen, O'Shea & Pfeiffer report a project in which they designed unfamiliar to the students tasks with the aim of promoting conceptual understanding and developing mathematical thinking skills. These tasks were employed with a group of first-year undergraduate Calculus students. In the paper, the authors draw on the evidence from the interviews with five students in order to highlight that the tasks were beneficial in the development of students' conceptual understanding and suggest that unfamiliar tasks are useful in the transition from school to university mathematics.

Finally, *Winsløw* focuses on students' transition from university (as students of mathematics) to secondary school (as mathematics teachers). The paper proposes a theoretical model for the analysis of this transition based on the Anthropological Theory of Didactics and analyses the changes of mathematical knowledge in relation to the individual who occupies a position (student or teacher) within an institution. The model is exemplified in the institutional context of a capstone course and in the case of exponents and exponential functions.

Affect

The number and quality of WG14 papers that focused on issues of affect is indicative of the upsurge of interest in research in this area. The five studies reported under this cluster focused on a range of socio-affective issues that included: student perceptions of and expectations from mathematics-related studies; the concept of interest in university studies of mathematics; university students' preferred ways of working and learning in mathematics; differences between beginners and experienced students' approaches to learning mathematics; and, students' perceptions of themselves as capable mathematics learners. The theoretical perspectives underlying these studies ranged across sociological and psychological perspectives and the research paradigms espoused included an equally broad assortment of qualitative, quantitative and mixed methods approaches. Specifically:

Bergster & Jablonka deploy Bourdieu's three forms of cultural capital (embodied, objectivised, institutionalised) to explore the rationale and expectations underlying engineering students' choice of study. They report that mathematical studies are seen as a means to access economic power and master the use of a versatile and potent thinking tool.

Liebendörfer & Hochmuth illustrate the potency of using Self Determination Theory (autonomy, competence, social relatedness) to analyse interview data that explore perceptions of interest in university studies of mathematics. They do so through the illustrative case of data from one student.

Sikko & Pepin present a survey of university students in their second year which explored the teaching and study methods from their first year that the students found most effective. They report the proliferation of active and collaborative ways of working in the student responses; and, the limited appreciation for lectures.

Stadler, Bengmark, Thunberg & Winberg deploy Stadler's triad (mathematical learning objects; mathematical resources; students' actions as learners) to explore differences in approaches to learning mathematics between beginners and experienced students. The statistical analyses reported here (descriptive and inferential) show that beginners rely heavily on the teacher, while experienced students re-orient themselves from the teacher to other kinds of mathematical resources, such as peers and Internet-based resources.

Finally, *Toor & Mgombelo* explore how undergraduate mathematics students perceive themselves as capable mathematics learners and whether differences exist between male and female students' perceptions. The quantitative analyses establish the existence of gender differences. The qualitative data analyses reveal that participant construct perceptions of themselves as capable mathematics learners from two positions: direct experience and ideal images of mathematical ability.

Teacher practices

Research on mathematics teaching has increasingly focused on teachers' roles, methods and knowledge, as evidenced in CERME congresses by the growing number of papers dealing with teachers' practices and teacher education (WG17). It is only natural that UME research develops a similar direction, with a focus on everything that could make teachers at this level specific – engagement in research, relative autonomy in terms of methods and contents to teach, etc. WG14 received five papers dealing explicitly with small-scale studies of university teachers.

In two of these papers, the teacher involved was the paper author, who adopts a reflective practitioner (insider research) approach:

Biza reflects on her role as a novice teacher of statistics in a British engineering programme. Based on three episodes from practice, she explains how involvement in different communities of practice – including her activity as a mathematics education researcher – has contributed to shape her approach to this teaching task.

Nardi analyses the challenges of teaching a graduate course on mathematics education to students with a variety of backgrounds, including bachelor degrees in pure mathematics and native languages other than the language of instruction. She also outlines key didactic techniques and principles to cope with these challenges.

The three remaining papers offer small-scale studies of how teachers prepare and deliver their teaching, involving more ethnographic and sociocultural approaches which include observation, interviews and other forms of data collection:

Jaworski explains the use of Niss' competency framework to describe the aims and effects of an inquiry-oriented development project, and in particular the design of a specific set of tasks aiming to deepen engineering students' grasp of the notion of function. Observations and analyses of classroom practice illustrate this further.

Gueudet studies a university mathematics teacher's use of digital resources as he prepares and delivers a course on mathematics for computer science students, and points out the importance of internet-based platforms to enable *shared* documentational work related to university teaching of mathematics.

Pinto examines two lessons on calculus given by two different instructors, based on the same lesson plan. Based on observations and interviews, he demonstrates the relevance and importance of teachers' resources, beliefs, attitudes and goals, for explaining the substantially different development of the two teachers' lessons.

Mathematical topics

In a group like WG14, some of the research works presented naturally a focus on the teaching and learning of specific mathematical topics. We observed it here once again, with topics that were already present in CERME7: functions, proof, and linear algebra. Furthermore, new topics appeared in CERME8: infinite series and abstract algebra. Some of these papers are written by mathematicians, using a mathematical, epistemological, or historical analysis, and drawing on their teaching experience. Others present research that makes use of different theoretical frameworks, and methodological tools, to analyse students' difficulties with these specific topics, to better understand the teaching of a specific topic and the consequences of this teaching, or to formulate propositions for the design of teaching to overcome these difficulties. The range of approaches varied from developmental ones (such as concept image – concept definition), to models for abstraction (such as the RBC model), to analysis of discourse (commognitive approach) and the consideration of institutional matters (anthropological approaches).

González-Martín retains a theoretical perspective articulating Anthropological Theory (an institutional perspective, Chevallard) and the concept of didactic contract from the theory of didactic situations (Brousseau). Textbooks, and teachers' practices, shape the institutional didactic contract and in particular its rules about the teaching of infinite series. When confronted with tasks which do not obey these rules, the students produce inappropriate answers.

Schlarmann's work concerns the concept of basis. Referring to the work of Skemp, she considers that the conceptualisation process, taking place during the learning of the concept of basis, lead to the development by the students of mental structures. She draws on the triadic sign model of Peirce to build a methodological tool to study these mental structures. She illustrates the use of this tool for the analysis of data concerning two students with different mental structures.

Hausberger focuses on the concept of homomorphism. This concept, he argues, is abstract and difficult to learn for students; a test, for third year students, confirms this statement. He proposes an epistemological analysis, evidencing that the learning of homomorphisms requires a clear view of what a "structure" is, in the context of abstract algebra, in order to understand that homomorphisms are structure-preserving functions. He formulates propositions for teaching supporting this point of view.

Berman, Koichu and Shvartsman also develop a precise epistemological analysis, concerning the concept of equivalence of matrices. They articulate this analysis with a conceptualisation of understanding, drawing (similarly to Schlarmann) on the work of Skemp, and distinguishing between formal, instrumental, representational, relational and applicational understanding. They propose different tasks, which should contribute to the development of each form of understanding. They consider in fact that equivalence relation is a "threshold concept" for linear algebra.

The notion of *threshold concept* is at the heart of the work by *Pettersson, Stadler and Tambour*, who analyse the development of students' understanding of the concept of function. A commognitive approach is deployed towards the analysis of the variations in the discourse of the students, and the identification of the discursive elements employed by the students – mainly an expansion in the use of mathematical words and visual mediators.

A commognitive approach was also used by *Viirman* to study the concept of function, this time from the perspective of teaching. The discourse of three different teachers is analysed, and inferences about the learning that can be achieved are made. The main differences in the discourses appear in the emphasis put on the use of different representations of functions, as well as on the use of more process-oriented or more object-oriented discourses.

The concept of function was also the main focus of *Hyvärinen, Hästö and Vedenjuoksu*. Using a developmental approach, and through a longitudinal study making use of questionnaires, their work focuses on students' development of the concept of function and on their awareness of this development. Their theoretical approach combines the concept image – concept definition construct with tools that emphasise self-awareness. Their main results indicate that, during the first year at university, students' construction of the concept of function evolves. However, many students are not aware of this development, and, in some cases, students' estimation of their proficiency did not match their proficiency as evidenced in their performance.

Finally, the notion of proof, essential in Calculus, was addressed by *Alvarado Monroy and González Astudillo*. In particular, they studied the construction of definitions as a prerequisite for proof. Their paper presents the analysis of an activity developed in the classroom, making use of interactions and group work. The authors consider the RBC-model to describe the process of reconstruction of a definition. Their main results indicate that students had difficulties to manage mathematical language correctly, and in many cases the definitions they produce are erroneous. In this case, the teacher's help was needed to propose constructive definitions.

A concluding note on rigour and quality

Across presentations, reactions-to and discussions of the WG14 papers a range of issues emerged that concerned issues of *rigour and quality* in UME research. We noted that often in UME practitioner research, subject and object of research can be precariously *and* potently close and that the collection of data may often rely exclusively on practitioner reflection. This generates the need for triangulation, for transparency of data interpretation, and for acknowledging, and aiming to eliminate, bias. We also noted that results are often heavily dependent on the context and vary significantly across local institutions.

We also identified the challenge of theoretical coherence: 'is the choice of framework adequate for addressing the research issues in question?' we asked. And, in the cases where a combination of frameworks was selected, we asked 'is this choice underlain by coherence and compatibility?'. In the case of quantitative approaches we often recommended caution in the interpretation of statistical data. We also acknowledged that UME data frequently provide the ground for apparently self-evident claims, or simply a more nuanced account of empirically known facts. We noticed that authors, particularly of brief papers such as the ones presented in CERME, understandably choose to present an isolated slice of data from larger studies and that, in these cases, a broader contextualisation of the paper in the larger study is necessary.

We felt that results from small-scale studies usually need further verification and that in CERME papers results sometimes are emerging, therefore are not yet ready for more formal dissemination. Apart from smallness of sample, UME studies often rely on opportunistic sampling and researchers need to acknowledge the potential limitations of this reliance. In this sense there was broad recognition of the need to verify conjectures in other contexts. Overall however we were content with the way in which UME researchers choose WG14 as a platform for presenting analyses from ongoing and often daring and innovative work.

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INTERACTIVE RECONSTRUCTION OF A DEFINITION

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For future mathematicians to be able to construct proofs, they must understand what a correct mathematical definition is and how to use it properly. In this sense, a task was proposed to college students for reconstructing the definition of even numbers through discussion in small groups and then a socialization guided by the teacher was carried out. Both interactions were recorded and analysed entirely with the RBC-C model (Schwarz, Dreyfus & Hershkowitz, 2009). The results show that the students had difficulties to manage mathematical language correctly and that the definitions they produce are erroneous; they needed the teacher's help to propose constructive definitions.

INTRODUCTION

One of the most important aspects in the process of doing proofs is language accuracy, which means realizing how arguments are presented and how concepts are defined (Vinner, 1991; Mariotti & Fischbein, 1997). This requires students to have a clear idea of what a mathematical definition is. Pinto & Tall (1999) indicate that students handle formal definitions in two different ways: by giving them meaning through consideration of examples or by extracting their meaning through manipulation and reflection.

Regarding ways of defining, De Villiers (1998) classified definitions considering ones that define *descriptively* (a posteriori), if "the concept and its properties have already been known for some time and is defined only afterwards… A posteriori defining is usually accomplished by selecting a subset of the total set of properties of the concept from which all the other properties can be deduced" (De Villiers, 1998, p.250). The other way is to *define constructively* (a priori). This takes place when a "given definition of a concept is changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process" (De Villiers, 1998, p.250).

These forms of defining are distinguished by their function. Whereas the function of a posteriori defining is to systematize existing knowledge, the main function of a priori defining is the production of new knowledge.

When students define, they appeal to two types of generalisation: result pattern generalisation (RPG) or process pattern generalisation (PPG) (Harel, 2001). The first one is based on the regularity in the results, obtained, for example, by substitution of numbers in a formula whereas the other is a way of thinking in which one's conviction is based on the regularity found in processes.

CONCEPTUAL FRAME

Schwarz, Dreyfus and Hershkowitz (2009) defined Abstraction in Context (AiC) as "a vertical activity for the reorganization of previous mathematical constructs within mathematics and by mathematical means so as to lead to a construct that is new to the learner"(p. 24).

A process of abstraction has three stages: the need for a new construct, the emergence of a new construct and its consolidation. The need may arise from an intrinsic motivation to overcome obstacles and contradictions, surprises, or uncertainty. In the second stage the new construct emerges by means of three observable epistemic actions: Recognizing (R-actions) that a specific prior construct is relevant to the problem or situation at hand; Building-with (B-actions) acting on or with the constructs recognized for achieving the goal of understanding a situation or solving a problem; Constructing (C-actions) to fit and integrate previous constructs by vertical mathematisation to produce a new construct. Constructing refers to the first time the new construct is used or mentioned by the learner. In this process, Recognizing is nested within Building-with actions, and these in turn in the Constructing actions that can be nested in other Constructing actions of higher level. Finally, the third stage, corresponding to Consolidation, is a long-term process that occurs when the construct is mentioned, constructed or used after a constructing action is observed. This last stage is characterized by personal evidence: self-confidence, immediacy, flexibility and care when working with the construct (Dreyfus & Tsamir, 2004) and also when the language is increasingly more accurate (Hershkowitz, Schwarz & Dreyfus, 2001), although Kidron (2008) and Gilboa, Dreyfus and Kidron (2011) consider that the increase in language precision is characteristic of the construction stage itself and not just the consolidation stage.

In AiC these epistemic actions are known as the RBC model (Recognizing, Building with, Constructing) and the RBC-C model with the second C corresponding to the consolidation phase.

Below we present an activity about the process of reconstructing the definition of simple mathematical concepts as a prelude to using these concepts in a proof. Our objective was to describe how this process is developed in order to reach a minimal mathematical definition (Zaslavsky & Shir, 2005). Although the mathematical concepts proposed to the students were elementary, this was necessary because the teacher considered that some preliminary work with definitions was needed, in view of the difficulties faced by the students with the use of mathematical language when they were proving.

We analysed the production of various small-groups regarding different definitions. Specifically, in this paper we present the productions of one of these groups concerning the concept of even number, analysed according to the AiC model (Hershkowitz, Schwarz & Dreyfus, 2001). To get an overall picture of the entire process followed, we also present a whole class interaction leaded by the teacher.

Thus, the question this study aims to answer is: What are the epistemic actions that arise in the course of small group and whole class interactions or during the process of reconstructing a minimal definition of even numbers?

METHODOLOGY

The activity described below is part of a broader research about the introduction of mathematical proofs to university undergraduate students of mathematics. In this context sets of activities have been designed and experimented in a university classroom during eleven sessions. The activities were designed by the researchers, who explained their purpose and aims in advance to the teacher. The teacher was neither of the researchers and he put the activities into practice in his regular classroom. The activity in this paper was developed in the first session, in order to know how definitions are handled and generated.

The construction of proofs requires an understanding of the concepts involved and the appropriate use of their definitions. Different researchers have found that students do not understand the content of relevant definitions and how to use them for writing proofs (Moore, 1994). In an exploratory study (Alvarado & González, 2010) it was found that students do not use the definitions of mathematical objects to prove a statement in which they are involved; rather, they use an example of the object, a formula that represents it or some characteristic of the object. Edwards and Ward (2004) suggest that the special nature of definitions should be treated in introductory courses about proofs as a content in its own right and suggest that its effectiveness must be investigated in future research.

Specifically, in this activity, students had to reconstruct definitions for already known mathematical concepts or objects through negotiation in small groups and then they had to share their definition in a whole class discussion mediated by the teacher. The construction process carried out in each of the teams allowed the students to verbalize their ideas, thus making it possible for us to analyse the evolution of this process in the light of the RBC-C framework. This experiment involved 23 students that were studying the first semester of Applied Mathematics in the University of Juarez in Mexico. The group was divided into 9 small-groups of 2 or 3 students.

DATA ANALYSIS

Below we describe, characterize and analyze the interactions that took place in one of those small-groups (team X) as they tried to reconstruct the definition of even numbers followed by whole group socialization influencing their small group work. During this task we could see that although some mathematical concepts are familiar to the students, they have difficulties to pose a correct definition; instead, they generate examples as a help to reconstruct the definition.

Small-group interactions

In the dialogues included in this section, only students participate. The contributions of each of them are consecutively numbered to refer to them easily in the text. At the beginning of the interaction, the students suggest a rather poor definition [1]. They evoke the concept of division (R-action) and relate the concept of even numbers to composite numbers using some examples to reconstruct the definition of even numbers to show their truth.

[1] Then an even number is one that can be divided by itself.

[2] And others.

[3] And others?

[4] Yes, because for example 10 can be divided by itself, by 5, by 2.

[5] ... the bigger the number the more numbers it can be divided by.

[6] Look, another even number.

[7] 18.

[8] 18, by 18, by 9, by 2.

[9] What else?

The generalization process being used is based on patterns of results (RPG) and not on processes (PPG) (Harel, 2001). The above examples led them to guess a more appropriate definition [10] which is verified again (B-action) with other numbers.

[10] All even numbers can be divided by 2, right?

[11] Yes, I think so.

[12] By 2 yes, 4 yes, 6 also, 8, 14 also.

By linking (B-action) the above [1 and 10] with the fact that 2 is a prime number [17, 18 and 20] they produce a “definition” [14, 19] in which similarities with the definition of prime number are seen: both of them have two divisors [19 and 20]. This is caused by their difficulties with the use of quantifiers.

[13] So even number. [he writes]

[14] Is that which is divided, which can be divided by ... oh no!

[15] By itself.

[16] And another number.

[17] The number 2 we did not say.

[18] By 2, right?

[19] By itself and by 2.

[20] By 2, isn’t it? It’s a prime number.

The definition produced [21] is obtained by a B-action to verify similarities between the examples found earlier.

[21] Yes that's right [pointing to the examples above], all numbers can be divided by themselves and by 2.

Another student suggests another definition [22] which is considered a C-action (C_{X0} as this is the first C-action shown by group X), but their peers did not take it into account.

[22] Even numbers are those who go two by two. Because 4, 6, 8 go two by two.

Similarly, we include the wrong definition they proposed of odd numbers, which is accepted [23] without discussion by the other peers. It can be seen the similarity with the definition of even numbers and the confusion with prime numbers.

[23] You can only divide it by itself and by 1.

Whole-group interactions

Small-group work allowed for an interaction that simultaneously provided some progress, but also some limitations. Some of these limitations would be overcome during the whole-group discussion because it addressed aspects or views not suggested during the small-group interactions, such as an input from a student of small-group X which was not taken into account, and would now be taken up by the teacher. During the socialization the teacher played an important role supporting the development of a mathematically productive discourse and also progress in the learning processes. Sometimes the teacher constructed the definitions by himself without allowing the students to modify their proposals.

To organize the discussion, the socialization starts by presenting the definition contributed by small-group Y. Then the definition of X is discussed, and while extending it, the production of T comes into play. Finally, we discuss the productions of small-groups V, W and Z which allow the construction of another definition.

This began with a spontaneous and intuitive notion: a list of numbers from zero that is increased by 2. This C-action C_0 [24] built by small-group Y is equivalent to a C-action that occurred during the interaction in small-group X (C_{X0} [22]). Demanding an explanation of the meaning with the purpose of clarifying the language used [25] constitutes an R-action. Students will refine the language using different words and examples that serve as B-actions giving rise to a C-action [31] C_1: a definition of even numbers that is neither accepted nor rejected pending further "definitions".

[24] Student: They are the numbers that ascend from zero 2 by 2. [small-group Y]

[25] Teacher: "Ascend", what does this mean?

[26] Student: That they go up.

[27] Teacher: Okay, explain.

[28] Student: That we are adding and adding.

[29] Teacher: For example.

[30] Student: 2, 4, 6...

[31] Teacher: Okay. So they are the numbers to which you keep adding 2 beginning with zero. Any other input?

The teacher shows that this definition can be completed and formalized. It should be noted that while the original idea of the students is respected, the teacher used the role of authority to extend the mathematical knowledge in a *constructive way* of defining (De Villiers, 1998).

[32] Teacher: It turns out that we only get positive even numbers, so to complete the definition with the negative even numbers, I have to subtract to the left.

[33] Teacher: But that thing, of using "ascend" – it's a very ambiguous word. Let's get used to the fact that in mathematics we must be very precise.

[34] Teacher: In this case we can define it as the sequence of integers where starting from zero we add and subtract two units successively. This would look like this [he writes on the blackboard] {…-8, -6, -4, -3, -2, 0, 2, 4, 6, 8…}.

[35] Teacher: It can also be formalized as $\{0 \pm 2n \,/ n \in Z\}$. This means that n is an integer or that n varies within the integers.

An R-action takes place when it is recognized that negative numbers must be included and that appropriate language must be used. This is concluded with two C-actions, C_2 [34] and C_3 [35]; the second as an extension of the first but the teacher does not encourage the interaction with students to build the definition.

Returning to the definition produced by X, the teacher tries to adjust it to produce a more economical definition because it is not necessary to include the number 1 in the definition.

[36] Student: They are the numbers that are divisible by themselves and by 2. [small-group X]

[37] Student: And by 1.

[38] Student: And 1.

[39] Teacher: Let's go over that again. Do we need to say that they "are divisible" by themselves and by 1? In fact, any number can be divided by itself and by 1. So this is not an exclusive feature for even numbers. Let's see, what distinguishes the even numbers?

This last statement and the students' answers to the teacher's suggestions are an attempt to clarify the language to formulate a more operational definition.

[40] Student: That even numbers can be divided by 2.

[41] Teacher: Right, but then we have a problem, for example, I can divide 3 by 2, can't I?

[42] Student: Yes.

[43] Teacher: Then the word "divided by" must be refined.

[44] Student: The matter is that we must get an integer.

[45] Student: Yes, a number that has no decimal.

[46] Student: That gets me zero as remainder. [small-group V]

[47] Teacher: ...A number that when divided by 2 the remainder is...

[48] Student: Zero.

A B-action [40] is activated when an exclusive feature of the even numbers is mentioned. During this time [41-43], with R-actions, a negotiation of the meaning of the term "can be divided" is elicited because for some students it means that the quotient must be an integer [44 and 45], which does not guarantee that the remainder is zero [46]. This expression appears repeatedly in the interactions of most of the small-group work with the idea of expressing that the remainder of a division must be zero. This can be attributed to the fact that when 3 is divided by 2, for example, it is common to hear young children say "can't be done" or "it won't fit", and thus the spontaneous use of language is maintained until university level.

Moreover, although the contributions of small-groups X and Y are discussed, the teacher takes advantage of this discussion to include another definition produced in small-group T; when this happens a new C-action C_4 [47 and 48] takes place.

Finally we discuss what small-groups V, W and Z did to build another definition refining the language.

[49] Teacher: Let's see, what happens if I divide 10 by 2, what will it be? How much is left over?

[50] Student: 5 and zero as remainder.

[51] Teacher: So I say that it is an even number. Remember when you were in elementary school you were told: "check your calculations to see if they are correct". What operation did you have to do?

[52] Student: The inverse operation of division is multiplication, adding what is left over.

[53] Teacher: For example here, in this case $2\times5+0$, and since the remainder is zero, we just write 2×5 and say that 10 looks like 2×5. So, what form do the even numbers take?

[54] Student: 2 times "something".

[55] Teacher: Does 12 take the form 2 times "something"?

[56] Student: Yes 2 by 6, so it is even.

[57] Teacher: Does 13 take the form 2 times "something"?

[58] Student: No.

[59] Teacher: So, even numbers, what do they look like?

[60] Student: 2 times an integer.

[61] Teacher: Well, then we have another definition, even numbers are those that take the form $2\times n$ where n is an integer [writing on the blackboard] Even numbers look like $\{2n / n \in Z\}$. The $n \in Z$, this is interpreted as n ranging over the integers or n belonging to the integers. This is a clear way to say it and if this doesn't happen, what occurs?

[62] Student: Well, is not an even number.

Although previously were obtained definitions of even numbers accepted by the group, now the idea is to achieve a correct definition obtained constructively. This is how the C-action C_5 [61] takes place, built on B-actions oriented and provoked by the teacher presented as a string of examples in which transitions are made from a particular example [49], to a process [51], to generic constructions [59, 60] via a check that ensures that non-examples are identified and ruled out [57, 58].

The following diagram summarizes the progression between the different definitions produced by the teams to reach more operational definitions.

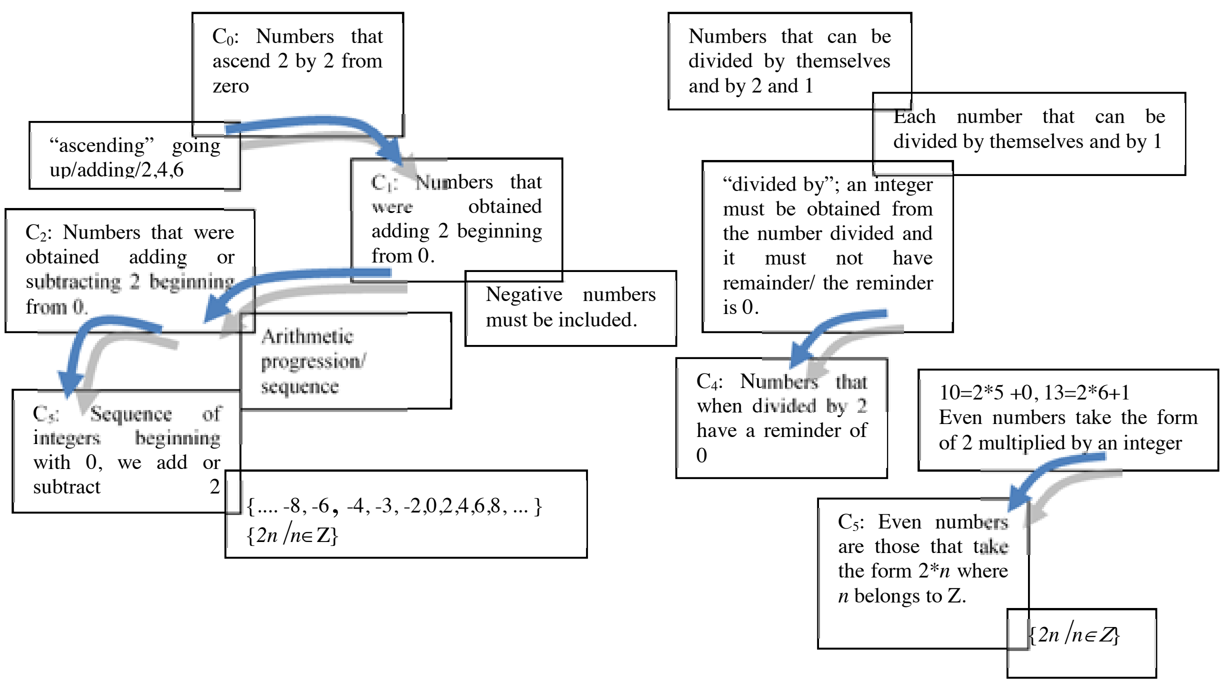

Diagram 1. The process of generalizing the definition of even numbers in whole-class.

The confusion between different types of numbers and the difficulties managing their definitions is present when solving other activities. For instance, when these students had to prove that "For any whole number n, $n^2 + n + 41$ is a prime number" they related prime numbers with odd numbers saying things like "3 times 3 equals 9, and 3 are 12, plus 41 are 53, 53 is an odd number so is prime".

CONCLUDING REMARKS

The RBC model is a tool for analysing the process of reconstructing a definition from a more informal view of mathematics to make definitions that require more formal reasoning (Gravemeijer, 1999). Even in the simplest situations, students use examples as verifiers and conjectures derived from the results rather than processes (Harel, 2001). This is shown essentially in the B-actions. Vinner (1991) states that the development process of a concept involves two cognitive mechanisms: identifying similarities and distinguishing differences. However, students look only at the former and disregard the latter completely. Some difficulties with quantifiers are perceived, resulting in the formulation of inconsistent definitions in the small-group interaction.

When the teacher leaded whole-class discussion it was possible to produce a definition focused on the generalization of patterns based on the regularity (PPG) and not obtained from each example (RPG) (Harel, 2001). In this sense Harel (2008) points out that only the students that generalize across PPG suggest the required arguments for the construction of proofs by mathematical induction. In this sense, the definitions proposed by students in small-groups were mainly obtained descriptively whereas during the whole-class socialization sometimes the definitions were obtained descriptively and other times in a constructive manner (De Villiers, 1998). The analysis of the role of the teacher during the whole-group interaction will form the basis of another study.

Finally, we argue that negotiated defining in small-groups seems to help the students to extend their understanding of the meaning of the terms they use. On the other hand, our results suggest that defining should be taught and learned in school classrooms in order to improve the mathematical communication, instead of assuming that students already have those skills. During all the teaching process the students realised the importance of having correct definitions to develop new concepts and to manage proofs and they learned to act more and more autonomously.

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MATHEMATICS AS "META-TECHNOLOGY" AND "MIND-POWER": VIEWS OF ENGINEERING STUDENTS

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The paper reports an exploration of first year undergraduate engineering students' rationales for choosing engineering programmes at master's level and the potential gains for their future professions they see from studying mathematics. As a means for organising the data, we used some notions of Bourdieu's theory of the economy of forms of social practice. The study aims to contribute to understanding differences in the students' experiences of and interest in university mathematics.

INTRODUCTION

In informal conversations at the mathematics departments where the authors have worked, engineering students from different programmes often are attributed different study habits, different approaches to mathematics and different chances to reach higher achievement levels, in particular on the more theoretical parts of the examination. This observation invites an investigation into the "reality" of such differences, but also asks for explanations of their origin. While descriptions of differences by referring to students' attitudes and beliefs might help to investigate similarities and differences within "categories" of students (e.g. Berkaliev & Kloosterman, 2009; Hassi & Laursen, 2009), an approach including student background and professional aspiration may contribute to further understanding students' experiences of and interest in university mathematics. Based on the observations outlined above, this paper investigates the following research question(s):

What are the rationales of students from different engineering programmes for choosing university studies at master's level in general, and for the choice of particular programmes? Are there differences in the potential gains they see from studying mathematics for their future professions?

CONTEXT

University education in Sweden encompasses professional training of, for example, pre-school teachers and nurses, besides traditional academic programmes. Despite educational reforms to open admission to university studies for students from less theoretical upper secondary programmes, statistics show no increase of students from families without university education. The pattern over the period from 1999 till 2009 instead shows decreasing numbers of these students, in particular for civil engineering, where in 2009 only 23% of the beginning students come from a family where none of the parents has higher education (Högskoleverket och SCB, 2009, p. 39). Policy discussions about the structure of the work force in Sweden continue to emphasise the need of more qualified engineers to secure that the country does not drop in economic development, which is seen to rely heavily on a strong development of technologically based industries. In this scenario mathematics and science education, at all school levels and in higher education, have been given key roles. Statistics of the number of university entrants and completed exams of engineering master's in Sweden show, however, very small changes during the last years, with around 6000 new students and 3500 exams per year (Högskoleverket och SCB, 2012, p. 89).

THEORETICAL CONSIDERATIONS

Paton's (2007) analysis of literature on educational decision-making identifies three main models of students' 'choices' for higher education: economic or instrumental rationality assuming strong individual agency, social and economic structures with weak individual agency, or a hybrid of both. Linked to a structural or hybrid model for their study and career choices, some notions provided by Bourdieu's theory of the economy of forms of social practice seemed relevant and useful for organising our data. Bourdieu (1983) assumes that even practices seemingly unrelated to the economic system can be analysed in terms of maximising material or symbolic profit. Capital and profit appear in a range of forms. In different fields, these forms of capital possess different values and thus contribute differently to the chances of gaining profit. According to Bourdieu (1983), capital is accumulated labour in the widest sense that exists in material form or in an embodied form, 'incorporated' within the individual. It is a *force* inhabiting both objective and subjective structures, as well as a *principle* shaping the structural regularities of the social world (p. 183). Bourdieu (1983) discusses three forms of capital (economic, cultural and social capital) and their interrelations. As Bourdieu notes a dominance of the economic field, economic capital seems to be more relevant than the other forms, which can be transformed into economic capital, or acquired through economic capital, including more or less transformational work for producing the kind of power relevant in different fields. Economic capital comprises all things that are institutionalised through "possession" and are more or less directly transferable into money. Cultural capital refers to different kinds of cultural productions. Even if under certain conditions it can be exchanged into economic capital, the reproduction of cultural capital follows its own logic. Social capital includes actual and potential resources that rely on an affiliation with a group and it is realised through exploitation of social networks.

In mathematics education research, notions from Bourdieu's work have been productively employed in diverse contexts, such as by Gates (2001) to explore how teachers' beliefs are grounded in the social fields they operate, and are maintained and reproduced, by Teese (2000) to account for secondary students' achievement, or by Zevenbergen (2005) to account for middle-class students' advantage in classroom interaction. In the context of studying the school-university transition, Bourdieu's notion of cultural capital has been used in reports from the UK based 'TransMath' projects, e.g. by Hernandez-Martinez and Williams (2013), who link it to the notion of resilience, and by Williams (2012) in discussing the relation between the use and exchange value of mathematical knowledge.

Most relevant for this paper is the notion of cultural capital, which according to Bourdieu (1983) exists in three forms, an *embodied*, an *objectivised*, and an *institu-*

tionalised. The notion of objectivised cultural capital shows some overlap with the one of economic capital, as it comprises all reified forms of cultural capital, available for example in the form of books, machines or pieces of art. Embodied cultural capital (also referred to as *cultivation* or *Bildung*, ibid., p. 187) comprises all forms of skills and cultural orientations that can be acquired through education and socialisation and it is realised as a habitus of the one who possesses it, for example in the form of knowledge, competence or taste. Accumulation of this form of capital needs time and personal investment, and also a form of a socially constructed *libido sciendi* (ibid., p. 186). Institutionalised cultural capital includes all forms of certificates or titles that formally accredit a person's possession of some specified form of embodied cultural capital. Further, Bourdieu states that symbolic capital is the potential of the other forms of capital for gaining social status within a system of struggle over symbolic power. Along with the more obvious transmission of economic capital, cultural capital and social capital play a major role in social reproduction. Bourdieu (1983) notes a dynamic relation between all forms of capital acquired by a group, as the possession of one form reinforces the acquisition of the others. We have taken up the latter idea in our research question, as we assume that the students' previously acquired cultural capital might account for differences in the potential gains they see from studying mathematics for their future professions.

We were thus interested in how engineering students, depending on their background and career aspirations, think they would make use of their mathematical knowledge. It is not evident that the cultural capital acquired in more theoretically oriented mathematics studies is seen by the students as directly applicable in the field they want to enter, but there are often gains in symbolic capital. The symbolic value of mathematics is in the context of symbolic production kept high by means of policing, most prominent with strategies that draw on knower characteristics, such as being a good problem solver or being able to think logically (cf. Bergsten, Jablonka, & Klisinska, 2010). We were also interested in whether the students from different programmes see their academic achievement differently, as exchangeable into economic capital, or primarily useful as institutionalised cultural capital or as symbolic capital.

METHOD

The empirical basis for this study is interview data from students enrolled in five different engineering master's programmes at two Swedish universities. Within a larger study (funded by The Swedish Research Council) about the transition from upper secondary to university mathematics education, three series of audio taped individual semi-structured interviews, each lasting around half an hour, were conducted during the students' first year at university. Among the questions asked in the last of these interviews, conducted at the end of the year, some focused on the students' backgrounds and reasons for their choice of studies, as well as on their views about the use of the mathematics studied at university at a possible future workplace. In particular, the students' answers to the following four questions will be analysed:

Q1. Why did you choose to study at university?

Q2. Why did you choose to study to become an engineer (master's programme), and to start the particular engineering programme that you follow?

Q3. What do you think influenced you to make these choices?

Q4. Do you think that you will use the mathematics you are studying at your future work place, and if so how?

As data for the reported investigation, interviews with 20 and 13 first year engineering master's students from the two universities, respectively, were selected to represent the five main different study programmes (see "Outcomes" below) as well as different achievement levels within each programme (low, medium, high). For the analysis of the transcribed interviews, similarities and differences among the responses from students across the different engineering programmes were investigated. The study followed the ethical guidelines of the Swedish Research Council, granting participant information, openness, freedom of participation, and anonymity.

OUTCOMES

In the interview quotes below, the code 1XY3 (for example) means student number 3 from university 1 in programme XY. H stands for 'Parents with higher education'.

Computer Technology, CT (4 students: 4 male, 1 H)

Q1: Most of the students referred to the possibility of getting a good job, while some saw it as the only reasonable option after having completed a theoretical programme in upper secondary. Many also referred to their parents:

> Nice to get a little higher salary (1CT7, H)
>
> It felt well like the only real alternative… didn't know what else to do after leaving upper secondary… and then yes the parents always encouraged you to go on studying (2CT1)

In addition, one student mentioned his living environment, a middle class area, as a contributing factor. *Q2*: All emphasised their interest in computers (or programming), while two students also said that owing a master's in engineering is a very strong educational merit, and one that he is good in mathematics. *Q3:* While two students did not say much, the other two saw their family as a main influence. *Q4*: Two students were hesitating about the use value of mathematics for their future profession, two said mathematics is useful, one of them emphasising the understanding of mathematical relations as a route to a more general ability.

> You learn to take in to understand mathematical relations then you learn to understand relations kind of generally (1CT6)

However, one student did not think mathematics is of any use:

> No, to be completely honest I don't think so (1CT7, H)

Energy and Environment, EE (5 students: 4 female, 1 male, 3 H - 1 unclear)

Q1: Four students said that it was the natural option after the theoretical programme at upper secondary; the fifth mentioned the influence of her parents and that she would not be satisfied with herself if she did not get a university education. *Q2*: All uttered aspirations to do something good for the world, the environment or the future.

> You do want to influence and make the world to something better in some way (1EE3, H)
>
> You kind of feel that you do something … good you contribute with something that is good … meaningful (1EE8)
>
> If we are going to take care of the next generation I guess we will have to do it (1EE4, H)

Two of these students also mentioned a good job as a reason to become an engineer along with an interest in energy management. *Q3*: Three mentioned family influence.

> Both parents have studied at university … that surely has an influence (1EE3, H)

Two reported experiences from upper secondary, and one also mentioned an interest in nature and a wish to help. *Q4*: Three pointed to a general learning experience from (mathematical) problem solving as most useful for their future professional work:

> That is I guess what it means to be an engineer master's to be a problem solver (2EE1, H)

One student mentioned "learning to learn" through mathematics, and one said that mathematics is "there in everything".

Industrial Economy, IE (6 students: 2 female, 4 male, 5 H)

Q1: To get a good job was mentioned by four students:

> You can get a somewhat better job … that was the main thing (1IE1, H)

Two students said that it was natural to them to go on with their studies. An additional reason given by two students was the influence from their parents:

> It was quite natural /…/ I did the science programme at upper secondary so I liked math and I wanted to go on studying and also both my parents have higher education /…/ but then all my relatives kind of my aunts and uncles also (1IE6, H)

Q2: Three students referred to their parents and three to the wide scope and the strong combination of, or an interest in, economy and technology of the programme, while two again pointed to the possibility to get a good job:

> I think I was very much influenced by my father to become a master's engineer … yes that he thought it was good (1IE9, H)
>
> … quite a lot about leadership in this programme … you kind of want to become a boss and then it felt like a good alternative (1IE6, H)

Q3: All four students from university 1 pointed only to the influence from their parents, one emphasising the values.

> Yes I don't know at least the values … so that technical subjects are pretty good … yes of course I am inspired by my parents it would be hard to say something else (1IE12, H)

Q4: Only one student saw mathematics as clearly useful but then mainly very basic mathematics. However, almost all students very strongly emphasised the role of mathematics for developing a general problem solving ability:

> If not doing the calculations it is more this way of thinking … the problem solving ability (1IE9, H)
>
> You more easily see through the problem somehow … you can put it down into small pieces somehow so that it doesn't get so big … I will probably I will definitely use this I think … I already feel that one has kind of changed as a person by the math (1IE12, H)
>
> One always has a structured way, in math, to solve a problem and then one can bring that way of thinking into many other things in life (2IE2)

Mechanical Engineering, ME (10 students: 10 male, 3 H)

Q1: Six students mentioned the opportunities for a good job as a reason to study at university. That it is natural and rewarding to go on studying after secondary school was mentioned by four, while only one referred to an interest in technology. *Q2*: In contrast, as much as eight students pointed to such interest, most often with the specification of "construction". Some also mentioned the wide scope of the programme.

> The interest in technology has always been there … more interesting [when comparing with energy] with CAD and construction (2ME3)
>
> To construct … cool to understand what is happening (2ME4)

Q3: Eight students attributed the main influence of their study choice to their family. In some cases practical work was emphasised, while one student explained that in his family it was not a tradition to enter higher education.

> It became like that my daddy has an engineering master's and my mother a physician … so I was directly accepted for the engineering master's (1ME5, H)
>
> Actually I am the first one in the whole family (1ME7)

Q4: Doubts were raised about the usefulness of mathematics at a future workplace. It was most common (six students) to underline the value of understanding the foundations, for example what is behind automated calculations:

> I don't think one will use it so much but yet I think it is good for the understanding (1ME5, H)
>
> You have to understand the basis, otherwise there is no purpose. If you don't understand what you are doing you can't understand if something gets wrong (2ME5, H)
>
> The calculations are done by the computer but the very understanding of what you are actually doing, the process, is perhaps important but not calculation ability itself (2ME1)

Also the general ability to solve problems, or to think analytically, was seen as an important outcome of their mathematics studies:

> Yes … that is maybe not the mathematics itself but … the analy- analytical thinking and … yes but maybe not just sitting calculating but … you learn to analyse the problem in a somewhat different way than you did in high school (1ME1, H)
>
> Yes I think so definitely … eh … partly that one learns to solve a problem, not … not just to calculate but also … one does learn a certain way of thinking and to break up a

> problem into smaller parts and then solve each part kind of separately … and so finally solve the whole thing (1ME9)

Technical Physics and Electric Engineering, TE (8 students: 2 F, 6 M, 4 H)

Q1: The answers were quite varied for this group. Some said they did not want to work in an "ordinary" job, wanted to "become something", and that it was natural to continue the studies after secondary school. *Q2*: Seven students referred to an interest in mathematic and/or physics, and some argued that the engineering master's will provide a strong background, especially with technical physics. Only one student explicitly mentioned to get a good job as the reason, and one seemed to aim at doing research. *Q3*: Family influence on the choice of study was mentioned by four.

> My father had an engineering master's and my mother was a teacher … and then I always heard shall you be like your mother or your father … and it was no I shall not be a teacher I shall get an engineering master's (1TE2, H)

One student emphasised that the parental influence was not experienced as forcing, while another one said "probably" and one "not much". Two students said they did not know. *Q4*: The potential use of mathematics at a workplace was seen mainly as indirect, through the ability to understand what is behind the calculations (five students), while a problem solving ability or an ability to "learn to learn" was mentioned by two students. As one student formulated this aspect:

> It is easier to formulate the problems if you can do it mathematically (1TE3)

One student was wondering and two hoping that they will have an opportunity to use their mathematics knowledge later on, for example to develop mathematical models.

Patterns in interview data

There are some patterns visible in the data, which are relevant for the overall research question set up for this paper. These concern similarities across the five engineering programmes as well as some noteworthy differences between the programmes.

Students from different programmes provided some similar rationales for starting to study at university, such as to get a 'good' job (where 'good' referred to a job that pays well and/or is not so boring) and that it was a 'natural' continuation of their choice of study at upper secondary school (or that it was the only real option). The first rationale, however, was less common amongst EE students. Linking to this the reasons for choosing an engineering master's, and a particular programme (study direction), an interest in this kind of study programme was mentioned by most of the students in all programmes except the EE and IE. The answers to the question about the use of mathematics in a future profession varied and developed in some cases into much detail and discussion. However, a very strong common view on this issue was that a general problem solving ability, or way of thinking in problem solving, was the outcome of their mathematics studies which they thought would be the most useful one (but not any specific mathematical skill, such as to calculate integrals 'by hand'). It was also common, across programmes, to point to the importance of an 'understanding' one has developed through the work in the mathematics courses included in the engineering studies, as for example to be able to evaluate mathematical inputs and outputs in computer software. However, this view on understanding was much more dominating in the ME and TE groups.

Some very clear differences can also be seen between the programmes. The aspect of getting a good job was much more emphasized by students from the IE and ME programmes than by students from other programmes. However, the reasons given by IE and ME students for their choices seem to be of a different character, the former acknowledging a stronger influence of their parents and expressing a stronger focus towards a career (this is also the group with the largest proportion of parents with higher education), while the latter provided reasons of a more pragmatic character, including an interest in the kind of work normally done by a mechanical engineer (this is the group with the smallest proportion of parents with higher education). A strong individual "will" behind their choice of study was expressed in particular by the TE students, also the group that most explicitly pointed to an interest in mathematics as one reason for their choice. However, the most programme-dependent rationale for their choice was given by the EE students, who all expressed a very explicit wish, or even need, to change the world into something better (related to sustainability). The CP students were the least 'optimistic' group in relation to how much they would be able to use the mathematical knowledge acquired during their studies.

DISCUSSION

Not surprisingly, the outcomes clearly show how the cultural capital possessed by the families influence the choice to enrol at a university [1]. However, in the group of students interviewed there were about a half from families without higher education, which are more than suggested by the Swedish statistics. Many talked about 'a natural continuation', suggesting that during their studies in the science or technology programme at upper secondary school, these students have acquired a cultural capital that has developed into a disposition towards academic studies.

When differentiating between positions the students aim at in social space (from an intellectual towards an economic pole), then it is visible that the EE students see the embodied cultural capital resulting from their studies as most important, and to some extent also the TE students, while the IE students hope for accumulation of economic capital. As to their position towards engineering, they see different roles for themselves. The EE group sees their future field of engineering more independent from the economic field and more related to the political field, as they think of themselves as an autonomous group who aim at transforming technology. The ME students talk about constructing technology, and for the students in the CT and TE programmes there is no clear tendency, except that the TE students also see their field as not very much linked to economy, possibly due to its clear theoretical nature.

Even if we started by expecting differences in their views of mathematics as more or less helpful for further gains in economic or cultural capital, respectively, or differences in the type of cultural capital they think mathematical studies provide, we did not find very clear patterns, with the only exception that most of the CT students did only see a use in its institutionalised form as symbolic capital provided by the certificate. The others referred to very general notions of problem solving competences or to an ability of structured thinking, including one who states that mathematics has changed him as a person, and two who say it helps learning to learn. These general abilities reflect an essentialising discourse about mathematics as enhancing individuals' problem solving abilities. The focus is less on particular mathematical skills as a form of cultural capital, but rather on a habitus to which engagement with mathematics amounts, seen as a general habituation to solve problems in a rational way. Only one TE student mentions that mathematical knowledge would help to build mathematical models, which constitutes it as a technology-related cultural capital. However, students from this programme generally expressed a disposition towards studying mathematics. A couple of students referred to mathematics as a means for understanding the principles behind other technology. The IE students did not mention any relation between mathematics and economic models, despite the tradition in their field to use theoretical models and statistics.

All of these views reflect a view of mathematics as unlinked to the social context of its use, despite the differences in social positions the students aim at. Mathematics is either conceptualised as *mind-power*, a kind of universal thinking tool in the form of a general embodied cultural capital contributing to the formation of a person (a habitus), or as a universal method, a *meta-technology*, that helps to access otherwise hidden principles [2]. We did not employ notions of ideology or discourse, to which the engineering students' views could be linked. However, the outcomes show some unexpected uniformity in their views, which did not depend so much on their different career aspirations or backgrounds, that is, on different forms of cultural capital, as we had anticipated. It is rather a discourse about the value of mathematics that might have shaped their conceptualisations, which foregrounds the values of rationality and objectivity that are associated with technological and economic progress. The students' views match very well the general discourse about the role of mathematics education in secondary schools that aim at channelling more students towards engineering careers to sustain economic development, which is seen to rely heavily on a strong development of technologically based industries.

NOTES

1. For another context, see for example Kleanthous and Williams (2013).

2. The term *meta-technology* is being used in several different contexts, e.g. in the philosophy of technology or in bio- and information technology, with a meaning differing from that given here.

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THE UNDERSTANDING UNDERSTANDING EQUIVALENCE OF MATRICES

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The title of the paper paraphrases the title of the famous paper by Edwina Michener "Understanding understanding mathematics". In our paper, we discuss what it means to "understand" the concept of equivalence relations between matrices. We focus on the concept of equivalence relations as it is a fundamental mathematical concept that can serve as a useful framework for teaching a linear algebra course. We suggest a definition of understanding the concept of equivalence relations, illustrate its operational nature and discuss how the definition can serve as a framework for teaching a linear algebra course.

INTRODUCTION

One of the major dilemmas that teachers of linear algebra face is whether to start with abstract concepts like vector space and then give concrete examples, or start with concrete applications like solving systems of linear equations and then generalize and teach more abstract concepts. Our personal preference is to start with systems of linear equations since they are relatively easy-to-understand and are connected to the students' high school experience. Unfortunately, for some students this only delays the difficulty of abstraction (Hazzan, 1999). The students also often tend to consider less and more abstract topics of the course as disjoint ones. Since the concept of equivalence relations appears both in concrete and abstract linear algebra topics, we think of equivalence relations as an overarching notion that can be helpful in overcoming these difficulties.

In this paper we first suggest a review of topics, in which the notion of equivalence relations appear in high school and in a university linear algebra course and then theoretically analyse what it means to understand this notion, in connection with the other linear algebra notions. For this purpose we suggest a definition of understanding the concept of equivalence relations in linear algebra and argue, by means of presenting mathematical tasks aimed at testing particular aspects of the definition, that it can be operationalized. The paper is concluded with suggestions for future empirical research.

EQUIVALENCE RELATIONS

Examples of equivalence relations known, latently, to high school students include equality of numbers and algebraic expressions, and congruence and similarity of geometric shapes. Enriched mathematics high school curriculum may also include congruence modulo and equivalence of (systems of) equations.

Equivalence relations between matrices are ubiquitous. Equivalence of systems of linear equations is usually the first time when a university linear algebra student explicitly encounters the concept (e.g., Berman & Kon, 2000; Carlson, Johnson, Lay & Porter, 1993; Hoffman & Kunze, 1972). The concept of row equivalence of matrices is introduced in this connection. The concept of column equivalence of matrices is introduced for the sake of symmetry, and an experienced lecturer would emphasize that elementary row operations transform a system of equations to an equivalent one, whereas elementary column operations do not. Matrix equivalence naturally appears in connection with rank, matrix similarity – in connection with eigenvalues, and matrix congruence – in connection with quadratic forms. Figure 1 describes the logical-hierarchical connections between these types of equivalence relations. An arrow in the figure between a relation α and a relation β means that if relation α exists between two matrices, then so does relation β; the matrices P and Q are invertible.

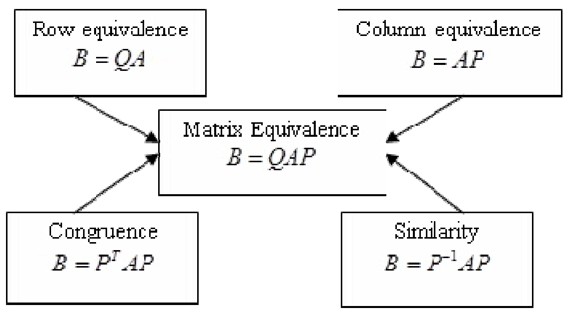

Figure 1: Examples of matrix equivalence

Other types of equivalence relations between matrices are restricted to complex matrices. These types are presented in Figure 2.

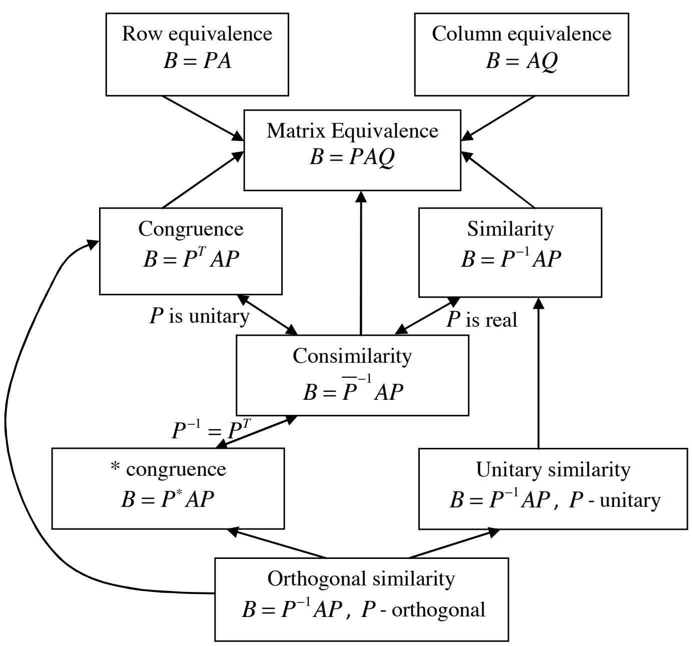

Figure 2: Equivalence relations of complex matrices

As in Figure 1, the matrices P and Q are invertible, and a one-side arrow between a relation α and a relation β means that if relation α exists between two matrices, then so does relation β. A two-side arrow in Figure 2 between a relation α and a relation β, accompanied by condition γ, means that under condition γ the relations are the same. By an orthogonal matrix we mean a real matrix P satisfying $P^{-1} = P^T$. Note that in the condition $P^{-1} = P^T$ that accompanies the arrow between consimilarity and *congruence, the matrix P is not necessarily real.

UNDERSTANDING EQUIVALENCE RELATIONS

In CERME 8, Gila Hanna (Hanna, 2013) talked about the concept of "memorability" discussed by Gowers (2007). She mentioned that for Gowers "easy to memorize" is intimately connected with understanding. Michener (1978) accounts a mathematician's perspective of what it means to "understand" as follows:

> When a mathematician says he understands a mathematical theory, he possesses much more knowledge than that which concerns the deductive aspects of theorems and proofs. He knows about examples and heuristics and how they are related. He has a sense of what to use and when to use it, and what is worth remembering. He has an intuitive feeling for the subject, how it hangs together, and how it relates to other theories. He knows how not to be swamped by details, but also to reference them when he needs them. (p. 361)

We apply this perspective to conceptualizing students' understanding in linear algebra. As will be evident shortly, our conceptualization of understanding is also stimulated by the work of Skemp (1976) on *instrumental understanding/relational understanding*. According to Mousley (2005), Skemp (1976) treated understanding as forms of knowing (cf. also Harel, 2008, for the compatible approach to conceptualizing *ways of understanding*).

Generally speaking, we refer to *understanding* a concept in a given mathematical subject as knowing to provide different representations of the concept, link it to other concepts by logical-hierarchical relations, and apply it in central issues of the subject. Specifically, in the context of equivalence relations in linear algebra, we suggest the following definition.

Understanding of an equivalence relation of matrices consists of:

- *Formal understanding* – knowing to recall (on demand or when needed in problem solving) its formal definition(s).
- *Instrumental understanding* – knowing to transform a matrix to an equivalent one.
- *Representational understanding* – knowing to recall (on demand or when needed in problem solving) properties of an equivalence class and knowing to find simple representatives of the equivalence classes.

- *Relational understanding* – knowing to relate the relation to other concepts, including other equivalence relations (e.g., Figure 1 and Figure 2 above).
- *Applicational understanding* – knowing to identify (not necessarily to solve) problems in which the relation may be useful.

Understanding is manifested by all, or some, of the above mentioned forms of knowing. Note that there is no hierarchical relation between the different types of understanding. For example, it is possible that a student may possess representational understanding without instrumental one, or vice versa. Also it is clear that each type of understanding has its own spectrum of deepness, from a basic to an advanced one. Deepness of understanding can be characterized in terms of available arsenal of relevant proofs, generic examples and problem-solving strategies. For operational reasons, we consider the following levels of understanding. A basic level of *formal understanding* is when the student can recall the relevant definitions. In case that there are several definitions, the ability to prove that they are equivalent demonstrates an advanced understanding. The levels of *instrumental understanding* can be characterized by the fluency of performing the transformations. A student at a basic level of *representational understanding* can only recall simple representatives. A more advanced student can prove existence and uniqueness of the representatives. A basic level of *relational understanding* presumes that a student knows how the equivalence relation relates to other concepts. A more advanced level is expressed by the ability to prove these relations. An advanced level of *applicational understanding* is when a student not only knows in which problems the concept may be useful, but also knows how to solve some of the problems.

These definitions can be operationalized since each of the above types and levels of understanding can be evaluated by means of appropriate tasks. In the next section we give some examples.

UNPACKING THE DEFINITION BY MEANS OF TASKS

As examples we explain the five types of understanding for row equivalence, matrix equivalence and matrix similarity. In addition, for matrix equivalence, we show how formal, representational and applicational understanding can be evaluated by tasks. For matrix similarity we discuss the pedagogical consequences of the lecturer's decision to stress some types of understanding more than the others. We also discuss the potential of teaching orthogonal similarity for creating an overall picture of the course through promoting its relational understanding.

Row equivalence

Formal understanding: the students are capable of recalling that a matrix B is row equivalent to a matrix A if B can be obtained from A by a finite number of elementary row operations. They should also know that this is the same as $B = QA$, where Q is invertible.

Instrumental understanding: the students can perform elementary row operations and know how to transform a given matrix to a row equivalent one.

Representational understanding: the students know that every matrix can be reduced to a row echelon form and is row equivalent to a unique matrix in a row reduced echelon form, and thus A and B are in the same equivalence class if and only if they have the same row reduced echelon form.

Relational understanding: the students are capable of associating row equivalence with systems of linear equations, can recall that row equivalent matrices have the same rank, but that the converse is not true, and that row equivalence implies matrix equivalence. The students also know that matrices of the same order are row equivalent if and only if they have the same row space.

Applicational understanding: the students can recall that a system of linear equations can be solved by reducing the augmented matrix to a row equivalent row reduced matrix (Gauss elimination) or to its row reduced echelon matrix (the Gauss-Jordan method). More advanced applications include vector independence, finding a basis and matrix inversion.

The concept of row equivalence is a basic concept and thus it is important that all students will develop all the five types of its understanding, at least at the basic level.

Matrix equivalence

The concept of matrix equivalence is less basic and it is not necessary to emphasize it in a very basic linear algebra course. However, in a more advanced course it makes sense to develop some of the following:

Formal understanding: the students recall that matrices A and B are equivalent if one can be obtained from the other by a finite number of elementary, row or column, operations, or, equivalently, if $B = QAP$, where P and Q are invertible.

Instrumental understanding: the students can perform row and column elementary operations.

Representational understanding: the students know that every $m \times n$ matrix of rank r is equivalent to an $m \times n$ matrix of the form $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$, where I_r is the $r \times r$ identity matrix.

Relational understanding: the students know that $A, B \in F^{m \times n}$ (The $m \times n$ matrices over the field F) are equivalent if and only if they have the same rank, that equivalent matrices represent the same linear transformation, and that the row equivalence, column equivalence, similarity and congruence are special cases of matrix equivalence (see Figure 1).

Applicational understanding: the students know that reducing a matrix to its simple representation $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$ can be useful in problems involving rank.

As an example, consider the following classical proof of the fact that for A, $B \in F^{m \times n}$, AB and BA have the same characteristic polynomial:

Proof: Suppose $rank\, A = r < n$ (If $r = n$, AB and BA are similar and thus have the same characteristic polynomial). Then A is equivalent to $\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$, i.e., there exist invertible matrices P and Q such that $PAQ = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix}$. Let $Q^{-1}BP^{-1} = \begin{pmatrix} C & D \\ E & F \end{pmatrix}$, where C is $r \times r$. Then $PABP^{-1} = PAQQ^{-1}BP^{-1} = \begin{pmatrix} C & D \\ O & O \end{pmatrix}$, and $Q^{-1}BAQ = Q^{-1}BP^{-1}PAQ = \begin{pmatrix} C & O \\ E & O \end{pmatrix}$. Hence AB is similar to $\begin{pmatrix} C & D \\ O & O \end{pmatrix}$ and BA is similar to $\begin{pmatrix} C & O \\ E & O \end{pmatrix}$. The matrices $\begin{pmatrix} C & D \\ O & O \end{pmatrix}$ and $\begin{pmatrix} C & O \\ E & O \end{pmatrix}$ have the same characteristic polynomial, and thus so do AB and BA.

In terms of understanding matrix equivalence, knowledge of the above proof requires basic level of formal and representational understandings, and an advanced level of applicational understanding. A lecturer interested in developing the applicational understanding of matrix equivalence may include this proof in the course. A lecturer who needs time for other purposes may use other proofs. We remark that the proof can be used for evaluating different types of understanding when divided into sub questions or stages.

Similarity

Formal understanding: the students can recall the definition of similarity.

Instrumental understanding: the students can implement the definition of similarity; they are aware that elementary operations do not preserve similarity.

Representational understanding: at the basic level, the students know that two diagonalizable matrices are similar if and only if they have the same characteristic polynomial and, in particular, the same determinant and trace. At a more advanced level they also know that triangulable matrices are similar if and only if they have the same Jordan form, and, in general, two matrices are similar if and only if they have the same rational form.

Relational understanding: at the basic level, the students know that similar matrices are equivalent. At a more advanced level, they also know that that similar matrices represent the same linear operator: if T is a linear operator on a finite-dimensional vector space V and if A represents T with respect to a basis α of V, and B

represents T with respect to a basis β, then $B = P^{-1}AP$, where P is the transformation matrix from α to β.

Applicational understanding: the students know that similarity to a diagonal matrix (similarity to a matrix in a Jordan form) is very useful in differential equations, difference equations and computing polynomials of matrices.

The concept of similarity is a fundamental concept and thus it should be taught in all linear algebra courses. Ideally, the lecturer should aim at developing all types of understanding similarity in all students. The least demanding linear algebra course should aim at developing basic formal, instrumental, representational and relational understandings of similarity. In between the ideal and the least demanding scenarios, there is a room for the lecturers' trade-offs. The trade-offs may be decided upon accordingly to characteristics of the class, and it is important that the lecturer will be aware of the pedagogical consequences of the choices. For instance, for engineering and applied mathematics students the applicational understanding should be emphasized, and for pure math majors, for whom the course is an introduction to more advanced algebra courses and to functional analysis, achieving advanced level of representational and relational understanding of similarity is crucial. Another example of a trade-off is teaching the Jordan form but giving up teaching the rational form and, in this way, gaining some time for applications. This decision would be appropriate for courses of mixed audience. We are aware, of course, that in many cases the trade-off decisions depend on the teacher's preferences, and hope that this discussion may result in better grounded decisions.

Orthogonal Similarity

Orthogonal similarity has an important role in highlighting the connectedness of the course. For this reason, we focus here on the relational understanding of the concept. Relational understanding of orthogonal similarity presumes the knowledge that orthogonal similarity is both similarity and congruence, that it is a special case of unitary similarity and that two matrices are unitary similar if and only if they represent the same linear operator with respect to different orthonormal bases. In addition, orthogonal similarity is related, by means of Sylvester Inertia Theorem, to comparison of different methods of diagonalization, and, in turn, to Givens Method for computing the eigenvalues of real symmetric matrices.

DISCUSSION

Equivalence relations play an important role in mathematics, in general, and in linear algebra, in particular. Halmos (1982, p. 246) points out that the concept of an equivalence relation "is one of the basic building blocks out of which all mathematical thought is constructed". In CERME 8, Kerstin Pettersson (Pettersson, Stadler & Tambour, 2013) studies the concept of a function as a "threshold concept" (Meyer & Land, 2005). Motivated by her talk we observe that equivalence relations is a threshold concept in linear algebra since by understanding it the students get a general comprehension of the whole topic. Skemp (1986) notes that the idea of equivalence relations helps to form a bridge between the everyday functioning of intelligence and mathematics. Many researchers and lecturers pointed out that constructing such a bridge is not an easy endeavour (e.g., Asghari & Tall, 2005; Mills, 2004), in particular, because the notion of equivalence relation is mathematically and epistemologically complex.

Stimulated by the famous paper "Understanding understanding mathematics" by Michener (1978), we try to understand and explain what it means to "understand" the concept of equivalence relations, and in particular equivalence relations between matrices. This is in line with and in continuation of the extended effort that has been made so far to promote students' conceptual understanding in linear algebra (e.g., Day & Kalman, 2001; Dorier, 2000, 2002; Dreyfus, Eisenberg & Uhlig, 2003; Harel, 2000; Jaworski, Treffert-Thomas & Bartsch, 2011).

In this paper we suggested a multi-facet definition of understanding equivalence relations between matrices and exemplified how the definition can be operationalized by means of mathematical tasks sensitive to its particular facets. We have also argued that an understanding of different types of equivalence relations between matrices, when taken as a central objective of a linear algebra course, can embrace most (if not all) topics usually taught in linear algebra courses. This also may educate the students to appreciate the applications of linear algebra and, at the same time, the mathematical structure and beauty of the subject. Consequently, the presented definition can be used as an organizational framework for planning, teaching and evaluating a linear algebra course.

More specifically, the choice of the equivalence relations, for teaching, depends on the level and the purpose of the course. Most one-year courses include row equivalence, column equivalence, matrix equivalence, similarity, congruence, unitary similarity and orthogonal similarity. A least demanding course can deal only with the first four relations, and a more advanced course can include also *congruence and consimilarity. The suggested definition of understanding equivalence relations may guide the lecturer in establishing feasible goals, in planning the course to achieve these goals and in evaluating the results. In some cases, a lecturer may be content with teaching aimed at particular types of understanding of the concepts at different levels of deepness. Reasons for this may be, for example, time constraints or the students' needs. Experienced teachers will decide what the minimum level of understanding of each type they want to achieve should be. We hope that our paper will help not only experienced teachers in doing the same. In addition, our operational definition may be useful as a part of a theoretical framework in a study dealing with students' learning of linear algebra or with development of lecturers' pedagogical and epistemological knowledge.

Although our paper is theoretical, we would like to mention here that we used the operational definition of equivalence relations as a framework for teaching a first year linear algebra course at the Technion. In addition, the tasks presented in the paper were tried in a series of informal interviews with the students who took part in the course and volunteered to participate in the interview. The interviews helped us to get an initial impression to which extent our conceptualization of understanding a concept is compatible with what students mean by understanding a concept and in particular, which types of understanding the concept of equivalence relations can be identified in students' mathematical performance. The interviews seem to confirm that the types and levels of understanding a concept of equivalence relations, described in the paper, are plausible. The validity of the suggested definition of understanding and the helpfulness of the suggested didactical approach will hopefully be tested in future empirical studies.

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TEACHING STATISTICS TO ENGINEERING STUDENTS: THE EXPERIENCE OF A NEWLY APPOINTED LECTURER

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In this paper I will adopt the reflective practitioner approach (Schön, 1987) to analyse the experience of a newly appointed lecturer in teaching statistics to engineering students. The lecturer is seen as entering a new for her community of practice (Wenger, 1998), the community of teaching mathematics at university level, which does not necessarily have universal characteristics. Different (sub)communities of students, lecturers and researchers are involved and in some sense affect teaching decision-making. Observations indicate that these communities interact to each other with potentially contradictions and affect the experience of the new lecturer.

INTRODUCTION

The last two years I have been offering a module on Statistics to engineering students in a well-regarded university in the UK. I am a mathematics graduate, I used to be a mathematics teacher and currently I am a researcher in mathematics education. When I started, two years ago, it was the first time I offered a module to a big group of engineering students and the first time I taught Statistics. In this paper I will offer an account of my reflection on this experience. I will adopt the *reflective practitioner* approach (Schön, 1987) to analyse this experience in two directions: the experience of teaching Statistics to engineering students and the first year experience of a newly appointed lecturer. I can see a newly appointed lecturer as a person entering a new community of practice (Wenger, 1998), the community of university lecturers. This community does not necessarily have universal characteristics. It seems that there are different (sub)communities that are involved and affect teaching decision-making and practices. In the experiences I am going to discuss in this paper I will consider the context and the different groups of people involved – engineering students, lecturers of Mathematics or Statistics, statisticians, etc. These groups can be seen as communities with characteristic practices. Drawing on my teaching practices and the feedback I received from the students, I will present three incidents related to: introduction to theory; creation of adequate intuitions; and, critical use of formulae.

In what follows I present the theoretical construct on which my analysis was based, a brief literature review on the topics mentioned above, the methodological approach and some observations related to these topics. Finally, I discuss some potential implications of this reflection to teaching practices at university level.

THEORETICAL CONSTRUCT

According to Wenger (1998) communities of practice are formed by people who engage in a process of collective learning in a shared domain of human endeavour. In these communities people are involved in activities that have same objectives and share a concern or a passion for this and learn how to do it better as they interact regularly. Learning is not necessarily the reason one community comes together as it might be the incidental outcome of member's interaction. On the other hand not any community or a group of people (e.g. a club, a neighbourhood or a company of friends) can be called a community of practice. A combination of three elements constitutes a community of practice: the *domain*, the *community* and the *practice*. The community needs to have a shared domain of interest that defines an identity of its members, who commit themselves to this domain, engage in joint activities and discussions, help each other and share information. A relationship is built between members and enables them to learn from each other. However, common interests are not enough to establish a community of practice – as, for example, in a film club. Members need to have a shared practice, namely a shared repertoire of resources such as experiences, stories, tools or ways of addressing recurring problems.

Research discusses several communities of practice related to teaching and learning of mathematics especially at university level: undergraduate students, mathematicians or mathematics education researchers. These communities are characterised by particular practices and ways of communication and potentially interact with each other. Solomon (2007) investigates potentially conflicting communities of practice within which undergraduate students find themselves, and presents a typology of their related learner identities. Additionally, Wenger's community of practice construction has offered a theoretical base for the analysis of students' beliefs about mathematics and their self-positioning within the school classroom or university lecture theatre community (for students beliefs about proof, see Solomon, 2006).

In a developmental study on engineering students' conceptual understanding of mathematics, Jaworski and Matthews (2011) regard university an institutional environment with "norms and expectations which can be seen to form an established community of practice" (ibid, p. 179). The elements of this community regard the curriculum, the assessment, teaching arrangements, student culture and expectation and teacher culture and expectation. Lecturers are "aligned with all of these to some extent and there are differing degrees to which change is possible" (ibid, p. 184).

Nardi (2008) describes mathematicians and researchers in mathematics education as two different communities with overlaps and conflicts that tantalise their relationships. She discerns the fragility – but also the importance – of the relationship between these two communities. Mathematicians admit the benefits of pedagogical practice being informed by mathematics education results, they reflect – often with scepticism – on research in mathematics education practices, and they acknowledge the influence of stereotypical perceptions on mathematics, mathematicians and educational research in their relationship (ibid, p. 257-292).

In my reflection on teaching Statistics to engineering students I can see different groups of people being involved: engineering students; lecturers of Mathematics or Statistics for specialist or non-specialist students; statisticians; users of statistics; and, researchers in mathematics education (see Figure 1). These groups can be seen as communities with characteristic practices. There are overlaps between these groups and this distinction might be seen not so realistic. However I found this construction very helpful on my teaching reflection and I will refer to some of these groups in my account in the paper. Putting myself in this spectrum of communities, at the time I was appointed, I was a mathematician (graduate of mathematics but not researcher), I had been practitioner in mathematics teaching, I was an active researcher in mathematics education, and I was a user of statistics in educational research.

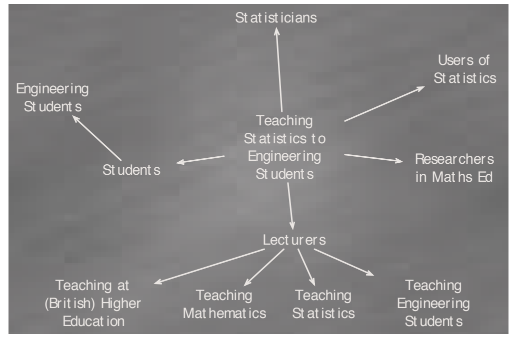

Figure 1: Sub-communities

ISSUES RELATED TO THE TEACHING AND LEARNING OF STATISTICS

Lecturing is a widespread mode of teaching at higher education in the UK, especially in mathematics and engineering departments. There is a scepticism and sometimes critique on lecturing as an effective teaching method for students' learning (Biggs, 2003). However we cannot ignore that numerous cases of lectures have been highly rated by students (Morton, 2009) and that both students and academics see value in this type of teaching (Folley, 2010). In the case discussed in this paper the big group lecture was the only teaching option and this characterise unavoidably the interaction between lecturer and students.

Statistics is "often regarded as being difficult to understand" (Kyle & Kahn, 2009, pp. 258). Several challenging aspects of the statistical concepts have been highlighted in research: the formulation of hypothesis; the distinctions and the application of different types of tests; the interpretation of the results (especially regarding the recognition of the significance level); and, the understanding the terminology used in stating a decision (Batanero et al., 1994). Garfield (1995) proposes five scenarios to support students' understanding of statistical concepts: activity and small group work; testing and feedback on misconceptions; comparing reality with predictions; computer simulations; and, software that allows interaction. Especially for non-specialist of statistics students, data-driven approach to the subject without insight to the mathematical foundation of the concepts is recommended (Kyle & Kahn, 2009).

Statistics teaching had changed dramatically in the last two decades. Older instructions used to be dominated by probability-based inference and abstract approaches with emphasis on memorising of formulae and techniques. A modern approach offers more opportunities for students' engagement with authentic activities. Also, this approach includes a more balanced use of the steps of data production, data analysis and inference. The transition between these steps has a lot of *back and forth* critical movements and it is not anymore straightforward as it used to be in the traditional statistical calculations (Moore, 1997). This modern approach is at an epistemological conflict with the formalist mathematical tradition and the persistence of students' difficulties in statistical reasoning might be the result of the continuing impact of the formalist mathematical tradition (Meletiou-Mavrotheris, 2007).

I discuss some of these issues regarding the teaching and learning of Statistics in the examples from my experience I present in the following section.

METHODOLOGICAL APPROACH

Throughout the academic year I had been keeping reflective notes on my experiences, annotated lecture notes with my comments just after each session and notes on my occasional communications with my colleagues. The observations I present in the next section draw on these resources and my background readings. I can consider myself as a *reflective practitioner* in the terms of Schön (1987). There are studies in which practitioners develop their reflection on practice while they are working with researchers in mathematics education (Jaworski, 1998; Nardi, 2008). In the case presented in this paper the arrangement is different as the researcher in mathematics education (myself) becomes a practitioner and develops her practice in parallel with her interaction with other practitioners.

THE EXPERIENCE OF FIRST YEAR TEACHING

In this section I will discuss a Statistics module that was offered to a group of 132 second year engineering students. This module is one of the service courses the Department of Mathematical Sciences offers to other departments. The students were from five different engineering programmes. Although all of them are engineering students their studying habits, needs and background vary across the subgroups. The teaching approach of the module is a combination of 11 two-hour lectures and 4 one-hour lab sessions using statistical software. As the module has been running without problems in a similar structure for several years, I used the existing module specifications without having any involvement to their design. The module is assessed through a coursework (20%) and written examination (80%).

When I started, two years ago, I had not taught statistics before and moreover I had not offered any module to engineering students. So, using Shulman's language the first challenge for me was to build a pedagogical content knowledge Shulman (1986) appropriate for this kind of teaching. This knowledge had to do not only with the statistical content that was necessary for these students but also had to do with the pedagogy that should be adopted in this particular teaching

I was a newcomer in this *world* and I had a lot to learn from the long experience of my colleagues. I aimed to have informal contacts with lecturers who had similar modules or researchers on statistics. I also participated in informal discussion organised in my department for lecturers who offer mathematical or statistical courses to engineering students. In parallel I was attending the "New Lecturers' Course" of my institution and regular seminars on "How we teach" offered by other lecturers of Statistics and Mathematics. I based my lectures on the plan and the lecture notes used in previous years with the support of the lecturers who had designed these notes in the first place. Recently, colleagues (including myself) have been forming a forum for members of staff at our institution with an interest in teaching statistics or quantitative methods, to enable us to meet and share ideas, experiences and resources. As a result I felt myself communicating with different groups of people (academics and students; cf. Figure 1). In the next sections I present three examples from my practices in teaching, as I experienced them in my lecturing and my reflection, especially in relation to the groups of people who are involved or related to these teaching practices.

Example 1: Theoretical explanations: Are they necessary and to what extent?

In the first session I had to introduce students to basic concepts of probability theory. According to Kyle and Kahn (2009), students with a strong mathematical background would be able to cope with a theoretical foundation of the subject. On the other hand non-specialists of statistics would find a data-driven approach to the subject more beneficial. In agreement with this recommendation, the lecture notes from previous years included a very brief introduction to elementary probability without any reference to theoretical parts, such as set theory. When I started designing my lectures, I found talking to my students about probability without introducing them to the basic concepts of the set theory impossible. How, for example, could we discuss the *addition law of probability*: $P(A \cup B)=P(A)+P(B)-P(A \cap B)$ without talking explicitly about sets, union and intersection of sets and their representation through Venn diagrams? For this reason, I added a brief section on sets (Figure 2a). Later on, I realised that the sets after their theoretical introduction turned out to be implicit components of the probability laws without any explicit reference to them. For example, no tasks on sets were included in the students' practice and exam questions. As a result, students ignored the sets and focused directly on the section on probability that was necessary for the tasks and the assessment. So by the end of the year, I decided that, given the restricted time-schedule of the module, this choice would be a *waste of time*. The year after I deserted this section by keeping only the corresponding Venn diagrams illustrating each of the probability laws (Figure 2b), and I used more time on examples.

Thinking about this choice retrospectively I can say that it was my mathematics teaching background that favoured a lecture that had all the necessary theoretical elements. I was looking for a structure that was theoretically complete without considering the students' needs and the practical constraints. In the language of communities of practice, I was coming from a community that practises the teaching mathematics for specialists, to a community related to teaching applications of statistics to non-specialists.

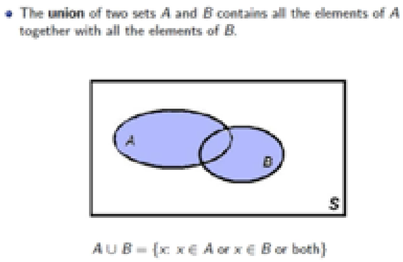

Figure 2a

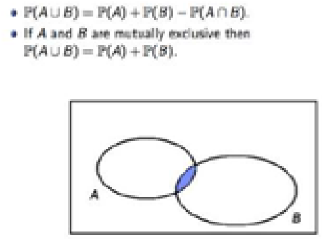

Figure 2b

Figure 2: Venn diagrams, sets and probability laws

Example 2: How can we help students create correct intuition about the processes they apply?

It is usual practice for users of statistics to interpret the results from statistical tables or outputs without necessarily understanding their meaning. In anecdotal conversations with colleagues who use statistics in their research or offer research methods courses, I heard that this is enough to make an accurate decision. Other colleagues believe that students need to be offered more explanations about these techniques in order to apply them and interpret the results properly. I see myself in the second group, as I believe that students will be more flexible and safer in their decisions if they know what is behind the procedures they apply. An unconscious use of a technical procedure without alternative confirmation methods may result in erroneous responses. A characteristic example is the interpretation of the *p*-value in the decision making of hypothesis testing: when the *p*-value is less than the significance level (usually in practice 0.05 or 0.01), there is a statistically significant result and therefore there is enough evidence to reject the null hypothesis. However students have trouble to interpret this result (Batanero et al., 1994). There are cases in which students, although aware of the need to compare the *p*-value with the significance level (e.g. 0.05), are not sure if the *p*-value needs to be *more* or *less* than this level. Alternatively, they would be more confident in their decisions if they were relying not only on their memory, but on knowing the meaning of the *p*-value. With the above in mind, I decided to offer students different aspects of the same concept as backing to their decisions.

The expectation in the mathematical community is that warrants of arguments are based on theoretical foundations. In the community of engineers who use statistics and mathematics in their applications, warrants should be based on appropriate intuitions of the concepts. With this aim in mind for my teaching design, I tried to highlight the connection of different aspects of the same concept towards the creation of correct intuitions. In this endeavour, I acted by putting together my personal intuitions, the content knowledge as well as my pedagogical content knowledge from mathematics education. According to the latter, research suggests that different and interconnected representations facilitate mathematical understanding (e.g. Presmeg 2006). Also, the provision of different representation and the manipulation of different aspects of a particular representation appear to help students learn basic statistics concepts (Garfield, 1995). In this spirit, and regarding the example of the *p*-value I mentioned earlier, I tried to put together the result of a statistical test and the corresponding critical value in the distribution graph and explain on it the meaning of the significance level and the *p*-value (see Figure 3). I did this not only in the first introduction of the test, but I kept using it in each of the examples we discussed in the lectures. I aimed the graph to become a validation tool for the students in their statistical interpretation of the *p*-value.

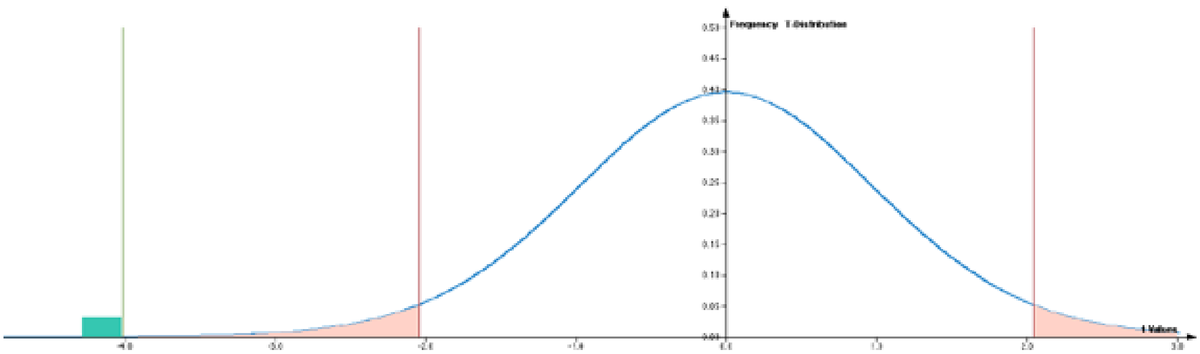

Figure 3: The statistical test and the critical value in the distribution graph

I do not have evidence that this approach was helpful to the students. In some cases I noticed little graphs made by students next to their responses in the exams. However, there were still students who misinterpreted the outcomes of the processes (e.g. by saying that there is significant difference when the *p*-value is more than 0.05).

In this incident, my actions were strongly influenced by the community of mathematics education. However, students who act under the norms of their community did not necessarily follow my intentions. Although lecturers may believe that representations are helpful for student understanding – especially us who are strongly influenced by mathematics education research – it's not always clear whether students notice these representations, what they see in them, and how they connect them. There are cases in which students interpret a representation differently from what the lecturer intends, and cannot see the mathematical meaning that is embedded in these representations. This is an area that calls for further investigation in two directions. One is on what students really notice and how they connect different aspects of the same concept. Another direction is whether students' practices are related to their community's practices, e.g. if their community is interested only in the application of methods, students might not put enough effort on the understanding of the meaning of this method.

Example 3: How can we help students to use formulae critically?

In Statistics many complicated formulae are used. Usually, remembering these formulae is not necessary and very often more emphasis is put on the correct use of a formula instead of its memorisation. To this aim, formulae sheets are used for students' practice and their written class assessment. In the module I describe in this paper students' practice both in applications of the formulae with calculator and in statistical analysis with software in the lab. The former is assessed through their final written examination and the latter through a written coursework in a form of essay. Throughout the term and in the exams, students use the tables and the formulae list from the Tables for Statisticians (White, Yeats & Skipworth, 1979). Additionally, they practice on examples from the HELM textbooks (Helping Engineers to Learn Mathematics (HELM) workbooks, http://helm.lboro.ac.uk/). The two resources use different versions of the tables and different notation in the formulae. For example, the formula I used in the lectures for the calculation of the *slope* in the *simple linear regression model* (the same as the one used in the HELM workbook, HELM workbook 43, p. 6) and the formula students had in the exams (White et al., pp. 65) are presented in Figure 4.

$$b = \frac{\frac{\sum xy}{n} - \frac{\sum x}{n}\frac{\sum y}{n}}{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$$

HELM workbook 43, p. 6

$$b = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

White, Yeats & Skipworth, 1979, pp. 65

Figure 4: Simple linear regression formulae

The mechanical application of a formula without understanding its meaning may result in trouble. This trouble may be arising from different notations or because of the implicit algebra of a formula. I have been raising these issues in my lectures and let my students know from very early on what formula sheet they would get at the exams. This is technical information that might be overlooked by the students as the following incident exemplifies:

> One of the students just after the exams emailed to me that the formula of the slope in the simple linear regression model at the lecture notes was *not correct* and although it took him some time to *memorise* it he was unable to give a correct answer to the exam question. Also, he said that this was *unfair*.

There are a couple of issues I would like to highlight in this incident. Firstly, it seems that for the student there was only one version of the formula and he was not flexible enough to see that the two formulae were equivalent. This might be because of the notation or because of his inadequate algebraic knowledge. Also, he claimed that he spent so much time *memorising,* indicating a practice that was in conflict with what I have been trying to encourage my students to do. Finally, he assumed that limited learning resources (in this case lecture notes or/and HELM workbook) are enough for his preparation and he defended himself by saying that the formula in the notes was incorrect. It would appear that there is a conflict between lecturer' and student' anticipations. Lecturer expects university students to be individual and critical learners. However, the student expects specific study guidelines and transfers the responsibility for his learning (or his mistakes) to the lecturer. The two communities have different expectations regarding this responsibility as well as approaches to university study.

DISCUSSION

In this paper I presented examples from my inaugural experience of teaching Statistics to engineering students. While this account is not a systematic study, I draw on my reflections in order to discuss the complexity of the experience of a newly appointed lecturer especially when she comes from different teaching and research communities.

A first observation is that, although I tried to adopt a more modern approach in teaching statistics (Moore, 1997) with less focus on memorising techniques and formulae, students were still keen on a rather mechanical approach to learning. A second observation is the different identities through which I experienced my first year of teaching: mathematician, lecturer of statistics, university lecturer, lecturer to engineers and mathematics education researcher. The relationship of these identities was not always smooth and many contradictions occurred between them.

It seems that although we may consider ourselves as part of the broad community of practice (Wenger, 1998) of teaching undergraduate mathematics with shared practices and ways of communication, in practice other (sub)communities are involved and establish their rules (cf. Figure 1). One of these communities is the community of students, which has its own regulations and interacts with the teaching community of lecturers – with some potential conflicts. Other (sub)communities are related to the mathematical content (e.g. mathematics or statistics) and the type of study programmes (e.g. prospective mathematicians or engineers). My enculturation to this new environment was not straightforward and the interaction with my colleagues was crucially helpful. In this paper I report observations on how communities interact and how this interaction affects the experience of a new lecturer. Further, systematic study is needed, firstly on the existence and the characteristics of these communities and secondly on the ways these communities affect the lecturer and student experiences. Finally, I can envisage the potentials of this study for new lecturers' training programmes.

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THE USE OF UNFAMILIAR TASKS IN FIRST YEAR CALCULUS COURSES TO AID THE TRANSITION FROM SCHOOL TO UNIVERSITY MATHEMATICS

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Research has shown that mathematics courses at university often focus more on conceptual understanding than those at secondary school (Clark & Lovric, 2008). Moreover, the literature reports that the types of tasks assigned to students affect their learning. A project was undertaken by the authors in which tasks were designed and presented to first-year undergraduate Calculus students with the aim of promoting conceptual understanding and developing mathematical thinking skills. Here we present data from interviews with five students; they reported an increased emphasis on conceptual understanding at university, and found the tasks assigned beneficial in the development of conceptual understanding. We suggest that unfamiliar tasks are useful in the transition from school to university mathematics.

INTRODUCTION

The transition from school to university has been the object of much research in recent years. Gueudet (2008) conducted a review of the literature and found four broad categories of research activity in this area. Two of these are research on thinking modes and research on the organization of knowledge and reasoning modes. In this paper, we will consider the views of students on their experiences as they deal with types of thinking and organization of knowledge new to them at the beginning of their university studies. Views of the students were collected as part of a project in which the authors designed a range of tasks for use in first year differential calculus courses. The aim of the design process was to create tasks that would promote conceptual understanding by encouraging students to develop some of the practices and 'habits of mind' of research mathematicians. According to Cuoco, Goldenberg, and Mark (1996) these include finding patterns, experimenting, conjecturing, arguing, using mathematical language, visualizing and inventing. Mason and Johnston-Wilder (2004) also mention exemplifying, generalizing, justifying, convincing and refuting as processes and actions that mathematicians employ when tackling problems. A selection of tasks was trialed in a first-year calculus module, following which five students were interviewed. The data from these interviews form the basis of this paper. We will present the students' views on the difference between mathematics at school and at university and on the effects of the tasks on their ways of working and on the promotion of understanding. We will then discuss the effect of the tasks on the students' experience of the transition process.

LITERATURE REVIEW

Clark and Lovric (2008 and 2009) have described the secondary-tertiary transition as a ‘rite of passage’ and detail the changes that students experience as they commence their mathematical studies at tertiary level. These include changes in the type of mathematics taught and changes in the way mathematics is taught. They contend that compared to the mathematics taught at school, mathematics at university involves an increased emphasis on conceptual understanding, multiple representations of mathematical objects, advanced mathematical thinking, proof, abstraction, and the importance of precise mathematical language. De Guzman, Hodgson, Robert, and Villani (1998) had previously reported on the difficulties that first year university students faced and had categorized them into three types: epistemological/cognitive difficulties; sociological/cultural difficulties; and didactical difficulties. The cognitive difficulties related to the transition from elementary to advanced mathematical thinking but also to students’ ability to organize their mathematical knowledge and to develop connections between concepts.

Mathematical understanding has been characterized by many different authors (for example, Pirie & Kieren, 1994). Skemp (1976) spoke about instrumental understanding (or ‘rules without reason’) and relational understanding (knowing both what to do and why to do it). In the US, the National Research Council (2001) described ‘conceptual understanding’ as the “comprehension of mathematical concepts, operations and relations” (p116). Much of the literature in this area refers to the understanding of school mathematics; however, Sofronas et al. (2011) asked 24 experts the question ‘What does it mean for a student to understand the first-year calculus?’. They were able to construct a set of four core goals that defined student understanding in this context. These goals were: mastery of the fundamental concepts and-or skills of the first year calculus; construction of connections and relationships between concepts and skills; the ability to use the ideas of first-year calculus; and a deep sense of the context and purpose of the calculus (Sofronas et al., 2011, p132).

Gueudet (2008) observed that many studies on transition compare the practices of students with those of mathematicians. In this regard, Gueudet discusses the work of Lithner (2000) on mathematical reasoning. He noted that first year university students often rely heavily on past experience when solving mathematical problems, while mathematicians usually display more flexibility in their thinking and reasoning. Gueudet (2008) reported that in order to deal with this issue, researchers have called for changes in the teaching methods both at school and at university. In particular, a wider range of tasks which would allow students to develop autonomy and flexibility have been proposed. Boesen et al. (2010) also contend that the types of tasks assigned to students affect their learning and the use of tasks with lower levels of cognitive demand leads to rote-learning by students and a consequent inability to solve unfamiliar problems or to transfer their mathematical knowledge to other areas competently and appropriately.

In Ireland, research at secondary level has shown that teaching in Irish mathematics classrooms tends to be focussed on the use of algorithmic procedures, with very little emphasis on conceptual understanding, and that students appear unable to apply techniques learnt in unfamiliar contexts (for example, Lyons, Lynch, Close, Sheerin, and Boland, 2003).

In discussing how students might accomplish a successful secondary-tertiary transition in mathematics, Clark and Lovric (2009) state “what we certainly can claim is that the success depends, in great measure, on the robustness of certain parameters in secondary education (attitude, motivation, approach towards work, and, in particular, learning styles and cognitive models) that might need to be significantly modified at tertiary level" (p.759). The paragraphs below consider to what extent the series of unfamiliar tasks assigned did modify some of these parameters in the view of the five interviewed students.

THE TASK DESIGN PROJECT

The first two authors are mathematics lecturers in different third level institutions in Ireland. In the academic year 2011/12, both were teaching first year differential calculus modules. Given the procedural nature of mathematics instruction at second level in Ireland, they endeavoured to design a series of unfamiliar non-procedural tasks for their students in an effort to give students opportunities to develop their thinking skills and their conceptual understanding. (In this paper, the National Research Council’s (2001) description of conceptual understanding has been adopted.) An ‘unfamiliar task’ is one for which students have no algorithm, well-rehearsed procedure or previously demonstrated process to follow. Following Lithner’s (2000) observation that students often rely heavily on past experience when solving problems, the authors hoped, by presenting the students with unfamiliar tasks, to discourage such reliance and help them to develop the flexibility in their thinking and reasoning characteristic of mathematicians.

The tasks designed in this project required students to make use of definitions, generate examples, generalise, make conjectures, analyse reasoning, evaluate statements, or use visualisation. A sample is shown below (note that students were asked to think about this problem after they had encountered the concept of continuity but before they had seen the Intermediate Value Theorem):

[1] Do you believe the following statement is true or false?

If $f(x)$ is a continuous function on the interval $[a,b]$ and k is a number between $f(a)$ and $f(b)$, then there is at least one number c in $[a,b]$ such that $k=f(c)$.

If you think the statement is false, provide a sketch as a counterexample.

A selection of other tasks designed along with a rationale for the task framework used can be found in Breen and O'Shea (2011). Each problem set (and the final examination) contained unfamiliar non-procedural tasks as well as some more procedural tasks. For example, the following procedural task appeared on the same problem set as question 1 above:

[2] Use the definition of continuity to determine whether the function $f(x)=\dfrac{x^2-1}{x+1}$ is continuous at $x=-1$.

In this paper we will concentrate on the module taught by the first author at St Patrick's College, Drumcondra. There were 35 students registered on this module, who had chosen to study Mathematics as part of a BEd (Primary) or BA (Humanities) degree. (BEd (Primary) students at St Patrick's College to date must choose to specialise in one Humanities subject; this subject accounts for 40% of credits awarded for the degree.) The designed tasks were assigned to the students either as homework (for students to work on independently) or as tutorial problems (for students to work on in small groups). A number of tutorial sessions were observed by the third author as part of this project. Towards the end of the module, the fourteen students who had previously attended the tutorials observed were invited to volunteer to be interviewed. (These students were selected in order to maximise later collation of the data collected.) Five students volunteered and were interviewed individually by the third author, as she was not involved in the teaching of the module nor in the design of the homework and tutorial tasks. The interviews were semi-structured and lasted between 15 and 25 minutes; they were audio-recorded and fully transcribed. The identity of the interviewees was not revealed to the first author (module lecturer) unless the interviewees chose to do so themselves. The students were assigned pseudonyms A, B, C, D, E.

Students were asked about their impressions of mathematics at university, how their experience of mathematics at school differed from that at university, how their study habits or ways of working had changed and about the tasks that they had worked on. In particular, the interview schedule produced in advance of the semi-structured interviews outlined the following questions: *What do you think of your maths course (at university)? How is it different from school mathematics?; Different types of questions were used on the problem sheets in the Calculus course, were you aware of the difference? Can you give examples?*; [Showing students two tasks on the same topic (one familiar/procedural, the other not):] *Which of these tasks is familiar to you? Why this one? How did you feel when you first saw the task? What are the differences between the tasks? What was the purpose of each of the tasks? What did each of the tasks help you learn? Did these tasks aid your development of understanding in the same way?*

Our research questions for this paper are:

1. What are the students' views on the difference between mathematics at secondary and tertiary level?
2. What are the students' views on how the designed tasks have impacted on their practices, learning and conceptual understanding in the transition to tertiary level?

The authors coded the transcripts separately in line with these research questions. Each author grouped the codes into broad categories and then all three met to discuss the codes and agreed on common categories. This paper reports on the main themes that emerged from this analysis.

RESULTS - STUDENTS' VIEWS ON DIFFERENCES BETWEEN MATHEMATICS IN SCHOOL AND UNIVERSITY

The students were first of all asked what they thought of their university mathematics course; two of them (B and E) immediately volunteered that their experiences of mathematics courses at university were very different to that of school. The other three students were asked if they had found differences between second and third level mathematics and they were all able to point to specific differences. The differences that the interviewees spoke of were categorised and two themes are reported here: a change in emphasis from procedural fluency to conceptual understanding, and a move to independent learning.

All the students interviewed described a change in emphasis from a focus on instrumental understanding or procedural fluency in school to a focus on relational or conceptual understanding at university. They all mentioned the importance of procedures in school. For example, students A and B said:

> Student A: In school it was kind a lot of procedural – you just, when you saw something you knew you did this. Like there were bits you'd understand and other bits you'd just take for granted.
>
> Student B: It's just procedure…You learn the methodologies [sic] rather than learning why you are doing what you are doing

The students also spoke about working with formulae at second level in a procedural manner.

Two students (B and D) spoke about memorisation or 'learning off' as being important at school. Two students (A and D) also felt that there was a lack of linkages between different topics in the second level curriculum:

> Student D: For the [senior cycle of secondary school] they were separate questions and they didn't really tie together at all.

Student A mentioned a specific example of the disjointed nature of her mathematics course at secondary school:

> Student A: It was never explained to us that limits had – dealt with functions. They were nearly kind of separate.

This was in contrast to the five students' perception of the importance of conceptual understanding and connections between topics at university. All of the interviewed students spoke about the emphasis on conceptual understanding, for example

Student C: The emphasis in the college course is about actually understanding the principles

Student D: It kind of explains why you were doing it before

The interviewees also noticed an increased emphasis being placed on connections or relations between mathematical topics and ideas in university. Both Student A and Student D asserted that links between concepts were made explicit in their university mathematics course:

Student D: It kind of ties together really everything from the [senior cycle of secondary school] … It kind of interlinks them more.

The five students also alluded to a change in teaching style between second and third level. This change seems to concern a move from a teacher-led classroom environment to one where more independent learning is required. For example:

Student B: It's hard to change from that mindset I find within a few months…Like from being given all the information to then having to find it yourself.

One student also indicated that, although unfamiliar tasks had been assigned at university, she would have been unlikely to encounter such tasks in school:

Student B: We hadn't done that in class, so we had to try to figure it out for ourselves. Whereas in school the teacher would have done that with you.

In addition, the students spoke about having to think for themselves and to schedule their own study timetables without the framework of daily homework assignments. However, the students interviewed acknowledged that the transition from school to university mathematics can be difficult and that it can be hard to make this transition in a short timeframe.

RESULTS - STUDENTS' VIEWS ON THE IMPACT OF UNFAMILIAR AND FAMILIAR TASKS

The five students were asked a series of questions concerning the tasks assigned to them during the module. We categorised the responses and report on the main themes that emerged: the students' views on the effects of the tasks on their conceptual understanding and the impact of the tasks on their mathematical thinking and ways of working.

Students' views of the effects of the tasks on their understanding

The students were asked what they gained from the tasks, and in particular which types of tasks helped them gain conceptual understanding. Four of the five students chose unfamiliar tasks designed for this study in answer to this question. Student B referred to unfamiliar tasks in general:

Student B: The ones I haven't seen before, definitely… in those ones you have to like completely understand it to get the answer.

Student D referred to less procedural types of tasks:

Student D: When you weren't just procedural the whole way, when you just had to stop and be like ok what do I know about it, is it continuous, differential and all this.

The comments of some students indicate that to perform unfamiliar tasks they were forced to apply and find relationships between previously learned concepts. The quote below from Student A arose from her response when she was asked to compare two particular tasks, the first one familiar (evaluating limits) and the second one unfamiliar to the student (an example generation task). Her answer suggests that she has learned more from the unfamiliar problem.

Student A: So you're kind of bringing together what you know from other things whereas in Question 1 you kind of — you're told what you have to do. So you're literally just kind of following a procedure really...whereas for the second one you kind of actually are more thinking yourself...it is more difficult, ya, but it kind of helps you understand it better. You see the relationship between them.

Other students said that some of the designed tasks involved 'actually explaining the process involved' (Student C) and that this led to better understanding. Student B also acknowledged that she had to understand and apply previously learned knowledge to perform unfamiliar tasks:

Student B: you had to go back on what continuity means and then try and apply it to the different ones. So it's making us see like when something is continuous and when it wasn't.

Two students also spoke of the benefit of unfamiliar tasks for assessing their own understanding of concepts. For example, Student E when referring to an example generation task, stated she found it beneficial because

Student E: it kind of proves you understand it more.

Students' views of the impact of the tasks on their mathematical thinking and ways of working

The interviews give some insight into how the tasks assigned encourage or stimulate thinking practices and ways of working. Some interviewees described certain tasks, which they had identified as unfamiliar or non-procedural, as encouraging habits such as thinking, analysing, questioning or exploring patterns. Four of the five students interviewed (A, C, D, E) asserted that the unfamiliar tasks made them think more or think for themselves. Student D claimed that, whereas for familiar tasks she would "just like write" and "plough through", the unfamiliar tasks "really make [her] think".

Considering specific types of tasks, Student C said about a conjecturing task:

Student C: You have to think about it and then, it's not actually a procedure, it's about you analysing the pattern and stuff.

This student also commented on a task which involves evaluating a statement:

Student C: about whether is the statement true or false, analysing it, ya it gets you thinking more than just actually using the knowledge that you learned like.

Student B described how performing an analysing reasoning task prompted her to ask herself questions:

Student B: Ahm, because it makes you analyse the proof and ask yourself questions like why you do things like that. Whereas if you were just given the proof, the correct one, you just take it for granted that that was correct.

A number of the students (A, B, C) also mentioned that, when they encounter an unfamiliar task, they refer to the basic definitions or theorems on the course and think about what they know (in relation to the task) and how they can apply it. One student (A) explained that she approached an example generation exercise by breaking it down or taking it step-by-step but also “drawing on other things” that she knew and bringing them together. Student C described using the following techniques (sometimes in combination) when confronted with various unfamiliar tasks: sketching a graph, examining different cases, generating examples, generalising, working backwards. When speaking more generally about their study habits in relation to the Calculus module, Students C and D mentioned focussing on understanding the concepts and Student B claimed a lot of independent work is required.

DISCUSSION

Research literature on the transition from second to third level mathematics seems to agree on the requirement at university level for more relational and conceptual understanding and more flexibility in thinking or approaching mathematical problems in comparison to second level mathematics (for example, Clark & Lovric, 2008; De Guzman et al., 1998). The responses of the students interviewed show that they are aware of the different requirements and, moreover, most of them welcome the change in focus. Various authors have made recommendations about the type of tasks that should be assigned to undergraduate students in order to gain the required relational or conceptual understanding and flexibility (for example, Geuedet, 2008; Boesen et al., 2010). Though the students interviewed in the study reported here struggle with the unfamiliar tasks assigned to them, they find such tasks beneficial regarding the development of conceptual understanding and their learning more generally. All of the interviewees stated that some of the unfamiliar tasks encouraged them to think more, or to analyse or question the information given. Some interviewees explicitly described how unfamiliar tasks they encountered led them to connect ideas met previously. Furthermore, the comments of the students interviewed indicate that unfamiliar tasks may not only have the effect of stimulating the development of conceptual understanding, but also may have some potential to raise an awareness that more than instrumental understanding is required at university, thereby easing their transition to university practices.

While all the interviewees reported using the practices or ‘habits of mind’ (generating examples, generalising, visualising etc.) of mathematicians when specifically called on to do so by a particular task (an example generation or conjecturing task, say), one student, Student C, described drawing on such practices in a broader sense when confronted with an unfamiliar task. Given the assertion by Cuoco et al. (1996) that it is these practices which ‘give students the tools they will need in order to use, understand and even make the mathematics that does not yet exist’ (p.376), this is a very positive development.

However, it should be noted that some of the interviewed students also described significant advantages of familiar and/or procedural tasks on their learning and confidence. In practice, the sets of problems assigned to students in this project also contained procedural-type questions, to aid the development of procedural fluency (the “ability to carry out procedures flexibly, accurately, efficiently and appropriately” as described by the National Research Council, 2001, p116). In combination, these two types of tasks helped to address the first three end-goals of a first-year calculus course as described by Sofronas et al. (2011): the mastery of fundamental concepts and-or skills; the construction of connections and relationships between and among concepts and skills; the ability to use the ideas of calculus.

The data analysed here consists of self-reported views of what the students gained from the tasks assigned. The next step in this project is to conduct a series of task-based interviews to explore the practices engaged in by students as they attempt the tasks and to investigate progress in their understanding of particular mathematical concepts related to the tasks.

While developing conceptual understanding requires a suitable teaching and learning environment as well as attention to task design, our findings to date suggest that the inclusion of unfamiliar tasks is beneficial in helping students negotiate the secondary-tertiary transition in terms of the changes in the type of mathematics taught and the way in which mathematics is taught. Clark and Lovric (2009) mentioned that certain parameters, such as a student’s attitude, approach towards work and learning style, may have to be modified to successfully accomplish the transition. We suggest that the tasks assigned to students can be used as a vehicle through which these parameters might be changed.

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STUDENTS' PERSONAL RELATIONSHIP WITH SERIES OF REAL NUMBERS AS A CONSEQUENCE OF TEACHING PRACTICES

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Infinite series of real numbers are a topic that students usually encounter in their first courses of Calculus, however this notion has not received much attention by research, and research has focused mostly on learning difficulties. Our research focuses on the teaching of series and on the consequences of institutional choices on students' learning. After having analysed textbooks and teaching practices, we conjectured the presence of some contract rules in the existing praxeologies to teach series which might have an impact on students' learning. The analyses of the responses to a questionnaire seems to indicate that institutional choices lead the students to learn series without being able to define what a series is, and without being able to identify any application of this notion.

INTRODUCTION

Infinite series of real numbers (*series* in what follows) have been at the heart of the development of Calculus and appear in the programs of the introductory Calculus courses in many countries together with the teaching of sequences, limits, derivatives, and integrals. Series have many applications within mathematics (such as the writing of numbers with infinite decimals, or the calculation of areas by means of rectangles), and also outside of the field of mathematics (as the modelling of situations such as the distribution of pollutants in the atmosphere, or the growing of interests in bank accounts), which may justify their position in Calculus courses.

In Canada, the organisation of education and official curricula is under the jurisdiction of each province. In Québec, compulsory education finishes at the age of 16. For students who wish to pursue university studies, the completion of two years of pre-university studies, called *collégial* is required. For students who want to pursue scientific or technical careers, Calculus is introduced during the *collégial* studies, and it is at this time that series first appear.

Our PhD research (González-Martín, 2006), about the learning of improper integrals, led us to conjecture that students' learning of series could be mostly based on routine aspects. As we discuss in the next section, literature led us to see that most of the scarce research about series has focused on their learning, but not on their teaching. For this reason, we decided to analyse how series are presented in *collégial* textbooks (González-Martín, Nardi & Biza, 2011), and to study how they are taught by *collégial* teachers (González-Martín, 2010), while adopting an anthropological approach (Chevallard, 1999) and acknowledging the key role that the *collégial* institution and its choices play in the learning of series. After our analyses, our results led us to conjecture the existence of some *contract rules* [1] which might have an impact on students' personal relationship with series as a consequence of teaching practices. The purpose of this paper is to discuss our results regarding some of these *contract rules* and their impact on students' learning about series.

BACKGROUND

Research literature in mathematics education focusing on series is scarce. Regarding their teaching, series appear implicitly in Robert's (1982) work, where she states that inadequate conceptions of convergence of sequences found in university students in France could be, in part, due to the exercises used in teaching. More recently, Bagni (2000) identified two levels in the construction of a notion in someone's mind (the *operational* and the *structural levels*), and observed that this distinction is not usually considered in the teaching of series.

Other results focus on the learning of series, such as Kidron's (2002) who identifies some difficulties linked to series themselves (such as the use of the potential infinity, for instance), or the confusion between sequences and series. A more exhaustive summary of literature can be found in González-Martín et al. (2011).

As we are interested in the teaching of series and in some possible consequences linked to their teaching, we undertook a research program on different stages. For the first stage, we analysed a sample of 17 textbooks used in *collégial* studies in Québec over a period of 15 years: from 1993 to 2008 (González-Martín et al., 2011), paying special attention to the praxeologies (see next section) privileged by textbooks. Our main results can be summarised as follows:

- R1. Series are usually introduced through praxeologies which do not lead to a questioning about their applications or *raison d'être*. They do not seem to solve any specific problem.
- R2. Praxeologies tend to introduce series as a tool in order to later introduce functional series, but the importance of series *per se* is usually absent.
- R3. Praxeological organisations tend to ignore some of the main difficulties in learning series identified by research.
- R4. The vast majority of tasks concerning series are related to the application of convergence criteria, or to the application of algorithmic (or previously exemplified) procedures.

The second stage of the research consisted in analysing the use of textbooks that *collégial* teachers make, and whether through their practices they attempt to do something different from what is usually presented in the textbooks (González-Martín, 2010). This is, we tried to analyse whether there are important differences between the *knowledge to be taught* and the *knowledge actually taught*. Our interviews with five teachers revealed that they considered their textbook as adequate for the teaching of series, and that their practices tended to mostly reproduce what was presented in their textbooks. We could even identify some *gaps* between their *didactic intentions* and their practices: for instance, some teachers said that during the teaching and learning of series it is important to feel that the arithmetic of infinity is different; however, in the tasks they privileged, it was not possible to see how these tasks could help their students to achieve this.

As a consequence of the results of these two stages, we conjectured the existence of some implicit *contract rules* in the teaching of series in the *collégial* institutions in Québec. For the purposes of this paper, we will only discuss the two following ones:

Rule 1: "To solve the questions about series that are given, their definition is not necessary".

Rule 2: "Applications of series, inside or outside of mathematics, are not important".

These two rules have been chosen for this paper because they are related to two main activities in mathematics: defining [2] and modelling. *Rule 1* could be a consequence of *R2* and *R4*; as series seem to be presented as a tool to introduce other notions, and as the tasks seem to be organised around the application of criteria, students might develop the idea that they do not need to be able to define series in any way, since this knowledge is not required to succeed in the tasks which are proposed. *Rule 2* could be a consequence of *R1* and *R4*; the praxeologies tend to introduce series as a notion which does not solve any particular problem and the focus is established on the application of convergence criteria, without giving any importance to the utility that knowing that a series converges or diverges could have. We believe that these contract rules participate in the characterisation of the institutional relationship of the *collégial* institution with series, which might have consequences for students' personal relationship with series and for the learning of other notions. We do not advocate that being able to define series in some way, or knowing applications of series, are indicators of a *good* learning of series. Our intention is to better understand the personal relationship of the *collégial* students with series as a consequence of institutional practices.

To verify whether these rules have an impact on *collégial* students' personal relationship with series, we created a sample of students and applied a questionnaire (for other examples of this type of work, see for instance Kouidri, 2009).

THEORETICAL FRAMEWORK

As we have said before, our research follows an anthropological approach (Chevallard, 1999), as we recognise the important role of institutional choices in the learning of mathematics, and the repercussions of these choices.

Chevallard's (1999) anthropological theory attempts to achieve a better understanding of the choices made by an institution in order to organise the teaching of mathematical notions. This theory recognises that mathematical objects are not absolute objects, but entities which arise from the practices of given institutions and that every human activity consists in completing a certain type of task. These practices can be described in terms of tasks, techniques used to complete the tasks, technologies which both justify and explain the techniques, and theories which include the given discourses. According to this theory, every human activity generates an organisation of tasks, techniques, technologies and theories which Chevallard designates as *praxeology*, or *praxeologic organisation*. A praxeological analysis allows us to characterise the institutional relationship with mathematical notions within given institutions. This institutional relationship is mainly forged through the exercises (or tasks), and not only through the theoretical explanations (Kouidri, 2009). Praxeological analyses are useful to describe praxeological organisations, but also to identify the existence of (sometimes implicit) *contract rules*, which are rules that the institution fosters through its practices around a mathematical notion and which contribute to determine the institutional relationship with a mathematical notion. This institutional relationship and its *contract rules* play an important role in the development of the learners' personal relationship with the mathematical notions s/he learns within the institution.

In our case, our praxeological analysis of the teaching of series (both in textbooks and in teaching practices) led us to identify some praxeological organisations in the teaching of series (see González-Martín et al., 2011), and to identify some implicit *contract rules* which may have a direct impact on the development of the students' personal relationship with series.

METHODOLOGY

To verify the impact of the *contract rules* 1 and 2, among others, on the students' personal relationship with series, we created a sample of 32 students in their second year of *collégial* studies (where series are introduced) after the teaching of series had occurred (so all the usual contents about series had already been taught). These 32 students come from three different teachers, who we name teachers *A, B* and *C*. Our sample consists of 4 students from teacher *A* (referred to as students A1 to A4), 14 students from teacher *B* (referred to as students B1 to B14), and 14 students from teacher *C* (referred to as students C1 to C14).

We constructed a questionnaire with 10 questions, aiming to assess the students' learning about series, as well as to verify our conjectures about the impact of the *contract rules* on their learning. The questions were designed taking into account: 1) our results of the textbooks analysis; 2) our results of the teachers' interviews about how they teach series; 3) research literature about the learning of series. The questionnaires were administrated in May 2011 during one of the courses (approximately 55 minutes in duration), and the students participated voluntarily (the students who did not want to participate just did not attend the course on that day).

In this paper, we discuss the students' responses to the three following questions:

<u>Question 1</u>:
Define the notion of "numerical series" or "infinite sum".
<u>Question 3</u>:
Name at least two different applications of series in the field of mathematics.
<u>Question 4</u>:
Name at least two different applications of series in a field different than mathematics.
For one of these applications, create a realistic problem whose resolution requires the use of series. Solve the problem.

In the next section, we present and comment on the results obtained to these three questions.

DATA ANALYSIS

Question 1

The distribution of responses to this question is the following:

Correct definitions with her/his own words explicitly mentioning a sum or the addition of terms.	B4*, B13
"Let $\{a_n\}$ be a sequence, we write: $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$"	A1, A4 B1
"An infinite series has the form: $\sum_{k=1}^{+\infty} aK^{k-1}$. We multiply a constant by a number till infinity. We add all the terms."	C11
Other definitions mentioning the necessity of defining an equation, or finding some logic among the terms. Some of these definitions illustrate confusion between sequences, series and other notions.	A2 B3, B12, B14 C3*, C4, C5, C8, C9, C12, C13, C14
Definitions showing some confusion between sequences and series.	B2, B6, B9 C1, C6*
Other incorrect definitions.	A3 B5, B7, B8, B10, B11 C2, C7, C10*

Table 1: Responses to Question 1

As we said before, the usual contents about series had already been taught, including the definition of series and the application of convergence tests. In spite of this, only five students (A1, A4, B1, B4, B13) provide definitions with no erroneous elements. However, mentioning that the sum could converge or diverge does not seem to be an important thing to mention for the students, as only the students marked with * mention this fact. Other students (particularly those from teacher *C*) seem to associate series with the existence of a formula or regularity, which might be a consequence of the praxeologies used. For many students, in their discourse, there seems to be some confusion between sequences and series, or in the use of some vocabulary: “it’s the sum of sequences” (B2), “infinite series, it’s series in which we add the terms indefinitely” (B6), “numerical series: a sequence of numbers having some logic. Infinite sum: when you add numbers infinitely” (C4)...

We also wish to highlight the fact that most of the students provided informal definitions in their own words, and only A1, A4 and B1 seemed to be able to provide a correct, symbolic definition, although none of them mentioned the possibility of the sum being finite or finite.

These responses seem to indicate that *Rule 1* is effective and that it has an important effect on many students’ learning, shaping their personal relationship with series. The praxeological organisation necessary to solve many tasks is usually structured around the application of known criteria (*R4*), therefore the students are not required to remember or to understand what a series is to apply these criteria.

Question 3

The distribution of responses to this question is the following:

No response or a rhetorical response (“good question”)	B10 C4, C9
I don’t know	A1, A2, A3, A4 B1, B8, B11 C1
“It’s too abstract. I just apply rules”	B11, B12
Helps to find some values (like e, π, $\ln x$, $\sin x$)	B4, B9, B13
Mention of some convergence criteria as applications	B5, B7
Helps to make approximations (to curves, in a calculator, or through Taylor polynomials)	B6, B12 C14
Helps to calculate areas	B8 C3, C10

Other	B2, B3, B14 C2, C5, C6, C7, C8, C11, C12, C13

Table 2: Responses to Question 3

Our analysis of textbooks led us to state that textbooks seem to teach students to determine the convergence or the divergence of some series, but that this task has no utility or purpose (*R1, R2*) (González-Martín et al., 2011). The responses to this question seem to confirm our initial impressions as most students do not seem able to clearly state any mathematical applications of series. The students who are able to give some applications (like finding some values, or making approximations), do not seem to be able to give details about how series are used (so, probably, they just heard their teacher quickly mention some applications).

One application which appears in certain textbooks (modelling the distribution of medication in the blood) is vaguely remembered by just one student: "I am not sure, but we can use them to determine or to know the amount of medication present in the organism of a person" (B14).

We believe it is also significant that the four students of teacher *A* acknowledge not knowing any application. We can also see that almost all the students who mention the use of series to calculate some values, or to make approximations, are students from teacher *B*. In sum, the students' responses lead us to believe that possibly, teacher *A* did not mention any application during his teaching while teacher *B* probably mentioned some applications without giving details, and finally, that teacher *C* made some links with integrals and Taylor polynomials.

The possible consequence of *Rule 2* is that students are unaware of the applications of series. Data seem to indicate that this consequence has occurred among the students in our sample, and the responses to question 4 seem to confirm this conjecture.

Question 4

The distribution of responses to this question is the following:

No response	B6, B10, B12 C7, C9, C10
I don't know	A1, A2, A3, A4 B4, B9, B11, B13 C1, C4
Example of the ball which bounces indefinitely	C2, C8

Example of the ball which bounces indefinitely (developing some calculations)	C6, C11, C14
In Chemistry: "every second, half the number of molecules having reacted the second before, react..."	C3
"The digestion of a substance is made in a specific way. After one hour, half the substance is digested. Two hours later, only a quarter of the substance remains..."	B3
Pharmacy: "A medication which is taken every day. Calculate the amount of the medication in the body of a person in this context".	B1
Besides mathematical applications, I don't know	B7 C5
Other	B2, B8, B14 C6, C12, C13

Table 3: Responses to Question 4

None of the students were able to create a realistic problem, to model it and to solve it using a series. As in the previous question, we observe some regularities in the responses: none of the students from teacher *A* can give a response; all of the students who mention the ball with infinite bounces are from teacher *C*; again, the example of the distribution of medication comes from a student from teacher *B*.

Just as the textbooks analysis and the teaching practices analysis suggested, applications of series to model some situations seem to be generally absent from praxeological organisations, which seems to strengthen the development of *Rule 2* and its impact on students' learning. Even if some students seem to be aware (in a vague way) of some applications (inside or outside of the field of mathematics), they do not seem to be able to further develop them or to give specific details, probably because tasks requiring them to do so are absent from the praxeologies privileged in the *collégial* institution.

FINAL REMARKS

In spite of the importance of series for the development of modern mathematics, the teaching of series in the *collégial* institution seems to reduce them to a set of criteria and algorithms used to solve tasks which do not seem to have any purpose ("once we know this series converges, what do we do with that knowledge?").

The analysis of the praxeological organisation of the teaching of series both in the textbooks and in the teaching practices at the *collégial* level led us to conjecture the existence of some implicit *contract rules* having an impact on students' learning. Our results seem to support our conjecture, meaning that students learn to solve some questions concerning the convergence or the divergence of given series, without:

- Clearly being able to state what a series is.
- Clearly knowing what utility solving this task could have.

Even if the definition of series appears in the textbooks (and usually this is the first encounter of students with series), and some textbooks present certain applications (in a very vague way), praxeological organisations do not seem to clearly place any importance on these elements. However, students are later required to construct other mathematical notions (like power series and Taylor polynomials) from a notion they are barely able to define, and one key mathematical activity (modelling) is completely absent from these practices.

We do not have the intention of making generalisations from our sample (32 students from three teachers). As we said before, the anthropological theory allows characterising the institutional relationship with mathematical notions within given institutions, which has a very important role in the shaping of the individuals' (as institutional subjects) personal relationship with these notions. Our analyses of textbooks and teachers practices led us to a characterisation of the existing praxeologies (and implicit *contract rules*) in the *collégial* institution to introduce series (González-Martín et al. 2011). This characterisation made us establish conjectures about the significance given to series within the institution, and guided us to design our questionnaire. Even if we do not have a representative sample of students, we can say that the analysis of their responses seems to give some strength to our conjectures about the impact of existing praxeologies on the development of the students' personal relationship with series.

Our paper discusses the responses to just three questions from our questionnaire. We expect to continue our analyses of the questionnaire and to provide a more detailed portrait of the learning achieved by students within the *collégial* institution, as well as to identify other *contract rules* that praxeologies might be developing and their impact on students' learning.

An awareness of the consequences of practices being privileged at the *collégial* level could be helpful in order to begin a discussion about how series are taught and whether the students' difficulties in learning series are taken into account in teaching practices and finally, the possible consequences of current practices. We hope that our results can lead to the development of more research focusing on the teaching and learning of series, as well as to stimulate a discussion among members of the teaching community at the post-secondary level about the practices which are privileged and their consequences in the learning achieved by the students.

NOTES

1. "*Règles de contrat*" in the French literature.

2. Some authors have argued the importance of definitions to learn mathematical notions. See, for instance: Harel, G., Selden, A. & Selden, J. (2006). Advanced Mathematical Thinking. Some PME Perspectives. In A. Gutiérrez & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education. Past, Present and Future* (pp. 147-172). Rotterdam: Sense Publishers.

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DIGITAL RESOURCES AND MATHEMATICS TEACHERS PROFESSIONAL DEVELOPMENT AT UNIVERSITY

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For university mathematics as for other levels, a profusion of digital resources is available for teachers. This evolution impacts the teacher's practices, and contributes to teachers' professional development. Retaining here a focus on the interactions between teachers and resources, I study the use of digital resources for the teaching of mathematics at university, and how this use articulates with teachers' professional knowledge and its evolution. I also investigate the consequences of digital resources for teachers' collective work. I consider these issues, drawing on a case study concerning a mathematics lecturer working with computer science students.

TEACHER RESOURCES AND PROFESSIONAL DEVELOPMENT, IN THE UNIVERSITY CONTEXT

The profusion of digital resources for the teaching of mathematics is a well-known fact. Its consequences for teacher professional development have been, and are studied in a number of research works (Gueudet, Pepin & Trouche, 2012), in the context of primary and secondary school. This issue has not been studied at university yet.

We retain here the theoretical perspective of the documentational approach (Gueudet & Trouche, 2009). This approach considers that teachers interact with resources: textbooks, websites, software; but also students productions, discussions with colleagues. Teachers combine resources, design their own resources, set them up in class, modify them: we name this process the teachers' documentation work. It holds a central place, in teachers' professional activity. Interestingly, a recent work (Mesa & Griffiths, 2012) considers textbooks mediation of teaching at university, in mathematics. The authors identify several types of mediation, and observe that the use of textbooks by lecturers changes over time. The perspective adopted here is very similar, and textbooks are part of our study, which considers more generally all teaching resources. Drawing on the instrumental approach (Rabardel, 1995) we consider that teachers, along their interactions with resources, for a given teaching objective, develop a document: a mixed entity, associating resources and utilization schemes of these resources. This process is called documentational geneses. For the sake of brevity, we do not give here a detailed description of schemes. We retain that documents encompass resources and, in particular, professional knowledge, of different kinds: subject matter knowledge, pedagogical knowledge, but also, and this is essential in our study, pedagogical content knowledge, if we refer to Shulman's (1986) categories. The documents of a given teacher constitute a structured set, his/her documentation system. This documentation system associates a resource system, and professional knowledge.

Teachers' professional knowledge evolves, along the use of resources. For example, using a piece of software can lead to evolutions of the teacher's knowledge, concerning this software, but more generally also concerning mathematics and their learning by students. Thus teachers' documentation work is central, for professional development; moreover, the interactions between teachers and digital resources are nowadays an essential aspect of this documentation work. We propose here to study teacher documentation at university, with a focus on the role of digital resources.

Our study takes place in France, where teacher education, at university level, is very limited. In most universities, short training programs are proposed to university teachers, to discover a new software in particular; but there is no in-service training program concerning pedagogical issues. In this context, I claim that teacher documentation work is the principal source of teacher professional development at university. I study here two main questions, related to this issue:

- How do university mathematics lecturers use digital resources, to reach their pedagogical objectives? How does this use articulate with their professional knowledge, and its evolution?
- Which evolutions of mathematics lecturers' work at university can arise from the use of digital resources, in particular concerning collective work?

My aim in this paper is not to provide complete answer to these general questions, but to propose possible answers, and to open directions for further research, through the study of a particular case.

STUDYING DOCUMENTATION WORK: METHODS

Studying documentation work means to consider long-term processes, that happen in several places: the classrooms, but also the teacher's office, or even his/her home. Interviewing the teacher, asking her/him about her/his use of resources and the evolutions of this use is essential, but not enough. An essential methodological tool, associated with documentation, is the collection of the teacher's resources (Gueudet, Pepin & Trouche, 2012): extract of the books, the websites she/he uses; the students' sheets he/she writes; e-mails exchanged with colleagues etc. During the interview, the researcher asks to investigate the teacher's computer, to see his/her cupboards, and all possible places where material resources are kept. This complete data collection process corresponds to an ethnographic approach, which requires an important mutual commitment of the teacher and the researcher. Naturally, the conditions of the research work do not always permit to reach such a commitment.

The data used here have been collected in the context of a European Research project, Hy-Sup (Burton *et al.,* 2011), aiming at studying the use at university of distant platforms. In the context of this project, I followed in 2010-2011 two mathematics lecturers, who used in particular a Moodle platform for their teaching. I focus here on the case of one of these two lecturers, Peter.

I collected his resources, and met him for an interview that has been recorded and transcribed. This interview comprised several parts, concerning respectively: his working context and use of resources in general, digital resources in particular; a detailed presentation of his use of digital resources in the context of a precise course – he retained his linear algebra course -; the evolutions he retained of his own practice, linked with the use of digital resources.

The interview is firstly analysed by noting all the resources, and the agents (students, colleagues etc.) mentioned. Then I retain the teacher's comments about the use of each resource, in particular: his aims, for the use of a given resource; his expression of beliefs about the role of these resources in his teaching of mathematics; the mention of evolutions. All these elements are confronted with the main resources collected: for example, when the teacher declares using a textbook, the content of the book is confronted with the students' sheets designed by using this book.

A CASE STUDY

Peter has a PhD, about partial differential equations, obtained in 2001. Since 2001, he teaches at university; he obtained a position as full-time lecturer, and stopped his research activity. Peter teaches mathematics at a Universitary Technological Institute, specialized in computer science, in France. These institutes, inserted in Universities, offer a two-years training with professional objectives. They deliver professional diploma after these two years; nevertheless, nowadays more than 80% of the successful students continue their studies afterwards, to graduate or even to obtain a Master degree.

The teaching of mathematics, in Peter's institute, is organised in several courses, each of them concerning a given mathematical topic (for example: “linear algebra”). Each course lasts 40 hours: 5 hours a week, over 8 weeks. One hour is the lecture; two hours are a paper-and-pencil tutorial; two hours take place in the computer laboratory, using a computer algebra system (CAS, in what follows) – namely Scilab.

All the material used for a course is available for the students on a Moodle platform. The students also upload their own productions with Scilab on this platform, for correction by the lecturers.

Peter's resource system: an overview

For the preparation of his courses, Peter uses classical mathematics textbooks, written for students of this level (for example, Ramis, Deschamps & Odoux, 1979). These were the textbooks he used himself as a student in mathematics. He also uses books about the history of mathematics (Hauchecorne & Surrateau, 1999).

Peter appreciates the use of technology – he used different software himself as a student, and developed this use as a lecturer. He can be considered as a technology enthusiast (Monaghan, 2004). He uses various software: Scilab, the CAS used in the technological institute courses; LaTeX, to compose his students' sheets, and his courses; different software for composing web pages, and wikis; and Moodle. He also visits many websites: websites associated with given software, personal websites of colleagues, Wikipedia etc. Peter is an experienced lecturer; the courses he produced during the previous years are now central resources, in his documentation work.

His professional webpage (figure 1) gathers many of these resources, produced during previous years. The texts of his courses, exercises lists (ordinary mathematics exercises, or exercises for working on the computer), exam texts. It also proposes technical notices, about Scilab, or about non-mathematical tools, like wikis.

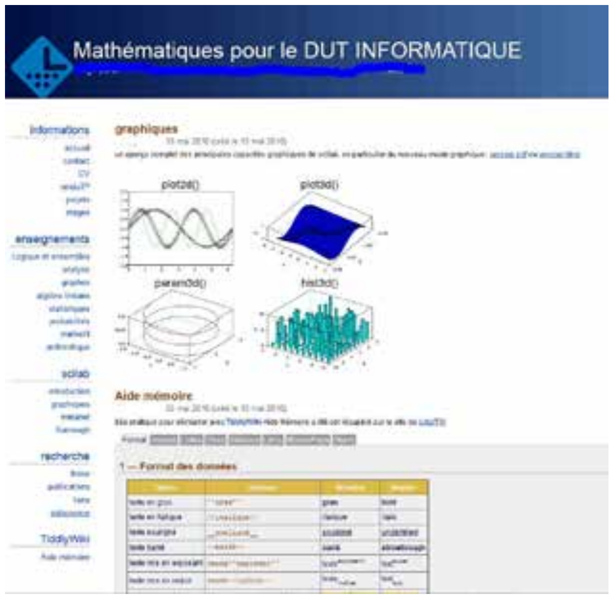

Figure 1. Peter's professional webpage.

Mathematics for computer science students, and use of digital resources

One difficulty, emphasized several times by Peter in his interview, is that his students are not specialized in mathematics. He mentions mainly two kinds of consequences of this fact: the students have limited mathematical skills; and they are not interested in theory, only in mathematics as a tool for computer science.

> P: We adapt the content, because it is very complicated, for such students […] They are students in computer science, mathematics is not their central subject.[1]

Another difficulty of his teaching context, that he emphasizes, is the heterogeneity of his students' skills in mathematics. He has indeed students from several origins. Some of them come directly from secondary school; amongst those, some were specialized in mathematics (which means receiving 8 hours of mathematics courses each week in grade 12), and others not (which means only 4 hours of maths each week in grade 12,

1 Our translation.

with a very technical content). Other students have spent one year at university before, possibly in mathematics major. Moreover, the students' aims can also be very different, between obtaining the two-year diploma, or trying to go further, to obtain a master degree. “We have a very broad spectrum of students”, declares Peter in his interview, describing all these possibilities.

Peter himself was a student in mathematics, specialized in a very theoretical research subject in partial differential equations. During his PhD, he gave courses at university for mathematics majors. Discovering this new teaching context, at the technological institute, was a important change for him, which raised professional questions. His answers to the interview, and the analysis of his resource system, indicate that he developed several kinds of solutions, linked with digital resources.

Scilab, supporting the learning of mathematics

Scilab holds a central place in Peter's resource system. One reason for this is that it contributes to answer to the professional questions mentioned above. Peter (together with his colleagues, at the technological institute) considers it as very important, for raising the students motivation, and helping their comprehension.

P: Programming the topics taught in maths gives a practical aspect […] when they implement Gauss method, they appropriate it.

Peter considers that programming a mathematical method with a software helps to understand the mathematics. Research works about the use of CAS to teach mathematics, at the beginning of university (e.g. Weller *et al.,* 2003, Gyöngyösi *et al.,* 2011) indicate that such possibilities indeed exist, under certain conditions, since the software can also constitute a difficulty. Another important aspect, not mentioned in the interview but evidenced by Peter's resources (figure 1), is that he extensively uses the possibilities of visualization offered by Scilab.

Scilab also intervenes in the management of students' heterogeneity. Firstly, almost all the first year students discover Scilab, the differences existing in mathematics do not exist about this software. Moreover, Peter composes texts, for the Scilab sessions, which comprise a common, minimum basis; and many complements. Thus the high-achieving students have something to do, working by themselves on a computer; at the same time the lecturer can support students having more difficulties.

P: I see that in the computer lab session, a weak student, not motivated, and a good motivated student, their work is completely different, a 1 to 3 ratio.

Here the text proposed by the lecturer, and the computer with Scilab are combined resources, for the management of students' heterogeneity.

I asked Peter, about his use of Moodle for the same objective (he could, for example, propose to some of the students an out-of-class work). He does not use it for this objective, because he considers that most of the work in mathematics must be done in class, in particular for the students who have difficulties.

Raising students' motivation by using mathematics for computer science

Peter tries to present mathematics as a useful tool for computer science, in order to raise the students' interest. For example, the text of his linear algebra course finishes with the example of Google searching process: the notion of PageRank, and its links with vector spaces, and eigenvectors (figure 2).

> En 1996 Larry Page et Sergey Brin, deux étudiants en doctorat à l'université de Stanford, eurent l'idée d'une nouvelle définition du PageRank : *"la pertinence d'une page web est proportionnelle à la somme des pertinence des pages qui pointent vers elle"*. Cette définition peut se traduire par un système d'équation linéaires dont les inconnues sont les pertinence des pages $x_k > 0, k = 1, 2, \dots$ et le coefficient de proportionnalité λ :
>
> $$\forall i, \quad \sum_{j \in P_i} x_j = \lambda x_i, \quad P_i = \{j | \text{ la page } j \text{ pointe vers la page } i\} \qquad (1)$$
>
> En d'autre terme : *les PageRank forment un vecteur propre (de coordonnées positives) associé à une valeur propre (non-nulle) d'une matrice représentant les liens entre les pages web.*

Figure 2. Extract of Peter's linear algebra course, Google definition of PageRank[2]

Peter also proposes to his students to implement well-known algorithms – he calls it "real applications". A "real application", for him, is for example the RSA algorithm[3], in cryptography; the students implement it, in the context of his arithmetic course.

Digital resources and evolving collective dimensions of teachers' work at university

Peter often works with his mathematics colleagues at the technological institute. For each course (e.g. "linear algebra"), a lecturer is responsible of the course. This lecturer gives the lecture; he/she builds the content of the lecture -the corresponding file is available on this lecturer's webpage, and on Moodle, for his/her colleagues and for the students. The texts of the tutorials and of Scilab sessions are usually collectively composed. Sometimes Peter also works with computer science colleagues, to propose and follow "students' projects" (for second year students; the students have to design a software, over six months).

Peter learned by himself to use various technologies, ranging from Scilab to Moodle. The university offers some technical training sessions, but they were proposed too late for Peter who trained himself before! He reads the software notices; but the most useful tool for him is always the "Frequently Asked Questions" pages. We consider

2 In 1996 Larry Page and Sergey Brin, two Stanford University PhD students, had an idea for a new definition of the PageRank: "the relevance of a web page is proportional to the sum of the relevance of webpages offering links to it". This definition corresponds to a system of linear equations, whose unknowns are the relevance of pages $x_k>0$, and the coefficient λ. […] In other words: the PageRank constitutes an eigenvector (with positive coordinates) associated with an eigenvalue (non-zero) of a matrix representing the links between web pages.

3 RSA stands for Rivest, Shamir and Adleman, who first publicly described this coding algorithm in 1977.

this as a collective dimension: Peter's knowledge evolved, by reading the answers to questions raised by colleagues on these websites.

As mentioned above, on his webpage he proposes, for students and colleagues, technical notices, about the use of different software: Scilab, but also wiki tools. He works with his mathematics colleagues, in particular, for the use of new digital means.

P: My colleagues have started to use Scorn, to program multiple choice questionnaires... We can insert them in Moodle, for the assessment.

Peter's colleagues have learned to use a software, for building online tests in Moodle. Peter himself did not learn it yet, but he can use the questionnaires. Here the collective work is needed, because of technical difficulties, in discovering new software. Moreover, it will probably lead to a change in the students' assessment: instead of written tests, they will fill in online tests on the Moodle platform.

DISCUSSION

Initial research questions, and the case of Peter

Our first set of questions concerned the use of digital resources by mathematics lecturers to reach their pedagogical objectives, and its articulation with professional knowledge.

Peter's resource system comprises a large range of digital resources of different kinds. This means that these digital resources belong to documents, developed by Peter, which associate sets of resources and professional knowledge.

Scilab (considered here as a resource) is involved in several documents, developed by Peter, with the objective to teach university mathematics to computer science students. Along his use of Scilab, Peter developed professional knowledge, about the possible use of a CAS to support the learning of university mathematics, for students who have a low interest in theoretical issues. Working with CAS permits an approach of the mathematical concepts through practical tasks (Gyöngyösi *et al.,* 2011); this professional knowledge is part of the document he developed.

Scilab is also involved in another document, developed for an objective of management of the students' heterogeneity. Other resources intervene in this document: the computer lab naturally, and also the texts composed by Peter, with a core content, designed for all the students, and complements for gifted students. Professional knowledge, intervening in this document, is formulated by Peter in his interview as: “the students have very different skills, some of them can be very fast”; “if a high-achieving student finishes the task to early, he/she will disturb the other students in their work”. As noticed above, he does not use Moodle, for the same objective, because he considers useless to propose distant work to students who have difficulties. He developed this knowledge along his observation of students at the technological institute. A direct consequence is that Moodle is not inserted, for Peter, in a document for the management of heterogeneity.

In fact Moodle is mostly used as a place where students can find information about the courses schedule, and can upload their productions (for downloading, Peter's webpage already offers most of the files that the students can find on Moodle). We consider that Moodle is not really integrated yet in Peter's resource system. This situation can naturally evolve, in particular if Peter develops his use of the Scorn questionnaires designed by his colleagues.

Besides all these digital resources, surprisingly Peter's resource system comprises a mathematics textbook, published in 1979 – the textbook that his own lecturers used, when he was a student. This textbook is for him the reference, for the mathematical content of his courses; he uses it to build his courses texts, and complements it with “cultural information”, about the life of mathematicians, or the actuality of computer science. It leads to an important gap, between a theoretical course, and the practical applications proposed to the students. The use of these “old” resources, in some cases, can contribute to explain why university mathematics teaching can seem to be “congealed”- resources also yield a kind of inertia (this has been evidenced, for assessment texts at university, by Lebaud, 2009).

Our second set of questions concerned the evolutions of mathematics lecturers' work, with a focus on collaboration.

Peter works with his mathematics colleagues, and digital resources play an important part in this collaboration. The files designed for their teaching are exchanged, via e-mail, or using the webpages, or Moodle. Moreover, since the university does not propose teacher training, about new software, a mathematics lecturer who has learned how to use a particular software writes information, suggestions, offers texts on his/her webpage, for colleagues.

University teachers' resources and documentation: directions for research

The case of Peter is certainly very special. He is a technology enthusiast, teaching mathematics to computer science students; under these conditions, the importance of digital resources in his system is not surprising. Moreover, Peter has no research activity; this fact certainly impacts his resource system. Nevertheless, Peter's case informs us more generally about the use of resources by lecturers at university, and suggests directions for further research. We retain here questions about teaching resources, digital resources in particular, and about collective teachers' work.

University lecturers, like all teachers, use many resources to design their teaching. They use in particular textbooks; the work of Mesa & Griffiths (2012) identifies needs for further research on this topic, in particular about how teachers' use and students' use articulate, how textbooks articulate with lecture notes, how textbooks use evolves over time. All these issues are essential, I add here a fourth direction: How does teachers' textbook use articulate with their use of digital resources?

These digital resources deserve a specific study. Many researchers have already studied the use, at university, of specific software for mathematics, like CAS, or visualization software. Nevertheless, it has mostly been considered in terms of students' learning. The integration of mathematical software by university lecturers, the professional development associated with it, has not been investigated yet. Moreover, the use by lecturers of digital resources like online exercises, distant platforms, and virtual learning environments is also a new, interesting direction. These resources open possibilities, in particular for assisting diversely able students with a diversity of tasks. Is this potential effectively used by the teachers, and if not, which kind of teacher education could foster and support this use?

Concerning collaborative work, in the case of France we can claim that lecturers' work is more collective at university than, for example, at secondary school. Lecturers have to work together, when a lecture is given to a large number of students, who will then follow tutorials in sub-groups. Teachers giving tutorials have to know the content of the lecture, to coordinate for the content of the tutorials etc. They also generally write the exam texts together. Digital resources have fostered these aspects of collaborative work: the lecturers can indeed easily exchange files, via e-mail, or on webpages. Which are the consequences, for teacher knowledge, of this collaborative work? Do university lecturers, working together, develop similar beliefs, ideas about their teaching?

Investigating the interactions between teachers and resources, at university, requires specific studies. These interactions are a major factor, for teachers' professional development, and studying them could enlighten specific features of university teacher's professional knowledge. Such research works could contribute to the design of resources, for the teaching of mathematics at university, and to propositions, for teacher education.

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ON THE CONCEPT OF (HOMO)MORPHISM : A KEY NOTION IN THE LEARNING OF ABSTRACT ALGEBRA

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This article is dedicated to the investigation of difficulties involved in the understanding of the homomorphism concept. It doesn't restrict to group-theory but on the contrary raises the issue of developing teaching strategies aiming at gaining access to structuralist thinking. Emphasis is put on epistemological analysis and its interaction with didactics in an attempt to make Abstract Algebra more accessible.

I. INTRODUCTION

In our context, Abstract Algebra means the discipline devoted to the study of algebraic structures, according to the new paradigm established after the publication of van der Waerden's textbook *Moderne Algebra* (1930):

> This image of the discipline turned the conceptual hierarchy of classical algebra upside-down. Groups, fields, rings and other related concepts, appeared now at the main focus of interest, based on the implicit realization that all these concepts are, in fact, instances of a more general, underlying idea: the idea of an algebraic structure. The main task of algebra became, under this view, the elucidation of the properties of each of these structures, and of the relationships among them. Similar questions were now asked about all these concepts, and similar concepts and techniques were used, inasmuch as possible, to deal with those questions. The classical main tasks of algebra became now ancillary. The system of real numbers, the system of rational numbers, and the system of polynomials were studied as particular instances of certain algebraic structures, and what algebra has to say about them depended on what is known about the general structures they are instances of, rather than the other way round (Leo Cory, History of Algebra, Encyclopaedia Britannica Online, 2007).

Abstract algebra is taught in France at third-year University level. The situation reflects the international one and can be summarized by Leron and Dubinsky's (1995, p. 1) statement: "*The teaching of abstract algebra is a disaster, and this remains true almost independently of the quality of the lectures*". If both students and experienced instructors generally agree upon this provocative claim, this shouldn't prevent us from investigating the nature of the obstacles and proposing alternative or complementary approaches (as Leron and Dubinsky did).

Our approach is based on an epistemological analysis of structuralist thinking (Hausberger 2012) with a view to introducing "meta" aspects in the teaching of abstract algebra. In other words, the goal is to build reflexive activities that can help students to make sense of this particular knowledge. This is justified by the identification, on an epistemological point of view, of the concepts involved as FUGS (formalizing, unifying, generalizing & simplifying) concepts (Robert 1987).

Obstacles regarding the built-up of teaching strategies, using traditional didactic tools such as Brousseau's “théorie des situations” (Brousseau 1986) or Douady's “dialectique outil-objet” (Douady 1986), can be analyzed as consequences of their epistemological nature as FUGS (Rogalski 1995). This approach has been conducted previously by Dorier, Robert, Robinet & Rogalski in the case of linear algebra (Dorier and al. 2000).

In Hausberger 2012, we underlined the fact that several levels of unification have to be distinguished in the context of algebraic structures: at level 1, a general theory applies to quite different objects sharing a common feature (for instance group theory), at level 2, the axiomatic presentation of structures is conducted in a uniform way (leading to structural questions and methods) and puts forward bridges between structures, at level 3, what has been previously a form (structures) is fully taken and studied as object in a superior level of organization (this is category theory or any other mathematical meta-theory of structures, the word “meta” being used here in a different context and with a slightly different meaning. Nevertheless, meta-theories and meta-activities meet in so far as they both introduce a reflexive point of view).

We made the assumption that meta activities of level 2 were worth being introduced to facilitate the access to modern structuralist expositions of algebraic structures. In the case of Linear Algebra, Dorier introduced in fact only meta of level 1. This is justified by the fact that a single structure is at play and that it is the first abstract structure that students encounter and theorize at University.

Moreover, unlike the impression conveyed by Cory in the previous quote, each structure has its own flavor. Indeed, it has its own history of problems, its own “typical objects”. There certainly is a tension in modern mathematics related to the articulation of abstract formalism and intuition (requiring a more direct grasp of objects). This tension is acknowledged by philosophers and is also visible in manuals[i]. This raises the following didactic issues and tasks:

–Find a right balance between formalizing and problem-solving involving more concrete objects.

–Think about strategies to gain access to structuralist thinking. This should be progressive: in this respect, the goals set in the teaching of group theory should be different from those regarding ring theory. We will make this statement more precise below.

In our view, the flavor of Linear Algebra resides in interrelating geometry and algebra (enlarging the notion of vector, vector space, or interpreting geometrically reduction theory) and its challenge is also to articulate computational and abstract theoretical aspects (for instance, matrices and linear applications). In group theory, the symmetric group is a paradigmatic example, classifying groups of small order is a very didactical moment and understanding the notion of isomorphism is fundamental to give access to the abstract group concept, thus accomplishing level 1. The emphasis on group action is also fundamental to make the concept fully operational. Ring theory has quite an arithmetical flavor: extending the unique-prime-factorization theorem to rings of algebraic integers has proved to be a motivating force in history for the development of an abstract divisibility theory in which the concept of ideal plays a major role and the main objects to unify, on an elementary level, are numbers and polynomial rings. On the structuralist front, ring theory (being preceded by group theory) should be an opportunity to discuss structural aspects shared in the modern exposition of both theories: the construction of quotients, the concept of homomorphism as structure-preserving function, the isomorphism theorems as tools to compare objects, decomposition theorems into simple objects, characterization by universal properties, etc. These aspects would certainly benefit from being made more explicit through level 2 meta activities. It is nevertheless a real challenge and it remains conjectural whether it is feasible or not at third year University level.

This article will now focus on the (homo)morphism and isomorphism concepts which are central in abstract algebra. Students often confess that they lose tracks when homomorphisms come in the foreground after the introduction of isomorphism theorems. Previous didactic studies also report on this issue, for instance Nardi (2000) titles p. 179: “Episode 3: The first isomorphism theorem for groups as a container of compressed conceptual difficulties”. We will contribute by deepening the epistemological analysis on morphisms, extending the picture to isomorphism theorems for other structures and conceptions about morphisms in general. This corresponds to our general philosophy of attacking globally the issue of teaching abstract algebra through epistemological, didactic, cognitive studies of processes involved in structuralist thinking.

II. A FEW CASE STUDIES

Previous studies on groups

According to Leron, Hazzan & Zazkis (1995), “the very concept of isomorphism is but a formal expression of many general ideas about similarity and differences, most notably, the idea that two things which are different may be viewed as similar under an appropriate act of abstraction”. It corresponds to the vague (but crucial and intuitive) idea that two isomorphic groups are essentially the same (on the group-theoretic point of view) and part of the proposed didactic strategy is to help students make sense of it before getting engaged in the formalization of the isomorphism concept. Indeed, such a formalization requires the function concept and reasonable understanding of quantification, which adds further difficulties.

This point of view contrasts with the standard exposition in modern manuals, in which structuralist conceptions have been naturalized. Interestingly, it matches van der Waerden's (1930) who didactically took great care in motivating the introduction of new concepts. If isomorphisms derive formally from homomorphisms, it should be pointed out that isomorphisms come first epistemologically.

Nardi (2000) carries on the work by analyzing tutoring sessions dedicated to retrieving and proving the first isomorphism theorem:

Let Φ: G→G' be a group homomorphism. If K=ker Φ then G/K $\simeq$ Im Φ. The isomorphism is constructed by setting $\psi(Kg)=\Phi(g)$.

She underlines that "the degree of complexity in a problem which requires a well-coordinated [linking ψ and Φ] manipulation of mappings between different sets is extremely high" and stresses the numerous "difficulties in the conceptualization of properties associated to the notion of mapping (homomorphic property, 1-1, onto, well-definedness)" as well as the high degree of abstraction involved in the definition of a mapping between the cosets of a subgroup and the elements of the group. Finally, she points out the impact of epistemological arguments that the tutor would put forward to motivate the newly introduced concepts. This certainly brings water to our mill.

Comparing objects through homomorphisms

Context. The following questions were asked to third year undergraduates as part of a mid-term examination for a second course in abstract algebra devoted to ring and field theory. The student is supposed to retrieve the statement and proof of a classical generalization of the first isomorphism theorem (in the ring context), which has been presented during the lectures as a **tool** for constructing homomorphisms from quotient rings (i.e. of type *A/I*→ *B*):

Let f:A→B be a ring homomorphism, I an ideal of A and π:A→A/I. Then f "factorizes through A/I" (i.e. there exists a homomorphism f such that $f=\bar{f}\circ\pi$) if and only if I⊂ker f.

The theorem is illustrated by the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\pi"'] & B \\ A/I \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Homomorphisms have been introduced as structure-preserving functions, emphasizing the condition $f(1)=1$[ii]. To facilitate the retrieval, the ring data has been denoted $(A,+,\cdot,1)$. The general idea has been developed that **homomorphisms aim at "comparing" rings**: if $f{:}A\to B$ is bijective then *A* and *B* are "essentially the same" and can be identified. If it is only injective (1-1), then *A* can be identified with a subring of *B* and the kernel measures the defect of injectivity, if it is surjective (onto), then *B* can be identified with a quotient of *A* through the first isomorphism theorem. As an application of the generalized theorem, named explicitly "factorization theorem for ring homomorphisms": Z→ Z/2Z factors through Z/4Z and induces an isomorphism $(\mathbf{Z}/4\mathbf{Z})/(\bar{2})\simeq\mathbf{Z}/2\mathbf{Z}$. The lecture then carried on with the third isomorphism theorem which generalizes such isomorphisms and is named "simplification theorem for quotients of quotients". To summarize the didactic intent, the tool-object dialectic (Douady 1986) is at play and epistemological insight was given to connect the formalism with cognitive processes of comparison and identification.

Here are the questions:

1.Recall the factorization theorem for ring homomorphisms and give a proof.

2.Can **Z**/4**Z** be identified with a subgroup of **Z**/8**Z** (in other words, does **Z**/8**Z** contain a subgroup isomorphic to **Z**/4**Z**)? Can **Z**/4**Z** be identified with a subring of **Z**/8**Z** (in other words, does it exist an injective homomorphism **Z**/4**Z**→ **Z**/*8***Z**)? Can you construct a ring-homomorphism connecting **Z**/4**Z** and **Z**/8**Z**? Can **Z**/4**Z** be identified with a quotient ring of **Z**/8**Z**?

Results. Out of the 13 students who took the test, only 5 managed to retrieve successfully the statement of the theorem (question 1). The diagram was often reproduced, indicating that it played some role in the memorization process. 5 students didn't answer question 1 and among the others, common mistakes concerned the omission of the condition on the ideal, or an inversion of the inclusion. Indeed, the student can have control on this only if he has understood that elements of I are mapped to 0 because the diagram commutes. Yet, only 4 students gave an attempt to retrieve the proof: 2 of them successfully demonstrated that the condition is necessary but couldn't carry on so that a single student gave a complete proof. This is a bit surprising since it was emphasized during the lectures that the proof can be worked-out exactly as in the group context. Moreover, the difficulty of articulating ψ and ϕ identified by Nardi (2000) is greatly taken in charge didactically by the formulation: "f factorizes through the quotient and induces $\bar{f}$". Even after completion of a first course in abstract algebra, the task of elucidating the properties that need to be checked (well-definedness, homomorphic property, 1-1, onto), thus reducing the problem to elementary tasks, is regarded as very complex by students.

A single student gave elements to answer question 2. He began by making the groups more explicit: **Z**/8**Z**={$\bar{0}, \bar{1}$,..., $\bar{7}$}, **Z**/4**Z**={$\bar{0}$, $\bar{1}$, $\bar{2}$, $\bar{3}$} and argues that:

G={$\bar{0}$, $\bar{2}$, $\bar{4}$, $\bar{6}$} is a subgroup of **Z**/8**Z** of order 4, therefore isomorphic to **Z**/4**Z**. The map ϕ*:* **Z**/4**Z**→ **Z**/8**Z**, $\bar{0}\mapsto\bar{0}$, $\bar{1}\mapsto\bar{2}$, $\bar{2}\mapsto\bar{4}$, $\bar{3}\mapsto\bar{6}$ is an injective morphism of groups but not of rings since $\phi(\bar{1}) \neq \bar{1}$.

Although the student uses the same notation for elements of both quotients, he doesn't mix them and has the right intuition on how to construct the 1-1 group-homomorphism. The teacher would expect a more abstract reasoning: to find a subgroup isomorphic to **Z**/4**Z** amounts to finding an element of order 4 in **Z**/8**Z**. The student doesn't justify that G is a group nor that it is cyclic because every subgroup of a cyclic group is also cyclic. It would be interesting to check him on isomorphism classes of groups of order 4. Finally, quantifiers are again handled with too little care: if ϕ is not a ring homomorphism, this doesn't imply such a homomorphism doesn't exist. For instance, the student could have argued that **Z**/8**Z** has a unique subgroup of order 4 or used the fact that the $\Phi(1)=1$ property completely determines Φ (which then isn't a homomorphism). The last part of the question remains unanswered and indicates that finding a map between two quotient rings is a conceptually hard task (and a bit tricky one since the arrow is reversed: $\mathbf{Z}/4\mathbf{Z}\leftarrow\mathbf{Z}/8\mathbf{Z}$), even if it consists in reproducing the reasoning made during the course on a very close example.

In fact, the two questions were asked before heavy use of the factorization theorem for rings was made during the tutoring sessions. The experiment shows that the test was premature even if the course was epistemologically-oriented and the students could rely on previously acquired conceptual knowledge (which remains very fragile). We will engage in further investigations of the difficulties in 2013 by asking the same questions during tutoring work. Nevertheless, this experiment already confirms that the homomorphism concept is much of an obstacle, even after the completion of a first course in abstract algebra devoted to group theory.

III. MORE EPISTEMOLOGICAL INSIGHT

By the way, what is a structure ?

Abstract Algebra teachers often speak about structures... but they never mathematically define any concept of structure! Let us explain this a priori abnormal phenomenon by giving a few details concerning the didactic transposition (Chevallard 1985) of the notion of structure.

In his attempt to give an historical account, Cory (1996) makes the distinction between “body of knowledge” and “image of knowledge”. Interestingly, the notion of structure takes its origin in the latter:

> This textbook [*Moderne Algebra]* put forward a new image of the discipline that implied in itself a striking innovation: the structural image of algebra. In the forthcoming account, it is this specific, historically conditioned image of mathematical knowledge that will be considered as implicitly defining the idea of a mathematical structure (Cory 1996, p.8).

We won't comment on historical methodology, but it is certainly the combination of Noether's mathematical ideas (see below) and a didactic intent to expose the recent advances in algebra in a systematic and clear fashion, in an organized and integrated whole, that lead to the idea of structure. It remained implicit in so far as van der Waerden didn't give any comment, formal or non-formal, on what he meant by a 'structure'. Bourbaki, on the contrary, gave a formal-axiomatic elucidation of the concept of mathematical structure in the first book of his treatise '*Elements de mathématiques*' dedicated to set theory and published in the 1950s. In parallel, he promoted (Bourbaki 1948) the structural image of mathematics within the noosphere (Chevallard 1985). Yet, Bourbaki's definition did hardly play any role in the exposition: it only provided a general framework which in fact didn’t prove to be mathematically functional (Cory 1996, p. 324, see also Mac Lane 1996), unlike category theory which is very advanced and a too hovering viewpoint for the present purpose. Therefore no definition at all is given in more recent manuals.

As a consequence, students are supposed to learn by themselves and by the examples what is meant by a structure whereas sentences like “a homomorphism is a structure-preserving function” is supposed to help them make sense of a homormorphism. Is that possible without any clarification on the notion of structure? The students certainly understand that when we talk about structures we refer to sets of axioms and we say that a mathematical object has a particular structure when these are fulfilled. This reflects the fact that the notion of structure is an outgrowth of the widespread use of the axiomatic method. In our view, meta-discourse on axiomatics together with activities devoted to building axiomatics and “playing” with the axioms would help students to make sense of them and memorize them. It is not straightforward to comprehend that axiomatics encode properties of mathematical objects which are analyzed abstractly as being made of elements connected by relations (therefore primitive terms of axiomatics are sets equipped with extra data encoding the relations such as laws of composition or binary relations). The abstract formalism (axioms and the language of set theory) is much of an obstacle and hides the simple ideas. Too little emphasis is made on the idea of relations: structure-preserving is synonymous with operation-preserving in contemporary manuals, unlike van der Waerden's (1930), in which examples are given of non-algebraic structures such as ordered sets when introducing the homomorphic property, defined as preserving relations. The idea of relation is also useful to understand quotients: making a quotient is equivalent to introducing more relations. This should help the students to interpret, for instance, the quotient $\mathbf{Z}[X]/(10X\text{-}1)$ as a ring isomophic to the decimal numbers. We will need of course to engage in further didactical studies in order to support our conjectural claims regarding conceptions and teaching strategies.

Coming back to the epistemological investigation, the word 'structure' is used in fact in two more contexts with a different meaning, which may induce some confusion: we also want to identify the different isomorphism classes for a given structure and we say for instance that the abstract group-structure of $\mathbf{Z}/3\mathbf{Z}$ is that of a cyclic group of order 3. Finally, we call 'structure-theorem' a result describing the way an object can be reconstructed from simpler objects of the same type.

Noether's set-theoretic foundation of Algebra

We have already distinguished the concept of isomorphism as an equivalence relation from that of a function (with properties) which relies on the notion of homomorphism. The equivalence relation is in fact a crucial concept with regard to the process of abstraction, which often amounts to selecting common characteristics of objects being thus taken as equivalent. A classification is a description of the corresponding partition: for instance, group theory will classify groups of a given order. We will now focus on the latter conception of isomorphism: it is the heritage of Noether who developed

> […] what she called her set-theoretic foundations for algebra. This was not what we now call set theory. […] Rather, her project was to get abstract algebra away from thinking about operations on elements, such as addition or multiplication of elements in groups or

> rings. Her algebra would describe structures **in terms of selected subsets** (such as normal subgroups of groups) **and homomorphisms** (MacLarty, 2006, p. 188).

We pretend that Noether's new conceptual approach (characterized in the last sentence of the quote) is a **major epistemological difficulty** in the learning of abstract algebra. It is also the key to level-2 structuralism (referring to the introduction): this indeed allows a unified treatment of structures. It proved in history to be a major breakthrough leading to a complete rewriting of algebra (in terms of newly-forged concepts that emerged from the new methodology: noetherian rings, principle-ideal domains,...).

Dedekind's theory of ideals in which divisibility relation between (algebraic) integers were replaced by inclusion of ideals certainly contributed to the transition from an arithmetical conception of algebra to the set-theoretic conception. But Noether's chief tools were isomorphism theorems and she made it obvious that this applied for different kinds of structures. She considered only onto homomorphisms, denoted $M \sim \bar{M}$ (the functional notation $\mathrm{f}: M \rightarrow \bar{M}$ comes from topology) and correlated them to distinguished classes of subsets through the first isomorphism theorem. As an illustration of the generality of the principle:

> Ideals bear the same relation to ring homomorphisms as do normal subgroups to group homomorphisms. Let us start from the notion of homomorphism (van der Waerden, 1949, p. 51).

Homomorphisms or morphisms?

On forums[iii] are taking place interesting discussions on the differences to be made between homomorphisms and morphisms. Some people argue that homomorphism is the old terminology and the shorter word should be adopted for pragmatic reasons. Others mention that morphisms come from category theory (which remains obscure to them). Finally, one of them argues that the difference between homomorphism and homeomorphism is clear, but not between homomorphism and morphism. This leads to a confrontation of two different definitions of a morphism in the context of topological spaces: open maps (preserving open sets) versus continuous maps. But this didn't allow to make the point underneath the morphism concept which derives from category theory and thus the difference to be made.

Indeed, the morphism concept is a relativization of the homomorphism concept as a structure-preserving function (which doesn't apply to morphisms in topology since they preserve open sets by *inverse* image): in a category, one is free to decide which maps are morphisms, these define the category together with a given type of objects. If morphisms tend to replace homomorphisms, beyond the pragmatic argument, it might be that algebra practitioners are implicitly assuming that they are working in a category.

IV. CONCLUDING REMARKS AND PERSPECTIVES

This study certainly contributes to break the illusion of transparency concerning the concept of homomorphism and the idea of a mathematical structure. It aims at reestablishing the rationale of this particular knowledge through the epistemological investigation of the concept and the engineering of epistemologically-oriented activities. In this spirit, a simple didactical situation has been given in part II. Unfortunately, it needs to be re-experimented in order to reveal its full potential. Meanwhile, we have engineered, on the basis of the epistemological analysis of the notion of structure presented in this paper, an activity dedicated to reflecting on the axiomatic method and the structuralist viewpoint in a simple context ("mini-theory"). We hope to report soon on the results of the pre-experimentation of this activity. On a more theoretical perspective, following Chevallard's idea that a "body of knowledge" is a complex of praxeologies (see for instance Chevallard 2002), we will need to track and analyze the "structuralist praxeologies" in order to support the idea, presented in part I of this paper, that there really is a "structuralist agenda" to meet in relation with the learning of abstract algebra. Finally, our analysis connects the transition problem that occurs at third year university level in relation to abstract algebra with epistemological transitions: the systematization of the axiomatic method, after Hilbert, and the transition, after Noether, from thinking about operations on elements to thinking in terms of selected subsets and homomorphisms. This analysis should be crossed with a didactical investigation of the transition problem: the analysis of praxeologies should again bring much light.

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[i] The preface of Artin 1991 is interesting in this respect and begins with an explicit citation of Hermann Weyl.

[ii] which isn't automatic since A^* is not in general a group for the multiplicative law

[iii] For instance, on the site http://www.les-mathematiques.net. We unfortunately have no space for a transcript here.

MATHEMATICAL ENCULTURATION – ARGUMENTATION AND PROOF AT THE TRANSITION FROM SCHOOL TO UNIVERSITY

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University students, especially in their first semesters, often lack specific mathematical learning and working techniques that are necessary to develop and apply mathematical notions, definitions, theorems and proofs. We consider this to be a key factor for problems arising in the secondary-tertiary transition in mathematics. Usually the foundations of mathematical proofs as well as questions of validity and justification are not an explicit aim of university teaching. In our approach we invoke philosophical aspects of epistemology. We show how our theoretical considerations lead to the development of a teaching design for the teaching and learning of mathematical reasoning, argumentation and proof, making the methodological foundations of mathematics explicit and accessible for students.

INTRODUCTION

Courses in mathematics at university level are often considered to be difficult. Unsatisfactory exam results and high drop-out rates seem to confirm this. Of course there are many reasons for this and the measures and projects to deal with it is also manifold. In Germany, for instance, a notable example is the council the "German Centre for Higher Mathematics Education" (http://www.khdm.de) that was founded in 2011, but there are also many projects like the already completed project SAiL-M (http://www.sail-m.de), in which one of the authors took part as a team member.

Many students fail because of their insufficient prerequisites, some just do not have the ability to work autonomously or do not have enough self-discipline to prepare and review lectures and exercise meetings efficiently. In fact "a shift between two institutional cultures happens when entering university" (Gueudet, 2008, p. 245). To attenuate the ruptures occurring at the transition to university most of the German universities have established so-called math-bridging courses for students who are about to begin first semester.

A lot of math-bridging courses at the secondary-tertiary transition are organized as blended learning scenarios aiming at the repetition of mathematical content and training exercises (e.g. the project http://www.math-bridge.org, Biehler et al., 2012). Within such courses mathematical learning and working strategies are mainly taught implicitly.

Talking about the secondary-tertiary transition, Gueudet (2008) named several related perspectives and issues. In our work we take an epistemological and didactical perspective, and we present a teaching design within our bridging course that focuses on argumentation and mathematical proof. Unlike many other bridging courses at German universities, we do not (only) aim at the repetition of mathematical skills and knowledge mentioning superior concepts and working strategies more or less explicitly. But we have a strong focus on uncovering, discussing and training the methodological foundations of argumentation and proof. In our approach mathematical skills and contents are used as occasions and examples for the development and reflection of comprehensive strategies in an explicit and general way.

The secondary-tertiary transition concerning mathematical argumentation and proof is characterized as follows:

Mathematical argumentation and especially mathematical proving at school is only taught in an exemplary way (Douek, 1999). Bringing the meta-theoretical aspects of mathematical proofs that legitimate the employed methodological means up for discussion is not envisaged or reflected explicitly. We think that this is a key factor for transition from school to university, since

> […] proofs provided during lectures at university play a new role: they are central in the building of the university mathematical culture, because they indicate methods, and also what requires justification or what does not. (Gueudet, 2008, p. 247)

The present article is meant as a theoretical contribution to the described topic although we present a concrete teaching design illustrating and underpinning our theoretical approach. In fact, we consider the science of mathematics education to be a ‚design science’ that

> […] presupposes a specific didactic approach that integrates different aspects into a coherent and comprehensive picture of mathematics teaching and learning and then transposing it to practical use in a constructive way. (Wittmann, 1995, p. 356)

In our article we focus on a learning activity exploring the epistemological aspects of mathematical proofs. In the next section we describe our *theoretical position* and give a *didactical analysis of the issue*. After that we shortly describe the *target audience*, *objectives and context* of our course. Following this, we present a learning scenario together with didactical and methodological comments (*mathematical reasoning and justification – learning scenario and didactical comments*). We close the article with a *summary and outlook on research questions guiding our future work*.

THEORETICAL POSITION AND DIDACTICAL PERSPECTIVE

One of the most important methodological principles of mathematics – and in fact of any science – is the principle of trans-subjective comprehensibility of its results. This means that there is a demand for the comprehensibility of the terminology and the notions, but also the duty to explain and justify its assertions. In particular, everybody – provided that she/he has the adequate prerequisites – should be able to check and understand mathematical assertions (at least this should be possible in principle) (Janich, 2002; Gatzemeier, 2005).

In other words, the main target of mathematical arguments and proofs is to convince ourselves or others of the truth of (mathematical) propositions that seem questionable or are uncertain (Thiel, 1973). In this sense the methods of mathematical proof could be considered as tools or as a guideline for the development of mathematical argumentations, and thus ensure the trans-subjective comprehensibility of scientific findings (Lorenzen, 1968).

Following the above epistemological and normative perspective of mathematical argumentation we are – at this point – only concerned with the minimum requirements for effective mathematical reasoning. Therefore, we abstract from special didactical differentiations between proofs as processes resp. proofs as products, or between (fully) formalized proofs resp. proofs "really performed" in textbooks (Duval, 1991; Douek, 1999). Moreover, we assume that our approach could lead to fruitful suggestions to close the gaps between the above differentiations in a theoretical and practical way.

What is our understanding of mathematical argumentation? Mathematical argumentation (as any argumentation) can be structured into the classical premise-inference-conclusion pattern written in linear form (Tetens, 2004). The verifiability of the conclusion is derived from certain propositions (premises). These do not require further justification or have to be accepted as having been justified already. In particular, the rules of inference are stated and justified explicitly. A mathematical argumentation that is modelled after this pattern can neither be accused of running into fruitless circular reasoning, nor of culminating in an infinite argumentative regress, nor of aborting the argumentation by appealing to a dogma, i.e., it does not lead into the notorious "Münchhausen-Trilemma" formulated by Hans Albert (1991).

The dialogical character of argumentation is a central idea. In some sense, it could be seen as a basis for the interpretation of (logical) implications. For a short overview from a didactical perspective we refer to Durand-Guerrier and Barrier (2007). Following this idea, logic does not deal with (everlasting) laws of verity (Frege, 2003), but logic is a means to create new knowledge from existing knowledge (Thiel, 1980).

Therefore, we want the students to critically discuss the ideal of trans-subjective comprehensibility as a minimum condition for mathematics as a science. The premises-inference-conclusion pattern should be recognized as a tool to reach this ideal. For this reason, we do not only want to teach some basic theoretical knowledge of argumentation and train the students to read and produce proofs basing on this pattern. In addition, we want the students to discuss these tools on a normative basis considering their usefulness to achieve the ideal of trans-subjective comprehensibility. The above logical pattern presented in a linear form has some didactical advantages in the sense of a didactical reduction.

The linear form is much easier to understand than the common Toulmin-scheme (as didactically analysed in Barrier, Mathé & Durand-Guerrier, 2009) or game theoretical concepts (as presented in Vernant, 2007; Marion, 2006; Durand-Guerrier & Barrier, 2008), since we reduce the number of notions (warrants, backings etc.) and the number of dialogical rules. However, the main advantage is that, beginning with Euclid's "Elements", proofs in mathematical literature and lectures have a monologist form of argumentation by presenting a proof as a step-by-step construction. Their dialogical character weighing the pros and cons and showing the decisions taken is usually not obvious. In this sense, mathematical texts emphasize the product-aspect more than the process-aspect of proofs.

We believe that the normative discussion could enable the students to critically reflect general objectives of mathematics as a science and to recognize methodological decisions as being appropriate. At least, we hope that this could lead to higher motivation and autonomous learning following an ideal of continuous rationality as part of the "mathematical culture".

TARGET AUDIENCE, CONTEXT AND OBJECTIVES

The math-bridging course is designed for first semester students with mathematics as a major subject, but also for future math teachers. A first course integrating the presented learning scenario was held at the University of Education Ludwigsburg in October 2012. In Ludwigsburg, future teachers (primary school, secondary school up to grade 10 and special school) are educated and have to take a variety of math courses depending on their choice of study programme. Especially here we hope that our approach has a positive impact on the future teachers' *idea of mathematics* and hence on their teaching at school.

We expect the target audience to be very heterogeneous and differ widely in their mathematical competencies, learning motivation and general academic ability. Nevertheless, we want to achieve some common learning goals.

At school the students usually come in contact with mathematical proofs in an exemplary form - for instance when applying geometrical theorems about the congruence of triangles in their reasoning or when proving the irrationality of $\sqrt{2}$. In the "educational standards" for mathematics in Germany (KMK 2003) *mathematical reasoning* is one out of six mentioned competencies within mathematics education at school. With our learning scenario we want to build on this basic knowledge of the students and achieve the following learning objectives:

The students should

- analyze mathematical proofs considering the example of the *proof by contradiction* and understand as well as describe their deductive structure, and
- realize that completeness and deductive derivation are necessary criteria for mathematical proofs and adopt the ideal of the premises-inference-conclusion pattern as a reasonable means for mathematical argumentation.

MATHEMATICAL REASONING AND JUSTIFICATION – TEACHING DESIGN

The presented teaching design consists of three phases:
First, a lecture (*Phase 1: Information*) in which the students are provided with information about the basics of "naïve logic" and the method of "proof by contradiction". Second, the first part of the exercise session in which mathematical proofs are analyzed and compared (*Phase 2: Cognition*). Third, the second part of the exercise session that provides an activity to think about argumentation and proof from a philosophical point of view (*Phase 3: Metacognition*). We will shortly describe each phase including didactical and methodological comments referring to our theoretical position.

Phase 1 (Lecture): In the first part of the lecture we introduce the usual logical operators including the quantifiers. Although we want formal interrelations to become transparent by discussing some logical riddles and parallels to everyday language, this part is organized in a stringent way and mainly directed and executed by the lecturer.

The second part of the lecture includes considerably more interaction with the audience. Here we analyze the "model" of a *proof by contradiction* by comparing two examples. First, we present an example of Cohors-Fresenborg and Kaune (2010, p. 31). Here, a judge reasons why a defendant is proved to be innocent following the scheme of a proof by contradiction in the following way:

> Assuming that the defendant *is guilty* (here: robbed the bank). Then he would have been in A-town at 16:00 h. This means he could have been in B-town no earlier than 17:30 h, since you need at least 90 minutes for this distance. Since the bank in B-town was robbed at 17:00 h, this contradicts the given facts. Hence, the defendant has to be innocent.

Together with the audience we develop the following scheme of the proof:

Logical opposite of the claim → conclusion from the first line → conclusion from the second line → fact → conclusion ("a contradiction appears") → conclusion ("the claim is true").

To obtain the pattern of a chain of deductions the students need to identify the premises and the implication steps. Moreover, this example allows resp. forces us to reflect on the character of mathematical propositions and leads us to the *law of the excluded middle* and the *law of non-contradiction* (Russell, 1912).

The second example is a proof of the infinity of the prime numbers as given by Euclid. The lecturer has cut the steps of the proof in lines and put the lines in a wrong order (*proof-jigsaw*). The audience is asked to find the right order and figure out the correspondences between the two examples. The results are collected by the lecturer.

Unlike many other bridging courses do, we do not aim at completeness in the sense that we try to communicate the complete range of methods of mathematical proofs. However, we have chosen the „proof by contradiction“ as being an exemplary method. By choosing the above approach we hope to implicitly suggest that formal (naïve) logic is a technical means for the proper formulation of mathematical propositions and everyday statements.

Phase 2 (Cognition): The above scenario is taken up again in the exercise session in the afternoon, where *two more proof-jigsaws* of the proof of the infinity of primes are given, differing widely in their level of detail. One proof-jigsaw is a very brief version of the proof similar to the one in Aigner and Ziegler (2009). The other one is a very long and detailed version including a high level of formalization. The students form groups and are again asked to put the proof-lines into a meaningful order, while at the same time comparing and discussing all three given proofs for the infinity of the prime numbers. We hope that this creates some sort of provocation resp. irritation in the following sense: at school, mathematics is usually done by learning and applying “recipes”. Usually there is only one way of execution and the result can be right or wrong. The above scenario allows the discussion of three different forms of argumentations, providing an action-oriented and suggestive access to the main objective of the learning scenario: The students discover and reflect the structure of mathematical proofs as chains of deductions. But they also implicitly discover the dialogical character of mathematics as a science through considering a proof to be an attempt to convince another person of the truth of my own statement. This attempt depends on the mathematical background or context of the reader/producer of a proof.

The learning activity leads directly to a higher level of abstraction through the thought experiment “What if a mathematical proof could never be formulated completely?” In the end, the question “What are the basic rules a proof must follow to be convincing?” – a question of validity – could arise, leading to the last phase of our learning scenario.

Phase 3 (Metacognition): The last phase is directed by the questions: *How do I know that it is true?* or *What is a perfect proof?* For this activity it is important to create an atmosphere of discussion and reflection to gain various ideas of the participants. Therefore, we put the students together in groups providing them with only few – but accentuated – impulses. The students are provided with some further material leading them to the question of justification and validity as follows:

> *The Münchhausen-Trilemma*
>
> If we ask of any knowledge: "How do I know that it's true?", we may provide proof; yet that same question can be asked of the proof, and any subsequent proof. The Münchhausen Trilemma is that we have only three options when providing proof in this situation:
>
> - The circular argument, in which theory and proof support each other (i.e. we repeat ourselves at some point)

- The regressive argument, in which each proof requires a further proof, ad infinitum (i.e. we just keep giving proofs, presumably forever)

- The axiomatic argument, which rests on accepted precepts (i.e. we reach some bedrock assumption or certainty)

The first two methods of reasoning are fundamentally weak, and because the Greek sceptics advocated deep questioning of all accepted values they refused to accept proofs of the third sort. The trilemma, then, is the decision among the three equally unsatisfying options.[1]

The students are asked to work in groups on the following tasks: *Formulate connections between the proof-jigsaws and the Münchhausen-Trilemma text. Name characteristics of "a perfect proof"*.

The results of the groups are collected in the end. Using this method we hope to get a larger variety of ideas as well as to involve all the students in the discussion process by generating commitment for everybody. But especially we want to allow a first step towards the establishment of socio-mathematical norms "which means in this context, criteria shared by students and teachers to decide whether a proof is valid or not, what is a satisfactory explanation, etc." (Gueudet, 2008, p. 243).

The didactical potential of this learning activity lies in the following:

Since most of the students have not thought about justification-theoretical aspects of mathematical proofs so far, this activity could create the awareness of the problem area. Therefore, we provide the students with three proofs that differ widely in the given details, which can be seen as an analogue to the "infinite regress" in the Münchhausen-Trilemma. We want the students to reflect their implicit convictions resulting from their previous mathematical experiences, and lead them to the question *"What is a perfect proof?"* in a suggestive way.

The reflection about the three options given in the Münchhausen-Trilemma could lead to the conviction that the premises-inference-conclusion pattern is a reasonable and adequate means for mathematical argumentation. But it also allows the recognition of "mathematical work": Defined properties can be used without proof, whereas all other properties must be proved by only using the definitions. This is also mentioned by Duval (1991) who pointed out the importance of the awareness of the logical status of propositions when trying to understand a mathematical proof – especially the difference between premise, theorem, conclusion etc.

SUMMARY AND OUTLOOK

The aims of the math-bridging course we present in this paper are twofold: the development of basic mathematical skills and the explicit training of strategies for the

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[1] The Münchhausen-Trilemma after H. Albert (1991). English formulation as in http://en.wikipedia.org/wiki/Münchhausen_Trilemma (14.9.2012)

learning of mathematics as a science. Our central method for teaching these learning strategies is to construct a teaching scenario that encourages the students to do two things. First, to implicitly apply these strategies in the course of an exercise. Second, to become aware of and to discuss their methodological status as norms or ideals that one should follow in order to justify knowledge as scientifically objective. As a concrete example, we outline a teaching scenario, including some comments on our didactical reasoning and teaching methods, that aims at clarifying the central inner- and meta-mathematical significance of deductive reasoning. This way, the students should not only value the importance of the concept of deductive reasoning in scientific mathematics, but also discuss the legitimacy of this concept as a helpful tool in reaching objective scientific knowledge in science in general.

In October 2012 a first bridging course integrating the above learning scenario was realized at the University of Education Ludwigsburg. This was meant as a first test of our approach and led us to further considerations and the formulation of research questions. In Ludwigsburg about 300 students took part in the bridging course. 150 of the students are future primary school and special school teachers who did not choose mathematics as a major subject, but will have to attend several mathematics courses. These students visited the lecture of one of the authors and a small part of these students took part in the exercise sessions conducted by her. Their attitude towards mathematics was mainly characterized by a procedural view of mathematics and by the fear of not managing the subject. Our experiences let us hope that our approach is practical and useful even with such an audience. Although the first realization is far from being empirically solid, we present some of the students' comments to give an impression of the students' "view of mathematics" after the bridging course:

> "I got another view of mathematics in the sense that things are not just as they are, but that there is a reason for everything and one could ask for the reason. The children will also ask for reasons at school."

> "I found it good to get motivated and to get another view of mathematics (questioning calculation rules: Why is it like that?)"

Therefore, we are confident that our approach could give the students "a new idea of mathematics" by initializing mathematical enculturation at the secondary-tertiary transition.

In the future we plan to develop further learning scenarios and a large pool of differentiated exercises for subsequent courses to allow the students to group themselves according to their individual needs. Another bridging course integrating our approach will be held at Humboldt-Universität zu Berlin in October 2013. For this course we will also implement assessment methods. Our future work will be guided by the following research questions:

- Will the "metacognitive" considerations lead to more comprehension concerning mathematical argumentation and proof and their relation to everyday language?

- Will our approach help the students not only to understand a proof as a product but to produce their own proofs?
- In which way is the students' view of mathematics as a science changed, and how does this change influence their learning of mathematics? E.g. will the students be enabled to work more autonomously on mathematical problems (including tasks that aim at argumentation, but also tasks to train skills)?

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DEVELOPMENT AND AWARENESS OF FUNCTION UNDERSTANDING IN FIRST YEAR UNIVERSITY STUDENTS

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This article presents results from a longitudinal study on the development of first year university students' function concept, and of their awareness of this development. We used questionnaires in the first and last quarter of the first year and had 38 students participating in both tests. We found that most students' function concept did develop, but many students did not notice the development. A small number of students had low proficiency with functions but high estimation of their proficiency; these students tended to show less development in their function concept.

INTRODUCTION

Too often our students complain that their mathematical training lacks relevance for their future careers as teachers, engineers, etc. Some student teachers value the mathematical component of their study as a personal intellectual pursuit of "real mathematics" before returning to "school mathematics". We, on the other hand, see mathematics studies as an opportunity to enjoy challenging yourself with mathematical problems; to learn to value critical thinking and argumentation over dogma; and to observe your own development, as well as that of your peers. In short, to develop mathematical habits of mind, as Cuoco, Goldenberg and Mark (1996) put it.

Goulding, Hatch and Rodd (2003) investigated what British students actually carry with them from their bachelor's degrees to the teacher training (PGCE) in a retrospective study. Of the seven themes emerging from the students' responses, only two pertain to the issues mentioned in the previous paragraph; and within these categories many responses were actually negative, in that the students' views of mathematics had changed in undesirable directions (e.g., "for exams only").

In 2010, we started a project to investigate whether standard pure mathematics courses do support the development also of concepts central to "school mathematics" and applications. We focus on the function concept in which the development is subtle and slow, and may go unnoticed to the students (Carlson, 1998). Further, it has been indicated as one of the central topics for mathematics teacher education in the *standards* endeavor (Conference Board of the Mathematical Sciences, 2001). Here we report our findings from the first year of follow-up.

BACKGROUND AND RESEARCH QUESTIONS

Several studies indicate difficulties when describing mathematical concepts in different, mathematically identical, ways (see, e.g., Even, 1998; Bayazit, 2011). Although most students can visualize simple functions by drawing their graphs, they often lack ability to link the graph and the algebraic representations of the function (Vinner &

Dreyfus, 1989). The intuitive notion of the function often relies on an explicit algebraic formulation (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Tall & Vinner, 1981). Functions and related notions are often treated as if they were only symbols and representations instead of proper mathematical objects (Eisenberg, 1991). We understand the function concept in the same way as Carlson (1998), see, e.g., Selden & Selden (1992) for a more detailed discussion.

Carlson (1998) found in a cross-sectional study that even A-students take a long time to acquire solid understanding of functions: second year undergraduate students still showed many flawed conceptions. The complexity of the concept and its slow development suggest that it might be quite likely that some or even most students fail to notice their developing function concept during their mathematics studies. This is especially problematic for student teachers since they then miss out on the opportunity to reflect on the difficulty of coming to terms with functions.

As far as we have been able to find out, there is little research about students' self-awareness of long-term development of mathematical concepts (for an exception, see Bjuland, 2004). In mathematics education, a typical time frame for reflection seems to be the solving of one problem (e.g. the "looking back" stage of Schoenfeld, 1992). Following Bjuland (2004), we see both long and short time frames as necessary for self-awareness: immediate and short-term reflection is needed to notice the different aspects of the concept whereas reflection over an extended period of time is needed to compare these observations at different points in time and thus become aware of the development. The scarcity of studies on long-term reflection in mathematics education led us to consider theories of reflection and metacognition in a more general context.

A general framework for addressing issues related to reflection and metacognition was proposed by Schraw and Moshman (1995). According to them, a person's *metacognitive theory* integrates metacognitive knowledge and experiences, and can be used to explain and predict his/her cognitive behavior. They argue that one aspect in which individuals' metacognitive theories differ is the extent to which they are explicit (i.e., the extent to which one is aware of possessing such a theory).

Schraw and Moshman (1995, p. 360) point out that explicit knowledge about your own cognition makes it possible to reflect on your performance and to use this information to modify your future performance and thinking. They also seem to imply that individuals' metacognitive theories gradually develop via awareness of changes and reflection on them.

In contrast, tacit theories are developed without conscious reflection, based on personal experience or adaptations from others, and therefore it can be difficult to notice and report one's own development to others. Since an individual with a tacit metacognitive theory is not readily aware of either the theory itself or evidence that supports or refutes it, it can be very difficult to change the theory even when the theory is maladaptive and the individual is explicitly encouraged to do so. Further, the lack of conscious reflection might cause metacognitive knowledge and regulation to lack transferability between task-types. (Schraw & Moshman, 1995)

Since the development of the function concept is likely to be slow and subtle, students might not be aware of it. However, it is also possible that some students may not notice any development since there simply is none to notice. Hence we divide our research question in two:

1. Do students' function concepts develop during 1st year university studies?
2. Are students aware of this development?

METHODOLOGY

Instruments

We used two questionnaires, both based mainly on Carlson's tests (1998). Our first test consisted of A1, A5, A6, A7, A14, B3 and a modification of A13 in which the linear graph was replaced by a piecewise linear one. The second test consisted of A2b, B2, A6, A7, and A8. The tasks were:

A1. Express the diameter of a circle as a function of its area and sketch its graph.

A5. Does there exist a function all of whose values are equal to each other?

A6. Does there exist a function whose values for integer numbers are non-integer and whose value for non-integer numbers are integer?

A7. Does there exist a function which assigns to every number different from 0 its square and to 0 it assigns 1?

A8. The given graph (Figure 1) represents speed vs. time for two cars. (Assume the cars start from the same position and are traveling in the same direction.)

a) State the relationship between the position of car A and car B a t =1 hr.

b) State the relationship between the speed of car A and car B at t =1 hr.

c) State the relationship between the acceleration of car A and car B at t =1 hr.

d) What is the relative position of the cars during the time interval between t =.75 hr. and t =1 hr.?

B2. Assume $F(x)$ is any quadratic function. True or false: $F\left(\frac{x+y}{2}\right) < \frac{F(x)+F(y)}{2}$

A13. Suppose that the given graph (in test) of height as a function of volume as a bottle is filling with water. Sketch the shape of the bottle.

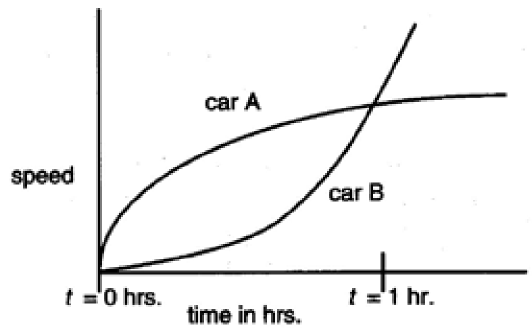

Figure 1. The graph from Task A8.

The questions had been translated into Finnish and tested by P. Hästö and M. Leinonen (unpublished). The tests were scored following Carlson's (1998) rubrics, and for A13 we developed our own rubrics. The number of tasks selected was dictated by time constraints, and the selection was based on our assessment of the relevance of tasks to the course in which the test was given, and on having sufficient variability in task types.

The second test also included six "self-awareness" claims which were answered on a five-point Likert scale and two other questions which are not analysed here. The claims were

Q1. Examples of functions at the university are similar to those in high school.

Q2. In high school it was clear to me what was meant by a function.

Q3. It is (presently) clear to me what is meant by a function at the university.

Q4. My understanding of functions has changed while at the university.

Q5. I have pondered over my understanding of functions during my university studies.

Q6. During my university studies there have appeared examples of functions contrary to my function concept.

It is, of course, clear that these questions will only provide a very sketchy picture of students' self-awareness and experience, which had to suffice for this pilot study.

Participants

The first test was administered in the second week of the first period in a class typically taken by first year students, both mathematics majors and other students with a math component (mainly physics and chemistry majors). The second test was administered in the first week of the fourth period (out of four) in an analysis class typically taken by first year majors and second or later year minors. Note that students did not follow any special courses on functions, only a standard university mathematics curriculum.

There were 98 participants in the first test and 64 in the second. Of these 38 participated in both tests. Two questions were repeated from the first to the second test; it should be noted that students got no feedback on the tests and solutions were also not distributed. A majority of these students are expected to become teachers, although the choice is not yet made.

RESULTS AND ANALYSIS

Our first observation is that students in our follow-up became significantly better at the tasks in the tests. Their scores in the two repeated tasks improved, on average, over 2.5 points (the maximum score being 5 points on each task). To compare the other tasks we use Carlson's Group 2 (A-students from a 2nd year calculus course) as a reference. In the first test our students' scores were consistently (and sometimes considerably) below the reference, whereas in the second test they exceeded the reference. This strongly suggests that there was on average much improvement in students' ability to answer these kinds of questions regarding functions.

Quantitative data reduction techniques, for instance concerning graphical and analytic components of the function concept, did not yield results.

To get a view on individual improvement, we cross-tabulated the sum of scores from the two tasks included in both our tests, displayed in Table 1. From the table we see that only three students got a lower score in the second test (in fact, all dropped from 1 to 0 points), while all others improved their score, many by more than 5 points.

1st test/ points	2nd test/points 0	2	4	5–7	9	10	Total
0	1	2	2	2	0	4	11
1–3	3	0	0	3	2	11	19
5–7					1	4	5
10						3	3
Total	4	2	2	5	3	22	38

Table 1. Number of students with given points in Tasks A6 and A7 of the two tests.

Based on the results in the two repeated tasks, we divided the students into three groups: LOW (less than 5 points in both tests), RISE (at least 5 point improvement between tests) and HIGH (at least 5 points in both tests). Three students belonged to none of these groups and are excluded from the next analysis. In other tasks the achievement level of the RISE group lied between the other two groups. In other words, they had not quite caught up with the HIGH group.

Group	*n*	Q1	Q2	Q3	Q4	Q5	Q6
LOW	8	.3	1.0	1.0	.3	1.0	–0.4
RISE	19	.3	.3	.8	1.0	.1	–0.3
HIGH	8	.0	.9	1.4	.8	.9	–1.3
Total	35	.2	.6	1.0	.8	.5	–0.5
All	64	.2	.6	1.1	.8	.5	–0.4

Table 2. Means of opinion scores. The shaded cells are discussed in more detail below.

For these three groups we calculated the means of the answers to the six questions mentioned on page 4. The results appear in Table 2. The answers were coded so that "strongly agree" has value 2 and "strongly disagree" –2. From the table we see that there were no radical differences between the groups. In fact, in a one-way ANOVA even the biggest between-group difference, in Question 6, was only approaching statistical significance ($p = 0.085$).

Nevertheless, we combine indications from several answers to form a tentative conclusion, or, if you will, a hypothesis for further study. Q4 (change of understanding) is consistent with Q2 (high school) and Q3 (university) in that groups RISE and HIGH now rate their understanding higher than in high school, and correspondingly rate it as having changed more that group LOW.

Group HIGH has consistently higher view of their function understanding than group RISE, which is consistent with test scores. However, students in group LOW find that they have had a fairly clear understanding of functions from high school, which has not changed. It appears that these students have an erroneous and overly optimistic view of the adequacy of their function concept.

Question 5 (pondering) is somewhat of a dilemma: the group with the least pondering (RISE) has improved most drastically. Finally, Q6 (new examples) is consistent with the groups in that the consistently high-scoring individuals found fewer examples which did not fit into their conception of function. Surprisingly, across all students there was a tendency to disagree with the statement that they had encountered new types of functions at the university. This may be due to the inclusion of the technical term "function concept" in the question.

Qualitative analysis

Based on students' responses to questions A5–7, we identified four students as major improvers and did a more thorough data-driven analysis of their answers. In the analysis we marked all interesting responses; since the data set was not large, all marked responses are presented.

Our first observation in the qualitative analysis of the two questionnaires was the obvious improvement in students' understanding of "the language of functions", as Carlson (1998) puts it. Initially the students demonstrated great difficulties in translating a verbal description of a function into algebraic notation: e.g., in task A5 one of them wrote "for example, $f(x):\ x = 2$", and another one started to solve the equation $x - y = y - x$, and after a couple of steps concluded that $x = y$. Apparently these students understood neither what is meant by "the value of a function" nor the usual notation by which functions are defined. However, three of these students showed very little difficulty in algebraic manipulation while expressing the diameter of a circle as function of its area (the fourth one left it blank), but all of them neglected to sketch the graph.

It also seems that these students thought that all functions must be defined by a single algebraic formula. In task A7, two of the students simply wrote "x^2" while two left it blank. In the second part none had difficulty in defining functions piece-wise.

Although there is little doubt about the students' improved writing in a formal mathematical style, it is unclear to what extent this indicates improvements in students' concept. For example, in the challenging task B2, one of the four students started the solution by defining $f(x)=x^2$ and then $f(y)=y^2$, as though the second did not follow from the first (a misconception also documented, e.g., by Sajka, 2003).

The difficulty in interpreting functional information from a given graph (rather than interpreting it literally) appears to be somewhat persistent. In Task 8 all the four students demonstrated very little difficulty in interpreting static graphical information (speed) or relatively standard dynamic information (acceleration), yet all of them failed in Task A8d, in which they were asked to describe the relative position of the two cars over a time interval. Although they noticed that the car A was driving faster than B for the whole time, they still concluded that B was catching up with A because B had accelerated so much whereas A had driven with nearly constant speed.

DISCUSSION

In answer to our first research question, we found that the function concept of most students does seem to improve. In particular, courses in pure mathematics then also support the education of student teachers and applied mathematicians. The answer to the second question, regarding students' awareness, is more complicated. Let us elaborate on this.

Our results are consistent with Carlson's study (1998). In our first test students had difficulties very similar to those in Carlson's Group 1 (A-students from a college algebra course): they had problems understanding the language of functions, for instance when defining functions piecewise. In the second test, however, our students seemed to have the same kind of difficulties as Group 2 (A-students from a 2nd year calculus course), e.g., interpreting dynamic functional information over an interval. Interestingly, although many participants in our first test apparently thought that a function must be defined via a single algebraic formula, this was not the case in the second test; in contrast, Carlson's Group 2 subjects had this misconception. We interpret this as meaning that our students were slightly more advanced in their general understanding of functions than Carlson's Group 2 students.

Between the two tests most of the students had taken introductory courses on both abstract and linear algebra, and advanced calculus. In these courses functions appear mainly through definitions and further properties (bijectivity, linearity, etc.) whereas little explicit focus is placed on development of intuitions. Nevertheless, the quantitative analysis showed that students made great progress in function tasks. This means that students were able to improve their function language without explicit instruction. Unfortunately, we cannot say how much of the improved scores is due to conceptual change and how much is a consequence of adopting the formal language used at university.

The qualitative analysis showed that this progress was tenuous in the sense that there was a tendency to relapse into old ideas when facing non-standard tasks with high cognitive demand. Apparently, students had least difficulty in tasks requiring algebraic manipulation of quantities. This is probably due to the Finnish curriculum in which a considerable attention is paid to solving equations via algebraic manipulation. On the other hand, in many high schools there is passing mention also of more abstract notions of functions, so the best students may have picked it up there. This may explain why they did not find examples at the university to be beyond their function concept. Altogether, these observations are consistent with Carlson's (1998) finding that students do not use the newest tools in their conceptual arsenal proficiently.

Our tests did not uncover a graphical and analytical component in the sense that there would have been greater correlation within the groups of graphical and non-graphical tasks. This may be due to floor and ceiling effects in the task scores which led to skewed distributions; we found several nonlinear relationships between tasks where only students with 5/5 points on problem X got non-zero points on problem Y. This means that the difficulty range of the problems was not optimal. On the other hand, it seems that also in most other studies the graphical and analytical components are postulated rather than recovered from the data. Our observations of graphical components are limited to the qualitative analysis. Here our findings mirror Bayazit's (2011) results of students' limited abilities to shift from a graphical expression to algebraic expression, with most depending on algebraic expressions.

Some answers indicate that the concept of variable is also problematic. For instance, one student introduced a function by defining it for several variables ($f(x)=x^2$ and $f(y)=y^2$) instead of just once. We do not know to what extent our students' problems are due to deficiencies in such prerequisite concepts as that of the variable.

About a quarter of the participants performed poorly and did not improve their performance between the first and second test. Interestingly, these students on average had a high estimation of their understanding of functions — in fact, they estimated that the function concept had been clear in high school and had remained clear since then. If they do not come to terms with the need to develop their function concept, this might cause serious problems for their studies and possible teaching careers. From course grades and other feedback these students must know that their studies are not progressing as well as they should be. Therefore they seem to be misattributing this lack of progress to some other factors than a weak function concept. This suggests that they do not have effective means of checking their metacognitive theories against reality, which indicates a persistent tacit metacognitive theory.

A second group, consisting of more than half of the participants, improved between the tests. Moreover, this group realistically estimated their function concept as having been average and then having improved. Unfortunately, we only asked participants to rate the clarity of the concept in the second test. Thus we have only a retrospective estimation of what the students thought after high school. Do they rate the conceptual clarity as low because they now understand it better? Or is it the case that they were unsatisfied with their understanding at the beginning of the studies, and were thus more open to new influences? This remains a question for further study. A methodological short-coming is the selection of the group based on only one repeated task: this might have introduced some bias. This will be addressed in the next iteration of the study.

If the retrospective rating is accurate, then it might allow us to design a diagnostic test for the beginning of studies which detects students at risk of belonging to the LOW category as those with low performance and high estimation of their function concept. Alcock and Simpson (2004) found that this kind of profile (low performance and high confidence) is typical for some graphically oriented students. Whether there is such a link in our case also remains to be determined. Another open question is whether this profile is context specific, or whether the students follow the same path in all areas of mathematics.

The results of the first year of our study in general agreed with our expectations. Students' concepts improved and most said as much in the questionnaire. Whether this group contains a subgroup of improvers who do not notice the change is a question for future study. Almost a quarter of the students did not improve, and showed low self-awareness of their situation. This pilot study had both deficiencies and strengths. In future versions we will diversify the data collection to get a better picture of the groups tentatively identified in the analysis above.

NOTES

1. The authors are listed in alphabetical order.

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MATHEMATICAL MEANING-MAKING AND ITS RELATION TO DESIGN OF TEACHING

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This paper addresses the design of teaching to promote engineering students' conceptual understanding of mathematics, and its outcomes for mathematical meaning-making. Within a developmental research approach, inquiry-based tasks have been designed and evaluated, through the use of competencies proposed for their potential to promote conceptual learning. A sociocultural frame draws attention to interactions between different cultural elements to address challenges to teaching relating to student perspectives and the mathematical meanings they develop. The paper recognizes tensions between design of inquiry-based practice and the outcomes of that practice, and demonstrates the need for new research to address mathematical meanings of a student community within a sociocultural frame.

SETTING THE SCENE

In this paper I focus on the ESUM project (Jaworski & Matthews, 2011) in which an innovation in the teaching of a basic mathematics module to first year engineering students (n=48) was studied. The ESUM innovation involved design of teaching using inquiry-based tasks and small group activity within a GeoGebra environment. The goal of teaching development was to promote students' *conceptual* understanding of mathematics rather than understanding that is instrumental or procedural (Skemp, 1976; Hiebert, 1986). The etymology of "understanding" (under-standing), as for example in comparison with the French word "comprendre" (taking together), is of interest. We have been challenged to declare what we mean by "understanding" or indeed by "conceptual understanding" and how we expect to recognise it. One finding from ESUM was the difficulty of discerning students' conceptual understanding, which we have expressed in terms of students' *mathematical meaning making*. We want to go beyond superficial indicators (like test or exam results) to find ways of revealing students' mathematical meanings. The focus of this paper is how to address this challenge. I consider a (draft) report from the SEFI (Société Européenne pour la Formation des Ingénieurs) Mathematics Working Group (2012) which recommends a *competence* approach (Niss, 2003). More precisely, this paper investigates the following research questions:

RQ1) *When a (developmental research) project seeks to enhance students' meaning making of mathematics, how can we gain insights to students' mathematical meanings?*

RQ2) *How can we characterise mathematical meaning making in ways which aid its creation?* (*In what ways can the SEFI/Niss competence framework aid characterisation?*).

To address these questions, I draw on findings from ESUM, using the SEFI framework to interrogate the design of inquiry-based tasks or questions, taking, as an example, the topic of *functions* which was a central topic in the mathematics module.

A literature search on innovative modes of teaching (in HE STEM-related subjects[1]), showed that the use of inquiry-based approaches is often conceptualised within a *constructivist* theoretical frame (Abdulwahed et al, 2012). As such, learning is considered from individual cognitive perspectives, possibly with a social dimension (e.g., Ernest, 1991). In ESUM, research findings have pointed to tensions and contradictions between the design of teaching and students' perspectives on learning and teaching (Jaworski, Robinson, Matthews & Croft, 2012). This has required us to deal with complexity within differing cultures and within institutional constraints, for which a sociocultural theoretical frame makes more sense than a frame of individual cognition. Thus, we see mathematics knowledge growing in social settings through mediational processes and the use of tools such as inquiry-based tasks and approaches to teaching (Schmittau, 2003; Wertsch, 1991).

A DEVELOPMENTAL AND INQUIRY-BASED APPROACH

The ESUM study employed a developmental methodology, incorporating an inquiry-based approach, in which research both studied developmental practice and contributed to development (Jaworski, 2003). A team of three teacher-researchers (insiders) designed and taught the module, with continuous reflection and review leading to modifications during practice and new insights for the next year of teaching. A research assistant (outsider) collected data and analysed data together with the teaching team. Analyses informed future teaching.

The developmental methodology involving nested layers of inquiry (A, B & C with $A \subset B \subset C$) with students' learning of mathematics at the centre: *Inquiry in mathematics* (A) involves students in learning and understanding mathematics through inquiry. *Inquiry in developing mathematics teaching* (B), involves questioning teaching approaches and the design of teaching, to understand the basis of teaching decisions and ways of improving teaching for better learning outcomes. Inquiry in layer C inspects the other two levels to gain insights to the developmental processes in both layers, and their outcomes (Jaworski, 2006). When inquiry practices are instituted or promoted within a group, an outcome can be the formation of an *inquiry community*, which can be seen to have all the hallmarks of a *community of practice*, as designated by Wenger (1998), except in one major respect. In Wenger's terms, those involved in the community can be seen to have joint *engagement*, *enterprise* and *repertoire*; and their identities can be conceptualised as encompassing the use of *imagination* in charting personal trajectories of engagement, and *alignment* with the norms and expectations within the practice (Wenger, 1998).

[1] STEM – Science, Technology, Engineering and Mathematics. The ESUM project was funded through the Higher Education (HE) STEM programme by the *Royal Academy of Engineering*. Two case studies from the project can be found at http://www.hestem.ac.uk/resources/case-studies.

While it is impossible to be a part of a community of practice without aligning with its norms and expectations, one does not have to align uncritically. Uncritical alignment can result in perpetuation of practices which do not achieve the goals of practitioners – for example, alignment with certain forms of teaching practice, within a community of mathematics teaching, can result in student learning outcomes which are instrumental or procedural in nature, lacking conceptual depth (Skemp, 1976; Hiebert, 1986). So, alignment needs to be critical – critical alignment – in which (established) practices are subject to critical questioning by the practitioners who engage with them (Jaworski, 2006). In learning mathematics, with inquiry in the three layers A, B and C. critical alignment involves asking *why*? Why do we do things in certain ways: why this formula, why this procedure, why these relationships? Inquiry-based tasks and questions are designed to get student to address these *whys*.

LEVELS OF COMPETENCY IN MATHEMATICS AND IN TEACHING

I turn now to competence and competency and their relation to the design and use of inquiry-based questions and tasks to address the given research questions and the ESUM main goal regarding conceptual understanding. Niss writes:

> Possessing mathematical competence means having knowledge of, understanding, doing and using mathematics and having a well-founded opinion about it, in a variety of situations and contexts where mathematics plays or can play a role (Niss, 2003 p.183)

A mathematical competency is a distinct major constituent in mathematical competence: eight competencies have been identified in two groups.

The ability to ask and answer questions in and with mathematics	*The ability to deal with mathematical language and tools*
1. Thinking mathematically	5. Representing mathematical entities
2. Reasoning mathematically	6. Handling mathematical symbols and formalism
3. Posing and solving mathematical problems	7. Communicating in, with and about mathematics
4. Modelling mathematically	8. Making use of aids and tools

Table 1: Mathematical competencies as expressed in SEFI (2012)

These competencies seems to have synergy with inquiry-based learning and what we aimed for in the ESUM project, and they offer starting points for design of tasks and evaluation of learning outcomes. Space here precludes a detailed account of each competency; I will rather clarify their meaning through application to task design. The authors emphasise three dimensions for specifying and measuring progress in learning with respect to competency: Degree of coverage: The extent to which the person masters the characteristic aspects of a competency; Radius of action: The contexts and situations in which a person can activate a competency and Technical level: How conceptually and technically advanced the entities and tools are with which the person can activate the competence. How to address these dimensions is an issue to consider.

TASK DESIGN AND ANALYSIS

When students emerge from schools and their A level[2] courses, we know that their mathematical learning has often been of an instrumental nature (e.g., Artigue, Batanero & Kent, 2007; Hernandez-Martinez et al. 2011). Thus, as part of the school culture, they know how to, for example, apply rules of differentiation and integration, but have little conceptual understanding of the nature of functions or of limiting processes. In both of these areas research has pointed to conceptual difficulties that students experience (e.g., Cornu, 1991; Even & Tirosh, 1995). So, the demands of design within the university course are to create tasks which engage students with mathematics, some of which is already familiar to them, in ways which take them beyond school practices and into a university culture in which it is hard to progress without deeper understandings.

The following two tasks (Table 1) were designed for these purposes. The first was used in a lecture at the beginning of our work on functions. In the second, the first part (a) was used in a lecture and the other parts (d-e) in a tutorial where students sat in groups of three or four each with a computer and access to GeoGebra software. In accord with design goals, and associated expectations of students' engagement, I have analysed the tasks in terms of the eight mathematical competencies (Table 1).

Analysis of Task 1 using the competencies

Task 1 (Table 2) was intended to open up discussion of functions. The lecturer offered the task and waited for students to write down two functions, meanwhile, walking round the lecture theatre and looking expectantly at students (and smiling, with eye contact) to encourage their engagement with the task.

	Task	Competencies
1	Think about what we mean by a *function* and write down two examples. Try to make them different examples.	1, 2, 5, 6, 7
2	In the topic area of real valued functions of one variable Consider the function $f(x) = x^2 + 2x$ (x is real)	
a)	Give an equation of a line that intersects the graph of this function (i) Twice (ii) Once (iii) Never (Adapted from Pilzer et al. 2003, p. 7)	1, 2, 3, 5, 6, 7
b)	If we have the function $f(x) = ax^2 + bx + c$. What can you say about lines which intersect this function twice?	1, 2, 7, (8)
c)	Write down equations for three straight lines and draw them in GeoGebra	1, 2, 5, 6, 7, 8
d)	Find a (quadratic) function such that the graph of the function cuts one of your lines *twice*, one of them *only once*, and the third *not at*	1, 2, 3, 7, 8

[2] *A level* courses means courses preparing students for "Advanced Level General Certificate of Education" a public examination qualifying students for study in Higher Education. These are high stakes examinations and schools are measured by their examination successes.

	all and show the result in GeoGebra.	
e)	Repeat for three *different* lines (what does it mean to be different?)	1,2,3,5,6,7,(8)

Table 2: Two tasks from ESUM, with associated competencies

The task is *open* in nature. Students could write down any example they could think of. Since most students had studied *A level* mathematics, they had certainly encountered the term "function" and used functions. So for most students the task was accessible. It encouraged them to *think* [1]. To write down the function they had to use symbolism to represent the function [5, 6]. I argue that in *writing down*, they were already starting to communicate, and, in deciding on *different* functions, to reason mathematically [7, 2]. After a suitable time, the lecturer, in plenary, asked students to offer one of the 'functions' they had written, and wrote these verbatim on the overhead projector. Initial contributions were made tentatively, the lecturer smiling encouragement and thanking the student, and many more then followed, thus overcoming some of the barriers to student contribution in a lecture. When a (long) list of offerings had been produced, the lecturer asked students to comment on the nature of what had been offered (importantly, a student who had offered any example was now anonymous). Some of the examples offered were as follows: $y=x+3$; $y=x^2$; $y=e^x$; $x+y=4$; $f(x)=x+1$. The majority were of the form "y=". When asked to comment on 'difference' some students mentioned linear functions versus quadratic functions, or exponential functions. Some queried $x+y=4$, stating that it is an *equation*, not a function. Very few used functional notation of the form '$f(x)=$'. When the lecturer added to the list $y=5$ and $x=4$, students were adamant that these are not functions. Thus, communication occurred between students and the lecturer [7], and students offered explanations and reasons for why an item was a function or not [2]. Students could see alternative offerings from their peers. For the lecturer, students' responses to the task provided insights to their current knowledge/thinking about functions, and allowed some immediate challenge – for example, "what is the difference between a *function* and an *equation*?", "why do you think $y=5$ and $x=4$ are *not* functions?"

In Task 1, students had to produce their own examples, leading to engagement, questioning, discussion and *inquiry*. *Inquiry* could be seen in the questioning which resulted, in consideration of what is a function and what is not a function, and in the mode of engagement in the lecture: students were expected to contribute, think, reason, argue, not to take some things for granted, and to deal with uncertain situations (not everything will be presented as cut and dried, right or wrong). We can see this episode as the beginnings of creating an inquiry community. We see here some starting points in addressing RQ1, and a start to characterisation of understanding using the competencies (RQ2). We can ask how the three *dimensions* relating to competency can be used to evaluate students' responses to the task.

Analysing Task 2 using the competencies

The first part of Task 2 (2a) was also presented in a lecture with a similar teaching approach to that described above. An analysis of this task suggests that:

- The function is easy to sketch for students who have reached A level in school – it is easy to see lines which cross it in the three conditions [5]
- Students have to talk to each other [7]
- They have to think about equations for their lines [1] [3] [6]
- They start to reason about the differences between the lines [2]
- They have to give feedback to the lecturer and others in the cohort [2, 7]

In the lecture, students were asked to write down the required equations and to discuss with a neighbour [1, 5, 6, 7]. After a short time, students' suggestions were written on the OHP by the lecturer. Some students offered equations of parallel horizontal lines, such as y=1, y= -1, and y= -3. Others offered non-horizontal lines. One question which arose was how one can know that a non-horizontal line will cross the graph (or not). This provided opportunity for discussion, with some students disagreeing with others as to which lines will cross or not cross [1, 2, 3]. Further graphical and algebraic activity resulted. GeoGebra allowed the possibility to experiment quickly changing coefficients in equations and scales on axes to gain insights into relationships. Some students were able to offer algebraic reasoning, but it was not certain that all were able to understand this [6].

Here the lecturer learns from students' responses and can consider how to plan differently for a future occasion, to give more time or not, to rearrange material or not. The lecturer also learned about interventions: where a question or explanation seemed to promote student engagement and where not; how to deal with incorrect assertions if no student offered a challenge. When students' themselves offered a challenge, mathematical communication between students provided corresponding opportunities for learning. The lecturer became aware of actions which promoted or inhibited students from offering such challenges. More time could valuably have been spent on such activity, encouraging questions and explanations from students, but there was much further material to address in the lecture, and so not enough time to give to continuing the discussion. These are examples of contextual constraints.

Tasks for use in a tutorial

Parts b) to e) of Question 2) were addressed in a tutorial. Typically a tutorial was related to material addressed in a recent lecture. Often a sheet of questions was provided, some questions offered practice and support in relevant areas of mathematics; others were inquiry-based questions in which exploration, questioning, discussion and justification were encouraged. Question (2) is an example of the latter: 2b) requires students to generalise from (a), [competencies 1, 2, 7]; in 2c) students have to invent their own mathematical objects and use a technological tool [1, 2, 5, 6, 7, 8]; in 2d) they have to tackle an open-ended problem [1, 2, 3, 7, 8], and in 2e) they are required to generalise mathematically [1, 2 3, 5, 6, 7]. In (b) and (c), use of GeoGebra can provide opportunities to visualize and to generate a range of possibilities for consideration. 2d) is seriously challenging – even with use of GeoGebra it is not simple to generate the function, analytical thinking is required. In

2e) the question about *what it means to be different* is designed to promote thinking at more general levels and encourages movement towards conjecture and proof.

In the tutorials, lecturer and a graduate assistant moved from group to group of students encouraging work on tasks and probing students' mathematical thinking. It became clear that different groups engaged very differently: some taking on the mathematical challenges and some seeking quick and easy solutions. GeoGebra was used variously as a graphical display (with a screen full of indistinguishable graphs), a source of quick/easy answers to questions, or as a help in tackling challenging questions. While tutor and assistant encouraged the latter, they were aware of the other uses. Although their questions encouraged a more meaningful, mathematically in-depth use, it could be seen, when the tutor left the group, that some students returned to other uses or were tempted to use social networking sites or engage with email. Critical alignment for the tutor is seen in how to promote deeper engagement when former school practice and current student cultures acted in other directions.

Data and Analysis

Data collected from these events included the lecturer's reflections: orally after a lecture or tutorial, and a written reflection each week addressing issues arising from the interpretation of teaching design in practice (critical alignment); the research assistant audio-recorded lectures and the oral reflections and kept observational notes from all events. After the end of the module (one semester) she and another colleague interviewed a selection of students. In addition, data was collected from student surveys and written project work. Data were analysed to address questions of students' engagement and their experiences of inquiry-based tasks and use of GeoGebra. Data from written project work showed that students were aware of ways in which GeoGebra could contribute to their understanding. However, the following two responses, from focus group interviews, are indicative of student attitudes.

- I found GeoGebra almost detrimental because it is akin to getting the question and then looking at the answer in the back of the book. I find I can understand the graph better if I take some values for x and some values for y, plot it, work it out then I understand it … if you just type in some numbers and get a graph then you don't really see where it came from. (Focus group 1)
- Understanding maths – that was the point of Geogebra wasn't it? Just because I understand maths better doesn't mean I'll do better in the exam. I have done less past paper practice. (Focus group 2)

How *dimensions of competency* might interface with such findings is hard to see.

MAKING SENSE OF STUDENT UNDERSTANDING

In the above I have focused on analyses of *the design of teaching*, principally the design of inquiry-based tasks and an associated teaching approach to engage students with mathematics for conceptual understanding. I have used mathematical competencies to qualify or start to characterise 'conceptual understanding'. I have suggested that students' responses to this careful design have not been what we would ideally have liked; factors identified being institutional constraints, time, students' school culture, students' social culture. Student remarks such as those quoted led us to characterise student responses as 'strategic' (Jaworski & Matthews, 2011). Students wanted the best possible grades and had clear ideas as to how this should be achieved; some of these ideas conflicted with the expectations of teaching, students expressed their own expectations on the nature of teaching (e.g., the teaching should focus more on graph *plotting*; there should be time given to practising past papers). Comments related to doing "better in the exam" suggested students valuing a more instrumental approach to understanding with a perception that a more in-depth understanding was unnecessary.

We are aware that the existence and nature of an exam (worth 60%), whose style had changed little from that before the innovation, was not exactly in the spirit of inquiry-based learning to encourage deeper understanding, although it might be seen to have synergy with dimensions of competency. We have considered replacing the exam with other forms of assessment, but institutional constraints have so far prevented this[3]. In written group project reports (worth 20%), understanding was demonstrated through responses to questions in which students had to pursue their own lines of inquiry and comment on the value of their use of GeoGebra. A typical response was:

> As a group we looked at many different functions using GeoGebra and found that having a visual representation of graphs in front of us gave a better understanding of the functions and how they worked. In this project the ability to be able to see the graphs that were talked about helped us to spot patterns and trends that would have been impossible to spot without the use of GeoGebra." [Group F – project report]

However, observational data showed some students not engaging seriously with the more demanding questions in tutorials, and many attending more assiduously to the more routine exercises. It was clear that where groups were taking seriously the inquiry-based questions, discussion with the tutor proved encouraging and motivating. Unsurprisingly, groups which responded best in tutorials gained the higher marks in the assessed group project.

In the above I have commented briefly on some of the key findings from our ESUM analyses. They reveal important insights into the sociocultural factors influencing the implementation of project design and its outcomes for students. Nevertheless, the nature of mathematical understanding remains elusive. Analyses using the competency framework have supported our design of tasks; apparent synergy between principles of inquiry and competency reinforce confidence in our didactic design. However, our research questions above are only partially addressed. The competency-based task analysis offers a form of characterisation (2a). The sociocultural analyses allow us to frame some of the obstacles to deeper insights into students' understanding (e.g., students' perceptions demonstrated in project writing in comparison to their views expressed orally in interview). The competencies and

[3] It is ironic that, in the exam at the end of the ESUM innovation, students' scores were on average 10% higher than those of previous cohorts. We are not able to link this directly to the innovation, since data was not available to compare intake grades with those of previous cohorts.

dimensions offer a framework for the design and evaluation of tests or examinations, but we believe this would give us little more than a summative evaluation of the sort we have already from exam and test scores, albeit perhaps more detailed and specific. RQ2 -- *How can we characterise understanding in ways which aid its creation?* – is only partially addressed, and perhaps we need a better-focused question. What is it, exactly, that we are trying to characterise? So far we have reinforced our design principles and the elusive nature of discerning students' mathematical understanding. We have juxtaposed design principles with sociocultural findings using activity theory to highlight inherent tensions or conflicts (Jaworski et al, 2012). Discerning tensions and conflicts is one step towards resolving them. Finding ways to characterise understanding is another. We still need to make the sociocultural findings active in our design so that we come closer to enabling the student understandings we seek. This requires us to go beyond competencies, while remaining aware of their contribution towards recognition of the mathematics for which we seek understanding. Since these are engineering students, discussion is taking place also with the engineering department.

A final consideration is methodology. We need to adapt our methodological approach with a deeper focus on mathematical meanings. The use of questions in lectures could be more focused towards revealing meanings; this would require the lecturer to pay more attention to generating student articulation of meanings which would be recorded and analysed within the sociocultural frame of the lecture community. We are designing further research to address these considerations.

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INTEREST IN MATHEMATICS AND THE FIRST STEPS AT THE UNIVERSITY

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Abstract: *First, we discuss interest in mathematics from a theoretical perspective. For this, we outline theory on interest and supplement it with reference to the specific context of the secondary-tertiary transition, constructing a specific theoretical perspective. We then use some interview data to demonstrate, how this perspective may help structuring practice reports and sketch an example for the interplay of the theoretical facets.*

Keywords: Interest, secondary-tertiary transition, self-determination theory

INTRODUCTION

Interest plays an important role in the experience and behaviour of students entering university, as numerous empirical studies have shown that interest (notwithstanding the way it was operationalized) is an influential factor in learning, being closely connected e.g. to the student's use of deep learning strategies, their effort and learning outcomes (Krapp, Schiefele, & Winteler, 1992). Such findings have not been replicated for the university, but can presumably be transferred from school. One might even speculate upon increasing influence at university, since students have more personal responsibility there (e.g. uninterested students might even not attend lectures). Therefore we believe it to be useful to learn more about interest, its structure and development, and present an approach to how to tackle this topic.

The study and thus our theoretical considerations refer to courses at the German mid-size university of Kassel where the mathematics is based on definition and proof from the beginning. The dropout rate is around 40-50 %.

THEORETICAL FOUNDATIONS

The context of the **secondary-tertiary gap** has many facets (see e.g. Gueudet, 2008). Personally, it often coincides with a new social situation and many more liberties and responsibilities. Much more than before, one may set goals for oneself. This is an opportunity to gain more autonomy, but may also lead to uncertainties. Then, the formal requirements of the study regulations are often taken as benchmarks, which unfortunately disregard important factors like learning goals, professional qualifications, the own vision of education and their contribution to science. These rather general issues are supplemented by some points which are specific to mathematics. Transition problems may be caused by a high degree of formalization, in which the focus is no longer given on calculations, but on proof (Tall, 2008). University mathematics requires skills (like language) which rarely correspond to the expectations of the students, and which play an important role in the student experience and behaviour within this new system.

Our conception of **interest** is based on the theoretical framework provided by Krapp (Krapp, 1992). Here, interest is conceptualised as a person-object relation characterised by value commitment and positive emotional valences. Individual interest is perceived to be a disposition that associates the object of interest with positive emotions (experienced and expected in future) and a personal value. Interest differs from motivation in the specific object. This can be a real object, but also something imagined, as long as it is known to the subject and perceived as one object. It can be located at different levels, e.g. school mathematics and university mathematics may be considered, but also specific aspects like proof or issues like calculus. According to Krapp, we distinguish individual interest from situational interest, which means that an individual acts with a concrete object motivated by external stimuli. However, this interest action requires the locus of causality to be perceived in the person itself (cf. autonomy, below). Interest development can be characterised by three stages: introjection, when interest actions are conducted for reasons which have nothing to do with the object, identification when the object has a value besides the aims of the interest action, and integration, when interest is integrated in the concept of self and doesn't conflict with it.

An important background for the analysis of origins and influences on interest is **Self-Determination Theory** (SDT) (Deci & Ryan, 2000). SDT posits the three basic needs for (perceived) autonomy, competence and social relatedness. All three factors are important, maybe necessary, for the development of interest (Krapp, 2005). This leads us to the following questions: When is competence experienced? How is autonomy experienced and secured? In which contexts do students feel relatedness? The very personal view is stressed in (Hannula, 2001), where the author revealed a strong interplay of needs and their satisfaction, goals and beliefs in school context, emphasising, that "what students want, has a strong influence on their experiences". Concerning university students, (Ward-Penny, Johnston-Wilder, & Lee, 2011) emphasised the importance of "students' view of what constitutes success and achievement" for the developing identities, which can be different on seemingly similar trajectories.

Our emphasis is on individual interest, which also requires situational interest to be considered, as both are closely linked. In the last decades, individual interest in mathematics has not been tackled often in depth. A notable exception is (Bikner-Ahsbahs, 1999) who investigated different facets of individual mathematics interest (like its history or cognitive problem solving). Recently, interest as outcome has received more attention, (e.g. Frenzel, Goetz, Pekrun, & Watt, 2010). In our work, we want to investigate interest in higher mathematics (HM), but also in school mathematics (SM), since it is assumed that the initial interest in HM and the attitude towards the subject are based on experiences and interest in school mathematics. The third major theme is studying in general. From the theoretical perspective, we would not believe that interest in HM is present from the first day, since it would require HM to be known to the beginners. Even interest in SM doesn't need to be present, since there is a whole range of additional motives for going to university. An important aspect of our work is the question of how the new objects are perceived in the light of the facets of interest emphasised by the SDT, and of which opportunities of action, belonging to these objects, are realised in which context. In this way we hope to obtain evidence on how and why the development of interest is hampered or supported by some aspects of the context or typically appearing as partially habitualised behaviour and patterns of argument. Thus, we have to capture the context and the way the subject deals with the context to understand the interest. Here, it makes sense to use a theoretical approach that reflects the role of the basic needs and includes pragmatic and habitual aspects, as offered in (Grotlüschen, 2010).

We could not find much literature on interest development during the transition. (Daskalogianni & Simpson, 2002) describes that entering university is a critical point for interest in mathematics. The authors interviewed students in school and later at university. The students showed a substantial loss of interest in the first six weeks, based on a mismatch between their beliefs and the mathematics they encountered. Some could recover as they managed to rearrange their belief system, whereas others not. Kenneth's case was illustrated, as he was one of the first kind. Here, we offer an alternative interpretation based on the presented framework: If we distinguish SM from HM, then we might see the early interest as referring to SM, as the first interviews were held in school and the authors mention school mathematics in this context. However, a possibly later interest might refer to HM, which the interviewers had asked for ("Why do you think, that Further Maths is more difficult?"). It seems that the students differentiate the two objects, as Kenneth states: "it's really quite a lot different to A-Level Maths". The "loss" would then be caused by changing the considered object of interest and the individual insight that the two kinds of mathematics differ (even from expectations). Inappropriate student expectations are well known to university teachers and e.g. documented by (Hirst, Meacock, & Ralha, 2004) concerning the importance of proof or technology use. The "recovery" would then simply be a new genesis of interest, which apparently requires the right kind of beliefs and some time for development. The importance of the student's beliefs and their adjustment had already been stressed by the same authors in (Daskalogianni & Simpson, 2001). From the SDT perspective on interest genesis, this again raises the question of the above mentioned interaction of beliefs and need satisfaction.

THE EMPIRICAL PART

We apply our framework to data from a focus group discussion and a subsequent interview at the University of Kassel, Germany. There, the typical course includes 2 hours of lectures a week with some 100 students, and a 2h tutorial in smaller classes (10-20 students). Each week, the students get a sheet with about four tasks as homework and have to hand-in their solutions. Successful participation (receiving at least 50% of the maximum score) is precondition for admittance to the exam.

The data is taken from a project where we investigate interest and its development in the first semester at university. In the study, we aim at reconstructing the subjective experience and behaviour, and want to relate it to interest. As a pilot study, we conducted a focus group since we believe that the more natural atmosphere of a group can help revealing student orientations and typical behaviour more directly. The participants were all taken from a lecture on linear algebra which normally is attended in the first semester (which was the case for two of the five students only). For about two hours we talked about transition problems and interest. The group discussion was audio-taped and transcribed. We first coded in an inductive way heading to different interests and related statements, and then recoded using theoretical categories emerging from SDT. All students agreed with anonymous scientific use of the data and were given the possibility to delete audio sequences. They reported very intense emotions concerning the homework assignment, like stress, anxiety and frustration. In their view, HM was quite different from SM. They also reported coping strategies like copying. We found that the students mostly didn't experience autonomy and competence. When asked about interest, they described themselves e.g. as "not uninterested" in HM, but couldn't specify this interest. It might be the case, that they had no interest in HM in the sense of Krapp, but didn't want to admit. As a disadvantage of the method, we couldn't easily ask for more details since the method requires the discussion to go on. Thus, we asked some students to come again for an interview (under the same ethical conditions), and it was a student we call Anna who decided to do so. The interview was based on her statements from the focus group and took about 60 minutes. It was again digitally recorded but not entirely transcribed. The analysis by the two authors was mainly done by discussion of the critical passages based on the theory sketched above.

Anna is an untypical case regarding the fact that her major subject is physics and also regarding her education biography. Anna dropped out of school before doing the Abitur (compares to UK's A-Levels) and went to schooling as chemical-technical assistant. She then decided to do her Abitur in evening classes. As a physics student, Anna attended the lectures on analysis in her first year, and those on linear algebra in the second year. (Mathematics majors usually attend them at the same time.) The focus group was conducted at the end of her 3rd semester; the interview was in her 4th semester. For us, her case was interesting since she reported mainly negative experiences, yet thought about switching major to mathematics, a seemingly contradicting behaviour. Other students also showed a discrepancy of interest in studying and interest in the subject. Anna's case, however, is a very clear one. Additionally, she spoke openly and reflected, although a self-report can only partly reveal her experience. We had first taped her talking in the more natural group and then went into detail in the individual interview. But still, we have methodological limitations some of which are inherent to any retrospective research method.

The transition experience

At school, Anna has always had good maths teachers and felt interested in this subject. For her, doing mathematics was mostly doing calculations. When Anna came to university with a friend, they both had decided to study together. After presentations of different courses, they agreed to study physics, although it wasn't Anna's first choice. It was more important for her to not start studying on her own. When she and her friend couldn't solve tasks no. 3 & 4 of their homework assignments in the first (!) week, they looked for someone she calls "private tutor" who solved the tasks for them. In the focus group she described herself as "desperate" since she "completely failed". One might wonder why she didn't see the two solved tasks as a success or give herself another try, especially since we know that neither the teachers expect the students to solve all tasks, nor many students do so. We hypothesize that her definition of success is adopted from school, where she usually solved a higher percentage, she always knew what to do, and copying homework is quite common. The tutor offered Anna and her friend to also explain the solutions, but in most times, they refused. The following interview passage illustrates her expectations and her experience with the homework:

Anna:	Well, we didn't understand the sheets at all. Well, yes, I think it was the first two tasks of the first sheet and there it stopped. And then I couldn't do it. Well I couldn't, I couldn't somehow handle the writing in mathematics, they have their own writing, e.g. that is sets, this and that, and I – I couldn't do this. I have never somehow worked on sets in school and thus it was difficult to read, what you want from the students. And therefore we were desperate and looked for a private tutor.
Interviewer:	Was it new, this feeling that there is something you can't do in mathematics, that it is so unfamiliar?
Anna:	Well yes, but I have – that is strange, but I have not really [related the sheets] to mathematics somehow – well I didn't really see them as mathematics. Well for me, that was something completely different. For me, mathematics had always meant calculating something, setting up something, and actually that's it. But not any strange proof. Maybe you have heard before a bit, that higher mathematics is something slightly different. But you have never really perceived what is really there.
Interviewer:	That's exciting, what kind of mathematics would you have expected, can you describe this?
Anna:	This is difficult now, because now I know what is to come. But if I – I do not know what I would have said if someone asked me when I was in school. (..) If someone had asked me, I don't know. Maybe – I think I would not have expected so many proofs. Well, calculations yes, also without using numbers, but simply just this general; but more

calculations are what I have thought. But what do calculations mean? Not silly replacement of variables, just setting up [equations], too. So, where does all this come from, e.g. we had proven in calculation methods [a course for physics students] why the volume of a sphere equals pi times r² – of a circle – and we have deduced it. And this is what I have thought, that we would do more things like this. But not, don't ask me, what we have done. But I have passed it!

For Anna, HM is very different from SM and she experiences many problems particularly in the theoretical parts of HM. Today, she still struggles with formalism and couldn't manage to adjust her beliefs on the nature of mathematics. ("How do they know 4-dimensional spaces exist?") She describes mathematics as "understanding, seeing, calculating". Although she failed at the two analysis exams and her exam on linear algebra (as well as some exams in physics) she still thinks she will manage her studies, albeit it will take her more time than usual. Anna's aims are dominated by achievement goals. In the interview, she often refers to exams, to the homework assignments and even to her CV. In contrast, she never talks about learning goals she would like to achieve, except for one statement on applications of mathematics (playing poker).

Basic needs

Anna experienced her first year as very stressful, and university didn't match her expectations. She couldn't solve the homework on her own and didn't pass the exam. Anyway, she managed to feel competent. Copying from her private tutor, she got the admittance for the exam (which is also valid the next years), what she calls a success. In addition, she learned some calculus techniques like integration by parts and feels competence in doing calculations (like she did in school). Concerning proof, she often didn't even understand the tasks, and showed no need to clarify this. Instead, she tries to fade-out or to dismiss such tasks (disvaluation of unaccessible tasks was also described in (Hannula, 2001)). She additionally excuses her problems by her education biography, often referring to her inferior prerequisites. Another explanation is the 'fact' that only gifted students can fully meet the requirements. Asked for feelings of success she reports that once she found a solution for a task in a book. This matches her goal orientation since it helps her receiving credits for her homework (but not achieving learning goals). When in the second semester she had success doing the homework with a colleague, she started solving the tasks without her private tutor. She reports she had learned how to deal with the symbols and now the tasks were more based on calculations. So, situational interest sometimes appeared when calculations were involved, at one point she even reported flow. Concerning autonomy, Anna felt forced to do the homework, but managed this by copying. Her own need for autonomy is strongly stressed:

Anna: "I don't surrender quickly, but if I don't understand it, then I won't do it. I won't sit down and do this for hours until I understand it. I won't do this for sure. I'm not proud of it, but yes…"

Unlike the other aspects, social relatedness is not a problem for her. She often mentions others (common decisions, work and experiences) and at no point reports feelings of unrelatedness. Anyway, the situation is unlikely to foster interest in HM.

Anna's interests

In both the interview and the group discussion, we couldn't find a statement of Anna that indicates interest in HM (such as positive emotions or valuations, see *Theoretical Foundations*), although we had explicitly asked for it. For instance, she couldn't name any topic of interest and also didn't want to learn the subject matter. This is consistent with our account of her SDT experience. However, rather than thinking about dropping out, in the focus group Anna had reported her thoughts about changing her major subject to mathematics, although her motives were unclear. In the interview, she said that she first wanted to see how physics is going, and since she has passed her last exam there, she sticks to physics, focussing on the more mathematical topics. We believe that her interest in studying mathematics has been preserved from the end of school. However, stressing this interest can also be an expression of an exit option from studying physics. Then it would be seen typical for a developing interest in studying physics (cf. Grotlüschen, 2010) and is an emphasis of autonomy. Concerning both study subjects, she reports that they are untypical for a woman but respected and important. From the SDT perspective, relatedness and perceived autonomy may be given, the latter restored by copying. The obvious lack of competence in mathematics needs not be necessarily an obstacle for Anna. First, she didn't experience it so much after she started copying and also when she repeated the 1st semester course. Second, she has the expectation that once she would start engaging more, she eventually would succeed: "I don't think that after years, when you do this, that you still don't understand it. I don't believe this."

The special case of proof

We chose the issue of proof because, although proof is the basic paradigm of university mathematics, Anna managed to study two years without acquiring a taste for proof. She starts with a special obstacle, since she sometimes struggles with language. Her parents presumably emigrated from Eastern Europe, which (besides her accent) sometimes causes phrasing problems. Again we take a passage from the transcript:

Anna:	[…] and I liked calculating everything nicely, writing down everything nicely, yes. – And still, well I still like calculating, and when we have the calculation tasks, but proof, maybe, you don't really see how to do these proofs yet. Maybe that's why it still is. But if I talk about it with someone or so, everybody says it: 'Oh I hope there is no proof [in the exam]' or so.
Interviewer	Yes ok. Is it the same with you?
Anna:	Yes. Proof is more difficult. Because, maybe you don't hm – have this learning effect for proof, because you have never had proven anything

	this in school. Well, we only have, the only thing we maybe proved is for example hm, oh how do you call it, hm this derivatives, well
Interviewer:	The derivation rules?
Anna:	Exactly, we have proven these derivation rules. However, that is also nearly calculating. You have to put in [variables].

Anna had never proven anything in school and she shows no meta-knowledge on doing mathematics. Thus, it is not easy for her to feel success or to act autonomously. At no point, she reveals a need for proof, which fits her belief system: If mathematics is about having the right formulae and doing the right calculations, proving doesn't help. Proof is also problematic regarding her goal orientation. Her performance goals are not compatible with understanding proof, since proof seems to be different every time. She doesn't notice a learning effect, except for simple calculation-based proofs. So she is missing, e.g., knowledge on what can be learnt by exercising proof and typical heuristics. One might even speculate about the question, if the repeated refusal of proof might grow into a habit.

On this basis, the requirements of the SDT can hardly be fulfilled. Anna feels incompetent in two ways: She doesn't know how to solve a given proof problem, and very often she even doesn't understand the problem itself (also strongly restricting autonomy)! This becomes clear e.g. when she joyfully emphasised, that in the exam she could understand all tasks. Meanwhile, she has learned to do proofs built on calculations, like checking subspace axioms, but not more. Even typical induction proofs seem unfamiliar to her. Concerning the need for autonomy, she doesn't mention any positive aspect, but some negative ones relating to the homework assignments. Social relatedness appears in avoiding proof only ("everyone says it: I hope there is no proof [in the exam]"). Sharing the same problems with proof, the students can exchange experiences. In summary, proof is a case where the lack of autonomy and competence is large and Anna neither has a good coping strategy, nor can construct a different view for herself. Consequently, she avoids proof whenever possible. Obviously, she has no interest in proving.

CONCLUSION

In this paper, we present an approach to investigate interest in Higher Mathematics (HM) using the interest theory by Krapp and Self Determination Theory (SDT). To this purpose we distinguish different objects of interest, namely School Mathematics (SM), Higher Mathematics (HM) and studying mathematics. Relating this approach to the work of (Daskalogianni & Simpson, 2002) revealed a new perspective of interest genesis at university. We then used empirical data to check plausibility of this new perspective. In a group discussion we could see that the participating students had different views on SM and HM and a problematic competence and autonomy experience. For an application of SDT we invited one student (Anna) for an interview to explore her subjective experience, beliefs and goals, and relate it to her articulated interests. Anna couldn't manage to feel competence or autonomy concerning proof within HM and didn't develop any interest in it. However, she could manage to keep her interest in studying alive by different strategies. Lack of perceived success on her homework assignments was compensated by a private tutor, which also restored autonomy. Lack of success in the exams was explained and excused by external factors. Besides situational interest, we see her individual interest in higher mathematics at the introjection stage, apart from identification with calculating, which form the only point in which she had success and thus feelings of competence.

The theoretical framework helped structuring the reports and identifying aspects which are presumably strongly connected to interest of different kinds. Her expectations and beliefs played a role in how she dealt with the new mathematics and her goal structure can help explaining her behaviour, both forming the basis for competence and autonomy experience. We could then observe that competence and autonomy were generated in different ways, sometimes leaving the promising ways behind. The conclusions are drawn from very few data, including problems of reliability of self-reports. Anyway, we believe our approach to be helpful to structure relevant aspects of interest development. It offered an explanation of contradictory interests in studying mathematics with respect to HM. In the on-going project, we want to apply the framework to a broader sample of first year students.

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SHIFTS IN LANGUAGE, CULTURE AND PARADIGM: THE SUPERVISION AND TEACHING OF GRADUATE STUDENTS IN MATHEMATICS EDUCATION

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A common case of a Masters or a doctoral student in mathematics education is that of a student doing so in a language other than their first, from a non-Western background and in a discipline other than that of their undergraduate studies. This student often needs a broadened understanding on how to read, converse, write and conduct research in largely unfamiliar ways. The intervention into the practices of post-graduate teaching and supervision that I describe here aims at fostering this broadened understanding and thus facilitating students' participation in the practices of the mathematics education research community. Here I exemplify the intervention through a brief discussion of an activity series designed to facilitate incoming students' engagement with the mathematics education research literature.

INTRODUCTION

In today's global and highly mobile educational community students arrive at their graduate studies often from different linguistic, cultural, pedagogical and scientific backgrounds. The case of an international student who embarks on study for a Masters or a doctorate in a language other than their first, from a non-Western background and in a discipline other than that of their undergraduate studies is quite common. This student needs a broadened understanding on how to read, converse, write and conduct research in academic environments that are unfamiliar in many ways.

The educational research literature has described this unfamiliarity as a key aspect of the post-graduate student learning experience; and the overcoming of this unfamiliarity as a key issue that the teaching and supervision of post-graduate students needs to address. The intervention into the teaching and learning practices of post-graduate teaching and supervision that I describe in this paper aims exactly at that: to facilitate students' gaining of aforementioned broadened understanding and thus facilitate their transition to post-graduate studies. In doing so I take cue from research in this area – some of which I summarise later in the paper – that calls for a reconceptualization of the Higher Education curriculum, pedagogy and assessment on this matter, and focus this small-scale trial, itself part of a plan for a larger study, on aspects of pedagogy.

Mathematics education, the discipline of my immediate expertise and the discipline within which the activity sets of the trial will be carried out, is a suitable Example Case discipline: mathematics education postgraduate students, especially those engaged in university-level mathematics education research, are likely to come from a background in mathematics. The shift from a Science to a Social Sciences milieu for these students is typically a very pronounced part of their transitional experience.

THE TRANSITION TO GRADUATE STUDIES: AIMS AND PRINCIPLES

Research into the challenges of the transition from undergraduate to postgraduate studies has been growing rapidly in recent years, often in connection with the move 'from elite to mass higher education' (Sharpham, 1993) or the 'widening participation' agenda that has been driving developments within Higher Education (HE) in several countries (e.g. http://www.hefce.ac.uk/whatwedo/wp/ in the UK).

Often this research includes direct, self-reporting studies of students' perceived needs and perceptions of effective supervisory practices (e.g. Egan, Stockley, Brouwer, Tripp and Stechyson, 2009; Pyhalto, Stubb and Lonka, 2009). Alongside a plethora of supervisory practice guides, much of this research tends to focus on generic issues. These include: interpersonal aspects of the relationship between supervisor and supervisee, accessibility and availability of the supervisor, academic compatibility, time management and expectations, etc. (e.g. Krauss and Ismail, 2010).

Findings concerning generic issues such as that 'especially international students and those in soft[1] disciplines, require a personal and holistic style of supervision' (Egan et al, 2009, p.337) are of utmost significance. However of at least equal gravity is the focus on practices that aim to foster skills and attitudes in postgraduate students which are *epistemologically* specific, namely specific to the discipline – in our case: mathematics education – they are coming into (Boaler, Ball and Even 2003). Attention on these is also part of a longer-term perspective on post-graduate studies as a stepping stone to a career in research (e.g. Shacham and Od-Cohen, 2009).

The project I describe in this paper aims to make a contribution in these respects (epistemologically specific, long-term) and builds on a relatively small body of work in this area – aptly summarised in (Boaler et al., 2003) and evident in several chapters in (Sierpinska and Kilpatrick, 1998; most explicitly in the chapter by Gione, p. 117 - 127). In what follows I outline the theoretical foundations on which the project is built.

Engaged pedagogy. As Pyhalto et al. (2009) identified there is 'an urgent need for more effective means of fostering PhD students' experience of active agency within scholarly communities' (p221). Much in the spirit of Gunzenhauser and Gerstl-Pepin (2006), the teaching and supervision practices trialled in this project reflect the prioritising of 'an engaged pedagogy, which represents a shift in emphasis from instrumental training in research methods to an approach in which students develop appreciation for complex possibilities' (p.319).

[1] The authors include in this term the social sciences, of which mathematics education is widely perceived to be one. I acknowledge that this assumption is not universal; in fact it is a culturally dependent assumption. For example, in continental Europe, chairs in didactics of mathematics are usually located in science faculties, often alongside those in applied or pure mathematics. However, regardless of whether mathematics education research is carried out by researchers whose affiliation is in a mathematics, education or another department, the epistemological differences between the two fields are profound. For extended accounts of these see, for example, (Sierpinska and Kilpatrick, 1998, Part VI, p. 445-548) and (Nardi, 2008, Chapter 8, p. 257-292).

Cultural sensitivity. Particularly for those whose background was shaped away from where much of the educational research dominating the publication venues was conducted in (e.g. graduate students of non-Western backgrounds), they, 'valued as knowing subjects, may enrich their investigations of educational problems and questions with epistemologies and theoretical perspectives that value their individual identities' (p.319) and inform their emergent research plans. Away from a 'dominant discourse' that 'appears to centre on what universities do to fit international students into their existing cultures' (Turner and Robson, 2008, p. 70), the project I describe in this paper aims to contribute to what the 2007 UK Higher Education Academy Report (Caruana and Spurling, p. 64) outlines as a much needed shift from merely 'awareness of difference' to 'valuing difference' and integrating this valuing into pedagogical practice in substantive ways – in other words the shift from 'symbolic' to 'transformative' internationalization (ibid. p. 126).

Independence, creativity and critical thinking. These are often described (e.g. Adler and Adler, 2005) as marks of the emerging membership to the scholarly community: decisions on what to focus on, the move from appropriating to creating knowledge, the growth of an epistemological perspective (for Adler and Adler's sociology students the 'sociological eye', p. 11); the flexibility of moving between immersion into the specificity of one's own research to contributing to abstract theory; and so many other features of what Baker and Pifer (2011) call 'transition to independence' (p. 5).

So far I have set out the foundations of the project as being built around the principles of: engaged pedagogy and participation; cultural sensitivity; and, independence, creativity and critical thinking. The spirit of the intervention is captured well by DeVita and Case (2003, p. 393) who outline the role of the HE teacher as 'helping students construct understandings that are progressively more mature and critical'. They thus propose 'the pursuit of didactic strategies aimed at facilitating processes of self-enquiry, critical reflection, mutual dialogue and questioning' that lead 'to a more participative and student-centred approach' in which students interact 'with the content and with each other' and are thus 'exposed to multiple perspectives and foster cultural understanding'.

Within mathematics education, working towards membership of the scholarly community often implies a rethinking of epistemological beliefs – as evident in the experiences of mathematics educators and university mathematicians engaging with collaborative research (see, for example, (Nardi, 2008, p. 257-292), including a review of literature on this matter). This is even more acutely true for those who arrive in mathematics education postgraduate study from a purely mathematical background. In a nutshell, the area where a shift of epistemological belief often emerges as necessary is towards what has been called in the literature (e.g. Boaler et al., ibid., particularly p. 497-516) a less absolutist, more contextually bound, more relativist and multiplist perspective on what constitutes knowledge (in mathematics education) and how it is constructed and shared.

METHODS AND SAMPLE OF PROJECT ACTIVITY

The intervention in the teaching and supervisory practices I describe in this paper is designed, implemented and evaluated, in collaboration with post-graduate students in my institution. I do so in consultation with the relevant literature and through drawing on personal and professional experience that I have accumulated over 20 years of my own post-graduate studies and post-graduate supervision and teaching.

Key to this intervention is also my on-going work with colleagues from my institution, whose own background (mathematics, international), similarly to mine, is a valuable resource to this initiative. Involving colleagues is crucial also in that the ways in which supervisors work with students is naturally filtered through their own interpretations of these activities – and, of how these activities can be tailored to address their students' specific needs. At the moment – for example, in the instances exemplified in this paper – the involvement of other supervisors is informal but a more systematic participation is envisaged for the larger study.

Furthermore, since the inception of the Research in Mathematics Education Group at UEA in 2003, the post-graduate student cohorts (on the Masters and doctoral programmes) have been steadily informing the formation of the practices and activities trialled in this project and exemplified in this paper. For example, of the 13 completed/current doctoral students of the Group, five come from a mathematics background (hold undergraduate degrees in mathematics) and 9 are non-UK students (EU: 2; non-EU: 7).

The intervention has been designed in the spirit of developmental research (e.g. Sierpinska and Kilpatrick, ibid., chapter by Gravemeijer: p. 277-295). Sets of activities are trialled in the course of the current academic year's post-graduate teaching and supervision. These aim to address key issues of the transition to post-graduate studies that I have observed as seminal over several years of experience. Realistically this small-scale intervention can only address some of these key issues. In subsequent phases of the project the list of issues to be addressed will be fine-tuned with further reading of the relevant literature and interviews with colleagues of analogous experience. I aim to carry out interviews with six HE teachers with substantial experience in post-graduate teaching and supervision (at UEA and elsewhere).

The activity sets address issues germane to the following three areas:

- *Engaging with Research Literature*
- *Forming the Conceptual/Theoretical Framework of a Research Project*
- *Choosing and Applying Data Analysis Methods*.

Data collected during the implementation and evaluation of the activities aim to:

- Describe and analyse the students' participation in the activities.
- Explore how subsequent versions of the activities can be amended to address students' needs more precisely and effectively.

The activity sets are trialled and evaluated with new cohorts of Masters and doctoral level students. These activities are fine-tuned versions of activities I already deploy – see example in Fig.1. I note that the treatment of issues germane to the clearly different needs of different groups of students (Masters and doctoral; British and international; mathematics and other backgrounds; with varying teaching or other professional experience) cannot be conflated into one single investigation. However, the profile of most participating students is such that a concurrent consideration of issues is often necessary, even potent. Any consideration of student data is alert to this variation of student profile and this variation of issues.

I trial these sets of activities during sessions of group and individual tutorials. In subsequent phases of the project the execution of the activities will be reported in field-notes produced by a collaborating doctoral student (not participating in the observed session) or me (drafted during the session and finalised immediately after).

Evaluation of the trialled activities takes place through evaluative questionnaires and interviews. The number of students who participate in the trials in 2012-13 is 6 (at either Masters or doctoral level). This participation is subsumed in the normal provision to the students. However, their written consent was obtained for using observations from the sessions, responses to the questionnaires and audio-recordings focused group discussions of their experiences of the intervention. Their participation in the evaluation phase was on a volunteering basis. Anonymity has been kept throughout, e.g. through the option to submit anonymised questionnaire responses and confidentiality secured through using pseudonyms in any quotations from the data.

An Example Activity: *Engaging with Mathematics Education Research Literature*

A post-graduate international student in mathematics education – or a student with a background in the *sciences* who arrives in the UK in order to complete post-graduate studies in the *social sciences* – is tasked with formidable challenges. Apart from carrying out their studies in a different language and learning the terminology of the field they are entering, this student faces novelty on several grounds. They would be required, for instance, to read the social sciences research literature that: is often *lengthier* than the research literature in the sciences; often uses a breadth of related, subtly *different but not equivalent terms* to describe similar phenomena; and, is typically rather more *open to multiple interpretations* than the bulk of scientific texts they are accustomed to.

This student is expected to identify, read, reflect upon, converse and write about this literature, often in a matter of months. In Figure 1 I sample some activities that I currently invite my Masters and doctoral students to participate in during the early months of their arrival. I then outline the empirical origins and rationale for each activity – in resonance with the observations on the type and scope of reading and writing that Boaler et al (ibid, particularly those on p.497-499 and p.512-513) highlight as pertinent in the transition to post-graduate studies in mathematics education.

The activities aim to facilitate incoming post-graduate mathematics education students' ***Engaging with Research Literature*** particularly in relation to:

- Searching: identifying relevant research literature
- Reading: critical reading of research literature
- Writing: reviewing research literature
- Conversing: presenting and discussing research literature

Of importance in the **outline of activities** below is to encourage students to draw upon the knowledge and experience they acquired in their own educational and cultural background and relate those to the reading of the novel research literature.

1. In an early session, students engage in a **discussion of** various ***types*** **of publications** (such as books, journal papers, reports, policy documents) and of their ***status*** **in research writing**. A significant part of this discussion is on the ways in which literature from the students' own educational and cultural context (often not published in English) relates to the (often English-dominated) literature that they are expected to engage with.
2. In the sessions that follow, the students are asked to prepare as follows:

   a) **Read** pre-specified texts, typically book chapters or journal papers.

   b) Produce/identify a piece of **writing** that illustrates how they relate their reading for the session with what they have read or experienced before. This can be a publication from their own cultural and educational context, or a short account of a mathematics teaching or learning experience that relates to the theme of the session.

   This is their ***Short Contribution I***.

   c) **Identify a journal paper** that matches the theme of the session from a particular journal, typically a leading journal in the field (such as *Educational Studies in Mathematics*).

   d) Write a short account of their chosen paper that provides

   - a summary of the paper,
   - their views on the paper, and,
   - how (if at all) the paper relates to their own research interests and plans.

   This is their ***Short Contribution II***.
3. In ensuing weeks the **range of sources** that the students are asked to draw on **broadens** (from one to several pre-specified journals, then non-pre-specified)
4. **Brief presentations of** Short *Contributions I* and *II* occur during the session.
5. Brief presentations are accompanied by **discussion** with the group.

Fig.1 **Example Activity Series on *Engaging with Research Literature***

Justifying Activities 1-5: ***Empirical origins and rationale***

1. **Discussion of *types* of publications and their *status* in research writing.**

The early sessions of the MA in Mathematics Education are partly dedicated to what we call *The world of mathematics education research*. In these, types of mathematics education publications are discussed in terms of intended readership and distinguished as *academic* (e.g. a paper in *Educational Studies in Mathematics;* a peer-reviewed, research-based monograph/edited book/book chapter; a *PME Research Report* etc.), *professional* (e.g. a paper in *Mathematics* Teaching, the official journal of UK's ATM, Association of Teachers of Mathematics) and *policy related* (e.g. a government-commissioned report such as the UK's *Smith Report* on post-14 mathematics of 2004). Students are encouraged to consider this distinction in terms of types of publication they are familiar with. Mathematics graduates appear to be more familiar with undergraduate mathematics textbooks and, to some extent, with publications in mathematics journals. Many overseas students typically put forward influential governmental reports as examples of what they perceive a publication in mathematics education to be. All along, the students are asked to offer counterpart information about analogous or equivalent activity from within their own backgrounds.

The discussion is interspersed with sharing information about key conferences and symposia in mathematics education, national and international, and a brief historical account all the way back to 1908 and the establishment of ICMI. Through this discussion the students are invited to perceive the launch of their mathematics education post-graduate studies as inauguration into the scholarly community of mathematics education. I note that these sessions are also attended by incoming doctoral students who are expected to already hold a Masters qualification in (Mathematics) Education. The experience of these students, typically small but non-negligible, operates as a helpful bridge in these early discussions.

2. **Reading pre-specified text and identify two related texts (insider, outsider)**

As we launch into the thematic sessions of the MA programme, preparation for each session tends to be highly regimented (2a) but in tandem with the expectations that:

- the reading is embedded into own prior readings and experiences (2b, 'insider');
- the reading will be enriched with further readings (2c, 'outsider'); and,
- a rationale will be put forward for the choice of this further reading (2d).

(2b) and (2d) require of students to produce a small piece of writing for each session. Intertwined with the students' inauguration into the world of mathematics education research, these exercises aim to foster the perception that writing is paramount; and, acquiring the skill to write with the rigour and sophistication expected at this level is feasible through constant and regular practice. This applies equally to the mathematics graduates on the course (who may not have written in this 'genre' for a long time, if at all) and the non-UK students (who are writing in a language other than their own).

An example

The module *Introduction to Research in Mathematics Education*, attended by MA in Mathematics Education students as well as Year 1 doctoral students includes five sessions on key theoretical constructs used in mathematics education research. Two sessions are on developmental / cognitive approaches; and, three are on sociocultural, discursive and anthropological approaches). In the first of the sessions dedicated to developmental / cognitive approaches in (2a) the students were expected to read two seminal papers: Richard Skemp's 1976 *Mathematics Teaching* article on *Instrumental and Relational Understanding* and David Tall and Shlomo Vinner's 1981 *ESM* paper on *Concept image - Concept definition*. In (2b) one student, a recent mathematics graduate (international, non-EU) wrote a short account in which she recollected her first encounter with the concept of limit. As part of (2c) she brought along a *PME Research Report* on a study that used the *Concept-image, concept definition* construct to explore students' understanding of limits in a university of her country. And, in (2d) she commented on the use of the construct in her paper of choice and related this to her own emerging research plans for her dissertation.

Within (2a) the students are asked to read two texts (Skemp's, Tall and Vinner's) that appeared at a time when mathematics education was at a turning point of its growth into an academic discipline – research was largely influenced by educational and cognitive psychology and PME was being founded. Within (2c) the students are asked to identify research texts, from that era or thereafter, that report research which deploys these theoretical constructs. They are thus encouraged to find out the *scope* and *impact* of these works in the field.

3. Broadening the range of sources

As the thematic sessions of the MA programme continue to unfold, the instructions for preparing for (2b) and (2d) gradually broaden and relax. In order to facilitate, and accelerate, the students' familiarisation with key publication venues in the field, initially they are asked to prepare for (2d)

- through searching for journal papers in *Educational Studies in Mathematics*;
- then, in a few more journals (*Journal for Research in Mathematics Education*, *Journal of Mathematical Behaviour*, *For the Learning of Mathematics*);
- then, a list of about ten international, peer-reviewed journals held in the UEA library. By the end of the module the list has opened up to include practically most peer-reviewed published work in mathematics education research.

The rationale for the **presentations within (4)** (for non-UK students it is perfectly acceptable that, at least to start with, this can consist of *reading out* to the group their *Short Contributions*) and the **discussion within (5)** is analogous. These activities aim to foster an understanding of how paramount these ways of engagement are and how regular participation in these practices can facilitate the acquisition of presentation and discussion skills.

A CASE OF EXPERIENCING ACTIVITY SET 1-5: SOPHIA

Sophia is a 2012 mathematics graduate, international student (EU) and a non-native speaker of English. Throughout the Masters module in question (see *Example* above) her participation was positive and enthusiastic and this is reflected in her written evaluation of the experience of Activity Set 1-5. Sophia comments on the gradation **of difficulty of the tasks** within the Set and acknowledges some **value in this gradation** ('the tasks gradually became more difficult, but we also became more familiar with the way of working on them…'). She recognizes that 'the freedom to choose a paper (more practical or looking at different aspects of the focus of the lecture) made the topic more clear'. Adding to this **clarity** was the **supplementary role of her peers' more substantial teaching experience**: '… we could see the views or the thoughts that the other classmates had regarding the topic and the practice of those theories, as they were more experienced teachers and had different background from me…'. Finally she encapsulates how the **network of reading, writing, presenting and discussing tasks assisted her gradual adoption of skills and practices** that are bound to be crucial in subsequent parts of her studies:

> '…coming from an environment where I didn't have to present anything in oral form, it was difficult for me to present a paper, even to express my thoughts on the readings, but with the way the tasks were designed, starting with reading something we wrote about the paper, and then slowly talking more about it in the group, and finally doing a presentations, made it easier'.

PROJECT PROSPECTS AND FURTHER WORK

As research in this area suggests (e.g. Pole, 1997; Boaler et al., ibid; Gione, ibid), there are issues in the training of graduate student supervisors and teachers that call for systematic investigation – what Boaler et al. identify as the search for an appropriate 'research curriculum' (p.518). As the bulk of my supervisory and teaching experience is in the discipline of research in mathematics education, this is the Example Case discipline of the intervention. I note however that the experiences and needs of students in this area are not untypical or substantially different to those of students in other areas, particularly those in transition from a science to a social sciences paradigm. I therefore see the potential of the project as transcending the disciplinary boundaries of mathematics education. I see this small-scale intervention as a precursor to a larger, longitudinal study that will include other institutions, involve a larger number of colleagues and extend the scope and range of the activity sets. The larger study will also allow the trial of the activities in several modification-and-improvement cycles.

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TRANSFORMATION OF STUDENTS' DISCOURSE ON THE THRESHOLD CONCEPT OF FUNCTION

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In this study of university students' discourse on the threshold concept of function, the aim was to examine what kind of changes of their mathematical discourse could be observed during a study year. Data was collected through interviews, questionnaires and observations during the students' first year of mathematics courses and was analyzed using the commognitive framework of Sfard. The result shows substantial differences in the students' discourse. The transformation of their discourse resulted in broader registers of words, mediators, narratives and routines. Students not transforming their discourse showed an unbalanced use of visual mediators and fewer connections between the interrelated features of the discourse.

THE THRESHOLD CONCEPT OF FUNCTION

For students who have recently entered university studies in mathematics the development of their mathematical conceptions seem to be a crucial issue (Thomas et al., 2012). Over the years, the meaning of mathematical understanding has shifted from focusing individual cognitive aspects to socio-cultural and situated perspective where the learning context and discursive elements are focused. Being aware of that the students do learn mathematics in a social setting, we have chosen to bring back focus on the relation between individual students and their understanding of mathematical concepts. More specifically, we are interested in novice university students' understanding of the mathematical concept of function. Limited understanding of the function concept has been shown to have an effect on the transition from secondary level to university (Thomas et al., 2012). Moreover, functions are used in all branches of mathematics and it is a concept that students encounter early during their school time. The first encounter with functions can be on a very informal and intuitive way but rather soon the students may work with functions as algebraic expressions and graphical representations. In university mathematics, students encounter a more formal definition. What distinguishes functions from other relations is that y in $y = f(x)$ is uniquely determined by x. The increasing abstraction and generality of functions may be one reason for that students' learning of the concept of function is troublesome (e.g. Artigue, Batanero, & Kent, 2007).

The notion *threshold concept* was introduced by Meyer and Land (2005) for initially troublesome concepts for which students' development of understanding of the concept involves a potential to transform the understanding not only of the concept, but also the understanding of the subject area where the concept is included. A threshold concept can be seen as a portal to a new and previously unreachable understanding. Meyer and Land characterized threshold concepts as initially troublesome, transformative, integrative and irreversible. To come to understand a threshold concept is troublesome for the students, it is a threshold to cross over, but then their understanding gets transformed. Understanding a threshold concept will bring a significant shift in students' perception of the subject. The transformation may be sudden but it often occurs over a long period. Integrating prior understandings is part of the transformation and the understanding is irreversible; when it has been transformed to a new one, it will not be forgotten or unlearned without a big effort. There are several threshold concepts in mathematics e.g. limit, function, derivative and integral (Pettersson, 2011). However, studies of threshold concepts in mathematics are rare and there is also a need for more general discussions about the transformation process of students' understanding of threshold concepts (Scheja & Pettersson, 2010). In this paper we focus the threshold concept of function and the transformation of students' discourse on this concept.

THE COMMOGNITIVE PERSPECTIVE

The notion of threshold concepts provides a theoretical perspective on the character of students' learning of specific mathematical concepts that play a particularly important role for students' access to university mathematics (Pettersson, 2011). However, to study students' development of their understanding of threshold concepts, we also need a theoretical framework that focuses students' learning of mathematics from a more general point of view. Threshold concepts are often presented by mathematicians or in the literature with a significantly different discourse than that of the students. Thus, to study students' learning of threshold concepts in terms of discursive development is relevant; and we have chosen to use Sfard's commognitive perspective to analyse the students' development of their understanding of functions. That means, we define learners' development of understanding as development of their discourse, and to study this we look for transformations of the students' discourses. In our analysis, we will focus on the characteristics that make a mathematical discourse distinct from other kinds of discourses; *mathematical words, visual mediators, narratives and routines* (Sfard, 2007). According to Sfard discursive development of individuals can be studied by "identifying transformations in each of the four discursive characteristics" (p. 575).

A mathematical discourse implies the use of *mathematical words*. There are many words that students almost only meet in a mathematical classroom context, but there are also colloquial words that get another meaning when used in a mathematical discourse. The concept of function is in itself a mathematical word and the very core of the notion of threshold concepts is that the meaning of the mathematical word function dramatically changes when a student crosses the threshold. *Visual mediators*, such as symbolic artefacts and manipulatives, can be used as a cursor on the objects of communication. Graphs are used as visual mediators for functions and can be used to show features and behaviours of different functions. The crossing of a threshold may be recognized by identifying new ways of using visual mediators that are related to functions. *Narratives* are descriptions or accounts of objects. It is any written or spoken text that is used within the discourse and can be subject to endorsement, i.e. narratives can be judged as true or false. In the interviews, the students make use of narratives to describe their conceptions of the function concept. *Routines* refer to repetitive patterns for actions and serve as important tools for the interlocutors to participate in any kind of mathematical discourse. Routines can be due to properties of mathematical objects. Concerning functions, a familiar routine from secondary level is to examine functions with tables of values and graphs.

We will use these four characteristics to analyse university students' gradual development of their mathematical discourse about mathematical functions. The research question is: What kind of changes of the students' mathematical discourse could be observed during the transformation of their understanding of the threshold concept of function?

METHOD

To capture the process of transformation of students' understanding of a threshold concept, a process that mostly is a development over a long period (Meyer & Land, 2005); the students have been observed during a study year. The students were student teachers at a Swedish university participating in mathematics courses. In total the study included 18 students, all the students that participated in the courses for prospective mathematics teachers in their second semester. These students also included a few students taking the courses a second time and one in-service teacher expanding her mathematical competence. In an earlier paper results connected to one student were reported (Pettersson, 2012). In the present paper four students, A, B, C and D, are presented. These four were chosen since they have participated in nearly all of the data collection occasions (only interview 3 is missing for C and interview 4 for A) and in that way have given a rich dataset. B and D were first year student teachers, A was an in-service teacher and C was taking the courses a second time. The four students participated in the same courses with the same lecturers.

During the first semester of their studies the students enrolled courses in general education and an introductory course in mathematics. The second semester the students participated in the courses 'Vectors and functions', 'History of mathematics', 'Geometry and combinatorics' and 'Calculus'. Data were collected through observations of lectures and tutorials in the two courses where functions are part of the syllabus. To get more specific data on the individual students' use of words, visual mediators, narratives and routines individual questionnaires and interviews were carried out. On three occasions questionnaires were distributed to the students in the tutorials. The two first questionnaires asked the students to explain what a function is and the third asked if given graphs (e.g. $y = 1/x$, the line $x = 2$, a discrete function) and algebraic expressions (e.g. $f(x) = x^2$ for $x \geq 0$; x for $x < 0$) represent functions. Individual interviews were conducted on four occasions with students who volunteered for interviews. The first three interviews used the student's answers in the questionnaire as a starting point. The fourth interview was conducted at the end of the next semester, five months after the third interview. During that semester the students were enrolled on courses in mathematics education, in probability and statistics, and general teacher education courses. The interviews were semi-structured and the duration was about 30 minutes. They were conducted on a time chosen by the student, audio recorded and later transcribed in full. All students were informed about the research and participated voluntarily. None of the authors were involved in the teaching or examinations. The students were informed that the data would not be presented to course leaders and examiners in a manner that would reveal individual identities.

In the pre-calculus course 'Vectors and functions' polynomial and exponential functions were presented. At the end of the course there was a discussion about the definition of function. The definition given in the compendium reads as follows: "A function is a mapping that for all numbers x in a specified set maps the number to another number which is called the value of x for the function and is noted $f(x)$" (authors' translation). In a lecture the students worked with an exercise, meant to practice reading of mathematical texts, where a more abstract definition of a function was given: "A function is a subset of the Cartesian product in which all elements x in the domain occur in exactly one pair (x, y)" (authors' translation). In the lecture this definition was discussed and the lecturer gave an example of a function with a domain consisting of five students and a codomain consisting of marks (A-F). The function was represented by the graph where marks are ordered to each of the students. The lecturer also pointed out that no rule or arithmetic formula was included in that example. In the course 'Calculus' a definition was given in the beginning of the book used: "A function is today understood as a rule or a process that in a well-defined and unique way remake (transform) some specified objects to new objects" (authors' translation). However, there was no discussion about the definition of the function concept in the lectures or tutorials. An interpretation of the absence of a formal definition and a discussion may be that this was expected as already known by the students. During the course, the derivative was used to analyse functions with respect to extreme values. These analyses together with studies of asymptotes were used to draw graphs.

The qualitative analysis of the questionnaires, interview transcripts and observation notes was focused on the students' ways of talking about their understanding of function. As mentioned above, according to Sfard (2007), the use of mathematical words, visual mediators, narratives and routines are characteristics of a mathematical discourse. To identify the transformation of the students' mathematical discourse, we read and re-read the data searching for instances where changes in the discourse regarding these characteristics could be recognized and made records on the development of each student's discourse on the concept of function.

RESULTS

The presentation of the results gives for each of the four students a review of the transforming of their discourse during the year.

Student A

Student A is a woman, about 35 years old. She is an in-service teacher studying math to get certified also as a mathematics teacher in secondary school. There was no fourth interview; at that time she was back to her work as a teacher. Student A used *mathematical words* already in the beginning of the semester. Her vocabulary was expanded during the semester and included correctly used formal words. In the first questionnaire she wrote that a function is "e.g. a variable a that depends on b" and gave an example: "The prize of a taxi journey depends on how far (s) I go and how long time (t) it takes, $p=as+bt$". We can notice the mathematical words 'variable' and 'depend' in her description of a function. As could be seen in the narratives below, during the semester she successively included more mathematical words.

She used formulas and graphs as *visual mediators*. The mediators seemed not to be crucial for her understanding, but are used in a convenient way. In the example above she used the formula for the prize as a visual mediator. In the second interview she spontaneously drew a graph with discrete domain, showing that she was comfortable with using graphs as illustrations. The *narratives* used in the beginning of the semester included everyday examples, such as presented above, but develop during the semester to formal expressions. In the middle of the semester she wrote in the questionnaire that for a function the key idea is "for every x there is a value y". She pointed out that the pairs (x, y) do not need to be connected and we can notice that the narrative at this time was more formal and included mathematical expressions such as 'for every'. In the interview she said that she did not remember the definition but "it was something with exactly one." During the semester the narratives she used changed from describing function as a rule to function as pairs. She told that her encounter with the abstract definition was important for this change. She talked about the change from functions as rules to accepting collections of points as functions, as an extension of her previous understanding. She used a narrative that is a reflection of the more abstract definition in the exercise earlier mentioned:

> Then I thought like that you have one rule that somehow determines what is going to happen, that was my relation, and now it feels like, no, there doesn't have to be any relation at all, we just have to say that they come in pairs.

During the semester student A developed a *routine* for deciding if a graph or a formula represents a function or not. In the third questionnaire she answered correctly on which of the graphs and algebraic formulas that are functions. In the interview she gave a definition of function: "For every value x there is a value y, or in other words, one value x can only give one value y, if you put in one x you will get one y, you can't put in one x and get two y". She said she used this as a routine to decide if the curves are functions. She also repeated what she said in the second interview, it does not need to be a rule; if one x gives one y then it is a function.

Summing up the transformation of the discourse of student A we can notice that she from the beginning included several mathematical words and that the vocabulary was extended during the semester. She used visual mediators in a convenient way. The narratives she used changed to more formal ones and she also developed a routine for deciding if a graph or a formula is a function or not.

Student B

Student B is a man, about 25 years old. Looking on the development of his discourse it could be said that he still, nine months after the first interview, was talking about functions nearly in the same way. During the whole study year he mainly used colloquial discourse and he included just a few *mathematical words*. In the first interview he, like student A, gave everyday examples as the prize of a taxi journey, but unlike her, student B did not use mathematical words in the examples. In the middle of the semester he used some mathematical words like ‘variable’ in his explanation of what a function is. At the end of the study year he said that a function is “like a connection, it is something that gives something, and maybe you could draw it graphically”, an example showing the absence of mathematical words.

Visual mediators, formulas and especially graphs, were important for student B all along the study year. In the first interview, he talked a lot about graphs and their interpretations. In the middle of the semester he said that he still mentally remains in the upper secondary school where, as he interpreted it, functions are regarded as graphs. In the end of the semester he talked about algebraic representations but still he also connected functions to graphs: “You feel a bit unsure, but if you can show something graphically then you feel that it is a function”. During the study year, the *narratives* did not include formal definitions. In the first questionnaire student B wrote that “from given data [a function] describe[s] a course of events depending on different factors”. In the middle of the semester traces from the more abstract definition, given in the exercise mentioned above, could be recognized but the content in that definition was not involved in the narratives. At that time he wrote:

> I regard functions as a relation of dependence where e.g. y depends on x. This can be shown graphically, but not necessarily. A function can express and maybe predict processes from given data.

In the narratives given at the end of the semester we once again found a trace of the more abstract definition, but still it was not made clear and established. Student B referred to the lecturer: “He said that even if you just have dots it could be a function”. A semester later, in the fourth interview, student B still could not give any formal definition.

Student B did not use any *routine* for deciding if a curve is a function. In the third questionnaire he answered wrongly that all the graphs were functions. He said in the interview that he did not have any routine for checking if a graph represented a function or not. He had not recognized, or could not make use of, that to each x there needs to be exactly one y. However, during the study year he developed routines for algebraic manipulations; he talked about routines of algebraic manipulations to get the possibility to “see it in the formula”.

The development of the discourse of student B included just a small expansion of use of mathematical words. Visual mediators were used frequently in the discourse during the whole study year. The narratives started with everyday examples, slightly changed but never included formal definitions. He developed routines for algebraic manipulations but did not use any routine for deciding if a graph or an algebraic formula represents a function or not.

Student C

Student C is a woman, about 30 years old. She was taking the courses this semester a second time because she did not pass the first time. In the beginning of the semester she was really happy about participating in the course with this, from her point of view, good lecturer. She did not take part in the third interview. At that time, in the end of the semester, she was really upset and frustrated. She had not passed the examinations and felt that the teachers did not help her. She said that she did not manage to talk about mathematics at all. However, five months later, at the end of the next semester, she by herself said that she wanted to take part in an interview. The passing of some examinations had given her strong self-confidence.

In the beginning of the year student C mostly used a colloquial discourse and she just included some simple *mathematical words*. The vocabulary slightly expanded during the year but mathematical words were partly used in a wrong way. *Visual mediators* were frequently used by her from the start. In the first interview the notion $f(x)$ was what she first mentioned when asked about functions. She also talked several times about the system of co-ordinates and curves, but she did not use the word graph. In the middle of the semester she did not mention curves or system of co-ordinates; neither did she use the notion $f(x)$. It seemed as if she had lost the connection to visual mediators without coming to a good use of mathematical words and more formal narratives. However, at the end of the study year she again frequently used visual mediators. In the last interview, as in the first, she talked about the notion $f(x)$, system of co-ordinates and curves, but at this time she also used the word graph.

The *narratives* were mostly connected to visual mediators and no formal definition was given. In the first questionnaire she wrote:

> A function describes a relationship between axes in a coordinate system and a line or a curve in the same co-ordinate system.

In the first interview she also talked about a function as a ratio between x and y. In the second questionnaire she wrote: “A function is an expression for conditions between values.” In the subsequent interview she could not mention any definition of function but remembered from the lecture: “...well, he said a very good definition a couple of lessons ago but I haven’t learnt it by heart so.” This is the only instance where a trace of the more abstract definition, given in the exercise mentioned above, could be found in her utterances. At the end of the study year she said that she now had matured in her understanding of mathematics and no longer was afraid of functions. But still she could not give any definition. She said that she did not grasp the key aspect. She never got any *routine* for checking if a curve represents a function or not. From the third questionnaire we can conclude that she, at this time, had not recognized that for an application to be a function it is necessary that for each x there is exactly one y. She answered wrongly that $x = 2$ is a function and did not give any answer on discontinuous graphs. The justifications given were in a colloquial discourse and some of them incorrect.

Summing up the development of discourse during the study year we notice that student C slightly expanded her vocabulary. She frequently used visual mediators in the beginning and at the end of the year. The narratives started in everyday examples and never developed to formal ones. She did not develop any routine to decide if a graph or an algebraic expression represents a function or not.

Student D

Student D is a woman, about 35 years old. Before she started her teacher education she had been working as an economist. In the beginning of the year student D correctly included several *mathematical words* and her vocabulary was expanded during the year. She did not frequently include v*isual mediators*, but when used it is in a convenient way. The *narratives* started with everyday examples and function as a rule and ended up with function as pairs without need for a rule. In the first interview she said that a function is a recipe. However, in her narratives she connected functions to everyday examples and did not give any algebraic examples. In the middle of the semester she talked about a new understanding of functions:

> …what's new in my understanding of functions is that there doesn't have to be a rule that defines this relationship; sometimes there's just a relationship between two… chosen things.

She continued with a reference to the lecture where she encountered the more abstract definition: "... it could be a bit arbitrary like in the example with the marks". She said this example was an eye opener for her, but also that she still could not use it by herself. "I think I'm still like in between two understandings of this and I suppose I'm trying to find a way to put them together." She also pointed out that the important characteristic of a function is that for every x there is one single y. At the end of the first semester she said that now it is not the functions that are the problem: "...but the functions, I think they've suddenly become, well I don't know, a table, a platform". At the end of the study year she again talked about her encounter with the more abstract definition and how it opened her eyes for a new understanding. Student D got a *routine* for deciding if a graph represents a function or not. In the third questionnaire she answered correctly on all the graphs. She was using a vertical test checking that it was just one y for each x.

The discourse of student D included several mathematical words from the beginning and the vocabulary was extended during the semester. Visual mediators were used in a convenient way. The transformation of the discourse included changing the narratives to more formal ones and developing a routine for deciding if a graph or a formula represents a function.

CONCLUSIONS AND DISCUSSION

The results show substantial differences in the transformation of the students' mathematical discourses. Student A and D developed their discourse on the threshold concept of function in several crucial ways. They expanded their use of mathematical words. They changed their narratives from everyday examples and function as a rule to function as pairs. The students' encounter with the more abstract definition seemed to be important for the transformation of their discourse. These students also got a routine for deciding if a curve represents a function. They used graphs and formulas in convenient ways and also connected routines and narratives to visual mediators. The other two students, B and C, did not transform their discourse in the same way. Through the whole study year they just used a few mathematical words. They frequently used visual mediators, predominantly curves in co-ordinate systems. The narratives were mainly connected to visual mediators and were given in colloquial vocabulary. No formal definition was included in the narratives and just traces of the more abstract definition could be found. These students did not have any routine for deciding if a curve represents a function or not.

The results show the complexity involved in developing the understanding of the threshold concept of function. For these four students it was important to expand their use of mathematical words. It seemed like the need of precision requires the use of mathematical words. Visual mediators are surely important for students' understandings, but it could be seen from the discursive development of these four students that it was not enough to base the narratives mainly in visual mediators. For two of the students, A and D, the encounter with the more abstract definition seemed to have had an important influence in the development towards a more formal level. When the students' view of a function as a rule developed to a view where functions are recognized as a set of pairs, they understood also discontinuous and discrete functions. It also helped these students to recognize the key idea that for each x there should be exactly one y.

Through this longitudinal study the complexity in coming to understand a threshold concept has been made visible. Since threshold concepts are crucial in students' development of their understanding of mathematics, it is important for university teachers to know how to support this transformation (Meyer & Land, 2005). An early development of solid conceptions may also make the transition from secondary level to university easier for the students (Thomas et al., 2012). The results of the study substantiate the importance of expanding the mathematical vocabulary and developing a balanced use of visual mediators and routines. Changing the discourse to include more formal narratives also supports the transforming of the understanding. Development related to all four features was needed for these four students to cross the threshold. Words, visual mediators, narratives and routines are interrelated features (Sfard, 2007) and the results indicate that students who developed connections between them transformed their understanding in a crucial way. To raise the level of awareness among university teachers in mathematics, on the impact of discursive development, can be regarded as a didactical implication of our study and a reasonable way to support the transformation of students' understanding of threshold concepts.

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VARIABILITY IN UNIVERSITY MATHMETICS TEACHING: A TALE OF TWO INSTRUCTORS

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This paper examines two lessons in an infinitesimal calculus course given by two instructors with different backgrounds and teaching agendas. The lessons were based on the same lesson plan but decisions the instructors made prior to class took the lessons in substantially different directions. Using Schoenfeld's framework for analysing decision-making processes we describe the resources, beliefs, attitudes and goals that guided and restricted the instructors. These findings provide an insight into the different agendas and considerations underlying the instructors' decisions and the subsequent course taken by their lessons. On the basis of this analysis we will reflect on the impact that background, teaching experience and orientations have on university mathematics teaching. This discussion is a contribution to the increased pedagogical awareness in university mathematics teaching.

INTRODUCTION

Students in university math courses are often divided into groups where the same curriculum is taught concurrently by different instructors. This is the case in a calculus course for first year math students at a leading university in Israel. Aside from the lectures, every week the students divide into small groups and attend a teacher assistant's (TA) lesson of their choice. These lessons are supposed to follow a lesson plan written according to the lecturers' needs and consequently the students are told they can attend any TA lesson because all the TAs teach more or less the same lesson. In this study we examine and compare two of these lessons given by two different TAs and based on the same lesson plan. These two lessons showed substantial differences that were in part a result of decisions each TA made while preparing for his lesson. These decisions are based on different interpretations of the lesson plan that led to the forming of different teaching goals and consequently to different adaptations of the lesson plan. The question this paper reflects upon is: *What led each of the TAs to interpret and implement the lesson plan in the way that he did.*

In recent years various members of the research community noted that there is a shortage of empirical research describing and analysing the practice of teaching mathematics at the university level. Speer et al. (2010) note that only little is known about what university math instructors do and think daily, in class and out, as they perform their teaching work. Weber (2004) states that most of the research that does exist in this topic consists of researchers' suggestions for improving pedagogy and that there are relatively few studies on how advanced mathematical courses are actually taught. In their survey, Speer et al. (2010) list (only) five empirical studies that analyze teaching practices at a sufficiently fine level of detail so that other instructors and researchers can inspect and learn from the instructional choices and reasoning described therein. One of these studies focuses on the beliefs of a novice instructor of college mathematics and their influence on his instructional practices, particularly on his in-the-moment decisions in class (Speer, 2008). In-the-moment decisions of instructors were also the focus of the case studies described in (Hannah, Stewart, & Thomas, 2011) and (Paterson, Thomas, & Taylor, 2011) where Schoenfeld's theory on decision making processes (Schoenfeld, 2011) was used to describe and analyse decision processes that took place during math lectures.

In this study we continue this line of employing Schoenfeld's theory and analysing the resources, orientations and goals (ROG) underlying instructors' decisions. The contribution of this paper is twofold. First, it examines decisions that were made prior to the first TA lessons in the course and that are thus unaffected by many factors which normally influence classroom in-the-moment decisions. Second, we have in front of us decisions made in very similar conditions by two TAs with very different backgrounds: TA1 is a math graduate student with little teaching experience, while TA2 is a very experienced calculus instructor with a PhD in mathematics, who is not engaged in active research. Thus in this paper we do not only describe the effect of the TAs' agendas on their decisions, we also reflect on the impact of the TAs' background and teaching experience on their stated and inferred resources, beliefs, attitudes and teaching goals and consequently on the way their lessons evolved.

Another focus of this paper is the substantial gap between the intended and the implemented curriculums in the TA lessons. While there are many studies about such gaps in the context of primary and secondary math education (e.g. Even & Kvatinsky, 2010), here this phenomenon is examined in the context of university math education.

We shall not discuss here the mathematics that each TA made available to his students, and the impact of the findings of this paper on students' learning. These two important aspects of the TA lessons are the subject of an ongoing follow up study.

THEORETICAL FRAMEWORK

To analyse the decisions made by the TAs we will employ Schoenfeld's theory on in-the-moment decision making processes. Schoenfeld (2011) asserts that what people do is a function of their resources (intellectual, material, and contextual), orientations (their beliefs, values, biases, etc.) and goals (the conscious or unconscious aims they are trying to achieve). According to Schoenfeld these categories are both necessary and sufficient for understanding decision making. Necessary in the sense that if any of these categories is left out of an analysis then the analyst runs the risk of missing a key factor in the decision making process, and sufficient in the sense that every root cause of decision making can be found within these categories. Thus, in order to understand the considerations underlying the TAs' decisions, we should aim to uncover the explicit and implicit orientations and goals that guided each TA as well as the resources they had at their disposal while preparing for class. We retain that the analysis done here does not make use of the full strength of Schoenfeld's theory. The theory addresses decision making processes and requires attending to the factors that shape the teacher's prioritizing and goal setting when potentially consequential unforeseen events arise. These factors are critical when dealing with in-the-moment decisions. However, the decisions we focus on are made prior to class, and are often planned, conscious and deliberate. We will therefore refer to the ROG categories as a framework that guides us where to look while analysing the TAs' decisions.

RESEARCH CONTEXT AND METHODS

This study focuses on the lessons of two TAs: TA1 is an exceptionally bright math graduate student. He has been teaching TA lessons for 4 years in various math courses and has taught calculus only once, 3 years prior to this study. TA2 has been the head TA of this course for more than a decade and he is a very experienced and a very popular calculus instructor. He has a PhD in mathematics but he has not been engaged in mathematical research for several years. A noteworthy fact is that TA1 is not only inexperienced as a calculus teacher but also as a calculus student since he taught himself calculus prior to his university studies and did not attend first year calculus courses as a student. The two lessons examined here were the first TA lessons in this course and revolved around the definition of the derivative that had been taught earlier in the lectures. The lesson plan for these lessons was prepared by a third TA according to the demands of the lecturers and under the supervision of TA2. The TAs in this course were expected to follow this plan but there was no supervision of its actual implementation during or after the TA lessons.

The author attended the lessons as a non-participant observer, audio recorded them and took notes. After each lesson the TAs reflected on their preparation for class. Each lesson was compared to the lesson plan and an initial analysis of the TAs' ROG was conducted by inferring from the TAs' actions in the classroom. Confirmation and expansion of this analysis was made in final interview with each of the TAs.

AN OVERVIEW OF THE TWO LESSONS

The lesson plan revolved around the following definition: *Suppose f is defined in a neighborhood of some point* x_0. *Then we will say that f has a derivative a at* x_0 *if the limit* $\lim_{x\to x_0}(f(x)-f(x_0))/(x-x_0)=a$. The plan had a theoretical part with several observations regarding the definition (e.g. differentiability implies continuity) and a practical part with three exercises, along with concise, formal, algebraic proofs:

A. *Find the derivative of the function* $f(x)=\sqrt{5x+1}$ *at* $x_0=3$.

B. *Show that the function* $f(x)=sign(x)\cdot x^2$ *is differentiable at every point and find its derivative at* $x_0=0$.

C. *Show that if f is differentiable at* x_0 *then* $\lim_{h\to 0}(f(x_0+h^2)-f(x_0))/3h=0$.

The plan did not specify time allocation or any teaching/learning goals.

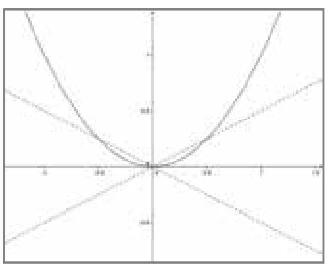
TA1's lesson

TA1 started his lesson by writing the definition of the derivative. Then TA1 stated that he would like to develop some intuition before getting into examples and departed from the lesson plan, initiating a teaching sequence that he designed. TA1 rewrote the definition in terms of epsilon and delta and with a few algebraic steps he obtained the following: *Suppose* $f'(x_0)=a$ *then for every* $\varepsilon>0$ *there exists* δ *such that* $0<|x-x_0|<\delta$ *implies* $(a-\varepsilon)(x-x_0)+f(x_0)<f(x)<(a+\varepsilon)(x-x_0)+f(x_0)$. TA1 then clarified to the students the geometric meaning of this statement:

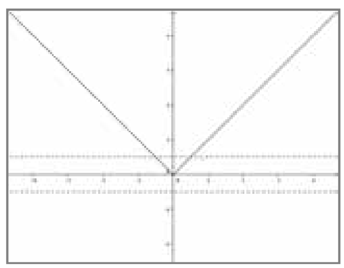
What we see is that if f is differentiable at x_0 *then for any given epsilon the (graph of) f is bounded, in some neighbourhood of* x_0*, from above and from below by two lines with slopes* $a+\varepsilon$ *and* $a-\varepsilon$ *crossing each other at the point* $f(x_0)$*. We are now ready to draw a picture.*

At this point TA1 drew **Drawing1** and used it as a visualisation aid while repeating his previous explanation. He then noted that differentiability implies continuity:

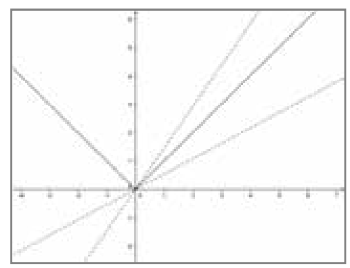
When I get closer to x_0 *(TA1's finger slides on the graph towards* $(x_0, f(x_0))$*) the graph gets closer and closer to the point* $f(x_0)$ *… I'm not giving here a full formal proof; I think that the fact that differentiability implies continuity is very clear from this drawing.*



Drawing1



Drawing2



Drawing3

After this explanation TA1 suggested using the geometric interpretation to develop an intuition as to why the absolute value function is continuous but not differentiable at zero and for this purpose he drew **Drawing2** and **Drawing3**. After discussing with the students what can be seen through these drawings TA1 emphasized that this type of argumentation does not qualify as a formal proof. He set aside the geometric interpretation, wrote on the board a rigorous algebraic proof and concluded:

This is a formal proof. Note however that it actually repeats what we saw in the drawings.

This remark ended the sequence TA1 initiated and he continued to exercise B in the lesson plan. Again he stated that he would like start by developing some intuition. TA1 drew the graph of the function and discussed the notion of "gluing" functions together (in this case x^2 and $-x^2$) and how in some cases (e.g. the absolute value function) gluing differentiable functions yields a non-differentiable function. While addressing the differentiability of the function he introduced the notion of a "local property of a function" and invested a great amount of time discussing why the derivative is indeed a local property. This discussion continued until the end of the lesson and TA1 did not address exercises A or C.

TA2's lesson

TA2's lesson remained close to the lesson plan. Like TA1, TA2 started his lesson by writing the definition of the derivative, while constantly making stops to clarify the mathematical terms he used. Then TA2 advanced quickly through the observations listed in the first part of the lesson plan using them as a springboard for exercise A:

The fact that differentiability implies continuity tells us that if f is differentiable at x_0 then the limit of $(f(x)-f(x_0))/(x-x_0)$ when $x \to x_0$ must be in the form 0/0. Pay attention! This means this limit cannot be found using the arithmetic rules of limits! Let's see an example.

At this point TA2 wrote exercise A on the board, turned to the students and asked:

Is it a legitimate question? Why? Why is f defined in some neighbourhood of 3? How do we start? How shall we find the derivative? Pay attention - Whenever we learn a new concept we always start solving problems by going straight to the definition. Only after some time we start developing theorems that we can use instead of the definition.

After this introduction, TA2 wrote the limit according to the definition and simplified it to obtain: $\lim_{x\to 3}(\sqrt{5x+1}-\sqrt{5\cdot 3+1})/(x-3)=\lim_{x\to 3}(\sqrt{5x+1}-4)/(x-3)$. He then commented:

The claim that the limit is of the form 0/0 relied only on the fact that f is continuous at x_0. Do you agree that f is continuous at 3? So this limit is of the form 0/0. How do we overcome this uncertainty? Recall what we did before in similar situations.

TA1 then showed that $(\sqrt{5x+1}-4)/(x-3)=5(x-3)/((x-3)(\sqrt{5x+1}+4))$, at which point he turned to the students and raised his voice:

A-ha! (Pointing at the two appearances of $(x-3)$ at the right-hand side) This is the culprit responsible for the fact that both the numerator and denominator tend to zero! We can cancel these expressions $(x-3)$ (which are not zero by definition) and now the limit becomes a simple exercise, since we can use the arithmetic laws of finite limits.

TA2 then continued to solve exercise B, again constantly reflecting on his actions. Unlike TA1, TA2 did not discuss the notion of the derivative as a local property. After exercise B TA2 temporarily left the course of the lesson plan telling the students that he would to review several concepts that were introduced in the lectures. After this review TA2 return to the plan, solved exercise C and concluded his lesson.

PREPARATIONS FOR CLASS

After their lessons the TAs described to the author their preparation for class. TA1 started by reading the lesson plan but after getting to the definition of the derivative he stopped and attempted to *try and make sense of the definition by playing with it and by drawing pictures*. This course of action has led to the geometric interpretation of the definition that TA1 considered interesting and valuable. He then decided to present this interpretation in class. TA1 explained that while reading the lesson plan he was asking in his mind questions like: *What do I find important and interesting in this topic? How do I approach this content? What would help me if I were in the place of my students?*

TA2's preparation was guided by different kind of questions. He explained that while reading the lesson plan he was thinking: *If I were a student, how would I see this topic? What might prove difficult for me? Where will I get confused?*

ANALYSIS

We now turn to describe and analyse some of the decisions the two TAs made.

Decisions

TA1 made on several occasions a conscious decision to deviate from lesson plan, for example when he initiated his original teaching sequence or when he ignored exercises A and C. By contrast, TA2 followed the lesson plan and restricted himself to content that had already been taught in the lectures, in what turned out to be a conscious decision on his part. A tacit decision of TA1 was to precede every rigorous proof with a visual intuition. TA2 did not develop any visual intuition in his lesson and presented to his students strictly rigorous proofs.

The interviews revealed several additional decisions. TA1 teaching was guided by a decision to concentrate on the theoretical aspects of the derivative on the expense of the applicative and procedural aspects. TA2 on the other hand made a decision to advance quickly through the theoretical part of the lesson plan and to concentrate on solving the exercises. TA1 saw in exercise B an opportunity for discussing the notion of the derivative as a local property. TA2 decided to use exercise A to emphasize to the students that the derivation is a property of a function at a point (rather than a property of a function as a whole). Below we expand on and analyse these decisions.

Resources

Both TAs had sound mathematical background and indubitable content knowledge and both relied on the lesson plan as a major resource. However, the TAs' reflections on their decisions uncovered additional resources that they had or lacked.

In his interview TA2 stated that in his opinion *the most significant element in preparation for class is the teaching experience*. Indeed, the rationale TA2 provided for his decisions indicated that his teaching experience played a central role as a resource of pedagogical content knowledge (Shulman, 1986). For example, knowledge of a potential learning obstacle has led TA2's to decide to highlight the notion that the derivative is a property of a function at a point (rather than of a function as a whole):

Students became accustomed over the years to differentiate functions as a whole. You start with $f(x)$ and differentiate it to obtain $f'(x)$. There is usually no mentioning of x_0 anywhere in the process! This approach often causes many difficulties for the students when they encounter functions which are not elementary. Thus it was very important for me to emphasize again and again that student are required to work with the definition, especially in the first exercise where most students would feel temptation to differentiate the function.

TA2's reliance on his experience as a resource was also evident in the rationale he provided for his decision to advance quickly to the second part of the lesson plan:

Keep in mind that in this particular lesson the students have just returned from a long semester break. For this reason I decided to get as quickly as possible to concrete examples, which I know are more effective than abstract theory for getting the students back on track.

In contrast, TA1's lessons were often based on what he himself perceived as interesting or challenging:

While preparing for the lesson I usually encounter something which makes me pause. It can be something I find interesting or maybe some difficulty I have with justifying a certain step in some argument. I often end up taking this something to class, thinking that if I found it interesting or if it got me confused then it would probably do the same to the students. I know it is naïve to think that I and the students would find the same things interesting or confusing, but the way I see it is an inevitable working assumption.

Later in the interview TA1 added that he cannot really say what students in his class need or want. These acknowledgments indicate that TA1's decisions were effected by his lack of pedagogical content knowledge.

Orientations

TA1's reflection on this decision uncovered also some of his beliefs and attitudes regarding how mathematics should be taught and communicated in the TA lessons:

The reason I opened the lesson with the geometric interpretation is that it is not standard. It is a good example of the things I'm drawn to. It is not just textual, it involves drawings and it requires a great amount of explanation. It is not something a student would get just by reading the lecture notes. I believe that the added value of the TA lessons does not lie in well-polished content but rather in the learning experience that it provides for the students.

An important factor in TA1's decisions is the image of the student he had in mind:

I wouldn't say that my teaching is directed at the brightest students but rather the hardest working students. I know there are students who prefer to learn in class just by writing everything that is on the board and then read their notes afterwards in the comfort of their homes or before the exam. I don't claim to provide very good service to these students.

Regarding his decision to focus on exercise B, TA1 expressed a dislike to exercises A and C, stating that he did not find them challenging or interesting:

In the lessons I prefer to focus on the more challenging exercises and let the students struggle with the simple exercises on their own. […] I don't think students benefit much from seeing me just performing algebraic manipulations in front of them. I prefer solutions which represent some mathematic depth, like in exercise B where you have the notion of gluing functions together and the notion of a local property of a function.

Like TA1, TA2's decisions were also shaped by the image he had of his students. In his interview TA2 noted that his decisions often reflect an evaluation of what these students could not manage or could not understand on their own:

I believe I have a responsibility towards all my students, not just those few who will later become mathematicians. The home assignments can be very difficult. I remember myself *struggling with them for hours. I think this is a good thing. However, from my experience this can be a breaking point, especially for the weaker students. I believe that it is the responsibility of the TAs to support these students and to make sure they are not left behind.*

Goals

TA1 reflection on the teaching sequence he designed uncovered an implicit goal**:**

I wanted the students to leave class with some intuition. The students probably heard in the lectures that the derivative is related to the slope of a tangent line to the graph. This is a nice intuition but there is a big gap between a tangent line, which is something you can draw or visualize, and the formal definition which express the derivative in terms of epsilon and delta. What I tried to do in this lesson is to narrow this gap.

Another goal can be inferred from TA1's actions in class: To model for his students the behaviour of a professional mathematician. This goal was confirmed in the interview, when TA1 reflected on the role and the goals of the TA lessons in general:

I think these lessons should develop the active aspects of learning. How do you approach a task? How do you start a proof? Often before I prove theorems in class I do some preparation. I tell the students that we should first gain some intuition, try and see the big picture of the proof. Same with definitions … As mathematicians, we constantly take actions to gain intuition and one of the goals of these lessons is help students learn how to do that.

An implicit goal, which seems to have had great impact on TA1's decisions, became evident when TA1 reflected on how the TA lessons should complement the lectures:

The lectures are restricted to content which is complete, precise and true. There is an unwritten law that the students can prepare to the exams by reading the lectures notes. I believe that this law does not apply on the TA lessons and that I am obligated to provide the students with a meaningful added value. I want the students to have a different interaction with the content, a mathematical experience that cannot factor through a notebook.

TA2 also addressed the role and goals of TA lessons and their relation to the lectures:

Ideally every new mathematical concept or term introduced in the lectures would be accompanied by several examples. This is not the case and thus one of the main goals of the TA lessons, as I see it, is to review the content taught in the lectures and to give the students an opportunity to get a firm hold on it. […] There are many things that students should hear more than once, preferably from different teachers with different perspectives and different terminologies. You cannot start to imagine how many times I have heard this "ahh" sound of understanding in class after repeating something that was already said in the lectures.

We note other explicit goals of TA2 already mentioned above: Improving students' capability to solve tasks, getting the students back on track after the long semester break, and emphasizing that formally the derivative is a property of a function at a point. The actions of TA2 in this lesson as well as other lessons suggest another goal which guided TA2: To model for the students the behaviour of an idealized student rather than the behaviour of an experienced mathematician. TA2 confirmed this goal.

SUMMARY AND DISCUSSION

There is a well-known myth stating that university math teaching is just a matter of accumulated experience, clear presentation skills and sound content knowledge. One may also speculate that within the community of professional mathematicians working side by side there cannot be much variability with respect to their resources, orientation and teaching goals. In fact, this assumption seems to be institutionalized by telling the students that all the TA lessons are roughly the same. This study not only challenges these assumptions but also highlights and analyses the differences between two instructors, both with sound mathematical background and indubitable teaching proficiency, implementing the same lesson plan. Using Schoenfeld's ROG framework we showed how different beliefs and attitudes, different goals and reliance on different resources resulted in two substantially different lessons.

A more specific contribution of this study refers to the ways in which the TAs' pedagogical content knowledge, or lack of it, affected the lessons. TA2 stated that his teaching experience was his most significant resource in preparation for lessons, highlighting the specific difficulties he expected from his students and ways of addressing them. For example, TA2's experience suggested that students will be better off with concrete examples and led him to decide to advance quickly through the theoretical part of the lesson plan. Similarly, TA2's decision to focus on the notion that the derivative is a property of a function at a point was his way to address what he considered a common mistake of students, and a potential learning obstacle. By contrast, TA1's lack of calculus teaching experience forced him to choose goals according to what he found interesting or difficult, led by his own introspective processes and by his aesthetics and values as a researcher trying to make sense of new concepts and ideas, designing teaching sequences accordingly. TA1 acknowledges that his perspective on the content may not necessarily coincide with the students' viewpoint or needs, but he sees it as his best possible approximation.

Paterson et al. (2011) suggested that lecturers who are research mathematicians bring different, at times conflicting, orientations into play. It is thus interesting to observe and compare the orientations of the two TAs, whose backgrounds represent these two identities. TA1 is first and foremost a researcher, and believes that it is in the best interests of his students that he will display before them the tools of the trade of a mathematician at work. At the same time TA2, who is first and foremost an instructor, believes that it is in the best interests of his students that he will model the behaviour of an idealized student and attend to their potential learning obstacles. TA1 directed his teaching at the mathematically oriented students, even though he acknowledged that by doing that he may not be providing good services to many of his students. TA2, on the other hands, felt a strong commitment to all his students and *not just those few who will later become mathematicians*, but he does so at the cost of not fully attending the needs of the brighter students in his class. While acknowledging these two identities as a potential source for conflicts regarding university math instruction, it is important to note that these two identities may also combine in fruitful ways. In this sense, it would be interesting to study a collaboration between TA1 and TA2, or any other two instructors representing their distinct identities, and examine how and when do their different orientations conflict or synergize.

An interesting by-product of this study was the way it influenced the TAs themselves to reconsider their beliefs and attitudes, simply by having a chance to talk with an outside observer about their teaching. TA1 noted that after reflecting on his lesson and discussing it with the author, he has decided to routinely ask questions and initiate dialogues during class and thus obtain a better understanding of his students' viewpoint on the content. TA2 expressed his worries on whether his teaching has strayed too far in favor of the weaker students at the expense of the rest of the class and especially the brighter students. These outcomes suggest that the approach taken in this study, where instructors reflect on decisions they and other colleagues make under similar conditions, may be used in development programs to enhance teaching.

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CONCEPTUAL UNDERSTANDING IN LINEAR ALGEBRA - ANALYSIS OF MATHEMATICS STUDENTS' MENTAL STRUCTURES OF THE CONCEPT 'BASIS' -

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Abstract. *In this paper I present the analysis of students' thinking processes and the reconstruction of their activated mental structure when they are dealing with a task. The analysing tool is based on Peirce's semiotics. The data is taken from clinical interviews, in which mathematics students were given tasks dealing with the concept of basis. The aim is to understand students' thinking processes and to reconstruct their activated mental structures.*

AIM OF THE PAPER

The theory of linear algebra is characterised by its high degree of interconnections between concepts. The concept of basis, for example, is connected to the concept of linear independence and the concept of a spanning set. Both are linked as defining conditions to basis (of course equivalent definitions that focus on one of these features are possible, too). These concepts refer to other concepts – linear combination and span – so that one can with good reason state that basis is a concept of higher order. Understanding the concept of basis implies that one has a high ability to connect ideas. As a complex theory linear algebra deals with concepts that have a wide significance. Furthermore several equivalences connect conceptual ideas.

The research question of this paper is as follows: Which individual mental structures of the concept basis can be reconstructed when students solve a particular task concerning this topic? When working on the challenging investigation task students will apply concepts they have learned before and in some cases students will broaden their conceptual ideas. Skemp (1973) explains that a mental structure is the "major instrument (…) for solving new problems, and for acquiring new knowledge" (p. 43). Before a person's mental structure can be reconstructed empirically, the thinking process should be analysed. In this paper, I describe the theoretical tool for analysing students' thinking processes and apply the tool via examples to data of students who are working on a task. Then, the analysed thinking processes are condensed in a net by focusing on the individual's mental structure of the concept of basis. Before doing so, I give a literature review of previous studies dealing with students' difficulties in advanced mathematics.

STUDENTS' DIFFICULTIES IN LINEAR ALGEBRA

Several studies are dealing with students' difficulties of the conceptual nature of linear algebra. Sierpinska (2000) states that concepts are not used precisely and are not well-conceived. Moreover, students struggle to interpret signs in definitions and to use them for the construction of contextual mental structures. Several researchers of the MAA-Notes (Carlson et al., 1997) report that their students are able to reproduce procedures in known contexts successfully, but that they often fail to understand the meaning behind the procedures. Stewart and Thomas (2010) tie their study in with this aspect by trying out a framework for teaching the concept of basis (and other concepts) that emphasises the embodied aspect of basis. Their aim was to help students enrich their understanding. “The results of this study show that a number of the students tended to prefer to work procedurally“ (p. 186). Maracci (2008) shows that students conceived of linear combinations more as processes than as objects (in terms of Sfard’s process-object duality). Britton and Henderson (2009) described students’ difficulties in dealing with the concept of closure. Students struggled with applying the formal concept of a vector space to the algebraic notation in a task. Furthermore, many of the difficulties that students experience in linear algebra are presented in detail in Dorier (2000). These studies all highlight that students struggle with the conceptual nature of the complex theory of linear algebra.

MENTAL STRUCTURES AND UNDERSTANDING

Several authors describe levels of learners’ construction of concepts (hierarchical or otherwise) (e.g. Winter, 1983; Harel, 1997). All have in common that it is essential to connect single ideas for understanding a concept. This corresponds to psychological views (e.g. Skemp, 1973; Sweller, 2006). According to Skemp (1973), pairing single ideas with concepts by connecting them results in a construction of a new idea, which he calls a “relation” (p. 37). A “transformation” (p. 37) is applied to an idea and describes a function of the idea. The entire process of connecting and transforming ideas results in a construction of a complex mental structure. The mental structure includes ideas of concepts and theories. Concerning a concrete thinking process, only a part of the mental structure will be activated. A mental structure is known as “schema” (p. 39). Skemp (1973) defines understanding something as being able to assimilate it into an adequately mental schema. Understanding is subjective. I assume that an individual’s mental structure has to be actively constructed by the individual. Information given to him or her in the form of external representations needs to be constructed by the individual. “The study of structures themselves is an important part of mathematics; and the study of the ways in which they are built up, and function, is at the very core of the psychology of learning mathematics” (p. 39).

THEORETICAL TOOL – ASPECTS OF PEIRCE’S SEMIOTICS

The triadic sign model of Charles Sanders Peirce (1931-1935) is taken as a foundation to analyse thinking processes and mental structures by using an interpretative approach. The triadic sign model has been applied successfully as a tool for analysing processes by some researchers in mathematics education (e.g. Hoffmann, 2003; Presmeg, 2006; Schreiber, 2012; Bikner-Ahsbahs, 2006).

The triadic sign model of Peirce

Peirce describes the triadic sign model in the following way:

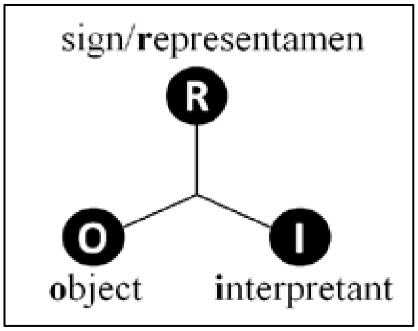

Figure 1: Triad of Peirce

"A sign, or representamen, is something which stands to somebody for something in some respect (…). It addresses somebody, that is, creates in the mind of that person an equivalent sign or perhaps a more developed sign. That sign which it creates I call the interpretant of the first sign. The sign stands for something, its object. It stands for this object not in all respects, but in reference to a sort of idea, which I have sometimes called the ground of representamen" (Peirce, CP 2.228).

According to Peirce, a sign is an external representation that is visible, hearable, or perceptible in some way. Peirce's sign has a wide range of use. It can be, for example, a word, a task or any material thing. The interpretant describes an internal representation that is induced by the sign and can become explicit. Concerning the term object, Peirce differentiates between the 'immediate' and the 'dynamic object' (Hoffmann, 2005). The immediate object is to be understood as the object to which the interpretant of the sign refers. Thus, the immediate object describes a special view towards the sign. "Immediate objects are neither obvious nor visible, but they can be reconstructed (as hypotheses) and verified through further data" (Bikner-Ahsbahs, 2006, p. 163). Unlike the immediate object, the dynamic object would emerge after all possible interpretations of its sign. These interpretations do not depend on individual views. Thus, the dynamic object is kind of a limit-object.

The frame as a foundation of Peirce's triad

Hoffmann (2005) describes the difference between a person working on a familiar sign and the same person working on a sign yet new to him. While interpreting a new sign, the focus is on the sign itself as problematic. While working on a familiar sign, persons "conflate them with the phenomenon itself" (Roth and Bowen, 2003, p. 468 as cited in Hoffmann, 2005, p. 35), as they possess "experience" (Hoffmann, 2005, p. 39) with the sign.

Hoffmann developed further the aspect of "ground of representamen" mentioned by Peirce (see quotation of Peirce above). He used the term "the general" (2005, p. 42, translated by the author) instead and explains that the general includes mentally or physically given concepts, theories, perceptions, habits, and skills of the interpreting person. The general is the foundation of Peirce's triadic relation. Schreiber renamed the general as "frame" (Schreiber, 2012, p. 7). It influences individual construction of interpretants because "each individual creates interpretants against the background of his or her own subjective interpretation experiences and under a specific perspective" (Schreiber, 2012, p. 7). The interpretants depend on the individual's activated frame. It is in this sense that Skemp's approach of mental structures can be linked to Peirce's theory. Mentally represented ideas of concepts and theories are included in the mental structure and belong to the frame in the triadic relation. The frame can be analysed by interpreting the students' interpretants.

The chaining of single triads

The strength of Peirce's triadic model lies in describing processes in which new ideas are constructed. This construction is an ongoing process in which an interpretant of a sign becomes explicit and serves in turn as a sign of a further triad (Peirce, CP 2.303). Presmeg (2006) called this process "chaining" (p. 169). While Presmeg refers to linear processes of chaining, Schreiber reconstructed processes that are not always linear. In the processes that I have reconstructed, the interpretants often do not only refer to the previous sign, but to groups of triads that previously occurred.

METHODOLOGICAL TOOL

Sample and data collection

The sample of the whole study consists of 15 mathematics students, whereby in this paper I only look at two mathematics student, Peter and Mike, to exemplify the analysis. Peter and Mike took part in the lectures and tutorials of the linear algebra course in 2010/11, but had to take part again in 2011/12 (their third semester) because they failed the final examination. They participated voluntarily in an additional workshop (four sessions, main topics: vector spaces, basis).

The data was collected about four weeks after the final examination in 2012. This time was chosen because I was interested in the concepts Peter and Mike were able to retain for a longer period of time. In a face to face situation Peter was given a task in written form (fig. 2). He had unlimited time to work on this task by himself. So, this was an individual and active process of construction in which Peter shows on which conceptual aspects he focuses. The interviewer did not influence this part. After working on the task Peter was asked to explain his procedure to the interviewer. This was a kind of retrospective think aloud. Thereby, Peter reflects on his approaches and develops further ideas. In the rest of the interview Peter goes on solving the task. The interviewer's role is to animate the students to verbalise their thoughts. The intention is not to arrive at a solution as quickly as possible. Thus, a semi-structured clinical interview was used to obtain data. The same procedure was carried out with the student Mike. The written survey part of Peter and Mike took about ten minutes; the interviews took each about 20 minutes. The whole session was videotaped. The same procedure was undertaken with the students working on two other tasks concerning the concept of basis in other domains and representations, but this is not taken into account in this paper. Peter and Mike are chosen from the whole sample because their solving processes of only one task show different kinds of connections in the mental structures, and because their ways of solving offer a comparison.

A mathematical task and possible approaches

A task given to the students is shown in figure 2.

$\mathbf{U} = \{(■(x@y@z)) \in \mathbb{R}^3 \mid 2\mathbf{x} = z\} \subset \mathbb{R}^3$ is given.
a) Declare a basis of U.
b) Describe in detail: How did you go on?

Figure 2: Mathematical task given to the students

In the following, approaches of ways of solving the task are briefly pointed out. An approach to solve the task is to choose a vector in U and complete it to a basis by focusing on linear independence. Another approach can focus on linear combination by separating $(\mathbf{x}, \mathbf{y}, 2\mathbf{x})$ according to variables. A starting point could also be to recognise the dimension of U and figure out the number of basis vectors. This could be done by noticing that the equation $2x=z$ geometrically describes a plane in IR^3 or by noticing that the condition $2x=z$ reduced IR^3 to dimension two. These are just examples or ideas; of course combinations of the ideas provide adequate approaches as well.

The task is particularly convenient for the reconstruction of mental structures because it complies with the following aspects: The task challenges the students to apply their knowledge conceptually, but also just deals with familiar aspects. The notation of the vector space U is typical and was used in lectures, tutorials, and in the workshop. The task offers various ways of solving. Several connections among ideas of concept can be applied. This is why every student should be able to get access to work on the task at all. There is not just one special idea that needs to be remembered. Moreover, a solution is not available by applying memorised calculations, but refers to conceptual ideas. To put it in a nutshell, the task is rich and simple at the same time.

DATA ANALYSIS AND RESULTS

In a first step, the students' thinking processes were analysed by using triads. Chains of triads arise from the whole analysis. They structure the students' ideas. The presented analysis consists of parts removed from the chains of triads. These examples show the reconstruction of conceptual aspects in a laudable fashion. In a second step, the chains become condensed. This process results in the reconstruction of the activated mental structures of the students, which are illustrated in a net. The focus is on conceptual understanding of the concept of basis.

In the original data, all vectors were written as column vectors. In the following I will use row vectors instead of column vectors because it saves space. In the following, the triads are consecutively numbered. T stands for triad, S for sign, I for interpretant and O for object.

The case of Peter

The first part of Peter's thinking process is analysed in table 1. The task is the sign (S1) in the first triad (see tab. 1). It causes the interpretant I1. The immediate object in triad 1 (see O1) is the aspect that basis vectors need to be characteristic of the subspace U. When Peter starts working on the task, he concentrates on the condition $2\mathbf{x} = \mathbf{z}$, which is given in U. He uses it when creating (1,1,2) and (2,0,4) as possible basis vectors.

T	S	I	O
1	task (see fig. 2)	Peter: "In U the condition $2x = z$ is given, and if I choose 1 for x then z must be 2 (writes: (1,1,2)). The same with the second vector (writes (2, ,4))."	- $2x = z$ is important for finding a basis. - basis vectors need to fulfill $2x = z$
2	↙	Peter: (Completes (2, ,4) to (2,0,4)) "Basis vectors need to be linear independent. This is why I choose one here and zero there (points at second components in both created vectors)."	- linear independence of a set of basis vectors is an essential condition for a basis - effect of 0-component concerning linear independence
3	↙	Peter: "Because we are in the $\mathbb{R}^3$ (in this task) I need three basis vectors"	to be in the three dimensional space ⇒ number of basis vectors is three
4	↙	Peter: "The third must be linear independent, too." Peter adds (1,2,2) to his set of basis vectors and immediately says: "This one is wrong because I can span it with the others." He eliminates (1,2,2) from the set of basis vectors.	- The set consisting of (1,1,2); (2,0,4) and (1,2,2) is not a basis. (1,2,2) is not an element of Peter's basis. - 'not linear independent' is used as being able to generate vectors from each other
5	↙	Peter: "I can span (1,2,2) with the other two vectors. If I multiply (1,1,2) by two, divide (2,0,4) by two and subtract this from this, I get this one." (points at vectors)	generate vectors from each other by combining them linearly $\begin{pmatrix}1\\2\\2\end{pmatrix} = 2\cdot\begin{pmatrix}1\\1\\2\end{pmatrix} - \begin{pmatrix}2\\0\\4\end{pmatrix}:2$

Table 1: Example of reconstructed triads from Peter's thinking process

The interpretant I1 serves as a new sign (I1=S2), which is a part of the next triad. This sign S2 serves to create a next idea, the interpretant I2. The reconstruction of the immediate object O2 is based on I2. Peter focuses the feature of linear independence of a set of basis vectors. In I2 Peter gives an indication of a part of his concept of linear independence. He applies the concept of linear independence by using the effect of a vector component that is zero. In the following interpretant I3, Peter assumes (wrongly) the number of basis vectors to be three. He refers to one of the symbols $\mathbb{R}^3$ written in the task (which one is not clear at this point) and concludes that he needs three basis vectors. It could be possible that he associates the dimension like "to be in the three dimensional space means to have the dimension three", and then concludes the number of basis vectors. The next interpretant is I4. Peter uses the 'linear independence'-feature again, but refers to another facet than in I2: The possibility to generate a basis vector from the others. I5 makes clear that generating vectors from others means to combine them linearly. This brings him to reconsider his preliminary set of basis vectors.

In the next part I will (because of the limited space) describe how Peter overcomes the crucial point concerning the call for three basis vectors in a summary. Peter adds another third vector and recognises again that the set of basis is not linear independent because a linear combination of the vectors is possible (T6). Then, he gets the idea that two basis vectors will suffice (T7). He establishes his idea by referring to different aspects: He reinterprets the subsigns $\in \mathbb{R}^3$ in the task, which mean that the vector is in $\mathbf{R}^3$ and then associates that U could be a plane for example, and that only two basis vectors are necessary to span planes (which are vector spaces) (T8). Thereby, he looks at U as an entity, for example a plane. Peter adds that a basis consisting of three vectors would always span the $\mathbf{R}^3$ (T9). He convinces himself that his basis consists of (1,1,2) and (2,0,4) by checking if he can span every vector of U with it. By combining them linearly, he focuses on relations among their components and on the structure among the elements of U (T10). Figure 3 shows Peter's activated mental structure when he is solving the task in figure 2. The focus is on the concept of basis because the task explicitly deals with this concept.

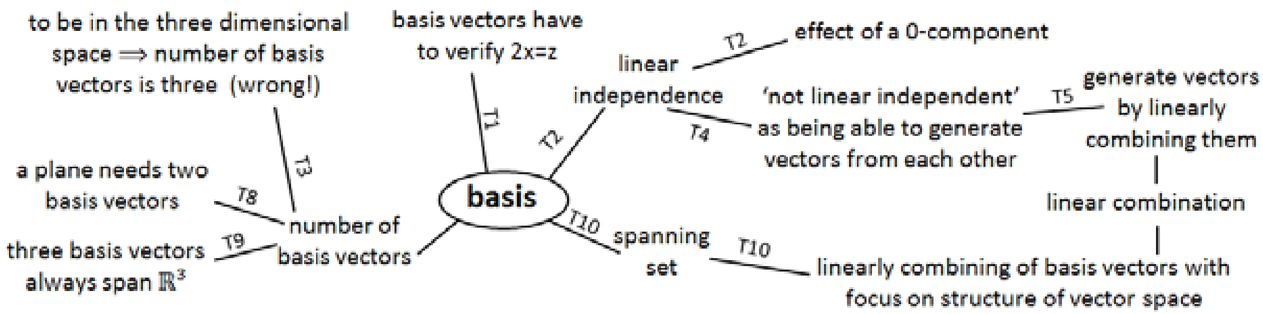

Figure 3: Reconstruction of Peter's activated mental structure

The inscriptions written at the connection lines declare the number of the triads from which the aspects result. Thus, the net offers a look at the order of the problem solving process and the conceptual priorities of Peter. In short, Peter focuses on linear independence, he argues by using aspects of the number of basis vectors, too, and convinces himself by referring to a basis as a spanning set. The connection lines based on linear independence (fig. 3) elucidate in which ways linear independence is applied in the context of the task. Peter refers to the concept of linear combination when he uses the feature of linear independence and the feature of spanning set. Thus, he uses the concept of linear combination flexibly to argue.

The case of Mike

In the following, I summarize the first part of Mike's thinking process (because of the limited space) and describe the second part in more detail by using triads. After reading the task, Mike mentions that a basis is a linear independent spanning set. In his work, he focuses on the part of the linear independence. He applies a relation between determinant and linear independence which he takes down as follows: "$\det(A) \neq 0 \Rightarrow$ linear independent" (T1). This relation implies A to be a *nxn*-matrix. In the context of Mike's idea A is a 3x3-matrix consisting of three column vectors. By choosing the vectors he carefully considers that *2x=z* is fulfilled (T2). Mike checks if a set of three vectors, which he assumes to be a possible basis of U, is linear independent. Mike applies the relation to four tries of vector sets consisting of (1,0,2), (0,1,0), and a third varying vector (T3, T4, T5, T6). His focus is on the carrying out of the procedure. Mike does not find three linear independent vectors (of course). He does not conclude a reason why he will never find three linear independent vectors, but he says that two basis vectors, namely (1,0,2) and (0,1,0), will suffice. The interviewer asks Mike to check if the two vectors build a basis (S8). Then Mike continues with I8 (see tab. 2).

T	S	I	O
8	…	Mike: The function of basis vectors is „to span U. (…) So, I have to prove if my vectors are a spanning set."	a basis needs to be a spanning set
9		Mike: "I don't know." (He writes: $\text{span}\left(\begin{pmatrix}1\\0\\2\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right)$). "This is the span. I don't know what can be spanned with it. That's beyond me."	- concretion of spanning set idea to the context by using span-notation - notation is not filled with meaning
10		Interviewer: "Linear combination, do you remember this?"	idea of linear combination is offered
12		Mike: "I heard about it. I think I should remember it, but I can't write anything down."	linear combination cannot be used in this context

Table 2: Example of reconstructed triads from Mike's thinking process

Mike refers to the 'spanning set'-feature of basis. In I9 Mike paraphrases I8 by using the symbolic notation of span. Moreover, he concretises his idea by inserting his own assumed basis vectors into the span notation. Mike cannot go on. It seems that the notation is not filled with meaning to him in this context. The interviewer offers the idea of linear combination in I10. I11 shows that Mike cannot associate ideas with this keyword.

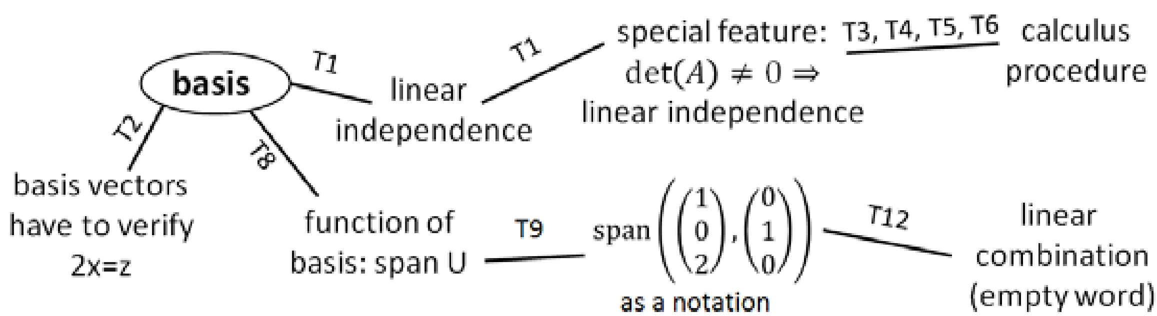

Figure 4: Reconstruction of Mike's activated mental structure concerning basis

Figure 4 shows Mike's activated mental structure in his process of solving the task. As well as Peter, Mike focuses on linear independence at first, but he concentrates his attention on another aspect, namely the calculation of the determinant to conclude linear independence. This is a special feature with a limited applicability. Concerning linear independence Peter refers to features that can be generalised like the aspect of zero components and the aspect of generating vectors from each other by using linear combination. This is why his activated connections are more substantial than the ones in Mike's reconstructed mental structure. Later, Mike addresses the defining feature of a basis being a spanning set, just as Peter, too. However, Mike has problems in applying it. He cannot remember linear combination and cannot justify his set of vectors to be a basis of U.

FINAL REMARKS

In this paper I present my analysis based on the semiotic theory of Peirce. The analysing tool allows describing processes of problem solving in a structured and detailed way. Moreover the order in which conceptual aspects are activated is indicated. The processes are described by chaining triads. The chains serve as a foundation for the reconstruction of students' activated mental structures that are represented in a net.

Outlook: This procedure is carried out with 15 students and three tasks each that are dealing with the concept basis. I am interested in describing and understanding the students' thinking processes in detail. Furthermore, I plan to make up types of activated mental structures explicit. By comparing the mental structures, it will be interesting to have a look at the differences between the mental structures of successful students and those who are less successful.

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APPROACHES TO LEARNING MATHEMATICS - DIFFERENCES BETWEEN BEGINNING AND EXPERIENCED UNIVERSITY STUDENTS

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In this paper, we report an on-going quantitative study of students' transition from school to university mathematics. The study aims at examining differences between beginners and experienced students' approaches to learning mathematics. Students were given questionnaires in the beginning and at the end of their first year at university. The results were summarized with descriptive and inferential statistics. The results show that beginners rely heavily on the teacher, while experienced students re-orient themselves from the teacher to other kinds of mathematical resources, for example peers and Internet based resources.

INTRODUCTION

In this paper, we report the first part of an on-going quantitative study of mathematics students in transition between secondary and tertiary level. We focus their experiences of mathematics studies in secondary school and after one semester of university studies. In particular, we are interested in the changes that take place during the students' first courses in mathematics at the university. The aim of this study is to examine students' approaches to learning at the beginning of their mathematics studies at the university and after completing their first mathematics courses when they are more experienced as university students in mathematics. After a short report of previous research that is related to our study, we give an account for the conceptual framework that has been used in our study. The results from descriptive and inferential statistics are discussed in the final section of the paper.

Previous research

Despite extensive efforts to make the transitional phase from school to university mathematics easier, entering higher studies in mathematics still seems to cause problems for many students (Gueudet, 2008). Entering university studies in mathematics put new demands on novice students' adaptability, both to a partly new mathematical content and to a new learning environment (Wood, 2001). Previous experiences of mathematics education from secondary level can have a crucial impact on the transition, for example if the pace of study increases significantly or if the relation to the teacher changes (de Guzmán et al., 1998). This may call for changes of students' approaches to learning and view on knowledge (Perry, 1970), which seem to be a crucial step for students to successfully undertake the transition into university mathematics (Stadler, 2009). Attempts have been made to develop quantitative measuring instruments to find correlations between students' view of knowledge and their learning approaches (Kemper & Leung, 1998), as well as between students' perception of changes when learning mathematics when entering university and their mathematics dispositions (Pampaka, Williams & Hutcheson, 2011). Also, the relation between affective variables, such as students' beliefs and self-conception, and students' success in the transitional phase from school to university mathematics have been examined, where a correlation between students' beliefs about mathematics and their approaches to learning has been shown (Liston & O'Donoghue, 2009).

In this study, we will focus on differences between beginners and more experienced mathematics students. Novice mathematics students at university have various experiences from mathematics teaching and learning from secondary level, which in turn influence their approaches to learning mathematics (Stadler, Bengmark, Thunberg & Winberg, 2012). Moreover, to have been subject to mathematics teaching is crucial for students' transformation from novice to expert mathematical problem solvers and their approaches to work mathematically (Schoenfeld, 1982). Consequently, we assume that students' approaches to learning are likely to change during the transition into university studies. The aim of our study is to examine students' approaches to learning mathematics with a focus on differences between beginners and experienced mathematics students at the university.

CONCEPTUAL FRAMEWORK

In this section, we present the concepts that we have chosen to use in our study to examine *students' approaches to learning mathematics*. In a qualitative study of students' transition from secondary to university mathematics, Stadler (2009) identified three concepts that describe students' approaches to learning mathematics.

The *mathematical learning objects* category refers to the students' view of the overall purpose of learning mathematics. It captures students' interpretation of what mathematics is and what learning mathematics is all about. Students' subject specific beliefs about learning and knowledge have been shown to strongly relate to their reasoning ability in the subject (Winberg & Berg, 2007). Perry's scheme (1970) describes stages of college students' intellectual and ethical development. It is regarded as a continuum between two views of knowledge and learning; from an absolute and transferable knowledge, which can be either right or wrong, to a relativistic view of knowledge, which is contextually dependent and where the students take responsibility for their learning. If university teachers teach mathematics in a way that corresponds to a higher level in the scheme, students can regard the teacher less useful (Stadler, 2009). Thus, from a student perspective, the transition brings about a need to re-orientate towards new and modified mathematical learning objects compared to secondary school.

Mathematical resources are objects and phenomena that students use to learn mathematics. Some examples are the textbook, the teacher, the peers, the students' pre-knowledge in mathematics and their logical thinking skills. For novice students in mathematics, modifications of the use of mathematical resources are an essential part of the transition (Stadler, 2009).

Students' actions as learners are closely related to their goals and aims of learning mathematics, are contextually dependent and may vary over time. Differences between mathematics teaching in secondary school and university call for new ways of approaching learning and actions to learn. A distinction can be made between independent and dependent actions as learners, where the former indicates that the students undertake actions they chose by themselves, whereas the latter means that even though students have intentions, they are not always able to undertake those actions by themselves that they find necessary to accomplish their intentions.

Choosing these concepts to examine students' approaches to learning is a methodological approach that combines qualitative and quantitative research methods. In the qualitative study, crucial aspects of learning mathematics for students in transition were identified. The quantitative study advances these positions by examining which aspects are more crucial than others, to what extent and for whom. For example, the qualitative study shows that the use of partly new mathematical resources and also the use of familiar mathematical resources in slightly new ways are important for mathematics students in transition (Stadler, 2009). The quantitative study gives supplementary information about to what extent new and other mathematical resources are used and by which category of students.

METHOD

We have developed a research instrument with two questionnaires consisting of 13 query themes (Table 1). The choice of themes was based on the conceptual framework of students' approaches to learning and their beliefs, motivation and self-concept regarding mathematics, as presented previously in the paper. The first questionnaire focused on the students' previous experiences of studying mathematics at secondary level and their expectations from mathematics studies at tertiary level. In the second questionnaire, almost the same questions were asked, but now with focus on their mathematics studies at university, after approximately one semester of mathematics courses.

In our study, we have chosen not to pose explicit questions about the students' mathematical knowledge. Instead, we have collected data about their previous grades from secondary level and study results on their initial mathematics courses at the university. To examine which *mathematical learning objects* the students are focusing on, we have examined students' orientations towards mathematics and the learning of mathematics (Stadler, 2009). Students' preferences about *mathematical resources*, known to be crucial for the transition, were captured by questions about their evaluation of different mathematical resourses and behaviour in relation to these mathematical resources. Students' actions as learners involve the two other categories and are operationalized in questions about what the students actually do when they aim at different mathematical learning objects and use different mathematical resources.

Questionnaire 1	Questionnaire 2
1. Entry requirements. 2. Lesson activities in upper secondary school. 3. Valuation of lesson activities in upper secondary school. 4. Help-seeking behaviour during mathematics lessons in upper secondary school. 5. Valuation of homework activities outside school. 6. Help-seeking behaviour during homework in upper secondary school. 7. Valuation of resources for the learning of mathematics. 8. Valuation of the mathematics teacher's actions 9. Valuation of working with peers. 10. Valuation of the textbook. 11. Orientations towards mathematics and the learning of mathematics. 12. Expectations concerning forthcoming studies of mathematics at university. 13. Expected requirements to succeed with mathematics studies at university.	1. Entry requirements. 2. Lesson activities at university. 3. Valuation of lesson activities at university. 4. Help-seeking behaviour during mathematics lessons at university. 5. Valuation of homework activities outside university. 6. Help-seeking behaviour during homework at university. 7. Valuation of resources for the learning of mathematics. 8. Valuation of the mathematics teacher's actions. 9. Valuation of working with peers. 10. Valuation of the textbook. 11. Orientations towards mathematics and the learning of mathematics. 12. Experiences of mathematics studies at university. 13. Experienced requirements to succeed with mathematics studies at university.

Table 1: Query themes in the questionnaires

Questions 3 to 13 were formulated as Likert scale questions with a five-step rating scale. For example, the initial questions about beliefs and attitudes towards mathematics and the learning of mathematics were formulated as follows:

11. Here are some questions about your views of mathematics and learning of mathematics.

	Strongly Disagree				Strongly Agree
a) It´s easy for me to learn mathematics.	1	2	3	4	5
b) I can solve most exercises by myself.	1	2	3	4	5
⋮					
w) I learn new concepts by solving exercises.	1	2	3	4	5

The participating students came from different universities and various study programmes. However, in this report, we have chosen not do discriminate between specific groups of students (see Stadler et al., 2012). All students were chosen according to availability. In total, 146 students answered the first questionnaire while 134 students answered the second questionnaire. Both questionnaires were distributed to the same categories of students. The first questionnaire was handed out during mathematics lectures and in the first two weeks of the first semester of the study programme. The second questionnaire was handed out when the students had studied two university mathematics courses. Each questionnaire took 15-25 minutes to answer.

The quantitative data have been analysed using two methods. Firstly, we have used *descriptive statistics* to summarize data in order to describe the main features of the participating students and to be able to compare the results from the first and second questionnaire. Secondly, *inferential statistics* (discriminant analysis with PLS, PLS-DA, a regression extension of Principal Component Analysis) was used to describe the relative importance of questionnaire items for discriminating between the beginner and experienced student group, that is, to investigate the distinguishing features of these two groups of students. The data from the first questionnaire has been separately analysed and reported in a previous paper (Stadler et al., 2012).

RESULTS AND ANALYSIS

With the descriptive statistics, we focus on changes of students' characteristics as learners of mathematics. For each Likert scale question we have calculated the mean value for beginners and experienced students and used a two-sided t-test to investigate possible significant differences between the two groups of students. In the first questionnaire the students' previous experiences of mathematics studies at secondary level were examined (Stadler et al., 2012). According to the students, a typical mathematics lesson begun with the teacher giving a short introductory lecture, which lasted for 10-15 minutes. In the rest of the lesson, the students worked with textbook exercises. In the questionnaires, the beginner and the experienced students were asked about the importance of the following lesson activities for their learning of mathematics at secondary and university level respectively (Table 2).

	Beginner	Experienced	95% CI for difference	p
Lectures	4.24	3.99	[0.01; 0.49]	.04
Demonstration		4.36	[-0.34; 0.09]	.26
Individual work with exercises	4.28	3.99	[0.05; 0.53]	.02
Individual help from teacher	3.92	3.07	[0.55; 1.16]	.00
Exercises with peers	3.37	4.02	[-0.96; -0.35]	.00
Discussions with peers	3.20	4.13	[-1.21; -0.65]	.00
Internet based resources	0.94	2.56	[-1.93; -1.30]	.00

Table 2: Evaluation of lesson activities, query theme 3.

The beginners value two kinds of lesson activities as the most important, namely lectures and demonstrations from the teachers, and work with exercises. The focus seems to be on working with exercises with the support of a teacher. The experienced students give higher ranking to peers, while the main contribution from the lectures are demonstrations of how to solve exercises. The shift from relying on the teachers to the peers is in accordance with Stadler's findings that the students in transition are forced to an increased independency and autonomy (2009). Another difference between beginners and experienced students is the use of Internet based resources.

The students' valuation of the importance of various mathematical resources is shown in Table 3.

	Beginner	Experienced	95% CI for difference	p
Teacher	4.29	4.16	[-0.11; 0.36]	.29
Peers	3.74	4.23	[-0.73; -0.26]	.00
Textbook	4.05	4.22	[-0.38; 0.04]	.11
Previous tests	3.20	4.25	[-1.29; -0.82]	.00
Book of formulas	3.60	3.04	[0.28; 0.84]	.00
Calculator	3.43	2.14	[1.04; 1.52]	.00
Computer for calculations	1.33	2.42	[-1.32; -0.87]	.00
Internet based resources	1.34	2.56	[-1.45; -0.98]	.00

Table 3: Evaluation of mathematical resources, query theme 7.

The beginners' evaluation of mathematical resources are in tune with the mathematics education that they have experienced at secondary level; the teacher gives a short introduction, which is followed by individual work with textbook exercises where the students are allowed to use the book of formulae and graphic calculators as resources. The appraisal of mathematical resources changes in favour for previous examinations, peers, computers and Internet based resources when the students become more experienced. According to Stadler (2009) the teacher as a mathematical resource changes at the university. Instead of giving instructions of how to solve exercises, the focus is on general mathematical ideas. To novice students, the information from the teacher becomes less useful. However, examples, previous tests and the peers can still provide information and instructions of how to solve exercises.

The students' beliefs about what will be or are the most important things to do to succeed with their mathematics studies at the university are shown in Table 4.

	Beginner	Experienced	95% CI for difference	p
Attend all lectures	4.32	3.72	[0.37; 0.84]	.00
Study outside school	4.29	3.55	[0.50; 0.97]	.00
Do as the teacher tells you to	4.24	3.61	[0.40; 0.86]	.00
Study theory in the textbook	4.23	3.92	[0.09; 0.52]	.01
Solve many exercises	4.18	3.90	[0.05; 0.51]	.02
Attend all tutorials	4.14	3.86	[0.05; 0.53]	.02
Get help from the teacher	4.05	3.12	[0.67; 1.9]	.00
Get help from the peers	4.03	4.02	[-0.21; 0.23]	.93

Table 4: Valuation of study activities, query theme 13.

The experienced students value the importance of almost all study activities lower than the beginners. Our interpretation of these results is that as a beginner, the students are highly motivated, but do not know what to expect. It is difficult for the students to discern which study activities may be important. When the students become familiar with the university, they acquire a more relaxed approach to their studies. Noteworthy is the decreasing importance of the teacher for the students' success in their studies and, in contrast to their expectations as beginners, the experienced students consider getting help from their peers as the single most important factor.

The students' actions as learners can be categorized as dependent or independent (Stadler, 2009). This is a crucial aspect of the transition because the students are to a greater extent forced to manage their studies on their own, relying more on their ability to read and learn and to use peers and resources on the Internet. Table 5 shows the differences between beginners' and experienced students' help seeking behaviour.

	Beginner	Experienced	95% CI for difference	p
Teacher	3.60	3.02	[0.35; 0.80]	.00
Peers	3.67	4.04	[-0.57; -0.16]	.00
Reading theory	3.61	3.81	[-0.42; 0.03]	.10
Solve examples	3.79	4.18	[-0.58; -0.19]	.00
Study notes from lectures	3.02	3.34	[-0.61; -0.02]	.04
Other peers' solutions	2.63	2.99	[-0.61; -0.12]	.00
Using the Internet	1.43	2.72	[-1.54; -1.04]	.00
Skip the exercise	2.33	2.56	[-0.43; -0.02]	.03

Table 5: Help seeking behaviour, query theme 4 and 6.

While students' help seeking from the teacher decreases, peers and the Internet are two interactive mathematical resources that increase in importance. The textbook, other notes and solutions of exercises are also available mathematical resources that can be used anywhere and anytime. Also, the intensity in the interaction between the teacher and the students usually decreases at the university (Stadler, 2009).

Unlike descriptive statistics, inferential statistics can be used to find correlation patterns between variables, and their relative importance for characterization of similarities and differences between groups of students. We have performed a PLS discriminant analysis, a regression extension of Principal Component Analysis, to further discern differences between beginners and experienced students. A model with three significant components was produced, describing 80% (R2) and predicting 72% (Q2) of the variation in group belongingness, using 22% of the total variation in the questionnaire items. However, as the first component was able to predict 65% of the variation in group belongingness and the second and last component 13% and 9%, respectively, only the first two components are presented in the loading plot (Figure 1). Items that are important for describing and predicting student group belonging are located far from the origin of the figure. This importance is also represented by the item's VIP (variable importance for prediction) value; i.e., the sum of each item's loadings on the components, weighted by the proportion of group belonging that is predicted by the respective component. This value is given for each of the items discussed below. Items that are close to each other in Figure 1 are positively correlated with each other, while items located on opposite sides of a line through the origin are negatively correlated. In the same fashion, items that are close to the experienced group in Figure 1, in particular in the horizontal direction, describe features that are typical for the experienced students and a-typical for the beginner students. The opposite is true for items that are close to the beginners group.

The interpretation of the loading plot and a VIP analysis revealed the most important features that discerned experienced students from beginners. Experienced students perceived a lower usefulness of calculator (item 7l; VIP 2.4), an increased value of internet-based resources to ask questions and find answers (7j; 2.2) and actual use of internet-based resources to seek information to support learning and problem solving in school (4g; 2.2, 3h; 2.2) as a calculating tool or task bank (7i; 2.1, 5e; 1.8). Internet based resources was also perceived valuable as general support (5f; 2.1) or problem solving assistance (6g; 2.0) when studying math outside school. However, the computer as a calculator was not considered as important for study success (13i; 1.3).

Furthermore, experienced students, to a greater extent than beginners, view peers as an important resource, for joint problem solving (9b; 2.1), discussion of theory and concepts (9e; 1.9) and question asking (9d; 1.9, 6b; 1.6) outside school. Getting help from the teacher (13l; 1.6, 3b; 1.3), as well as preparing well before going to the lectures (13a; 1.6), daily work with the course after school, doing what the teacher tells them (13e; 1.3), and attending all lectures (13b; 1.2) were considered less important for study success by experienced students than beginners.

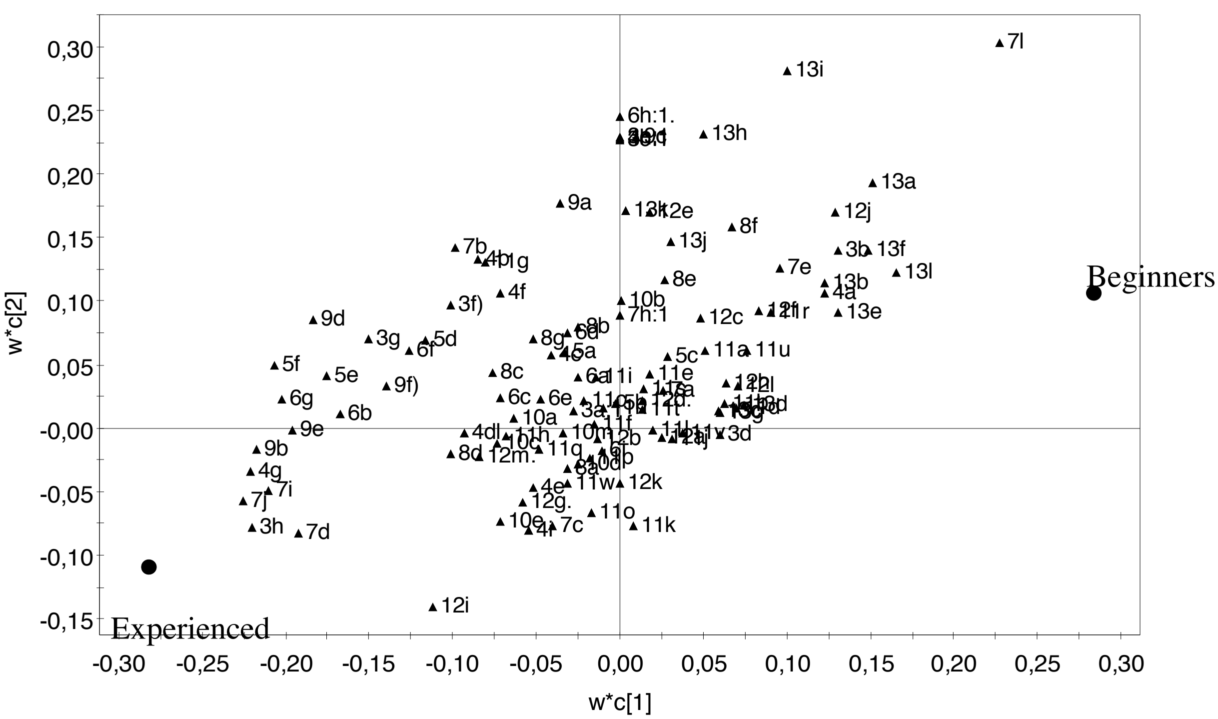

Figure 1: Loading plot from the PLS-DA, showing the relative importance of the questionnaire items to describe the distinguishing features of beginners and experienced students respectively.

The validation of the model was made through analysis of Hotellings T2 range, DmodX (distance to model), Response permutation testing, and Observation risk analysis. One student was outside the Hotelling 99% confidence interval. On deletion of this student, no substantial differences in the model occurred and it was decided the student should be included in the final model. No DModX outlier was detected. R^2 and Q^2 values were significantly lower than for the original model when group belonging values were permutated, meaning that the model's predictions were not spurious. No students had undue effect on the model, e.g. by being extreme in their responses to the questionnaire items and having large residuals when omitted and re-predicted into the model, built on the remaining students.

DISCUSSION AND CONCLUDING REMARKS

The aim of this study was to examine students' approaches to learning mathematics with a focus on differences between beginners and experienced mathematics students at the university. The results indicate that during the transition, the students' approaches to learning change. This may be due to the students' exposition to university teaching of mathematics (Schoenfeld, 1982) that may differ from their previous experiences of mathematics education from secondary level (de Guzmán et al., 1998). Shifting focus from the teacher to the peers can be interpreted as a way to adapt to a new learning environment, which is a crucial ingredient of the transition (Wood, 2001). The decreasing significance of the teacher may be due to limited availability of the teacher or that the university students found the explanations and help from the teacher less useful (Stadler, 2009). However, a higher value might have been predicted since the university teacher can be regarded as the best representative for the new epistemological beliefs and approaches to mathematics that the students have to adapt to in the transition. The decreasing valuation of teacher and the increasing valuation of peers may be interpreted as the students' attempt to handle the transition without changing their epistemological beliefs and approaches to mathematics.

Despite the complexity that the multifaceted transition implies for research in this area, our approach has rather been to take the complexity as a starting point and yield a result that shows which variables are the most crucial for students in transition. Previous studies have mainly adopted a bivariate approach to predict the outcome (Pampaka et al., 2011; Kemper & Leung, 1998), which limits the possibility to understand the impact of confounding variables and yields a lower predictability than multi-variate methods. In contrast to previous studies (Pampaka et al., 2011) we have also included questions about students' actions and behaviour so that we can correlate these with their performance and their beliefs and experiences.

Our results indicate important insights about students in transition. However, we are aware of the danger of jumping to too far-reaching conclusions. The sample is small and not representative. We also lack information about whether individual students have answered both the first and the second questionnaire. The results presented in this paper give an indication of what the crucial differences between beginners and experienced mathematics students can be. Also, the results can be used as an indication of what to expect from a larger sample and how to design forthcoming studies.

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STUDENTS' PERCEPTIONS OF HOW THEY LEARN BEST IN HIGHER EDUCATION MATHEMATICS COURSES

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This paper reports results from an on-going project investigating the transition from upper secondary to higher education mathematics. From a survey of students in their second year in higher education, we report the teaching and study methods from the first year which, according to the student survey, gave students the greatest dividends as learners of mathematics. Results show that work assignments and collaboration with fellow students in tutorial groups or informal groups were the methods students felt they learnt most from. Lectures were not perceived as equally important in the learning process, and ICT was perceived as having little importance for their learning. The implications of the results are discussed in the paper.

INTRODUCTION

The transition from secondary to higher education mathematics has been widely studied and it appears to be a difficult phase for many students (Gueudet, 2008). In addition to causing disappointment and distress for students personally, student failure and/or drop-out represent a significant financial loss for the university/college, a concern for teachers and a loss of potential for society as a whole (Gamache, 2002). It is said that students coming into higher education are more numerous and have more diverse backgrounds than previously, and they have different and often vague views of mathematics, its learning and its role in their future careers and lives (Kajander & Lovric, 2005). Students struggle with university studies because they have a distorted perception of what the acquisition of knowledge entails, and many students see knowledge as a collection of facts that can be absorbed passively (Gamache, 2002). Moreover, undergraduate mathematics does not yet seem to accommodate the diversity of its student body in its offerings and learning mode opportunities (Barton, Ell, Kensington-Miller, & Thomas, 2012).

In this study we investigate how students perceive their 'learning milieus', why and how students continue with mathematics and what could be reasons for dropping out. We examine how students develop their identity as mathematics learners, at transition and through their first year university mathematics, and how they perceive the 'use' of mathematics for their further studies and lives. In this paper we report the results from the second of two data points performed in the second year of the project. The question we ask is "Which forms of study did the students perceive as having learned most from in their first year of study?"

CONCEPTUAL FRAMEWORK

The larger study was conceptualised in collaboration with, and it is very similar to, the TransMaths project at the University of Manchester[1]. The project's aim is to develop a deeper understanding of how student experiences of mathematics education practices may interact with various (identified) factors to shape students' development as learners of mathematics, their dispositions and their decision-making at this crucial time. Students experience difficulties at different stages, and they develop different strategies to make these transitions successful (e.g. Brown & Rodd, 2003). Wenger (1998) contends that learning involves both *practice* and *identity*, that is learning develops as students engage and participate in a particular 'world' and in a *practice.* At the same time institutional practices afford, or hinder, students developing a mathematical disposition and an identity (Boaler, 2002) that supports their engagement with mathematically oriented subjects. The on-going project studied students' identities in relation to their experiences of different mathematics learning-and-teaching practices.

The literature (e.g. Wingate, 2007) argues that at transition to university students are often expected to become 'independent learners', hence the importance of "learning to learn". In a previous article (Pepin, Lysø, & Sikko, 2012) we argued that the strategies for learning to learn mathematics were not adequately addressed in the higher educational institutions we studied, except in elementary teacher education, even though, according to our survey, this was just what students said they needed most.

In terms of learning-and-teaching practices, face-to-face lectures remain a standard component of most higher education mathematics courses, despite being widely criticised, and even with advances in information technology and access to the internet, and it is claimed that lectures often are, in practice, where students' learning starts (Pritchard, 2010). These criticisms relate to the lecture as a mode of teaching (mathematics) which promotes superficial learning and 'transmissive' teaching, an environment where 'right-and-wrong' answers are encouraged, amongst others. However, there is evidence that lectures, appropriately 'conducted', are likely to provide opportunities for student learning, for students to take responsibility for their own learning and to engage in activities that are conducive to collaborative learning (e.g. Barton et al., 2012).

Research (e.g. Crawford, Gordon, Nicholas, & Prosser, 1998) shows that students' views of mathematical learning and knowledge relate to their experiences of learning as a whole. This indicates the need for shifting lecturers' attention from just focussing on the mathematical content (and presentation of their course) to a more systemic view of the learning environment. For example, Schoenfeld (1998) proposes an environment that fosters "a community of sense-making in which exploring ideas is

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[1] http://www.education.manchester.ac.uk/research/centres/lta/ltaresearch/transmaths/

highly valued" (p.61). In his mathematical problem-solving courses teachers encourage students to conjecture and propose solutions where the validity and accuracy of the solutions are decided by the group. Barton et al.'s study (2012) shows that lectures are crucial in establishing new social norms (Yackel & Cobb, 1996) where the lecture goals are changed from 'covering the content' to 'developing mathematical understanding'. This change includes active engagement of students where "students spend some time working in informal groups engaged in mathematical activity" (p.6). Hence the literature advocates student activity, collaborative learning and informal group work as having overall positive effects for the learning of mathematics, in the cognitive domain as well as the social and affective domain in higher education mathematics (e.g. Barton et al., 2012).

This change of focus is also supported by new technology. However more research is needed in this relatively new domain, and one could ask whether 'technology would be helpful in fostering novice students' autonomy by using appropriate online resources' (Gueudet, 2008, p. 252). E-learning is now advocated in the secondary-tertiary transition in mathematics (e.g. Bardelle & Di Martino, 2012), for the purpose of transforming practice and hence learning. Advocates believe that it can transform thinking and attend to individual students' needs for personalisation (of paths) and collaboration (in a single activity) (Bardelle & Di Martino, 2012).

RESEARCH DESIGN

The research design (of the whole project) was based on a theoretical framework of mixed methodology involving student longitudinal survey, student biographical interviews and case studies of practice; all at two data points (DP1 and DP2). The questionnaires were developed based on the Manchester examples, subsequently tested and calibrated for the Norwegian context (which included the validation of each question), and appropriately translated.

The data chosen for analysis reported in this article were the questionnaire/survey data from students at the following institutions: at City University (CU) the study was conducted with students in three different courses (Calculus 1 for civil engineers specializing in maths and physics; Basic Analysis; and, Mathematics for Applications). The Basic Analysis course was followed by students enrolled in the teacher programme in the sciences or mathematics, but also included students in various undergraduate programmes and a one-year mathematics programme. The Mathematics for Applications course is usually taken by students who need a somewhat less theoretically oriented mathematics course and focuses on applied mathematics. At River University College (RU), we conducted the survey in both the three-year engineering programme (RUE) and the teacher education (RUT) for years 1-10. RUE students from three different programmes were following the same math course. At DP 1 (autumn 2010) we collected questionnaires from 720 students (and interviewed a total of 49 students spread across the various programmes and courses -

see Pepin, Lysø, & Sikko, 2012). At DP 2 (autumn 2011/spring 2012) we collected 562 questionnaires.

The DP 2 Questionnaire had a total of 26 questions. In this paper we look at three of the questions, Question 16, Question 17 and Question 18. In Question 16 respondents were asked to consider 10 different statements regarding teaching and learning methods in the mathematics they experienced during their first year. For each of the statements they were asked to respond on a scale from 1 (Strongly disagree) to 5 (Strongly Agree). There was also an opportunity to check for Do not know / Not applicable. In Question 17, they were asked to give more detailed comments on which of the learning methods and study forms mentioned in Question 16 they believed they had benefited most from. In Question 18 they were asked to give reasons for possible non-attendance in lectures.

In order to develop deeper insights into students' experiences at transition from school to university mathematics education, the study applied robust methodological principles. These included:

- the principle of 'extended time' survey: in the case of the students, this involved following their development from their first weeks at university (DP1: case observations; individual student interviews; informal talks/interviews with lecturers) throughout their first year and into the second (DP2: as above);
- the principle of 'continuity': selected students were 'followed' inside sessions (e.g. lectures, tutorials) and outside these sessions (informal group discussions/sessions);
- the principle of 'seeing it through the student's eyes': when following the students and observing their 'work environments', data collection was conducted as far as possible through the 'lense' of the students' eyes and their work practices (e.g. observing them in lectures writing, listening, discussing with their peers, etc.);
- the principle of 'reflective investigation': this included discussing what they had written down, or submitted, or said before, in a reflective discussion.

In addition, and in order to counter threats to the validity of the data, and to further strengthen our rigorous data collection across different sites, the teams of researchers/investigators changed in the following ways: always one person was responsible for a particular case, but the second person changed from data point to data point. This allowed each investigator to see different cases 'in situ', and in turn to reflect on his/her own case and its students. In terms of ethics, we adhered to the code of conduct for surveys (observation and interviews) in Norway (NSD), which included student anonymity in questionnaires and voluntary 'opting-in' for interviews.

The methodology clearly has limitations, in particular our survey. Although we had two investigators at each data point, who both had experience with studies of this kind, and we had good rapport with the lecturers at the institutions, we were limited in terms of time and capacity: it was clearly not possible to understand and research the full range of students' experiences. Our survey data also has limitations regarding the longitudinal aspect. At DP 2 all students present at the lecture where the survey was conducted was allowed to take part, regardless of their participation at DP 1. Combined with the option they had to remain unidentifiable, this accounts for the fact that only around 40% of the surveyed students at DP 1 were identified at DP 2. This also has implications for the significance of comparison of results at the two DPs. Hence, the results cannot be generalised across other sites in Norway, or other subject areas. The extensive nature of our investigations could however deepen our understanding of what students might experience when studying mathematics courses in the selected institutions.

THE FINDINGS

The 10 different statements in Question 16 can be divided into four different categories related to a) Lectures; b) Organized tutorials; c) Informal self-selected groups; and d) Use of computer programmes.

Questions about the benefits of lectures were measured with three statements which are summarized in Table 1.

	Mean	Standard deviation
I benefitted a lot from lectures in mathematics	3.75	1.06
I generally feel that lecturers responded to my needs in the mathematics courses	3.30	1.09
I was able to understand most of what was being taught in the lectures	3.41	1.01

Table 1: Benefit of lectures. 1) Strongly disagree, 2) Disagree, 3) Neutral, 4) Agree, 5) Strongly agree

There were some differences between the institutions. RUE students (3 year engineering programme) seemed to report most positively on their benefit from lectures, as 42% strongly agreed with the statement "I benefitted a lot from lectures in mathematics", and they also strongly agreed with the statement "I generally feel that lecturers responded to my needs in the mathematics courses" (20%). However, only 14% of students at CU strongly agreed with the former (and 22% at RUT, and 7-8% to the latter). When asked about understanding what is going on in lectures, the percentages were more equal: 14% of the RUT students; 11% of the engineering students; and 6% of the CU students strongly agree.

How the students felt they benefitted from organised tutorials and working with obligatory exercise hand-ins was also measured through three statements. Results can be seen in Table 2.

	Mean	Standard deviation
I learned a lot of mathematics by working with the obligatory hand-in exercises	4.12	0.97
I learned a lot of mathematics from working with my fellow students	4.04	0.94
I benefitted a lot from the tutorials with teacher assistants	3.41	1.19

Table 2: Benefit of organised tutorials. 1) Strongly disagree, 2) Disagree, 3) Neutral, 4) Agree, 5) Strongly agree

It is noticeable that the mean score on the statements about obligatory hand-ins and working collaboratively with fellow students were considerably higher than the score for lectures. It is also noticeable that the score for benefits of the tutorials with teaching assistants was much lower than the two others. The standard deviation was also much higher. A closer look at the data makes it clear that the RUT (teacher education programme) pulled the score down; in fact, as many as 41% of the RUT students checked "Don't know" on this statement. In comparison, the "Don't know" percentage was only 5% at CU and 9% at the RUE programme.

Even if working with obligatory exercise hand-ins was given a high score at all education programmes, CU students were those who saw this as most beneficial, as 90% agreed or strongly agreed to the statement "I learned a lot of mathematics by working with the obligatory hand-in exercises". For RUC students the percentage was 78% and the percentage at RUE was 59%.

In addition to or parallel to the organised tutorials, students organised themselves in informal or self-governed groups (see Pepin, Lysø, & Sikko, 2012, p. 357). The benefits from working in such groups were measured with two statements (Table 3). The statement about working together with fellow students fits both in the category of organised tutorials and the category of informal group, and is therefore included twice.

	Mean	Standard deviation
I learned a lot of mathematics from working in informal groups with friends and colleagues	3.90	1.10
I learned a lot of mathematics from working with my fellow students	4.04	0.94

Table 3: Benefit from informal groups. 1) Strongly disagree, 2)Disagree, 3) Neutral, 4) Agree, 5) Strongly agree

RUE and RUT students agreed more with these statements than their peers at CU.

The benefits from working with computers appeared to be surprisingly small. The statement “I learned a lot of mathematics from working with different computer software” was given a mean score of 1.79; whereas the statement “I learned a lot of mathematics by communication on the LMS “It’s learning etc.” attained a mean score of 1.70. RUT and RUE students seemed to agree somewhat more with these statements, but across programmes and institutions it appears that computers did not add significantly to the students’ learning experiences.

It is interesting to note that the statement “I preferred the teaching at upper secondary school to the teaching last year at university/college” attained a relatively low score, the mean being 2.73. This indicates that the students did not want the higher education institutions to adapt more to the way teaching was conducted at school level. This can also be seen from the findings at Data Point 1 (see Pepin, Lysø, & Sikko, 2012), where it was found that students appreciated that they had to take more responsibility for their own learning at university.

The three statements that attained the highest mean scores were the following: (1) ‘I learned a lot of mathematics by working with the obligatory hand-in exercises’ - Mean 4.12; (2) ‘I learned a lot of mathematics from working with my fellow students’ - Mean 4.04; and (3) ‘I learned a lot of mathematics from working in informal groups with friends and colleagues’ - Mean 3.90. These results point unambiguously towards the fact that students perceive working in groups with hand-in exercises as meaningful activities for their learning. They gave statements concerning these much higher scores than statements about how much they learnt from attending lectures. In terms of references within the open comments, exercises, hand-ins and group work were mentioned 409 times.

DISCUSSION OF RESULTS

(1) The findings from the survey provide evidence for the claim that students perceive the most important learning to take place when they are working with the mathematics themselves, and in particular when they are working together with their colleagues in small groups. Comments from students at all three institutions support that. A student at CU claimed that he rarely went to lectures and that the “obligatory hand-in exercises …these were my main tool for learning mathematics”. In terms of seeking help, the tutorials and informal group work appeared to provide the main support.

> “I benefitted most from the tutorials, there one could ask and get help when you didn’t understand something.” (CU student 1).
>
> “I benefitted a lot from working in a small group with fellow students. To get help, and giving help back to others, this helped me a lot in my learning.” (RUE student 1)

Although the participants in our survey came from different backgrounds (strong mathematics; engineering; teacher education), there were commonalities in their 'discourse' and survey answers, in the sense that students largely welcomed collective work and team-based learning. It appeared that they felt safe in these groups where they could ask questions, seek help from and provide help to others. This supports previous research findings (e.g. D'Souza & Wood, 2003) in terms of benefits of collaborative learning for the creation of an environment of active, involved and goal-directed learning. It also allowed students to exercise a sense of control on the tasks they had to perform and was likely to enhance self-management skills. It appeared that collaborative learning has an overall positive effect in the cognitive/mathematics learning domain, as well as the social (and possibly affective), domain in higher education mathematics (Bardelle & Di Martino 2012)

(2) Many students claimed that their learning outcome of lectures was low. Asking students about their reasons why they did not attend lectures, a common thread in answers from all students, regardless the institution, was that students felt that the pace of lectures was too fast. As the pace was too high, they could not follow the teachers' explanations, and as a consequence they felt that they did not learn. Others claimed that they could not follow the lecture and take notes at the same time.

> "I felt the lectures went along at too high a pace, and I really could not follow the teacher. I have never understood so little in mathematics" (CU student 5).

Even those who persisted in going to lectures found themselves at a loss with the mathematics - resilience was not the route to success in terms of learning.

> "I attended all lectures, but I often had trouble understanding what was going on. Since then I have studied chemistry and I have had to use a lot of the maths from the maths course, but I soon found that I really had not understood the theories and the methods." (CU student 4)

As lectures were typically held in large auditoria, students apparently had little opportunities for asking questions. Only when they prepared the lecture, did it seem to make sense and provided positive experiences.

> "I attended the lectures, but I only benefitted from them when I had read about the subject prior to the lecture." (RUT student 2)

We argue that what students perceived as most useful for their learning may be seen as threefold: (1) Lectures could be seen as the place where learning started, in the sense that this was the place where students gained knowledge about what they had to learn. (2) In organized tutorials students received help with their hand-in exercises, so this was where they were provided with ways on 'how to solve' the problems at hand and with 'the answers'. (3) It was when working with their peers that students perceived that actual learning took place (likely through discussions and reflection).

For these students the social and socio-mathematical norms (Yackel & Cobb, 1996) of the lectures they experienced did not help them to understand and engage in the mathematics. Lecturers would need to develop their pedagogic practice, e.g. in terms of more skilled questioning and more active student involvement, in order to offer a diversity of learning approaches for their large and diverse audiences. This may also change student perception of learning in and through lectures.

In conclusion we claim that students are clear about how they can learn best – and this is collaboratively and actively engaged. Not ‘remedial instruction’ (e.g. Gamache, 2002), but innovative practices, either face-to-face (e.g. Barton et al., 2012) or by using technology (e.g. Bardelle & Di Martino, 2012; Borba & Llinares, 2012), are likely to be beneficial for ‘re-invigorating’ large-group pedagogic practice, such as lectures, or individual/small group tuition.

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UNDERGRADUATE STUDENTS' PERCEPTIONS OF THEMSELVES AS CAPABLE MATHEMATICS LEARNERS

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This exploratory study aimed to investigate how undergraduate mathematics students perceive themselves as capable mathematics learners and whether differences exist between male and female students' perceptions. A concurrent mixed method design was used in which quantitative and qualitative data were collected and analyzed separately. The findings from quantitative data suggest that gender differences exist with regard to participants' perceptions of their self-efficacy and their environment. Qualitative data analysis revealed that the participants describe their experiences of being a capable mathematics learner from two positions: from their direct experience and from their perceptions of ideal images of a capable mathematics learner.

Keywords: Post-Secondary Education, Gender, Mathematics, Identity, Perceptions

INTRODUCTION

Researchers have highlighted various aspects of the mathematics learning process and have presented it from a range of theoretical standpoints (Grootenboer et al., 2006). Recently, the notion of identity has been explored to bring an understanding of mathematics learning (Boaler & Greeno, 2000; Sfrad & Prusak, 2005). Learning mathematics involves the continuous development of a student's identity as capable mathematics learner. Identity can be defined as a perception of self in an academic environment that develops through relationships and experiences with peers, educators, family, and community, and an individual's own connections and meaning of mathematics in the broader context (Anderson, 2007; Solomon, 2007). Sfrad and Prusak (2005) refer to identity as the way in which one is defined by others and oneself.

Most of the studies on identity have focused on school mathematics learning. Current research has highlighted the importance of identity for understanding learning at an undergraduate mathematics level (Solomon, 2007). There is a need to recognize how mathematical identity contributes to the experiences and educational success of students at the undergraduate mathematics level. Undergraduate mathematics students' identity is essential to students' beliefs about themselves as capable mathematics learners and as potential mathematicians (Solomon, 2007). There is evidence from literature that gender plays a significant role on mathematical identity (Anderson, 2007; Solomon, 2007; O'Brien, 1999). This raises the question regarding whether male and female undergraduate students perceive their identity as capable mathematics learners differently. This study aimed at exploring how undergraduate mathematics students identify themselves as capable mathematics learners and whether gender differences exist. The study was guided by the following questions: 1) What are undergraduate students' perceptions of capable mathematics learners? 2) What does it mean for undergraduate mathematics students to be a capable mathematics learner? 3) How do their perceptions of self-efficacy, the environment and four faces of identity compare with their meanings of a capable mathematics learner? And 4) Are there any gender differences?

CONCEPTUAL FRAMEWORK

Literature shows various factors that may contribute to one's self perception o as a capable mathematics learner. Most of the studies on mathematical identity have focused either on self-efficacy, primarily from psychological perspective, or on environment, mainly from a socio-cultural perspective (Grootenboer et al., 2006). Grootenboer et al., argued that the plurality of theoretical perspectives may provide a richer and more comprehensive understanding on the issues of identity in mathematics education. Like Grootenboer et al., this study looked at identity as a capable mathematics learner at the undergraduate mathematics level from the plurality of the three approaches: self-efficacy, environment, and four faces of a learner's identity (Figure 1). In this study, self-efficacy refers to the way in which one may perceive him/herself and may ask, *Am I a capable mathematics learner?* According to Bandura (1986), self-efficacy is learned and self-efficacy expectations are acquired through various sources. The essential source of self-efficacy is one's accomplishments where one's experience on given task will either increase or decrease one's self-efficacy connected to that task (Bandura, 1986). A second source, vicarious learning, can affect one's self-efficacy where one sees others (peers and classmates) succeed or fail on a given task, assessment, or in class. In observing the behavior, performance, grade, etc., of others an individual may reflect on their experiences and make meaning of its relevance in a new situation. Other sources of self-efficacy are verbal persuasion and emotional arousal. In verbal persuasion, beliefs about oneself are influenced by the messages conveyed by others (e.g., what others are telling me about my capability to learn mathematics). Emotional arousal refers to the stress and anxiety in a given task and its effect on self-efficacy.

According to O'brien, Martinez-pons, and Kapala (1999) environmental factors may negatively affect self-perceived academic skills and career goals. Furthermore, Boaler and Greeno (2000) emphasize that identity is decisive in the belief that one can be a creative participant in mathematics as a social practice. Given this orientation, this study explored identity as a capable learner from the environment approach, where one's identity as a capable mathematics learner might be influenced by how others define him/her. For the purposes of this study, the environment refers to one's surroundings, which may compel one to question: *Do my peers, classmates, educators, etc. perceive me as a capable mathematics learner*? In addition, this study explores identity from Anderson's (2007) four faces: engagement, imagination/relativity, alignment, and nature. Engagement face refers to the experience and active involvement of an individual with people within their environment/the world around them. Imagination/relativity face refers to the images one has of him/herself and of how mathematics fits into the broader experience of life. The third face of a learner's identity is alignment. This refers to how one aligns personal energies within given institutional boundaries and requirements in response to their imagination face of identity. Finally, the nature face of a learner's identity looks at the connection one makes of their natural characteristics, which one has no control over, such as sex, and is dependent on their relationships and broader social settings (2007). Anderson's (2007) conceptualization of identity in terms of four faces of learner's identity, might lead a student to ask: *Why should I learn this*?

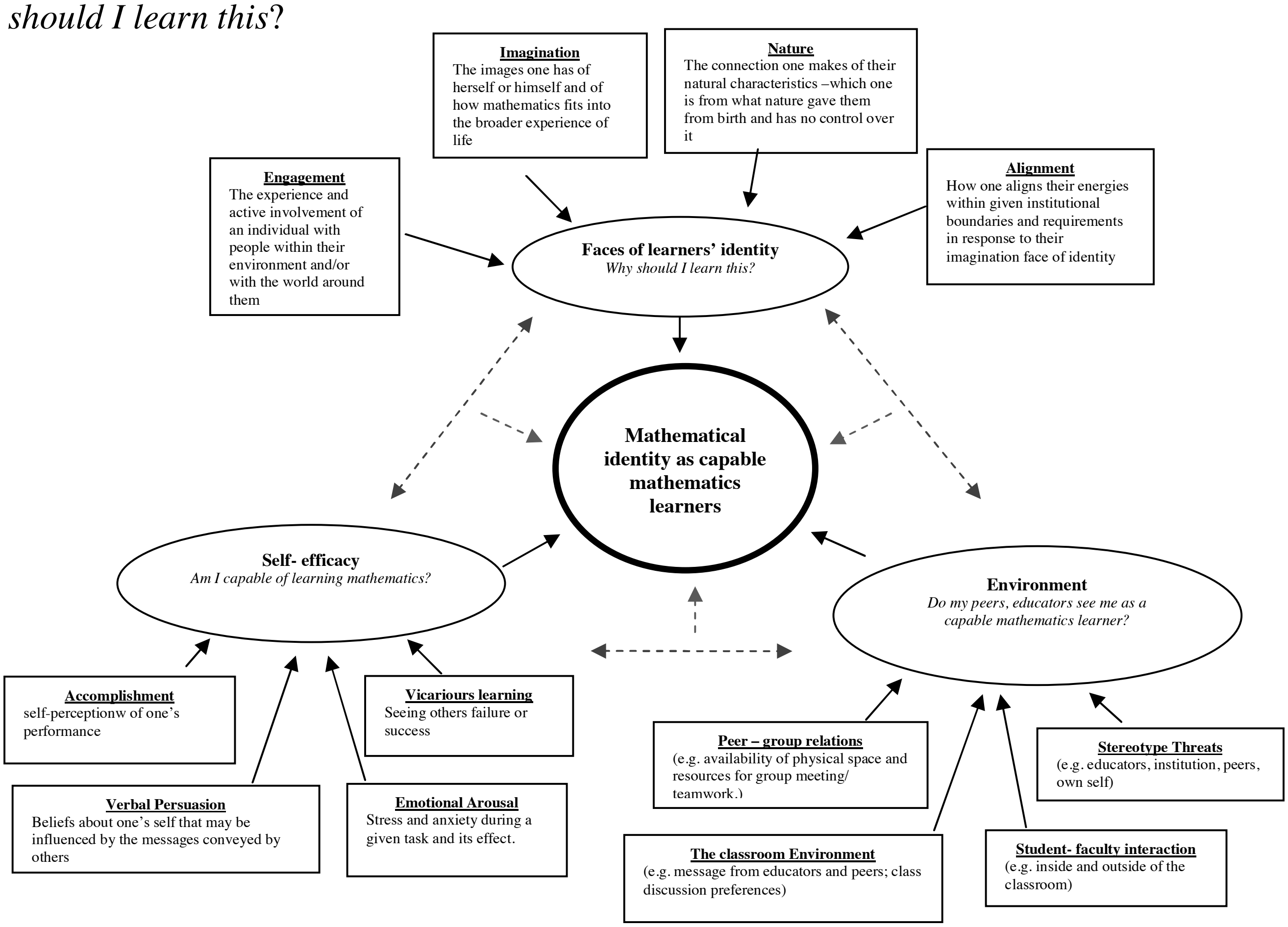

Figure 1: Conceptual framework: Mathematical identity as capable mathematics learners. The figure has two levels. First, the bolded centre circle is influenced by the three approaches in the elliptic shape (Self-efficacy, Environment and faces of learners' identity). Each of the approaches are influenced by the sources/ factors/each face, represented within the rectangular shapes. However, in the second level, these approaches may have an impact upon each other, represented by the dotted bi-directional two-way arrow, which shows the possible relationship among the approaches.

METHODOLOGY

Both quantitative and qualitative methodological approaches were taken in this study. The two methods were not intended to serve as a contrast point to one another; rather each method was utilized for a different purpose. The quantitative method was utilized to get a snapshot of the ways in which undergraduate mathematics students see themselves as capable mathematics learners and whether gender differences exist. The purpose of the qualitative analysis was to provide an in-depth understanding of students' experiences at the undergraduate mathematics level influenced their perceptions of themselves as capable mathematics learners.

Quantitative data was collected from an online questionnaire. A total of 30 participants, 10 males and 20 females responded to the online questionnaire. These participants were comprised of undergraduate mathematics students majoring in mathematics and those enrolled in the concurrent mathematic education programme at a Canadian University. Of the 30 participants, four were in First Year, five in each of Second, Third and Fourth Year of undergraduate mathematics programme, and 11 in the "others" category (including graduates and students who were in the programmes for more than four years). Twenty-seven participants were from the honors programme (a 20 credits base programme) and three from the pass programme (a 15 credits base programme). The online questionnaire consisted of 24 statements where 10 statements related to self-efficacy, eight statements related to environment, and seven statements related to four faces of learners' identity. These statements utilized the 5-point likert scale and were designed using the three approaches of understanding mathematical identity as a capable mathematics learner (see Figure 1). Both parametric and non-parametric measures for descriptive statistics were used to analyze the data. Given the assigned rating of 1-5 of the Likert scale level, the means of response rate of each statement, along with corresponding standard deviations and associated modes, were calculated using SPSS software in terms of all the participants and, male and female participants. The Mann-Whitney U test was used to test significant gender differences at $p < 0.05$ level.

The qualitative data were obtained from individual semi structured interviews with three males and three females. The six participants were given the pseudonym of Adam, Brent, Craig, Andrea, Britney and Carole. Adam and Andrea are graduates of a mathematics undergraduate programme. Brent and Carole are in the last year of mathematics undergraduate programme. Craig is in his third year of undergraduate intermediate/senior concurrent education with mathematics as his first teachable. Britney has completed a junior/intermediate concurrent mathematics education programme. The questions for the interviews were designed using the three approaches of understanding mathematical identity as a capable mathematics learner (see Figure 1). The interview consisted of eight questions and took about 25-30 minutes. In the invitation letter sent out by email, the interested candidates were asked to fill out a form and reply through email. The form asked for their name, gender, year of study, and contact information. Six participants volunteered for the interview component, which ensured a balanced sample to gender, and variation of their year of study. The data analysis followed John Creswell's (2008) general principals of qualitative data analysis. These procedures entailed preparing and organizing data, exploring data, describing emerging themes from the data, representing and reporting the findings, interpreting the data and, validating the accuracy and credibility of the results through prolonged engagement and persistent observation.

QUANTITATIVE FINDINGS

The results from quantitative data indicate that participants had strong perceptions of their self-efficacy, and strong perceptions of the influence of their environment. A majority of participants responded, 'Strongly agree'/'Agree' to the statements related to self-efficacy and environment. With respect to the four faces of a learner's identity, the results indicated that participants had strong perceptions regarding preferences for dispositions to nature face and strong perceptions of the influences of imagination face. A majority of participants responded, 'All of the time'/'Most of the time' to two out of three statements related to the nature face of a learner's identity. In relation to the imagination face, 76.6% of the participants responded 'Strongly agree'/'Agree' to the statement *Mathematics will help you further your career goals*.

		Rating				
		Strongly agree or all of the time	Agree Or most of the time	Neutral or some of the time	Disagree or few of the time	Strongly disagreed or never
Self- efficacy		25.7%	34.3%	18.7%	5.6%	0.7%
Environment		21.7%	37.0%	14.2%	5.8%	1.3%
Four faces of learners identity	Engagement	26.7%	33.3%	16.7%	3.3%	0.0%
	Imagination	63.3%	13.3%	3.3%	0.0%	0.0%
	Alignment	3.3%	26.7%	20.0%	23.3%	6.7%
	Nature	26.7%	23.3%	38.9%	11.1%	0.0%

Table 1: Average of Participants' Responses to the online questionnaire

Furthermore, the quantitative results show that there were significant gender differences in the perceptions with regards to the statement related to self-efficacy, *if someone oppose to your idea, you can find the means to prove your idea* (p=0.003), and a statement related to environment, *there is enough physical space available and mathematics resources in your institute for you to work by yourself* (p=0.014).

QUALITATIVE FINDINGS

Results from the qualitative data suggest two positions regardless of gender from which the participants described their experiences of being a capable mathematics learner: from direct experiences and from their ideal images of a capable mathematics learner. Within these two positions are the six emerged themes. These themes suggest that participants perceive themselves as capable mathematics learners at the undergraduate mathematics level: (a) when they contribute and fit in, (b) if they can teach others and others understand it, (c) when they have opportunities to interact with their peers, (d) when they are recognized by their professors, (e) if they have previous knowledge of the course content, and (g) if they fit in with their image of a capable mathematics learner.

Participants described their perceptions of their mathematical identity from their own, direct personal experiences. For example, in relation to the theme 'when they contribute to and fit in the class', Britney expressed:

> I usually see what other students got…how other students did. So I try to compare myself with others and I try to see, well, okay if the rest of the class did bad …okay, I guess we are all just bad together. If I see that I am on the lower end, I know something …you know something is up like: I need to, myself, do something.

In relation to the theme, 'if they can teach others and others understand it' Craig stated:

> For me, I learn a lot by teaching and so if and when I have a study group, I can maybe help out some of the other students that might struggle a little bit. And through that, I learn better. And because by teaching you sometimes have to think of it [problem] in different ways, and through thinking different ways you are learning yourself new ways, new techniques to do these problems.

For the theme, 'when they have opportunities to interact with their peers' Britney reflected on her experience as an undergraduate mathematics student and stated that it was important,

> when I had support of friends who were able to basically be the professors for me. And they were the ones who were able to explain things in a different way, interpret it for me, give me concrete examples and spend the time, work with me, problem solve with me. Because I am not the type where I can just have a professor standing in front of me. They do their own thing on the board, and I don't try out much in the class with them.

In relation to the theme, 'when they are recognized by their professors', Britney commented,

> The professors would influence the course the most for me. If the professor can teach and they can actually…they take the time, and you know I feel respected in the class they'll take the time to teach me something, then I would take their class, and I will struggle and I will try hard…

In relation to the theme, ‘if they have a previous knowledge of the course content’, when asked the question, *How do you see yourself succeeding in a course*, Craig responded,

> Some courses I see that I might not do so well in and for that, it would be based on my prior experience with the course. So, for instance, when I went into the second year statistics, I knew that I would do well in the course because it is very much similar to the data management course.

In contrast to participants’ descriptions of their direct experiences of capable mathematics learners, in the second position, the participants used their ideal images to describe their experiences of capable mathematics learners. Participants described their experiences by giving definitions or characteristics of their ideal images, which they then compared with themselves. Carole’s response to the interview question demonstrates this way of perceiving. She said, “Once I am in the range on As …that’s very happy…like for the outcome. But once I am not As but Bs and Cs, like which is lower… I start to question myself if I am doing my work…” Carole’s response shows that she perceives a capable mathematics learner to be someone whose grades are in the A range. However, if she does not measure up to her image of a capable mathematics learner, then she begins to question herself as a capable mathematics learner.

Britney also framed her perceptions relative to her ideal image of a capable mathematics learner. In her interview, she pointed out the importance of dialogue in mathematics classes for learning. She commented, “In mathematics courses there isn’t that much room for dialogue, and if there is dialogue it’s with the smartest students in the class and the professor. So I am kind of cut out of that part.” Britney noted that given a limited dialogue in mathematics courses, it would seem that the only dialogue that occurs is between “the smartest students in the class and the professor.” This situation provides her with an image of a capable mathematics learner as someone who dialogues with the professor in the class. She then compares herself with this image where she notes that she is “cut off” from the dialogue. This suggests that Britney perceived herself as a capable mathematics learner if she fits in with her image of those students who dialogue with the professor in the class. Further, in the same interview Britney revealed some other ways in which she perceived an image of a capable mathematics learner and how she compared herself with that image. She added:

> I could tell that there were students who, somehow I guess, had a math gene in them but I didn’t, and they were the ones who could teach math at a higher level, understand at the higher level and do more mathematical things.

Here, in this response, Britney’s image of a capable mathematics learner is someone who is born with “a math gene.” Therefore, she sees herself as a capable mathematics learner if she fit in with her image of someone who has “math gene”.

Brent's response to how he sees himself as a mathematics student gives further indication of participants' ideal images of a capable mathematics learner. During the interview, Brent mentioned that right after high school he enrolled in a chemistry undergraduate programme. Due to his relationship with his professor in his First Year mathematics course, he decided to switch his undergraduate programme from chemistry to mathematics in his second year. The interview revealed how Brent negotiated his image of a capable mathematics learner in his undergraduate mathematics studies. He noted:

> I didn't like that impression that everyone is like you are in math, you are nerdy. Well, like I am not…that is not necessarily the case. So I didn't like to associate myself with other math majors…like I guess because of that. So that didn't last too long.…first I was like okay it really doesn't matter…I don't identify myself as a nerd so it doesn't matter…I can still talk to people and talk to them…so then now I am like, yeah, I guess I fit into the nerd category.

Brent's response indicates that he did not want to be perceived as a "nerd" by others just because he was an undergraduate mathematics student and hesitated from associating himself with the image of a "nerd". However, upon interacting with others, he realized that he fits with his ideal image of a capable mathematics learner, "nerd". In other words, he sees himself as a capable mathematics learner if and when this fit in with his image of a "nerd".

DISCUSSION OF RESULTS

The discussion of results is guided by the research questions and by the conceptual framework discussed earlier. The two positions and six themes, in which participants describe their experiences of being a capable mathematics learner, point to participants' perceptions and their meanings of a capable mathematics learner (research question 1 and 2). An attempt is made to weave together the results from both quantitative and qualitative data to discuss about how participants' perceptions of their mathematical identity in terms of self-efficacy, environment and four faces of a learner's identity compare to their meaning of being a capable mathematics learner (research question 3). The results from quantitative data pointed to the existence of gender differences. For example, results from qualitative data show that participants perceived themselves as capable mathematics learners if they can teach others and others understand it. They noted that teaching others provoked one to think of different ways of solving a mathematical problem and validated their own understanding of the concepts. These participants' perceptions and meaning of a capable mathematics learner might lead to their perceptions of self-efficacy in terms of verbal persuasion. In verbal persuasion, beliefs about oneself can be influenced by the messages conveyed by others (Brown, 1999). Further, teaching others may provide opportunities for participants to assess their confidence in their ability to succeed/accomplish a variety of tasks and problems. Being able to teach others requires one's ability to deal with questions and different ways of thinking by their peers. Most of the participants strongly agreed/ agreed that *they are confident that they could deal efficiently with the unexpected* (70.0%), *they can solve most problems if they invest the necessary* effort (66.7%), *they can always manage to solve difficult problems if they try hard* (53.4%), and *they can find the means to prove their ideas if someone opposes their idea* (66.6%). This suggests that these participants might have strong perceptions of self-efficacy due tasks related to teaching others.

Results from qualitative data suggested that participants perceived themselves as capable mathematics learners when they contributed to and/or fit in the class. For participants, in order to see if one was a capable mathematics learner in terms of contributing to the class, one would assess their performance to see whether their grade is above or below the class average. This would then show, whether their performance is contributing positively or negatively to the class average, and consequently determine whether one is a capable mathematics learner. In regard to fitting in the class, participants assess their performance to see if it belongs to a majority or a minority of overall class performance. For instance, if majority of the class did not do well on a test and one's performance is part of the majority then one judges one's performance as not something to do with one's capability as a mathematics learner, but rather to do with other factors out of one's control. In contrast, if one's performance belonged to the minority that did not do well on a test then one saw oneself as not being a capable mathematics learner. Participants' perceptions of contributing to and fitting in the class might contribute to their perceptions of self-efficacy in terms accomplishment and vicarious learning as well as to their perceptions of their classroom environment in terms of their readiness to participate in class (Shunk & Pajares, 2004). For example, seeing oneself as contributing negatively or positively to the class average might lead one to feel worthy/unworthy or competent/incompetent (i.e. (not) a capable mathematics learner), which in turn might contribute to their timid or passive behaviour in class (Shunk & Pajares, 2004). Quantitative results suggested that the majority of participants (60%) agreed or strongly agreed that their participation in class depends on their comfort level in that class. Following these results from both quantitative and qualitative data, one might postulate that one of the contributors to one's feeling comfortable/uncomfortable to participate in class may well be their perceptions of contributing to and fitting in the class.

IMPLICATIONS

This study utilized a mixed method design and was conducted at one university. Hence, the implications of the study should be read not in terms of generalizability but of transferability to other cases (Creswell, 2008). The results suggest that the participants' perceptions of themselves as capable mathematics learners are influenced also by the perceptions held about them by their professor. Subsequently, while teaching/providing explanations, educators might benefit from taking undergraduate mathematics students' understanding of mathematics into account, by spending the necessary time that a learner may require for clarification and by providing explanation for mathematics concepts in multiple ways. Furthermore, this study indicates that participants' learning experience deepens when they develop personal relationships with professors. Consequently, there is a need for educators to provide learners with a comfortable learning environment where both educators and other learners invite one another's ideas, and where an individual feels comfortable to pose questions and discover answers. Hence, educators should provide opportunities where undergraduate mathematics students have opportunities to interact with educators and other learners at the undergraduate mathematics level.

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WHAT WE TALK ABOUT WHEN WE TALK ABOUT FUNCTIONS – CHARACTERISTICS OF THE FUNCTION CONCEPT IN THE DISCURSIVE PRACTICES OF THREE UNIVERSITY TEACHERS

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The study reported in the present paper forms part of an ongoing project regarding the teaching of functions in undergraduate courses at three Swedish universities. The theoretical framework underlying the project is Sfard's commognitive theory. In the present study, three lectures in calculus from two different universities are analysed, focusing on the characterizations of the function concept presented by the discursive practices of the teachers. All three teachers are found to make extensive use of the principle of variation, but differences are found for instance in the emphasis put on different realizations of functions, and on the role of domain and range. Also, according to the type of content of the lectures, differences between more process-oriented and more object-oriented discourses of functions can be seen.

INTRODUCTION

The teaching of mathematics at tertiary level has attracted growing interest within the research community over the last decade. However, studies of the actual teaching practices of university mathematics teachers remain relatively rare. My thesis project, focusing on the teaching practices of university teachers regarding the function concept, contributes to this area of research. The main part of the project, reports of which have been presented at PME 35 (Viirman 2011) and ICME 12 (Viirman 2012), aims at describing and categorising the discursive practices of the teachers regarding functions. The present paper, however, investigates how these discursive practices serve to characterize the function concept.

THEORETICAL FRAMEWORK

Underlying the project is the view of mathematics as a discursive activity. According to this view, mathematical objects are discursively constituted, and doing mathematics is engaging in mathematical discourse. Central to this approach is the commognitive theory of Sfard (2008). Taking the assumption "that *patterned, collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual*" (ibid, p. 78, emph. in original) as a starting point, Sfard defines thinking as "an individualized version of (interpersonal) communicating" (ibid, p. 81). The notion of communication is thus central to commognitive theory, and it is used to define what Sfard means by discourse. Different types of communication are called discourses, and they can be distinguished through four characteristics: *word use*, *visual mediators*, *narratives* (sequences of utterances regarding objects and their relations, subject to endorsement or rejection within the discourse) and *routines* (repetitive patterns characteristic of the discourse) (ibid, pp. 133-135). The rules of discourse act on two different levels. Object-level rules regard the properties of the objects of the discourse, while meta-level rules govern the actions of the discursants. A routine can thus be seen as a set of meta-rules describing a repetitive discursive action (ibid, p. 208). These discursive actions are also central to the notion of concept. From a commognitive standpoint, a concept is defined as a symbol together with its discursive uses (ibid, p. 111). How mathematical concepts are discursively characterised of course has great influence on what students might learn. Within the commognitive framework, learning is viewed as change in the individualized discourse. Learning can then take place on two levels, related to the levels of rules of discourse. Object-level learning is expressed through expansion of existing discourse, whereas meta-level learning involves changes in the meta-rules of the discourse (ibid, pp. 255-256). Central to meta-level learning is the notion of commognitive conflict, a term used to describe a situation where different discursants act according to different meta-rules. Most often, such conflict can be detected through the fact that different discursants endorse contradicting narratives.

Discussing the possibilities for learning afforded by teachers through their teaching practices, Runesson (2005) looks at dimensions of variation in the critical aspects of the object to be taught. She claims "that exposure to variation is critical for the possibility to learn, and that what is learned reflects the pattern of variation that was present in the learning situation." (ibid, p. 72) Since these patterns of variation are made visible through routines in the teaching discourse, the notion of patterns of variation fits within the commognitive framework.

The question the study aims to answer is: How is the function concept characterized, on both object- and meta-level, by the discursive practices of the teachers?

PREVIOUS RESEARCH

The function concept has been widely studied within mathematics education research (see e.g. Harel & Dubinsky 1992). Much research has focused on student difficulties with the concept. Of relevance to the present paper is for instance Even (1998), illustrating how knowledge of different representations of functions is dependent on several other factors, for instance the context of the problem, and the ability to consider both the global and point-wise behavior of a function. Sfard's (1991) notion of process-object duality, while not directly related to the function concept, has still been highly influential on later research on the topic. Research (e.g. Breidenbach, Dubinsky, Hawks & Nichols 1992; Viirman, Attorps & Tossavainen 2010) has shown that students mostly possess process conceptions of function, and that attaining a structural conception is very difficult for many students. The process-object duality is a critical aspect of meta-level learning regarding the function concept, as is the notion of arbitrariness (Even 1990) associated with a structural view of functions. Critical object-level aspects of the function concept highlighted in the literature are for instance the role of domain and range and defining rule in the definition (Breidenbach et al 1992), the importance of one-valuedness (Even 1990), and the different representations of functions – e.g. algebraic, graphic and numerical (Even 1998; Schwarz & Dreyfus 1995).

The teaching of mathematics at university level, while nowhere near as well-researched as the function concept, has still grown as a research field in later years. Studies of particular relevance for the present paper are for instance Pritchard (2010), arguing the continued relevance of lectures in university mathematics teaching, as well as Weber, Porter & Housman (2008), discussing the different roles examples play in conceptual understanding. There are also a number of studies focusing on the actual teaching practices of university mathematics teachers (e.g. Weber 2004; Wood, Joyce, Petocz & Rodd 2007). In Sweden, research on teaching at tertiary level is still rare. Apart from my own work (Viirman 2011, 2012) on the discursive practices of university mathematics teachers, one example is Bergsten (2007), using a case study of one calculus lecture on limits of functions to discuss aspects of quality in mathematics lectures.

METHOD

The empirical data in my project consists of video recordings of lectures and lessons given by teachers in first year mathematics courses at three Swedish universities – one large, internationally renowned; one more recently established; and one smaller, regional university. The participating teachers were chosen among those willing to participate, and giving relevant courses during the time available for data collection. A total of seven teachers participated in the study – four from the large university (labelled A1-A4), two from the younger (B1-B2) and one from the regional university (C1). The topics taught by the teachers included calculus, introductory algebra and linear algebra. The present paper focuses on the lectures covering calculus topics, limiting the number of teachers to three – A1, A4 and B1. Of these teachers, one is female (A1) and two male. All three are quite experienced teachers, having taught at tertiary level for more than 20 years. Two have doctoral degrees in mathematics, while the third (B1) was educated as an upper secondary school teacher. In all three cases, the courses were aimed at engineering students, and the lectures were given to fairly large groups, between 50 and 100 students. The topic of teacher A1 is an introduction to the function concept, part of a course preparatory for calculus, while teacher A4 covers continuity and teacher B1 the inverse trigonometric functions and their derivatives, both as parts of courses in single-variable calculus. In all three cases, the textbook used was Adams (2006).

For each of the three teachers, I have about two hours of videotaped lectures, which have been transcribed verbatim, speech as well as the writing on the board. The transcribed lectures were analysed, with the aim of distinguishing the discursive activities characterizing the teachers' respective discourses of functions, paying special attention to repetitive patterns (for more details, see Viirman 2012) and also patterns of variation regarding the critical aspects of the function concept mentioned in the previous section. I first analysed each lecture separately, and then compared them, searching for differences and similarities. I have intentionally chosen an outsider perspective, trying to view the unfolding discourse in as unbiased a way as possible. At the same time, I am of course making use of the fact that my mathematical knowledge makes me an insider to the discourse. However, I have specifically tried to avoid making references to what is *not* present in the discourse, except in contrasting the teachers' discursive activities, or making comparisons with previous research.

RESULTS

Looking at the discursive practices of the three teachers in this study, we find that all three teachers use various patterns of variation, highlighting different aspects of the function concept. For instance, the examples used by teacher A1 when introducing the function concept make the various components of the definition visible through variation. The examples she uses are $f(x)=(x-1)^2+2$, $g(x)=\sqrt{x}$, and $h(x)=\sqrt{|x|}$, varying rule, domain and range, as well as indicating the connection between them. She then uses the example $x^2+y^2=1$, using this as contrast, making the one-valuedness requirement visible. As seen in the following excerpt, she even uses the same defining formula while restricting the domain, to highlight the fact that all three parts of the definition are necessary (all excerpts are translated from Swedish by the author. Text within [square brackets] indicates writing on the board):

Teacher A1: [On the board two graphs are drawn: the function $f(x)=x^2$ defined on the intervals [-1, 1] and [-1, 2] respectively.]

And what I want to get at is that this proper definition of what a function is.

It is important that you state both the sets and the rule, not just the rule, really, because these are of course different functions, you can just look at them.

Later, when introducing trigonometric functions, she investigates the behaviour of the functions through variation of angles, periods and amplitudes. We can note, however, that all these functions are real-valued functions of one real variable, making the notion of arbitrariness of domain and range impossible to discern. In fact, with one exception, all examples of functions given by the three teachers are of this type. But since all data are taken from courses in single-variable calculus, where the general set-theoretic concept of function is not included in the syllabus, this is not very surprising. It should be mentioned that teacher A1, whose aim is to introduce the function concept, on a few occasions refers to a more general version, hinting that the elements of domain and range need not be numbers, and that the rule need not be given by a formula. However, as soon as she actually uses the concept, it is in the more restricted version, so it is fair to say that functions, as characterized by her teaching practices, are real-valued and defined on (some subset of) the reals.

Teacher B1, when introducing the inverse trigonometric functions, highlights the requirements for the existence of an inverse by varying the functions, as well as their domains, showing how too large a domain results in loss of injectivity:

Teacher B1: [the graph of sin(x) is drawn on the board]

You sense that we get a problem here, I think. Because you can't exactly say that this is everywhere increasing, nor that it is everywhere decreasing, and it simply must, it can't have much of an inverse, this one, because if we choose one *y* here.

[He marks a point on the *y*-axis, a short distance above the origin, and draws a line through the point parallel to the *x*-axis.]

There will be a whole lot of possible *x*s, right?

Student: You could define an interval.

Teacher B1: Yes, exactly!

He then goes on to show how different restrictions in domain need to be done for the different functions.

Teacher A4, finally, in a lecture on continuity, uses variation in function graphs to make the possible types of discontinuity visible. He draws a graph which is nice and continuous up to a certain point, marked by a dotted vertical line. He then says:

Teacher A4: Let's assume for example that the function is reasonable in this way, it might go there and then approach some line here.

What can happen when we approach this line from the other direction? What possibilities do we have? And they turn out not to be very many.

He then asks the students to give suggestions, and together with them he constructs a catalogue of possibilities. He introduces further variation through describing the different cases both geometrically and analytically, through limits. He also indicates the importance of domain through variation, showing how a continuous function whose graph has a jump discontinuity can be made discontinuous through assigning it a value at the jump point. Later, when discussing one of the basic theorems about continuous functions, the boundedness theorem, he highlights the necessity of the conditions (continuous function on a closed, bounded interval) by removing them one at a time and presenting counterexamples. In conclusion, all three teachers use patterns of variation to characterize functions through rule, domain and range, with the domain and range subsets of the reals.

Regarding different realizations [1] of functions, for teacher A1 the algebraic realization of a function through a formula is given prominence. New functions are introduced as formulas, while for instance graphs are spoken of as pictures of the function, as in this typical example:

Teacher A1: [$f(x)=(x-1)^2+2$]

It is a function; it is the function *x*-squared that I move one step to the right and two steps upwards. Then we can for example draw it.

This pattern occurs nearly every time an example of a function is given. It can be found also in the practices of the other two teachers, but not to the same extent. Also, for teacher A1, the graph is rarely used to gain knowledge of the function. For instance, when determining the range of the function in the excerpt above, she uses the formula to determine the minimum, instead of using the graph. For teacher B1, the formula is still central, but the graphical realization is not just seen as a picture, but as a means of gaining information about the function, as in the example above about the non-injectivity of the sine function. The relationship between graph and formula is also used, for instance to find the graph of the inverse given the graph of the function.

Teacher A4, finally, is less reliant on the algebraic presentation of function. In fact, he explicitly states the need for both the algebraic and the geometric view:

Teacher A4: [CONTINUITY.]

> And that is a property that functions can have, which on the one hand has an arithmetical, algebraic way of expression through inequalities and equalities and so on, and a geometric way.
>
> It is always reflected in what the graph of a function can look like.

He often introduces functions through graphs, as in the example of the classification of discontinuities described above. Of course he also uses formulas, but his more varied approach makes the connections between different realizations more visible. This is also achieved through his use of words. For instance teacher A1 speaks mainly in terms of “what does this function look like?”, giving the impression that the graph is just an illustration. Although teacher A4 occasionally expresses himself in this way, he generally speaks of “the graph of the function”, indicating that the graph is a realization in its own right.

On the meta-level, the move from viewing functions as processes on mathematical objects, to viewing them as mathematical objects in their own right, is known to be problematic for students. This duality is seen in various ways in the discourses of the teachers, and the distinction is not always made clear. For instance, all three teachers speak of functions as objects, which can for instance grow, be moved around or be split into smaller parts:

Teacher A1: It is the function *x*-squared which I move one step to the right and two steps upwards.

Teacher B1: There was a word collecting functions which are either just increasing, increasing, increasing or just decreasing, decreasing, decreasing.

Teacher A4: This function really approaches a whole lot of different points here, right.

Teacher A4: We have this function, this kind of patched-together function that consists of two pieces, and what I’m asking is: can you give a value for b such that f becomes continuous?

In their working with functions, however, distinct differences in the discursive practices of the teachers can be seen. Teacher A1 speaks of functions as objects performing processes on numbers:

Teacher A1: Well, it was all right putting in all real numbers here, all real numbers we can subtract one and take the square and then add two, and what comes out are also real numbers.

Teacher A1: Well, you can say that *f* itself is the machine, and *A* is the set of things you are allowed to put into the machine, and when you put a thing from *A* into the machine, then something comes out that is in *B*.

In the teaching of teacher A4, the transition from process to object is taken a step further. He often makes no distinction between the function and its values, but still doesn't treat the image as a totality, instead referring to it using metaphors of moving, like in the following example (concerning drawing the graph of the function $f(x)=\frac{1}{x-1}$):

Teacher A4: What happens to this function when *x* is bigger than one?

(...)

It goes down, yes, and then it will wander here, and become bigger and bigger and bigger, and when we approach one, this one is still positive and really big.

This way of handling functions is also apparent in the example referred to above, about the classification of discontinuities:

Teacher A4: In what way can it go wrong when I come wandering here towards this line? One way is that I hit the line, there is a limit but it is the wrong limit. Is there something else?

Student: Oscillation.

Teacher A4: Yes, it could swing, right, so that we don't have a limit when we approach here, that is, that it starts getting the shivers here.

For teacher B1, finally, functions are treated very much like objects, where the global behaviour of the function affects invertibility, and where you can create new functions from old ones, for instance by restricting the domain, or by differentiation. He still speaks of the value of a function at a point, but the function as a process acting on numbers is no longer so prominent.

DISCUSSION

As mentioned above, the lectures of the teachers in the study cover different topics – the definition of function (teacher A1), continuity (teacher A4) and derivatives of the inverse trigonometric functions (teacher B1).These topics typically occur at different stages of a calculus course, with functions defined early, continuity introduced somewhat later, and derivatives of inverse functions covered even later. This fits with the move towards a more objectified discourse observed in the teaching of the three teachers. Since I have no way of accessing what has occurred outside of the video-taped lectures, I can merely make conjectures about the effects of this. To validate these, I would have to observe the same teacher at different stages in the same course. However, even given these limitations in my data, it seems to me as if there is room for commognitive conflict here, with potential for meta-level learning for the students. Teacher A1 speaks of a function as an object performing a process, acting upon numbers. For teacher A4, however, the word 'function' signifies something slightly different. This can be seen in the way he sometimes conflates the function with its values, indicating a view of the function as a totality, comprising domain, range and rule. This description is given by teacher A1, but it isn't apparent in her discursive practices the way it is for teacher A4. However, there is a slight tension in his lecture, seen for instance in the way he uses metaphors of movement, indicating a process view. It is known from previous research (Sfard 1991; Breidenbach et al 1992) that attaining a structural view of the function concept is difficult, and that such a view is rare even among university mathematics students (Viirman, Attorps & Tossavainen 2010). Making this tension more explicit might help the students change their own meta-rules.

On the object level, the role of rule, domain and range indicated by Breidenbach et al (2002) is central to the discourse of all three teachers, but as mentioned above it is characterized somewhat differently, where teachers A4 and B1 see them as an integrated whole, whereas teacher A1 speaks of domain and range more as necessary complements to the rule. Furthermore, research has shown that the use of multiple representations contributes to students' conceptual understanding, and that the ability to move between different representations is dependent on other factors, like the use of a global or point-wise approach (Even 1998) or the use of different kinds of language (Wood et al 2007). In the teaching observed in the present study, the two main realizations of functions are formulas (algebraic) and graphs (geometric). The relation between these is made more or less explicit in the teaching, with teacher A1 mainly treating the graph as an illustration of the formula, and teacher A4 using graphs independently of formulas, to investigate the qualitative behaviour of functions. Also, teacher A1 mainly takes a point-wise view of functions, whereas teachers A4 and B1 use both point-wise and global approaches. It would appear as if teacher A1 is less successful in integrating the use of algebraic and geometric realizations, giving prominence to the algebraic language of formulas.

I also wish to discuss some observations regarding the use of examples. Weber, Porter & Housman (2008) consider two types of example usage in university mathematics teaching – worked examples and examples aimed at concept image building (see also Pritchard 2010). The teachers in the present study display both types in their teaching, but perhaps most interesting to discuss here is the use of examples to develop discursive objects through variation of critical features

(Runesson 2005). The way teacher A1 introduces the function concept through a series of examples (and non-examples), and the way teacher A4 develops the notion of discontinuity at a point through a systematic use of examples are cases in point.

Finally, I want to mention one topic of further study suggested by my data. This paper has focused on the function concept, but a very interesting aspect of the discursive practices of the teachers in the study concerns how their views of mathematics as a more general practice are made visible. The scope of this paper is too limited for me to be able to present such an analysis here, but this is planned as the topic of a future publication.

NOTES

1. Within the commognitive framework, Sfard (2008, p. 155) uses the term 'realization' rather than 'representation', signifying the fact that the realization and the signifier being realized belong to the same ontological category.

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THE TRANSITION FROM UNIVERSITY TO HIGH SCHOOL AND THE CASE OF EXPONENTIAL FUNCTIONS

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University mathematics programmes cater, in various ways, to future teachers of mathematics. The gap between the university contents and methods on one hand, and on the other hand secondary school mathematics, is investigated in studies of how students experience the beginnings of university mathematics programmes. Here we consider the gap as it appears at the other end of the students' experience at university, as they try to relate their new mathematical knowledge to contents and methods in secondary school teaching. The main points of the paper are to introduce a theoretical model to analyse the changes in relation to mathematical knowledge which may be operated at that point, and to show its use on a significant example (the construction of exponents or, more generally, of the exponential function).

AN DOUBLE TRANSITION PROBLEM

Most research on university mathematics education has been devoted to the study of the specific challenges students face at the beginning of a study programme with a strong or exclusive mathematical component (see Gueudet, 2008 for a review of such research, as well as several papers in the CERME7 version of this working group). These challenges are *institutional transition problems* since they arise as students change institution, and several challenges appear to be related to major differences between emphases and methods of teaching in high school and university. At the opposite end of the university programme, the students face another type of institutional transition, namely that from academic mathematics to various professions. In this paper we explore the nature of this second kind of institutional transition, and we also discuss how a mathematics programme may seek to "smoothen" it in view of the profession of secondary mathematics teacher.

Already Klein (1908, p. 1) observed that students taking this direction may face a "double discontinuity" as they move from high school to university, then back again:

> At the beginning of his studies, the young student is faced with problems that in no way remind him of the [mathematical] things he worked with in school; naturally he then forgets these matters quickly and thoroughly. If he becomes a teacher after having finished his studies, he must suddenly teach this time honoured elementary mathematics in a school like fashion; and as he cannot by himself see the connection between this task and university mathematics (...) his university studies become just a more or less pleasant memory which has no influence on his teaching. (Translated by the author)

Indeed in many countries, secondary school teachers receive their mathematical education in study programmes geared towards academic "pure" mathematics. The second discontinuity or gap, from university mathematics to secondary school mathematics, appears as a problem to the extent the students' outcome from the programme does not function as a resource for teaching mathematics at the secondary level. While there is extensive research on how teachers experience the transition from teacher education into teaching practice (see e.g. Winsløw, 2009, for an overview), there remains a practical need for systematic didactical research on how standard undergraduate mathematics is, or could be, developed in view of facilitating its use by students in inquiries related directly to high school mathematics.

This problem is clearly at the borderline between research on mathematics teacher education and research on university mathematics education. But it is of relevance to the latter for at least two reasons: university mathematics programmes often include courses specifically designed to treat elementary mathematics "from a higher viewpoint", as Klein (1908) put it in his book title; and also one may speculate that the two "smoothing problems" might usefully be approached together.

In this paper, we present a new theoretical model of the second transition, with a concrete case to illustrate it. We first introduce the institutional context in which we have considered the problem (a capstone course). Then we explain the theoretical model in order to situate this somewhat arbitrary context within the general problem of institutional transition and the corresponding changes in relation to mathematical knowledge. Finally, we explore a case which is both illustrative and important in its own right, namely the work with exponential functions in a capstone course.

We emphasize that the primary aim of this paper is to present the problem (of transition from *U* to *HS*) and our theoretical model of it, while the case serves to illustrate both through a concrete didactic design, its premises, and some observations of its effects. The case is not an empirical study; it helps to substantiate the model.

CAPSTONE COURSES AND THEIR COUSINS

Universities operate with different traditions and methods across the world. In American colleges and universities, there is a long standing tradition of "senior year programs", which include the so-called "capstone courses":

> The capstone course typically is defined as a crowning course or experience coming at the end of a sequence of courses with the specific objective of integrating a body of relatively fragmented knowledge into a unified whole. As a rite of passage, this course provides an experience through which undergraduate students both look back over their undergraduate curriculum in an effort to make sense of that experience and look forward to a life by building on that experience (Durel, 1993, 223).

In mathematics programmes, capstone courses are not always aimed at preparing students to a life as mathematics teacher, but many are, as evidenced by the offer of text books such as Usiskin, Peressini, Machisotto and Stanley (2003) or Sultan and Artzt (2011). These texts contain exposition and exercises related to critical areas of secondary mathematics such as trigonometry, number systems or plane geometry, but from the "higher viewpoint" of academic mathematics, just as in Klein's (1908) treatise which had similar goals (for its time). In fact, a related feature of the German

Stoffdidaktik (content didactics) tradition is to make use of academic mathematics in the development and study of secondary school mathematics. One could mention also the *concours* system, a kind of preservice teacher examination in France and other Romance countries, as an example of transition measures of this kind which are more or less integral parts of the university curriculum within a discipline.

Our context

At the University of Copenhagen, almost all mathematics courses focus on inducing students into the methods and theory of the academic disciplines in question. Only a minority (~40%) of the students become teachers, and many students are undecided about career plans during most of their study. Even the options to specialize in the history or didactics of mathematics (at the master level) are aiming to develop familiarity with research and its methods, rather than professional skills as a teacher.

It should be noted here that to become a high school teacher in Denmark, one needs to study two disciplines, normally a major (for 3-4 years) and a minor (1½-2 years). Students who do a minor in mathematics get to study the first parts of the bachelor programme in mathematics, including abstract algebra, analysis and differential geometry. For some of these students, that material appears both relatively advanced and rather disconnected from what they perceive as relevant to teach mathematics in high school. Then, a few years ago, a course on "Mathematics in a teaching context" (UVmat) was introduced as an option at the bachelor level, primarily in view of students who take mathematics as a minor (but open to others interested in teaching). In recent years it was co-taught by the author and a colleague; in the version of 2011-2012 it was based on Sultan & Artzt (2011). The course has gradually adopted the idea of serving as a capstone course with the following aims:

- Identifying and filling serious gaps in the students' knowledge of select areas high school mathematics which remain after taking more advanced courses;
- Treating selected topics – mainly from high school analysis and algebra – but from "from a higher viewpoint" (cf. Klein, 1908), in an attempt to illustrate how the material learned in university can serve also when working with more elementary topics in view of teaching.

To sharpen the meaning of these points further, we introduce some elements of the anthropological theory of didactics as well as a more general model of the double transition mentioned in the introduction.

THEORETICAL MODEL

It is a basic contention in the anthropological theory of didactics (Chevallard, 1991, 206-207) that the relation of an individual to an object o of knowledge is strongly conditioned by the institution I in which this object of knowledge lives. For a didactic analysis, we usually consider the individual as occupying a position p within the institution, for instance as a first year student, and study the relation $R_I(p,o)$ of this position in the institution to o, where the subscript stresses that p and o depend on I.

This abstraction is by no means artificial. In fact, data may inform us, with more or less precision, about the relation of smaller number of student to the practice and theory related to Taylor series, or a similar object of knowledge *o* that of course must be more precisely delimited. But our real aim is usually more general, such as investigating the relation to *o* of first year students in the University of Copenhagen – or perhaps a generic university of a similar kind.

Such a relation can, indeed, exhibit many variations, which are not exhausted by simple measures of "mastery" of target knowledge. The relation involves also, for example, the *status* assigned to *o* with respect to other knowledge objects, which may depend highly on *p* and therefore also on the institution *I*. For instance, to Spanish high school students, *limits* of functions may end up with an exclusive and marginal status as a formal preliminary to defining *derivatives* of a function (Barbé et al., 2005). Another important variable in the relation is the *modality of access* which *I* enables *p* to have to *o*, for instance, solving exercises, study of a textbook, autonomous inquiry etc.

To model the "objects of knowledge", Chevallard (1999) introduced the notion of *praxeology* (do read Barbé et al., 2005 if you are unfamiliar with this). In this model, an object of knowledge *o* is modeled as praxeologies which consist of two related parts: *practical knowledge* (abbreviated *P*) such as a method to estimate the error of a given Taylor approximation of a given function, at a given point; and *theoretical knowledge T*, such as a way to explain and justify *P*. In fact, the knowledge object *o* which we consider may be formed by a whole collection of praxeologies (P, T).

When considering the double transition outlined in the previous section we are actually investigating transitions of the type $R_{HS}(s,o) \rightarrow R_U(\sigma,\omega) \rightarrow R_{HS}(t,o)$ where the institutions are that of a more or less well defined high school (*HS*) and university *U*; also, s and σ refer to more or less delimited positions as student in these institutions, and t to the position as teacher in *HS*. Finally o and ω are knowledge objects of a more or less comparable nature. Chevallard (1991) developed the notion of didactic transposition to explain how a knowledge object is transformed in view of enabling students in an institution such as *U* or *HS* to establish relations with it; an important case is the transposition of praxeologies ω developed and taught at universities into praxeologies o taught in schools. The second transition above is, clearly, of another nature, and takes place at another pace; still, the success of the didactic transposition certainly depends on the relation $R_{HS}(t,o)$ and this, in turn, relies in part on a past relation $R_U(\sigma,\omega)$ with the transposed object ω, and with teachers now in position t after having been in the position σ.

In capstone courses, we focus on students in position σ and consider how $R_U(\sigma,\omega)$ could be developed in view of a future situation of the students in a position t, and we try to prepare bridges between relations of type $R_U(\sigma,\omega)$ and a more or less hypothetical relation of type $R_{HS}(t,o)$. As a matter of fact, the relation may not be hypothetical at all. Due to an increasing shortage of graduates to fill positions in

Danish HS, some of the students in UVmat are already part time high school teachers. This clearly is a special situation worth particular attention as these students are already experiential in some of the challenges transitions of type $R_U(\sigma,\omega) \rightarrow R_{HS}(t,o)$ which are our topic here. However, we do not focus on this situation here.

CASE STUDY: EXPONENTS AND EXPONENTIAL FUNCTIONS

We now consider, as knowledge object o, the approaches to exponents a^b currently or potentially found in Danish high school (HS), and we consider the set of knowledge objects ω more or less close to o, which are taught in the bachelor programme in mathematics at the University of Copenhagen (U). Our focus is on how a capstone course such as UVmat may contribute to (and draw on) $R_U(\sigma,\omega)$ in order to prepare the transition to the relation $R_{HS}(t,o)$.

High school approaches to exponentiation

There is a certain variation in the approaches to exponentiation found in text books and, conceivably, therefore in the relation students and teachers in *HS* will develop to o. Most textbooks define a^x for rational x, using more or less elaborate justifications of the formula $a^{m/n} = \sqrt[n]{a^m}$ based on the definitions of $a^{1/n}$ and a^n. Given that real numbers and limits are not rigorously treated in HS, the variation lies in how the books explain the passage to real exponents. Here are some typical examples:

> The power is calculated by approximating the exponent by a finite decimal number. How many decimals you include depend on the required accuracy (Timm & Svendsen, 2005, 26; translated from Danish by the author)
>
> In Chapter 3 we saw how to calculate powers where the exponent is integer and positive, 0, integer and negative, and rational (fraction). Strictly speaking we have not explained the meaning of a symbol like $7^{\sqrt{3}}$ but we assume CAS will take care of this. (Carstensen, Frandsen & Studsgaard, 2006, 82; translated from Danish by the author)
>
> It is also possible to extend the notion of power to the case of arbitrary exponents like for example π and $\sqrt{11}$ but it will take us too far to do that here (Brydensholt & Ebbesen, 28; translated from Danish by the author)

After this, all books operate with exponential functions as functions defined on $\mathbb{R}$. There is nothing surprising about this and one should not overestimate the impact on $R_{HS}(s,o)$ of the differences like the above. Judging from informal questioning at lectures over the past few years, few students recall the definition of powers with rational exponents, and even less wondered how to define a^x properly and in general.

Exponentiation in the mandatory bachelor curriculum

All texts used in U simply *assume* the existence of exponential functions as part of the prerequisites from HS. Students are also assumed to know certain basic properties of exponential functions, such as continuity, derivatives and other specific rules, like $a^{x+y}=a^xa^y$ for $a \geq 0$ and x, y real. Exponential functions appear frequently in the first courses on calculus and analysis – for instance, in examples of calculating Taylor series, as solutions to differential equations, as building blocks in functions of several variables and complex functions, as a tool to study complex numbers (polar form) or functions, etc. Briefly speaking, in $R_U(\sigma,\omega)$ the meaning and basic properties of a^x (for real a, x) are certainly available, but they are neither questioned nor explained.

Exponentiation in UVmat

There are several classical ways to approach the construction of a^x for $a>0$ and x real, some of which are elegant but out of reach in the first year of HS when this object is first encountered. In UVMat, it is a specific point to develop $R_U(\sigma,\omega)$ so as to know and relate five of these, based on

(1) "Direct" construction, starting with the case $x\in\mathbb{N}$ and extending first to $\mathbb{Q}$, then to $\mathbb{R}$;

(2) The inverse function to $\log_e$, itself constructed via $\log_e(x) = \int_1^x \frac{dt}{t}$;

(3) The initial value problem $dy/dx = y$, $y(0) = 1$;

(4) The functional equation or "property" $f(x+y) = f(x)f(y)$;

(5) The power series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

Here, (4) can be used to show uniqueness of a certain form, detailed in the next subsection, while existence must be based on other methods. We notice that (1) is similar to the brief explanations found in o (first year) as explained above, but it still relies on deeper properties of $\mathbb{R}$ that are may be identified in ω but not in o, while (2)-(5) are all entirely beyond the scope of o.

Indeed, giving a complete account of (1) is somewhat challenging even based on $R_U(\sigma,\omega)$. The textbook by Sultan and Artzt (2011, pp. 242-250) provides such an explanation up to rational exponents. For the last step, it provides an insufficient "proof" while acknowledging that "there are some serious issues with this, and to get into all of them would be beyond the scope of this book" (p. 250). The serious issues concern the equation $\lim a^{q_n} = a^{\lim q_n}$ for a sequence (q_n) of rational numbers, assumed in the proof with no warranty even of the existence of the first limit and its independence of a choice of sequence (q_n) tending to a given real number. While one may argue that this could be sufficient for a high school teacher, $R_U(\sigma,\omega)$ can be developed to give a complete explanation by making use of the following property of $\mathbb{R}$, taught in first year and equivalent to the completeness property : every ascending sequence of real numbers with an upper bound, has a limit.

From a didactic viewpoint, the real challenge lies of course in the effective extension of $R_U(\sigma,\omega)$ and its possible consequences for $R_{HS}(t,o)$. In a recent version of the course (2011), we chose to present (1), (2) and (5) in lectures, while leaving the approaches (3) and (4), and the links between them, to an exercise. We now take a closer look at the tasks left to students and the outcomes.

Students' turn

The 25 students worked on the following exercise (part of a weekly assignment for group work; translated from Danish and slightly rephrased to be self-contained):

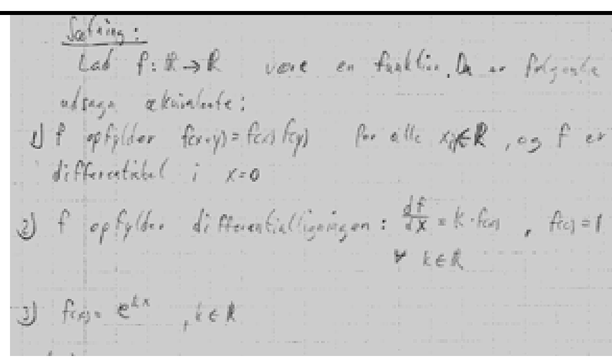
a) Show that if a function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ then $f(0) = 1$ or f is the zero function. Give a (non-zero) example of such a function.
b) Show that if a function as in a) is differentiable in 0, then it is differentiable on all of $\mathbb{R}$. [Hint: look at $(f(x+h)-f(x))/h$ for $h \neq 0$.] What is the derivative of f?
c) You know from calculus that the initial value problem $dy/dx = ky$, $y(0) = 1$ has the unique solution $y = e^{kx}$. Use this together with the results obtained in a) and b) to provide a characterization of exponential functions.

The point in b) is that the condition implies that f' exists and equals $f'(0) \cdot f$ in all cases. It is the *existence* of a solution, mentioned in c), which the "real meat" of the result assumed, as a special case of Picard's theorem which is treated in first year calculus. The uniqueness, which is the main point here, is quite easy to prove.

The first questions are technical questions and most groups were able to solve them on their own, with a few minor lapses like forgetting the case $f = 0$ in b). However, the third point proved to be challenging for almost all groups. The challenge seems to be the word "characterization" and also to link the form e^{kx} to the notation a^x privileged by the text book. In fact, the theoretical point of view involved with recognizing and formulating a "theorem" is not frequently required from students in $R_U(\sigma,\omega)$ at least in the work prior to UVmat. On the other hand, with the result recalled from calculus and the results proved before, there is – from a technical point of view – a small step to realize and prove that a function f is an exponential function (i.e. $f(x) = a^x$ for some $a \geq 0$) if and only if f is differentiable at 0 and satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Of course there are other possible formulations, and in this case we made a point of leaving the formulation open so that alternatives could appear (in fact they did). For instance, the term "exponential functions" could be defined as including the zero function, or not.

The students have access to supervision during their work, and many needed help to even get started on part c) of the task. Looking closer at students' final formulations of theorems, we find a number of shortcomings that are of a more logical nature. Here is one example (translation to the left, the original in Danish to the right):

<u>Theorem</u>: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Then the following propositions are equivalent:
1) f satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$ and f is differentiable at $x = 0$
2) f satisfies the differential equation
$df/dx = k \cdot f(x)$, $f(0) = 1$
$\forall k \in \mathbb{R}$
3) $f(x) = e^{kx}$, $k \in \mathbb{R}$


<u>Sætning:</u>
Lad $f:\mathbb{R}\rightarrow\mathbb{R}$ være en funktion. Da er følgende udsagn ækvivalente:
1) f opfylder $f(x+y) = f(x)f(y)$ for alle $x,y \in \mathbb{R}$, og f er differentiabel i $x=0$
2) f opfylder differentialligningen: $\frac{df}{dx} = k \cdot f(x)$, $f(0)=1$
$\forall\, k \in \mathbb{R}$
3) $f(x) = e^{kx}$, $k \in \mathbb{R}$

For instance, we note the strange appearance of the letter k in part 2) and 3). In 2) an existential quantifier ($\exists$) would be more relevant than the universal ($\forall$), while in 3), the unspecified status of the letter k is just as problematic. On the other hand, it appears from the students' proof that these points are really mainly of a formal nature. Indeed, many students produced essentially sound proofs despite occasional lacks of clarity in their statements, as above. For these students, we can focus on the formal features of mathematical expression which are, of course, of special importance to $R_{HS}(t,o)$ regardless of the piece of knowledge o involved, as the teacher should not only be able to express formal relations with a variation of formality while retaining basic correctness, but should also be capable of assessing less correct ones.

For other groups of students – representing roughly a quarter of the 25 students – there appear to be real challenges even in the parts of the reasoning that are completely within the kind of arguments that appear in HS text books. As an example, one group did not only formulate a "Theorem" which is formally incorrect, but also provided a proof that is flawed from the first lines, as the following excerpt (translated into English) shows:

> **Theorem:** An arbitrary function $f : \mathbb{R}\rightarrow\mathbb{R}$ which is differentiable in $f(0)$ and satisfies $f(x+y) = f(x)f(y), f(0) \neq 0$ for all $x, y \in \mathbb{R}$, is of the form e^{kx}
>
> **Proof:** A function f satisfying $f(x+y) = f(x)f(y)$ could be the exponential function. We therefore let $f(x) = e^{kx}$, $f(y) = e^{ky}$ in the rest of the proof. (…) f also satisfies $f(0) = 1$ since this is well known for the exponential function and can be easily seen by:
>
> If $y = -x \Rightarrow f(x-x) = e^{k(x-x)} = e^{k\cdot 0} = 1 \neq 0$
>
> (…) [*The group goes on to show, by a lengthy calculation repeating parts of the argument from b) in the special case f(x) =* e^{kx}*, that f satisfies the differential equation df/dx = kf. They conclude as follows*]
>
> From Picard's theorem we know that for all real numbers k there is one and only one solution to $dy/dx = ky$ satisfying $y(0) = 1$. As we have shown the exponential function satisfies this we know that the exponential function is the only one satisfying this, and the theorem is proved.

For students who produce "proofs" like this one, even formulating autonomously a proof within the core contents of o is likely to be a serious challenge, and one could say that the task c) is perhaps missing the level at which they need to improve their relation to basic HS knowledge. Fortunately, UVmat also offers many much simpler tasks and discussions, focusing more directly on o, such as a) and b), which in fact were solved almost perfectly by all groups, including those who failed on c).

PERSPECTIVES AND CONCLUSION

All of the five approaches (1)-(5) involve technical work that is to some extent beyond $R_{HS}(s,o)$; and as power series are not treated in HS, (5) is perhaps of less importance to $R_{HS}(t,o)$. The other four can strengthen $R_{HS}(t,o)$ in the sense of providing extensions or alternatives to standard presentations. In fact, the above tasks do not go beyond the formal boundaries of the HS curriculum which does include a formal definition of derivative as well as simple differential equations. The main point is to supply $R_U(\sigma,\omega)$ with an alertness to the non-triviality of *o* from a mathematical point of view, which could certainly be relevant to $R_{HS}(t,o)$; it could, , be achieved just by working on (1). However, to include alternatives such as (2), (3) and (4), which are strongly related to more advanced parts of the HS curriculum, is clearly of merit to $R_{HS}(t,o)$ as well. For instance, a rigorous approach to integrals has been part of the HS curriculum and might become part of it again, with the increasing use of computer algebra systems to take care of calculations; and then (2) would be a reasonable and elegant way to make up for the shortcomings of *o* based on (1), including the ease with which fundamental properties of exponential functions may be derived using this construction.

It is interesting to reflect on the difficulty which students had with c) even after solving a) and b) correctly. This exemplifies a phenomenon which we have noticed also in regular, semi-advanced analysis courses, namely a transition which occurs *within* U, and not as a result of changing institution. These transitions have to do with the kind of relation $R_U(\sigma,\omega)$ students need to develop to a university situated mathematical praxeology ω. In the terminology of Winsløw (2008), the transition is said to be of type I when theory is to be used and operated with by students; it is said to be of type II when tasks require students to autonomously develop theory.

In a) and b) students are faced with relative precise tasks, which – although they do involve the theoretical level of ω – can be carried using familiar techniques (operations with the definition of derivative etc.). However, in c) the task is to formulate and prove a "characterization of exponential functions"; it is less "precise" and students have no familiar technique to solve it.. This kind of task is indeed relevant to the position $R_{HS}(t,o)$ of a teacher relative to a piece of knowledge *o*, such as explaining the sense in which a property (to be precisely defined, as part of the task!) characterizes a class of mathematical objects (equally to be defined). Thus, while the relation $R_U(\sigma,\omega)$ necessary to solve a) and b) is already different from $R_{HS}(s,o)$, this is not so much due to a difference between *o* and ω, but because of the fact that theory (definitions, proof based on limit operations) has to be drawn upon by σ but usually not by *s*. On the other hand, going from b) to c) represents an instance of the transition $R_U(\sigma,\omega) \rightarrow R_U^*(\sigma,\omega)$ where the asterisk indicates autonomy with respect to the theoretical level of ω; notice that this is a transition of type II in the sense of Winsløw (2008). Typically, we find that $R_U^*(\sigma,\omega)$ is more relevant for a "mature" teacher relation $R_{HS}(t,o)$ than $R_U(\sigma,\omega)$. While a capstone course may have other agendas as well, this is an important point in many of the tasks proposed to students in UVmat, the main difference from other mathematics courses in the programme being that ω remains elementary and close to a corresponding *o* in HS.

By way of conclusion, we hypothesize that in general, the second institutional transition $R_U(\sigma,\omega) \rightarrow R_{HS}(t,o)$, as referred to in the introduction, is not only very different from the first institutional transition $R_{HS}(s,o) \rightarrow R_U(\sigma,\omega)$, it is also closely linked to transitions of type II within the university programme (in the above notation, $R_U(\sigma,\omega) \rightarrow R_U^*(\sigma,\omega)$). These are in general very hard to accomplish (cf. Winsløw, 2008). The aim of a capstone course could be – among other things – to achieve it, not in general, but for as selection of mathematical praxeologies ω close (if not identical) to praxeologies *o* which are, or could be, developed in high school. In short, a capstone course could strive deliberately to get students started in autonomous work with theoretical parts of the mathematical praxeologies ω that are most directly related to school mathematics praxeologies *o*.

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TEACHING AND LEARNING COMPLEX NUMBERS IN THE BEGINNING OF THE UNIVERSITY COURSE

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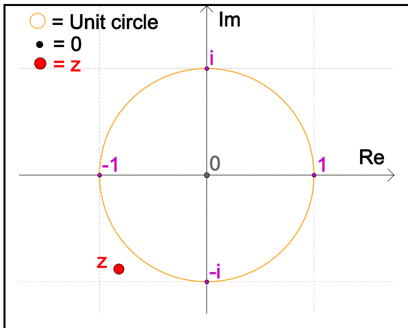
This contribution relates to the secondary-tertiary transition and to the new flexibility requirements (changes of frames or registers…). We especially study the teaching and learning of complex numbers, at the beginning of the university course. First, we present experimental teachings using the online basis of exercises WIMS [1], to work on different representations of complex numbers (Rennes, France). Secondly, we analyse students' productions working on complex numbers inside a geometrical frame (Namur, Belgium). It seems that students willingly move from a geometrical to an algebraic frame, but the choice of the most appropriate one remains a challenge.

COMPLEX NUMBERS AND TRANSITION

Different transition forms take place at the entrance of the university course (Gueudet, 2008). There is a clear need for greater flexibility and greater autonomy for the students to change from one frame to another (Douady, 2008), from one register to another (Duval, 1995)… As far as complex numbers are concerned these requirements are crystal clear. A complex number can be represented under its algebraic, trigonometric or even exponential form. It also has various interpretations in the geometrical frame, where it can be associated with a point, or with a vector, or as a transformation within a plane. Different studies point out difficulties associated with these requirements of flexibility, particularly with respect to geometric representations (Panaoura et al., 2006). We consider that they require specific teaching.

Our research questions are then: which teaching devices need to be set up in order to assist students to learn complex numbers, according to the above flexibility requirements? Why choose an on-line device to help students to learn complex numbers? Which strategies are used by students to carry out a task involving complex numbers in a geometric frame? Which difficulties do the students meet in performing such a task?

To answer these questions, the poster first presents two experimental teaching approaches put in place in Rennes (France): a face to face teaching on complex numbers given at the beginning of the year, and an online course prepared by a team of IREM (Research Institute on the Teaching of Mathematics), associated with some specific exercises available on the web server WIMS[1]. WIMS can propose exercises focusing on the technical aspects, as well as exercises requiring transfer from one register to another (like « complex shot », Figure 1). The choice of this online platform enables to meet two specificities of the work at university: acceleration of didactic time and greater availability of knowledge.



The figure on the left represents the plane of complex numbers, with a number $z=x+iy$ in the plane. Please find the position of the number $w = -i\bar{z}$.

To answer, click on the figure where you think that *w* is located.

Figure 1: Example of a "Complex shot" task – WIMS (Vandebrouck 2006)

Secondly, the poster presents a brief analysis of students' productions (Namur, Belgium) working on a task involving complex numbers in a geometric frame (exercise of a « complex shot » task). We have noticed that, for example, even if the geometrical frame is sufficient to solve the problem (by the addition of vectors for example), many students convert the problem under algebraic form, which requires more calculations and sometimes approximations. The poster presents extracts of several students' copies. It seems that students willingly move from a geometrical to an algebraic frame, but the choice of the most appropriate one remains a challenge.

NOTES

1. WWW Interactive Multipurpose Server, http://wims.unice.fr/wims/en_home.html

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OPEN ACCESS MATHS TEXTBOOK: STUDENTS' PERSPECTIVE

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Student perceptions (attitudes, behaviours, beliefs) regarding the use of an open source text were collected through pre- and post- surveys and student journals before and after major assignments and tests. This paper highlights student's characteristics related to their attitudes towards open access text/materials, course outcomes, and service learning projects.

RESEARCH QUESTION

What are students' attitudes, behaviours and beliefs regarding the use of published vs. open access textbooks for a maths course? What is the impact of the use of an open access textbook on student learning (self-reported)?

CONTEXT OF THE STUDY

Engaging non-STEM (science, technology, engineering, or maths) major students in an undergraduate maths course can be daunting. Students may be taking the class to check off a graduation requirement. Interest in the class may simply be passing a maths course and not "learning" maths. On many occasions students may not even purchase the \$100 - \$140 dollar text book. The cost of students' textbooks has increased faster than inflation (Office of Program Policy & Government Accountability, 2008, April).

Evidence of student engagement in learning has been linked to the use of open textbooks (Doering, Pereira, & Kuechler, 2010; Petrides, Jimes, Middleton-Detzner, Waling & Weiss, 2011). Open textbooks were found to support inquiry-based, interactive learning and pedagogy in a series of studies by Baker, Theirstein, Fletcher, Kaur, and Emmons (2009) and Hilton and Laman (2012). The hybrid introductory statistics course that is the subject of this paper was revamped with the purpose of: (a) using a free open access book with open access applets for exploring statistical content/concept; (b) infusing a social justice theme (Atweh, Forgasz, & Nebres, 2001; Solomon, 2009); (c) using Microsoft Excel with shareware add-ins for statistical analysis; and (d) requiring students to participate service learning with a non-governmental or non-profit organization in the design, implementation and analysis of data important to the service agency (Hadlock, 2005). The on-campus sessions were conducted within a computer classroom. The off-campus sessions were conducted utilizing Moodle and engaging with service agencies. Given the choice, 11 students selected an open source text and 9 students selected a traditional text.

RESULTS

The quantitative and qualitative data collected revealed the following: (a) A statically difference on the post survey between open source (M=1.6, SD=.70) and traditional (M=2.5, SD=.76) text students on the question: I believe that digital resources did increase my understanding of the course content; t(14.56)=-2.6, p=.021. (Scale: 1=strongly agree to 4=strongly disagree); (b) A statically difference on the post survey between open source (M=1.4, SD=.52) and traditional (M=2.25, SD=.46) text students on the question: I believe that open access textbooks provided the same level of quality as a traditional published print textbook; t(15.74)=-3.68, p=.002. (Scale: 1=strongly agree to 4=strongly disagree); (c) Reflective comments by students frequently reported on the value and challenge of service learning and infrequently commented on the value of an open source or traditional text; (d) More consistency in students' response to "use of learning materials several times a week" with the open access vs. traditional text; (e) More non-white (60% non-w vs. 40% w) students, more first-year students and more female (65% f vs. 35% m) students chose the open access over the traditional text; and (f) No statistical differences between the groups in attendance, grades, or overall course academic performance.

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STUDENTS' PERSONAL WORK
A CASE STUDY: BUSINESS SCHOOL PREPARATORY CLASSES

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The poster displays the skeleton of an ongoing research study pertaining to students' personal work in the learning of mathematics at an undergraduate level. It presents the tools used to explore students' personal work, and focuses on the main results found through the analysis of the data collected for students enrolled in three different tracks of business school preparatory classes (CPGE).

Keywords: mathematics learning, student personal work, CPGE

DESCRIPTION OF POSTER CONTENT

Section 1- Research Background

The study targets students enrolled in specific French higher-education institutions (*classes préparatoires aux grandes écoles de commerce* - CPGE). It focuses on the development of students of these institutions over one year, with regard to their acquired knowledge and methods, as well as the factors influencing this learning process.

Section 2- Research Questions and Rationale

- What are the forms of study expected by the teachers, and what are the study gestures exhibited by "successful" students on their own initiative, in addition to those that are taught, as opposed to those who fail?
- How does the personal work of the students evolve throughout a preparatory year, in terms of quantity, modalities, and acquired knowledge and methods? What factors influence this learning process?
- What is the impact of the institutional context on the personal work of students? What is the impact of social relationships on student work, in particular the relationships that are built between the students and those with the teachers?

We attempt to answer the above questions while focusing on two main aspects of the students' activities: problem solving and studying math lessons. We hope that this would help us understand how the work that is carried out, or not, by the students contributes to the difficulties which they face. The institutional context plays a major role in this study, hence the choice of the CPGE. These institutions are known for their selectivity and supportive culture, which favours student collaboration and close follow-up by teachers, in a relatively rigid high-school-like system. In fact, students enrolled in these institutions seem to be achieving better results than those in regular universities, where failure during the first years seems to be a serious widespread problem particularly in France. These institutions are viewed both in terms of the constraints they weigh on students and the resources they offer them, hence they constitute a rich and interesting field of study and observation.

Section 3- Theoretical Framework

The theoretical framework is still in the process of elaboration. This study comes as a continuation of the work of Castela (2011) on this topic. Hence, it is possible to include certain elements of the Anthropological Theory of the Didactic: first to consider the institutional dimension while analyzing the activities of students, who are subjects of particular institutions, which impose on them ways of doing and thinking; and second to account for the knowledge which is necessary for mathematics learning while emphasizing the idea that this type of knowledge is not taught, it is rather constructed by the students themselves. As for the cognitive dimension, other more psycho-social frameworks, yet to be defined, will be used.

Section 4- Data Collection

The study uses a combination of quantitative and qualitative methods which allow close follow up of several volunteer students throughout the year. So far, extensive data has been collected about teachers' practices and expectations, as well as student work habits, through several informal discussions, classroom observations and extensive email communication. In addition, three questionnaires have been designed, pilot-tested and filled out by the students of three different branches (S scientific, E economics and T technology) from four preparatory business schools involved in the study. They all include similar items organized into six categories, which were mainly inspired from prior research on the topic (general work habits, inside the classroom, taking notes, studying the lesson, solving exercises, evaluation of work and results), but contrast two moments of mathematics learning during a student's path (pre/post type): end of high school and end of first year of a preparatory class. The last phase of the study which has started in February 2013 is more clinical. It includes interviews with several volunteer students, one last questionnaire, and videotapes of student group work sessions.

Section 5- Some Results (until end of January 2013)

The focus of this section is on the data collected from the questionnaires. The main objective is to compare the results of the pre/post questionnaires in order to identify common ways of student work, and how these evolve throughout this first year of preparatory classes, in the particular context of these institutions.

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THE FIRST ACADEMIC YEAR – STEPS ON THE WAY TO MATHEMATICS

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To enable students who are becoming math teachers to provide a process related image of their subject in the future, they need to be made familiar with the subject's methods and – probably even more vital – they need a positive attitude towards mathematics. To this end, a program for the first academic year consisting of several measures, each of them complementary to the others, has been designed at the University of Hildesheim. Studies have been and will be conducted to evaluate the accomplishment of our goal. In this poster the program is presented and discussed.

STARTING SITUATION

Over the past decades there has been growing complaint about increasing problems of first year math students, including larger gaps between school and university with regard to subject-specific skills. Consequently, questions of supporting academic starters have become topic of research within the last few years (proceedings of conference about courses for mathematic newcomers are to be released soon, [2]). For beginners studying mathematics to become school teachers, additional aspects occur, there is on the one hand lack of motivation to do university math whilst on the other there are specific needs for this group of students. German curricula include the teaching of process related competencies such as problem solving, presenting and mathematical communication (as laid down e.g. in [4]), thus students aiming for being a teacher e.g. need to acquire the ability to communicate mathematics on very different levels. Further information on subject-specific requirements for math teachers can be found e.g. in [1].

INTRODUCTION OF THE PROGRAM

Within the frame of the new approach to the first academic year for those studying mathematics to become a teacher for primary or secondary school at the University of Hildesheim, modules were developed in order to help the students in being more successful and gaining a clearer view of their subject during their first two semesters. These should support math students taking their first steps at university towards subject-specific methods.

A preliminary two-week course focussed on repeating fundamental basics of school mathematics and a scholastic assessment test before the beginning of academic studies are the first two steps on the way. Those were evaluated in [3]. The third module is a weekly "tutorial-market" during the first semester, which gives students the opportunity for self-reliant group work on mathematical tasks and problems with assistance from lecturers and tutors if necessary. Within this arrangement students will be given the opportunity to communicate about mathematical contents and methods of problem solving. The fourth step, evaluated in ([5]), is a joint three-day study trip ("Mathe-Hütte") in the second semester where students work independently and literature-based in small groups on a mathematical subject. On the third day they present their findings to the other participants in the course of a poster session, thus getting an opportunity to communicate and present mathematical contents to others – skills particularly crucial for a school teacher. In order to support students assessing their own proficiency and approach towards mathematics after the first stage of their studies, there is a fifth and final step: a mathematical talk in which lecturers and students meet to discuss the students' abilities and attitudes. This talk completes the first academic year.

PERSPECTIVE

Further research has been initiated to evaluate the program with respect to the goal formulated in the introduction. Particular questions are whether the program helps to bridge the gap between school and university math and if it increases the teacher-specific skills in mathematics.

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ENHANCING BASIC SKILLS IN MODERN INTRODUCTORY ENGINEERING MATHEMATICS WITH HIGH IT INTEGRATION

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At a technical university, the freshmen meet technical textbooks and instructions that require mastery of basic pre-university mathematical skills. In this project we are testing these skills, and propose a curriculum redesign for introductory mathematics aimed for bridging the gaps. Mathematics 1 at the Technical University of Denmark (DTU), a course with high IT and Maple integration, now opens with a four-week paper and pencil course in complex numbers and functions. Since this topic is essential for the subsequent instruction in linear algebra and differential equations, we claim that this is a forward-looking and motivating method.

INTRODUCTION

Many research questions within the relation between mathematics and ICT is following this template: How can we in a paper and pencil teaching environment remedy the observed problem X by a dedicated and delimited use of the software programme Y. In this project the scenario is upside down. The large introductory two-semester course in mathematics (Math1) at DTU has since many years been in the frontline of the pedagogical-didactical development within teaching and learning mathematics at university level, especially regarding the integration of ICT (Markvorsen and Schmidt, 2012). Of special relevance here is that the advanced CAS-software program Maple for more than 10 years has been a fully integrated part of the course. As a response to the well-known worry of *the black box effect*, we introduced rules of thumb for a cautious CAS use in 2007-2010 (Schmidt, Rattleff and Hussmann, 2010). Recently, the CAS debate has been replicated at DTU, but now also as a high school-to-university-transition problem. It has been claimed that the widespread use of CAS in Danish high schools has caused a lack of basic mathematical skills, and that consequently many students are unable to follow the reasoning behind introductory technical textbooks (internal DTU paper, 2011). To counter this new challenge we have redesigned the first semester of the course.

CURRICULUM REDESIGN AND INQUIRY DESIGN

In 2012 we redesigned Math 1, so that it now opens with a four-week paper and pencil sub-course in complex numbers and simple complex functions. It includes precisely what is needed for linear algebra and the subsequent presentation of (systems of) linear differential equations in a frame of linear transformations between spaces of functions. But to be able to cope with complex numbers and functions the students need to deepen their understanding of (or learn for the first time!) a big range of pre-university topics from parenthesis, fractions and equations to elementary real functions and differentiation rules. We claim that this method to enhance the basic skills is more motivating and efficient than a traditional and back-looking repair/brush up course in high school topics.

The **inquiry data** consists of the following components: 1) At the very first day of the math course the students were assigned a 30 minute test in real numbers and elementary real functions (Test1). 2) After the end of the four-week sub-course the students received the same 30 minute test with small parameter-changes (Test2). 3) One week after Test2 the students were asked to answer an electronic questionnaire (EQ) about their experiences of the four-week complex number course.

SOME RESULTS AND CONCLUSIONS

In Test1 the students had in average 12.7 correct answers out of 24 (simple) questions. This indicates a lack of basic skills. As a majority of the students were unsatisfied with their performance and at the same time a vast majority found it highly relevant for "a new-coming engineering student" to be able to answer the questions (EQ), we conclude that the new course design is a motivating factor. The figure below shows that there in general was a significant progress from Test1 to Test2. We underline that the students' individual progress was very different.

<table>
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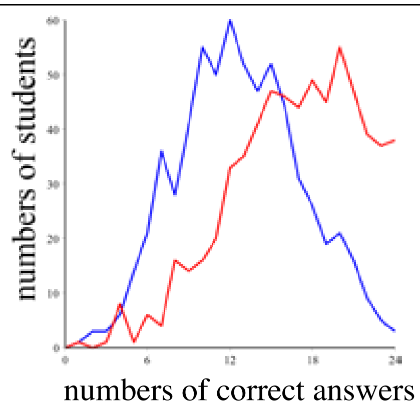
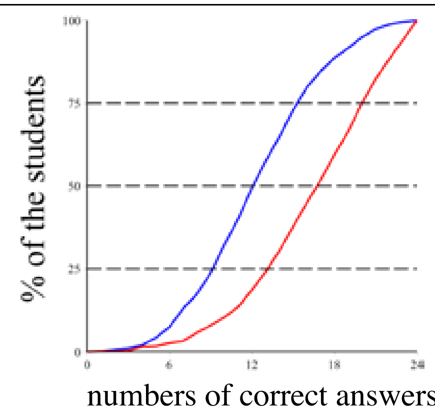
<tr><td>
</td><td>
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USING "IRDO" MODEL TO IDENTIFY ERRORS MADE BY STUDENTS IN DIFFERENTIAL EQUATIONS EXAMS

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In this action research report, we investigated the errors made by students in their Differential Equations (DE) exams and classified them in order to develop a model that describes the causes of students' low scores as a result of those errors. Knowing the causes would help the lecturer to be more cautious about the roots of students' errors and it would also allow the researchers to help students overcome their errors and enhance their understanding about DE problems. This could improve students' abilities to perform better on their DE exams and develop their self-confidence too. In this study, four types of errors were identified as Identifying, Recalling, Doing and Overviewing (IRDO).

This paper reports on the results of a PhD study of the first author that has several phases. Here we report on the preliminary findings of the first phase of the pilot study. Literature review on teaching and learning of Differential Equations (DE) helped us to realize that teaching and learning of the DE goes much beyond memorizing a set of rules, algorithms, and procedures to solve a set of routine problems. For instance, the DEs could be studied through the use of graphical, numerical and contextual systems of representations as well.

Research studies have investigated the ways in which university students understand DE concepts and become skilful in solving related problems (Arslan, 2010; Camacho-Machín, Perdomo-Díaz & Santos-Trigo, 2012; Kwon, 2005, 2002). The research findings are useful for educators and researches in this field and help them to learn more about new approaches on teaching and learning of DE. However, students' performances in DE exams have not been the subject of much scrutiny. Specifically, students' errors in DE tasks and the causes of these errors have not been investigated so far. Thus, identifying and categorising the errors could help the educators to implement the appropriate teaching strategies that might be useful for students to improve their performances in DE problems.

This study aims to contribute in the categorisation of the errors in DE tasks. To do this, the study focused on students' exam papers. The authors, firstly, tried to understand the roots of the students' errors based on the evidences of their responses on the exam papers; then they categorised these errors accordingly; and, finally, they put together these categorisations in order to generalise into a model for identifying errors that students might make while dealing with DE tasks. For example, some students were not able to realize that $x(1-x)y'' + \frac{1}{2}(x+1)y' - \frac{1}{2}y = 0$ is not a

"Second Order Exact DE" and it is not a "Second Order Cauchy–Euler" either, but it is a "Second Order Homogeneous Linear Equations with non-Constant Coefficients" instead. When students study DEs, they need to learn to look at a DE and classify it into one of the above three groups. The reason is that the techniques for solving differential equations in each group are common within that group. Our study showed that the "Identifying" errors usually happen, when students fail to distinguish between different classifications of DEs; namely "First Order", "Second Order", "Linear vs. Non-linear", "Homogeneous vs. Non-homogeneous", etc. Further, "Identifying errors" happen when students are not able to identify specific kind of DEs such as "Separable", "Exact", "Bernoulli", "Cauchy–Euler", "Abel's formula", "Riccati", etc. Additionally, we named "Recalling" errors the cases in which students fail to use an easier solution techniques; "Doing" errors the usually mathematical errors when students work out a mathematical expression; and "Overviewing" errors when students do not pay attention on overviewing their work and identifying potential mistakes. These four types of errors comprise the elements of the IRDO (Identifying, Recalling, Doing, Overviewing) model of students' errors in DE tasks.

Throughout the DE course that the first author taught, he witnessed the benefits of introducing the IRDO model to the class and using it to help students to improve their performance in the exams. Additionally the IRDO model enhanced the mathematical communication in class and allowed the lecturer to assist the students to have better control in the exam questions; to organize more efficient their work; and, consequently, to improve their exam scores.

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG15: TECHNOLOGIES AND RESOURCES IN MATHEMATICS EDUCATION

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Technologies in mathematics education have been a topic of a working group since the CERME 1999. The last two conferences confirmed the relevance of considering information and communication technology (ICT) in mathematics education within a range of various resources, such as software, hand-held devices, online classroom activities, but also more traditional geometry tools or textbooks. A number of important issues related to the design of technologies and resources and their use by students, teachers and teacher educators emerged from discussions within the working group at the previous conferences, such as the importance of coping with a strong interconnectedness of mathematical and technological knowledge in student-tool interactions, or the necessity of including the users' feedback in the ICT and resource design processes (Trgalová, Fuglestad, Maracci, Weigand, 2011). Concerning teacher education questions appeared about the complexity of ICT and resource integration in teachers' practices due to a double instrumental genesis, both personal and professional one (Haspekian, 2011). The idea of teachers' communities, sharing resources and practices in using ICT, emerged as a powerful means to favour teachers' professional development (Wenger, 1998; Jaworski, 2005). These issues require further theoretical and methodological developments.

The call for contributions proposed to deepen these issues in the following three themes: (1) Design and use of technologies and resources, (2) students' learning with technologies and resources, and (3) technologies, resources and teachers' professional development. The group work combined plenary sessions where common issues were discussed, such as theoretical frameworks used in the field of ICT in math education or conceptualization of mathematics with ICT, and parallel sessions addressing various topics, such as software design, task design, teachers' professional development toward the ICT integration, or reports of empirical studies of ICT use. In what follows, we give a brief overview of the working group and the discussions within it, and we outline a few perspectives for the next conference.

WORKING GROUP IN A FEW NUMBERS

The group involved 44 participants from 15 countries: Brazil, Czech Republic, Denmark, France, Germany, Greece, Iceland, Ireland, Israel, Italy, Mexico, Norway, Sweden, Turkey, and United Kingdom. The 27 accepted papers and 6 posters were distributed according to the three themes as follows:

(1) Design and use of technologies and resources	4 papers	
(2) Students' learning with technologies and resources	13 papers	3 posters
(3) Technologies, resources and teachers' professional development	10 papers	3 posters

Table 1: Distribution of papers and posters in the three themes.

REPORT OF THE WORKING GROUP DISCUSSIONS

In spite of several attempts toward building an integrative theoretical framework allowing to address the issues of ICT in math education (e.g. EU project ReMath, Artigue & Mariotti, to appear), the discussions within the WG15 highlighted a ***variety of theoretical approaches*** used, such as constructionism (*Kynigos & Moustaki*), theory of didactic situations (*Joubert*), instrumental approach (*Misfeld*), variation theory (*Attorps et al*.). *Abboud-Blanchard & Vandebrouck* proposed a new theoretical frame for studying the teachers' professional development, which integrates some of these theories, and *Tabach* developed a general framework to describe instrumental orchestration. Necessity of networking of theories appeared as an issue. Suggestions on how to proceed were formulated: look at the same set of data with the lens of different frameworks or analyse a given instrumented task from different theoretical viewpoints.

Particular importance was given to ***software*** and ***task design***. *Lagrange & Psycharis* explored the potential of a learning environment and *Mackrell et al*. designed tasks for a specific environment, both studies drawing on two complementary theoretical frames. *Libbrecht & Kortenkamp* highlighted the importance of metadata in the design of learning activities and *Libbrecht & Zimmermann* brought forward didactical design patterns that can impact software construction processes. *Pilet et al.* designed a piece of software allowing teachers to automatically generate exercises for differentiated instruction adapted to learning needs of various students' groups. Task design was explored by *Robová & Vondrová* in relation with the integration of netbooks in math classes. *Müller* proposed an instrument, drawn from history of mathematics, allowing to document changes in the type of tasks proposed in technology environment.

A number of contributions addressed the issue of ***teacher professional development***. They described and analysed various means of teacher development, formal (e.g., *Akkoç; Santos-Trigo et al*.; *Sollervall*) or informal (*Trgalová & Jahn*), in presence (*Balgalmis et al*., *Dullius*) or on-line (*Fredriksen*; *Gravina et al.*). The efficiency of these different means need to be evaluated, and for this, specific methodologies have to be developed. The participants raised a need for a model allowing to analyze the evolution of teachers' practices related to the ICT use. *Clark-Wilson* identified perturbations when teachers use technology due to discontinuities within teachers' knowledge which she calls "hiccups".

Students' learning of mathematical concepts with specific software was discussed in numerous papers. Students' conceptualisation processes with dynamic geometry were explored by *Henning & Hoffkamp* (the concept of limit), *Kilic* (elementary geometry), *Kaya et al.* (transformational geometry) and *Pettersson* (linear functions). The use of CAS and handheld calculators was investigated by *Persson* (development of problem solving skills with TI-Nspire), *Rieß & Greefrath* (lower achieving students' learning of functions with CAS-calculator), *Storfossen* (primary school mathematics with graphic calculator) and *Grønbæk* (use of professional CAS to teach upper secondary mathematics). *Weigand* explored the interplay between mental, digital and paper representations in a CAS environment in tests and examinations.

A general issue of a ***capitalisation of research results*** was raised in the light of an important number of research studies on ICT in mathematic education. First steps toward this was done by *Bray* who provided an overview of recent technological interventions in mathematics education and examined the educational affordances of the technology, and *Scheffer* who proposed a survey of research conducted by undergraduate, master and doctoral students, aimed at promoting reflections about the ICT integration.

CONCLUDING REMARKS AND PERSPECTIVES

The papers and posters presented and discussed within the group show a big variety of research topics, theoretical and methodological approaches, which is undoubtedly a sign of a richness of this scientific domain. However, the feeling is that the research presented in these contributions is rather local, focusing on a particular aspect of teaching and/or learning mathematics, rather short-term, not allowing to draw general conclusions about the benefits of ICT in mathematics education, and quite often conducted in controlled, laboratory conditions.

Thus, the participants expressed a need to know more about the "real" use of ICT in mathematics classrooms and outside, but also why ICT is not used. Long-term studies focusing on "ordinary" teachers and "ordinary" classrooms are necessary to explore the impact of the ICT use on students' performances and on teachers' practices. Such studies require developing specific methodologies enabling to assess the effectiveness of ICT in learning processes.

Moreover, research working toward a definition of recommendations or guidelines for teachers suggesting how to use efficiently ICT is missing. For this, surveys of international comparative studies highlighting best practices with ICT use could/should be conducted.

Little or no studies focus on how mathematics as a school subject matter is impacted by the ICT use. We know since many years that computers can perform complex calculations and that educational software provides multiple representations offering reliable visualisations of mathematical concepts. But some "old" questions are still

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Little or no studies focus on how mathematics as a school subject matter is impacted by the ICT use. We know since many years that computers can perform complex calculations and that educational software provides multiple representations offering reliable visualisations of mathematical concepts. But some "old" questions are still present and valid: Do we still need to teach and learn the same mathematics as before the rise of technology? How can we define the educational added value of ICT use for the learning of mathematics given the explosion of new tools and functionalities provided in technology-rich environments? How can we connect the mathematics embedded within particular ICT tools and curricular mathematics?

In addition, there are emerging research themes in education which are still underrepresented in CERME: the design and use of innovative technologies such as Web 2.0 or mobile technologies, or the design and use of technologies and resources for learners with special educational needs.

Clearly, these concerns could define a research agenda for the coming years. On a CERME working group scale, especially the following two tasks can be carried out:

- Working towards common understanding of theoretical constructs and research vocabulary used in various national contexts. This can be done for example by organizing during the sessions a group work on the analysis of concrete material (e.g. a task, an ICT tool, a set of data) from different theoretical perspectives;
- Working towards capitalising research outcomes, especially from previous technology working groups at CERME conferences. Each contribution to the working group should draw, when relevant, on the results of studies presented at previous conferences.

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Pilet Julia, Chenevotot Françoise, Grugeon Brigitte, El Kechaï Naïma, Delozanne Elisabeth - Bridging diagnosis and learning of elementary algebra using technologies

Rieß Michael, Greefrath Gilbert - Results on the function concept of lower achieving students using handheld CAS-calculators in a long-term study

Robová Jarmila, Vondrová Naďa - Pupils' role and types of tasks in one-to-one computing in mathematics teaching

Santos-Trigo Manuel, Camacho-Machín Matías, Moreno-Moreno Mar - Using dynamic software to foster prospective teachers' problem solving inquiry

Sollervall Håkan - Threshold constructs instrumenting teachers' orchestration of an inquiry with GeoGebra

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Müller Matthias - A new instrument to document changes in technological learning environments for mathematical activities drawn from history

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Scheffer Nilce Fátima - Information and communication technologies and mathematics teaching: researches conducted from elementary to higher education

GENESES OF TECHNOLOGY USES: A THEORETICAL MODEL TO STUDY THE DEVELOPMENT OF TEACHERS' PRACTICES IN TECHNOLOGY ENVIRONMENTS

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This paper introduces and discusses a new frame aiming to describe and characterize the evolutions of teachers' practices related to technology uses. Articulating several theoretical developments of Activity Theory enables us to conceive a model to interpret the evolutions in terms of "geneses of technology uses". We consider these geneses as movements which articulate three levels of organization of practices related to the temporality of actions and to the goals/motives of the activity: a micro level, a local level and a macro level.

Research studying mathematics teachers' practices when using technology were very fruitful in the two last decades. Authors investigated this field by using diverse theoretical frames. In the European sphere, Monaghan used the Saxe cultural model (Monaghan, 2004) to study the emergent goals of teachers in ordinary technology-based lessons. Ruthven (2007) introduced five key components structuring the classroom practices in the context of technology uses. Drijvers et al. (2010) develops Trouche's concept of instrumental orchestrations (2004), by defining several type of orchestrations in the technology-rich mathematics classroom. Even if these studies have considered some evolutions in teachers' practices, they have not adopted a theoretical perspective to investigate it.

We conducted two studies of teachers practices, carried out on the long term. The main common issue of these studies was to observe the teacher's activity in classroom and its impact on the students' learning. At a methodological level, we have particularly analyzed students' tasks, teachers' options regarding these tasks and the corresponding management of students' work on these tasks. The first research (Abboud-Blanchard, Cazes & Vandebrouck 2007, Abboud-Blanchard & Cazes 2012) dealt with specific tools, called here Electronic Exercises Bases (EEB). These are software applications that mainly consist of classified exercises with an associated environment. The research provided a set of data about thirty teachers over 3 years. The analysis was both qualitative and quantitative, and related to: lessons preparations, class observations and answers to questionnaires and interviews. The second research (Abboud-Blanchard & Lenfant-Courblin 2009, Abboud-Blanchard et al. 2008) investigated the first professional uses of technology by pre-service mathematics teachers in order to understand the conditions in which these uses take place. The data was of two types: professional dissertations about using technology in classroom and interviews carried out with them at the end of their first year of teaching practice. The analysis enabled an exploration of the uses of technology in two phases of the teacher's work, which are preparation work and classroom work. The analysis carried out within these two research studies led us to question the evolutions of teachers' practices and to conceptualize them by constructing a *model of geneses of technology uses*. This construct articulates several theoretical frames deriving from Activity Theory, mainly the instrumental approach and the double approach of teachers' practices. The aim of this paper is not to provide a detailed presentation of our studies but rather to introduce our model and to show how it can give meaning to the observed evolutions.

In the first section, we present the fundaments of our frame starting from elements related to Activity Theory (AT). We introduce "geneses of technology uses" as movements which articulate three levels of organization of practices. In sections 2 and 3, we describe these movements between the levels and give examples to illustrate our purpose.

THEORETICAL CONSIDERATIONS

Activity Theory takes the object-oriented-artifact-mediated-collective activity system as its unit of analysis, thus bridging the gulf between the individual subject and the societal structure (Engestrom, 1999). In this paper, AT enables us to study mathematics teachers' technology uses, by considering the unit of the context of the teaching activity, while being particularly sensitive to the mediations between the subjects (teachers) and the object of their activity (the students and their mathematical work). However, we address AT from a cognitive individual perspective, taking into account the context surrounding the teacher's activity.

More precisely, we use the concepts of AT in the context of French research, as developed within the field of ergonomic psychology (*i.e.* cognitive ergonomics) (Leplat 1997). This development highlighted the fundamental distinctions between, on the one hand, subject and situation and on the other hand, task and activity. Within this French approach, the activity of the subject is indeed developed in situ. The task corresponds to the goal the subject must achieve, taking into account a double system of determinants which relate to the subject or the situation. The development of AT within cognitive ergonomics research brought Verillon and Rabardel (1995) to introduce the Instrumental Approach (IA) which distinguishes artifact and instrument, the artifact being on the situation's side and the instrument on the subject's side. IA consists, for a given subject, in a process (instrumental genesis) of appropriation and transformation of a given artifact, in order to accomplish a given task, within a variety of context uses and a same class of situations. IA was then developed in the field of French mathematics education by Artigue (2002), Guin and Trouche (2002) and Lagrange (2005).

In our work, we articulate the cognitive individual perspective of AT with the cognitive aspects of the ergonomic approach. This allows introducing a developmental approach not specifically highlighted in the AT, which enables us to characterize the evolutions of teachers' technology uses that we call *geneses of* *technology uses*. In particular, we consider that geneses of technology uses *encompass* instrumental genesis of specific artifacts. Indeed, the instrumental approach, applied to the teacher, does not deal with the practices as a whole. It does not take into account the context surrounding the teaching activity, neither the connection between teachers' instrumental geneses and students' activities, which are key elements that we aim to investigate in our work. This fact leads us to articulate the above concepts of AT with the double didactic and ergonomic approach of the teaching practices (Robert and Rogalski 2005, Robert 2012).

The double approach adapts AT to the teaching of mathematics in school situation. It aims at the study of the teacher's practices starting from the study of various forms of his/her activity in class and for the class. It introduces five components of teaching practices. The didactical components, cognitive and mediative, translate recurring choices in the activity of teachers that are related to the activity of students. The cognitive component translates choices related to the design and organization of tasks for the students. The mediative component translates choices during sessions related to the management of students' work on the tasks and interactions through verbal communication. The ergonomic components complete the former and are related to the professional context of teaching. Personal, institutional and social components respectively translate the personal (beliefs, knowledge and experience) determinants and the determinants related to the teaching situation (social constraints, curricula…).

The double approach postulates the stability of practices. This stability is, at the same time, a postulate, in the double approach, and a result of a set of research work relating to the teaching practices, with or without technology (see for example Vandebrouck (ed.) 2008). In our study, we consider that AT takes into account, in a global way, for the study of the geneses of technology uses, the complex articulation between stability of practices and evolutions of the activity in situ.

To achieve a degree of stability concerning the evolutions of the activity, we introduce three levels of organization of practices (Abboud-Blanchard & Vandebrouck 2012). These levels are related on the one hand to the temporality of the action and on the other hand to the goal/motives of the activity:

- The micro level is about "automatisms", for example elementary gestures of the teacher for the preparation of lessons' management. One can bring this micro level closer to the short-term of the action and the level of the operations within the AT (Galperine 1966, Léontiev 1978). However, the construction of the micro level of teaching practices (constitution of the actions' schemes or the operations' schemes (Vergnaud, 1982) can be done only over the long-term of the action.
- The local level refers to everyday actions, where co-exist preparations and improvisations; the level of the adaptation of the teacher to the students' work.

The goals of the teacher's activity are basically related here to management's issues and to teacher-students interactions. Studying this level allows to better understand what precise activities the students could have developed, related to the task, the teacher's discourse and help and also the nature of the students' work.

- The global level refers to scenarios, preparations, assessments… It can be reached by studying the motives of the teacher's planning of his/her teaching project. Some of these motives may explicitly occur after a reflective view on the action already accomplished.

We consider that a teacher who begins using a technological tool does not have sufficient "automatisms" and routines related to this use, neither a holistic view on the organization of a coherent teaching approach that integrates this tool. In response to this unfamiliar "overload situation" at the local level, several phenomena would take place:

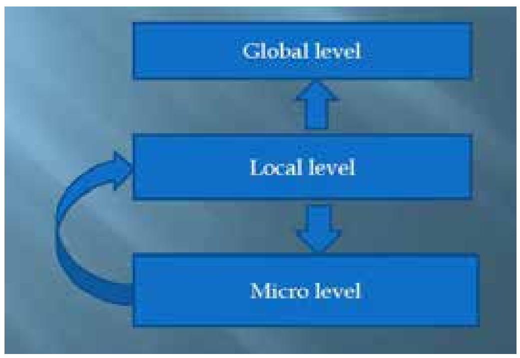

Figure 1: the three phenomena modelizing the geneses of technology uses

- a first phenomenon reflects the way that the teacher makes use of his/her 'traditional' paper-and-pencil practices for the context of technology uses: the micro level of practices supporting the local level. We describe this phenomenon in the second section;
- two other phenomena are involved in the geneses of technology uses. We interpret these as « *movements* » going from the local level of practices towards both the global level and the micro level. We describe in the third section theses movements and their articulation with the IA.

FROM THE MICRO LEVEL TO THE LOCAL LEVEL

A teacher's first session in the classroom using a new tool reveals phenomena that lead the teacher to manage his/her session at the local level within an « improvisation mode ». Even if the session is well-prepared, some difficulties occur that relate to classroom management. It seems that the teacher bases his/her management on his/her traditional practice, which allows him/her to better deal with the technology session at the local level. In other words, it seems that the automatic regulation of teaching practice at the micro level allows the teacher to cope with difficulties emerging during the technology based session at the local level.

For instance, a teacher using EEB provides different help to her students depending on whether they are more or less in difficulty. She gives more procedural help if the student is really in trouble (reinforcing one method) and she provides more constructive help (giving other possible methods…) if the student is more comfortable with the task - Robert (2012) distinguishes teachers' procedural and constructive help for the students: procedural help allows students to perform their mathematical tasks and a constructive help promotes more directly the students' constructive activity. However, during the interview post-observation, it appears that the teacher is not aware of this differentiated practice. This can be related to a characteristic of the mediative component of her practice.

Nevertheless, the use of such micro level of "automatisms" does not ensure efficiency at the local level. For instance, a teacher gives the same individual help to each student, as he does usually during traditional setting with the whole class group, considering that students may be at the same level of task resolution. Therefore some students do not succeed to overcome their difficulties even after the teacher's help. Indeed this help is not adapted to their personal resolution trajectory. For other students, the teacher answers questions they have not yet asked.

Thus, faced with difficulties at the local level, some teachers feel the need to build new specific practices with technology, while others tend to reduce the role of technology within their teaching. As an example of this last case, the students of a beginning teacher had not understood the task during a technology based session. The teacher realizes only at the end of the session that the students were really in trouble with the technological aspects of the task. This observation led her to give a posteriori explanation, during a traditional session, in order to be sure that the task is correctly understood by students, without although considering changes of the technological aspects of the task.

MOVEMENTS STARTING FROM THE LOCAL LEVEL

The phenomenon described in the previous section concerns teachers performing classroom activities in order to manage the classroom on the short term; whereas geneses of technology uses are mainly related to developments at the medium and long-term of the action. Our analysis reveals an evolution of the whole teaching project, including the uses of technologies, starting from their uses at the local level. We interpret this evolution as a movement from the local level of practices to the global level. In parallel, there is a development of technology uses at the micro level entailed by frequent uses at the local level.

Towards the global level

The main movement seems to be an evolution of scenarios. It concerns what we might call, referring to the double approach, the cognitive component of the geneses of use. There is a new balance between traditional sessions and technology sessions, between collective work and individual phases of students' activity or between old and new mathematical knowledge in students' activity.

As an illustration of this type of evolutions, we note that all our studies mentioned above emphasize that teachers promote quickly the use of 'paper notes' within the students' activity involving technology. For instance, teachers using EEB insist that students use a sheet to keep notes and some of them promote use of a specific notebook devoted to technology sessions. This use of paper evidences is an aspect of the integration at the long-term of technology activity within ordinary activities of students.

Another aspect of this movement is the fact that some teachers evolve towards a systematic implementation of collective moments such as phases of institutionalization at the end of technology sessions. When observed, these collective phases are often based on the use by the teacher of students' written notes. This observation shows particularly the articulation with the above evolution and the relative complexity of the movement. However, the nature of the technological tool or the kind of technology session can lead in some cases to opposite phenomena. For instance, we observe the complete absence of such phases at the end of training sessions with EEB. These sessions seem to be too individualized and too heterogeneous to permit such collective moments.

This movement, from local to global level, explains some difficulties for the integration of technology within ordinary practices as a consequence of the stability of practices. Indeed, for some teachers, established practices at the global level are incompatible with such a development of practices.

Towards the micro level

In this paragraph, we deal with the development of technology uses at the micro level, starting from their uses at the local level of practices. This development is particularly related to teachers' instrumental genesis of specific artifacts. Indeed, the identification of schemes of uses which characterize instrumental geneses is related to both the micro level and the local level of practices, i.e. during the process, the teachers builds operations schemes of use of the artifact as well as actions schemes. However, in the model we provide, the identification of schemes is not a central issue. We rather examine the evolution of technology uses through the lens of teachers' practices dealing with students' activities. Moreover, Abboud-Blanchard and Lagrange (2006) have shown that there exist two dimensions in instrumental genesis of teachers: the personal dimension and the professional dimension.

Understanding geneses of technology uses as a whole, needs to consider these two dimensions.

Personal instrumental genesis concerns teachers' activity with technology for their personal uses, as well as for teaching preparation. For instance, a beginning teacher was meeting dynamic geometry for the first time. Starting from a position of apprehension ("*it is something that scared me*"), she firstly evolved with the development of some practices related to the preparation phase. She used dynamic geometry to draw geometric figures to be integrated in students' worksheets. Then, she developed some practices in which she feels confident without however really reflect on the educational aspects of practices with this technology: a whole class video projection, where students are simple spectators of "technological images".

Professional instrumental genesis concerns both the personal appropriation by the teacher of the software in order to be efficient when using it with students and the actual use in classroom. For instance, an experimented teacher using EEB for the second year, has felt the need to master all the functionalities of this technological tool in order to be able to better adapt his assistance to students for each given exercise. He developed three types of actions to manage the students' activity in computer room. When he arrives to help a student, he first tries to understand the path of the student by questioning him/her, secondly he provides the appropriate help and finally he gives advices enabling the student to pursue his work without the presence of the teacher.

In both these cases, what teachers experienced at the local level, led them to develop micro actions which might enrich their routines when using technologies. Moreover, such development sometimes goes further and impact traditional practices involving both mathematical tasks and classroom management. For instance, a teacher who uses spreadsheet in both private and professional sphere on regular bases has improved her way of introducing algebra. She built new tasks, not necessarily using spreadsheet but although inspired by her instrumentation of spreadsheet. In addition, when performing algebraic tasks, she authorizes students to have tinkering procedures as they would have done with the spreadsheet.

DISCUSSION

The frame that we presented in this paper aims to describe and characterize the evolutions of teachers' practices in their connections with students' activity. The instrumental approach, applied to the teacher, goes also in this direction but does not however deal with the practices as a whole (as we explained above).

We think that our model can better grasp the complexity of the emergence in situ and evolution at the middle and long term of teachers' practices related to technology-based mathematics lessons. It articulates three levels of the organization of practices, and it opens to new meaning for these complexity and evolution. The variety and multiplicity of uses at the local level leads to:

- developments at the global level, mainly evolutions of teaching projects and scenarios, which go beyond the instrumental geneses of specific artifacts;
- developments at the micro level, i.e. the teacher developing new "automatisms" related to the use of technology in the classroom. These automatisms concern of course the schemes of uses of specific artifacts but can also concern other types of automatisms about mathematics tasks or class management.

Moreover, the model also leads to new meaning for the somehow observed stability of practices in technology based environments. Indeed, the stability of teaching practices seems to result in "automatisms" at the micro level scaffolding the local daily practices. It seems to prevent some teachers who have difficulties to manage technology sessions at a local level from creating new "automatisms" specific to technology at the micro level or seeking to change globally their practices. The micro level of ordinary practices (not involving technology) could thus act as a barrier to the integration of new tools.

In the double approach, the stability of practices is based theoretically on the consistency between the five components of practices: cognitive, mediative, personal, institutional and social. In our studies we have identified geneses of technology uses related mainly to cognitive (tasks and scenarios) and meditative components (class management) of teachers' practices. We also investigated the determinants of these geneses, related to personal, social and institutional components (Abboud-Blanchard & Vandebrouck, 2012). In this short paper, we chose only to consider the geneses of technology uses through the lens of the two didactical components, in order to better exemplify the dynamic phenomena between the three levels of practices, in the "real technology-classroom life".

Finally, the model draws originally on features of the double approach of practices, yet it enriches this general frame by adding a dynamic modeling of the practices' development. Hence, we wonder if it can be generalized to the evolution of any given practice (not specifically addressing technology uses).

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PRE-SERVICE MATHEMATICS TEACHERS' PRACTICE OF QUESTIONING IN COMPUTER LEARNING ENVIRONMENTS[1]

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This paper focuses on a specific aspect of formative assessment, namely questioning. Given that computers have gained widespread use in learning and teaching, specific attention should be made when organising formative assessment in computer learning environments (CLE's). A course was designed aiming to develop knowledge and skills of questioning in CLE's for the purpose of formative assessment. This case study investigates how a pre-service mathematics teacher used questioning in the classroom to introduce the derivative concept using Geogebra and Graphic Calculus software. The findings indicated that the course provided a guideline for pre-service mathematics teachers in planning and using effective questioning in CLE's.

INTRODUCTION

Assessment plays an integral role in teaching. However, as Heritage (2007) point out, assessment and teaching have been traditionally seen as reciprocal activities as a result of measurement concerns such as high-stakes accountability of testing. Many researchers mention that good practice yields from a recognition of both summative and formative purposes of assessment and use them accordingly (Dwyer, 1998).

Despite its importance for learning and teaching, assessment has not been a main focus of teacher training courses. Furthermore, administrators "also lack training in assessment and therefore do not have the skills to support the development of assessment competencies" (Heritage, 2007, p. 4). Dywer (1998) mentions that, courses on evaluation of learning have been disappearing from teacher education programs. However, she claims that it is well understood by experienced teachers and assessment is well targeted in many professional development programmes for in-service teachers (Danielson, 1996 as cited in Dywer, 1998).

Given that computers have gained widespread use in learning and teaching, specific attention should be made when organising assessment in computer learning environments (CLE's). A successful integration of technology into instruction requires an integration of technology into assessment. On the other hand, there is little research on how to organise assessment as an integral part of teaching in computer learning environments (Kissane *et al*., 1996).

Considering the need to incorporate assessment component into pre-service teacher education programs and the importance of integration of technology into instruction as suggested by the relevant literature, we designed a course for pre-service

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mathematics teachers, which aims to develop their assessment skills as a component of TPCK. This paper focuses on how a pre-service teacher developed the knowledge and skills of formative assessment in computer learning environments. Particular attention is given to questioning which occur during classroom assessment.

THEORETICAL FRAMEWORK

In teacher education research, assessment has been considered as an important component of teachers' knowledge of pedagogy. After Shulman (1986) suggested pedagogical content knowledge (PCK) as a separate domain of teacher knowledge, many researchers such as Tamir (1988) defined assessment as a component of PCK. Pierson (1999) and Mishra & Koehler (2006) has added the technology component to PCK framework and defined Technological Pedagogical Content Knowledge (TPCK) framework. Although, in the literature, the components of the TPCK framework have been defined as parallel to the components of PCK framework, assessment as a component of TPCK has not been sufficiently dealt with.

The theoretical perspective of this study is situated within the distinction between summative and formative purposes of assessment. Summative assessment is used for the purpose of grading or certifying students. On the other hand, formative assessment intends "to monitor student progress during instruction to identify the students' learning successes and failures so that adjustments in instruction and learning can be made" (Gronlund, 2006, p. 6). There is research evidence of the extraordinary effectiveness of formative assessment (Black & Wiliam, 1998). Despite its importance, most pre-service teachers use assessment for summative purposes while a minority uses for formative purposes (Volante & Fazio, 2007). There are various aspects of formative assessment: uses of tests to diagnose what students have already known or using the evaluation of homework in decision making for the next lesson and classroom assessment which occurs in the classroom on a daily basis. Among those, researchers point out classroom assessment as an area of difficulty which is encountered by pre-service teachers (Mavrommatis, 1997).

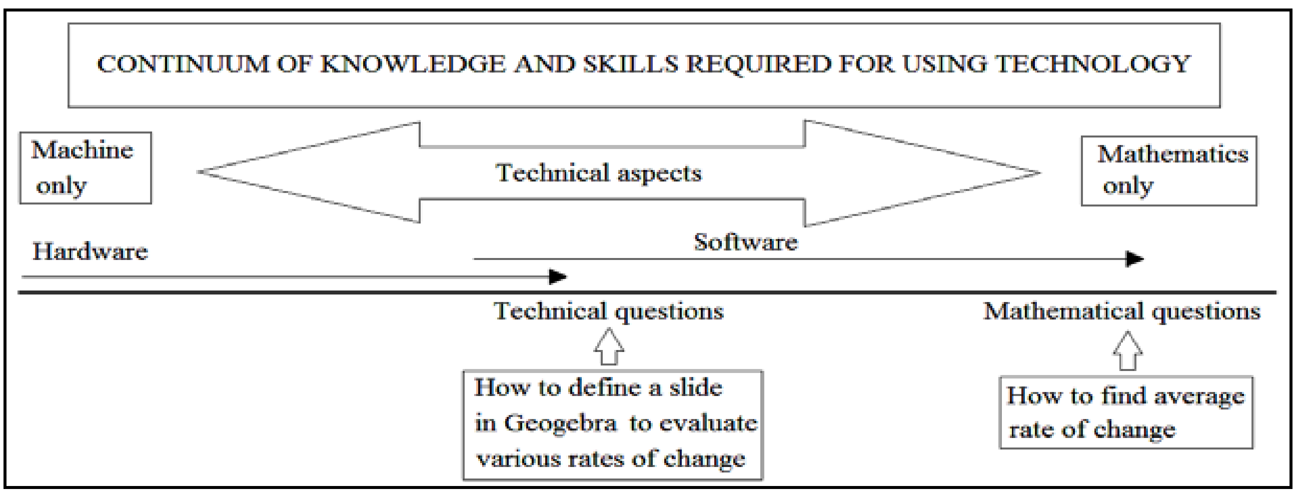
Classroom assessment refers to the processes of collecting information and making interpretations and decisions based on this information on a daily basis in order to improve teaching and learning (Airasian, 1991 as cited in Mavrommatis, 1997). In that process, questioning is an important information-gathering technique by which teachers can monitor student learning. Airasian & Jones (1993) claim that pre-service teachers are not given adequate training in developing questioning strategies and, indeed, that some receive no training at all. Therefore, questioning that can facilitate formative assessment for the purpose of learning should receive more attention in the preparation of teachers.

Given that computers have gained widespread use in learning and teaching, specific attention should be made to questioning in CLE's. Therefore, this study focuses on the design of a course aiming to develop knowledge and skills of questioning in

CLE's for the purpose of formative assessment. To analyse pre-service teachers' questioning, Pierce & Stacey's (2004) framework is adopted. Their framework identifies the main characteristics of students' interactions with CAS technology. They specify aspects of effective use of CAS which they suggest to adopt to other mathematical software tools. Using a CAS in particular or any other software in general to do mathematics requires both traditional mathematical knowledge (which could be constructed through the use of paper and pencil methods as part of a lesson in CLE) and knowledge of the machine. These two requires a constant interplay which Pierce & Stacey (2004) defined as *technical aspect* of effective use of technology which could be described as the knowledge and skills related to the software rather than the hardware of the machine. It is where mathematics meets machine (e.g. fluent use of software syntax, ability to systematically change representation or interpreting the software output) as mentioned by Pierce & Stacey (2004).

In the framework, two types of questions will be distinguished. The first is *mathematical questions* which aim to assess what Pierce & Stacey (2004) call traditional mathematical knowledge. The second will be called *technical questions* which attend to technical aspects of using technology. Although questions in this category seem to focus on what the software perform, there is a constant interplay of mathematical knowledge and knowledge about the technology (See Figure 1).

Figure 1: Continuum of knowledge and skills required for using questioning in CLE's



The aim of this paper is to explore what kinds of issues come into question in CLE's in terms of questioning for formative purposes. In this respect we formulate the following research question: "How do pre-service mathematics teachers use mathematical and technical questions for formative purposes in CLE's?

COURSE DESIGN

In a wider context, this study is part of a research project for which we designed a course guided by TPCK framework. In this paper, the description of the course is restricted to its assessment component. An eight-hour workshop was conducted on assessment. During the first phase, which we call PCK workshop, general information on assessment, its integral relationship with learning and teaching, and examples of summative and formative assessment was given. This is followed by activities during which forty pre-service teachers worked in groups. In the first activity, pre-service teachers were asked to specify objectives of a lesson which introduces the concepts such as function and derivative. They presented their objectives to their peers and discussed each group's objectives in an interactive way. For the second activity, they designed lesson activities to achieve their objectives. As they began to structure their activities, they were asked to prepare questions to provoke student thinking for the purpose of attaining their lesson objectives. The aims of such questioning in a lesson were explained in relation to classroom assessment for formative purposes. At the next phase, which we call technological knowledge (TK) workshop, pre-service teachers learnt how to use computer software and did hands-on-activities in a computer lab in groups of twenty. They used Graphic Calculus, Geogebra, Probability Explorer, Excel and Cabri Geometry software. This phase focused on the technical knowledge of the software. The last phase, which we call TPCK workshop, focused on the pedagogy of using technology with specific attention given to the assessment component. Focusing on the content, that is function and derivative, pre-service teachers were asked to re-consider their lesson activities and how to attend to assessment of their lessons. They also practiced various computer based assessment tools such as dynamic worksheets of Geogebra and Inspiration software for making concept maps. During this phase, we focused on the following questions with regard to assessment in general and questioning in computer learning environments in particular:

- How would assessment and evaluation techniques/tools change when concepts such as function and derivative are taught using technological tools?
- How can technology be used for summative/formative assessment to achieve lesson objectives which you specified for the lessons for function and derivative concepts?

Questions above were discussed with pre-service teachers during the workshops considering a specific lesson objective as shown below:

Let us consider the following lesson objective: • Students will be able to express derivative at a point as instantaneous rate of change. To assess whether this objective is achieved by students, ask questions with the following purposes: • What kinds of questions could be asked during a lesson in CLE's for summative/formative purposes to promote thinking in accordance with lesson objectives?

Table 1: Points of discussion concerning questioning during the workshop

Following this, workshops focused on classroom assessment in CLE's and how to evaluate students' understanding when they use technology. We emphasised that the nature of probing questions will be changed as a result of change of media in the classroom. Pre-service teachers were encouraged to ask questions on what were performed by the software and their mathematical meanings to promote purposeful use of technology.

METHODOLOGY AND CONTEXT OF THE STUDY

This study is part of a research project which aims to develop a programme for pre-service mathematics teachers guided by TPCK framework. The research has been carried out in a mathematics teacher education program in a state university in Istanbul, Turkey.

Following the TPCK workshop which was explained in detail above, pre-service teachers were asked to prepare lesson plans which introduced the concepts of function and derivative at a point as the first part of the program. In these lesson plans, they were also asked to explain what kinds of assessment they plan for their lessons. Ten pre-service teachers taught these lessons as part of micro-teaching activities and discussed their assessment approaches with their peers. This way, pre-service teachers had the chance to put their knowledge of assessment into practice. In the second part of the program, pre-service teachers planned and conducted their own workshops of TPCK on various mathematical concepts such as limit, continuity, integral, probability and radian and did micro-teaching activities.

For the current exploratory study, a case study was conducted to investigate a pre-service mathematics teacher's practice of questioning for the purpose of formative assessment in CLE's. The pre-service teacher, Güven, is male and twenty-two years old. He completed mathematics courses which lasted for three and a half years and started to take education and mathematics education courses. The data was collected during "Mathematics Teaching Methods II" and "Instructional Technologies and Material Development" course. Pre-service teachers participated in the program were asked to prepare a lesson plan with detailed teaching notes to introduce the concepts of function and derivative and they were interviewed on their lesson preparations. Semi-structured interviews, which included a section on how assessment is planned, were conducted. In addition to that, pre-service teacher's lesson and his reflections at the end of the lesson were video-taped. This paper focuses on the analysis of Güven's lesson plan, verbatim transcripts of his interview on the preparation of his lesson plan, and video of his micro-teaching lesson on derivative.

FINDINGS

In this section, findings will be presented in two sub-sections. The first sub-section focuses on how Güven planned to use questioning for formative purposes in his second lesson plan on derivative at a point which he prepared after the TPCK workshop. In the second sub-section, findings from the analysis of Güven's lesson will be presented with excerpts demonstrating his questioning in the classroom.

Güven's planning for questioning

Güven included the following problem in his lesson plan to start his lesson:

> *Engineers who design car templates are working on the highest velocity that the template is going to reach after two seconds. They evaluated the distance during the first five minutes and they represent it with the function $f(x) = x^2$.*

During the interview, he mentioned that he chose this problem to create a cognitive disequilibrium. He also added that he would use a lot questioning to start a discussion on the problem in the classroom.

Güven's lesson plan draft included two sub-sections on assessment: assessment during the lesson and assessment at the end of the lesson. Güven wrote a few questions to be asked during the lesson for two different purposes: diagnostic purposes and formative purposes. For formative purposes, he mentioned that he would check whether students (that is their peers) had learnt what he intended to teach using these questions. Some of these questions in his plan were specific to the software he used, namely Geogebra and Graphic Calculus. One example of these is the following: "How does Graphic Calculus calculate the values for rate of change? Find one of these values with paper and pencil". During the interview he said the following:

> Güven: The formative questions that I prepared were related to the activities that were performed on the computer. Students (his peers) performed these activities by looking at the computer and making calculations.

As can be seen from the question in his lesson plan and excerpts above, Güven purposefully planned for formative assessment. The question above can be considered as a technical question since it requires both the knowledge of how to evaluate the values of rates of change and the knowledge of the software.

Güven's practice of questioning

In practice, Güven used a lot of questioning for formative purposes during his micro-teaching lesson which he taught to his peers. Below, a detailed account of his questioning approach during his lesson is presented. Strength and weaknesses of his pedagogical approach to using mathematical and technical questioning will be discussed below.

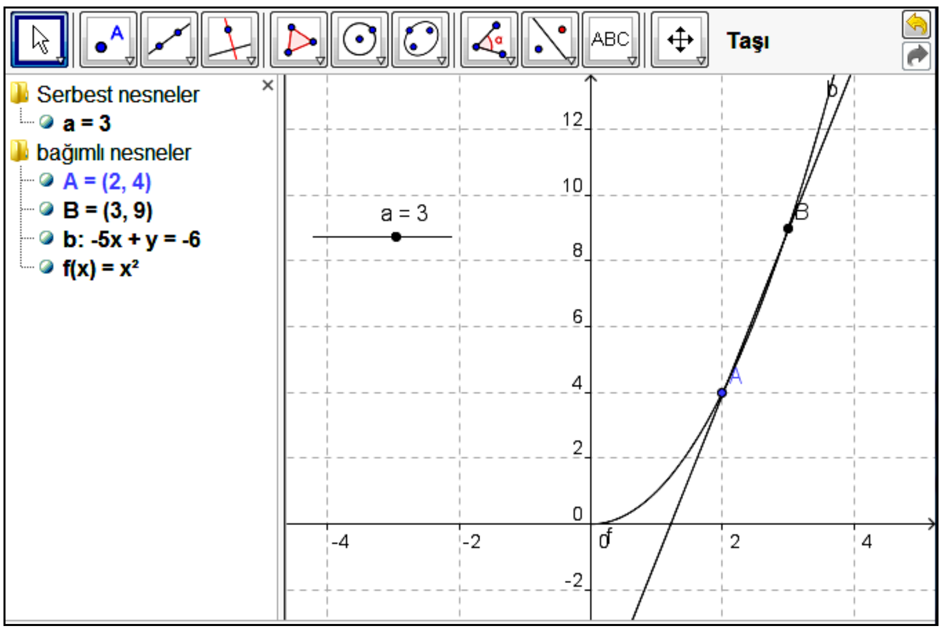
In the computer lab his peers were in front of the computers in pairs. Güven started his lesson with the problem above. After asking questions about velocity and instantaneous velocity to assess their prior knowledge, Güven asked his peers to find the average velocity in the first two seconds which is [*f*(2)-*f*(0)]/(2-0)=2. Following this mathematical question, he asked how to represent the average velocity using

Geogebra software which is a technical question. At this point, it should be mentioned that this question can be either solved graphically (finding the slope of the tangent line in the equation of the tangent) or numerically using spreadsheet view. Instead of letting his peers to chose the representation to find the average velocity, he preferred to explain it on the graph using Geogebra. He then asked how to represent the average value on the graph by plotting two points on the graph which is again a technical question:

Güven: Did we specify two points, both (2,4) and (0,0). Are these, change in y divided by change in x? Let us check it again using Geogebra. But how?

He then mentioned that he would define rate of change using "slide" feature of Geogebra. Although he asked technical questions about how to do it, he immediately demonstrated it in a step by step manner without waiting for the class to do it in front of their own computers.

Figure 2: Geogebra activity used by Güven to explain graphical meaning of derivative



After obtaining the graph as shown in Figure 2 above, he asked the following questions to help his peers discover the relationship between average velocity and the slope of the chord:

Güven: What else can velocity between A and B be equal to? Let's think about it on the graph. Let's draw a chord from A to B. What would velocity be equal to in terms of the chord?

Student: Rate of change, slope

Güven: Well, we can draw a straight line through two points using Geogebra.

After that point he went back to the problem he asked in the beginning of the lesson and asked his peers to find the highest velocity in the first two seconds:

Güven: We chose two points for the average velocity. How can we find the velocity at a point? For instance, let's move your slide in Geogebra.

Student: We can't find it. It becomes undefined.

Güven: How did you choose the points to approach? Where does the point *A* approach to?

Student: To the point *B*.

Güven: When the points are on *B* it's undefined. Let's see it on the table.

As can be seen from the excerpts above, Güven asked questions to help his peers find the instantaneous velocity and told them to move the slide. In other words, using the slide feature of the software he wanted them to interpret the outcomes of the software and find instantaneous velocity. Therefore, his questions above can be considered as technical questions where the slide feature of Geogebra interplays with the knowledge of instantaneous velocity.

After that point, he focused on the table as well graph to explain the instantaneous velocity. To do that, he used the spreadsheet view of Geogebra and evaluated $\Delta y/\Delta x$ and asked his peers to interpret different values obtained on the table by moving the slide:

Güven: (Pointing out slide a). Is "a" at 0? Let's trace this point on the slide and see what happens in the table. What happened now? Let's interpret these values (He moved around the class, observed what everybody did in front of their computers and helped them when they needed).

Student: When does the point A approach to the point B, it's $\Delta y/\Delta x$

Güven: Well, what would be the velocity of the car template at 2? What is your guess?

As can be seen from the excerpts above, Güven used technical questions which focus on the outcomes on the screen and promotes an understanding of instantaneous velocity. After getting the answer for instantaneous velocity, Güven went back to the geometrical meaning of it and asked his peers to find out where the chord approaches to. After getting "tangent" as the answer, he focused on the relationship between velocity and tangent with the following question:

Güven: Fatih, could you find a relationship between the velocity at the 2nd second and the tangent? I'm asking this question to everybody.

After that, he used the properties of the slide in Geogebra to get closer points by changing the increment from 1 to 0.1:

Güven: It becomes 3.99. OK. Is this enough for you?...Can we get closer values?

After that, Güven asked his peers to start Graphic Calculus software and to find out how the software calculate the rate of change ($\Delta y/\Delta x$) for smaller Δx. Some of the pre-service teachers mentioned that it approached to 4 when Δx is very small.

Güven: 0.0001 and this gives us 4. Can the slope be equal to 4?

Student: It can't be.

At this point he mentioned about the limitations of the software and that the slope of the chord can never be equal to 4 but the software makes an approximation. He then questioned the idea of limit and explained the mathematical definition of derivative at a point. To do that, Güven asked questions to promote an intuitive understanding of limit using the software and moved to the mathematical definition of derivative as the limit of rates of change.

DISCUSSION

Findings above indicated some strength and weaknesses of the pre-service teacher in integrating technology into his formative assessment practice. Güven was successful at asking technical questions which have two purposes. First of all, he focused on technical aspects and how the software perform certain tasks e.g. how to get smaller values of rate of change using the slide feature of Geogebra. Second, Güven used these questions to focus on the mathematical meaning behind what is observed on the computer screen e.g. the rate of change being 3.99 as the increment of the slide becomes 0.1, in other words, the idea of limit. At this point, Güven used a lot questions to promote an intuitive understanding of limit and its relationship with instantaneous velocity. In that sense, it can be claimed that his technical questions successfully focused on the interaction between technical and mathematical aspects. More importantly, interview data indicated his awareness of using the technical questions to promote an understanding of derivative.

Although Güven used extensive questioning during his lesson, he had some pedagogical weaknesses. For example, he did not give enough time to his peers to interpret the outcomes of the software and discover mathematical ideas. He performed some of the tasks by himself in a step by step manner which might yield to loosing the purpose of the task and dismiss the potential interplay between technical and mathematical aspects.

The study had some implications concerning the courses designed for pre-service mathematics teachers. As mentioned in the literature, assessment in general and assessment in CLE's in particular have not been a main focus of teacher training courses (Dywer, 1998; Heritage, 2007). This study aimed to help pre-service teachers equip with the required knowledge of formative assessment in CLE's. The workshops focused on how to use questions during a lesson in CLE's and pre-service teachers were encouraged to ask questions on what were performed by the software and their mathematical meanings to promote purposeful use of technology. The workshop was effective in the sense that it provided a guideline for pre-service teachers to use questioning in CLE’s.

This study has also implications at a theoretical level. Theoretical framework adapted from Pierce & Stacey (2004) provided a theoretical lens to analyse questioning practice for formative purposes in CLE's. For a further study, this framework could also be used to guide the programs for pre-service or in-service teachers in terms of how to use questioning in CLE’s.

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TEACHING INVERSE FUNCTIONS AT TERTIARY LEVEL

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This study is a part of an ongoing research that attempts to explain the relationship between the teachers' instructional practise and students' learning in the context of functions and function inverses. The question in this paper is how the use of technology as a pedagogical tool may contribute to the understanding of the inverse function concept. An engineering student group (n =17) was taught functions and inverse functions with the assistance of GeoGebra. In our theoretical framework we apply Variation theory together with the theory of Concept image and Concept definition. The data were gathered by doing a pre and post test concerning inverse functions. Our experiment revealed that students' concept images in the post test were more developed compared with the results in the pre test.

Keywords: Concept definition, Concept image, Inverse functions, Technology, Variation theory

INTRODUCTION

The function concept is a central but difficult topic in mathematics and therefore it has received considerable attention in mathematics education (Akkus, Hand & Seymour, 2008; Ponce, 2007). Strong understanding of the concept of function is crucial for any student hoping to understand calculus, which is a critical course for the prospective teachers, engineers, and mathematicians.

Functions have different faces, and to make students aware of that is a pedagogical challenge for teachers in mathematics. A number of studies have been conducted concerning students' understanding of the functions at the tertiary level, confirming a frequent inconsistency in students' conceptions of function and the definition of function (e.g., Thomas, 2003; Thompson,1994; Vinner & Dreyfus, 1989). Vinner and Dreyfus (1989) conducted one study, showing that tertiary students during a course in calculus, even when the students were able to correctly formulate the definition of function, could not apply the definition of function successfully.

Even (1992) conducted an investigation of prospective secondary math teachers' understanding of inverse functions. She found that many students conceptualized a function inverse using the notion of 'undoing'. "'Undoing' is an informal meaning of inverse function which captures the essence of the definition" (ibid., p. 557).

Bayazit and Gray (2004) investigated student learning of function inverses from two teachers, Ahmet and Mehmet. Ahmet focused his instruction on the idea of inverse "undoing" operations, whereas Mehmet on algorithmic and procedural skills (Bayazit & Gray, 2004). Students were given pre test and post test to evaluate their understanding about inverse functions before and after the classroom instruction.

Results from the post test indicated that more students from the class of Ahmet were able to answer a question regarding the domain and range of inverse functions correctly using verbal explanation. In Ahmet's class 25% of the students chose to take a global approach in reflecting the function across the line $y = x$, while no students in Mehmet's class used this method. The authors conclude that in order to grasp the concept of inverse function, students would be given the opportunities to experience conceptually focused tasks (Bayazit & Gray, 2004, p. 109).

Can technology as a pedagogical tool help students to understand different faces of the concept of the inverse function? Technology is becoming increasingly used at teaching of university mathematics but there are still few studies which have examined technology-assisted teaching at the university level, even though university mathematics teaching has been changing quickly during the past two decades (Attorps et al., 2011; Lavicza, 2006; 2007; Zimmermann, 1991).

THEORETICAL FRAMEWORK

In our study we apply two theoretical frameworks, the Variation theory and the theory of Concept image and Concept definition. We consider these two theoretical frameworks to be complementary. The theoretical constructs of concept image and concept definition have proven to be a useful analytical tool for nearly three decades (Tall & Vinner, 1981). According to Vinner (1991) and Tall (1999), each mathematical concept is associated with concept definition and concept image. The concept definition can be the stipulated as a definition assigned to a given concept. The concept image, on the other hand, is a nonverbal representation of any individuals understanding of a concept. It includes the "visual representations, the mental pictures, the impressions and the experiences associated with the concept name" (Vinner, 1991, p. 68). We agree with Vinner in believing that many mathematics instructors generally would imagine that their students' concept image is growing out of a delivered concept definition in class and normally supplied with a textbook definition. In our study, it is our strong belief that by using the GeoGebra software, we have affected the student's concept image of functions and function inverses.

Another interesting framework in our study is Variation theory. Teaching and learning research has found that ways of experiencing something are essential to what learning takes place (Shulman, 1986). Marton & Booth (1997) stated that qualitatively changed ways of experiencing something is the most advanced form of learning. If we can describe learning as coming to experience something in a changed way, we should also acknowledge that experiencing something must require the ability to discern this new way of seeing the experience. Central in Variation theory is an assumption that variation is needed to discern aspects of object of learning not previously distinguished by learners. According to this theory the most powerful factor concerning students' learning is how the object of learning is handled in a teaching situation. Marton et al. (2004, p. 16) have identified four patterns of variation in a learning object: contrast, generalization, separation and fusion. They are described as follows:

Contrast: … in order to experience something, a person must experience something else to compare it with.

Generalization: … in order to fully understand what ‘‘three’’ is, we must also experience varying appearances of ‘‘three’’…’’

Separation: In order to experience a certain aspect of something, and in order to separate this aspect from other aspects, it must vary while other aspects remain invariant.

Fusion: If there are several critical aspects that the learner has to take into consideration at the same time, they must all be experienced simultaneously.

According to Leung (2003), these patterns of variation create opportunities for the students to understand the underlying formal abstract concept. In order to generate the patterns of variation, we use the dynamical nature of the GeoGebra software, which has the “ability to visually make explicit the implicit dynamism of ‘thinking about’ mathematical, in particular geometrical, concepts” (Leung, 2003).

THE PURPOSE OF THE STUDY

The aim of this study is to investigate if the technology-assisted teaching of functions and function inverses at the university level can contribute to the development of engineering students’ understanding of the concept of function and function inverse. The study investigates the following research questions:

1) How can the patterns of variation be visualized by using GeoGebra when teaching the concept of function and function inverse?
2) Which qualitative differences between the students’ concept image of the concepts function and function inverse could be distinguished in pre and post test results?

METHOD AND DESIGN OF THE STUDY

The study took place during one teaching session in mathematics at a Swedish university. A total of 17 students were involved. They were all students at the engineering program, studying the course Calculus in one variable. The data were gathered by analysing the teaching sequences during the lecture and by doing a pre and post test. In the analysis of the test results we started by making an easy quantitative overview. Then we continued with a qualitative analysis of the outcomes of the students’ answers.

The pre and post test

The test contained five questions, including both conceptual and procedural ones. Students had maximum 30 minutes to do the test. It was not allowed to use any technical facilities. In this paper we focus on the following three questions:

1. How would you explain if someone asks you: What do you mean by the concept of function? You may like to explain by drawing a picture.

2. How would you explain if someone asks you: What do you mean by the concept of inverse function? You may like to explain by drawing a picture.

3. How would you explain if someone asks you: What do you think are necessary conditions for a function to have an inverse? You may like to explain by drawing a picture.

In order to categorize the answers to the questions above based on their quality, we coded the answers as following: no explanation, incorrect explanation, acceptable explanation, good explanation and excellent explanation.

RESULTS

We used the pre test results as a starting point to design our lecture. In order to create different teaching sequences that could encourage students to discern varying aspects of the object of learning, we applied the ideas of the variation theory in the context of the free dynamic mathematics software GeoGebra.

Teaching sequences

Teaching sequences were implemented in an ordinary lecture with a teacher manipulating the computer and students observing the screen. In the first application of GeoGebra (Figure 1), we visualize, by using dynamically the vertical test, that the graph is a function (the most left picture in Figure 1). By applying the horizontal test, also dynamically, on this function we can see that it doesn't possess an inverse (the picture in the middle in Figure 1). However, by shrinking and moving the domain of the function we can find an interval where the function is invertible.

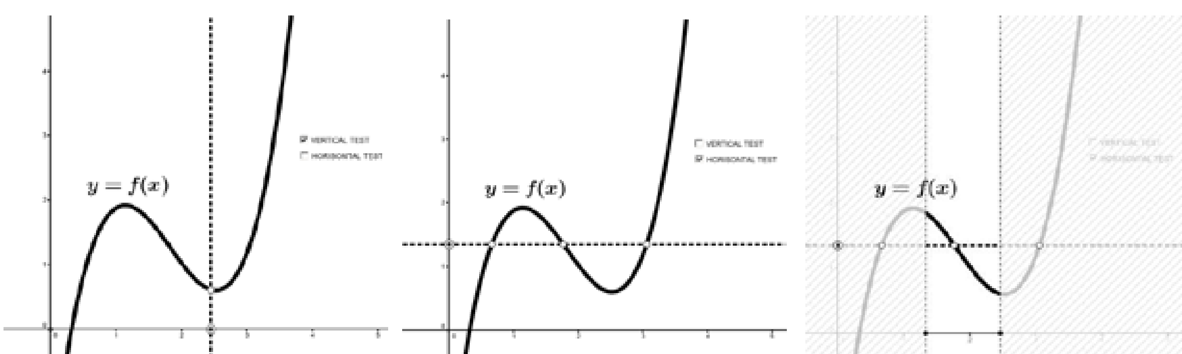


Figure 1: The domain of the function whose graph passes the vertical test has to be adjusted in order to make the graph to pass the horizontal test.

In order to experience the pattern of variation, *generalization*, i.e. to experience that the vertical test works in all positions for the given function, in Figure 1 (the left picture) we moved the vertical line through several points in the domain of the function. In the teaching sequences related to Figure 1 (the middle picture), the students were given opportunities to experience a *contrast*, i.e., to discern that the horizontal test both works and for some points doesn't work on the same graph depending on the position of the horizontal test line. In the right picture in Figure 1, we illustrated the pattern of variation called *separation* by changing both the length and the position of the interval representing the domain of the function. In this way the students were given the opportunity to experience one of the necessary conditions for a function to have an inverse, namely, being strictly monotonic.

The second example (Figure 2) should help the students to understand how to plot a given invertible function and its inverse in the same coordinate system. We use here the dynamical nature of GeoGebra to show how an arbitrary point on the function graph is reflected in the line y = x.

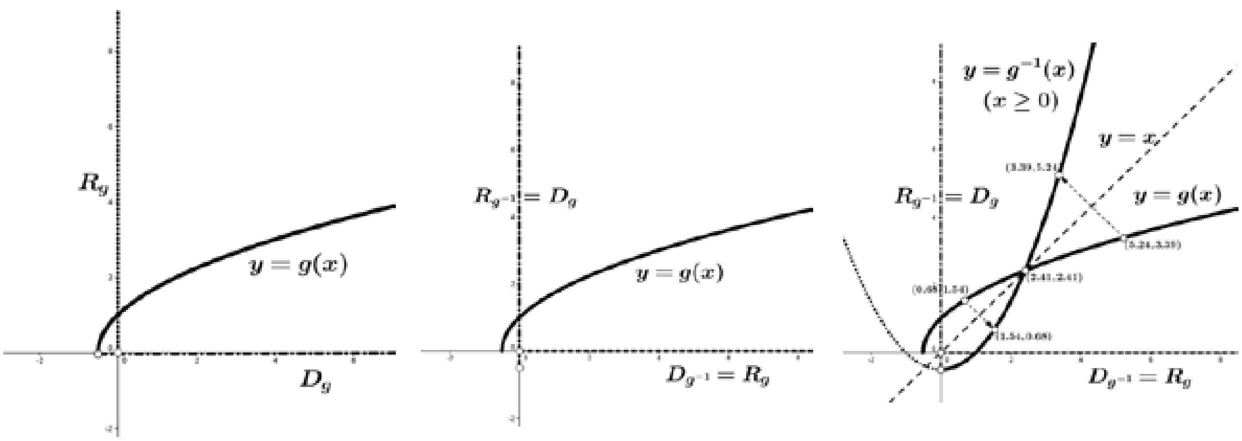


Figure 2: Plotting the graph of function g(x) and its inverse g^{-1}(x).

The teaching sequence illustrated by Figure 2 should help the students to understand how to plot a given invertible function and its inverse in the same coordinate system. They were given the opportunity to experience the pattern of variation – *fusion*. The critical aspects that they could discern simultaneously were the following three: reflecting of the function graph through the line y = x, in the corresponding points x- and y-coordinates switching the positions and observing that the domain of the inverse must be restricted.

In the third presentation (Figure 3) we wanted to illustrate an informal conception of inverse function, "undoing", which captures the essence of the definition.

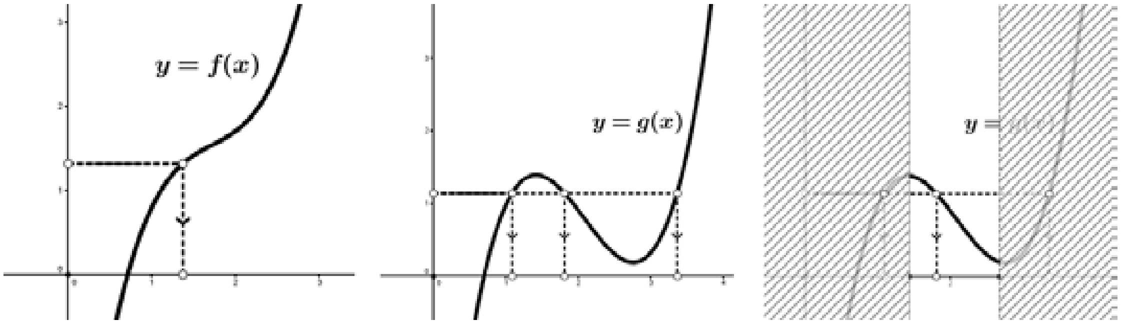


Figure 3: Illustrating of the "undoing" process.

In Figure 3 we wanted to illustrate the concept of "undoing". In the first two pictures to the left we created the opportunity for students to experience the pattern of variation – *contrast*. In the third picture to the right we separated the crucial condition concerning the existence of the inverse, namely, strictly monotonicity of a function connected to the informal notion "undoing".

Qualitative differences distinguished in pre and post test results

The Geogebra lecture began with a brief review of the concept of function, familiar to students from previous mathematics courses.

Figure 4 shows an overview of how students responded to the questions in the pre and post tests according to our criteria described in the Methods section.

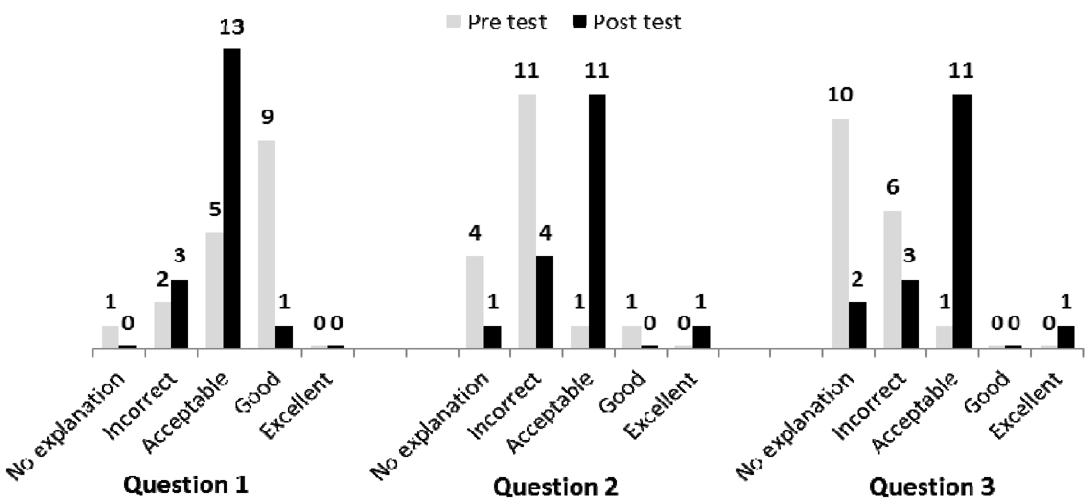


Figure 4: Students' pre and post test responses to the test questions.

In order to find qualitative differences between pre and post test results we selected sex students' responses to analyse the changes in their concept image more deeply.

The first question in our test focused on the perceptions students had about the function concept. It turned out that we could not notice any direct qualitative differences between pre and post test results. The students already had adequate pre knowledge of this notion.

The second question in our test centered on the students' conceptions of inverse functions. When we analysed the students' responses in the pre and post tests we noticed qualitative differences in their concept image. Some of the students responded in the pre test as follows:

Student 1: The inverse function is a function that is undoing another function.

Student 2: It is a reflection of a function

Student 3: It is a mirror function

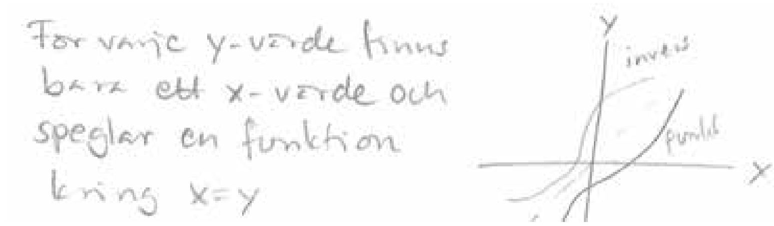
Student 4:

In the post test the same students' conception was:

Student 1: The inverse function is a function that is undoing another function or taking back an original function. If I take the starting value x, expose it to a function to form a final product y, so I can take this final product y, expose it to the inverse of the first function and get x that was my start value.

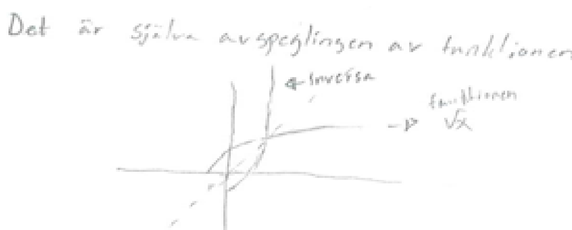
Student 2: An inverse function reflects the function in the line x = y

Student 3:



Student 3 says: For each value y is only one value x and reflects a function around x = y

Student 4:



Student 4 says: It is the reflection in itself of the function

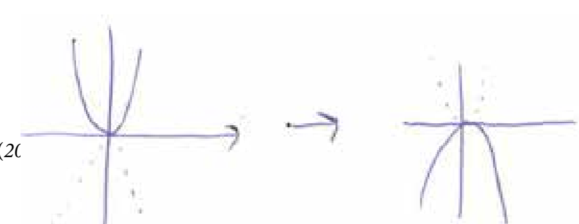
The students' pre and post test responses to the second question reveal that they (S2, S3, S4) often have an intuitive conception about inverse functions as some kind of mirroring. However they often lack the full comprehension of why and where the mirroring should be performed. The results above also show that one of the students (S1) became able to completely explain the notion "undoing".

The third question in our test focused on conditions that must be satisfied for a function to have an inverse. Some of the students responded in the pre test as follows:

Student 1: No explanations.

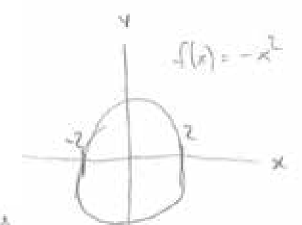
Student 2: What is required is that the function is really a function and it must be symmetric.

Student 5: That you should come back to



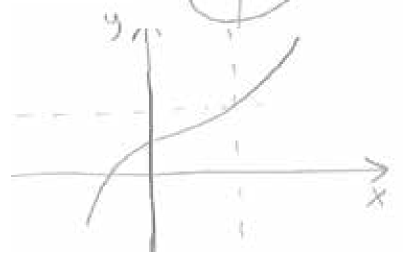
the original position and that it is "vice versa".

Student 6: A function can only have one value on y and x axis because otherwise you cannot come back to the same place again.



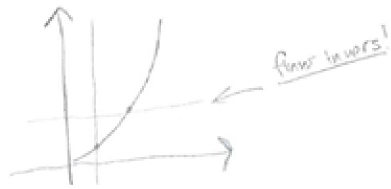
In the post test the same students responded this way:

Student 1: One value x must correspond to one value y and one value y should correspond to one value x. If one draws a function in a coordinate system the graph of the function is neither to cut an imagined horizontal line more than once, nor to cut an imagined vertical line more than once.

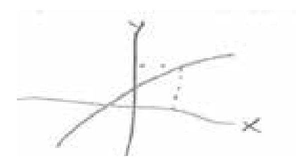


Student 2: Each value x should only have one value y, and each value y will only have one value x.

Student 5: One horizontal line and one vertical line will only be allowed to have one intersection in the graph.



Student 6: There should be only one value x and one value y on the graph. In the figure there is both an inverse and a function.



Unlike the poor pre test responses, the students' answers in post test show that most of the students after the lecture could give a rather good explanation about the conditions that are necessary for a function to have an inverse.

DISCUSSION

Patterns of variation i.e. generalization, contrast, separation and fusion create learning space for the students to understand the underlying formal abstract concept.

According to Leung (2003), when engaging in mathematical activities or reasoning, one often tries to comprehend abstract concepts by some kind of mental visualization of conceptual objects in hope to discern patterns of variation.

By continuously moving the vertical line through several points we could present the general idea of function in terms of "vertical line test" (generalization). In the similar way we created opportunities for students to experience a contrast by using the standard horizontal line test on a given function. We moved the horizontal line continuously so that students could experience the contrast between functions having an inverse and functions not having an inverse by simply counting the number of intersection points (one or more). We illustrated separation by changing the length and the position of the interval representing the domain of the function, one at a time. In this way the students were given the opportunity to experience one of the necessary conditions for a function to have an inverse, namely being strictly monotonic. In order to get the students to experience fusion several critical aspects could be illustrated simultaneously for instance by reflecting the function graph through the line y = x.

Analyzing the pre and post test results we could notice that the students' concept image of the inverse function had developed. For example, they were able after the lecture to explain why and where the mirroring should be performed. Furthermore, they were able to completely explain the meaning of "undoing". We could also notice that most of the students after the lecture could give a satisfactory explanation about the conditions that are necessary for a function to have an inverse.

As already mentioned, this study is a part of an ongoing research that attempts to explain the relationship between the teachers' instructional practise and students' learning in the context of functions and function inverses. Our ambition in the future research is to further explore already collected data which also involve the pre and post test results for a control group, as well as qualitative results from the final exam.

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REACTIONS OF PRE-SERVICE ELEMENTARY TEACHERS' TO IMPLEMENTING TECHNOLOGY BASED MATHEMATICS LESSONS

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The results reported in this paper focus on the reactions of three pre-service elementary mathematics teachers (PSTs) after designing and implementing three GeoGebra lessons. The analysis of the data revealed that focusing on the mathematical concept more than technology and using technology when it is really necessary were the basic criteria for effective technology based lesson. PSTs defined GeoGebra as a tool that assisted them to promote students' learning and a helper for students to discover features of mathematical concepts. Pre-knowledge of students, classroom management and time management are important components of successful lesson implementation. The PSTs reported increased self-confidence and valued having a mentor to help them reflect on their lesson implementation.

***Keywords**: Mathematics Teacher Education, Integrating Technology in Mathematics, GeoGebra*

INTRODUCTION

Educational technologies have become a significant part of the teaching and learning process in Turkey. The current elementary mathematics curriculum developed by the Ministry of National Education (MoNE) in Turkey emphasizes using technology effectively in teaching to provide students the opportunity for expressive mathematics teaching (MoNE, 2009). Dynamic technology-supported instruction presents an opportunity to enhance mathematical reasoning. For instance, with the help of dynamic geometry software, students explore various conjectures by constructing geometric shapes and making connections between them (MONE, 2009). In order to be an effective teacher, pre-service teachers need to learn fundamental concepts, knowledge, skills, and attitudes for applying technology in educational settings (NETS•T, 2008). It is our claim that the field experience is a crucial element in the development of TPACK. The purpose of this report is to examine three pre-service teachers' (PSTs) reactions to implementing three technology-based mathematics lessons in the context of a field experience.

LITERATURE REVIEW

Technology plays an ever-increasing role in the lives of elementary school students. Dynamic software packages, such as GeoGebra and Geometer's Sketchpad are vital in raising student awareness, challenging their conceptual understanding and motivating the synthesis of mathematical notions (Hollebrands, 2007; Kaput &

Thompson, 1994; Peressini & Knuth, 2005). Construction of mathematical objects, creating models and conducting interactive explorations are available via GeoGebra by dragging objects, tracing points, changing parameters and measuring objects.

According to the National Council for Accreditation of Teacher Education standards (NCATE, 2002), the new professional teacher who graduates from a department of education should be able to integrate technology into instruction to effectively enhance student learning. To achieve the technological goals stated by NCATE, teachers have to be prepared for their new roles in a technological environment (Thompson & Kersaint, 2002). To design and structure learning environments, researchers have suggested that teacher educators need to integrate technology into their teaching and their technology integration should go well beyond teaching technical skills (Kim & Baylor, 2008). Therefore, many teacher training programs and professional development initiatives integrate technology, with educational aims, into the courses to develop pre-service teachers' knowledge of technology (Koehler & Mishra, 2005; Katic, 2007).

Examples of undergraduate courses intended to promote Technological, Pedagogical, and Content Knowledge (TPACK) are found in (Kersaint, 2007; Ozgun-Koca et al., 2009/2010; Powers & Blubaugh, 2005). These three courses share a few common characteristics. All of the courses include activities that focus primarily on pedagogical and content-related tasks. For example, PSTs engaged in technological training, and then they examined the activities by discussing how and when to use appropriate technology in mathematics instruction. A second way in which the three courses are similar is that they include student-centred teaching methods such as guided discovery (Powers & Blubaugh, 2005), collaborative teaching (Kersaint, 2007) and inquiry-based learning with open-ended questioning (Ozgun-Koca et al., 2009/2010). Microteaching is found in the methods course developed by Ozgun-Koca et al. By providing these types of experiences, PSTs are challenged to reorganize their subject matter knowledge and use educational technology on the development of that subject itself (Niess, 2005).

FRAMEWORK

Teacher training courses should provide PSTs a rich technological environment to develop TPACK. Unfortunately, it is our experience that most pre-service teachers are unable to integrate what they learn in teacher education programs into their future teaching practice. Consider balancing the technological/pedagogical training with experiences of classroom observation, implication and reflection. A methodology used to support the development of TPACK is found in the Situated Technology Integration (SiTI) model (Hur, Cullen & Brush, 2010). It is the authors' opinion that the recommendations found in this model are well crafted due to the focus on didactical modeling, tool use, and skill practice. The SiTI model includes aspects of TPACK that occur within a broader classroom context. An emphasis on technology implementation in the classroom is emphasized in the guidelines (italics were added by the authors). Three out of five SiTI guidelines were used to frame this study.

SiTI Guidelines:

- **Provide concrete experiences**: To assist pre-service teachers in understanding the relationship between theory and practice, various examples and concrete experiences should be provided.
- **Assist in application**: To help pre-service teachers apply knowledge learned in real situations, opportunities to observe expert teacher's classrooms and chances to utilize knowledge in actual classrooms should be included.
- **Develop Technological Pedagogical and Content Knowledge**: To successfully integrate technology into their future classes, pre-service teachers should be called on to develop plans to use their technological knowledge in a meaningful way in relation to their content and classroom teaching knowledge. (p. 167)

To position these guidelines in their teacher preparation program, Hur, Cullen and Brush (2010) developed a three-phased approach (preparation, exploration and implementation). They readily acknowledge difficulties in providing pre-service teachers with a field experience as part of a mathematics methods course. These three guidelines provide information about the role of the teacher educator in pre-service teacher field experiences.

METHODOLOGY

The Elementary Mathematics Education Teacher Program is an undergraduate program of the Department of Elementary Education in the Faculty of Education at Middle East Technical University (METU) in Ankara. The population for the study were elementary mathematics majors who had taken ELE 430 and were enrolled in ELE 435 (the course descriptions follow).

ELE 430 – Exploring Geometry with Dynamic Geometry Applications (Spring 2011): Introduction to Geogebra software with emphasis on technical and pedagogical skills needed to teach geometry in grades 6-8.

ELE 435 – School Experience (Fall 2012): Introduction to grade 6-8 learning environment through classroom observations, and planning and implementation of a learning-centered activity.

During the ELE 430 course, pre-service teachers were not only presented with the features of GeoGebra software program, but also with how they can support students' mathematical understanding via this program. The instruction provided was aligned with the SiTI guideline recommendation to **provide concrete experiences**. An example used by the instructor was "Activity Exterior Angles of Polygons" [1]. The goal of the activity was to explore this topic within the dynamic geometry environment. In this activity, an exterior angle of a polygon was formed by a side and an extension of an adjacent side. PSTs were guided to create the dynamic sketches. In the first sketch, PSTs created and used a slider to resize a triangle. They discussed the sum of the exterior angles of a triangle. They observed that changes in the angle measures did not affect the sum of the exterior angles.

In the class sessions, the instructor's primary role was to facilitate discussion by asking PSTs pedagogical questions. She also asked them to answer and ask new questions about the activity. There were two types of pedagogical questions. The first type of questions asked PSTs to discuss the mathematical reasoning they encountered when constructing figures in GeoGebra. Some examples of the first type of questions are: *Why is it important to construct this object first? Can you show us your thoughts by constructing via GeoGebra? How can you represent your thinking? Is there any other way to construct this activity? What would you do if...?* After finishing the activity, the instructor asked a second type of questions. These questions required that the PSTs discuss implementation of the geometrical activity in an actual classroom environment. Some examples of the second type of questions are: *How might this activity be useful for you as a teacher with your students? What kind of "what if" questions can you ask students on this task to facilitate their learning? Do you think the use of this sketch can somehow change the learning environment? Which difficulties do you think can be encountered when conducting this activity?*

A total of sixteen PSTs had taken ELE 430, and from this group, eight were enrolled in the ELE 435. Six of the eight PSTs agreed to participate in the study and three of these participants were chosen as subjects for the post hoc case study analysis; Meltem, Pelin, and Ali (pseudonyms). Meltem exhibited an especially strong knowledge of technology, Pelin showed substantial change in the success of implementation from the first to the third and Ali showed a strong technological knowledge but claimed not to feel the need to use technology to teach mathematics unless it was required. These three PSTs were chosen by a purposeful sampling method because this sampling strategy provides access to people who will provide rich information about the research question (Creswell, 2007). A small stipend was paid to all six of the student volunteers.

Three limitations that can affect the generalizability of the reported results include:

- The population of students accepted into the teaching program at the METU is inherently biased. These students possess strong content knowledge;
- Researcher bias is possible because the lead author was a mentor to the PSTs;
- A case study methodology was used. The results that follow focus specifically on the data gathered in the final interview (completed January 2012).

DATA ANALYSIS AND RESULTS

To analyse the final interview data, the researcher's focus was especially on the **assist in application** and **develop TPACK** guidelines of the SiTI Model. The PSTs had developed and implemented their plans in actual classrooms. The purpose of the final interview questions was to promote and document reflection on their experience. The 50-minute interviews were transcribed and translated into English by the researcher in order to collaborate with an associate professor from the USA. From each transcript, significant sentences directly related to the interview questions were identified. For example, the 18^{th} question was about time: Because of using educational technology in your teaching, did you have any problems in terms of time management?

A total of 21 theme items emerged and were numbered as they arose. Sorting the theme items into nine distinct categories followed this analysis. While analysing the data, an associate professor experienced with qualitative research and analysis, reviewed the data, validated the emerging themes and collaborated in the assignment of categories. In what follows, each of the identified categories is followed by two to four theme items and selected quotes (with time stamps) from the interviews.

The first category, **Content Criterion**, is important because it indicates that the PSTs were focused on mathematics content and not just on technology. The PSTs stated that to integrate educational technology into mathematics education there should be some criterions to follow. The three theme items that emerged related to content criterion to teach a geometrical concept via GeoGebra were:

- Use technology when it is really necessary,
- To implement lesson in computer lab or classroom environment make a decision based on the lesson objectives,
- Focus on one concept; focus on concepts not technology.

> *I think, we should use technology when it is necessary to teach a geometrical concept. We need to focus on one objective; otherwise, as we see from my implementation, it makes students confused. [Ali.13:25-14:00]*

Ali was referring to his third lesson implementation where he taught three objectives related to factorization. That lesson covered greatest common factor, factoring by grouping and factoring quadratic polynomials. His GeoGebra activity was constructed to teach factorization of quadratic polynomials. He reported during the interview that some of the students appeared to be confused because of learning three objectives in a single lesson and had some difficulties to focus on a GeoGebra activity at the end of the lesson.

Time management is the second category identified. PSTs stated that, if there is no technical problem, educational technology helps teacher use time economically. The two theme items categorized as time management were:

- Technology helps PST use time economically,
- Have a back up plan in case of any technical problems.

I think when we use technology in math education, it provides time saving. [Ali.14:03-14:27]

All the time we need to have a plan B in case of any technical or other problems. [Pelin.35:20-35:51]

In some cases, the technology saved time. For example, in Meltem's first implementation she was able to demonstrate how to make a triangle with the same area by dragging one of the vertex points of the polygon (instead of drawing on the board for each situation). In other cases, technology problems wasted class time. In Pelin's third implementation, the electricity cut off and she couldn't project the screen onto the board. Nevertheless, her laptop battery was full and she displayed the GeoGebra file via laptop screen by grouping students into the three parts.

Classroom Management is the third category. PSTs said that educational technology makes classroom management harder. The two theme items that came out related to classroom management were:

- Technical problems make classroom management harder,
- Following students is more difficult in technology-based classroom.

Most of the classroom management problems arose from technical problems. [Ali.12:15-12:24]

To follow students at the same time is more difficult when I was using GeoGebra. [Meltem.13:05-13:34]

The fourth and fifth categories that generated strong reaction and appeared in each of the interviews were named **Tool Use**. PSTs defined educational technology as a tool for teachers and students separately in order to have effective teaching and learning. **For Teachers**, technology is a tool to promote students' learning. The two theme items that came out related to tool use for teachers were:

- Technology is a tool to help teachers support their students reasoning,
- Technology gives opportunity for teacher to implement student-centered lesson (Problem solving, questioning).

It is a really helpful tool for mathematics teachers if they want to make students discover geometric objects. One of the GeoGebra features that can be used was hiding objects. It gives an opportunity for teachers to hide something and let students discover … teacher can show the objects step-by-step to confirm students reasoning. [Ali.24:25-25:20]

For Students, technology is a tool to help them learn easier. The theme items were:

- Technology is a tool 1) used for discovering features of geometrical concepts, 2) to help students transfer their mathematical knowledge, 3) used for testing students' answers,

• With the help of technology, students learn geometrical rules conceptually without memorizing.

GeoGebra helps students to observe and discover features of geometric concepts… GeoGebra can help students make abstract concepts concrete. [Ali.23:30-24:22]

As you observed in my third implementation, students could transfer their knowledge, they learned in previous lesson, to another problem situation while drawing a circle when only three points to pass through are given. [Meltem.16:52-17:20]

For example, in Ali's third implementation, he modelled algebra tiles via GeoGebra and taught factoring quadratic expressions. With the help of GeoGebra, students could visualize $2x^2 + 5x + 2$ and its factors. Another example is from Meltem. In her second implementation, she asked the students to find the circumcenter of a dynamic triangle first by dragging a point (that was connected to the three vertices). This was followed by an exploration of the use of the perpendicular bisectors of each side of the triangle to find the circumcenter of the given triangle. In Meltem's third implementation, she asked students to create a circle when only three points to pass through were given. In the second step of the activity, a few of the students made a connection with the previous lesson and found the correct solution for the problem (this was observed in the video tape of this lesson).

The sixth category that also generated strong reaction when it appeared in each of the interviews was named **Implementation**. During the interviews, PSTs reported contributions of the implementation experience to their future teaching. For example, they reported an increase in confidence in teaching and predicting what students might ask. The theme items that related to implementation were:

• The more they implement lessons in a real classroom context, the more the PSTs have self-confidence in teaching,

• The more implementations in real classroom contexts, the more PSTs interpret what will happen in the lesson,

• The more implementation in real classroom contexts, the more the PSTs see their weaknesses,

• Personal reaction to lesson implementations.

The more we have teaching experiences in real classrooms, the more we have knowledge about students' reactions…. As a teacher, I feel more comfortable in real classroom environment now. [Meltem.34:57-35:20]

In the third implementation I could predict what kind of questions students will ask me, about the task. [Ali.08:30-08:45]

Luckily all of the class activities that I implemented in real classroom context were really close to what I imagine while designing… Normally I thought that I could not implement what I planned, since I am looking for ideal students that have mathematical content knowledge very deeply as a pre-knowledge and they will answer whatever I asked to them related to content [Meltem.04:09-04.30]

There is an improvement in my opinion. My best implementation was the last one. [Ali.05:41-05.52]

The seventh category was **Recommendations**. The last question of the interview was: do you have any recommendation about training teachers in terms of using technology? PSTs gave recommendations to the teacher training programs. The four remaining theme items that came out related to recommendation were:

- Teacher training programs should have more technology courses,
- Integrate technology into methods course,
- PSTs should improve their techno pedagogical skills before graduation,
- PSTs need more lesson plan implementations in real classroom contexts to have more experiences before graduation.

There should be more obligatory courses about educational technology in our national teacher training programs. [Pelin.46:04-46:20].

Teacher training programs need to integrate technology into method course [Ali.35:37-35:48].

I need more experiences in real classroom context before graduation [Ali.35:03-35:21].

Two remaining categories were identified by one theme item each;

- Need to know student's pre-knowledge,
- Mentoring is helpful to improve lesson plan.

The need to know students' **Prior Knowledge** appeared in each of the interviews. All three of the PSTs commented on the impact of not having a good understanding of the students' knowledge when designing and implementing a lesson.

Therefore, just because I didn't know students' pre-knowledge, my lesson implementation was not effective. Without any reasoning, they answered the questions. It was not a challenging activity for them. [Ali.04:49-05:15]

We need to have information about students' pre-knowledge to prepare effective lesson plan. [Pelin.33:57-34:06]

The last category that emerged was **Mentoring**. While it was not anticipated, this category is in alignment with the SiTI guideline "assist in implementation." The PSTs commented on the feedback and advice offered by the researcher in her role as a mentor.

While I was creating GeoGebra activities, I took your opinions about in which part of the lesson I need to use that activity ... Based on our discussion during the interviews I made some changes in my lesson plan. [Ali.05:54-06.15]

DISCUSSION AND CONCLUSION

The purpose of this report is to examine three pre-service teachers' (PSTs) reactions to implementing three technology-based mathematics lessons in the context of a field experience. Three of the five the SiTI Guidelines provided a framework for the PSTs coursework and the additional field experience that participation in this study provided. The overarching goal of developing the PST's TPACK can be observed in five of the categories identified in the final interviews. Issues that emerged with regard to classroom management and time management could be considered technological pedagogical knowledge (TPK). Issues that emerged with regard to content criterion and tool use (teacher and student) could be considered as technological content knowledge constructs. Pre-knowledge could be considered as necessary to a teacher's pedagogical content knowledge, which for these PSTs affected the overall success of their lesson implementation. What is interesting is that the PSTs did not specifically reflect on the technological aspects of their lesson implementation. This might be due to the questions asked in the interview or to the PST's past experience with regard to using GeoGebra and/or preparing the files used in the lessons.

The remaining two categories provide important information about the field experience as a whole. The PSTs valued the implementation experience, recommended that this type of experience be included in a teacher education program, and reported positively on the role of the mentor.

NOTES

1. GeoGebra Wiki, http://www.geogebra.org/en/upload/files/english/Barbara_Perez/Triangle_Exterior.html

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HOW TEACHERS LEARN TO USE COMPLEX NEW TECHNOLOGIES IN SECONDARY MATHEMATICS CLASSROOMS - THE NOTION OF THE HICCUP

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This paper reports the outcomes of a longitudinal doctoral study which sought to illuminate the process through which secondary mathematics teachers learned to use a complex new multi-representational technology, the TI-Nspire handheld and software. The research examined the trajectories of fifteen teachers, with a focus on the pedagogical approaches that privileged the exploration of mathematical variance and invariance. Analysis of the data reveals the importance of the notion of the 'hiccup'; that is the perturbation experienced by teachers during lessons stimulated by their use of the technology, which illuminates discontinuities within teachers' knowledge.

Keywords: Classrooms, Handheld technology, Hiccup, Mathematics, Teachers.

INTRODUCTION

Most studies concerning the appropriation of technological tools within mathematics classrooms have been approached from the students' perspectives and far less is known about teachers' epistemological developments, i.e. the process through which their mathematical, pedagogical and technical knowledge develops over time as a result of their use of mathematical digital technologies. The opportunity provided by a funded project in which a group of English secondary school teachers were introduced to the TI-Nspire handheld and software environment provided the context for this study. For the teachers, this involved both learning about the affordances of the new technology and then devising teaching activities and approaches that utilised these affordances in ways that had educational legitimacy in their classroom settings. The outcomes of these classroom activities led to the development of 'instrument utilisation schemes', which provided the platform from which to observe and evidence teacher learning (Verillon & Rabardel, 1995).

The TI-Nspire handheld and software is described throughout this paper as a 'multi-representational technological' tool as it is an environment which incorporates numeric, syntactic, geometric and graphical applications that can be dynamically linked through the definition of variables. These variables can either be defined by the user or captured from existing objects within an application and hence the multi-representational technology (MRT) offered a new condition for organising teachers' actions.

THEORETICAL FRAMEWORK

The research was underpinned by Verillon and Rabardel's theory of instrumented activity, which seeks to explain the process through which humans interact with technological tools (Verillon & Rabardel, 1995). Their 'triad of instrumented activity' was adapted for the context of the study, resulting in the diagram shown in Figure 1 below. Consequently, the instrument (in a Vygotskian sense) incorporated the use of the MRT, the subject was considered to be 'teachers as learners' and the object was 'teachers' learning about the teaching and learning of mathematics through the exploration of mathematical variance and invariance'.

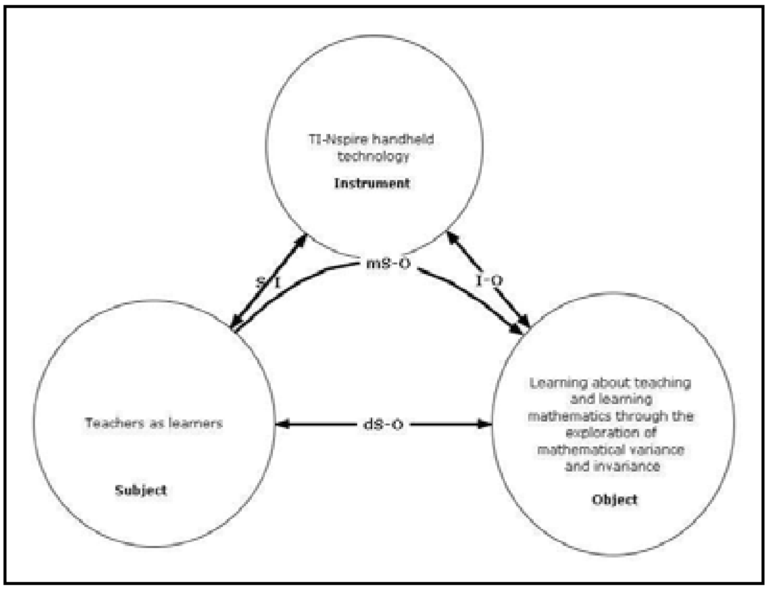

Figure 1: The adapted triad characteristic of Instrumented Activity Situations. The arrows indicate the interactions between Subject and Instrument (S-I), Instrument and Object (I-O) and, in the case of the Subject and the Object, the direct interaction (dS-O) and the mediated interaction (mS-O) (Verillon & Rabardel, 1995).

The research was also influenced by three key themes from the literature, which concerned: the appropriation of technological tools and the role this plays in teachers' subject and pedagogic knowledge development (Guin & Trouche, 1999; Pierce & Stacey, 2009; Ruthven & Hennessy, 2002; Stacey, 2008); the development of representation systems for mathematics (Kaput, 1986, 1989; Mason, 1996) and the interpretations of knowledge and the processes involved in mathematics teachers' professional learning (Polanyi, 1966; Rowland, Huckstep, & Thwaites, 2005; Shulman, 1986).

RESEARCH METHODOLOGY

As a researcher, I adopted the perspective that reality is socially constructed and I sought to privilege the voices, actions and meanings of the teachers as the main data sources and ensure the reliability of the study through a robust and systematic process of data analysis leading to a valid set of conclusions. (See Clark-Wilson (2008b) for more detail). The research was carried out in two phases, (Jul 2007 – Nov 2008 and Apr – Dec 2009) and, in each of these phases, a group of teachers was selected and a series of methodological tools developed to capture rich evidence of their use of the technology in classrooms to enable the aims of the study to be realised. Hence this was a situated exploratory study in which the unit of analysis was (secondary mathematics teachers + mathematics + activity design) that sought to expand the discourse on teachers' appropriation of technology tools; that is how they adapt and mould the tool for their own use. The research lens was trained on the trajectory of the teachers' interpretations of variance and invariance.

Phase one of the study

During the first phase, fifteen teachers were introduced to the technology and encouraged to develop activities for the classroom, which they then trialled with their students and reported the outcomes of these trials to the study. The teachers reported a total of sixty-six activities and the research data comprised: teacher questionnaires, which included a mandatory detailed lesson evaluation; teachers' lesson plans, pupil resources and software files; and teachers' presentations during project meetings. The data analysis of the first phase of the study (using Nvivo8) revealed the trends in the teachers' interpretations of variance and invariance and the emergence of nine different instrument utilisation schemes (IUS), within their 'intended' lesson activities. As these lessons were not observed, it cannot be assumed that they correspond to the students' actual utilisation schemes.

Instrument utilisation scheme (IUS)	**Frequency of use (n=66)**
IUS1: Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic, tabular or graphical form.	37
IUS2: From a given set of static geometric objects, make measurements and tabulate data to explore variance and invariance within the measured data in numeric and tabular forms.	4
IUS3: Vary the position of an object (by dragging) that has been constructed in accordance with a conventional mathematical constraint and observe the resulting changes. (Use another representational form to add insight to or justify/prove any invariant properties).	13
IUS4: Vary a numeric input and drag an object within a related mathematical environment and observe the resulting visual output.	2
IUS5: Vary a numeric or syntactic input and use the instrument's functionality to observe the resulting output in numeric, syntactic,	8

Instrument utilisation scheme (IUS)	Frequency of use (n=66)
tabular or graphical form. Use another representational form to add insight/justify/prove any invariant properties.	
IUS6: Vary the position of an object that has previously been defined syntactically (by dragging) to satisfy a specified mathematical condition.	1
IUS7: Construct a graphical and geometric scenario and then vary the position of geometric objects by dragging to satisfy a specified mathematical condition. Input syntactically to observe invariant properties.	1
IUS8: (Construct a geometric scenario and then) vary the position of objects (by dragging) and automatically capture measured data. Use the numeric, syntactic, graphical and tabular forms to explore, justify (and prove) invariant properties.	2
IUS9: (Construct a graphical or geometric scenario and then) vary the position of a geometric object by dragging to observe the resulting changes. Save measurements as variables and test conjectures using a syntactic form.	3

Table 1: Summary of the instrument utilisation schemes in the lesson activities designed by the teachers (n=66)

There were three outcomes of the first phase of the study: a clearer understanding of the ways that the teachers used the multi-representational environment to emphasise different conceptions of variance and invariance; the identification of the teachers who would become the subjects of the research in its second phase; and the emergence of my interest in the instances in the classroom where the teachers were perturbed (in an epistemological sense) as a result of using the MRT with their students. I use the word epistemological to mean that the teachers underlying knowledge in relation to mathematics, technology and pedagogy was being reviewed and reorganised as a direct result of their classroom experiences with the technology.

Phase two of the study

The second phase involved the focused case studies of two teachers, Tim and Eleanor, who had demonstrated the desired attributes (technical competency and a diversity of IUS) and had adopted pedagogical approaches that placed the students' mathematical experiences at the centre of the classroom environment. As this phase of the study aimed to elicit the nature and process of the teachers' learning through a close observation of them in their classrooms, the research methodology used in phase one was developed further to include audio-recorded lesson observations and interviews.

In addition, the opportunity for the teachers to use TI-Nspire Navigator technology in their classrooms resulted in additional data such as students' files, handheld screens and screen capture views being collected, providing an unanticipated additional rich resource for the study.

In all, eight of Tim's lessons and six of Eleanor's lessons were observed and, as previously, the data was imported as a synchronised set into Nvivo 8 software, which facilitated the 'replaying' of the lesson in the fullest sense. The analysis of each set of lesson data led to the development of a detailed, accurate and complete lesson narrative, by a broad analysis of the lesson that utilised Stacey et al's 'pedagogical map' (Stacey, 2008) and the emergence of the 'hiccup' as an organising principle.

HOW DO TEACHERS LEARN TO USE COMPLEX TECHNOLOGIES? – THE EMERGENCE OF THE 'HICCUP'

As I began to observe the teachers in their classrooms, my attention was increasingly shifted towards the existence, and opportunity to analyse, what I refer to throughout the study as lesson 'hiccups'. These were the perturbations experienced by the teachers during the lesson, triggered by the use of the technology that seemed to illuminate discontinuities in their knowledge and offer opportunities for the teachers' epistemological development within the domain of the study. They were highly observable events as they often caused the teacher to hesitate or pause, before responding in some way. At the time of the study, the teachers were not aware of the concept of the hiccup as I would refer to them as 'surprises' or 'unanticipated moments'.

For example, the Nvivo coding summary for one of Tim's classroom activities is shown in Figure 2.

Name
TP6 Hiccup1 - Difficulties over identification of dynamic objects
TP6 Hiccup2 - Students' mis-interpretations of task - 'different way around'
TP6 Hiccup3 - Instrumentation (T) 'your c has gone off the screen'
TP6 Hiccup4 - Instrumentation (S) - grabbing and dragging
TP6 Hiccup5 - Jump from MRT task to trad paper and pencil problem

Figure 2: Tim's coded hiccups for the activity 'Pythagoras exploration'.

The analysis of all of the lesson data enabled all of the hiccups to be identified and there were sixty-six in total. A constant comparison method, led to the definition of seven categories of 'trigger'. These are detailed later in the paper. What follows immediately is a detailed description on one particular hiccup and a justification of why its occurrence provided evidence for the teacher's epistemological development.

An example of a hiccup and its relationship to Tim's situated learning

In the lesson activity 'Pythagoras exploration', Tim had designed an activity in which his students were dragging the vertices of a geometric construction and it was his intention that the students would conclude that <u>when</u> the triangle was dragged such that it appeared to be right-angled, the areas Tim had defined as *a* and *b* would sum to the area he had defined as *c*.

The initial screen that the students encountered is shown in Figure 3.

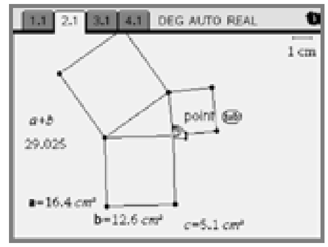

Figure 3: The students' opening view of the task

What follows is the detailed analysis of one of these hiccups (coded TP6 Hiccup2 from Figure 2), and an articulation of how this event may have contributed towards Tim's situated learning during and soon after the lesson.

This hiccup was observed during a point in the lesson when Tim was clearly reflecting deeply on the students' contributions to the shared learning space and 'thinking on his feet' with respect to responding to these. It coincided with his observation of an unanticipated student response. The chosen hiccup came about when a student had found a correct situation for the task, that is the two smaller squares' areas summed to give the area of the larger square, but the situation did not meet Tim's activity constraint of $\boldsymbol{a + b = c}$**.**

Tim commented about this in his personal written reflection after the lesson,

> One student had created a triangle for which a+b did not equal c, but (I think) a+c=b. This was also right angled. This was an interesting case because it demonstrated that the 'order' did not matter... when the sum of the smaller squares equalled that of the larger square, then the triangle became right angled.

Tim revised the TI-Nspire file after the lesson, providing some convincing evidence of his learning as a result of the use of the MRT. Tim gives an insight into his learning through his suggestions as to how he thought that some of these perceived difficulties might be overcome by some amendments to the original file (see Figure 4).

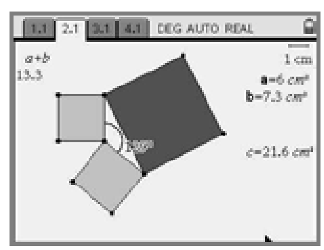

Figure 4: Tim's revised MRT file for the task

The squares whose areas were previously represented by 'a' and 'b' have been lightly shaded and the square represented by the area measurement 'c' has been darkly shaded. Tim also added an angle measurement for the angle that is opposite the side that was intended to represent the hypotenuse.

Both of these amendments to the original file suggest that Tim wanted to direct the students' attentions more explicitly to the important representational features. He wanted to enable the students to connect the relevant squares to their area measurements and 'notice' more explicitly the condition that <u>when</u> the condition for the areas was met, the angle opposite the hypotenuse would be (close to) a right angle. This seemed to suggest that Tim was still trying to overcome the inherent difficulty when using mathematical software concerning the display of measured and calculated values in the hope that students would achieve an example where the areas were equal and the measured angle showed ninety degrees. This seemed to suggest a conflict with his earlier willingness to try to encourage his students to accept an element of mathematical tolerance when working with technology with respect to the concept of equality.

CATEGORIES OF HICCUPS

The research concluded that the teachers were engaged in substantial situated learning, prompted by their experiences of lesson hiccups. In this sense the hiccups are an epistemological phenomenon as they are the manifestation of a rupture in the fabric of the teacher's knowledge. The seven categories of hiccups are detailed below alongside a brief exemplification from the research data.

1. Aspects of the initial activity design

Hiccups in this category were attributed to aspects of the teacher's choice of initial examples, the sequencing of the examples, the methods for identifying and discussing objects displayed on the MRT or unfamiliar pedagogical approaches. The hiccup described previously within Tim's lesson is an example of this type as the lack of any on-screen labelling made it difficult for the students to interpret the initial instructions for the task.

2. Interpreting the mathematical generality under scrutiny

This category concerned the acts of relating specific cases to the wider generality under observation, the appreciation of the permissible range of responses that satisfy the generality or failing to notice the generality at all.

Figure 5: The students' handheld screens displayed publicly during the plenary.

For example, in a lesson designed and taught by Eleanor, the large set in functions that the students were asked to plot within the MRT led to diverse set of screens on which it was difficult for the students to notice the generality that was common to the functions that had all been transformed by a 'sideways shift' of ±a (Fig. 5).

3. Unanticipated student responses as a result of using the MRT

There were several instances where the response from the student differed from that which the teacher had anticipated in his or her original design, leading to occurrences of hiccups. For example, the students' prior understanding was below the teacher's expectation, the students' interpretations of the activity objectives differed from that of the teachers or the students developed their own instrument utilisation schemes for the activity.

For example, in a lesson created by Eleanor, in which she has asked her students to construct linear functions through the given co-ordinate point (3, 6), one of the students produced the screen in Figure 6.

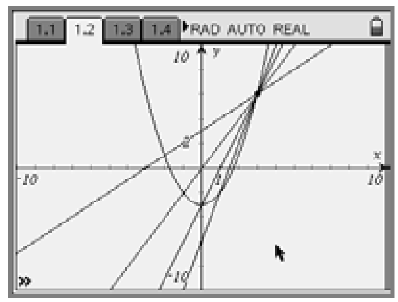

Figure 6: Emily's response to the task

4. Perturbations experienced by students as a result of the representational outputs of the MRT

A number of observed hiccups resulted from the students responses to a particular syntactic or geometric output or their doubt of the 'authority' of the syntactic output from the MRT.

An example of this is shown in Figure 7 where Tim was required to make sense of a student's response to a task in which the student had questioned the output of the MRT.

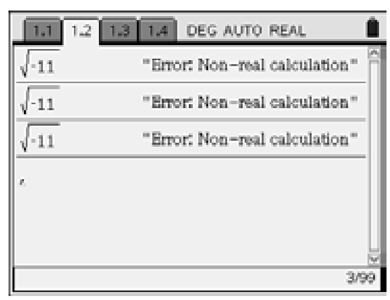

Figure 7: A student's screen in which he repeats his entry to the MRT.

5. Instrumentation issues experienced by students when making inputs to the MRT and whilst actively engaging with the MRT

The hiccups within this category resonate with much of the research concerning students' uses of complex technologies and they were related to: entering numeric and syntactic data; plotting free coordinate points; grabbing and dragging dynamic objects; organising on-screen objects; navigating between application windows; enquiries about new instrumentation and the accidental deletion of objects.

6. Instrumentation issue experienced by one teacher whilst actively engaging with the MRT

The high level of experience and confidence of the two teachers with the MRT most probably accounts for the low incidence of hiccups relating to their own instrumentation issues. In this case, the teacher 'forgot' how to reveal the function table at a key point in one activity.

7. Unavoidable technical issues

The teachers were using prototype classroom network technology which did result in some equipment failures during some lessons. Although these occurrences were definitely classed as hiccups, they were considered to be outside of the domain of the research study.

CONCLUSIONS

The evidence from the study strongly supports the thesis that teachers were engaged in substantial situated learning, which was prompted by their experiences of lesson hiccups, as they designed and evaluated activities using the MRT. These activity designs privileged explorations of variance and invariance in some way and most also involved multiple mathematical representations. In this sense the hiccup is considered to be an epistemological phenomenon, that is, a rupture in the fabric of the teacher's knowledge. All of these hiccups provided opportunities for the teachers to at least interrogate, if not develop their knowledge. It is not suggested that all hiccups would lead to a clear learning outcome for the teachers. However, the research evidence from my study is rich with examples of how individual hiccups (and combinations of hiccups) have prompted the teachers to rethink the subtle aspects of their activity designs evidencing them to be learning sites for the teachers. The discussions within the CERME conference working group raised a number of interesting questions such as: how the concept of the hiccup might be incorporated into the design of professional development for teachers? and, if so, whether the theory should be introduced first or whether teachers should need to be allowed to experience them first? Other questions related to how the notion of the hiccup might be incorporated into existing theories about teacher knowledge development and whether the teacher's experience might be an important factor.

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MATHEMATICS TEACHING ON THE WEB FOR STUDENT TEACHERS: ACTION RESEARCH IN PRACTICE

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The article summarises experiences gained through my practice as web-based pedagogue in mathematics at the University of Nordland (Norway)'s teacher-training institute. It has been my wish to make continuous improvements to my practice and I present two of the most important improvements that I investigated on the basis of practical action research. Data collected from students' feedback indicates that video lectures produced by filming whiteboard teaching yields a greater educational result than so-called screen video of a slideshow run on a PC. In addition, we present and discuss significant findings regarding the use of Skype in web-based teaching.

INTRODUCTION

The purpose of this article is twofold. Firstly, I wish to describe some of my findings in respect of the new web-based maths teaching in the teacher education at the University of Nordland. I then wish to present some suggestions for possible initiatives against the background of empirical observations collected during my first term of web-based teaching.

My professional background is as university lecturer in mathematics. During the spring term 2011 I taught the first class of student teachers at the University of Nordland in statistical analysis, basic geometry and probability calculation in a web-based version. In addition to the web-based classes, this teaching has been held in equivalent classes on campus. The goal of the web-based program is to enable a flexible form of tuition in which the bulk of the teaching per se is conducted over the internet instead of face to face, as was the case in the old general teacher-training program. One of the primary intentions is that the university to a greater extent should accommodate the students' desire for freedom from a teaching form that entails constraints on time and place. Traditional campus-based tuition forces the student to be present at the time and place in which lectures and group seminars are held. By putting lectures out as video clips on an established web address, as well as arranging web-based meetings with voice and webcam, an attempt has been made to liberate participants from location-bound tuition. The video clips are accessible 24 hours a day so that the student in principle can view them whenever it is convenient and on any suitable medium that has a modern web browser installed. The idea is that this provides far more students with an opportunity to complete teacher training – primarily those who for one reason or another are unable to move to Bodø during the period of study.

I believe that it may be of interest to others who are involved with web-based tuition in general, and web-based maths teaching in particular, that I share these experiences.

Issue to be examined

In the course of this article I will address the following questions through action research (Schmuck, 1997), without arriving at definitive answers: What form of lecturing in mathematics using films published on the web has the better learning potential? Are there lessons to be learned about being present online as a math tutor in a web-based learning environment?

THEORY AND METHOD

Theoretical basis

What makes action research attractive as a theoretical backdrop for the survey is that it gives the researchers a more participatory role in their own research. It is not only permitted but even encouraged to be both a participant in the experiment that is being carried out, as well as to carry out research on it. The expression "experiment with" rather than "experiment upon" can be an expression of the general philosophy behind this scientific theory (Schmuck, 1997). The criticism often expressed by traditionalists is that such an approach will produce problems in terms of repeatability and objectivity. It may be replied that even with a traditional socio-scientific approach it is often impossible to guarantee an adequate degree of objectivity. The results that are presented are often derived from issues and interpretations of results that are not necessarily repeatable. One can also view action research as a systematic means of describing or interpreting personal interaction with the practice field, often entailing a quest for "improvements" after one or more actions have been carried out (Coghlan & Brannick, 2001).

In the methodology section of this article we will look more closely at our own approach to this general way of viewing action research. In particular we will attempt to employ it to challenge our own practices within web-based pedagogy.

The concept of web-based pedagogy

Web-based pedagogy can be viewed as one of the building blocks in a web-based education, in which the pedagogic interaction between teacher and student takes place via a data network as opposed to face-to-face contact in a teaching room. The term "web-based teaching" can perhaps be understood as a more general concept of which the actual web pedagogy forms one of the parts, alongside other elements such as how the courses are adapted to the internet, which attendance requirements should be for a web student as opposed to a campus student, etc. In this study I will deal particularly with one aspect of the pedagogic part: *how changes in presentation form used in the online lectures on the internet changed the perception quality of the students*.

Maths teaching for student teachers

For student teachers one wishes especially to promote a type of mathematics of a more practice-related and didactic variety than is found in other higher education. Teacher training is primarily a professional study that should be firmly rooted in practice (Eraut, 1994). How can mathematical connections form a didactic backdrop for a teaching program? Different teaching theories, amongst which social constructivism (Bråthen, 1996) is perhaps the most prominent today, should shine through in the teaching.

Method

In this article I have chosen to employ a responsive action research form in which I collect data in advance of the actions in order to be able to form an instantaneous picture of how my own practice is working. This forms a basis on which to determine adequate measures, often with contributions from others such as the year coordinator and colleagues. These had a central role in the assessment stage in terms of making decisions about which initiatives to include in the action.

All in all, I have adapted the familiar action-research cycle to my own research in the following manner (Fig. 1):

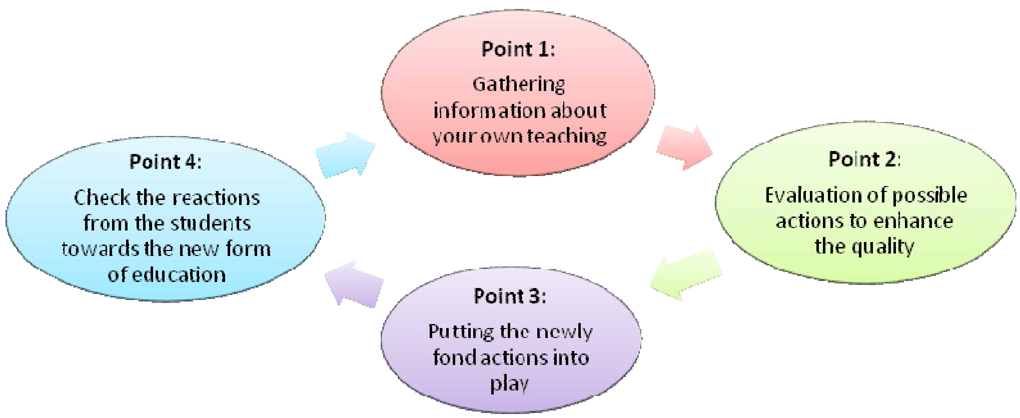

Figure 1: Action-research cycle in personal teaching practices

The purpose of action research

The concept of web-based education contains, naturally, no detailed definition of how the pedagogic scheme should function with regard to each individual course. It is naturally up to each individual web tutor to find his preferred way of doing this, according to the available resources and the terms of reference that are set by administrators, the director of studies and others (Paulsen, 2001). There does however exist some principal guidelines for the setup and construction of web-based education. In the background paper from Europace (2001) for instance, there is given a division in pedagogical principles, functions and variables in their theory of online pedagogy, and it highlights the importance of a teacher attaining the role of a support person for the learner.

In light of this, the goal of the research can be stated as: A search for the form or forms of web-pedagogic mathematics program that provides the best possible learning outcome for student teachers.

Actions

The actual actions will be dealt with in point 3, where I employ the new teaching reform. I have chosen to highlight two actions in particular, even though I undertook many minor adjustments to the teaching program which will not be described here:

1. Turning from static slide-show-based teaching to filming whiteboard presentations.
2. Using Skype for assistance in solving mathematical tasks.

In addition to these primary actions I constantly made small adjustments involving, for instance, the use of technical aids and how the material was presented.

Data in the survey

Data collection is built on concrete responses from students regarding how they perceived the different teaching forms used in the courses that I held. More precisely, I issued a simple question regarding this form in an e-mail. There were only 10 students attending this course, from which I received 7 replies. See the section "Results and discussion" for more details about the question asked and some representative samples of responses.

Description of electronic aids

To understand the pedagogic context in which the web study has been carried out I have chosen to include a presentation of the technical environments that were tested during this one term, as well as some lessons drawn from their use. This is in order to understand more easily the backdrop for the chosen actions. In terms of action research this description will be involved in points 1 and 2 described in Figure 1.

Camptasia Studio is a computer program that provides the teacher with the opportunity to combine a Power Point slideshow with sound/video of the lecturer which can be edited and put out on the internet for the students. Many lecturers at the University of Nordland, including myself, use this program to record lectures which in turn usually is uploaded on the virtual learning environment (VLE) "Fronter".

Fronter is the VLE used by all students and staff at the Faculty of Professional Studies in the University of Nordland at Bodø. The platform offers a common structure with functionality for document sharing, news wheel, submissions and forum. As much of the teaching in a web and practice-based study is carried out over the internet, a VLE is the most natural portal for making teaching material accessible.

In addition to the purely static one-way communication involved in Fronter, I have also attempted to persuade the students to use the forum in Fronter for presenting their questions. In practice, however, it was difficult to motivate students to use the specially-designed forums to any great extent. We have not had time to investigate all the reasons for this, but we believe that one of the reasons is that it takes "many clicks" to find out whether there is any activity in a particular forum. I believe that a learning portal such as Fronter should incorporate means of notification using media such as email, RSS and social network. In addition, I believe that editing of mathematical text must soon be taken seriously in such portals. At the moment, only poor-quality formula editors are available. In addition, support for LaTeX should be investigated in order to simplify entering formulae without having to resort to massive clicking in a Word-like WYSIWIG (”What You See Is What You Get”) editor.

The use of GeoGebra provides web teachers with many opportunities to demonstrate principles and analysis within geometry and function teaching. We used GeoGebra to demonstrate theorems such as Thale’s proposition by making screencasts in Camptasia Studio together with voice recording. The screencast combined with GeoGebra provides an opportunity to make dynamic geometric constructions, something that is not possible to put directly into Power Point without inserting film clips. In addition, the constructions are very precise and it is possible to demonstrate to the students how this important learning resource can be used.

As an extension of this it is possible to imagine the use of this tool in internet meetings, if it were possible to share one’s own desktop with the students. If concrete problems are brought up in which the students are struggling in geometry, solutions for these can be shown very effectively to everyone in the internet meeting by demonstrating the construction live, with everyone following. In this way one can attain a virtual blackboard teaching in geometry. Conferencing programs such as Skype offer just such a system of sharing the desktop that can make this possible.

Skype is a type of computer program that enables real-time communication in speech, text and image. If logged in, contact between different participants in a learning community can occur spontaneously. The service is free of charge, apart from the conference section.

This program is particularly straightforward to use in a web-based teaching context as it is not necessary to keep attention on the program itself all the time, as is the case for instance when using a chat function in a web browser, because when someone wishes to take contact, the user’s attention is drawn by means of sound/notification.

In a social-constructivist view of teaching, it is clear that a direct exchange of opinion and a joint problem solving will have a fairly central part in teaching math as well as in other subjects (Barron & Darling-Hammond, 2008). To my experience, the discipline is usually regarded so abstract that students can quickly find themselves frustrated by “getting stuck” in their own understanding of it. Regular contact with other students and with the teacher can be regarded as an essential part of mathematics study for teachers (Lampert et al., 2010). Skype can be a good alternative in this respect since it is easy to contact other students and teachers who are online in order to get help.

It was observed that individual students who did not often speak out in a normal classroom situation found it easier to do so when using Skype. Email may also seem more formal than Skype, which assumes a more chat-like form.

PREPARING AND PERFORMING THE ACTIONS

Students' reactions after two months of internet tuition in Camptasia

Between the two physical meetings of the web-based class a period of eleven weeks elapsed. During that period, new lectures and teaching on statistical methodology recorded with the help of Camptasia Studio were regularly issued, along with accompanying assignments and guidance with the help of NTR Meeting. The whole statistical analysis syllabus was gone through before the first meeting and we were very anxious to hear the students' reactions. Up until then there had been a more or less one-way pedagogic transmission of the syllabus.

The first hour with the group after the meeting was used for a repetition of the greater part of the syllabus that had been gone through on the internet. After this session we received feedback such as "I learned more during these two hours with you and the blackboard than by watching all the online lectures combined".

Why was this the case? A great deal of time had been spent in making an entire lecture series on statistical methods, in which emphasis was placed on using illustrations to simplify understanding of the fundamental concepts. The slideshow on which the online lectures were based was also made available so that the students could look at it in their own time as well as watching the actual video, optionally pausing/stopping/rewinding it.

The students pointed out that the material had too high a threshold to be put out on the internet. Analytical statistics was a topic of which few of them had any experience in advance and it required a good deal of assimilation time to understand all the new concepts that were introduced during the course. The students also pointed out that they had few "pegs to hang the material on". The perception of having learned a great deal during the blackboard teaching could well have been the result of a subconscious maturing that had taken place in the period leading up to the meeting.

In addition to the material on statistical analysis we had also put out some lectures on geometry. For these, films of demonstrations, prepared with GeoGebra on an office PC, were used, with an explanatory voice recorded in parallel. These lectures were given a better reception. Reactions received suggested that "the only thing we managed to follow was the practical material on GeoGebra, because then we could see things done step by step".

Another issue highlighted in students' feedback was that one had been too little accessible as a teacher. The communication was too much in one direction and the sessions using NTR Meeting were only partially successful: the sound/picture was too poor and the use of time too little flexible.

Planning a new teaching form

Some hints had also been given during the course that the chosen teaching pattern was not entirely suited to the fairly heavy statistics material. We reviewed some web-based lectures that were produced by another institution (NKI) which combined filming of blackboard-based lectures, simple statistical attempts and slides. Short and simple lectures demonstrated with actions and words what the statistical formulae entailed. Even though the material's level of difficulty was far lower (descriptive statistics) and filming the lectures demanded a great deal of preparation and resources, there were ideas to be gained here. It seemed rather alarming that one required a cameraman to actively zoom in on formulas written up on the board, as well as that the final result contained edited-in film clips from other presentations of the material. The resources demanded by this type of arrangement clearly exceeded those available to me.

We were however interested in how others had designed web-based teaching in mathematics and conducted internet searches about this. I then found via YouTube a commercial resource from the American firm Classmate. This showed how a teacher, just through filming a static section of a whiteboard, was able to show mathematical procedures in a very understandable manner. I had the idea to try out the same thing. With a film camera mounted on a tripod I found out that this was possible without the help of an assistant, in other words, with far less planning and use of resources.

In order to accommodate the students over the second issue – a greater degree of presence – we decided to begin to use Skype far more actively. Skype has the opportunity to publicise accessibility by setting one's status in the program. If actively used, this can give an efficient way for students to know about your virtual availability for consultation.

The actions themselves

Having completed the whole syllabus before and during the final session, we only had the repetition lectures to try out the new lecture form. When the camera and external microphone were rigged up it was possible to concentrate fully on just the pedagogic conduct of the lecture. Personally I found this much simpler as a teaching method than using Camptasia. The reason for this may be that the teaching took place in a more familiar context, in front of a board.

In addition, as intended, we set up the use of Skype. Accessibility was ensured on the occasions when it was possible to help the students with problem solving. This led to a number of enquiries, either in the form of groups of students taking contact or of individual contact.

RESULTS AND DISCUSSION

Students' reactions to the actions, as well as follow-up assessments

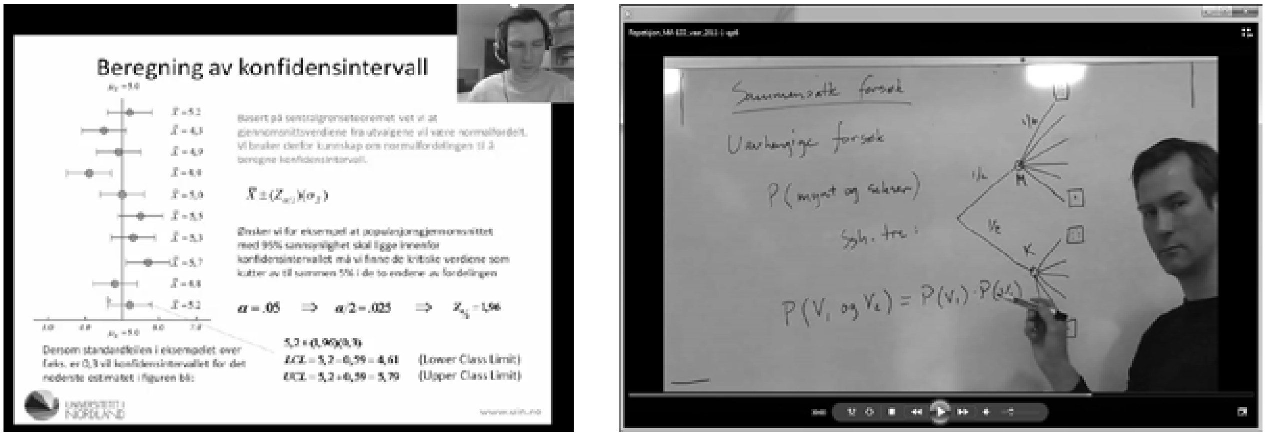

Figure 2: Web-lecture before and after the action. On the left the Camptasia studio recording, on the right, filming my own lecture.

Following the one repetition lecture that we had time to hold for the students we got in touch to ask for reactions to the new lecture form. We asked the students to draw a comparison between the previous form, with the Power Point presentation, and the new filming of the blackboard-based teaching. We did not use a questionnaire but instead sent out an email to which all the web-based students were asked to respond. The text was as follows:

> I am working on writing up the web-pedagogic concepts of NP 5-10, and in this connection wonder whether I could have your opinion about the new form of web lecture, involving filming of the whiteboard, which I demonstrated in the last repetition lecture. Your opinions can contribute to shaping the future layout of the maths teaching in the NP classes. Are there other things that you feel could improve the web-pedagogy?

As one can see, the investigation is very loose in its form; there were not enough time to present and analyse a detailed questionnaire on how and why the new teaching approach had improved the learning. In all, the email was sent to ten internet students who had followed the course. I received seven replies. The reactions were unanimously positive. One of the students says:

> Hi Helge! The last web lecture with the whiteboard worked well (…) I believe at any rate that this method of conducting web-based teaching is better than Powerpoint lectures (…) I think, by the way, that the sessions in which you used Geogebra were very good. The fixed times on Skype are a good idea.

Another wrote:

> Hi Helge! I liked the whiteboard presentation a great deal and would be very glad to have this type of web-based lecture in future. It becomes immediately easier to grasp and one gets concrete examples for comparison.

Regrettably, there was no time for an in-depth study on how this change in lecturing affected the students learning outcome and quality, I only recorded the unambiguous positive response on my single-question e-mail survey. I see of little point in using tools like Grounded Theory to extract meaning from these data due to the small amount of text in the feedback from the students.

Skype also worked very well in the sessions we had with the students. We believe that using Skype creates a more synchronised presence on the net. The dynamic contact list with availability status creates a proximity to the student that is not achieved to the same extent via more asynchronous media such as Fronter.

Justification in mathematics-didactics research

Raymond Duval has researched the characteristics of the cognitive processes on which a good learning process in mathematics is based. In his article "A cognitive analysis of problems of comprehension in a learning of mathematics" (Duval, 2006) he highlights the term "transformations between representations". In terms of mathematical objects, it is important for learning to at least achieve two types of cognitive representations of these. In geometry for example, one representation can be geometric, i.e. consisting of a picture (a right-angled triangle, for instance) and the other can be numerical showing the numerical relationship among the sides of the triangle (the property of Pythagoras, for instance). Duval believes that the essential prerequisite for learning is the transformation between these systems; in other words, to understand a mathematical object the purely formal symbolic representation is not sufficient but must be supplemented by an understanding of how the object is significant in other contexts.

Using the whiteboard as a learning tool as compared to a static slideshow, it is my opinion that this kind of representation dynamics is easier accomplished with a whiteboard. As the teacher explains verbally, she/he can sketch the connections, draw a graph, make a diagram or a figure, etc., to illustrate how a mathematical object will behave in different circumstances. There is also evidence that teacher's gestures can be a rather critical ingredient in the mediation of mathematical concepts and relations (Alibali & Nathan, 2011). All these various "analogue tools" that can be used for meaningful mathematical teaching are clearly not available in the static slideshow context.

SUMMARY

On the basis of students' feedback we can gain an indication that the action led to a clear improvement of my own web pedagogy. Naturally, critical comments are possible in relation to such a conclusion. For instance, there may arise a more or less conscious desire to appear positive towards me in order to increase their chances of a better grade at the end of the course. Another critical comment to the final result may be that the nature of the actions themselves leads to a positive attitude:

- Modern humans often have a positive image of change, at least in areas in which problems exist. The saying "in need of a change" is a significant one.
- The simple fact that one has attempted to address a problem may in itself be enough to ensure that the students have a positive reaction.

Also, one could raise critical remarks as to if such a small number of participants in a survey could lead to a definite conclusion.

Even if purely objectively we cannot say that this sort of change in web pedagogy will lead to a significantly improved learning process, it is possible, on the basis both of a theoretical consideration of maths learning (Duval, 2006) and of the unanimously positive public response, to assume that ground has been gained.

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CONTINUING PROFESSIONAL DEVELOPMENT AND DIGITAL MEDIA IN MATHEMATICS EDUCATION

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This paper presents the results of a study in a continuing professional program for mathematics teachers that had as main objective the integration of digital media in their teaching practices. The program was offered in a distance education modality. The didactic material was designed taking into account cognitive aspects that can transform the digital media into a powerful math learning tool. The analysis the didactic material is based on the semiotic potential of an artifact and the analysis of the learning process makes reference to the instrumental genesis theory.

INTRODUCTION

Studies on mathematics teaching and learning in digital environments have been developed since Papert´s work (the turtle geometry in the 80's). Since then, a considerable development of theories has happened (Gueudet & Trouche, 2010). One important subject focus has been the technology potential to enhance the cognitive process in mathematics learning (Moreno-Armella, Hegedus, & Kaput, 2008).

However, the integration of technologies in the practice of mathematics teachers has been happening very slowly, as reported in different studies. This is because a considerable amount of teachers has graduated prior to the widespread dissemination of digital media. Therefore, it is understandable that they prefer to keep away from practices that make use of digital media. As a result, programs of continuing professional development are necessary to overcome the absence of digital resources in mathematics classrooms.

In this paper we present the results of a study in a continuing professional program for mathematics teachers offered by Mathematics Institute at Federal University of Rio Grande do Sul (UFRGS/Brazil). The "Mathematics, Digital Media and Didactic (MDMD)" is a distance education program that had its first version in the period 2009-2011. Its main objective is to prepare mathematics teachers for using digital media in their classrooms. We will outline the program design and analyze the teaching and learning process in its first course. This course had focus on dynamic geometry software.

Our study is based upon theories developed to understand the complex process of integrating technologies into teaching practices. The idea of semiotic potential of an artifact (Bartolini Bussi & Mariotti, 2008), associated with the theory of registers of representation (Duval, 2008), helps to understand the mathematics learning process.

The theory of instrumental genesis helps to explain the dialectic between conceptual and technical work accomplished by the teachers in their first experience with dynamic geometry (Artigue, 2002; Trouche, 2004).

SEMIOTICS REPRESENTATION, DIGITAL MEDIA AND THE LEARNING OF MATHEMATICS

The theory of registers of representation emphasizes the importance of systems of representation in the learning of mathematics (Duval, 2006). Duval makes clear that the systems used in mathematics not only convey concepts and ideas, but they also have operating rules that allow to perform processes that lead to new concepts and ideas – those are the registers of semiotic representation. The concept of transformation highlights the mathematical process that occurs within a register (a treatment) or that occurs between registers (a conversion). The coordination of registers creates mathematical comprehension and enlarges cognitive abilities.

Digital media enlarge the possibilities of registers of representation. Indeed, nowadays there are a number of dynamic representations that can be manipulated directly on the computer screen. Different registers are synchronized - text, graphs, figures, equations and also metaphorical objects – offering different aspects of the mathematical object. This makes the digital tools powerful in the development of cognitive abilities for the learning of mathematics (Moreno-Armella et al., 2008).

However, digital systems of representation might not be enough to ensure the learning of mathematics (Bartolini Bussi & Mariotti, 2008). According to these two authors, the learning process depends on the design of didactic situations that can provoke the development of utilization schemes that convey mathematical ideas – in a vygotskian perspective, a digital system of representation is an artifact to support and develop the mathematical reasoning. But the authors say that "*the link between artifacts and signs is easily recognized, yet what needs to be emphasized and better explained is the link between signs and the content to be mediated* (p. 752)".

We also rely upon the theory of instrumentation that explains the dialectic between conceptual and technical work when using a digital artifact (Artigue, 2002; Trouche, 2004).The differentiation between artifact and instrument is clear in this theory: an instrument is a mixed entity that involves an artifact and cognitive schemes of utilization. In this theory, the development of cognitive schemes related to software is called instrumental genesis. On the one hand, this development depends on the subject's actions towards the artifact (the instrumentalization process). On the other hand, it depends on the subject's actions that are provoked by the feedback of the artifact (the instrumentation process).

The interesting expression *toolforthougths*, coined by Clinton and Shaffer (2006), highlights that subjects and artifacts (as digital systems of representation) can interact in an action and reaction process that produces cognitive attitudes towards new knowledge. In this paper we will use the expression *toolforthougths* as synonymous of *instrument*, because it conveys better the notion of "a tool to think with".

The theories helped us to understand that teachers need to experience their own *instrumental genesis* prior applying digital media resource as *toolforthougths* in their classrooms. So the didactic material was designed to provoke their instrumental genesis. The experience with the teachers showed that they need to be supported by professional development programs in new practices in order to think creatively about a curriculum that integrates the digital media. We look at these issues in this paper.

THE DESIGN OF HYPERTEXT WITH INTERACTIVE ANIMATIONS

The program MDMD was designed to prepare teachers to use different mathematics software in their classes. Particular attention was given to the software semiotic potential and to the process that makes an artifact become a *toolforthougths*. The courses' contents emphasized the mathematics concepts and technical skills required to use a particular software. Pedagogical strategies were also presented.

The design of the didactical material focused on the role of the registers of representation, as well as on the utilization scheme to be developed in the learning process. Knowing beforehand the semiotic potential of the software, we prepared activities requiring utilization schemes related to transformations of registers, especially the conversions. For instance, activities using discursive and geometric registers were planned through geometric constructions with GeoGebra. Real variable functions activities requiring conversions from geometric register to algebraic one were also prepared with GeoGebra. An interesting conversion from geometric to algebraic register was proposed with software GrafEq – the goal is to produce geometric shapes using algebraic registers.

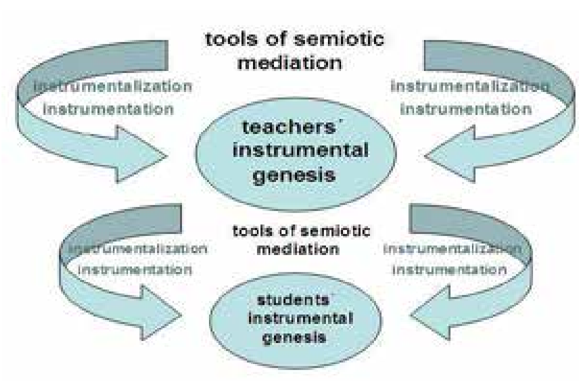


Figure 1: Cascade of instrumental genesis processes

The Program predicted a *cascade of instrumental genesis processes* represented in Figure 1. The activities, designed according to the principles of theory of semiotic mediation, were organized to provoke teachers´ instrumental genesis and it was expected that the same material would be used by them to promote a similar instrumental genesis process in their classrooms. This cascade idea is supported by the theory of documentation (Gueudet & Trouche, 2010); the theory highlights the important role of the resources used by teachers while preparing a topic to be taught.

Each course used its own website and focused on different software. The websites have a similar design in order to make the navigation easier (Figure 2). In the left frame there are modules and each one is divided in sub menus: (i) objective - gives the answer to the question "what we are going to study in this lesson?"; (ii) contents – presents the mathematics contents and the didactic background theories; (iii) activities – brings the activities to be developed; (iv) resources – presents tutorials for the software to be used; (v) complements - suggests additional material to enhance the contents and media used in the module. We will discuss a specific course of the program, Digital Media I, which emphasizes GeoGebra as a math learning tool. As undergraduate courses in Brazil, even nowadays, still do not prepare teachers for integrating digital media in their teaching (Jover, 2008), the website was designed to provoke the instrumental genesis in dynamic geometry environments. The main purpose was to provoke the utilization schemes in order to properly explore dynamic geometry, especially the "drag action" scheme.

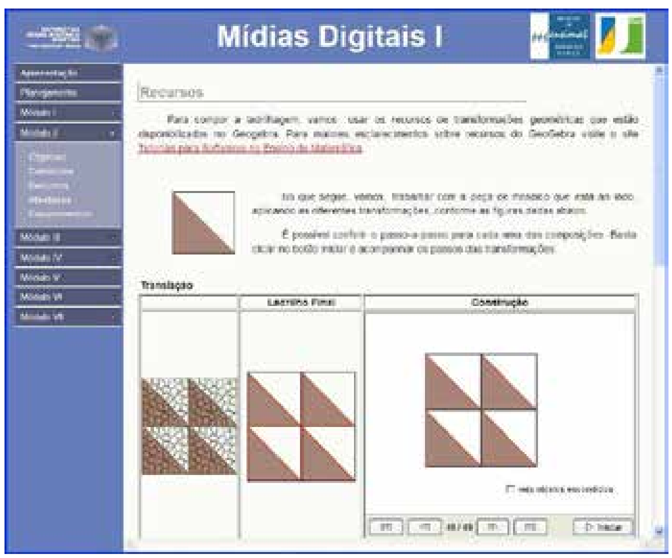
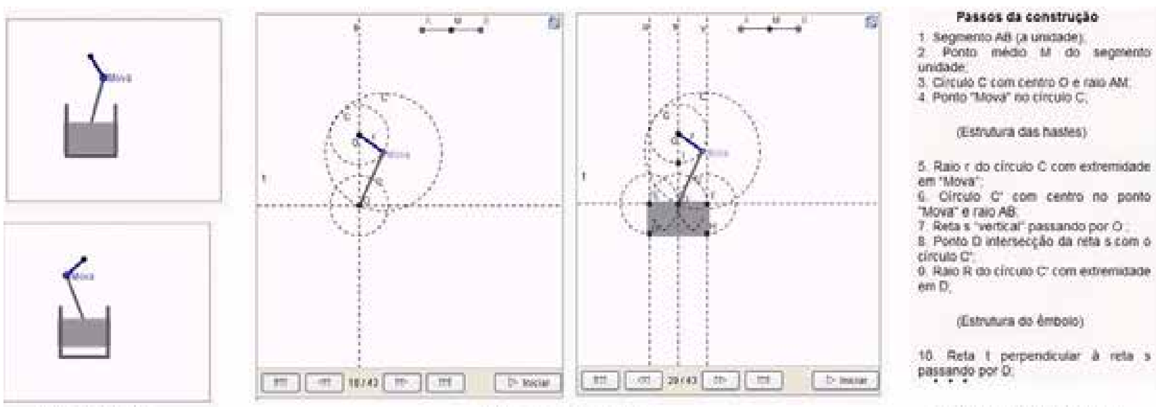


Figure 2: one of the course website interfaces. In the detail, the submenus objective, contents, resources, activities and compliments.

The theme of the module I and II is mosaics and tiling patterns and the module III is about geometric modelling. We are going to observe the instrumental genesis process through the work realized in those three modules. The whole site material can be seen at http://www.ufrgs.br, at the link ´Disciplines´.

In module I the teachers are invited to observe different mosaics in their everyday life and produce a similar one with GeoGebra. A simple construction like this is not easy for beginners in dynamic geometry. The site offers interactive animations that explain how to produce mosaics and the construction procedure can be followed step-by-step in the user's learning pace. The module II focuses on the geometric transformations: the task is to produce different tiling patterns using the same mosaic but different transformations (reflection, rotation, translation). As in the mosaic activity, interactive animations are available to help the beginners.

In module III, geometry is explored through modelling. Now the task is to produce a virtual model of an everyday-life mechanism – as a fan, a car piston, a hydraulic jack or a scale. Some virtual models that can be manipulated are available in the submenu *Contents*. Interactive animations and the protocol of construction that will help in the understanding of the construction procedures are presented in the submenu *Resources*.



(a) virtual model (b) interactive animation (c) construction protocol

Figure 3: different registers of the model piston in the website.

Figure 3 illustrates the resources available in module III related to the model “piston”: (a) shows two instances of the virtual model; (b) shows two different moments of the interactive animation of the construction; (c) shows the construction protocol.

The material offers different registers of the model to be learned. The virtual model can be manipulated (the “drag” of the blue point produces the piston movement) and the dynamic geometric register might be enough to proceed with the construction. Otherwise, the user can observe the step-by-step construction using the interactive animation and, at the same time, manipulate the model. If this second geometric register is still not enough to proceed with the construction, the user can read the discursive register.

One of the characteristics of the interactive animations is the navigation bar allowing going back and forth in the geometric construction. If necessary the user can also change the steps speed. Those two features make the interactive animation adaptable to different learning rhythms, which is an important aspect to be considered in the process of *instrumental genesis*. The teachers learned to use GeoGebra in the distance education program MDMD using those resources which were organized in a website.

A TEACHER LEARNING TO USE THE SOFTWARE GEOGEBRA

One hundred and eighty teachers attended the course and were accompanied by a professor and seven tutors. Teachers were distributed in seven cities in the state of Rio Grande do Sul, forming groups between 15 and 30 participants. A tutor, under the supervision of the professor, supported each group. The didactic material used was the website “Media I” presented in the previous section. Guidelines for the activity to be performed were posted weekly in the virtual environment Moodle. The teachers published their production in a Moodle Database consisting of a GeoGebra file and a text about the difficulties and the progress during the week. The interactions between groups and tutors happened in a Moodle Forum.

As most teachers had no knowledge of dynamic geometry software, it was interesting to observe their instrumental genesis process. In what follows we will analyze the process experienced by one teacher. Through the production presented in the three first modules of the course, we will bring evidences of the instrumentation and instrumentalization processes that were experienced by the teacher. The analysis uses the material that she posted weekly on Moodle. In her first attempts to produce the mosaic activity, proposed in module I, she says:

> I did not know the software, so I had great difficulty.... I made several attempts because I was not having success to move the vertices of the mosaic without any deformation in shape.... I felt that it is needed a specific training with GeoGebra, so that we can explore constructions and then get the desired results.

When the teacher talks about the need of specific training to explore the software to accomplish the construction, she stresses the importance of recognizing the potential of the software (the instrumentalization process). When she refers to the need of practice with the software to obtain a figure in dynamic geometry, one can see a manifestation towards the development of utilization scheme (the instrumentation process). The tutor´s comment shows that the teacher is still developing the utilization schemes:

> I realized that you explored the software quite a lot and you've already done a nice job ... still there is a problem with the construction related to the figure stability ... but this is common to most of those who are dealing for the first time with dynamic geometry... congratulations for the many construction attempts.

It was observed that the interactive animations of mosaics, available in the website, had an important role in the awareness of the two aspects of the instrumental genesis. The mosaic activity provoked an evolution of attitudes: in the beginning the teacher talks about things she does not know and difficulties with GeoGebra. She explores the software menus and makes many essays of construction – we would say that she is developing the utilization schemes that will transform the artifact into an instrument.

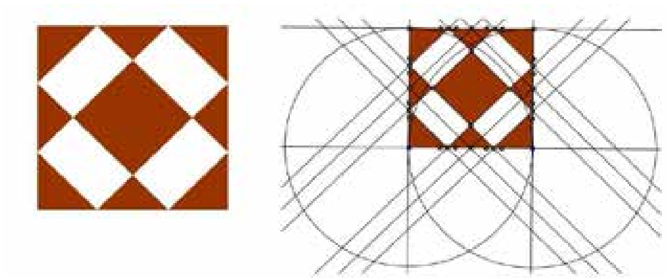


Fig 4: The mosaic produced by the teacher

Figure 4 shows her final mosaic and the geometric procedure. The square, made with perpendicular lines and compass, is a dynamic figure; for decoration she used several times the menu "midpoint" instead of the efficient menu of transformations available in GeoGebra – after constructing the first white rectangle, the others can be done through rotation. The teacher´s action indicates an instrumentation process in progress.

In the tiling pattern activity of module II, the tutor observed that the teacher did not make so many attempts as before. She was quite confident to produce a tilling pattern in dynamic geometry. Figure 5 indicates: a) the tiling pattern produced; b) the construction procedure; c) the basic elements of the pattern. At the end of the activity the teacher says:

> As mentioned in the previous task, I did not know the software ... It was still very hard to carry out the new activity ... For reaching to the end of the task I spent several hours in front of GeoGebra... I am very concerned about the time that will be necessary to perform the next tasks.... I need to learn how to get the desired result faster ... I did my best at this moment.

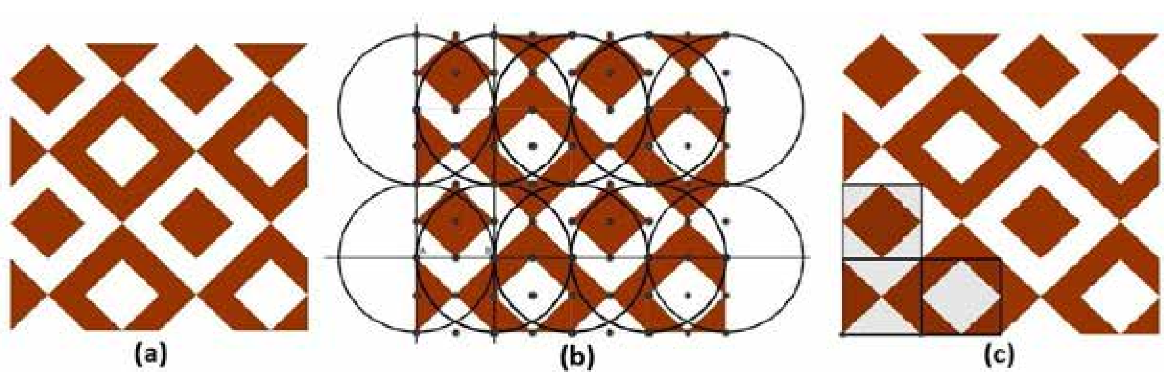


Figure 5: The tiling pattern produced by the teacher

The teacher advanced in her instrumentation process – the figure was made stable under "dragging" without the need of many attempts. However, it is possible to observe that the utilization schemes related to geometric transformations are still quite incipient even though this was the utilization scheme to be developed through the activity. The central reflection transformation was the only one present in the construction protocol and it was applied only to points. The tutor´s comment is:

> As I have mentioned, you already have understood the spirit of dynamic geometry. In the construction of the tilling pattern I noticed the frequent use of the reflections, which is very interesting. I invite you to explore other geometric transformations in the task that is coming.

The construction procedure, illustrated in Figure 5b, still shows several circles, lines and segment midpoints - the construction has about 250 steps. In fact the teacher "spent several hours in front of GeoGebra" to accomplish the task and we would say that she was not yet using GeoGebra as a powerful toolforthougths. She did not pay attention to the basic shapes that are highlighted in Figure 5c. With those shapes and using the geometric transformations (rotation, translation, reflection) it would be possible "to get the desired result faster" (a comment made by the teacher herself).

In the modelling activity of module III, the teacher produced a virtual blind window (Figure 6). The construction required more mastery of the GeoGebra menus. To obtain the "open-close" virtual movement, a point on a circle arc was used and the corresponding radius was the initial segment of one of the blue parallelograms. The reflection transformation was used in different moments to produce a second copy of the articulated blind window. In this third activity the teacher reveals a progressive recognition of the GeoGebra potential and also shows more confidence with the schemes of utilization.

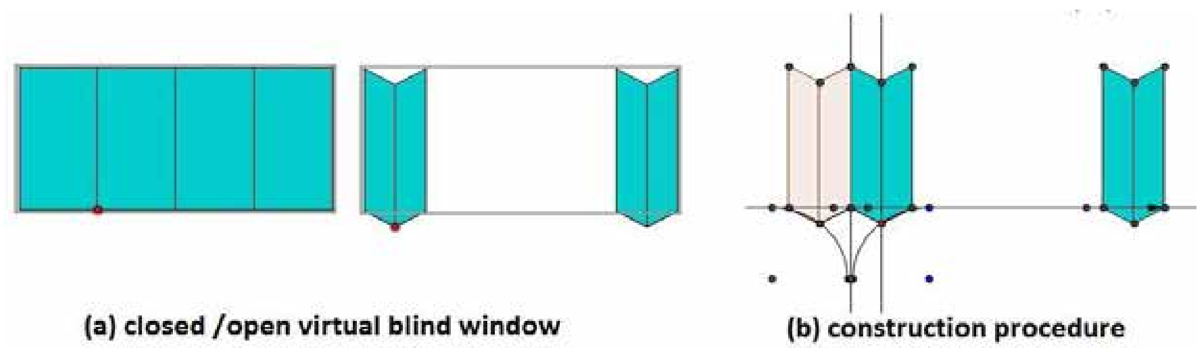


Figure 6: The geometric modelling activity of the teacher

The teacher's production and her interactions with the tutor illustrates that simple geometry activities can be a source of difficulty for those who are starting a process of instrumental genesis. Artigue (2002) points out that the instrumental genesis involves the interweaving of mathematical knowledge and technical abilities to use the software features. In fact, it was observed that some of the teacher's difficulties were related to the ability of using GeoGebra features, but she also had difficulties with the geometry knowledge required in the activities. In the whole group, similarly, we observed that the processes of instrumentalization and instrumentation happened completely intertwined with the geometry content. For instance, it took time for teachers to feel confident with the use of the geometric transformation menu and the main reason was the lack of knowledge, since geometric transformation is not a school subject in Brazil.

CONCLUSIONS

The analysis of the production of one of the teachers following the first course of the Program MMDD shows that the *instrumental genesis* is a complex process. As a general remark we would say that the process that will transform an artifact into *toolfortoughts* depends on many experiments. As a whole group, teachers did many experiments with GeoGebra and after two months, they carried out teaching experiments in their classrooms, as part of the course activities. Even though teachers had experienced their instrumental genesis (in fact, the instrumental genesis is always a process in progress), the use of GeoGebra that they did with their students was still quite modest. But even so, they spoke enthusiastically about the new teaching experience. One teacher said that "in the first meeting my students explored the GeoGebra tools... then they started the construction of polygons that do not deform, with motivation, interest and cooperation... they were very excited"; another one said that "the activities carried out with my students were special... I felt satisfaction and delight of the students with their geometric constructions". The teachers' experiments were not enough to evidence a cascade of instrumental genesis processes, as had been predicted in the design of the Program.

We might say that for the happening of such a *cascade*, a new perspective about the teaching and learning of mathematics must be also envisioned by teachers during their instrumentalization and instrumentation processes. Besides attention to the instrumental genesis process, professional development programs also need to pay attention to what could be called the genesis of a digital pedagogy. The teachers´ experiments with their students showed us that to think creatively about a school curriculum that integrates *toolfortoughts* is not an easy task. One important research issue to be addressed is: "how can the digital technology transform the school curriculum?"

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DEVELOPING AN INTUITIVE CONCEPT OF LIMIT WHEN APPROACHING THE DERIVATIVE FUNCTION

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The central idea of calculus is the concept of limit. German secondary school curricula claim to introduce the concept of limit in an intuitive way refraining from a rigorous mathematical definition. However, it is unclear what can be regarded as a "good intuitive basis". After the discussion of typical student problems we present a DGS-based activity to support the introduction of the derivative function fostering a dynamic concept of limit. Aside from the step from derivation as a local phenomenon to the global view of the derivative function we also show how the activity can be used to visualize and talk about the variety of limit processes.

Keywords: conceptualization processes, DGS, interactive learning activity, concept of limit, teaching and learning of calculus

INTRODUCTION AND RATIONALE

The problems of contemporary calculus courses at secondary school mainly result from the tension between learning or teaching of routines and the development of a structural understanding of the underlying concepts (Tall 1996, p. 306).

In fact this problem exists since the introduction of calculus to German school curricula: for example Toeplitz (1928) urges to remove calculus from school if teachers are not able to bring out more than the teaching of mere routines.

The fundamental concepts of calculus (e.g. the concept of limits, derivation and integration) are mathematically advanced. Therefore teachers are always required to make didactical decisions about what to teach in a visual-intuitive way and what to teach in a mathematically rigorous way.

For example the secondary school curriculum in Berlin, Germany, explicitly demands an intuitive or propaedeutic approach to calculus concepts at the end of grade 10. This is established through introducing a special "Modul" called "Describing change with functions" (Jugend und Sport Senatsverwaltung für Bildung, 2006a). In this context Hoffkamp designed and investigated several DGS-based activities to support such a qualitative approach (Hoffkamp, 2009, 2011), which could lead to a sustainable intuitive basis of certain calculus concepts.

Considering the central role of the concept of limit within calculus, the described situation needs particular attention. On the one hand the Berlin secondary school curriculum for grade 11 mentions that the concept of limit can only be taught in a propaedeutic way since the necessary exact notions (series, convergence tests) are not available to the students. On the other hand, the teachers are required to introduce the derivative as limit of the difference quotient (Jugend und Sport Senatsverwaltung für Bildung, 2006b).

Sfard (1991) describes the process of mathematical concept formation. Starting with operational conceptions (e.g. functions as computational processes, rational numbers as results of division of integers) structural conceptions (e.g. function as set of ordered pairs, rational numbers as pairs of integers) develop, the latter leading to the establishment of abstract objects.

Fischbein (1989) offers another perspective on the issue of intuitive approaches as well as the process of mathematical concept formation. He describes how, during the process of mathematical abstraction, mental models of the abstract concepts develop in the learners mind. According to him these intuitive models, tacitly or not, influence the way we conduct mathematical reasoning processes. These tacit models are one reason for the difficulties students are facing in the process of learning mathematics. He suggests allowing the students to consciously analyze the influence of those tacit models and in this way to allow them to avoid the development of misconceptions.

Taking the above into account means to realize the necessity of a profound intuitive basis or cognitive root (Tall, 2006) for the limit concept that could lead to the development of an object view of the derivative. Cognitive roots as described by Tall (2006) are not mathematical foundations but approaches that build "on concepts which have the dual role of being familiar to the students and also provide the basis for later mathematical development" (Tall, 2006). According to him they require a combination of empirical research and mathematical knowledge in order to find them. In our opinion, DGS appears to be a good means in this context, therefore we present an activity using DGS-based interactive visualizations fostering a dynamic idea of limits and leading to an object view of derivation.

In the following we will describe the conceptual change approach and the potential of DGS-based activities. We will give an example of an activity that is related to our rationale. After that we will present our research questions and the results of a video-study.

Spontaneous conceptions of limit and conceptual change

The above considerations conform to a genetic view of learning as described by Wagenschein (1992). Especially for the process of conceptualization, a theoretical perspective like the conceptual change approach is helpful to design activities like the one described in this article, and to understand the students' learning processes. The conceptual change approach itself is a genetic learning theory. A description can be found in Verschaffel and Vosniadou (2004). Conceptual change does not mean to switch from one concept to another by replacing the old concept by a better new one. It is the process of reintegration and reorganization of cognitive structures in order to develop mental conceptions and to activate the appropriate conceptions dependent on given contexts (Verschaffel & Vosniadou 2004, p. 448).

For the process of conceptualization it is important to know about the students' spontaneous conceptions and to build on them. In fact the students' spontaneous conceptions can be considered a learning opportunity and a starting point for further development by dealing with them explicitly (Prediger 2004).

Therefore we tried to find out about the students' spontaneous conceptions of limits. We asked students at the end of grade 10 and 11 to write a letter to an imaginary "clueless" friend explaining the mathematical notion of limit. We present two excerpts from letters here. One student wrote:

> The limit is the outermost value of a number range. For example if one says "all numbers from one to five" then one and five are limits.

Another student wrote:

> Considering the graph of a function over a large interval one can observe that some functions come closer and closer to a certain value. The function tends only to this value, without drifting away again. However, the function does not reach or exceed it.

The first excerpt shows that the student considers limits as bounds of intervals and formulates a static conception of limit. The second excerpt reflects the student's experiences with limits in connection with asymptotic behaviour. Although the student formulates a dynamic conception of limit, his conception is not elaborated since limit processes seem to be always "monotonous" and limits "cannot be reached or exceeded".

These observations are confirmed by the work of Cornu (1991) who described typical spontaneous conceptions of limits. In fact all limiting processes like the concepts of continuity, differentiation or integration contain similar cognitive problems: To overcome or integrate the spontaneous conceptions in the learner's individual concept.

THE USE OF DGS-BASED ACTIVITIES

As already mentioned we think that DGS is a good means to establish an intuitive basis of the concept of limit. With respect to the mentioned spontaneous conceptions and the conceptual change approach, DGS-based activities could not only help to develop a dynamic view of limit and limit processes, but also help to develop a more elaborated conception of limit by visualizing the variety of limit processes. Therefore we combine interactive visualizations with special tasks stimulating verbalization and exploration processes. Especially the role of verbalization as a mediator between the representations and the students' mental concepts when working with interactive visualizations was pointed out in the work of Hoffkamp (2011) and based on Janvier's work (1978).

Which role of the computer do we focus on? At first, we benefit from the various possibilities of visualizing mathematical concepts. Therefore, we make use of the possibility to visualize a holistic representation in contrast to a linear order of mathematical content (see also Sfard 1991). Moreover we add interaction to enable learners to explore the interactive activities without negative consequences.

While giving the opportunity for exploration we use the computer to “restrict the actions of learners and thus help them to develop appropriate mental models of representation” (Kortenkamp 2007, p. 148). In this sense visualizations play a heuristic role and can be used before exact mathematical notions or concepts are available.

THE ACTIVITY “TOWARDS THE DERIVATIVE FUNCTION”

In the following we present a DGS-based learning activity. It introduces the derivative function as an object and its relation to the original function. This activity is meant to be exemplary. It shows how conceptualization processes in the context of the concept of limit can be supported by using special DGS-based activities. Our idea of using the difference quotient and its extension by continuity for an object-based approach to obtaining the derivative function has already been mentioned by Mueller and Forster (2003) quoting Yerushalmy and Schwartz (1999). However, they did not explicate the full didactic potential of this approach and did not use the dynamic approach towards limit that can be fostered by a DGS-based activity.

The two focal points for our activity are emphasizing the difference quotient and permitting a look on various limit processes that constitute the analytical step. The difference quotient is not only a (theory generating) precursor for the differential quotient and the derivative (the way it is often used in schools). It is the concept that has a direct relation to reality through the concept of average change (of speed etc.). Therefore it is evident for students and is thus worth taking a closer look at.

The activity is meant to be used when the derivative of a function at a certain value is already known to the students. However, the students have no elaborate limit concept so far. This is usually the case at the start of grade 11 in secondary schools. The derivative function, however, is not known to the students yet. The way from a local (derivative at a point) view on derivation towards a more global, object based view (derivative function) shall be supported, while several limit processes get examined on the way. This is in line with our idea of broadening the view on limits as well as bringing about an object view on the derivative function.

The activity can be found at

http://www2.mathematik.hu-berlin.de/~hoffkamp/Material/ableitungsfunktion.html.

Description of the activity and didactical analysis

The activity consists of three separate worksheets or tasks. It is very rich concerning mathematical concepts. We will however only describe and didactically analyze selected parts that have a direct relation to the presented results and aims of this article. Figure 1 gives a general idea of what one of the tasks looks like. The text above the applet gives an introduction to the task. The text next to the applet poses special questions and tasks that also ask students to verbalize their thoughts and observations and are meant to help the students go through the conceptual change process we intend to initiate. This also relates to Fischbein. Verbalization makes tacit models and intuitive concepts accessible to a conscious process of reflection.

The tasks are consecutive. Different predetermined functions f can be chosen in every task to make sure there is not just one but many graphs available. The restriction on predetermined functions allows a broadening of the view while at the same time focusing on certain sustainable examples (see also Kortenkamp, 2007). The functions were chosen as examples of certain classes of functions - symmetric and non-symmetric, polynomial and trigonometric functions and in tasks 2 and 3 also the (at the origin) non-differentiable absolute value function.

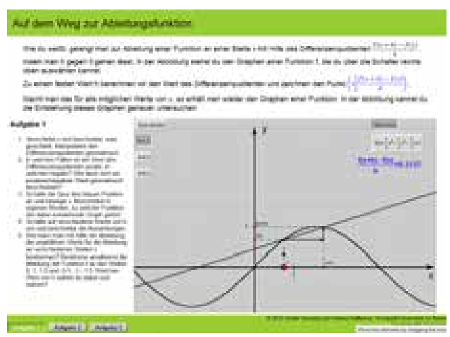

Figure 1: General overview of a worksheet.

The first task takes up the so far local ideas of the derivative at a point and differential quotient as the limit of the difference quotient. In the second task a first object view on the derivative function is reached. Also the limit process observed is changed. The third task introduces yet another variation of the limit process while the object view on the derivative function remains in focus.

The first Applet (see figure 2) offers a process view on the functions we will take an object view on in tasks 2 and 3. For three fixed values of h the term $\frac{f(x+h)-f(x)}{h}$ (the difference quotient) is evaluated at a certain x-coordinate. The x-coordinate can be changed by dragging the big red point on the x-axis. The point $(x|\frac{f(x+h)-f(x)}{h})$ is always printed in blue. The students already know the difference quotient. So far they only evaluated it for a single fixed value of x and varying values of h to gain the derivative at a point. If tracing is activated one gets a trace of the resulting points which forms the graph of the function g with $g(x)=\frac{f(x+h)-f(x)}{h}$ (for fixed h). This way we have a point wise (process) view of the construction of the graph. We change between processes by varying h. The resulting function becomes a better approximation of the derivative function for smaller values of h.

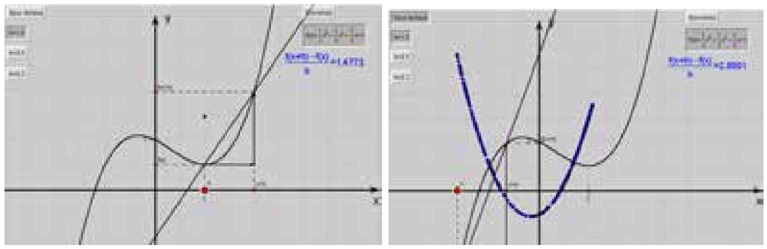

Figure 2: Applet 1 without and with tracing within the first task.

The second and third applets are pretty similar in their construction, which is why only one of the two is pictured here in figure 3. They both show graphs of functions that can be manipulated. In applet 2 there is only one function namely g with $g(x)=\frac{f(x+h)-f(x)}{h}$. In addition to that applet 3 also visualizes the function k with $k(x)=\frac{f(x+h)-f(x-h)}{2h}$.

Switching over to applet 2 we have a new situation. The function that developed as the result of a process (of changing the x value) in applet 1 now exists as a single entity. We evaluate the difference quotient for all values of x simultaneously now (note that this is not the difference quotient function as that would be $g_x(h)=\frac{f(x+h)-f(x)}{h}$ for a fixed value of x). At the same time we no longer have fixed values of h but can change those using a slider. If the value of h is changed, the graph of g moves as a whole. The students observe a family of curves that converges to a limit function. According to Sfard (1991) working with families of curves or equations with parameters is already a step towards reification. Limit becomes something more dynamic in this context as the students observe and describe the limit process that leads to the derivative function for differentiable f.

The third applet is aimed at the development of a more elaborated conception of limit. In contrast to task 2 we now take a look at not only one but two different limit processes. We show that different limit processes can lead to the same limit. The idea is to prevent or change a very restricted view on limits as could be seen in the student letters described above. An object view on the functions involved is necessary now.

One observed object is the function g, already known from task 2, the other is the above mentioned function k with $k(x)=\frac{f(x+h)-f(x-h)}{2h}$. k represents a symmetric differential quotient. It has some interesting characteristics. For example, if f is a symmetric function, k has a local extremum where f' has a local extremum for any given value of h. For g this is obviously not the case. The students are asked to describe and compare the functions g and k and to explain their observation geometrically (by using secants) and algebraically (by comparing both forms of difference quotients). One other observation is that k converges faster to f' than g. Therefore the symmetric difference quotient is better for numerical computations. For polynomial functions this can even be proved with students. The students may discover that convergence is not always the same and that different processes may converge at different speeds.

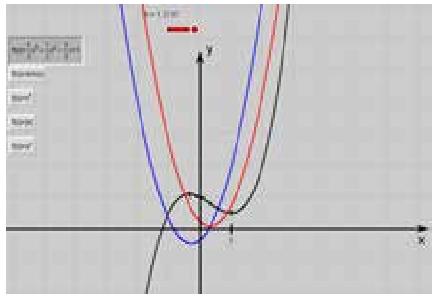

Figure 3: Applet 3 within the third task.

METHODOLOGY AND RESEARCH QUESTIONS

A first video study was conducted in August 2012. The two authors acted as researchers and teachers respectively. The activity was introduced in an 11th grade advanced course in mathematics. Students were video-taped during their work with the activity. The students' worksheets were collected for further study. Classroom interaction was noted by one of the authors while the other lead the course in classroom discussion and students' presentation of their results.

Our analysis of the collected data is led by the following research questions:

In what way can a dynamic-visual approach as depicted in this paper support the development of a sustainable conception of limit and related mathematical concepts?

What conceptions of limit do students develop if confronted with activities as the one presented?

What spontaneous conceptions can be uncovered through analysis of students' work and how can they be used as learning opportunities?

FINDINGS

One of the tasks for the first applet was "In which cases is the value of the difference quotient positive, negative and when is it equal to zero? Explain the geometric meaning of a positive/negative value or a value equal to zero." Many students answered in the way of "Positive if the graph increases; Zero if the central point of x+h and x is similar to the local maximum; Negative if the graph decreases." Their reasoning however is only possible through the analytical step; the core achievement of school analysis. From a mathematical point of view, what we see here is a misconception: the difference quotient is interpreted as the differential quotient. The students spontaneously assumed linearity. For linear functions the slope of any secant equals that of the tangent in any point of the graph. Therefore we may infer the behaviour of linear functions from the slope of any secant. In general, this is no longer true for non-linear functions, so the slope concept of the secant needs to be evolved to that of the tangent. The students' spontaneous conception of linearity became apparent and could be productively and explicitly used directly within the classroom in order to emphasize the fundamental difference between the difference quotient and the differential quotient. The second part of the students' answer was taken up in class discussion to develop Rolle‘s theorem starting in an informal way.

If we take another look at the above excerpt, we make another interesting discovery. The task was taken up again in later class discussion. It lead to a theorem of monotonicity. What became apparent here is that mathematical logic and quantification constitute an important obstacle for students. It was difficult for them to express that the function is not increasing or decreasing for every x between x and x+h but that there is at least one x with an increasing tangent.

The third applet offered a view on different limit processes that lead to the same limit. The students were asked to verify that the usual difference quotient and the symmetric one tend to the same limit. Or rather that the limit of the two series of secant-slope-functions actually is the derivative function. It was very hard for the students to verify any of this geometrically, which sheds light on the fact that geometric representations need to be focused on more, and the geometric meaning of the limit process when approaching the differential quotient should be made more explicit in a dynamic way. However, the students were able to compute the limit for certain values of h and this way convinced themselves that the symmetric difference quotient actually leads to the same limit as the usual one. To give an example of what they did, we present the work of one of the videotaped students. He said:

> “We are not supposed to know, what function this is, but if we let h go to zero in this task, we shall only show, that it becomes roughly the same. They should move towards each other now... [uses the slider to let h go to zero]... Because it does not matter if we have plus zero or minus zero here or two times zero.”

He used the dynamic possibilities of the applet to support his assumption and realized that when he made h very small, the two graphs were almost indistinguishable. Finally he computed (in a longer way, but essentially) $\lim\limits_{h\to 0}\frac{(1+h)^2-1^2}{h}=2=\lim\limits_{h\to 0}\frac{(1+h)^2-(1-h)^2}{2h}$ to convince himself and the other students that his earlier observation was true. This way the students became aware of the fact that there is a variety of comparable limit processes leading to the same limit. The insight into the necessity of proving their assumption was directly evoked by the use of the dynamic possibilities provided through the applet.

CONCLUSIONS

The presented observations give an insight into the polymorphism of possibilities within the described individual cases and may raise one's awareness for the perception of specific students conceptions and the own didactical action.

We are also confident that activities like the above are valuable means in the conceptualization process and may serve as cognitive roots respectively as an intuitive basis, as can be seen in our study. They allow the development of a dynamic conception of limit that appears to be a good cognitive root. Additionally, when using the applet there can be a clearer focus on the analytical step as a core achievement of analysis.

We have seen that a qualitative-empirical analysis as depicted in our paper can give impulses for further didactical and mathematical analyses. The search for appropriate cognitive roots necessitates further inquiries into the nature of such roots. More work in the field of ICT-based concepts and activities, also in the way of mathematics education as a design science (Wittmann 1995), is necessary here.

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A FRAMEWORK FOR EXAMINING STUDENT LEARNING OF MATHEMATICS: TASKS USING TECHNOLOGY

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This paper, taken directly from the author's doctoral thesis, (Joubert, 2007) develops a theoretical and methodological framing for examining student learning in the context of mathematics classrooms where computers are used. The framing, drawing particularly on the Brousseau' theory of didactic situations (1997), takes into account not only the student interactions with the environment but also the crucial role played by the feedback from the computer. This approach focuses on the processes in which the students are engaged and suggests the sorts of interactions that might provide evidence of student mathematical learning. The paper concludes with a section which analyses an episode of student mathematical activity using this framing.

Keywords: modes of production, computer, feedback, task, graphing

BACKGROUND AND CONTEXT

Mathematical activity, like all human activity, is mediated by tools, which include not only symbol systems, mental representations, algorithms and representational systems (such as functions and co-ordinate graphs) but also physical tools such as pen and paper, calculators, measuring devices and so on. Computers can be seen as a particularly interesting tool because of their 'intrinsically cognitive character' (Balacheff & Kaput, 1996, p. 469). For some software used in the teaching and learning of mathematics, the implication is that, to a greater or lesser extent, the software can perform mathematical processes (or 'do the mathematics') (Hoyles & Noss, 2003; Sutherland, 2007) for the user. For example, in traditional mathematics classrooms producing a bar chart from a set of data is seen as a valid mathematical activity, but a data handling package is able to 'do' this mathematics (as described by Ruthven and Hennessey (2002)). This has important implications for the design of classroom mathematical tasks.

A second important characteristic of software, related to its cognitive character, is the fact that software provides feedback for the user;

> 'The interaction between a learner and a computer is based on a symbolic interpretation and computation of the learner input, and the feedback of the environment is provided in the proper register…' (Balacheff & Kaput, 1996, p. 470).

This feedback, together with the ability of the software to 'do the mathematics' as described above, are perhaps the most compelling reasons for seeking to understand the way these tools are used in ordinary classrooms.

The current literature focuses more on teachers' perceptions of the actual use of computers in mathematics classrooms (Monaghan, 2004; Ruthven & Hennessy,

2002) than on the student processes (Lagrange et al., 2003). However, as Tall (1995) argued over fifteen years ago, there is a need to focus our attention on the students' thinking processes in situations where computers have become established in classrooms, to find out 'what is *really* happening under the surface'(p 11, italics in original). It seems that this research agenda has still not been fully met (Lagrange et al., 2003) and that there is still a need to develop a detailed understanding of the students' mathematical activity and learning in these authentic situations.

The data analysed in this paper is taken from a study which aimed to address this research agenda. The study aimed to contribute to the body of research conducted in a naturalistic research paradigm with a particular interest in the use of computer software in mathematics teaching and learning. The paper provides a theoretical and methodological framing for the observation of student interactions in these contexts with the overall aims of a) establishing an understanding of the relationship between observable student mathematical activity and their mathematical learning and b) understanding the role of the computer. It lays out the framing in detail and then provides an example to demonstrate how it has been used.

THEORETICAL FRAMING

To theorise the classroom situation, the paper draws on the *theory of didactic situations* (Brousseau, 1997) which identifies the 'didactical contract' between teachers and students and is based on detailed observations of authentic mathematics classroom settings. (This is explained in detail in my doctoral thesis (Joubert, 2007)). Brousseau uses the notion of the *milieu*; 'everything that acts on the student or that she acts on' (p 9) to describe these settings. It is through interacting with the *milieu* and with the tools of the *milieu* (including the task set for them by the teacher) that the students engage with classroom mathematics. The student interactions can be conceptualised as a 'dialogue' between the student (or group of students) and the feedback from the *milieu* (Brousseau, 1997). The feedback from the *milieu* can take many forms; for example verbal feedback from other students and the teacher, an outcome of a game or a graph produced by computer software. The importance of feedback should not be underestimated; as pointed out by Balacheff (1990) :

> 'The pupils' behaviour and the type of control the pupils exert on the solution they produce strongly depends on the feedback given during the situation. If there is no feedback, then the pupils' cognitive activity is different from what it could be in a situation in which the falsity of the solution could have serious consequences' (p. 260).

Student activity and mathematical learning

Brousseau (1997) uses the notion of 'modes of production' to describe the different types of dialectic interactions between students and the *milieu*; he suggests that as they work through a mathematical task, they will engage in all or some of the dialectics of action, formulation and validation.

Brousseau (1997) describes a dialectic of **action** as the student constructing an initial solution to the problem straight away, informed by her current knowledge. He explains that, in dialectics of action, students use 'implicit models', making decisions based on rules and relationships of which they may not yet be conscious; he suggests that the strategies the student uses 'are, in a way, propositions confirmed or invalidated by experimentation in a sort of dialogue with the situation' (p. 9).

It is possible that *all* dialectics between the student and the *milieu* in a given didactical situation are dialectics of action; if, for example, the student knows what to do and how to do it in order to complete the task. This would mean that, although she completes the task, she does not need to extend her mathematical knowledge or understanding to do so; 'simple familiarity, even active familiarity … never suffices to provoke a mathematization' (Brousseau, 1997, p. 211).

There may be an argument that, in some cases of situations of action, the feedback from the *milieu* seems to have little or no role. For example, in a lesson where the student works through a set of examples, it could be that the only feedback they receive is when the teacher reads out the answers. However, as Brousseau argues, the student can be seen to be anticipating the results of her strategies, and in this sense the *milieu* provides feedback, which can perhaps be seen as unrequested and as expected; it does not require the student to adapt her strategies. On the other hand, feedback may occur from time to time as the student works. For example, students may use self-checking methods such as multiplying out factorised functions. In these cases, the dialectical nature of the student interactions is clearer, and the feedback from the *milieu* can be seen as requested and as 'a positive or negative sanction relative to her action' (Brousseau, 1997). A negative sanction might prompt the students to formulate new strategies, but it may also result in a sort of 'guessing' behaviour, where they simply try something different but do not use the feedback to inform the guess.

Dialectics of **formulation** occur when students meet a difficulty or problem as they engage in mathematical activity; Brousseau explains that when a solution to a problem is inappropriate, the situation should feed back to the students in some way, perhaps by providing a new situation. That means that the student may become conscious of her strategies and begin to make suggestions. Brousseau includes in this category 'classifying orders, questions etc.…' (p. 61). He goes on to say that in these communications students do not 'expect to be contradicted or called upon to verify … information' (p 61). In making these formulations the students construct and acquire explicit models and language, which, as Christiansen and Walther (1986) argue, serves to make the learner conscious of strategies: 'actions become conscious for the learner' (p. 268).

In the discussion above ('Action'), the possibility of trial and error cycles of student behaviour was proposed. In these trial and error dialectics, the role of the feedback was seen to be only to inform the student that the strategy she had tried was incorrect. However, depending on the nature of the problem, it is possible that the feedback may also provide some clue for the student about how to improve her strategy and she may formulate a new strategy; this approach can be seen as 'trial and improvement' or 'trial and refinement' approach (Sutherland, 2007).

Both action and formulation involve manipulating 'moves in the game' or mathematical objects; **validation** however involves manipulating 'statements about the moves' (Sierpinska, 2000, p. 6). Validation therefore takes place when an interaction intentionally includes an element of proof, theorem or explanation and is treated thus by the interaction partner (or interlocutor) 'this means that the interlocutor must be able to provide feedback…' (Brousseau, 1997, p. 16). Brousseau argues that this interaction should be seen as a dialectic because of the presence of the interlocutor. Examples of dialectics of validation include justification (perhaps of a procedure, a word, a language or a model), organising theoretical notions, 'axiomization' (ibid., p. 216), and developing proofs.

Brousseau, while suggesting that all three modes of production are 'expected from students', (ibid., p. 62) argues that it is through situations of validation that genuine mathematical activities take place in the classroom. There seems to be general agreement with this within mathematics education (for example, see Lakatos, Worrall, & Zahar, 1976; Romberg & Kaput, 1999).

Brousseau suggests that situations of validation do not occur very often and are unlikely to occur spontaneously and it is probable that validation will not take place unless it is explicitly called for.

A final comment in this section about validation concerns the role of feedback. The implication from Brousseau's 'interlocutor' (above) is that this interlocutor provides feedback. It is in discussion with this interlocutor that the individual develops his or her arguments; it is unlikely that feedback will come from any source in the *milieu* other than classmates or the teacher because of the need to convince someone else.

METHODOLOGICAL FRAMING

The focus of the investigation was on what takes place in authentic classrooms. The implication is that the classroom situations should reflect, as far as possible, the everyday practice of teachers and students; teachers should teach and students should react as they normally do, all the teachers' teaching decisions are their own; the choice of topic, software, approach and task. One possible constraint required by the research was that the students should work in small groups or pairs so that interactions between them (what they said and did) could be observed. This constraint is not a major concern; it is common for students to work in pairs in computer rooms.

The teachers taking part in the study were asked to choose one small group of students as a 'focus group'; the request was that they chose a group whom they perceived to be of average attainment and who might be expected to talk as they worked (to provide verbal data). When the students worked on the task the teacher set, this focus group was observed.

Collecting and analysing data

The focus on processes required the researcher to be able to see and hear what the students did and said as they engaged with the activity, and these were captured on video. Screen activity was recorded using software which 'grabbed' the screen automatically every 30 seconds.

The perspective adopted uses the interactions between students and the setting or *milieu* (in other words, the dialectics) as the unit of analysis, and pays attention to the 'flux of ongoing activity' (Nardi, 1996). However, also as argued above, each interaction is situated in time, and related to all the other interactions taking place as the students work. The implication of this point of view is that, to make sense of interactions, it is necessary to understand the unfolding, or narrative, of all these interactions.

Creating the narrative

The interactions were coded using a scheme derived from the theoretical arguments developed above; to represent the type of student mathematical interactions (action, formulation, validation). These dialectics provide a useful way to investigate the data; mathematical thinking and acting requires all three, and the degree to which each is present will provide an initial understanding of the students' mathematical learning as it relates to the feedback from the *milieu*. Further interactions (technical and other) were also represented. In order to create a narrative, the coded interactions were placed on a timeline.

Figure 1 below shows part of a timeline and explains how the interactions are represented. On the left are the six categories, and on the right is part of a coded timeline. The section of coded timeline shows when interactions occur in relation to others and the length of each block is directly proportional to the length of time the interaction lasts. This part of the timeline begins with a formulation, which is followed by an action, some computer feedback, another formulation, a technical interaction and one 'other' interaction. It does not include any validation interactions.

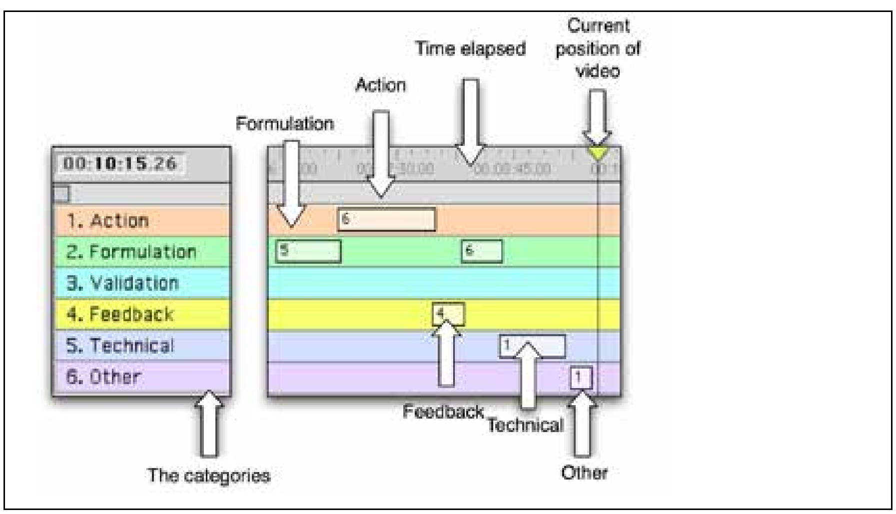

Figure 1: Explains the layout and elements of the timeline

GRAPHS WITH AUTOGRAPH

The data discussed below was taken from one of a series of five lessons for 14 to 15 year olds. From the class of approximately 25 students, the teacher chose a focus group of three girls; Claire, Alice and Charlotte. (These names are pseudonyms).

According to the teacher, the overall aim of the five lessons was for the students to develop an understanding of the relationship between the algebraic and graphical forms of quadratic functions of the form $y=x^2 + bx + c$, where b and c are integers. (The initial interview with the students established that they knew that these graphs would be 'u-shaped' but they had no experience of constructing the graphs by hand.) In some lessons, the software *Autograph*[1] was used to produce graphs of quadratic functions. Some of the lessons took place in the classroom, and some in a computer room. The first of these lessons is analysed below.

The task for this lesson, set out on a worksheet handed out to the students, was to use *Autograph* to create graphs of quadratic functions and then to sketch the graphs onto a paper-based worksheet. The teacher's stated aim (both to the researcher and the students) for the lesson was for the students to '…notice the intercepts' *[with the axes, but this is implicit]*. (It is argued in the thesis (Joubert, 2007) that the task is unambitious and is unlikely to lead to significant student learning. However, the methodological approach adopted in the study was non-interventionist, and the researcher had no influence on the task chosen by the teacher (perhaps, retrospectively, mistakenly).

The worksheet consisted of six similar questions, in which students were given an equation, such as $y = x^2 - 6x + 8$ (which the students had already factorised for homework as $y = (x - 2)(x - 4)$).

The students began by turning on the computer and starting *Autograph*. This technical interaction is the first on the timeline (see Figure 2 below). They opened the *Enter Equation* dialogue box and typed in the first function $y = x^2 - 6x + 8$. This action is the next interaction on the timeline. The students went on to complete the question, and Figure 2 below summarises the activity taking place as the students worked on this question.

The timeline shows the various coded interactions. The zigzag (black line) traces the computer activity, and the feedback interactions represent the computer outputs and the students' reactions to these.

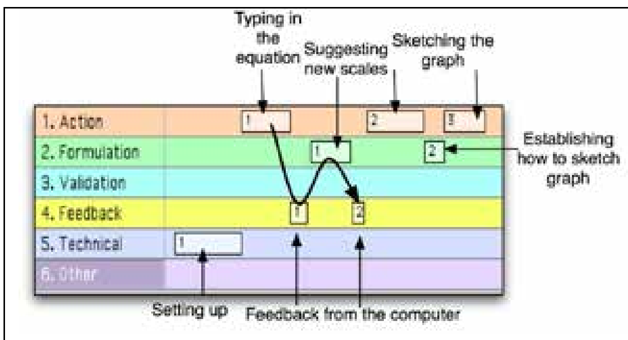

Figure 2: Summary of student interactions Q1

This analysis begins to unpick the student interactions with the *milieu*; there are three action dialectics, two formulation dialectics and two feedback dialectics. As will be seen below, the first formulation dialectic is promoted by the computer feedback. It is perhaps unsurprising that there are no dialectics of validation as these were not demanded by the task. However, the dialectics of formulation, and in particular the relationship of these to the computer feedback, are worth exploring.

The first formulation dialectic occurs in response to the computer feedback, an on-screen graph (see Figure 3 below). Alice remarked:

Can we see it all? Can we change the axes?

Without further discussion, she clicked on the Axes > Edit Axes menu commands. This opened a dialogue box which allowed the students to enter values for *x* and *y* 'max' and 'min' values. Claire suggested the changes needed:

(*Points to the y minimum*) Change this one to minus two and (*points to the y maximum*) that one to, say, eight. (*Points to the x minimum*) And this one to minus two and, yeah, that will do.

The second graph (see Figure 4) shows the computer feedback after this change. One of the students commented:

That's better, OK now we can draw that.

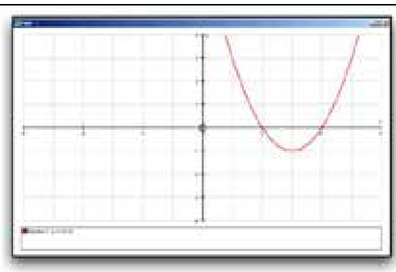

Figure 3: First graph, Q1

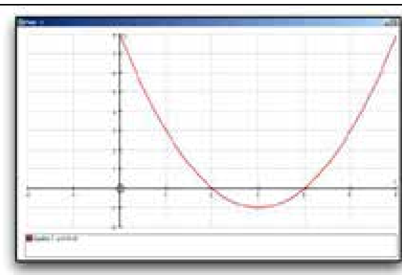

Figure 4: Second graph, Q1

The students then went on to sketch the graph on their worksheets. The second formulation was an unremarkable discussion about how to sketch the graph.

The first point to note about this student activity is that the students did not seem to find any difficulty in using the software. There was no discussion between them about how to create the graph or how to make changes to the scales on the axes. The first computer output was a graph page with the graph of the function drawn on it. However, as Alice's comment indicates, the students wanted to change it. The reason, it seems, was that the *y*-intercept could not be seen. Alice's reaction suggests that the feedback was unexpected, and in response to this unexpected feedback the students changed the graph page.

The students did not discuss the changes, and there is no way of knowing the basis for their decisions. However, it is likely that it was based on the visual appearance of the graph on the page because of Alice's reference to not being able to 'see it all'.

In the remainder of the lesson, the students followed much the same pattern of activity. In general, they tended to type in the equation of the graph, then adjust the graph page by guessing maximum and minimum values (as above) based largely on the appearance of the graph. Only in one case did they sketch the graph on paper before entering the equation into the software (see below).

Two dialectics of validation took place. The first occurred as the students attempted the fourth question. The teacher had encouraged them to sketch the graph first, and then to try it out on the computer but they found a mistake in the worksheet; whereas they had been asked to solve the quadratic equation $x^2 - 8x \boldsymbol{+ 15} = 0$ by factorisation, they had then been asked to sketch the graph corresponding to y = $x^2 - 8x \boldsymbol{- 15}$ and in the dialectic of validation the students explained (neither very clearly nor very confidently) why the y-intercept could not be -15. After some further actions and discussion they called the teacher and pointed out the mistake. The teacher asked how they found the mistake. In response one of the students suggested:

> We worked out that it would cut the axes there and there (*pointing at predicted intercepts on the screen)* because those numbers are the same as those (*pointing at the zeroes on the sheet)*.

Through her questioning, the teacher had drawn out a justification (second dialectic of validation), by encouraging the students to articulate what they had noticed. However, she did not question **why** the roots correspond to the intercepts, and it seems that both she and the students were concentrating on the visual connections between the algebraic and graphical forms of the function, rather than drawing on the theoretical notions underpinning this connection. Further, it seems that the teacher was relying on the computer to draw out connections by concentrating on noticing rather than explaining.

It could perhaps be expected that the students, having made the connection explicit, might use their understanding to adjust the graph page in further questions, or to sketch the graph before entering it on the computer. Their approach began as follows:

Alice: OK let's do the next one then.

Claire: What, draw it before?

Alice: No, I'm not doing that – too much hassle.

As the dialogue indicates, it seems that they preferred a trial and improvement approach rather than a more theoretical approach. Their talk indicates that they were deliberately choosing this approach, even though they knew that it was likely that the graph page would need to be changed later (by entering new maximum and minimum values for the visible parts of the page).

In the second dialectic of validation, one of the students justified her choice of scales to the others. Once again, however, she referred to the factorised equation but did not attempt to explain why the factors were related to the roots.

Overall student learning

The brief analysis above suggests that there was some student learning as evidenced by the presence of dialectics of action, formulation and validation. However, it seems that many of the dialectics of formulation were visually based, and were prompted by the computer feedback. The students used a trial and improvement strategy based on the visual appearance of the graph, but there is evidence that on several occasions they made the connections to the equation (initially suggested by the teacher). However, they chose to return to the former approach in later questions. Although they connected the numbers in the equation to the graph but there was no attempt to explain why the connection existed. There is some evidence of mathematical learning, in terms of the theoretical framework used, but the students never explained the connections they found and their learning can be seen as relatively superficial.

CONCLUSIONS

Space has not allowed a detailed analysis of the entire lesson, or of the place of this lesson within the series of five lessons concerned with the topic, but the example has served to demonstrate the use of the chosen framework. This analysis has, however, suggested that to complete the task the students were not required to engage in dialectics of validation, and it was primarily the teacher's intervention that prompted one of the two dialectics of validation observed. The difficulty, perhaps, lies in a confusion related to the role of the computer, which does the mathematical work of creating the graph, and the question then is, what mathematics will the students do? Further, although the feedback from the computer prompted dialectics of formulation, it appears that it was easier for the students to guess the changes needed and to try these out than to adopt any systematic predictive trial and improvement strategies.

The study overall set out to understand the use of computers in authentic classroom situations; the analysis above has confirmed the importance of understanding the role of the computer. It particular it has emphasised the importance of having a clear idea of the mathematics the computer will do and the mathematics the students will do, and of how feedback can be used so that tasks take advantage of the computer's potential to provoke situations of validation as well as action and formulation.

NOTES

1. http://www.autograph-math.com/

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THE EFFECTS OF INTERACTIVE WHITEBOARDS ON TEACHING TRANSFORMATIONAL GEOMETRY WITH DYNAMIC MATHEMATICS SOFTWARE

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The studies about the determining the effects of using Interactive whiteboard (IWB) have gained importance since the installation of IWB in almost every school by government funded project "FATİH" in Turkey. The purpose of this paper is to share some ideas about an experience of teaching transformational geometry with dynamic mathematics software on interactive whiteboards. The participants of this study were 31 tenth grade students of one of the public high school in Ankara. There were 16 students in experimental group and 15 students in control group. A test was developed to assess students' transformational geometry achievements, which included translation, reflection and rotation tasks. The effects of IWB are also investigated by qualitative and quantitative analysis.

Key words: Interactive whiteboard, dynamic computer software, geogebra, geometry, transformations.

INTRODUCTION

The qualifications asked for from people have changed in line with the needs of our era and thus individual profile aimed to be created has also changed. It is aimed not to raise individuals who do memorize the information word by word but to raise individuals who can reach at the information s/he needs and who can use this information and synthesize it. In order to raise individuals having the required qualifications, it is necessary for them to pass through good education process. Including new technologies in education and training institutions will provide easiness to address learning needs of the individuals. These qualified individuals can discover fundamental concepts and use knowledge which they need. To educate individuals with the desired qualifications, it is important to integrate the new developing technology to education. New technology like interactive whiteboards (IWB) can help individuals to discover and use knowledge.

Recently, the use of new technologies becomes more and more widespread in teaching and learning environments (Tate, 2002). Thus, there is the need of further research aiming to investigate the effects of the use of these technologies in teaching and learning environments. Similarly, NCTM (2000) emphasizes the use of new technologies in mathematics education, and it has been reported that using new technologies in mathematics education can enhance the learning of students. Using technology in mathematics education can help students to focus on mathematical ideas, make sense of them and solve problems whose solution would be impossible without using technological tools. It can also enhance students' learning by giving them chance to discover further level (Van de Walle, Karp & Bay-Williams, 2010).

"One can make easy drawings, make measurements, and drag elements of a drawing while those elements maintain the dependency relations that exist based on the initial construction in the environment by using dynamic computer software (DCS) programs" (Hollebrands, 2007). Dragging which is called "real-time transformation" is one of the most defining features of DCS programs. There are several DCS programs, but it can be said that Geogebra is one of the most popular and widespread. It is popular not only because its development is always in progress but also because it is a freeware software. Thus in this study geogebra was chosen as a DCS program.

The touch-sensitive board allows users to interact directly with applications without having to be physically at the computer which is projecting the image onto the board (Beeland, 2002). In this context, IWBs give teachers the opportunity to interact with teaching materials in every time during the teaching sequence by its increased functionality. This increased functionality introduces substantial benefits to teachers in the sense of flexibility and variety compared to video projector (Haldane, 2010). The use of IWBs is increasing day by day in education because of the benefits and innovations it brings to teaching and learning process. In Turkey, IWBs are going to be delivered to four thousand schools, and were given to several schools along with FATİH project. So it can be said that the IWB will supersede the classical white board with this project.

Several researches were carried out to examine the effect of IWB in teaching and learning process (Beauchamp & Parkinson, 2005; Kennewell, 2001). It was found that IWB enhances students' motivation and interest towards learning. IWB also expands interaction between students, meets the wide range of student needs through the use of multimedia and varied presentation of ideas. By using IWB, students can answer easily such questions like 'Can you explain?' 'Why?' that asked students to clarify points in their mind and to help them to enhance their own learning (Glover & Miller, 2002). But the use of the full potential of the IWBs in teaching and learning process depends on how the instructors use them. In researches it was found that if instructors use the IWBs without considering the interactivity features of IWBs, and use it just for writing and drawing like the classical board, the IWB use will make no difference in teaching and learning process (Glover & Miller, 2002). Teacher and researchers can benefit from this research on using the IWBs that will be installed to almost every school by government funded project FATİH in Turkey.

IWB involves three modalities of learning: visual, auditory and tactile (Beeland, 2002). Stimulating the one or more sense can provide effective learning and also by this way, students make sense of concepts and ideas. On the basis of Becta's (2003) analysis, the main research findings for general benefits of IWB has been summarized

- Versatility, with applications for all ages across the curriculum
- Increases teaching time by allowing teachers to present web-based and other resources more efficiently.

- Provides more opportunities for interaction and discussion in the classroom especially compared to other ICT.
- Increases enjoyment of lessons for both students and teachers through more varied and dynamic use of resources with associated gains in motivation.

IWBs are effective educational tools, which provide dynamic learning environment for students. These dynamic learning environments can help students develop positive attitude towards learning. IWBs have been being used in some countries (e.g. England, USA; Canada; Brazil, Portugal). Thus there are many researches about the effectiveness of IWBs (Beeland, 2002; Glover & Miller, 2002; Beauchamp & Parkinson, 2005). However, there is little research in Turkey. Therefore, we aimed to investigate the effects of using IWB in mathematics classrooms.

The purpose of this study was to determine the effect of the use of the IWB as an instructional tool on student academic achievement. In particular, the following research question was investigated:

Does the use of Geogebra via IWB as an instructional tool affect students' academic achievement on transformational geometry?

METHODOLOGY

The participants of this study were 31 tenth grade students of one of the public high schools in Ankara in the second semester of 2011-2012. There were 16 students in experimental group and 15 students in control group. Data were collected through eleven open ended questions about transformational geometry. Translation, reflection and rotation were chosen from tenth grade geometry curriculum. Posttest-only control group design quasi-experimental research model was used in this study (Fraenkel & Wallen, 2008). While preparing activities, Turkish national geometry curriculum and common core standards were considered. In the Turkish curriculum, the educational gains are "Students can do translation; rotation and their composition on the coordinate plane." and "Students can do reflection and glide reflection on the plane". In the common core, the standard is "Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another." The same activities were used in both groups. During teaching process, dynamic mathematics software GeoGebra is used via IWB. In the teaching process constructive, collaborative and interactive learning strategies were used. It wasn't used any technology in the control group. One of the activities used with IWB in the experimental group and its' classical board version used in control group is presented below (Figure 1). During this activity, worksheets including rotation tasks, grid papers were used and students were asked to fill the worksheet. It was intended to make clear the relation between the translated shape and the vector by changing translation vectors dynamically and interactively on IWB. It was asked to students to find the relation between the coordinates of translated shape and untransformed shape of triangle and to explain the rule mathematically that students asked to find. All the students' inferences are discussed with class. In the control group, unlike IWB, the same activities were done traditionally only by drawing shapes without changing dynamically or interactively anything on the classical board. A test was developed to assess students' transformational geometry achievements, which included translation, reflection and rotation problems by consulting experts. The test involved 11 open ended questions. The developed test was applied to both experimental and control groups as post-test. The study was carried on for 4 weeks. Experimental and control groups were matched with the analysis of geometry scores of the last semester. The developed test was also applied as pre-test to experimental group. Data were gathered through participants' written responses. Quantitative data is analysed with non-parametric statistics Mann-Whitney U, Wilcoxon rank test and qualitative data were cleaned, coded and analysed. Students' responses in the pre-test and post-test were grouped, summarized and analysed using a content-based analysis approach to gather qualitative data. Students' responses were coded in three categories: "Completely wrong", "Completely true" and "partially true". The qualitative result was consistent with the quantitative results. Percentages of each response the students gave were computed. Frequencies and percentages were displayed in tables.

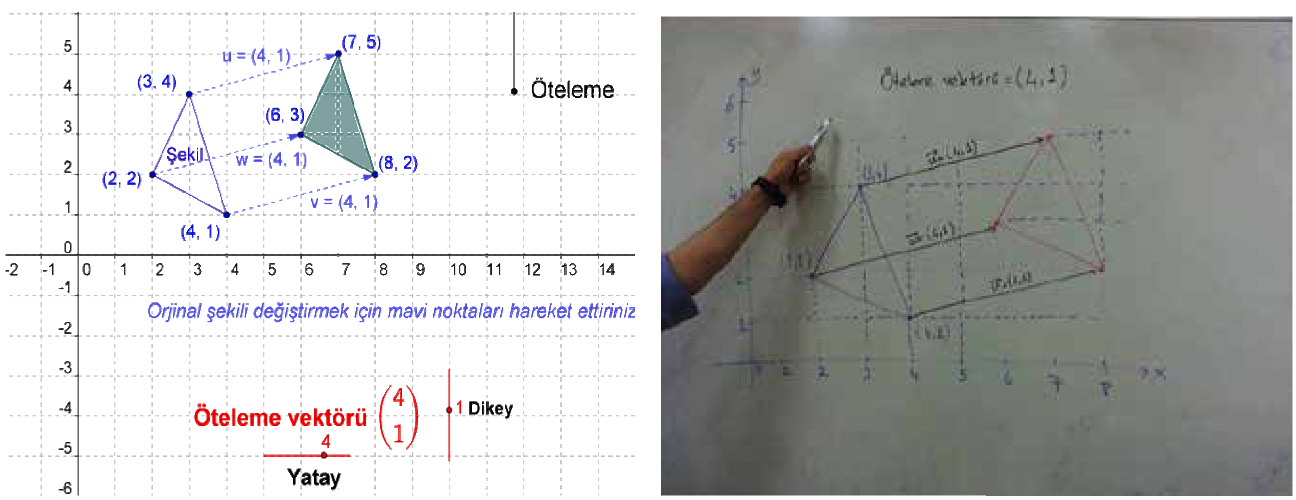

Figure 1: The Same Activity on IWB and Classical Board

RESULTS

In this section we show the results of the comparison between the achievements in the post-test of the experimental group and of the control group, and of the comparison between the achievements in the pre-test and in the post-test of the experimental group. Also some qualitative results of students' responses are shown. Three of eleven questions' analysis is shown because of limitation of the pages. The results were analyzed according to learning objectives about transformational geometry.

Groups	N	Mean rank	Sum of Ranks	U	p
Control group	15	14.47	217.00	97.000	.363
Experimental group	16	17.44	279.00		

*p<.05

Table 1 The results of comparing geometry academic scores of last semester with Mann-Whitney U Test

It is seen in Table 1 that average rank of geometry academic scores in last semester of the students in the experimental group is 17.44, while the average rank of score for the control group is 14.47. According to Mann Whitney U Test which was conducted to experimental and control groups students' geometry academic scores of last semester, it is observed that there is not a statistically significant difference between the geometry academic scores in last semester of the students (U=97.000; p=.363>.05). According to this result, it can be said that these groups are appropriate to be chosen as control and experimental groups before the treatment.

Groups	N	Mean rank	Sum of Ranks	U	p
Control group	16	9.50	152.00	16.000	.000
Experimental group	16	23.50	376.00		

*p<.05

Table 2 The results of comparing post-test academic achievement test scores with Mann-Whitney U Test

It is seen in Table 2 that average rank of post-test scores of the students in the experimental group is 23.50, while the average rank of score for the control group is 9.50. According to Mann Whitney U Test which was conducted to experimental and control groups students' post test scores, it is observed that there is a statistically significant difference between the academic achievement test scores of the students in favour of the experimental group (U=16.000; p=.000<.05). According to this result, it can be said that the students, who have been taught with the IWB, understand better than students who have not been taught with the IWB.

Post-test-pre-test	N	Mean Rank	Sum of Ranks	z	p
Negative Ranks	0	.00	.00	-3.519*	.000
Positive Ranks	16	8.50	136.00		
Ties	0				

*Based on negative ranks

Table 3 The result of comparing pre-test and post-test with Wilcoxon test

A Wilcoxon test was conducted to evaluate whether students showed greater success. The results indicated a significant difference, z=-3.519. p<.01. The mean of the ranks in favour of pre-test was .00, while the mean of the ranks in favour of the post test was 8.50. First learning objective is doing translations, rotations and their composition on two-dimensional figures using coordinates. Second learning objective is doing reflection, glide reflection on plane.

Question 1	Pre-Test	Pre-test (%)	Post-Test	Post-Test (%)
Correct Ans.	2	12.5	12	75
Wrong Ans.	13	81.25	3	18.75
Partially Correct Ans.	1	6.25	1	6.25

Table 4: The Change of the Students' Performances Related to Question 1b

In question 1, students were supposed to rotate ΔABC 90° around origin. While 12.5% of students give correct answers to Question 1, 81.25% of them gave wrong answers and 6.25% of student made some mistakes but the answers were not completely wrong. After the treatment the rate of students that gave correct answers increased to 75%, and the rate of wrong answers decreased to 18.75% in the post-test. On the other hand partially correct answers remained same. The most common mistake done by the students in question 1 is rotating around wrong point. Most of the students rotate the shape around the corner A of the triangle. After the treatment with IWB some students corrected their answers as shown below.

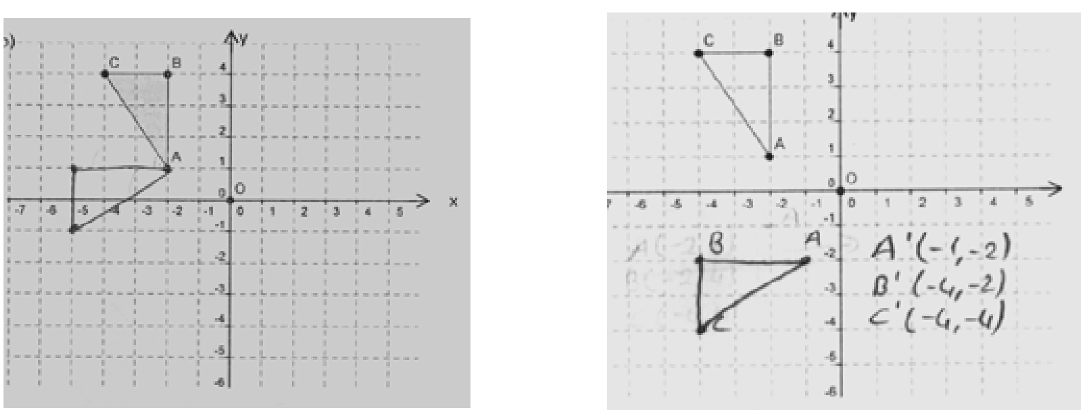

Figure 2: Example of pretest and posttest response of a student for question 1

Question 3	Pre-Test	Pre-test (%)	Post-Test	Post-Test (%)
Correct Ans.	0	0	4	25
Wrong Ans.	16	100	2	12.5
Partially Correct Ans.	0	0	10	62.5

Table 6: The Change of the Students' Performances Related to Question 3

In question 3, the shape below on the left was obtained by rotating ΔABC with particular and constant angle. Students were asked to find the rotating angle in option a, the coordinates of point F in option b and the angle $\widehat{EOB}$ in option c. In pre-test all the students answered fourth question wrong and any of the students didn't even try to draw a line to obtain answer. After implementing the teaching process with the IWB, the correct answer rate increased to 25% and the wrong answer rate decreased to 12.5% and the rate of the partially correct answers were 62.5%. Most of the students answered the question similarly with using formula. İn partially correct answers, most of the students could find the rotating angle in option a and the angle $\widehat{EOB}$ in option c. But only four of the participants could use the formula correctly in option b. A precise correct answer is displayed below on the right and the shape of the question 3 is displayed on the left.

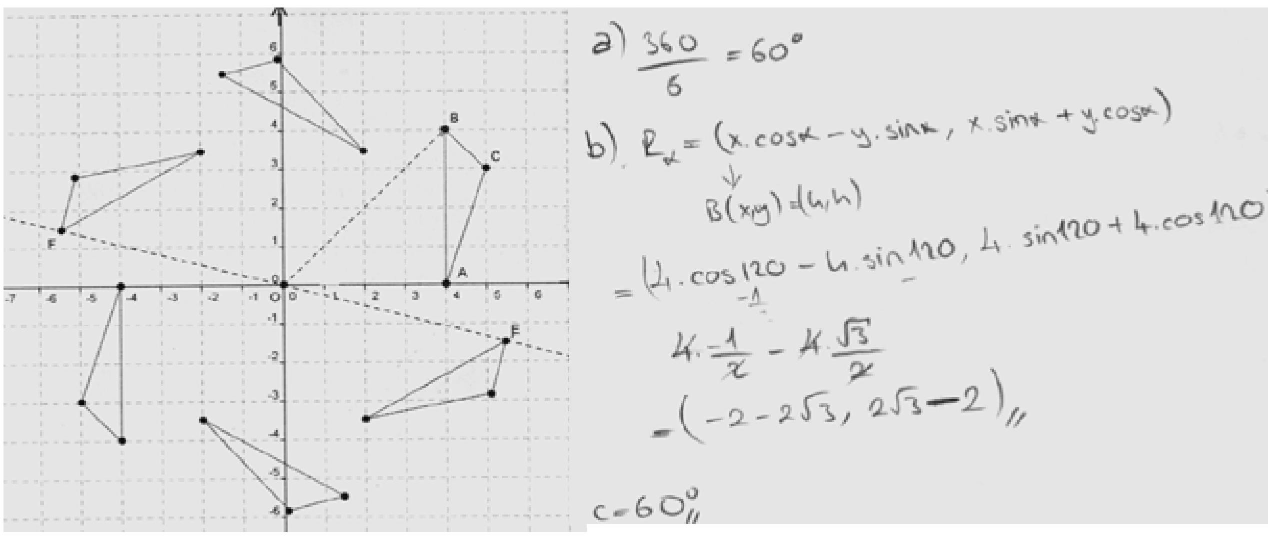

Figure 3: An example of posttest response of a student for question 3

Question 4	Pre-Test	Pre-test (%)	Post-Test	Post-Test (%)
Correct Ans.	3	18.75	10	62.5
Wrong Ans.	10	62.5	1	6.25
Partially Correct Ans.	3	18.75	5	31.25

Table 7: The Change of the Students' Performances Related to Question 4

In question 4, students were asked to obtain ΔA"B"C" by just reflecting ΔABC. Question 4 in pre-test was answered correctly by 18.75% of students and it was answered wrong by 62.5% of students. After the treatment, the rate of correct answers of Question 4 increased to 62.5% and the rate of wrong answers decreased to 6.25% in post-test. İn pre-test most of the students couldn't reflect ΔABC to obtain ΔA"B"C". After the teaching process most of them do the reflection correctly. There is a correct answer below.

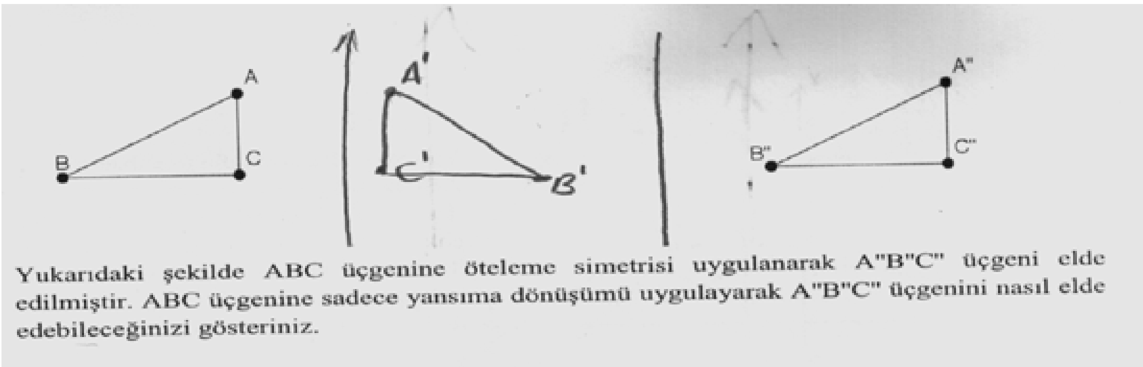

Figure 4: Example of pretest and posttest response of a student for question 4

CONCLUSIONS

The purpose of this study was to provide a view of the impact that using Geogebra with IWB has on academic achievement in transformational geometry. Understanding the concept of transformation is an important topic for students. Because patterns can be described with opportunities that geometric transformations provide. It also helps students make generalizations, and develop spatial competencies (Yanık, 2011). The improvements in the post test can be caused by both IWB and Geogebra. Effective using of software by teachers makes the lesson more efficient. The IWBs cannot facilitate or enhance anything about students' learning just by standing in front of the class and by being used as a classical whiteboard (Glover & Miler 2002). Necessarily it is to be used with software and method. Thus the effectiveness of IWBs depends on how the instructor uses it and with which software the instructor uses. Therefore the interaction between the IWB, software and method cannot be completely isolated and cannot be separated from each other. Two groups can be chosen, using projector in one and using IWB in other, to distinguish the interaction between them but it can be only partially.

It can be said that effective using of interactive whiteboards (e.g. with geogebra or other DGS program) can be used to increase student academic achievement during the learning process. Because IWB supports three modalities of learning: visual, auditory and tactile and it allows interacting with learning contents for students (Beeland, 2002). Thus, stimulating more senses the use of IWBs can provide effective learning and, by this way, students can retain longer what they learnt. Teaching and modeling transformational geometry via classical board is a difficult process (Harkness, 2005; Harper, 2003). Therefore students can understand transformational geometry better with the visual and unique features of IWB like drag and drop, manipulating images easily on the board. However software design by itself cannot provide opportunities automatically for effective learning (Wood & Ashfield, 2008). By controlling the software on IWB can offer to read display easily and offer an opportunity for a teacher to control the material by his hands. Thus, the pace of the lesson can be enhanced by the teachers' personality and dynamism by using IWB.

There is a statistically significant difference between the pre-test and post-test results of the experimental group in favour of the post-test. The post-test results were significantly higher in experimental group in comparison to control group. This result is consistent with the researches of Flacknoe and Thomson (2000), Swan, Kratcoski, Schenker and Hooft (2010), Zittle (2004), Dhindsha and Emran (2006), and BECTA (2007). Also there was an increase in corrects answers between the pre-test and post-test in favour of the post-test. Moreover, in the post test students tried to respond the task with more visually than the pre-test. This study cannot exactly figure out how IWBs affect students' mathematics achievement so sustained studies should be done about the pedagogy of IWBs. For this purpose, further researches can be carried out on designing appropriate materials and developing adequate software to enhance the efficiency of IWB.

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THE EFFECTS OF DYNAMIC GEOMETRY SOFTWARE ON LEARNING GEOMETRY

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The aim of this paper is to present the findings of a pilot study which was designed to collect data about the effects of using dynamic geometry software in the tenth grade geometry lessons on the students' geometric thinking, geometry achievement and ability of doing proofs in geometry. It was a quasi- experimental study in which treatment groups were given dynamic geometry activities during the geometry lessons. The findings revealed that the students' scores from geometric thinking test and proof test increased significantly with respect to the students' who did not experience those activities.

Keywords: dynamic geometry, geometric thinking, problem solving, proof skills

INTRODUCTION

Geometry is "a complex interconnected network of concepts, ways of reasoning, and representation systems that is used to conceptualize and analyze physical and imaged spatial environments" (Battista, 2007, p.843). Thus, learning geometry entails visualization and construction of images of geometric concepts and making appropriate relationships between the concepts. Van Hiele also identified visualization as an indicator of one's geometric thinking level. He noted that visualization is the first level of geometric thinking such that everybody should possess accurate concept images to attain higher levels of geometric thinking (Battista, 2007). Research findings show that students fail in geometry because they have difficulty in visualization of geometric concepts such that they are unable to analyze geometry problems and draw appropriate figures for the problem (Clements & Battista, 1992; Healy & Hoyles, 1999; Yerushalmy & Chazan, 1993). The conceptualization of visual objects, in other words having a valid matching of concept image and concept definition in mind (Vinner & Dreyfus, 1989), is vital to understand geometry (Battista, 2007). Therefore, teacher-oriented teaching strategies are not effective enough to help students understand geometry and improve their geometric thinking and proof skills (Reiss, Heinze, Renkl, & Gross, 2008).

Improvement in information technologies arise questions about how to use those technologies for educational purposes and how they affect students' understanding and learning. It is noted that using technology for educational purposes increase students' motivation for learning, helps for understanding and gives opportunity for repetition of the subject matter (Healy & Hoyles, 1999; Knuth & Hartmann, 2005). The studies indicated that although students can solve algebraic problems by following some algorithms they have difficulties solving problems including images and shapes (Healy & Hoyles, 1999). Because dynamic software enables students to observe the properties and the relationships by drawing figures and manipulating them easily, it has potential to decrease such problems (ibid.). Indeed, the studies investigating the effective ways of teaching geometry suggest that dynamic geometry software (DGS) helps students visualize geometric concepts and understand geometric rules, generalizations and relationships between the concepts (Healy & Hoyles, 1999; Jones, 2000; Marrades & Gutierrez, 2000). Hence, using DGS in geometry lessons may be effective in terms of increasing students' geometry achievement and developing their geometric thinking and problem solving skills.

DGS not only contributes to the development of geometric thinking and problem solving skills but also facilitates understanding and proving hypotheses and conjectures (Jones, 2000). Doing proofs entails making a good plan for proof and following it by justifying each step (Heinze, Cheng, Ufer, Lin, & Reiss, 2008) and similar pattern is required when doing constructions with DGS (Scher, 2005). For instance, to construct a square with DGS students cannot just draw equal segments and connect them together but they need to use the idea of perpendicular lines, right angles and transformations. As students engage in fundamental constructions with DGS, the better they appreciate the fact that they have to follow an order to make constructions correctly. However, the studies about the effects of DGS on students' proof skills are limited, more research are needed to support that relationship (Hollebrands, Laborde, & Straesser, 2008).

Although the effectiveness of using DGS on learning geometry was investigated by many scholars, most of them were limited in terms of sample size, context, investigated variables, way of integrating DGS into geometry lessons or data collection tools. Still, there is a need for large-scale experimental studies supported by both quantitative and qualitative data (Battista, 2007). Therefore, the aim of this study was to investigate the effects of using DGS on students' geometric thinking, geometry achievement and proof skills. However, the results presented in this paper emerged from the pilot study of that large-scale investigation.

METHODOLOGY

In this pilot study, quasi-experimental research design was used to investigate the effects of using DGS on the tenth grade students' geometric thinking, problem solving and proving skills. It was conducted in the second term of 2011-2012 academic year in Turkish high schools.

Sample

A total of 227 students from 6 schools which are administered by the same foundation participated in this study. In these schools, the teachers follow the same yearly plan and they use the same or quite similar teaching materials. The schools were assigned into groups in terms of their preference. Three of the schools preferred to be in the experiment group and two of the schools preferred to be in the control group. In one school, one of the tenth grade classes was assigned to the experiment group and the other class was assigned to the control group. Then 145 students were in experiment group and 82 students in the control group. However, because not all students were available during the testing days some of the data was lost. Furthermore, in two experiment schools all DGS activities were completed but in the others they were not. Therefore, the data analyzed for this paper is based on 12 students from the experiment group who experienced all DGS activities and took all the pretests and posttests and 37 students who took all the tests from the control group.

Data collection

In both experiment and control schools geometry is taught three hours per week. Five DGS activities were determined in line with geometry curriculum and they were applied during the geometry lessons in the experiment group. Because the activities were aligned with the objectives of the geometry lesson, the teachers did not fall behind their schedule. However, they assigned more homework problems to their students for exercise because in the control group, the teachers had time to give more paper-pencil exercises to the students during the lesson.

Before the study, the teachers in the experiment group were given a workshop about how to use DGS and how to apply the activities. They were provided a worksheet (manual) for each activity and they used these manuals for their students as well. In the experiment group, the teachers used the computer laboratory only for the activities and used their regular classrooms otherwise. The activities were about triangle inequality, angle bisectors, medians and perpendiculars, transformations and Euclid, Menelaus, Ceva and Carnot theorems. Geometers' Sketchpad Program (GSP) was used as dynamic software. The students were given three types of tests, namely Geometric Thinking Test (GTT), Geometry Achievement Test (GAT) and Geometry Proof Test (GPT) prior to and at the end of the study. The tests consisted of questions related to triangles and transformations which were covered during the second term in the curriculum. The items in all tests were in supply type form such that students were expected to solve the questions and explain and/or justify their answers. GTT test consisted of seven items such that the items were written in the line of first four levels of van Hiele's geometric thinking levels (recognition, analysis, order, deduction and rigor) and Driscoll's (2007) geometric habits of mind (reasoning with relationships, generalizing geometric ideas, investigating invariants and balancing exploration and reflection). By GTT, it was aimed to measure students' knowledge of geometric concepts (how they describe given concepts), ability to interpret the relationships in geometric constructions and ability to transfer their knowledge into application and make inferences. GAT consisted of ten items such that they were chosen from the 10th grade geometry textbooks and it was aimed to measure students' geometric knowledge. GPT consisted of six items such that they were the proofs recommended to be covered in the curriculum. It was aimed to measure students' proof skills (i.e., choose an appropriate type of proof, make a plan to use it, write conjectures and justify them). Although during the pilot study qualitative data was not collected, for the main study qualitative data in terms of interviews and videotapes will be collected.

Data analysis

The test results were analyzed by using statistical analysis software. For the content validity of the tests, a specification table was prepared. In addition, for the GTT items van Hiele's and Driscoll's (2007) ideas for geometric thinking were taken into account. The tests content validity was agreed by one academician and three experienced geometry teachers. The tests concurrent validity was also checked. For the reliability test-retest reliability analysis was held. Furthermore, rubrics for each test were prepared for scoring and the interrater reliability was checked.

RESULTS

The content validity of the tests was agreed by one academician and three experienced geometry teachers. For the concurrent validity the correlation between each test was found. Then, for the pretest the correlation between GTT and GAT was .63 (p=.000), GTT and GPT was .38 (p=.001) and GAT and GPT was .51 (p=.000). For the posttest the correlations were as follows: .50 (p=.000), .38 (p=.000), and .47 (p=.000), respectively. Test-retest reliability was sought and Pearson's r for GTT, GAT and GPT tests calculated as .56 (p=.000), .67 (p=.000) and .68 (p=.000), respectively. The tests were out of 100 points. Two academicians rated the tests. The interrater reliability for GTT was .96, for GAT was .98 and for GPT was 1.00.

The descriptive statistics about the tests are given in Table 1.

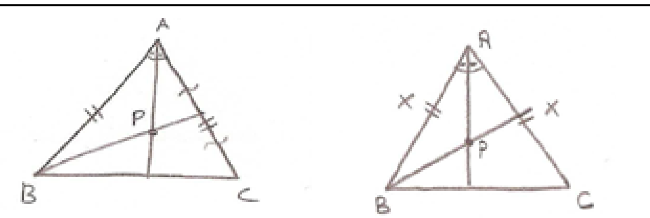
	Experiment Group					Control Group				
	N	Min.	Max.	Mean	SD	N	Min.	Max.	Mean	SD
GTTPre	12	7	55	30.08	14.34	37	7	67	39.73	14.50
GTTPost	12	22	63	43.42	14.59	37	7	81	44.70	19.89
GATPre	12	0	33	9.25	10.50	37	0	47	16.68	13.14
GATPost	12	0	48	21.17	11.51	37	0	82	31.27	23.51
GPTPre	12	0	13	3.17	4.99	37	0	60	15.73	14.97
GPTPost	12	0	20	7.92	5.42	37	0	45	12.11	12.20

Table 1: The descriptive statistics about GTT, GAT and GPT

The pretest results for each group were compared. There were no statistical difference between GTT ($t_{47,\ .05}$=2.007, p=.050) and GAT ($t_{47,\ .05}$=1.778, p=.082) but there was significant difference between GPT ($t_{47,\ .05}$=4.405, p=.000). The posttest comparisons showed that there were no significant difference between the groups in terms of GTT, GAT and GPT results ($t_{47,\ .05}$=.206, p=.838, $t_{47,\ .05}$=1.982, p=.055, and

$t_{47,\ .05}$=1.639, p=.109 respectively). The pretest and posttest results for each group were compared. For experiment group significant differences were obtained in terms of all types of tests but for control group there was significant difference in terms of geometry achievement. For the experiment group the following t-scores were obtained for each test: For GTT, $t_{11,\ .05}$=2.945, p=.013, for GAT, $t_{11,\ .05}$=3.856, p=.003, and for GPT, $t_{11,\ .05}$=2.844, p=.016. Because the number of participants taken into account was small, Wilcoxon Signed Ranked Test was also applied for the experiment group. The findings were aligned with t-test results such that for GTT, z=-2.432, p=.015, for GAT, z=-2.701, p=.007, and for GPT, z=-2.174, p=.030. For the control group the following t-scores were obtained for each test: For GTT, $t_{36,\ .05}$=2.025, p=.050, for GAT, $t_{36,\ .05}$=5.562, p=.000, and for GPT, $t_{36,\ .05}$=1.980, p=.055. The following results were obtained from Wilcoxon Signed Ranked Test: GTT, z=-1.753, p=.080, for GAT, z=-4.275, p=.000, and for GPT, z=-2.021, p=.043. Except GPT, t-test and Wilcoxon test results were compatible in the control group.

The items for each test were analyzed. The maximum points for each item in GTT were 30, 9, 9, 8, 12, 12 and 20, respectively. In the first item of GTT the students were asked to write the definitions of fifteen geometric concepts including line, angle, median, incenter of a triangle, transition and reflection axis. In the second item, the students were asked to explain the relationships between the sides of a triangle (triangle inequality). In the third and the fourth items they were asked whether given information is enough to construct a triangle and find the measures of its sides and angles. The fifth question was about transformations and in the sixth question they were asked to distinguish congruent triangles among the given set of triangles. Finally, in the seventh item the students were given four different descriptions of geometric figures and they were asked to draw them by using appropriate labels and symbols. For instance, students were asked to draw a triangle ABC such that $|AB| = |AC|$ and the intersection of angle bisector of the angle A and the median of the line segment AC is P. In the figure 1, two examples from the students work are given. For the group taken into consideration, nobody deserved full credit for the first, the second, and the seventh items in GTT. Furthermore, although the credits for each question differed in GTT, the results showed that the weighted mean for the second item was the lowest (2.80 out of 6 points) while the weighted mean for the last item was the highest (11.41 out of 20 points).



Student work 1 Student work 2

Figure 1: Typical student answers to the construction item of GTT.

In GAT, all items were worth 10 points. The test items were about triangle inequality, angle bisector, median, Pythagorean, Euclid and Menalaus theorems, congruency, similarity, area, and transformations. In GAT, there were students who received full credit from the items. However the mean of the tenth item (transformations) was the lowest (1.73 out of 10 points) and the mean of the sixth item (similarity) was the highest (5.61 out of 10 points). In the sixth item the students were given the following problem: *As shown in the picture, a bridge will be built up over a river. The segment* $\overline{BD}$ *represents the bridge and* $\overline{AE} \perp \overline{AB}$ *and* $\overline{CE} \perp \overline{CD}$. *Use the given information in the figure and find the length of the bridge*. In the figure 2, a typical student answer is given.

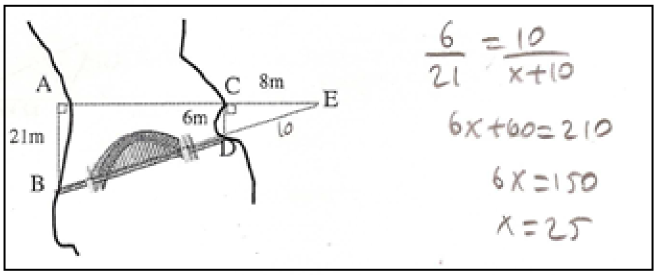


Figure 2: Typical student answer to the sixth item of GAT

In GPT, the third and the sixth questions were out of 20 points while others were out of 15 points. In the first item of GPT, the students were asked to prove that an exterior angle of a triangle equals to the sum of nonadjacent interior angles. The other items were about proving angle-side relationship in a triangle, area of a triangle, Sinus theorem, Carnot's theorem and Euclidean theorems, respectively. In GPT test the mean of the items were generally low but the first item had the highest mean (6.53 out of 15 points) and there were students who got full credit from the first and the sixth items.

DISCUSSION

The aim of this pilot study was to determine the reliability and validity of data collection tools and effectiveness of GSP activities. The data analyzed in this paper was based on 49 students who took all pretests and posttests. The mean scores of the tests and test items provide some information about the students' geometric thinking, geometry achievement and proof skills. It was apparent that students' geometric thinking levels, geometry achievement and proof skills were low.

Although comparing the means of the test items may provide information about the students' weaknesses and strengths in geometry, such inferences should be supported by some qualitative data in terms of structured interviews or other tests. In this pilot study, qualitative data was not collected but a detailed analysis of students' responses for each item is likely to provide some concrete data about students' geometric knowledge, geometric thinking and proof skills. Although such detailed analysis is not the scope of this paper, an instance from GTT is noteworthy to be discussed. In GTT, the seventh item in which the students were asked to draw geometric figures by using information given about them had the highest weighted mean. In the first item, the students were asked to write the definitions of some geometric concepts. Although some of the students failed to provide complete and valid definitions for the geometric concepts in the first item, they were able to use those concepts to make some of the constructions in the seventh item. For instance, when they were asked to define angle bisector, some of them wrote "it divides an angle in two" or "it is a segment that divides an angle in two equal parts". In the seventh item, there were descriptions in which angle bisector could be considered as a line segment and those students used appropriate symbols to indicate that the line segment was the angle bisector. Hence they received full credit from that construction if other parts of it were valid. However, they failed to draw another figure in the same item because they failed to extend the angle bisector, that is, they did not know that angle bisector is a ray not a line segment. Therefore, having higher mean score with respect to other items may not be an indicator of students' ability to analyze geometric figures and definitions. Furthermore, this instance supported the importance of the consistency between concept image and concept definition (Vinner & Dreyfus, 1989). In this case, the students probably wrote the definitions of the concepts in terms of their images in their mind. However, DGS provides opportunities to eliminate such students' misconceptions because the images and tools in DGS are compatible with their mathematical definitions such that an angle bisector is drawn as a ray or a line. Then, experiencing constructions with DGS may enable students to define geometric concepts correctly.

Furthermore, the mean scores of some of the items in GTT and GPT indicated that students' proof skills were low. In the most cases, the students who attempted to prove the given statements gave examples to show that it worked rather than proving them deductively. One of the reasons for poor skills in proving is probably the fact that teachers did not spend enough time for proof activities. Therefore, students did not know much about how to prove theorems and they skipped most of the items in GPT. Since deduction is one of the levels in geometric thinking, for the main study, teachers will be encouraged to cover some of the proofs in the curriculum. GAT scores revealed that students do not know geometric concepts and relationships and they have difficulties to understand transformations. However, since DGS enables students to manipulate geometric figures and do transformations, more practice with DGS may help students understand and visualize transformations. Thus, higher scores from GAT may be obtained during the main study.

Because only posttest comparison was not enough, the pretests and posttests within the group were compared. The results revealed that there was significant increase in terms of students' geometric thinking, geometry achievement and proof skills. These effects of DGS are also supported in the literature (e.g., Jones, 2000; Marrades &

Gutierrez, 2000). This result was noteworthy because although improvement in skills requires more practice and longer time, even five GSP activities created such a difference for the experiment group. Therefore, ten GSP activities which were planned for the main study will be likely to create such difference between the groups. However, because not all students took all types of tests during the administration of the posttests, the majority of data was lost. Furthermore, some teachers in the experiment group could not do all GSP applications, thus the data from those schools was excluded for this paper. Therefore, the number of students who experienced all GSP activities and who took all the tests was low. This was the limitation of the study. For the main study, there will be ten GSP activities and they will be done throughout the academic year. The exam weeks will be determined for each group and students will be encouraged to participate in the tests and do their best. Therefore, the completion of all activities and minor loss in data is aimed to achieve.

The values obtained for the reliability and validity of the tests were acceptable for supply-type teacher made tests (Miller, Linn, & Gronlund, 2009). It indicates that these tests can be used for the main study for the next academic year. However, the test may be revised for the main study for several reasons. First, the mean scores for GPT were quite low with respect to other tests. This was compatible with findings of many studies about students' proof skills (e.g., Clements & Battista, 1992; Stylianides, 2008). Although there was statistically significant increase in the GPT scores in the experiment group, the GPT may not be considered as a single test for the main study such that some of the items may be replaced with the items in GTT and proofs may be asked during the interviews. Second, some items in each test were aiming to measure the same or similar learning outcomes. Those items may be excluded or replaced with other items. For instance, the second, the third and the fourth items in GTT, which required from students to explain an identity or a rule and justify their reasoning, were aimed to measure the fourth van Hiele's geometric thinking level. Therefore, integrating some of the items from GPT into GTT may not cause any loss in terms of content validity of the tests. Similarly, the items with the lowest mean scores will be changed in GAT. Briefly, the tests will be revised for the main study although the values for reliability and validity were at acceptable levels.

In this study, the homogeneity of the groups was not satisfied because the assignment of the schools into the experiment and control groups were based on their choice due to some administrative reasons, and the pretests were administered after forming the groups. In fact, students' mathematics achievement in two of the control schools was higher than the achievement of the students' in two of the experiment schools. Therefore, comparing each group's posttests did not provide valid information about the differences between groups. For the main study, some demographic information about the students and the schools will be collected prior to the study and the pretests will be administered before assigning groups as experimental or control. If possible, the classes will be assigned into groups rather than the schools to achieve homogeneity between the groups. Furthermore, since national curriculum is applied in Turkey and free textbooks are distributed for all subjects and grade levels, the teachers should cover the same content by using more or less the same teaching materials and activities. Therefore, the teacher as a threatening factor for the external validity of the study was at minimum, but it should be controlled. For the main study, the lesson plans and all teaching materials will be requested from both experiment and control teachers prior to and during the study and they will be asked to change anything that is seen as a threat for the external validity. Moreover, in the experiment group, as it was done for the pilot study, each DGS application manual will be provided to the teachers and a two-day workshop about the activities will be given before the study. Thus, the homogeneity in terms of application of DGS activities will be satisfied.

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ON-LINE DISCUSSIONS ABOUT EMERGING MATHEMATICAL IDEAS

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Collaborative knowledge-building within an on-line community of learners has for several years been at the core of several studies both for the CSCL and the technology-enhanced exploratory learning approaches. In the Metafora Project, Exploratory Learning Environments (ELEs) are combined with CSCL tools, all in one web Platform. In this paper, we study how the students' meaning making processes were shaped by their on-line discussions in the Platform as they interact with a half-baked microworld. As the Project moves further from the social aspects of learning and touches socio-metacognitive ones, the focus shifts to how on-line learners learn how to learn with and from each other. Thus, we attempt to also study if and how students' meaning making processes are influenced by learning to learn together aspects that come forward as they share and discuss their ideas.

EXPLORATORY LEARNING AND CSCL ENVIRONMENTS

The idea of discussing and sharing artefacts within a community of users through emails, fora or repositories (and nowadays through Web2.0 tools), has long been an issue for technology-enhanced exploratory learning (Resnick, 1996). One of the explicit attempts to develop specific technologies to support social aspects of exploratory learning was made some years ago through a "WebLabs" Project (Mor et al., 2006). At this Project, Noss and Hoyles studied groups of students as they collaborated through a web-based system that allowed them not only to share their experiences in a textual form, but also to co-construct and share models embedding mathematical and programming ideas. Apart from the exploratory activities in which the students engaged in their attempt to make sense of "how does this model work", emphasis was also given to the social interactions among the members of the community and the ways these interactions shaped mutually constructed artefacts. In this case (as in others before), math meaning making was viewed as a process also taking place when students shared and discussed their ideas, argued about the validity of their models, reflected and redesigned their constructions, while working together in groups. Collaborative "knowledge-building in communities" (Bereiter, 2002) has also been at the core of studies regarding the use of computer supported collaborative learning (CSCL) environments. Stahl (2009) at the Virtual Math Teams (VMT) Project studied small groups of students meeting in chat rooms to discuss on-line about their ideas as they worked with complex math problems. The design of the system also included a shared whiteboard, a wiki for the common artefacts and a portal for social networking. The Project focused on the students' "knowledge-building" processes as a result of their in-group interactions (which they termed as "group cognition"), while working with this system. The social practices emerging as the students worked together were considered to be crucial not only for making sense of what they were jointly doing, but also for making sense of how to work together as a group. Thus, the effective collaboration was defined as the one in which the students not only "produced knowledge artefacts" to give to the broader community, but also the one in which each member made sure that everyone in the group understood and progressed as they should.

In this paper we discuss our attempts (in a 'Meta-fora' project) to study students' math meaning-making as they worked in groups with one of the system's Exploratory Learning Environments (microworlds) which were integrated with a CSCL tool in an on-line Platform (Mavrikis et al., 2012). The microworld in question, called the "Twisted Rectangle" (TwR), was designed as half-baked for the students to make changes to (Kynigos, 2007). The CSCL tool was a well known discussion-argumentation tool called 'LASAD' (Dragon et al., in press). The TwR is a 3d Turtle Geometry programmable microworld that is based on a buggy procedure given to the students to figure out how to fix. The task is to make the procedure construct a parametric rectangle which swivels in space as one of its sides swivels on a plane vertical to the opposite side. The procedure given to the students is pedagogically engineered to be buggy (half baked) in the sense that it is incomplete, challenging students to deconstruct it and make sense of the math behind the reasons for its buggy behaviour. These microworlds have been perceived as 'boundary objects' (Kynigos, 2007), i.e. questionable and improvable objects engaging members of communities in meaning making emerging from the joint de-bugging effort. Thus, they may operate as a tool around which the members of the community structure their activities. In this case, meaning generation processes are considered to emerge and be shaped both by the students' mathematical activity as they interact with the half-baked microworld and their social activity as they discuss on how to make it work, change and customize it. LASAD, in our study, is the tool that allows on-line communication and collaboration among the members of the community.

The Project, however, brings a new strand to integrating ELEs with CSCL tools as it views computer-supported learning in groups as a complex task that requires from students -as they collaborate- to also become aware of elements considered to be important for successful learning in collectives and to learn how to put those elements in use (Wegerif & Yang, 2011). Thus, the group members need to be able to show distributed leadership, plan and coordinate the tasks to be carried out, motivate one another, ensure everybody engages, reflect on the quality of the work through peer reviewing and reflect on the overall progress of the groupwork (Wegerif et al., 2012). All those elements constitute the key components of the "learning to learn together – L2L2" pedagogical approach adopted by the Project.

In this paper, we put emphasis on how the students' math activity was shaped as they interacted with the TwR: a) by their need to explicitly articulate their own ideas so as to share them through a discussion tool and b) by the ideas brought at the table by the other group members. Moving between on-line group discussions and microworld actions, we sought to identify manifestations of L2L2 skills such as organizing and coordinating the work so as to proceed as a group, discussing and evaluate findings from others, reflecting on own findings.

THE DIGITAL TOOLS

The "3d Math" Authoring Tool (http://etl.ppp.uoa.gr/malt) is a programmable constructionist environment (Harel & Papert, 1991) that allows the creation, exploration and dynamic manipulation of 3d geometrical objects through the use of Variation Tools (Kynigos & Psycharis, 2003). Building and manipulating geometrical objects in 3d Math is not restricted to solely looking at the 3d world form static 2d views. A Camera Controller gives students the affordance to navigate around, inside and through their constructions, offering the potential for new ways of visualizing 3d space and conceptualizing mathematical notions, especially ones related to stereometry (Moustaki & Kynigos, 2011). We view 3d Math as an authoring tool for developing half-baked microworlds, such as the TwR (Figure 1).

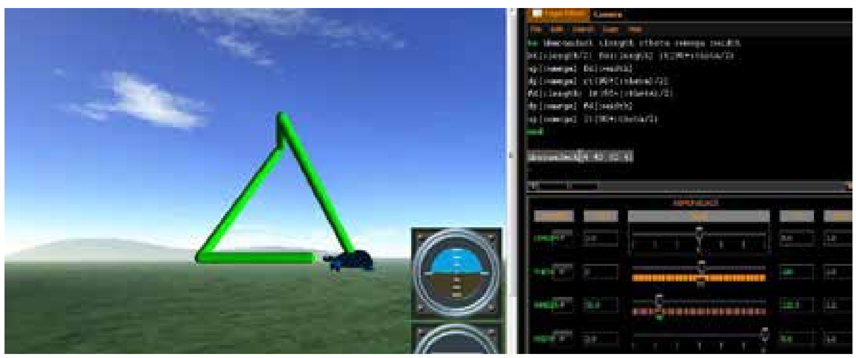

Figure 1: The Twisted Rectangle half-baked microworld in 3d Math

Being a web-based environment, 3d Math is fully embedded in the Metafora System (Dragon et al., submitted), an on-line software platform that offers a set of Exploratory Learning Environments (microworlds) as well as shared workspaces that allow communication among individuals or groups of students. One of those shared workspaces is the LASAD Discussion and Argumentation Tool. The users place in LASAD's UI text boxes with their ideas (we call those "contributions") and link them with existing ones, forming in this way a kind of a structured discussion map (Fig. 4). To further tag each contribution with respect to its content (e.g. as "a suggestion" or as "a claim"), a dropdown list is available for each text box. LASAD is designed to function as a tool in which the students may discuss, argue, negotiate their ideas and as a reflection space as it depicts how the group discussions evolved.

RESEARCH DESIGN AND METHODOLOGY

Our research approach was based on the idea of studying learning in authentic settings through "design experiments" (Cobb et al., 2003). "Design experiments" aim to contribute to the development of grounded theories on "how learning works" and intent to shed light on relationships between the material designed for the experiment (usually innovative technological artefacts having added pedagogical value) and the learning processes within a specific implementation context.

The study described in this paper took place in a Secondary Vocational Education School in a small island near Athens (1st Vocational High School of Salamina) with four 10th grade students (15 years old). The students worked together for 8 hours (2 sessions) in two types of social orchestrations: all four of them as members of just one Group in face to face meetings, and divided in two Subgroups of two members each when working on-line with the microworld and the LASAD Discussion Tool. The researchers adopting a "participant observation" methodology chose not to intervene in the experimentation to give out specific instructions or to provide the "correct answer" to the students on how to address the challenge and proceed. They preferred to pose meaningful -often intriguing- questions at certain time points, so as to encourage students to continue their explorations, elaborate more on their thoughts, share and discuss their ideas collaborating with the other students. The researchers in this study had a dual role as they also acted as the class teachers.

Tools and Tasks

Phase 1: Making one side double the other

For this Phase of the Study, we designed in 3d Math the "L" letter microworld (Figure 2a). This was meant to be a preparatory activity for the TwR work which required rather difficult concepts for these students such as consecutive projections on a plane and trigonometrical relations. So this preliminary task, the students were given a 4-line Logo program to construct the letter 'L', including two variables (one for each of the "L"'s sides).

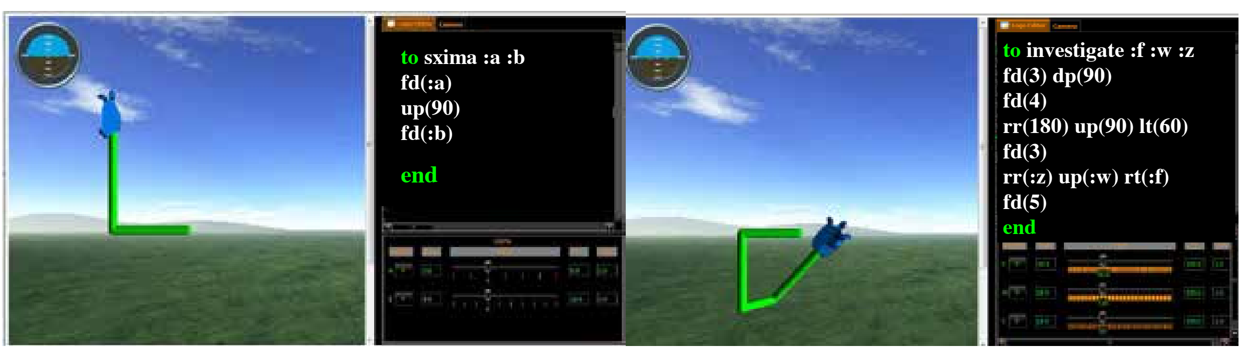

Figure 2a and 2b: The microworlds for Phases 1 and 2

The shape can be dynamically manipulated using the Variation Tool, which allows attributing sequential values to those two variables. The challenge that the students had to address working in two Subgroups (Subgroup A and Subgroup B) was to "make the vertical line be twice as long as the horizontal one", so that the "L" shape changes proportionally as one letter-shape.

Phase 2: Closing an open shape

For the Main Phase of the Study the students worked with the TwR buggy procedure. The task was to construct a skewed quadrilateral as one of its segments swivels along a plane vertical to the one in which the original rectangle belongs. In this version of the TwR, the figure depicted is purposefully not a closed shape, but an open one, as the end of one of the rectangle sides is not attached to the rest of the shape (Figure 2b). The students working in two Subgroups of two students each (Subgroup A and Subgroup B), were asked to try to “make the shape close”. Since, we didn’t intend to provide an answer on how to work with variables to do so, but ask them to discuss any ideas within their Subgroup and with the other Subgroup, we had prepared a discussion space in LASAD in which the two Subgroups could meet and share their findings as they explored this issue within the TwR microworld.

DATA COLLECTION-METHOD OF ANALYSIS

A screen-capture software (HyperCam2) was used to record students’ interactions the Metafora Tools together with their verbal interactions. Since previous work with 3d Math had shown an extensive use of gestures as means to explain and communicate turtle movements and turns, a Camera was added to record students’ hand and body movements. The corpus data is completed by the students’ LASAD maps and the Researchers’ Fields notes. The video-recorded data from the screen-capture software were verbatim transcribed, while the rest of the data were used for providing additional details. In analysing the data, we searched for verbal exchanges between the students and interactions with 3d Math and through LASAD that indicated that learning to learn together aspects were brought forth as they students attempted to address the challenge when working with the half-baked microworld.

RESULTS

The episodes of this section are selected so as to highlight the students’ interactions at the Main Phase of the experimentations and describe: 1) their discussions within their Subgroup as they explored the idea that less variables than the ones that appeared on the Logo program were needed so as to close the “TwR” and 2) their discussion with the other Subgroup in LASAD around this same issue.

We focus, however, on how the students’ mathematical activity was fuelled by these discussions and specifically by: a) the fact that they needed to articulate their own ideas in LASAD and explain them to the other Subgroup as clear as possible and b) the fact that they are receiving an idea from the other Subgroup which they needed to try out and decide on its feasibility and usefulness in the process of closing the “TwR”. In these instantiations, we also look for manifestation of L2L2 elements

Subgroup’s B idea: One of the values is redundant

The students of Subgroup B, in their attempt to close the TwR, manipulate the three sliders of the Variation Tool. Through this action, they attribute each time different values to the three variables of Logo program that generates the figure on screen.

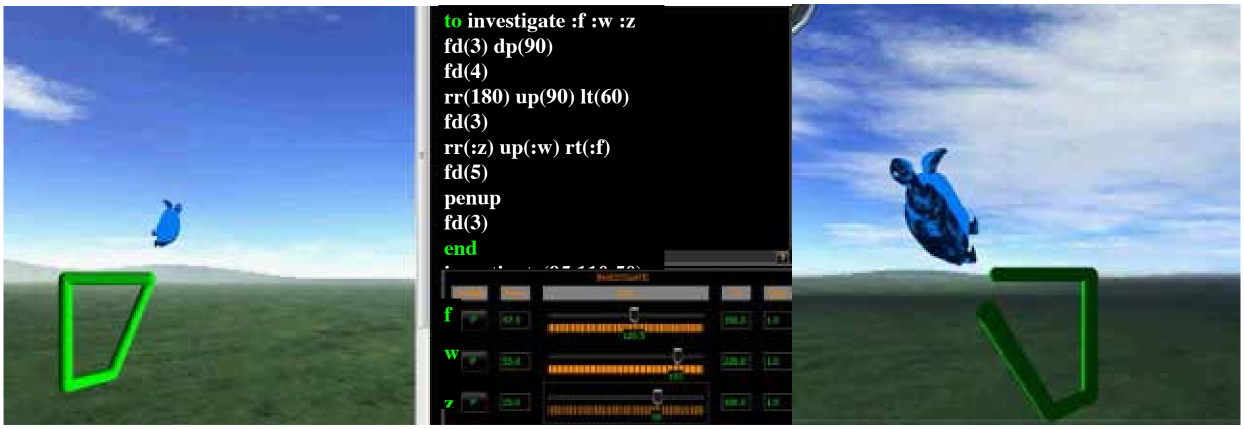

Figure 3: Attempting to close the figure for f=126.5, w=193 and z=78

As multiple times they have almost closed the figure, but haven't really managed to do so yet (Figure 3), the students come up with the idea that *one of the values is probably not needed for closing the figure* and thus they should manipulate only the sliders corresponding to the other two. As they believe this could bring the Group closer to achieving the common goal, they share this idea with Subgroup A through their LASAD discussion map.

S3: I believe that one should go…. It's….. how do we say that? Redundant?

S4: Redundant… We may remove the one line [refers to the slider's numberline]

S3: [Types in LASAD]…have just two values, because may be the one is redundant and makes the shape becoming larger. A suggestion.

Having entered their contribution in LASAD (Contribution no 2 – Figure 3), the students of Subgroup B move back to the microworld in an attempt to solidify the idea of "removing one of the values". Being quite focused on the manipulation of the sliders for closing the figure, the students find it easier to eliminate the effect the third value has on their figure, by simply placing the third slider's pointer on the zero value. Their explorations from this point on, move to a more specific level as for what needs to be done ("have a value equal to 0", instead of "removing it") and share the results of these explorations with Subgroup A by entering a "Claim" in their discussion map (Contribution no 4 – Figure 3).

S3: [manipulates one slider at the time] This one should go. Because with this one you can do that and with this one you can make it come closer. I believe this one is redundant.

R2: So you say this is redundant. How should we make sure? What should we do?

S3: Let's make it 0.[change slider value]

S4: Now it is almost closed

S3: Let's go to the camera. Ahh!! [Laughters – it's not closed] I know what we'll do

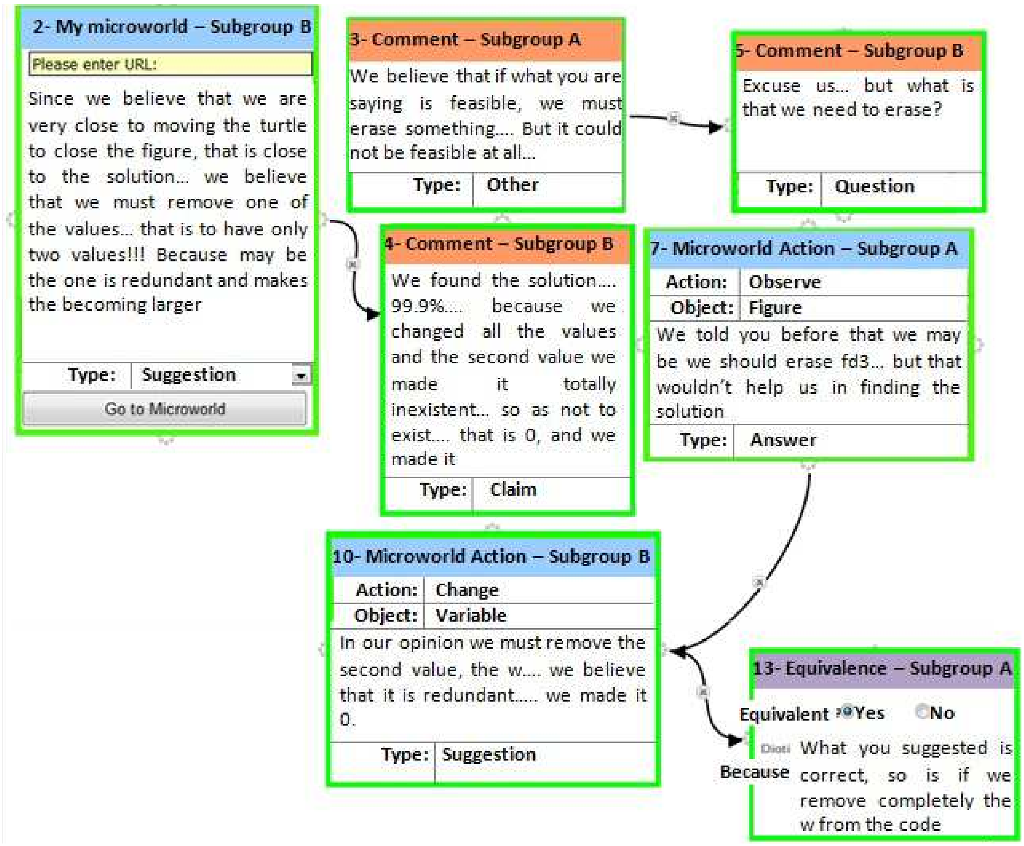

Figure 4: The LASAD map Subgroup A and B use to discuss their ideas on how to close the shape (translated in English)

Subgroup's A idea: Erasing a variable

After reading Subgroup's B contribution (no 2) about "removing one of the values", the students of Subgroup A move to their microworld to explore if this idea is feasible. However, it seems that while trying to give an answer to Subgroup's B suggestion, they come up with another idea as for how to make two instead of three variables have an effect on the figure they are trying to close.

R3: What do you suggest to do?

S1: Let's try to erase something from here… [the Logo code]

R3: What do you suggest to erase?

S1: A letter to start with… a letter… one of the commands.[they explore which variable/command controls which turtle movement]

S2: Just tell them what is that WE believe and then we will try answering them. But I still don't know what answer to give them…

These students are less focused on the way the manipulation of the sliders affects the figure and pay more attention to the Logo program and to the way each Logo command corresponds to specific turtle's moves and turns that construct the 3d figure. Thus, they perceive the "removing one of the values" strategy proposed from Subgroup B as one to be implemented in the Logo Editor and interpret it as an "erasing something" action (Contribution no 3 – Figure 4). As the Logo command for the TwR's side that the students attempt to attach to the rest of the shape contains variables instead of constant values, the students of Subgroup A go further with their assumption and explain the need to "remove a value" as a need to erase a "letter" (a variable) from the Logo program or a whole command that encompasses a variable. However, as they don't feel confident about their answer, they choose not to expose this new idea to its full extend to Subgroup B. As a result, the students of Subgroup B, post one more contribution (Contribution no 5) demanding from the Subgroup A students to explain what needs to be erased.

Subgroup's B idea revisited: Make one variable's value equal to zero

Just few moments before that, the same students posted the results of their explorations as they had already revisited their initial idea of one value being redundant and suggest making one value equal to zero (Contribution no 4). However, they omit in their Contribution that the variable to which they gave the zero value was the "w" variable. Realizing that the reason for the misunderstanding is the fact that they hadn't efficiently explained to others how to implement their idea, they insert a "Microworld action" contribution (Contribution no 10 – Figure 4) that offers more details as for which "Action" to be carried out ("change") and for which microworld object needs to be manipulated ("variable").

Subgroup A: Evaluating both ideas

The students of Subgroup A, coming to view the symbolic representation (Logo program), the dynamic Variation Tool and the figure graphically generated on screen as three interconnected representations, validate the idea Subgroup B offers as an equivalent to their own and insert Contribution no 13.

S1: "In our opinion we need to remove a variable"…. Ahhhh….. we agree!!!!!

S2: The "w"? [the "w" variable]

S1: Yes the "w", yes… because we also said that it should be the "w"…

S2: Yes! Because it was the "w" that just rolled the turtle!!!

S1: Yes… I'll tell them that we also found that it's "w".

S2: "f" and "z" are more important. They make it [the turtle] go up, right and left

S1: So should we tell them that we agree?

S2: It's the same if we totally remove "w" or we make it zero.

DISCUSSION

The students of Subgroups A and B were both given the TwR half-baked microworld and were asked to figure out the math required to close the figure. The Subgroups worked independently with the microworld but shared an on-line mutual workspace in LASAD in which they discussed their ideas on how to achieve their common goal as one Group. Taking a close look at the students' activity, we tried to identify how meaning generation processes while working with the microworld where shaped by their social activity as they discussed both within their Subgroup and with the other Subgroup on how to make the figure close. Furthermore, looking at the students' moves between on-line discussions and microworld, we sought to identify specific L2L2 elements that may have influenced the students' meaning generation processes.

Our findings indicate that the interactions in the dialogue carried out between the Subgroups in the LASAD tool enhanced their awareness of involvement with math meanings and the ways in which these evolved. This dialogue was sustained by the fact that the students constantly moved between their discussion map and their microworld, trying out ideas and making them objects of discussion and reflection both for themselves and for other Subgroup students in LASAD. Each time a new idea was proposed by a Subgroup by means of sending a new version of the code or proposing variable values as potential solutions, the members of the other one tried it out so as to evaluate and check it for its usability with respect to the Group's goal.

At the same time, the members of the Subgroup initially suggesting the idea, revisited it and come to reflect on it so as to make it more explicit for the others, exemplifying the original idea and offering new insights on how to implement it. Reflecting on both approaches (their own and the one offered by the other Subgroup students), the two Subgroups came to put all ideas suggested not only under peer assessment processes but also under self-assessment processes. Moreover, Subgroup A, being proactive, used the feedback and experience from the other Subgroup's explorations and taking control of their understanding as a Group, decided that the two ideas were equivalent (Contribution no 13).

Meaning generation, in this case, was also fuelled by the fact that the students evaluated and monitored the progress they made as a Group towards the common goal. They assessed specific learning outcomes as important for the Group's understanding and re-organised their activities accordingly. All these L2L2 elements appeared to play a specific role in students' further explorations with 3d Math and discussions in LASAD. The outcomes of this small-scale pilot study were used to design the main study which was implemented with the participation of 10 9th grade students for 26 school hours.

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EXPLORING THE POTENTIAL OF COMPUTER ENVIRONMENTS FOR THE TEACHING AND LEARNING OF FUNCTIONS: A DOUBLE ANALYSIS FROM TWO TRADITIONS OF RESEARCH

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In this paper we aim to address the potential of computational environments offering integrated geometrical and algebraic representations for the teaching and learning of functions. We follow a 'double analysis' method to analyse learning situations of an experiment that took place in the French context through the lens of the original research tradition (Theory of Didactical Situations) and an 'alien' one (Constructionism). The analysis indicates that this method enhances our efficiency to capture aspects of research traditions which influence knowledge concerning the nature of learning situations for functions with computers.

INTRODUCTION

The notion of function occupies a central role in a wide range of mathematical topics, but engaging students in functional thinking is known as a demanding task. We note issues identified by research in relation to students' difficulties in understanding function as covariation (Carlson et al., 2002) and dealing with algebraic symbolism (e.g. distinguishing between independent dependent and variables, Thompson, 1994). The development of new modes of representation within specially designed technological tools has generated further interest as regards their potential to deal with the above mentioned difficulties. One of the prime affordances of such tools is the multiple linked representations designed with the aim of providing some sort of combination of Dynamic Geometry Systems (DGS) and algebraic multirepresentation, possibly including Computer Algebra Systems (CAS) (Mackrell, 2011). In this study we are especially sensible to the possibility offered by particular computational environments to connect the notion of function to dependencies and covariations between geometrical objects. Existing research indicates that geometrical situations in DGS can be a fruitful context to challenge students' intuitions and ideas about covariation and functional dependency come into play (Falcade et al., 2007). In addition, problem solving by way of algebraic modeling of geometrical dependencies, which includes also sensual experience of these dependencies, can provide a basis for students' understanding of the idea of function provided that students can work flexibly between the geometrical and symbolic settings (Lagrange & Minh, 2010). Here we report research aiming at shedding light on the potential of computational environments offering interconnected algebraic and geometrical representations to facilitate students' making of links between the qualitative experience of dependencies in the geometrical context and the algebraic notion of function. Yet, it seems difficult to really appreciate this potential, since it is needed to take into account the visions provided by specific theoretical frameworks in technology enhanced mathematics, and because of the fragmented character of these frameworks (Artigue, 2009). Here we consider fragmentation as resulting from the existence of different research traditions. By research tradition, we do not mean only a reference to a theoretical framework, but also all the research practice built jointly with a framework: reflection on a practice gives theoretical elements for a framework, and, in return, practice, constituted by design, observation and interpretation, is affected by the framework. While compartmentalized research traditions participate into fragmentation, we assume that different research traditions can be confronted and in some sense articulated in order to address a particular question, the potential of computer environments for the teaching and learning of functions. Testing this hypothesis is the general aim of this study.

Our choice here is to consider two research traditions, both dealing with functions and software, but different in many other aspects. One involves *Casyopée*, a piece of software that offers a dynamic geometry window connected to a symbolic environment specifically designed to support students' work on functions (Lagrange, 2005). Casyopée's design and experimentations occurred in a French context shaped by didactical theoretical frameworks and epistemological considerations. We focus here on a framework preeminent in the French context: the Theory of Didactical Situations (TDS) (Brousseau, 1997). The other research tradition involves *Turtleworlds*, a piece of geometrical construction software which combines symbolic notation (Logo) with dynamic manipulation of variable values (Kynigos, 2004). The design and the research on the use of Turtleworlds is inspired by Constructionism (Papert, 1980) and have been carried out in the Greek context. Constructionism and TDS share a common focus on the design of learning situations through devices – such as the "milieu"- providing affordances for interaction and knowledge construction. Our assumption is that divergent views of the two traditions on the contribution of milieu (e.g. the design and analysis of a session, the nature of the constructed knowledge and its relation to the official knowledge) provide a complementary way to address the potential of computational environments for the teaching and learning of functions.

We drew on data from two concrete teaching experiments taking place in France and Greece respectively. We consider here the teaching experiment designed and implemented with Casyopée in the French context. We carry out "double analysis" of this experiment by way of TDS (a priori and a posteriori analyses) and Constructionism in order to be able to tackle under an "integrated" perspective (in the sense of Prediger et al., 2008) the following question: what new insight about the potential of computer environments offering integrated geometrical and algebraic representations for the teaching and learning of function might be gained from the double analysis of research studies carried out in different national and didactic contexts? Prior joint research experience in cross-analysing teaching experiments with use of digital tools for function (project ReMath [1]) revealed that misunderstanding rooted in divergent views of functions and distinctive theoretical orientations could be addressed at two levels: one is the economy of learning situations, to which we refer in this paper, and the other is the process of conceptualisation of functions by students. Thus, the links between the issues involved in our research focus here are as follows: investigating the potential of computational environments for the teaching and learning of functions raises the issue of fragmentation of theoretical frameworks in the field of technology enhanced mathematics; in order to explore such fragmentation at the level of research traditions we adopt methods (such as double analysis) and theoretical tools (such as the economy of learning situations) that promote a deeper focus on the design, implementation and analysis of teaching experiments realised in different contexts.

THE ECONOMY OF LEARNING SITUATIONS

The notion of "economy" of learning situations helps to address the role of the many different components intervening in the classroom progression of knowledge: students, teacher, but also various artefacts which can be material (e.g. blackboard, disposition of the room) or not (e.g. tasks, rules, systems of notation, language). According to Hoyles, Lagrange and Noss (2006, p. 301): "… a learning situation has an economy, that is a specific organization of classroom components, and technology brings changes and specificities in this economy …".

Theory of Didactical Situations (TDS)

Brousseau (1997) presents TDS as a way to model mathematical situations in a learning context. In this model, a central notion is the "milieu", a device which justifies the use of knowledge objectively to solve a given problem. Student's acting on the "milieu" provokes feedback calling for modifying or adjusting action. Learning thus results from the student's adaptation to an antagonist "milieu". Teaching consists in organising these constraints and keeping optimal the conditions of the interaction. TDS considers adidactical situations designed in a way that the desired outcome can be obtained only by applying the knowledge aimed at in the situation. Researchers who refer to TDS in order to consider interactions with digital environments (e.g. Cerulli et al., 2008) propose to think of technological learning environments as means to provide students with an antagonistic milieu, offering tasks and feedbacks adequate for the knowledge at stake, under the condition that situations of use are based on a suitable a priori analysis.

Constructionism

Constructionism incorporates and builds upon constructivism's connotation of learning as "building knowledge structures" through progressive internalization of actions, in a context where students are consciously engaged in constructing (or de/re-constructing) physical and virtual models of situations on the computer (e.g. geometrical figures, simulations, animations): the notion of construction refers both to the 'external' product of students' activity as well as to the process by which students come to develop more formal understandings of ideas and relationships (Papert, 1980). The constructionist paradigm attributes special emphasis on students' construction of meanings when using mathematics to construct their own models during individual and collective 'bricolage' with digital artefacts, i.e. continual reshaping of digital artefacts by the students in order to complete specific tasks.

CASYOPÉE

Casyopée deals with various representations of functions. It provides a symbolic window (Fig. 1, left) with tools to work with functions in the three registers: numeric, graphic and symbolic. Casyopée also includes a dynamic geometry window (Fig. 2, right) linked to the symbolic window. The geometric window allows defining independent magnitudes (implying free points) and also dependant ones that can be expressions involving distances, x-coordinates or y-coordinates. Couples of magnitudes that are in functional dependency can be exported to the symbolic window and define a function, likely to be treated with all the available tools; this can be done automatically, a functionality that was expected to help students in modeling dependencies, and that we will refer to as "automatic modeling" below.

THE EXPERIMENT

The design of a session

The classroom session analysed in this paper was the fifth of a series of sessions in the ReMath project following three sessions by which students get familiar with the symbolic window, and one in which they were introduced to the dynamic geometry window and to problems about areas. A series of tasks was conceived in which the students had to make a choice of the independent variable as a key step to get an algebraic model of a geometric dependency, in order to solve the following problem: *ABCD being a rectangle, what can be the position of a point M in order that the area of the triangle BMC is one third of the area of rectangle ABCD (Fig. 1, right)?*

The sides of rectangle were parametric ($AD=a$ and $AB=b$) in order to ensure generality and a discussion on the fact that the solution does not depend on a. The solution is that the points satisfying the condition belong to one of two straight lines parallel to (BC) crossing (AB) respectively in M_0 and M_1 such that $BM_0=BM_1=2AB/3=2b/3$. It is possible to reach this solution geometrically, but the way the problem was proposed to students (in coordinate geometry) and their lack of knowledge in geometry, oriented towards using a function as a model of the variable area. Five successive tasks were then proposed to the students: (1) Build the figure in the dynamic geometry window, M being a free point in the plane (2) Create a geometric calculation for the area of $ABCD$, and moving M, conjecture positions of

M for which the area of *BMC* is one third of the area of the rectangle (3) Choose an adequate independent variable to get a model of the geometric function of the area of *BMC* (4) Use Casyopée's "automatic modeling" to get the definition of a function in the algebraic window (5) Use the algebraic window to get algebraic solutions, and then interpret these solutions in the dynamic geometry window.

Instructions were given in order that the side *AB* was parallel to the *y* axis, and the side *BC* to the *x* axis. So in task 3 the students had the choice to select for independent variable some length involving the point *M* or coordinates of *M*, but only calculations depending univocally of the *y* coordinate of *M* could be adequate variables. It was expected that students would observe that moving horizontally the point *M* does not change the area, and connect this observation with the fact that *xM* is not an adequate independent variable. After a user selects an independent and a dependent variable, Casyopée gives some feedback on whether it is possible to create a function with these data. Together with the observation of values of the variables when moving point *M*, this feedback was expected to create a milieu helping students understand the statute of variables in a function modeling a dependency.

A-priori analysis

For solving task 3, lengths involving *M* cannot be chosen as independent variables because they depend on the two coordinates. x_M can be an independent variable, but, as mentioned above, a change of x_M does not affect the value of the area of the rectangle. The version of Casyopée, still in development at that time, calculated a formula involving y_M, but after that refused to create a function. y_M is a suitable variable and the function calculated by Casyopée is $x \to a \times \frac{\sqrt{(x-b)^2}}{2}$. It was expected that the identifier *x* for the independent variable could be confusing for students. Casyopée offers other identifiers, but it was not likely that students will use this feature. In the preceding session, the independent variable was a length on the y axis, and the teacher insisted that this length could be labelled x in the function. After creating the function, the students could work in the familiar symbolic window to solve an equation. If *yM* has been chosen as a variable, the equation is $\frac{a \times |x-b|}{2} = \frac{a \times b}{3}$ but it can be different if the student choses another variable; for instance y_B-y_M is a possible choice and the equation is then $\frac{a \times |x|}{2} = \frac{a \times b}{3}$. Another difficulty was expected to emerge from the fact that Casyopée displays the solution in the non-simplified form $\frac{\sqrt{(x-b)^2}}{2}$, thus students had to interpret the two solutions in *x* as two values of y_M and to connect it to the geometric solution.

Data

The situation has been implemented in a 90-minute session in two classes. Data consisted of recordings of students' work via screen capture software, observers' field notes and students' written assignment. Below we present briefly the work of a pair (Elina and Chloé).

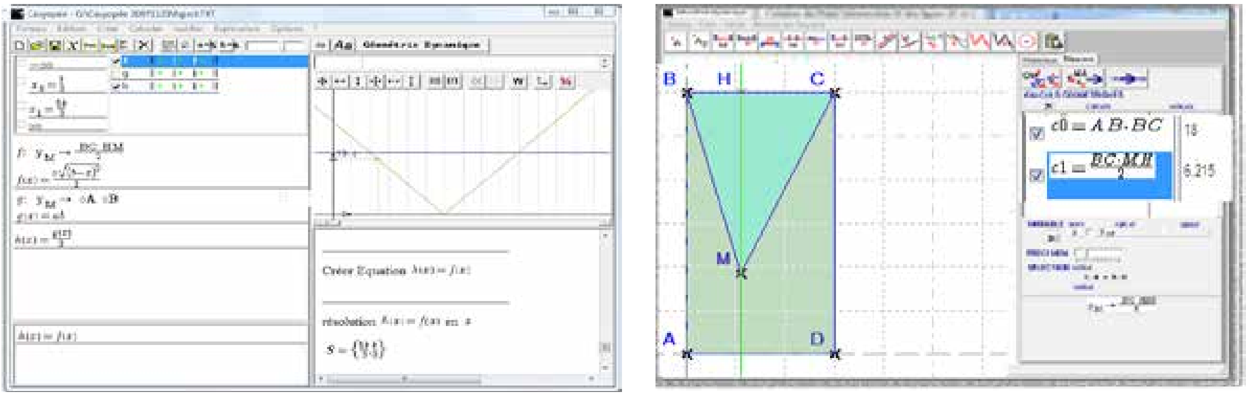

Fig. 1: Casyopée: The symbolic window and graphic tab (left) and the dynamic geometrical window and the geometric calculation tab (right).

The work of Elina and Chloé

In the first 20 minutes students built the rectangle, created a geometric calculation for the area of it and created a free point *M* initially in the plane and after some dialogue on a vertical side [*AB*]. By moving *M* on [*AB*] and evaluating numerically this area (considering the numerical values of the *a* and *b* parameters) for about 10 minutes, the students found a solution without taking numerical information of the software. They commented "This is good, this is one third of [*AB*]" and wrote their solution. After the teacher's prompt that M is in the plane, the students explored the figure again looking for a single position of *M*, but now on the perpendicular bisector of [*BC*] without using values of the area of *MBC* calculated by Casyopée.

Then Elina proposed to create a function, but Chloé stressed that an independent variable had to be chosen first. Thus they returned to the text of the given problem and tried to identify the requested variables. Reading the message after trying the constant measure *AB*, they moved *M* to this segment. Trying *BM*, Chloé commented: "here we cannot create the variable". After that, they tried x_B-x_M and y_B-y_M, reading the message of Casyopée ("the variable depends on *M*, it is defined over]-infinity, +infinity[") but not creating the variable. At 50mn, the teacher told them to choose a variable and they chose x_M. Then they defined the function: $x_M \to AB \times BC$ and got a function $x \to a \times b$ in the symbolic window. After that, they defined the function $x_M \to \frac{MH \times BC}{2}$. As indicated in the a priori analysis, Casyopée calculated a formula involving y_M, but after that, refused to create a function. At 65mn, they had to recreate the figure because of a technical problem and Chloé realized that the triangle area was constant and equal to one third of the rectangle area for every position of *M* on a certain horizontal line. They commented "it is always one third… then the *y*-coordinate is what is important". Surprisingly they again chose the variable x_M and got the same feedback as before. At 70mn, the teacher told them to test the variable y_M as indicated in the text of the problem. Casyopée indicated (*-infinity*, *+infinity*) for the domain. They were not happy and tried to find a way to redefine this domain into [0;3]. Giving up, they defined the function by way of "automatic modeling" and it was accepted by Casyopée. They tried a graphical resolution, but they were confused by the graphical window and needed to get help by the teacher. He showed them how Casyopée offers a dynamic link between a trace on the graph and the free point from which the function is built. The students observed that when they moved *M* horizontally in the geometry window, the trace does not move (Fig. 1, right).

The written report prepared by Elina and Chloé was divided in two parts: Dynamic geometry and Casyopée (i.e. the symbolic window). In the first one they describe their exploration for calculating the areas for the values a=3 and b=6 to the parameters: "in our case, the area of *BMC* must be 6. Thus we move *M* in order that the value displayed is 6. We see that there are two positions of *M* and only the *y*-coordinate has an influence on the area, the *x*-coordinate does not change the area". As for their work in the symbolic window they write: "we chose the variable x_M" [2]. And then "we draw the functions $y_M \to AB \times BC$ and $y_M \to \frac{MH \times BC}{2}$", copying the formula given by Casyopée and not mentioning the domain. They copy also the equation and the two solutions. They conclude: "To satisfy the condition, the *y*-coordinate of M should be y_M=5b/3 or y_M=b/3". About the difficulties encountered, the students mention: "Finding that only *y* has an influence on the area of *BMC*, and then choosing y_M as an independent variable".

THE ECONOMY OF LEARNING SITUATIONS: A DOUBLE ANALYSIS

A posteriori analysis from a TDS perspective

The situation was certainly productive in the sense that students could grasp the necessity of choosing an adequate independent variable in order that Casyopée will be able to express a function, but they were far from giving an algebraic signification relatively to this necessity and to the other algebraic objects involved, like for instance the parameters. Casyopée's feedback was generally not well understood. Eventually, it conflicted with students' views, for instance relatively to the domain. It happened that the teacher had to intervene to help going forward in the task. He tended to offer more than a technical help to students, steering them towards steps of the solution and then breaking the intended adidacticity. This was also the case in other Casyopée experiments conducted in ReMath. Thus, the influence of the provided feedback seems to be less productive than expected as regards the students' attempts to identify key steps in their mathematical work. Relatively to the question at stake of how students could appropriate the choice of the independent variable, as well as other functionalities already encountered in the preceding sessions, and through this appropriation progress in their understanding of functions, the appreciation is then mixed: the milieu highlights actions that students can identify as steps in the solution; then students are "pushed" towards these actions; however, this does not guarantee that they acquire an appropriate understanding of these actions.

Analysis from a constructionist perspective

The issue of design was mainly materialised through the preparation of a milieu facilitating - in constructionist terms- meaning generation for function as covariation. The reported episode reveals students' diverse views of the symbolic forms provided by the tool and difficulties to relate their selection of variables to the mathematical concept of function. To analyze this divergence, we refer to *two gaps at the level of design* and try to connect it to constructionism: one has to do with the design of the environment and, in particular, *the nature of the provided feedback* and the second with *the teacher's role*. As for the first, the level of design of Casyopée at that time did not provide students with opportunities to take some actions in relation to the provided feedback. Thus, we see that students could not experiment directly with notation in Casyopée. The correct symbolic form in mathematical terms appears as a 'closed' answer pre-supposing in some way students' understanding of the standard algebraic symbolism of functional dependencies. A constructionist view on design should stress that further development of meaning generation can be facilitated if students have at their disposal a mechanism to manipulate so as to take further action based on the provided feedback (i.e. to 'do something' with the tool). Learning activity within constructionist computational media very often consists of students' engagement in debugging intentionally designed 'buggy' behaviors of objects. These objects operate as means to challenge productive meaning generation and provoke further interpretations and actions by students. Thus, constructionism should emphasize the *expressiveness* of computational environments as a design principle, i.e. design based on the use of dynamic representations that make algebraic symbols and relationships more concrete and meaningful for the students through the ability to express mathematical ideas possibly in ways that may diverge from standard mathematics (see for example the idea of *autoexpression,* which privileges the role of a programming language as a mechanism to control objects by expressing explicitly the relationships between them, Noss et al., 1997). The second point has to do with the role of the teacher. In the episode we can see that the teacher seems to be reluctant to intervene and does it only when he realizes that the students face strong problems in coping with the provided functionalities and integrating them in their activity. In a constructionist perspective, in contrast, teacher's interactions are more participatory from the teachers' side and more strategic in encouraging students to elaborate emergent ideas and generalisations.

DISCUSSION

The motivations underlying the Casyopée experiment meet at a general level those involved for the use of multirepresentational, DGS and CAS software. Its specificity is that it focuses on key points of the transition from function in a DGS to symbolic functions, aiming to facilitate students' access to symbolic forms. The task in the experiment is to find a solution of a problem. It can be explored in geometrical settings, but can be really achieved only after the transition to symbolic functions.

The milieu is then inspired by TDS, the task being challenging and the transition being conceived as a non obvious step. Feedbacks are prepared in the software in order to ensure that interaction will actually put the aimed knowledge at stake, TDS rely on an a priori analysis and uses an a posteriori analysis to compare actual procedures of students to the a priori expected procedures to bring evidence that the milieu is adequate for the targeted knowledge. That is what it was aimed in the Casyopée experiment. However, as indicated above the appreciation is mixed: the interaction seems to produce effects in terms of action in the environment, but not to really make sense for the students. The constructionist interpretation points to an important fact: Casyopée is a mathematical tool, and most of the feedback it provides supposes algebraic knowledge, or coordination between geometry and algebra, that is precisely at stake. In this vein, constructionist analysis brings to the fore issues of tool design emphasising the importance of design choices allowing students' meaningful use of the available infrastructure by forging connections between students' action and tool formalism. The interventions of the teacher constitute a point of common interest in the double analysis. They are seen by constructionism as participatory and strategic in enhancing students' exploratory activity. In contrast, TDS cares for adidacticity that could be broken in these interventions, by way of "Topaze" effects. However, in the Casyopée experiment, it happens that total adidacticity would have led the students to an impasse.

Constructionist and TDS analyses of learning situations in the Casyopée experiment in part converge when they consider a milieu and in part diverge because they have different conceptions of this milieu. TDS analysis is oriented towards evaluating the reproducibility of situations of learning aiming a given knowledge, and constructionist analysis towards identifying occurrences of progression of meaning. However, this "double analysis" is clearly deeper and helps to look at the economy of learning situations about functions with computers as a particularly complex question. On the one hand, the multiplicity of interconnected representations of functions, of students' possible actions on these, as well as of students' understanding of these representations and actions is an obstacle to the possibility of a controlled milieu, and of adidacticity consistent with TDS. On the other hand, relying exclusively on uncontrolled meaning generation would question the extent to which connections can be made between knowledge built by interacting with the milieu ("knowing") and the standard mathematical knowledge ("knowledge").

NOTES

1. "Representing Mathematics with Digital Media", 6th FP, IST-4-26751-STP, 2005-2009 (http://remath.cti.gr)

2. Actually, *xM* was the label of the button allowing choosing a variable, which explains why the students mention this label, while being aware of *yM* being the right choice. This label changed in subsequent versions of Casyopée. The design decision at the time was to implement key actions at Casyopée's interface by way of buttons like in DGS. The difficulty was to find icons that could accurately represent the nature of the action.

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THE ROLE OF METADATA IN THE DESIGN OF EDUCATIONAL ACTIVITIES

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How can dynamic geometry documents be shared? There exist several repositories of learning resources where this can be done. Each repository has its own specificities. In the process of understanding how dynamic geometry resources are selected and shared, preparing for the Open Discovery Space federation [1], we attempt to describe the role of the metadata for the potential users of the learning resources: what purposes it serves, and when it is useful or not. We do this based on the log-books of actual maths teachers of the i2geo platform whose experience and professional practice of teaching sets particular utility of the metadata: how they find the resources, assess the resources' qualities, and evaluate the cost of possibly needed adaptations: all depends on the practice.

INTRODUCTION: METADATA IN EACH CONTEXT

Metadata is understood to be the data about data – a fairly generic concept, probably as general as the notion of a *resource*. Our basic concern in this paper is learning resources: electronic information sets, which can be used by an educator or learner to support their teaching or learning processes. More precisely, we are interested in the information that is encoded in repositories besides the resources itself: the metadata.

Multiple repositories of learning resources exist, including LeMill (lemill.net), i2geo (i2geo.net), Curriki (curriki.org), Merlot (merlot.org), Agrega (agrega.educacion.es), each with its specificity: for example, Curriki allows to search by educational levels for sections of ages, while Merlot does not allow such a search, and Agrega supports search by the exact educational levels (of Spain). Each of these repositories was created with a target population and a target set of learning resources in mind; based on these, the toolset and the metadata structure [2] were chosen. However, the impact of these choices of metadata schema and tool sets on the day-to-day practice of teachers is little explored. In this paper we aim at qualifying the role of the metadata in the teachers' usage practice based on log-books of actual mathematics teachers. The differences between the learning resource repositories above make it so that it is easier to search for resources for a given educational level using Agrega or Curriki than using Merlot or generic web search engines (such as Google): in the two latter cases, only words can express the query and they do so quite ambiguously when one considers simple examples such as the word *quatrième* representing two different ages depending on the country one is living in.

This paper aims at inspecting the explorative processes of the selection of learning resources. The usage of learning resources by teachers, as well as the social and professional development that follows this usage, have been studied in such works as the book of Gueudet, Pepin and Trouche (2012), but they assume that the resources are already found and study their impact on the teaching. In this paper, we study the selection process (searching, choosing, evaluating) preceding the use that seems to be little explored. The utility of the information that is displayed to the educators so as to decide on the re-use the relevance of a learning resource should be measured with a look to the professional life of teachers: we contend that re-using learning resources can help the teachers introduce innovative learning practices and support the usage of software in classroom. This paper does not discuss this hypothesis, but the question: Which information should a sharing environment display for each resource so as to trigger re-use? Differently said: What exchange vocabulary can be used between producers and recipients of learning resources?

While Ha Lee et al. (2013) investigate the role of the metadata fields of video games for the usages of different type of *personas*, we investigate the role of the fields of metadata based on the log-book of teachers preparing their mathematics courses.

Definitions

The *learning resources* we are interested in are any form of a digital artefact that is ready to be used by educators or learners to support the teaching and learning process. Learning resources can be found in published works such as textbooks or their supplementary materials, they can be found in the portfolio of the experience of each teacher, or they can be found in sharing platforms. The concept of "learning resource," which is equivalent to that of a "learning object" (Wiley, 2000), is used by all current *Open Educational Resources* repositories. These repositories create a context, which allows the learning resources to be found and, later on, to be re-used.

For this paper, we shall call *metadata* of a learning resource in a given environment any information recorded directly about a learning resource that is not included inside the learning resource itself. Thus, metadata includes a description or a caption, annotations indicating the target educational level or an instructional type or a snapshot of the learning resource. Partially standardized formats exist to encode metadata (LOM, DC-ED, LRMI) and may help the exchange between different platforms (container websites). Metadata records are generally split in sections such as: general, authorship, rights, pedagogical metadata, and technical metadata. Beyond the metadata, one often calls *paradata* the set of data about a learning resource that has been recorded following a particular view or usage of the resource. This includes ranking statements, records of how many students have succeeded, or comments on the resource. While paradata is not exactly metadata (and it often lies in separate places than metadata), it may often serve the same role.

Although our investigations have a potential of application beyond teachers of mathematics in Europe, they are our focus: tools, resources, and annotations vocabulary in our study are designed for them and by them. The paper first informs about the objectives of this research: the design of the Open Discovery Space platform. It then describes the principles the i2geo log-books approach which constitutes our experimental basis. Then it presents the roles of the metadata, computer-wise and didactic-wise. These roles are then instantiated in an interpretation of the log-books. General remarks form the conclusion.

OPEN DISCOVERY SPACE

The research described here is intended as a basis of the platform design process of the Open Discovery Space portal. This portal will be the result of the EU project of the same name, a broad project gathering 52 institutions across Europe and about 20 learning resources repositories in many subjects, including mathematics.

Open Discovery Space will federate multiple existing learning resources repositories already on the web. Among others, the i2geo platform (i2geo.net, Kortenkamp et al. 2009), the Cosmos portal (cosmosportal.eu), open-science-repository (openscienceresources.eu), organic edu-net (organic-edunet.eu), or edu-tube-plus (edu-tube-plus.info). A more detailed overview is made by Megalou et al. (2012).

This federation will be enriched by a social network, by students' delivery tools which should empower teachers to re-use learning resources including features as far as the analytics services that allow to know if and how learners have used the resources, and by optional extension-servers which support a deep integration into the school infrastructures. Open Discovery Space is a EU project running from April 2012 to March 2015; as of this writing, its design is being articulated.

The platform design process includes the elaboration of a vision of the usage of the portal in the design of educational activities for secondary school. The vision is to be complemented by scenario (or lesson plan) templates, which will support teachers in the application of alternative didactical approaches. The project gathers technology enhanced learning specialists in the field of science (notably biology and physics), language learning, and mathematics. It aims at serving the complete range of stakeholders involved in secondary school life.

In order to describe concrete scenarios of usages of the Open Discovery Space portal, user stories have been written and the design process of teaching activities using the expected platform is being sketched: this is where the learning resources sharing platform is expected to be used, and thus where metadata becomes important.

To understand this process, reports of the experience with other platforms are gathered. In this paper, the log-books of usages of the i2geo platform are discussed.

THE I2GEO LOG-BOOKS OF THE RESOURCING PROCESS

During the Inter2geo project (which ran between 2007 and 2010, see Kortenkamp, 2009), a team of active teachers attached to the INRP in Lyon (France) decided to gather to discuss and attempt the usage of the i2geo platform and to report about it. This effort was lead by Jana Trgalová and Sophie Soury Lavergne. The objective of this report was to help to guide the elaboration of the platform so as to make it useful for the work of teachers. Log-books [3] were filled tracing the discoveries made and the expectations felt. These log-books are all dated and sometimes represent the i2geo platform in a very preliminary state. Many of the issues have been dealt with in the meantime, be it on the platform level or on the level of resources. Moreover, some of the log-books mention resources which have been changed in the meantime. These log-books however should be read with the perspective of informing how these platform usages have an impact relevant to these teachers' teaching activities.

We shall review several of these log-books. They all involved a simple *resourcing process*: start with a need for a future teaching occasion, formulate search queries, skim through each of the probable results, identify the useful ones, try each of them, file a quality evaluation, attempt in class, file another quality evaluation.

THE ROLES OF METADATA

In this section, we describe the roles that we propose the metadata can have in the software activities of a learning resources sharing platform. We differentiate the technological functions and the didactical functions.

Technological functions of metadata

We are interested in the following functions that a computer program can perform with metadata within the activities around learning:

DISPLAY: When a person browses a learning resource within a collection of resources, metadata is presented. Parts of the metadata can then be read or seen by humans; this can help to recognise a resource. Good examples of rendered metadata include the title and description, the media-type (typically as an icon), and the educational level. The display can also include paradata.

SEARCH or FILTER: Using several retrieval methods, it is possible to find the resources that match particular metadata values. This includes browsing taxonomy and clicking the links or entering a text and showing its matches. The information of the metadata is the basic search ingredient. Search engines generally apply multiple levels of matching between queries and resources so that the results list appears to be sorted by *relevance*: e.g. a word found in the title is more important than in the description or a resource, a didactical function (e.g. reference, handout, demonstration…) matching a query in the metadata of a resource that only has one such function is more important than such a function in a resource that has dozens of such functions.

RECOMMEND: Based on recommendation algorithms, automatic searches can lead to suggestions of learning resources for users. Recommendation is similar to searching, but the search criteria are given by the software and search is usually not initiated by the user, but by the platform itself.

INPUT: A user that contributes a learning resource, and one who updates it, has the possibility to input or modify most of the metadata.

Didactical Functions of Metadata

Such basic functions as above help an educator to perform a number of actions that are useful for his or her teaching preparation and implementation process.

SELECT: Within a *resourcing* process, teachers routinely seek learning resources that could support their teaching. This generally involves cycles of search, preview, trial, and refinements until something applicable for their objectives and conditions is found. Selection involves an elaborate dialectic activity between the usage of search tools, the observation (and thus evaluation) of the displayed metadata, the available (or missing) resources, the attempts of usage, and the refinements of the search.

PUBLISH: When users feel that a learning resource would be valuable to contribute so that others can take advantage of it, a basic record of metadata is populated with information that the user considers to be useful. Doing this he or she has an idea how to present the source so that it will be displayed adequately and that expected search queries will show it.

ADAPT: Finding the right resource is most commonly an imperfect quest which needs a complementary adaptation process. For example, one needs to adapt wording, the technical conditions of use (e.g. make PDF out of Word, find the exact link, package into a different format, cut irrelevant pieces…). The cost of adaptation is generally compared to the benefit of re-use as discussed in Libbrecht (2011).

ORGANIZE: Course planning and resource publishing often require the resources to be grouped and labelled. This activity allows a collection of content to be presented along a structure that is practical to get an overview (for example a thematic grouping, or a lesson plan).

DEPLOY: When it is time to get to the classroom, a publishing process happens: a print, the creation of a resource in the learning management system, an indication on the blackboard, an assignment... These processes can end in the classroom (for in-class activities) or later. This is generally the time when the learning resource is ready for the students' use (e.g. when an interactive exercise is properly linked so that most learners will be able to just click and start it).

RATE: During the usage of the resource, and during the selection process, a constant critical eye is exercised. The output of this critical eye is a judgement of the quality that is published, typically, on the sharing platforms. Various forms of rating exist, from simple star-based judgements to elaborate multi-dimensional questionnaires such as that of i2geo (Trgalová et al. 2011). This creates paradata.

SUGGEST: A more general form of than rating, suggesting is commonly done in social networks of teachers and learners (for example via Facebook or Twitter). It involves transmitting the information about a learning resource from one person to another (or several). The suggestion should invite the recipient to explore the learning resource by formulating characteristics that are relevant for him/her. This can be done via email, for example, exchanging a URL and indicating or summarising the particularly interesting metadata facets.

THE ROLE OF METADATA REPORTED IN LOG-BOOKS

The roles we have described above appear in the INRP-log-books mentioned above. They show which metadata property and function has when led to a decision. We will summarize the analysis for several of these log-books below.

Triangular Inequality: Perfect but...

Log-book: JdB-inegalite-triangulaire.pdf.

Our teacher searches interactive geometry resources about the triangular inequality (*inégalité triangulaire*). No concept is registered for this, so he searches for these words and finds a page-full of search results.

Because some resources have these words inside the title, they are listed earlier. The teacher reads the metadata excerpts of the search results: a bit of the description, the ranking… (see Figure 1 for an example). Based on the title and the descriptions, the annotated level and topics, and the didactical functions, he can select a resource that seems appropriate: inégalité triangulaire [4]. This teacher has decided on this resource because of its title which matches exactly the expectations.

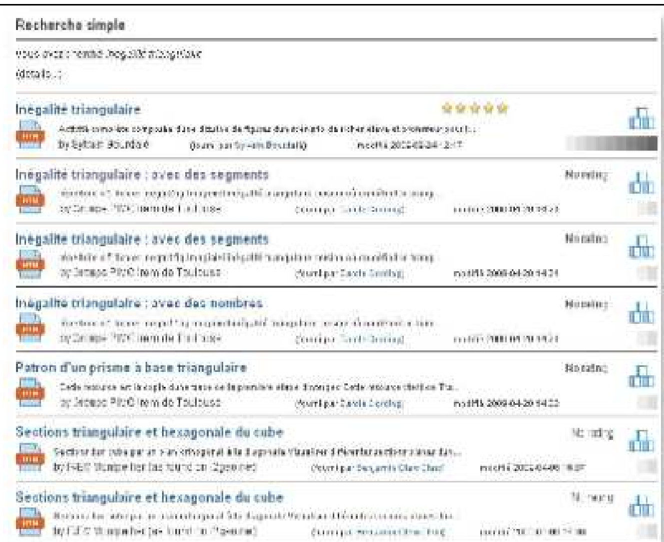

Figure 1: Excerpt of a list of resources

In this case, it is a linked page, which contains students' and teachers' sheets as well as 9 interactive exercises. Our teacher can test individual parts of the resource, making sure it is ready to try for the students. Thanks to the teacher sheet, he can plan that this activity will take two course-hours in lab and can book the rooms accordingly. He skips a part so as to save time. The second sheet is the starting sheet. During the course, he realises that some of the computers are missing a classical requirement (Java, Flash). Moreover, that day had a very low network bandwidth. Both of these technical issues lead to a loss of time of 15 minutes (of 50). The teacher notes that this resource is perfect, and he rates it highly, but he notes that he would wish his usage to be a bit different and that, since he cannot adjust the resources, he only can tell the students to follow things differently.

Metadata fields used: all that is displayed in the search results and in the resources' info (title, description, levels, instructional function).

Actions: done: select, deploy, rate; wished: adapt, organise.

One the most important criteria this teacher sets forth and has successfully encountered is the completeness of the didactical details (teacher's sheet, students' sheets, time estimates…). To our knowledge, no metadata structure encodes this completeness.

Corresponding Angles: Too Coarse Resources

Log-book: JdB-angle.pdf

Our teacher searches for *angle* using the search tool available then: a plain text search tool similar to that found on Curriki. She wishes to find resources allowing students to infer the relationship between parallelism and corresponding angles. As is usual with plain text search (Fig. 2), the result includes multiple unrelated results; only two seem to be closely related. She then tries to search for the occurrences of the plural word *angles*, which gives unrelated results. For these two, she looks deeper, opens the resource viewand the linked URLs. These URLs are large collections of resources (Fig. 3), such that inspecting each of them takes much time. One of the resources she finds in this big collection matches exactly in theme (Fig. 4) but the pedagogical approach bothers her: she wants the students to discover the relationship between the angle equality and the parallelism themselves but that interactive construction fixes the parallelism at start. At the end, she creates a new resource Angles correspondants [5] which corresponds to her intents. She expects to use it as a demonstration tool in classroom.

Figure 2: Search for "angle"

Metadata used: title, description, authors. Missing the use of a more precise topic.

Actions: done: select, deploy, publish, rate; wished: adapt.

Remarks: This log-book is a classical story of *poor metadata*: the topic our teacher expected is a relatively precise topic, a topic that needs two words hence is difficult to search for with precision. As a result, our teacher uses more general terms and has to sort through a pile of irrelevant resources which she can easily identify thanks to titles (such as the set of resources about the trigonometry).

She then meets another widespread issue for the central role of the learning resources platform (to make available learning resources findable to many): granularity.

Because the resource that is contributed is a broad collection encompassing multiple objects (Fig. 3), it cannot be finely annotated with topics and educational levels. Moreover, the description of each resource and the resources' content is not indexed. This is the reason why the best-practice guidelines of i2geo propose alternatives (Mercat et al., 2009).

Finally, one should note that the resourcing process is also made of simple *visits*: our teacher has created her resource at the end, which is less expensive for her than, for example, requesting the installation of new software.

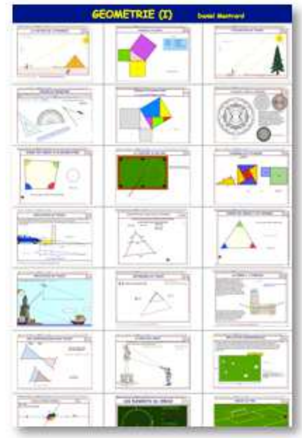

Figure 3. A collection of resources

That creation has been clearly supported by the resources that she has viewed before. A *dark re-use,* as coined by Wiley (2009), has happened.

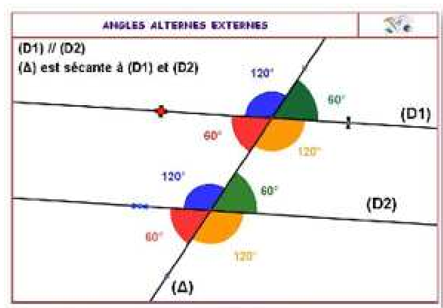

Figure 4: Resource chosen by the teacher

Exponential: Cross-lingual topic search

Log-book: JdB-fctexp-euler.pdf

Our teacher intends to find supporting material for the introduction of the exponential function applying the Euler method. First she searches for the *exponential function* topic.

This search finds the resources that have been annotated with that concept. She only finds two resources, only one is interesting (Exponentialfunktionen [6]) but it is in German (Fig. 5). Nonetheless, she attempts it and tries to understand it but grasping it enough from the geometric aspects alone was impossible, she gave up.

Figure 5. Two resources responding to the search for "exponential function"

She then searches in plain text in several attempts. At the beginning, the results list is big and full of unrelated results (because such words as functions are very common and words that match fuzzily are also included). Finally, she finds how to search for a "phrase" putting quotes around it. The resources she finds are, however, insufficient. She then uses a generic web search engine, Google, and operates the same resourcing process. She finds the tenth result to be appropriate. This resource fits her needs, she is easily able to deploy it to her students and adapt it as needed. She contributes it on i2geo (Introduction... [7]).

Metadata: topic, title, description (displayed, searched, input)

Actions: done: select, adapt, publish, deploy, rate.

It should be noted that the topic annotation is a very precise search ingredient: it allows to search with almost no error but often misses some resources which have not been tagged appropriately. Similarly the query for “fonction exponentielle” (as a phrase) is much more precise and misses results which, for example, do not contain this exact phrase. Such an ambiguity is recurrent and not fully solvable unless a considerable effort is made into polishing the annotations of the resources, for example by supporting is encoding by applying suggestions based on text analysis.

CONCLUSIONS

In this paper, we have proposed a coarse model of computer functions and of didactical functions of metadata. We have applied this model to the log-books reporting early activity of the i2geo platform successfully: the resourcing process described there is entirely dependent on the quality of the metadata records.

These log-books have shown the tricky role of the metadata: when read, it must be expressive enough for resources to be easily identifiable, still it must be easy to input. They also have shown that the criteria to choose a given resource to be applied in a teaching situation include all fields of the metadata that can be searched or displayed; these fields also include the didactical facets of the resource (in particular, the available documentation) and the compatibility to the technical environment.

These log-books have also shown a premise of the metadata that is often forgotten: its goal is to form a catalogue, and this catalogue should be informative. If a search result shows information that does not allow recognising the resources contents, it is likely to require extensive manual skim through all the results. This implies that a person that inputs a good metadata is one that knows the available content well.

This study has also shown a role of metadata which is completely different than that of enabling the *automatic assembly of learning resources* (as expressed in early visions such as those quoted in Wiley (2001)): the metadata display forms a step in the selection process, where the teachers’ expertise plays an important role.

Finally, these log-books have shown us interconnections between select, publish, adapt, and deploy actions: all teachers' log books demonstrate that previous actions have influenced the next ones, even if they were a selection that lead to a rejection.

NOTES

1. http://opendiscoveryspace.eu

2. The name *application profile* is generally used to describe a structure of metadata that partially follows and extends a previous metadata structure.

3. These logbooks can be read, in French, at http://i2geo.net/Coll_Group_IREM-INRP-AcademiedeLyon/LogBooks.

4. http://i2geo.net/xwiki/bin/view/Coll_sbourdal/InegaliteTriangulaire

5. http://i2geo.net/xwiki/bin/view/Coll_Group_IREM-INRP-AcademiedeLyon/Anglescorrespondants

6. http://i2geo.net/xwiki/bin/view/Coll_DRaab/Exponentialfunktionen

7. http://i2geo.net/xwiki/bin/view/Coll_ileyraud3/IntroductiondelafonctionexponentilleparlamethodedEuler

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DIDACTICAL DESIGN PATTERNS FOR THE APPLICATIONS OF SOFTWARE TOOLS

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We report on didactical design patterns for mathematics education with a special focus on the impact on the software construction process. We describe how such short texts, in the first instance developed for educators, can also be used by developers of learning software. We claim that such an application supports a more effective usage of the tools, thus raises their quality.

The paper presents one of the design patterns in details, Representation-on-Demand, and illustrates its applications through several tools.

INTRODUCTION

Numerous software learning tools have appeared in the last ten years. Just as technology found its way into society (personal computers, mobile phones, tablets,...), technology found its way into the learning places. But what is the best way to learn with learning software? This question has not been solved till now. Learning software is not yet commonplace in daily school lessons or at university lectures, even though there are many well developed learning tools.

In the authors' experience, from the view of university education, many students still prefer working with pencil and paper. On the one hand using technology and learning tools is a new concept to the students. Before working on a task they have to learn to handle the program. Therefore it seems faster for them to get a solution by hand. On the other hand, students argue that in the end the exams also have to be done by hand. Using learning software makes no sense, because no matter what, you have to work on a task with pencil and paper to prepare for the exam. Additionally, students in teacher education argue that these programs are not going to be used in school, therefore there is no need to use them. As a result, for many students the learning tools do not connect to the learning behaviour.

Hence, learning tools, which we shall understand as any software that helps learning, have to be designed so that students will use them. They have to follow the learning behaviour of the students. We claim that learning tools could be enriched, by using the same patterns that are used by teachers and educators. In this paper, we propose the dissemination of short texts describing patterns in the learning processes.

The didactical design patterns are the focus of this paper. They form a collection of short and readable texts, which members outside the (mathematics) pedagogy community can exploit. We are particularly interested in the applications of the didactical design patterns to the learning tools' development and documentation: we observed that the language to describe the learning tools may lack the concepts of the mathematics pedagogy and may sometimes be far from the users' day to day tasks.

A common language is needed for the technology enhanced learning and the mathematics pedagogy research and practice communities to understand each other. This has been noted by Noss (2009) who stresses the importance of interdisciplinarity for the field of technology enhanced learning. A few initiatives are emerging to help summarizing the branch in the form of encyclopedic knowledge. Among others, on the computer science side, the TEL Thesaurus (http://tel-thesaurus.net/) and the interaction-design foundation (http://interaction-design.org/) are both present, and on the mathematics education side, one finds MaDiPedia (http://madipedia.de/) or the ReMath scenarios (http://remath.cti.gr/).

The didactical design patterns we present here are a form of common language with a different perspective than the works above: they depict recurrent schemes which are backed by literature in pedagogy. Their *usage* by software designers or teachers should be simple and backed by a sufficient vocabulary that enables ease of discussion. We attempt to document possible application processes in the software construction and documentation.

Our contribution connects the three themes of the working group: it contributes to the design and use of technologies by proposing patterns to raise the quality of the learning experience (theme 1.1). It also provides means to raise the impact of the use of the technologies in their learning (theme 2.1). Finally, it proposes patterns that support best practices in using the technologies (theme 3.3).

OUTLINE: This paper first introduces the principles behind didactical design patterns and the concepts they relate to. It then describes *Representation on Demand*, the central pattern of this paper. Two of its exemplary implementations are then described. In the conclusion, future research directions are outlined.

DIDACTICAL DESIGN PATTERNS

In general, design patterns were first described in the subject of architecture by Alexander (Alexander., Ishikawa, & Silverstein, 1977). Even there, they were used long times before. The issue of design patterns was to give a general solution for recurring problems when constructing buildings. This approach was adopted in the 90s by computer scientists (Gamma, Helm, Johnson, & Vlissides, 1995) for recurring problems they encountered when writing programs, e.g. algorithm or programme concepts, which are used in many programs. For these, the design patterns are still used today in these fields. Therefore, design patterns are used to describe possibilities to solve challenges [1] on pedagogical or didactical [2] problems. This does not mean, that they are an instruction that says, after doing this all is fine. They provide (theoretical grounded) hints and outlooks in which way the challenges can be solved.

To describe these kinds of challenges with a possible solution as a pattern in education is not new. The "Pedagogical Patterns Project" (Bergin et al., 2012; www.pedagogicalpatterns.org) contains a lot of very broad patterns relating to pedagogy. Vogel and Wippermann (2004) as well as Niegemann and Niegemann (2008) introduced didactical design patterns as a possibility to document didactical knowledge. The advantages of patterns are that they provide named short and repeatable approaches to a (pedagogical) problem. Thus, they are easy to read, understand, and thus to use by everyone, especially teachers to optimize learning scenarios and processes.

Figure 1 depicts how patterns are situated to other texts in terms of their applicability, didactical diversity (the diversity of situations it can be employed in), and their coverage (how many dimensions does it take in account).

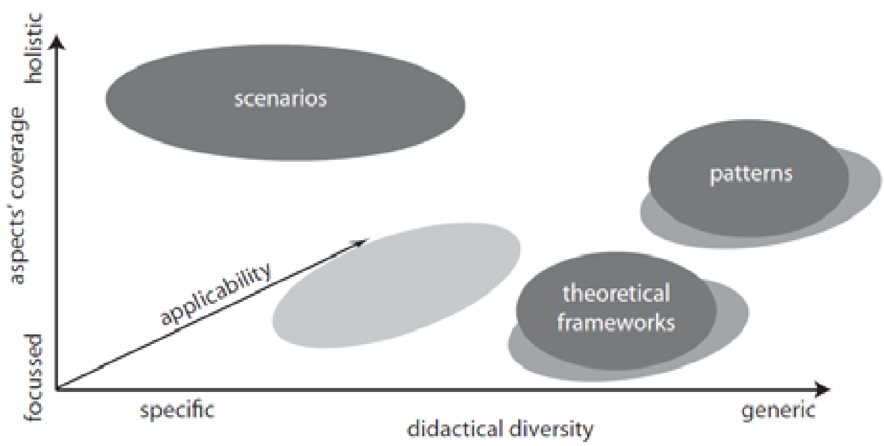

Figure 1: Localisation of didactical design patterns, compared to learning scenarios.

In contrast to the pedagogical patterns (c.f. Bergin et al., 2012), which describe challenges in learning scenarios in a very broad way, e.g. students should be active while learning (p. 17, ibid.), didactical design patterns described in this paper are somewhat more precise to the learning process. Didactical design patterns differentiate themselves from learning scenarios in that they are abstract and describe a more general situation. They more focus on didactical principles in learning scenarios, e.g. when does a learner need to get a hint on a problem (Zimmermann, Herding, & Bescherer, 2013). The patterns are the work of the SAiL-M project (Semi-automatic Analysis of individual Learning-processes in Mathematics). They all are available on the web at www.sail-m.de and at sail-m.i2geo.net. The availability of their text as simple web-pages with a short title, a direct URL, and under an open content license makes them considerably easier to mention in electronic communication, a fact of growing importance.

There are different and various styles to describe didactical design patterns the patterns developed in the project (Bescherer & Spannagel, 2009; Bescherer, Spannagel, & Müller, 2008) follow a structure made up of the following ingredients:

- **challenge/motivation** (problem): The issue intended to be solved is introduced.
- **forces**: Factors influencing the described problem and, therefore, no easy solution is possible.
- **solution**: One (general) recommendation is formulated.
- **rationale**: Theoretical reasoning on which the possible solution is based on. It helps readers to dig out justifications of particular aspects.
- **examples**: Precise situations where the pattern has been successfully used.
- **related patterns**: Connection to patterns that are relevant when applying this one.

DIDACTICAL DESIGN PATTERNS IN SOFTWARE DEVELOPMENT

As well as in school, didactical design patterns can be used in the develoment process of learning tools. Didactical design patterns can be interesting because they represents a compact set of guidelines for several types of applications, which focuses on the essential aspects of the learning processes. Software developers are left free to use them when creating a tool but should always be aware of an environment where the pattern can take place.

For example it is important in a learning process to get hints and/or feedback and learning tools often provide them. But the hints that are given by the tools should not reveal the whole solution at once. They have to be given in different levels of detail and have to be individual to the learner's problems. This is an approach a human tutor normally applies when helping students working on tasks. Hence learning tools should also provide hints and feedback at different levels and provide them when the learner demands it. The Hint on Demand Pattern (Zimmermann, Herding, & Bescherer, 2013) describes how this didactical approach can be implemented in learning tools.

Another question is, when the help or technical support should be given to the learner. Giving the help beforehand, because there are problems where learners normally get stuck, is disadvantageous, just as giving it in the time a mistake occurs. The Help on Demand pattern (Bescherer & Spannagel, 2009) describes that help should be given just in time and when the learner actively demands it.

THE REPRESENTATION ON DEMAND PATTERN

The didactical design pattern Representation on Demand gives advices to educators as well as learning tools developers to get the highest possible impact of learning. Most learning requires forms of representation such as written symbols or diagrams. For that reason, in lectures, lessons or even learning tools, multiple representations of the same (mathematical) objects should be provided. How this problem can be solved is described in the following didactical design pattern.

CHALLENGE / MOTIVATION (PROBLEM)	Contents, e.g. in mathematics, can be represented in many and different ways. For example, functions, in mathematical contexts, can be represented by algebraic terms, graphs, arrow-set-diagrams or a value table. E-learning tools only provide representations of the content, which fit to the context or the exercise. But learners are mainly on their own when working with e-learning tools, therefore the tool should offer more or all representations that the learner needs. With regard to theories of information reception, every learner should have the possibility to choose his/her best known and most comprehensible representation of the content according to their learning style. Only this way it is ensured that the learning potential is applicable.
FORCES	Offering all possible representations of a content to learners, e.g. in a lecture, costs a large amount of time. Usually, this time is not available or other contents have to be eliminated. Additionally, not all representations are needed. In many cases one or two representations are sufficient, and additional representations of the same content are boring for the learners. Multiple accesses to a topic are not always beneficial to all learners. In particular, the weak learners or students at the early learning stages get confused and overburdened by too many representations. Different descriptions of content require different approaches and perceptions, and therefore flexible handling of them. In computer tools displaying all of the representations of content would take too much working load and would take away “the view on the essential”. The GUI of a computer program would be too crowded and would move the main focus to the background. Users first have to get familiar with all of the representations, before he/she can start working or learning.
SOLUTION	In lectures one rarely has time to introduce more than two representations of a concept. One would, otherwise, loose too much time or having lectures only for showing representations. However, lecturers should provide two representation formats which are commonly used in the literature. Nonetheless, additional

	representations can be outsourced to bulletin boards or Learning Management Systems (LMS) and offered to students when needed. The learners can access these representations of content on demand. In computer learning scenarios or e-learning tools you can implement different kinds of representations. Beside the most commonly used presentation formats, the learner can enable additional representations when needed. In addition, the learner can try some of the yet "new" or unknown representations to obtain a new access to the content.
RATIONALE	When learning new content, several representations are often met. "Representations are any thing that stands for something else" (Schnotz, 1994). Manuals for technical products provide representations just as school content does. By connecting more representation formats of content the information level can increase (Kaput, 1989). Multiple representations can complement one another (Ainsworth, 1999) and contribute to a deeper comprehension. Different people need different representations concerning learning a new content (Vester, 1998; Bruner, 1968). As results of research on learning behaviours, learning contents should be presented with different representations for every type of learner. Hence, it can be ensured that the student can learn and assimilate the content optimally. The cognitive load-theory of Chandler and Sweller (1991) suggests that the working memory of our brain is limited. New information is first stored and processed in the working memory, and afterwards transferred to the long-term memory. Too many representations of content will overload the working memory and there is no space left for the learning content.
EXAMPLES	See following section "Actual Applications providing representation on demand".
RELATED PATTERN	Hint On Demand (Zimmermann, Herding, & Bescherer, 2013); Technology On Demand; Feedback On Demand (Bescherer & Spannagel, 2009)

Table 1: The representation on demand pattern

ACTUAL LEARNING TOOLS APPLYING REPRESENTATION ON DEMAND

Primarily, the didactical design pattern Representation on Demand was made to make learning for students easier by providing multiple representations of the contents of the lecture. For example, students can get additional materials through a learning management system (LMS) or in their weekly recitation groups, which contain a form of representation of a similar context to the lecture. Also, learning programs can be provided this way to show another perspective on the content.

Furthermore, learning tools can support the students learning if they follow the concepts of the didactical design pattern Representation on Demand. The learning tool has a chance to honour the learners' needs and demands by orientating on the patterns. Learning tools would feature more than one form of representations of the learning content but only make supplementary representations available if the students demand it.

The project SAiL-M (Semi-automatic Analysis of individual Learning-processes in Mathematics) has investigated applications on the didactical design patterns to the development and evaluation of the learning tools in teacher education in Germany from 2008 to 2012. The project has not only pointed out patterns (e.g. Bescherer & Spannagel, 2009; Bescherer, Spannagel, & Müller, 2008; Zimmermann, Herding, & Bescherer, 2013, also available on www.sail-m.de), it has applied them in an exemplary manner in the development of several learning tools and has evaluated the applicability of the patterns for them.

With the e-learning tool ColProof-M (Bescherer, Herding, Kortenkamp, Müller, & Zimmermann, 2012) students can verify simple geometric proofs, e.g. Thales' Theorem. The learner has to arrange a set of given logical propositions, and to state why each proposition is valid (fig. 2). On the one hand, the propositions are given in (mathematical) short notation as well as in plain texts. That means that those weak students can also work on the proof even if they do not feel confident with the mathematical notation. On the other hand, the students have the possibility to display the statement they have to prove via the dynamic geometry software (DGS) Cinderella (Richter-Gebert & Kortenkamp, 2012). Additionally, elements corresponding to the chosen proposition are highlighted. Every type of learner can select their way of representation when working on the task, using (short) mathematical notations or geometric manipulations.

SQUIGGLE-M (Fest, Hiob, & Hoffkamp, 2011) is a learning tool for the concepts and the properties of functions, also developed in the SAiL-M project. The software consists of several open learning laboratories. Each of them outlines a property of a function illustrated by one or more interactive forms of representations of the function. These representations also employ the DGS Cinderella. In some laboratories the function is represented as a term, a graph and a diagram (fig. 2), thus the learner can switch between these representations and get them connected or just choose the preferred representation.

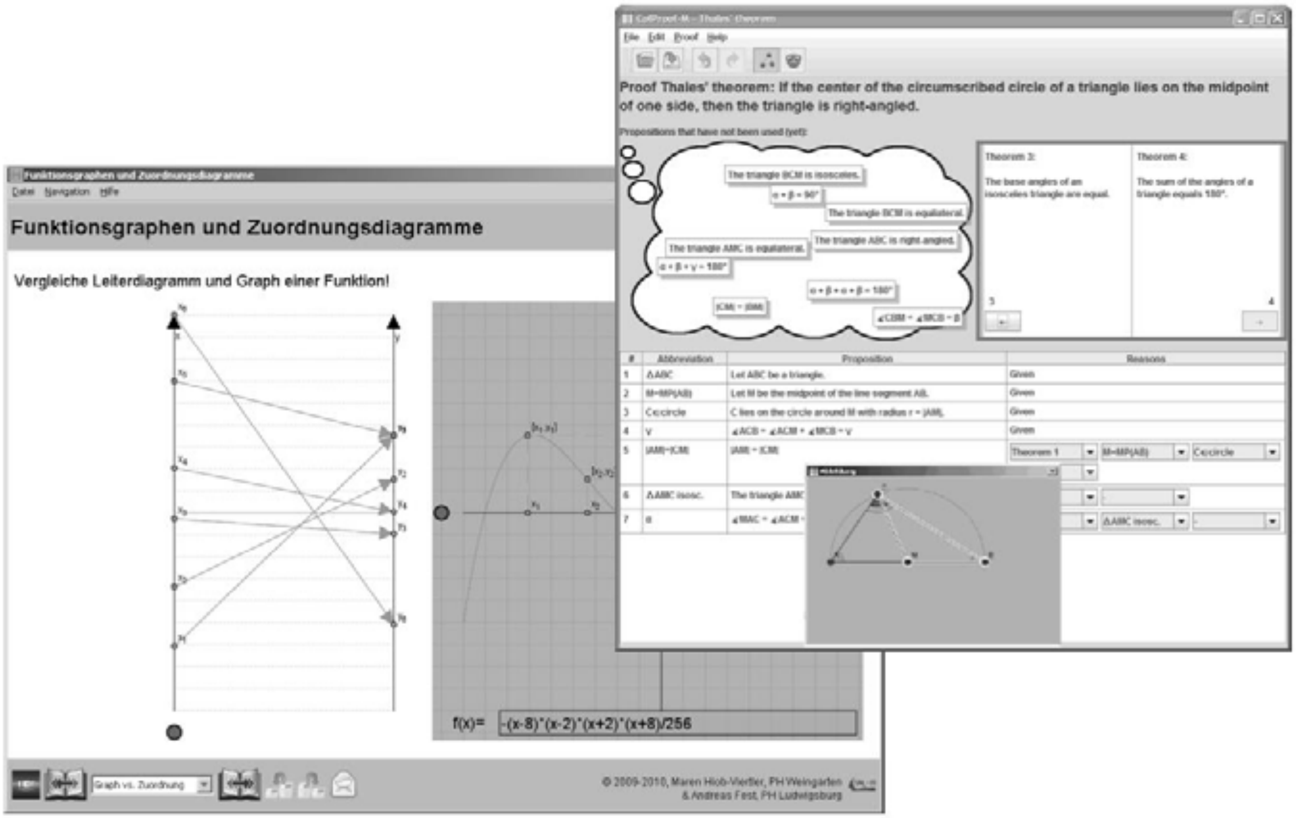

Figure 2: SQUIGGLE-M and ColProof-M with multiple representations

CONCLUSION

Several other didactical design patterns have been contributed within the SAiL-M project, most related to the usage of computer based learning tools. We refer to www.sail-m.de/sail-m/Patterns. Among others, Hints on Demand and Feedback on Demand are patterns that lie in the centre of the semi-automatic-analysis principles that have launched the project. They have been implemented in several learning tools which are run within a learning analytics architecture (Libbrecht et al., 2012). The Feedback on Demand pattern is implemented by a contact-teacher feature in the learning tools; feedback can be provided by the teacher because he can view the previous steps of the learning processes before responding.

This paper is a small contribution towards a greater visibility of the mathematics didactics to a broader public. The didactical design patterns we have outlined in this paper offer a simple and readable view of outcomes of the research in mathematics education. Their interpretation may support the software design process: the patterns may be embodied in user stories, their vocabulary may support the designation of software components or processes in as an interaction diagram. Examples of such user stories can be read at http://www.sail-m.de/sail-m/MoveIt-M_en but more research is needed into generalizing the application processes.

The contribution of this paper is, at the same time, an invitation for the mathematics education community to employ the format of didactical design patterns to describe mechanisms of the learning process as it appears to be appropriate to support the software construction process.

ACKNOWLEDGEMENTS

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NOTES

1. In this article, we replace the name problem by the name challenge, because in pedagogy there is never one problem which occurs in the same way and which can be solved in the same way all the time. Instead, when challenges occur, they have to be solved with respect to the persons and context.

2. The German idea of didactics (Didaktik) means the science of learning and teaching of a specific subject i.e. didactics of mathematics or didactics of foreign languages. This definition is more specific than the general concept of pedagogy. In German, as in many of continental Europe's language, the negative connotation of didactics is absent.

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THEORY OF DIDACTICAL SITUATIONS AND INSTRUMENTAL GENESIS FOR THE DESIGN OF A CABRI ELEM BOOK

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The contributions of two theoretical frameworks (Theory of Didactic Situations and Instrumental Genesis) to the design of a sequence of tasks in the Cabri Elem environment, where task and technology design are closely linked, are shown. Considering the potential for instrumental genesis as a theory of technology design reveals a fundamental difficulty in dealing with representations. It is hence suggested that the role of the artefact be broadened to include environments, tools, and entities.

INTRODUCTION

The first part of this paper consists of a summary of some aspects of our ICMI-22 submission on task design, in which we analyzed a particular sequence of tasks created using Cabri Elem in order to illustrate the interconnections between the affordances of the technology and the ability to implement particular didactic principles. Our choice of theoretical frameworks to use for this analysis was based on an analysis of tasks conducted as part of the Intergeo project (Trgalova, Soury-Lavergne & Jahn, 2011), which used Brousseau's theory of didactical situations (TDS) (Brousseau, 1998) together with instrumental genesis (IG) (Trouche, 2005).

In part 2 we begin a study of instrumental genesis as a theory of task/technology design with the aim of both creating greater links between TDS and IG and enabling IG to become a more effective framework for design.

PART 1: TECHNOLOGY AND TASK DESIGN

Cabri Elem technology was created to serve the needs of primary students and also to enable the creation of "applets" in order to support teachers to engage more confidently with technology (Laborde and Laborde, 2011). It is a task design environment in which "activity books" consisting of a succession of pages incorporating a sequence of tasks may be created, and a more restrictive task performance environment in which activity books may be used by teachers and students. Cabri Elem has the affordances of earlier Cabri technology for direct manipulation of geometrical objects and numbers, together with additional features such as 3D models and tools. A major difference between Cabri Elem technology and other dynamic geometry software, and also other generic technology, such as graphical calculators, CAS or spreadsheets is that the user interface of the task performance environment is under the control of the activity book designer, who must decide which objects (tool icons, images, text, geometric figures, etc.) to arrange on initially empty pages, and who may program control actions on these objects. Creating an activity book hence involves issues of both task and technology design.

The theory of didactic situations

In this theory (Brousseau, 1998), knowledge is a property of a system constituted by a subject and a "milieu" in interaction. Learning occurs through this interaction: the subject acts within and receives feedback from the milieu. Technology, or the part of technology relevant to the mathematics concerned, may form part of the milieu, and the milieu related to a student changes as student knowledge, both technical and mathematical, develops. With a learning task in a technology environment, the author determines the possible milieu and hence the potential for learning by creating all the elements the student will deal with: the objects the student will manipulate, the possibilities of actions on these objects and the feedback provided by the environment.

Key aspects of a didactical situation are the mathematical problem and the choice of didactical variable values to set for the task, where the task involves learning objectives and the mathematical problem. The teacher assumes that achieving the task will cause the student to learn. The goal of a task, whether teacher or student determined, should be clear, together with criteria for success or failure. A task is performed by concrete and conceptual student actions, with the existence of a space of uncertainty and freedom for the subject about appropriate action and strategy. This contrasts with the common dynamic geometry tasks such as "drag this point and observe" where the student has no choice of action and is uncertain about what is relevant to observe. The task corresponds to phases of the didactical situation and is related to different values of a set of didactical variables. Didactical variables are parameters of the situation, with values that affect solution strategies. The effects can be of three kinds: (i) a change in the validity of a strategy, where a strategy that produces a correct answer with a certain value of a didactical variable will produce an incorrect answer with another value, (ii) a change in the cost of the strategy (for example counting elements one by one is efficient for a small number but much more costly for a larger number) (iii) the impossibility of using the strategy. A combination of the different didactical variable values contributes to the task definition. The learning situation is a choice of different tasks that lead the students to construct the appropriate strategy. Thus task design will consist, for a part, in identifying the didactical variables of the situation and then choosing the succession of appropriate combinations of didactical variable values.

Instrumental genesis

Learning situations involving the use of technology may be modeled as instrument-mediated activity situations (Rabardel, 2002), wherein the subject acts upon an object either directly or with the mediation of an instrument, which consists of an artefact together with the subject's utilization schemes. Instrumental genesis (IG) is the main aspect of instrument-mediated activity situations to be considered in the literature, and is the process by which instruments are developed from artefacts, through instrumentation (the development of utilization schemes) and instrumentalization (using the artefact for new purposes). Instrumental genesis, originating in ergonomics (Rabardel, 2002), is well established as part of the instrumental approach (Artigue, 2002) dealing with the integration of technology in the mathematics classroom, but also, being derived from the work both of Vygotsky and of Piaget (Rabardel, 2002), has links to socio-cultural approaches and, less explicitly, to constructivist approaches and constructionism.

An issue with instrument-mediated activity situations is that the "*object*" can be interpreted either as the goal of activity, consistent with the activity theory of Leontiev (Kaptelinin & Nardi, 2006, p. 59), or as the "*thing*", not necessarily concrete, upon which the subject acts, which is the main sense in which Rabardel (2002) uses the word (e.g identifying object with wallpaper and ceiling (p. 43)). In considering learning tasks in technology environments, the *object-as-goal-of-activity* and the *object-as-thing-acted-on* are both of importance. We will refer to the former by the word *goal* and to the latter by the phrase. Note that the *object-as-thing-acted-on* will vary as the subject's actions change; nothing is intrinsically an *object-as-thing-acted-on*. We will use the word object without italics to refer to entities that may become *objects-as-things-acted-on* during the course of the activity: this is consistent with the way it has been used in describing TDS above.

The Cabri Elem Task

We will now look at an activity book and discuss the links between this sequence of tasks and the theoretical frameworks from which the tasks were generated. The "Target" activity book addresses the French primary school level CE1 (7 year old students) and the *goal* for the teacher is for students to learn about the representation of numbers using place value notation. The idea arose from comparing counters on a scoreboard, where the value of the counter depends on its position on the board, with the way that the value of a digit depends on its position in a written number. It was designed by a team of ten researchers (including two of the authors of this paper), teacher educators and teachers involved in a French national project [1] whose purpose is to create resources for the teaching of mathematics in kindergarten and primary school.

The process of creating the activity book involved elaborating the milieu by choosing appropriate objects, possible actions and resultant feedback. In our example, the objects are essentially the scoreboard with three different regions, the counters, the target number and the score, as shown below.

Figure 1. Title page and a task page from the "Target" activity book

The actions on the objects are simple: dragging the counters, clicking on a button to reset counters and get a new target number or to get an evaluation.

Didactical variables played an important role in the task design process. Some were identified a priori, while others emerged during the design process as the authors became more aware of what aspects of the situation could be changed. Once a potential variable was identified, an analysis of the ways in which this variable could be changed produced a better understanding of the possible tasks and their consequences. It also enabled the creation of strategy feedback.

Three kinds of feedback were essential to the activity book design. Evaluation feedback is related to the achievement of the task or part of the task. Strategy feedback aims to support the student in the course of task resolution, like scaffolding (Wood et. al., 1976). It is a response to the strategy used by the student. The authors needed to identify (i) configurations of objects that were typical of a strategy and hence enabled a diagnosis and (ii) new objects or actions that could be provided to help the student without changing the nature of the task. Such feedback could consist of help messages, or a graphic enlightening of contradictory elements. Another possibility is to modify the values of didactical variables in order to make the student aware of the current strategy limitations. Direct manipulation feedback is the response of the environment to student action, and may serve the function of either of the previous types of feedback.

The first page of the resultant activity book, shown in Figure 2, is a title page. In page 2, the main objects are presented. The student may interact with these objects, by dragging counters to different positions on the scoreboard and noticing how this affects the score. This is dynamically calculated: one, ten and one hundred for each counter in the green outside region, the purple intermediate region and the orange central region respectively. The aim of the page is to give time for instrumentation to both teachers and students. They can explore interactions with the objects that will constitute the milieu without the constraints of a particular task. It also contains a reset button which, when clicked, replaces counters in their initial positions, and a button which allows students to move on to the next page.

The changing score is direct manipulation feedback that shows students not only the effect of their action, but also that action on one object (moving a counter to a different region) will affect another object (the score). The score is always displayed in some pages, but displayed only after a specific sequence of actions in other pages.

On page 3 the student first receives evaluation feedback. A specific task is given: to reach a score equal to a target number, randomly generated between 1 and 999 (see Figure 1). Clicking on the reset button now in addition generates a new target number. Another new action is that the student may, in addition to comparing whether the score matches the scoreboard, click on a new button for evaluation feedback: a red frowning face if the answer is wrong, and a yellow smiling face if the answer is correct. In case of failure, the student can continue to drag counters and ask for a new evaluation: a new smiley will appear to the right of the previous one. It is important that new feedback is only generated at the student's request: otherwise a trial and error strategy not stemming from mathematical considerations could lead to success.

From page 4 to 7 students are no longer given the direct manipulation feedback of seeing the score. They hence need to take into account the value of the counters in the different regions of the scoreboard to determine the score. "Score" was identified a priori as a possible didactical variable, with two values: visible or hidden.

In page 5, the number of counters is reduced so that, if the target number is over 27, a strategy that consists in placing counters only in the green units region will fail. A strategy which takes into account that a single counter can have another value than 1, i.e. using the inside regions of the scoreboard, is necessary. Therefore, another potential didactical variable is identified: the number of available counters, with two values, 3x9=27 and >27. In page 6, the target number is a multiple of ten, between 10 and 990. As there are enough counters to either leave the green region empty or to fill it with multiples of ten counters, a change of strategy is not necessary. In page 7, however, a single counter is fixed in the green region. Therefore, new strategies are required, involving the placement of a multiple of ten counters into the units region of the scoreboard. The "fixed counter" didactical variable is identified, with four values: no fixed counters, or fixed counters in the units, tens, or hundreds region.

Page 8 contains input boxes for the student to enter the values of a counter in each region of the scoreboard. The aim of this task is to summarize the key idea of the activity book, i.e. that the value of a counter depends on the scoreboard region.

Other pages of the activity book are not devoted to student tasks. The first page is a title page showing an iconic representation of some of the main objects. Pages 9 and 10 contain commentaries for teachers, reporting the main aspect of the task, the evolution between pages, possible student strategies, and also the solution. The didactical variable analysis helps to determine what information is useful.

Trialing the Activity Book

This occurred in spring 2012 in two primary school classes: CE1 with the version presented here and CP (six year old students) with a version where the target number size was limited to 99. Teachers used the activity book as one resource for learning about place value and instrumentalized the book by printing pages to construct related paper and pencil tasks. They were enthusiastic about student engagement, mathematical reasoning and the evolution of strategies, but raised a number of issues.

It was expected that the strong metaphor between the task situation and real scoreboard situations would both provide a meaningful context and minimize the need for instrumentation. Students expected, however, that moving a counter would require tossing it in some way and were initially uncertain about how to do this using the software. Teachers also proposed that instrumentation would be enhanced by modifying page 2 to include a target number chosen either by the teacher according to the constraints of the class, or chosen by students in order to challenge each other.

Some students used the target number update not only to get a new number after finding a previous target but also, unexpectedly, to get a number they knew they were able to deal with, showing the ability to diagnose their level of expertise. This is an example of students' instrumentalization that has led the designers to modify the task in two ways: to provoke the task achievement for each target number, but also, in some pages, to enable the students to choose the target number. The possibility for students to adapt part of the task to their level of expertise is a new, generalizable element in activity book design.

The number of available counters was not a didactical variable for most CE1 students, who used each region of the scoreboard and limited the number of counters they needed to drag. Many of them did not notice the reduced number of counters on page 5 and were surprised to apparently have to solve the same task again. However, for many of the younger CP students who used only the units region of the scoreboard the number of available counters was indeed a didactical variable. The status of page 5 will hence be changed in further developments of the book. Instead of being automatically displayed to CE1 students, it will only be displayed as necessary, i.e. if the unit region is repeatedly filled with many more than 10 counters. The strategy feedback, resulting from our analysis in terms of didactic variables, will consist in reducing the number of counters to better fit the sum of digits of the target number and choosing a target number over 50.

Discussion

Both TDS and IG provide a useful lens to explore aspects of the design of the task. However, according to Prediger *et al.*(2008), there is not an integration between the two theories in our above analysis, but rather a coordination: for example student instrumentalization (described by IG) will affect solving strategies, and hence the milieu and the learning, as described by TDS. Identification of possible instrumental geneses should hence contribute to an a priori analysis in the framework of the TDS. A further issue is that technology design issues, crucial in the Cabri Elem environment, are not readily addressed within TDS.

PART 2: INSTRUMENTAL GENESIS TO DESIGN TECHNOLOGY

In this part, we will first address the potential for IG as a theory of technology design, and then show how an extension to the role of the artefact may both resolve some of the issues in its use in technology design and enable further integration with TDS.

In the field of human-computer interaction, IG is already recognised as a theory for the design of technology (Kaptelinin & Nardi, 2006). General design principles are that artefacts should be designed for efficient transformation into instruments through enabling flexible user modification and through taking into account the real needs of users while appropriating the artefact. It also explores user contribution to design, particularly through instrumentalization. An example of a design principle from Rabardel's (2002) analysis is that in a professional situation action should be easy, safe and reliable, but in a learning situation, action might be constrained in order to promote learning. This has connections to the use of constraint in TDS and is relevant in analyzing the types of action and feedback that should be enabled.

For fifteen years, IG studies concerning technology in the classroom have produced analyses of constraints and possibilities, that could provide a base for a constructive critique of the technologies being used. However, IG in mathematics education has primarily been used to describe the use of existing technologies and to contribute to the design of tasks, by using their constraints to promote student learning (e.g. Fuglestad, 2007, using spreadsheets). It was mainly used to analyze complex environments, like CAS on the TI-92 (Lagrange, 1999), where researchers had no control over the technology. Thus IG has not been much used to explore how such technology might be designed. For instance, Lagrange (2011) discusses the design of Casyopée with no connection to a framework that he was one of the first to use (Lagrange, 1999), and CERME 7 reports on the design and development of new technologies do not mention IG. An interesting exception is the current constructionist exploration of the significance of instrumentalization in design (Healy & Kynigos, 2010).

A means for connecting IG and technology design can be found in a list of artefact affordances that are perceived to enhance instrumentation: to constitute exploration spaces, mediate between formal and informal, provide executable representations, offer dynamic manipulation, evoke interplay between private and public expression and generate interdependent representations (Kynigos *et al.* 2007).

However, the model of the instrument-mediated activity situation has a major shortcoming: the *object-as-thing-acted-on* is typically taken as being independent of the instrumented artefact that mediates the action. Hence, with the focus on the development of the instrument, the *object-as-thing-acted-on* is not problematized. This is appropriate in the ergonomic context in which the model was developed, where typically the subject is construed as a worker using a machine to create or manipulate a product. It is also appropriate at the level of analysis of the teacher-as-subject, whose activity is directed toward the students rather than toward the technological artefact; the entire artefact is part of an instrument used in achieving the *goal* of facilitating student understanding of a particular concept. However, for the student-as-subject the *goal* is given by a task that involves using instruments to interact with objects that are screen representations - and both the means to perform the action and the *objects-as-things-acted-upon* are contained within the same technological artefact.

We will hence consider a technological artefact which provides representations (such as on a screen) as consisting of an environment within which are the means of action and also the objects which are acted upon. The need to consider the environment has been raised both by Hegedus *et al.* (2007) and Trouche (2005). The environment also gives feedback as to the way objects change through interaction.

There is a link to the TDS concept of milieu: milieu is the share of the environment that has a mathematical signification for the student. The counters, target and feedback within the "Target" activity book environment are clearly parts of the milieu, while the button to click to move to the next page is not. Note that the environment provided by an artefact does not constitute the entire environment within which student action takes place. Aspects of the wider environment may form part of the milieu, while aspects of the artefact environment may not be part of the milieu.

Identifying the object, the means of action and the feedback provided as distinct within IG clearly connects the theory more closely to TDS. A question is whether the unique contribution of IG, that of genesis (loosely considered as developing cognitive schemes in order to more effectively meet the goals of the activity) can usefully be applied to the artefact environment and objects as well as to potential instruments. We will address this question by again considering the "Target" activity book.

One aspect of "environmental genesis" is the ability to navigate to different parts of the environment, as required to be able to move between pages in the activity book The title page contains text and images aimed at enabling the student to connect to a familiar activity in a real-world environment: other aspects of environmental genesis might be assimilating the environment to previously encountered environments and developing expectations as to the task and the type of actions relevant to the task.

Objects have a number of roles in the activity book, with different associated geneses. Objects which are icons (such as the reset button) provide a means of action: clicking on such an object will cause a particular action to occur. This action does not affect the icon itself. The "genesis" of such an object is instrumental. A second type of object (such as the counters) may be manipulated directly by means of an instrument (such as dragging) with feedback resulting. Objects (such as the score) cannot be manipulated directly but give feedback as to the result of actions on other objects. Interpreting such feedback requires some form of "meaning" genesis. In contrast to icons, the counters and score will not immediately be experienced as a means of action. But such as these may be linked to form both instruments for action and at the same time representations of mathematical concepts.

During the course of using the activity book, the student progressively develops the instrument of "using counters and target together to change the score" (instrumental genesis). By doing so, the student develops the understanding of the way in which the counters, target and score together constitute a means of action on the representation of a number in place value notation and also develops an understanding of the place value representation of the score.

It should also be noted that the development of the counters, target and score instrument and understanding is specifically facilitated through forcing student strategies to evolve via different types of feedback and changes in the values of different didactical variables: perhaps a consideration of feedback and didactic variables could also form a more prominent part of instrumental genesis in general.

CONCLUSION

In part 1 of this paper we have shown that two theoretical frameworks, TDS and IG, even though coordinated rather than integrated, contributed effectively to the design of a sequence of tasks in the Cabri Elem environment where task and technology design are closely related.

In part 2, it was shown that a technology such as a computer is not only an artefact to be instrumented, but consists of an environment which contains objects, such as representations, which are acted upon and can become means of action. This enables closer links with the TDS concept of milieu, containing the objects which the student will manipulate, as well as the possibilities for action on these objects, when they generate a mathematical meaning.

Consideration of the geneses of the environment and of objects has also been shown to be useful in analyzing the original task and the learning outcomes.

Representation is a major concern in the use of technology in teaching mathematics. Expanding the focus of IG from potential instruments to include objects such as mathematical representations will enable useful links to be made.

The distinction is also important for the designer. Potential instruments have design considerations such as accessibility, whereas representations have different considerations, such as their appearance, behaviour and the feedback given when they are manipulated.

In most software there is much more scope for creating and designing the representations with which students will interact than creating and designing the tools which they will use; this expansion may also be empowering for many teachers and researchers who would like to more actively engage with the design of technology in mathematics education.

NOTES

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INSTRUMENTAL GENESIS IN GEOGEBRA BASED BOARD GAME DESIGN

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In this paper I address the use of digital tools (GeoGebra) in primary school children's open-ended design activities. I discuss how open-ended transdisciplinary design activities can support the development of instrumented techniques, by considering the extent that pupils address mathematical knowledge in their work with GeoGebra. Furthermore, I evaluate how the pupils relate their work with GeoGebra and mathematics to fellow pupils and real life situations. The results show that pupils' consider development of board games as meaningful mathematical activity, and that they develop digital design skills with GeoGebra. Additionally, during the design phase the pupils consider the potential use of their board game by classmates.

DIGITAL TOOLS IN PRIMARY EDUCATION

The use of digital technologies, such as symbolic calculators, computer algebra systems, and dynamic geometry systems are changing teaching and learning of mathematics at different levels of the educational system. These tools lead to new didactical possibilities and create new challenges as well (Borba & Villarreal, 2005; Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010; Guin, Ruthven, & Trouche, 2005; Kaput & Balacheff, 1996). Most research on the possibilities and pitfalls with digital tools for teaching mathematics deal with students at the secondary and tertiary level, but recent research suggests dynamic geometry tools are relevant at the primary level as well (Sinclair & Moss, 2012). Furthermore, curricular development in several countries suggests that Information and Communication Technology (ICT) should be included in the primary level curriculum (for example, its use in Norway cf., Saabye, 2008). Hence, trends in research and curriculum development suggest it is relevant to investigate possibilities and problems using dynamic geometry tools in primary education.

Innovation and technological development drives a substantial part of the economy, and the application of mathematical concepts, models and methods to the developing cultural artefacts are of increasing importance. Such development calls for a more entrepreneurial attitude to the interplay between mathematics and technology in educational settings. This entrepreneurial approach has been addressed by relating education more directly to innovative disciplines (Rangnes, 2011; Shaffer, 2006), and, with the use of robotics and programming languages (Resnick, 2012).

In this paper I will investigate what kind of mathematics learning occurs in a case where pupils in primary and middle school use GeoGebra to design their own mathematical board game. The analysis is based on the instrumental approach (Guin, Ruthven, & Trouche, 2005), and focuses on three aspects of the pupils work designing games with GeoGebra: 1)The degree to which GeoGebra is appropriated to fulfil the students own need, 2) the pupils use of GeoGebra for investigating and describing mathematical concepts, and 3) whether or not the pupils consider their work designing games as authentic in the sense that it relates to a future use situation where someone is playing the game.

GAME DESIGN AS A MATHEMATICAL ACTIVITY

The empirical basis of the current paper is a pilot project for the project *Creative Digital Mathematics*. The main project introduces GeoGebra to primary school teachers by supporting their collaborative development of digital teaching materials for open-ended work with GeoGebra. The purpose of the pilot was to develop and test a format for creating such learning materials (called 'crushers'), as well as obtaining a better understanding of possible student learning when using such collaboratively authored material.

The pilot project has been running between March 2011 and November 2011, and been through two cycles of design and intervention (grade five = age 11 and grade three = age 9). In both cases this intervention was the first time these pupils used powerful mathematical tools in their mathematics class. In each intervention, the pupils developed their own board game using the tool GeoGebra. GeoGebra is a dynamic mathematical software that provides a close connection between symbolic manipulation, visualisation capabilities, and dynamic changeability of geometrical constructions. In this project GeoGebra was mainly used for its geometric capabilities, and less for dynamic and algebraic capabilities.

The involved teachers collaboratively authored the teaching material, while I provided additional inspiration and technical assistance. The tasks are simple instructions inviting pupils to use GeoGebra for a number of aesthetic and mathematical activities. A translation of the first and simplest scenario ("the fraction crusher" – designed for grade five.) can be found at the following url: https://sites.google.com/site/fractioncrusher/fractions, and the second scenario ("the multiplication crusher", Figure 1) can be found in Danish at the following url: https://sites.google.com/site/spilfabrikken/.

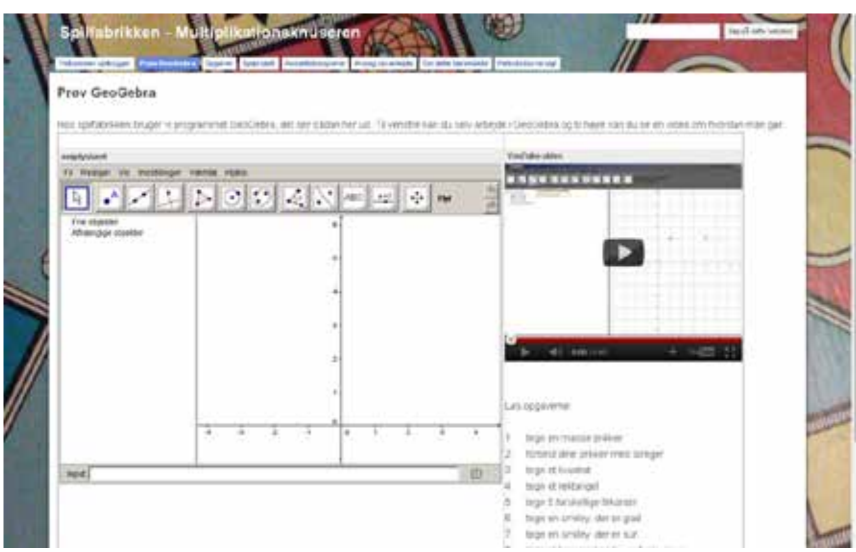

Figure 1: Screenshot from "multiplication crusher" with tasks (text field, bottom left), video-introduction (top left) and an embedded GeoGebra workspace

The board game design activity was an important part of the teaching material. The pupils made their visual layout of the board in GeoGebra, they were writing rules for the game, printing the game, and trying to play it with their classmates.

THEORETICAL FRAMEWORK

The intervention and research design is guided by two concerns with relates to different theoretical frameworks. The first concern relates to the didactical use of transdisciplinary, open-ended, and entrepreneurial practices that simulate professional activities, and the second concern deals with the use of digital mathematical tools in the classroom. The analysis that I present is based on the instrumental approach, and relates to the second concern. However, I briefly present the theoretical foundation relating to the first concern.

Shaffer has developed the concept *epistemic frame* to describe students learning in simulated work life situations. He describes an epistemic frame as a combination of values, knowledge, skills, and identity that people have when they are competent in such a work life situation (Shaffer, 2006). This suggests he views student's activities in light of professional skills, knowledge and values, and considers students learning as a result of adopting a certain epistemic frame.

The term *microworld* was first used by Papert (1980) to describe how the Logo software could reform primary and lower secondary mathematics education. Papert's approach was to use pupil's creative and aesthetic work, in a computer-based environment, as means to develop their skills in mathematics. He suggests that the combination of Logo's focus on computational procedures and geometric (aesthetic) output allows students to learn mathematics in interaction with computers when they are working to obtain their own goals. It is an important part of Papert's approach to teaching with technology that pupils produce (digital) artifacts as part of their learning process.

Kaput and Balacheff (1996) describes a mathematical microworld as a combination of a set of primitive objects and procedures that constitutes a formal system, as well as a domain phenomenology that determines the feedback that students receive from their on-screen work. In a GeoGebra environment the primitives are for example geometrical concepts such as lines, polygons, and circles, while the domain phenomenology relates to the dynamic aspect of geometric constructions, such as the dragging mode, as well as the consistent use of multiple representations.

Students appropriation of digital tools for solving mathematical tasks has been described within the instrumental approach to mathematics education (Guin, Ruthven, & Trouche, 2005). The instrumental approach builds an activity theory framework and views computational artifacts as mediating between user and goal (Rabardel & Bourmaud, 2003); additionally, use situations is seen as continuations of the design of the tool used. Hence, a pupil's goal directed activity is shaped by his use of a tool (this process is often referred to as *instrumentation*); simultaneously, the goal directed activity of the pupil reshapes the tool (this process is often referred to as *instrumentalization*) (Rabardel & Bourmaud, 2003, page 673). In order to relate the appropriation of tools in goal directed activities to learning of mathematics Luc Truche (in Guin, Ruthven, & Trouche, 2005, p.149), referring to Vergnaud (1996), introduces the concept of scheme as consisting of both a conceptual and a competence oriented aspect. Hence we can investigate the schemes in students' instrumented activity by studying the conceptual entities and involved competencies. However, such conceptual entities can be difficult to see empirically; hence, I will apply two more concepts from the instrumental approach. A distinction between *epistemic mediations* and *pragmatic mediations* (Guin, Ruthven, & Trouche, 2005; Rabardel & Bourmaud, 2003). Epistemic mediations relate to knowledge (Rabardel and Bourmaud uses the example of a microscope, and Lagrange (in Guin, Ruthven, & Trouche, 2005, ch. 5), refers to experimental uses of computers), and pragmatic mediations relate to action (Rabardel and Bourmaud uses the example of a hammer, Lagrange (in Guin, Ruthven, & Trouche, 2005, ch. 5) refers to the mathematical technique of "pushing buttons"). And finally I take from Rabardel and Bourmaud (p. 669) a sensitivity towards the *orientation* of the mediation. Instrumented mediations can be directed towards (a combination of) the object of an activity (the solution of a task) other subjects (classmates, the teacher) and oneself (as a reflective or heuristic process). Hence my theoretical framework consists of the concepts: *instrumental genesis* as consisting of *instrumentation and instrumentalization,* the concepts *of epistemic* and *pragmatic mediations* as well as a sensitivity towards the *orientation* of an instrumented mediation.

RESEARCH QUESTION

The research question that I address in this paper is:

How can the use of GeoGebra in pupils board game design activity, support pupils instrumental genesis with GeoGebra?

Furthermore, I will investigate *the types of mediations* that GeoGebra serves to the pupils:

> To whom are these mediations directed? To fellow pupils? The teacher? or Towards fulfilling the task?
>
> Are GeoGebra used for epistemic mediations, and what knowledge is involved?
>
> Are GeoGebra used as a pragmatic mediation and towards what actions?

These questions are guided by two hypotheses. The *first hypothesis* suggests the nature of GeoGebra as a mathematical microworld will support pupils use of it for epistemic mediations. The *second hypothesis* infers that engaging in open-ended design activities allows GeoGebra to act as a mediating artefact towards other subjects and not only towards solving specific tasks.

METHODS AND PROCEDURES

The methodology is design based, in the sense that we have been dedicated to an iterative approach and to the application of theoretically based analysis of learning goals and envisioned learning trajectory, as well as to the collection of empirical evidence (Cobb & Gravemeijer, 2008). The first game design scenario, the "fraction crusher", was developed with the mathematics supervisor from the partner school in the project, and used for a weekly ICT class that this supervisor taught together with a first language teacher. The intention of this intervention was to test the idea of board game design with GeoGebra as a mathematical activity among children in grade five. The data from this intervention consisted of the students' productions as well as reflections from the two teachers and the researcher who participated in some of the lessons.

The second scenario, the "multiplication crusher", was developed together with the mathematics supervisor of the school and two teachers in grade three who tested the design in their classes. The intention of this intervention was to further understand the mathematical learning potentials in board game design with GeoGebra in grade three and to test if the idea of using board game design scenarios and the "crusher" format would work with different teachers. The data consisted of minutes from meetings between the teachers, supervisor, and researcher as well as field notes and pictures from the researcher's participation in a total of 11 lessons in the two classes (six in one class and five in the other class). Both classes' spent 10 lessons working with the scenario. Furthermore, a research student conducted four in depth interviews with pupils participating in the "fraction crusher" and four interviews with pupils participating in the "multiplication crusher" (Rosenkvist, 2012).

DATA

In both interventions the pupils worked in pairs. All the (pairs of) pupils developed a game with a mathematical theme. For some of the pupils the mathematical theme was very weak, whereas for others mathematics was used very explicit in the game. The pupils accepted designing board games as a meaningful activity. The involved teachers found the pupils engagement and developed competence with GeoGebra as positive and valuable aspects of the intervention.

The interviews (Rosenkvist, 2012, p. 142-143, here shown in translated form) show that the pupils considered their work as mathematical work. Furthermore, the interviews revealed that the work they did in the interventions were very different from the normal mathematics lessons. The pupils in general felt that the GeoGebra classes where much freer, building more on their own ideas than normal classroom activities.

The following quote describes how one pupil experienced the relation between the activities in the intervention and mathematics:

Interviewer: What mathematics have you used when you made your game. Have you used math to do that?

Pupil: Yes, we have used mathematics. We have created the shapes of the game, we needed to make some shapes.

Interviewer: What kind of shapes?

Pupil: Mostly squares, we have also made some circles and some pentagons.

Interviewer: Have you used any other mathematics than shapes?

Pupil: Yes, we have also used calculations actually, when you landed on a field, then you might need to solve a task.

This example shows the two types of reasons that the pupils gave for considering their game design work as mathematical work. First, the use of GeoGebra for creating visual layout enforced the pupils to design through mathematical shapes, and hence to connect to mathematical concepts. Second, the task of creating a mathematical game did influence the students' design related discussions. The pupil in the transcript describes calculation tasks as a natural part of their gameplay. And from looking at the resulting games it is clear that adding tasks is a typical strategy for including mathematics in the gameplay.

Figure 2, provides two examples of the pupils' game-designs. In both examples the students are working in pairs designing a mathematical board game. In the first example the pupils are working with GeoGebra and in the second the pupils are working with pencil and paper to design their game before implementing it in GeoGebra.

In the first example the two students designed a game where you move around on tiles (the small circles), in different "worlds" (the larger circles and the corners). They point to a part of the board and say, "this is a multiplication world, here you have to solve three multiplication calculations and then you can fly on to the next world – which is an addition world."

In the other example two students sketched an initial game design with pencil and paper. They explain that in their game you have to throw a dice in order to get to a field; some of these fields have a multiplication task attached. If you land on such a field (for example, 5 x 5), the other player should count to ten while you calculate and say the result, and if you get it right, you can go on, otherwise the turn is given to the next player.

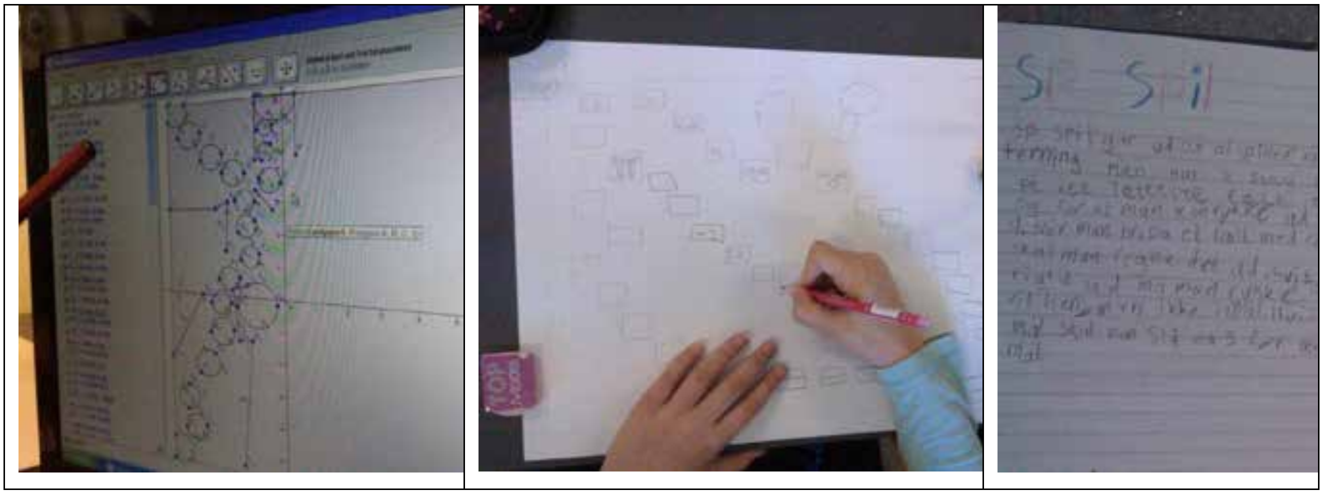

Figure 2 from left: (1) Two pupils have made a game where you move between different mathematical worlds. (2) Pencil and paper activities also play a role for some pupils; here pupils are sketching a game board, later to be drawn in GeoGebra. (3) Writing rules for the game.

This example shows another aspect of the pupils' activities; they write rules for the game. The rules written by the two pupils reads (translated from Danish): "The purpose of SP Game is that player number one throws a dice three times, and if you throw a '4' you can move to the nearest next field. When get to a field with a multiplication calculation, you should do the calculation right and then you can move all over the game board but not into the target zone. If you are next to the target zone you have to throw 5 with the dice in order to get into the target."

ANALYSIS

In this section I will analyze the presented data in order to answer the research questions.

The main research question of whether or not the board game design activity, supports instrumental genesis with GeoGebra, can easily be answered with a yes. All pupils were somehow able to use GeoGebra for something after the intervention. This observation is not entirely trivial. It could have been (but was not) the case that the software was too complicated or inappropriate to the age group or the task. However, it is contestable if the mere application of GeoGebra to a visual layout task can in any way be viewed as an activity that relates to the teaching of mathematics. Two aspects do suggest that this could be the case: First, the pupils were also doing a number of simple mathematics tasks with the software. We do not have data describing the development in the students' mathematical performance, but both observations and teachers' evaluation suggest that the pupils learned to use GeoGebra to visualise mathematical concepts and solve mathematical tasks. Hence, it is reasonable to conclude that the combination of board game design activity and mathematical tasks allowed the pupils to develop instrumented techniques with GeoGebra, which related to mathematical goals. Second, during the interviews the pupils described that the use of GeoGebra for developing the visual layout of the board game, forced them to reflect on aspects of mathematics. The piece of transcription provided in the data section is typical in the sense that the respondent points to the mathematical shapes as the way in which the software made the pupils design work more mathematical. This can be viewed as a process of instrumentation and seen as a result of choosing to work with GeoGebra rather than any other visual layout tool.

The makers of GeoGebra most likely have not considered the type of visual layout activity that the pupils engaged in when designing games. Therefore, the pupils often needed to find ways to make GeoGebra "do" various things such as change colour, fill figures completely, and remove points for aesthetic reasons. This can be viewed as a waste of time and as an example of a bad choice of software for the task. However, these activities also give the students an experience of appropriating a tool to their own need. Such an experience with instrumentalization can be of potential value to the pupils later since it suggests that mathematical tools are open-ended and can be appropriated to different situations in school and life. As an example, some of the fifth grade students on their own initiative chose to use GeoGebra as a part of an assignment in an English class where an illustration was needed (and proudly announced it to the mathematics teacher afterwards). In that sense data suggest strong signs of the process of instrumental genesis with GeoGebra as a result of the intervention.

It is arguable to what extent we see GeoGebra used for epistemic mediations in the board game design activity. The observed dialogue among the pupils and the questions posed to the teachers, were mainly of a pragmatic nature. By wanting the software to support the development of specific visual layouts, the involved mathematical concepts were not object of investigation in their own right. They were used to get GeoGebra to do what the pupils wanted. However one aspect of the game design can be viewed as a mediation of a more epistemic nature. Many of the pupils included mathematical tasks in their game. The SP game in figure 2 shows one example. When developing these tasks some of the pupils were explicit in their discussion about what a difficult mathematics problem is, and how such problems could make their game easy or hard. However, GeoGebra was not used as a mediating artefact in these discussions.

GeoGebra was used for epistemic mediations by some pupils in some of the mathematical tasks that were done as a part of the intervention before and after the board game design. The tasks that dealt with visualizing mathematical concepts task, such as, "Make a drawing, which compares 2/3 and 3/5: Which fraction is the largest? Draw ½, 5/6, 2/4, 6/8, 1/5, ¾, 4/6, 2/10" seemed to allow epistemic mediations.

When analysing the orientation of the use of GeoGebra as a mediating artefact, the situation of future use of the pupils' game became apparent. While the pupils' identification with professional designers was weaker than expected, their relation to the idea of their classmates playing their game was strong. In that sense the board game design did constitute mediation towards others. This mediation often did have explicit mathematical aspects because it included posing mathematical challenges as part of the gameplay, and because the appropriation of GeoGebra to create a functioning and aesthetic layout did involved working with geometrical concepts.

CONCLUSION

In this paper I have presented an analysis of how an open-ended design activity can support instrumental genesis with GeoGebra. The analysis suggests that board game design tasks support instrumental genesis, and allows GeoGebra to mediate to fellow students. Most use of GeoGebra as a mediating artefact for board game design can be characterized as pragmatic rather than epistemic mediation. We can conclude that board game design can be a way to introduce strong tools into mathematics teaching and learning in primary school. Such activities might lead to easy adoption of GeoGebra, familiarity with appropriating GeoGebra for different tasks, a positive attitude to mathematics among the pupils, and a re-scoping of primary level mathematics in direction where the discipline play a part in constructing cultural artefacts. However the fact that the pupils in the interventions reported here, mainly used GeoGebra for pragmatic mediations, suggests that open ended design tasks might not be well suited for as the students main activity, and should be complemented with activities that addresses the use of GeoGebra for epistemic mediation.

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A PROBLEM-SOLVING EXPERIMENT WITH TI-NSPIRE

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This paper focuses on a problem-solving experiment which was an important part of a study of how students and teachers use laptops in their classes with TI-Nspire CAS technology and software, with or without concomitant use of handheld devices. Of particular interest has been the use of this technology for improving students' mathematical learning, problem-solving methods and deeper understanding of mathematics. Eight classes of students in theoretical programmes at upper secondary level in Sweden had continuous access to TI-Nspire CAS in mathematics during a whole semester. They used the software, and in some classes handhelds, during a whole course and also implemented the national test for the course on their laptops.

Keywords: CAS, classroom practice, problem-solving, TI-Nspire, upper secondary

INTRODUCTION

Calculators as well as computer software have been used for some time in mathematics classrooms. Calculators have changed over the years, from basic calculators to graphing ones, and now advanced calculators working with computer algebra systems (CAS) that include dynamic graphs and geometry. Simultaneously, computers have changed from being large and rather rare in mathematics education into smaller, mobile units (laptops) that can more easily be used in instruction. At the same time, software has changed from mathematics programs for specific tasks to ones which are much more flexible. One such software for laptops is TI-Nspire, which can be used with or without CAS. It also can be combined with handheld units in a teaching and learning system and thus is possible to use in a variety of ways.

The data for this paper comes from a larger study of how students and teachers from 8 classes used TI-Nspire technology in classroom work in a regular mathematics course at upper secondary level. In total, 133 students and 11 teachers participated in the larger study. Two of the classes took their first course (Ma A) and six of them their second course (Ma B). The mathematics in both these courses included different aspects of algebra and functions. In this paper, I focus on the results of one of the methods used in the study; a problem-solving experiment in which the students were given a longer task with several parts, of which the use of algebra and functions was central. Some examples of students' solutions and reflections after working with the tasks are presented, along with a discussion of the implications of this experiment. The complete report of the whole study is given in Persson (2011).

THEORETICAL FRAMEWORK

Balling (2003) distinguishes between the use of software and calculators as *calculating tools*, *teaching tools* and *learning tools*. When they are used mainly for facilitating calculations, they function as calculating tools. When the teacher takes advantage of their possibilities to illustrate and show important features of concepts and methods, they are used as teaching tools. Finally, when students use them for exploring mathematical objects, to discover concept features and to solve problems, they have the role of learning tools.

A tool can develop into a useful *instrument* in a learning process called instrumental genesis (Guin & Trouche, 1999), which has two closely interconnected components; *instrumentalization*, directed toward the artefact, and *instrumentation*, directed toward the subject, the student. To utilise the affordances of these processes requires time and effort from the user. He/she must develop skills for recognizing the tasks in which the instrument can be used and then must perform these tasks with the tool. For this, the user must develop instrumented action schemes that consist of a technical part and a mental part (Guin & Trouche, 1999). In the present research project, TI-Nspire CAS calculators together with the emulating computer software constitutes the physical parts of the instrumentation process. It is important to realise that an instrumentation process occurs for each of Balling's (2003) aspects of a tool. For each individual student the pace of the process also varies between these aspects.

The term *resources* is used to emphasize the variety of artefacts that can be considered: a textbook, a piece of software, a student's worksheet, a discussion, etc. (Gueudet & Trouche, 2009). A resource is never isolated; it belongs to a set of resources. When the process of genesis takes place, a document is produced. The teacher and the students build schemes about the utilization of a set of resources for the same class of situations, across a variety of contexts. This process is called *documentational genesis* and also demands time and effort from the users (Gueudet & Trouche, 2009).

The TI-Nspire environment has been studied for example by Artigue and Bardini (2009). They list why this type of technology is considered novel and special, including: its nature; its file organizing and navigation system; its dynamic connection between graphical and geometrical environments and lists/spread sheets; as well as its possibilities to create variables that can be used in any of the pages and applications within an activity. Aldon (2010) has studied the use of TI-Nspire calculators, and assumes that the calculator is both a tool allowing calculation and representation of mathematical objects but also an element of students' and teachers' sets of resources (Gueudet & Trouche, 2009). As a digital resource, these handheld calculators possess the main functions required for documentary production.

Weigand and Bichler (2009) also have researched the use of calculators, and formulated some interesting questions for research, such as:

- When working with new technologies, polarisation occurs in that some students benefit greatly from symbolic calculators use, whereas for other students, symbolic calculator (SC) use inhibits performance or even decreases performance. Are there ways to get all students convinced of the benefits of the SC?
- The reasons for non-use of the calculator are on the one hand the uncertainty of students regarding technical handling of the unit and on the other hand a lack of knowledge regarding use of the unit in a way which is appropriate for the particular problem. Is there a correlation between these two aspects?
- The responses of the students confirm that familiarity with the new tool requires a very long process of getting used to it. It is surprising that it took almost a year to establish familiarity with this tool for students to use it in an adequate way. After one year of SC use, confidence in and familiarity with the SC grow. However there is still a large group of students who experience technical difficulties when operating the SC. Will there be ways to shorten this period of adjustment? (pp. 1199-1200)

AIMS OF THE STUDY

The larger study considered the teachers' and students' perspectives as well as the cognitive and affective outcomes. In particular, the ways this technology was used according to Balling's (2003) classification and the levels of students' development in the instrumentalisation process were the focus. The two specific questions addressed in this paper are:

1. What skills in using TI-Nspire technology for problem-solving and in exploring mathematical tasks do the students show after working with it for a significant period of time?
2. What examples can be found of how the instrumental and the documentational geneses have developed during the project?

The first question was partly researched through the problem-solving experiment which is described here, and the second through comparing the results with what the students were able to perform at the beginning of the study.

Interviews with students (2 from each class) had been made near the start of the study, and the end of the study both students and teachers were given separate questionnaires. Directly after the problem-solving experiment, focus groups of students were interviewed about what they had experienced working with the different tasks. Some of their responses will be presented below along with the results from their solutions to the problems.

THE PROBLEM-SOLVING TASKS

The students were presented with problems that were constructed on three levels. The first level involved the students doing ordinary calculations and/or readings graphs. The next level involved some more complicated calculations that required the students to compare different answers and make decisions. At the final level, an exploratory task was given to the students who had to write their answers in plain text. The intention was to create tasks that were close to Balling's (2003) three types of technology use – calculating tools, teaching tools and learning tools.

The problem provided to Ma A classes was solved by the two classes taking their first course (27 students). It was called “Holiday cabin” and described three holiday companies with different fee policies:

- ”Stugbyar AB” takes a basic fee of 1250 kr and then 100 kr more per day.

- ”Semestersol” takes no basic fee, but has higher cost per day, 250 kr.

- ”Strandängen” has a fee per week, no matter how many days you stay during the week. The price for the first week is 1500 kr, but you can stay another week for 950 kr, and after that 950 kr for each week or part thereof.

The students first calculated the fee for specific time-periods (3 days, a whole week and 10 days) and determined which alternative was the most favourable for each period of time. Then they were asked to represent the fee policies with functions and graphs, and finally they had to sort out which company had the smallest fee for all possible time-periods. They were also asked to explain why it is difficult to represent the fee policy of "Strandängen" with a function. A special interest in the study was to observe how the students handled this within the platform. This type of function, a "staircase"-function, was not specifically addressed in the syllables for the course. In the final task, the students' ability to make an overview of a rather complicated problem and to communicate this was tested.

The problem for Ma B was solved by six classes (96 students). It was called “Intersection points” and was based on two functions, one quadratic and one linear that intersected ($f_1(x) = x^2 + 1$ and $f_2(x) = 2x + 4$). These were initially provided through the Graph application. First the students were asked to read and note the points of intersection; then the constant term of the linear function was altered so that there was no intersection and they had to comment on what happened. Secondly, the students had to find out what the constant term in the linear function (instead of 4) was in order to get two, one or no intersections (see figure 1). Thirdly, they were asked to solve a non-linear system of equations that exactly reflected the graphs in the first part (the students were supposed to discover this). Finally a parameter *m* was introduced in the linear function for the constant term, and they were asked to solve the system again and explained why this general solution created two, one or no solutions for the system:

$$\begin{cases} x^2 - y = -1 \\ 2x - y = -m \end{cases} \text{, with the solutions } \begin{cases} x = -\sqrt{m} + 1 \\ y = m - 2\sqrt{m} + 2 \end{cases} \text{ or } \begin{cases} x = \sqrt{m} + 1 \\ y = m + 2\sqrt{m} + 2 \end{cases}$$

The students were asked to reflect on the two general solutions and explain why these created different types of solutions for varying values of *m*. Using parameters in equation solving and explaining the outcomes is not a part of the syllables for the course, and a special interest in the study was to see how the students would interpret this new experience.

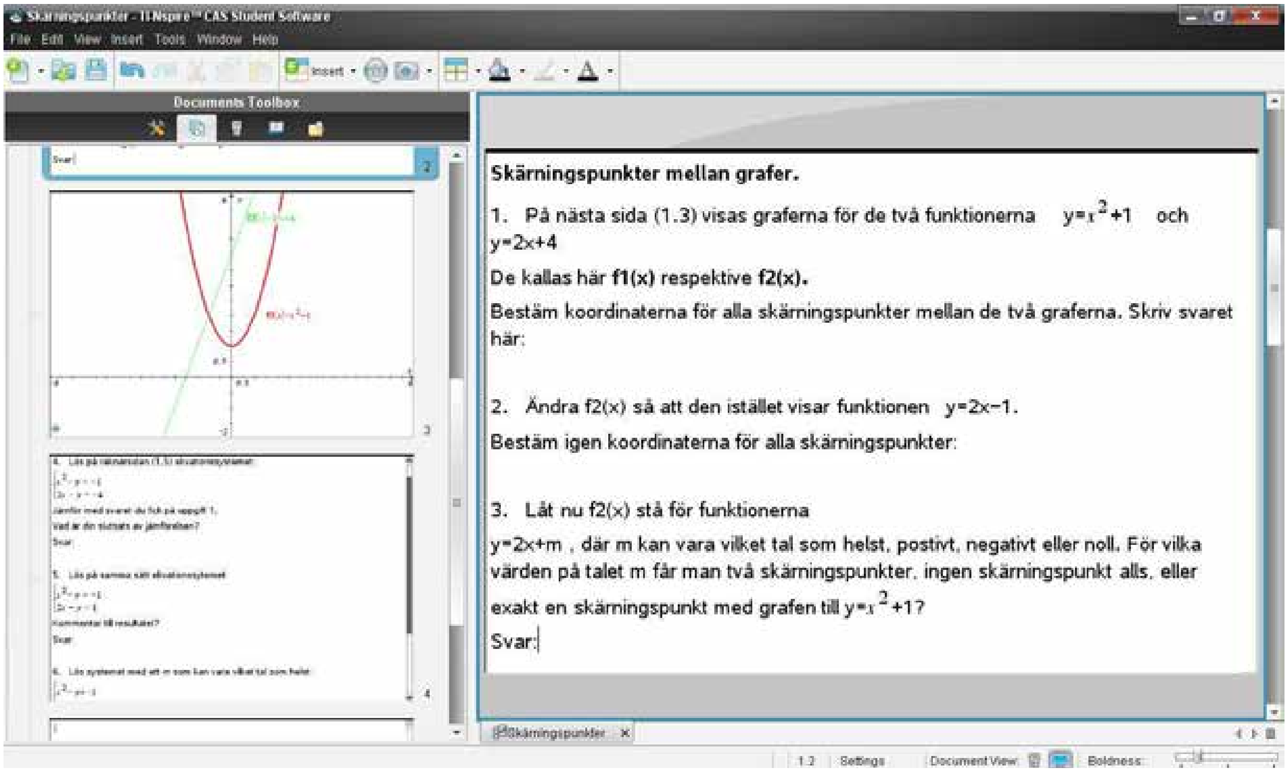


Figure 1: The first part of the problem for the Ma B students, concerning intersection points between graphs.

The problems, with all the instructions, were given to the students by means of the special TI-Nspire-files called tns-files. The students were asked to work individually with the tasks in the classroom. They also had to work within the given file and to present all their solutions in the Notes application within the file. The tns-files were then handed in, identified with the students' names. During the whole experiment the students’ work with the task was observed by the researcher.

SOME RESULTS

The observation of the classes showed that they handled TI-Nspire as a tool in a mainly productive way, both through calculators and on laptops. Some features of the technology were new for them, but in most cases they coped very well with these. The difficulties seemed to lie more with the mathematics that the students had to engage with in the problems, and this was connected to the students' general mathematical abilities. Their problem-solving skills with TI-Nspire were generally good, with only a few exceptions. Many also managed to give good answers to the more difficult parts of the problems. The analysis of the files that were handed in was built on an 'a priori' analysis of the mathematical structure of the problems, and on the anticipated possible ways to solve them. It also included the types of sub-tools that could be used in the problem-solving process, and the representations the students worked with. With these, students' versatility in using the technology as a tool within the three classified types could be established. In the final analysis of each student's file also 'post priori' considerations were used, such as the quality of the students' answers, how they communicated these and in what ways they showed signs of mathematical reflection. Each file was taken through the analysing process several times in order to secure the validity of the results, which are shown in table 1.

	Calculating tool	Teaching tool	Learning tool
Ma A (n = 27)	24	21	10
Ma B (n = 96)	93	83	57
Total (n = 123)	117 (95 %)	104 (85 %)	67 (54 %)

Table 1: Results of the problem-solving experiment, classified according to Balling's (2003) definitions.

The methods used by the students in the solving process varied. Some worked more with algebraic methods, using the CAS application, others more with graphic solutions. For example, some students demonstrated that they can manipulate graphs directly by dragging them up and down. Others instead worked with the equations for the functions and made changes in them, either in the entry line or in the formula inside the diagram. Some Ma B-students were familiar with the way equation systems were entered in CAS and could combine with the Solve-command. Others used more indirect methods. Most of the Ma A-students were quite unfamiliar with broken functions, especially the way these are possible to handle within TI-Nspire. These differences can be explained by the way that the teachers involved had, or rather had not, let the students work with this type of functions and objects in TI-Nspire during the study.

The problem for Ma A did not demand much use of CAS, but the one for Ma B did. Some students had specific difficulties, and asked for help with some of the practical components involved in using it. Some students also thought that the solution they got to the equation system with the parameter was ‘weird’ and therefore probably wrong. Data from the focus group provided further details about this:

Ma A:

Male student 1: It was things that we've done before.

Male student 2: Things that we knew, nothing particularly new.

Interviewer: Was the software well suited for working with such a problem?

Male student 2: I think CAS is great for this kind of task.

Female student: The first two questions were not so difficult, but the third one was pretty tricky. But we have had similar questions.

Interviewer: It seemed as if you were a little inexperienced at reading the intersections of the graphs?

Female/Male students: Mmm. We had not done that so much before. And it was difficult with "Strandängen" [the staircase function].

Ma B:

Female student 1: It was pretty similar to what we usually get in math books.

Female student 2: And sometimes we get ... and we will work in pairs. And you probably get one of those sent to the calculator.

Female student: We have not used this ‘solve’ in systems of equations before, but we have made it algebraically by hand.

Female student: The exercise was not so difficult in itself, but we do not usually do such exercises.

Interviewer: Can you mention something specific that was an obstacle for you?

Female student: Exercise 6, the penultimate. That kind of task we have not had. Like having an m besides x and y.

Nevertheless, most of the students had no substantial difficulties working with CAS. In analysing the solutions, having access to the calculations that the students had made in the Calculator application in the tns-files, gave extra information. Most students had usually not thought of deleting their mistakes, so it was possible to follow their ways to a solution. In Fig. 2, two examples of students' calculations are shown.

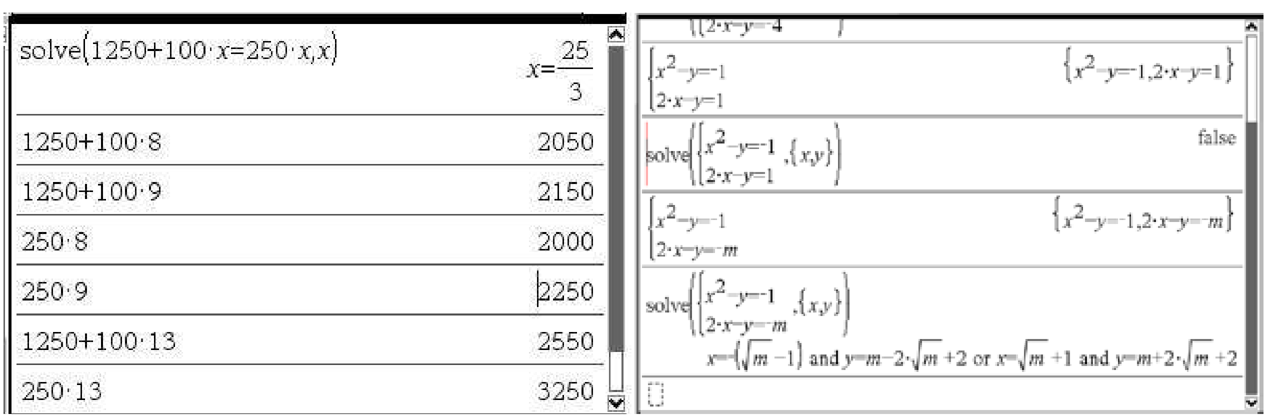
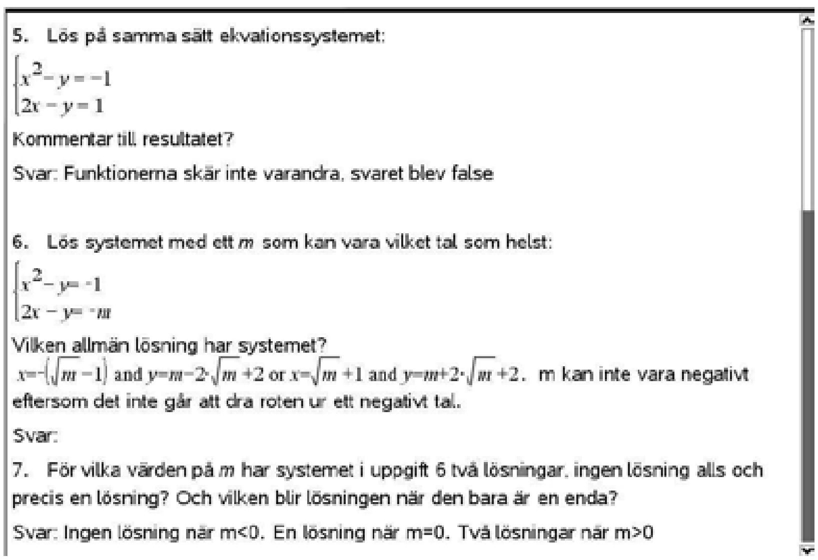


Figure 2: Excerpts from two students’ tns-files (CAS application). The problem for Ma A is given on the left and for Ma B on the right.

Writing text within TI-Nspire seemed to cause problems for many students, especially those with handheld units. They were generally not used to do so within the system. The written answers to the different tasks were with a few exceptions very short, using mostly symbolic mathematical expressions and very few words (Figure 3). When asked about this, the students explained that it was hard for them to write, for example because the keyboard was small. There was, however, one exception. In one of the Ma A-classes, the method of distributing tasks through tns-files and for the students to hand in their solutions using written text was a common daily procedure. For this class, which used laptops, writing text represented no obstacles.



5. Lös på samma sätt ekvationssystemet:

$$\begin{cases} x^2 - y = -1 \\ 2x - y = 1 \end{cases}$$

Kommentar till resultatet?

Svar: Funktionerna skär inte varandra, svaret blev false

6. Lös systemet med ett m som kan vara vilket tal som helst:

$$\begin{cases} x^2 - y = -1 \\ 2x - y = -m \end{cases}$$

Vilken allmän lösning har systemet?

$x=-(\sqrt{m}-1)$ and $y=m-2\cdot\sqrt{m}+2$ or $x=\sqrt{m}+1$ and $y=m+2\cdot\sqrt{m}+2$. m kan inte vara negativt eftersom det inte går att dra roten ur ett negativt tal.

Svar:

7. För vilka värden på m har systemet i uppgift 6 två lösningar, ingen lösning alls och precis en lösning? Och vilken blir lösningen när den bara är en enda?

Svar: Ingen lösning när m<0. En lösning när m=0. Två lösningar när m>0

Figure 3: Example of from a Ma B student's solution with a rather short text. The translated answers to the questions are: 5. The functions do not intersect; the answer was "false". 6. $x=-(\sqrt{m}-1)$ and $y=m-2\cdot\sqrt{m}+2$ or $x=\sqrt{m}+1$ and $y=m+2\cdot\sqrt{m}+2$. m cannot be negative since you cannot take the square root of a negative number. 7. No solution when m<0. One solution when m=0. Two solutions when m>0.

Laptops are normally not allowed when students sit the Swedish national tests, something that the students in the project had to do at least once. Thus special permission to use laptops for research purposes was applied for at the Swedish National Agency for Education. Permission was granted on two main conditions: First, any communication between students or through the Internet was forbidden, and second, unwanted files that could be used for cheating should not be accessible. Only TI-Nspire software was allowed for the students to use. These conditions were met by the teachers, and laptops were successfully used during the tests at the end of each of the courses, Ma A and Ma B. This showed that it is possible for students to complete the national tests using laptops. If this is implemented at a larger scale in the Swedish school system, solutions like turning off the Internet during the test or creating special 'test clients' for the laptops with USB memories will be possible.

DISCUSSION AND CONCLUSION

Several students had in their first interview explained how difficult and complicated TI-Nspire seemed the first time they used it. It contained so many ‘things’ that they hardly knew where to begin. However, most of the students also answered that after a short period of time, when they had become familiar with the software and/or the handhelds, it did not seem so complicated. During the project, the students’ versatility in using the technology had progressed substantially. Many of the difficulties they saw in the beginning disappeared, although even at the end of the study there still were a few students who had significant problems with the use of the software or the handhelds. In the teacher’s questionnaire 9 out of the 11 teachers answered that some students continued to have difficulties with the use of TI-Nspire, although most students had made progress in their ways of working with it. Five teachers also answered that some students enjoyed exploring TI-Nspire in order to find new functions, and they often shared what they found with other students and sometimes also with the teacher.

> Teacher: But then there are some students who understand a little quicker and can show the others. So suddenly you have a whole staff that is helping. And it's good for you to have that.

The ways in which students documented their work with tasks and problems showed very little progress during the project. To some degree this was due to the fact the teachers rarely used the possibilities to work with files with many pages or pictures, and that the students did not use the Notes application in TI-Nspire to really document their work or to hand in solutions to tasks, with the exception of one class mentioned above, for which it was a fairly common procedure.

The problem-solving experiment, and the reflections made by the students in the focus groups afterward, showed that the students could use the technology as a tool in their classroom work, but also that there were possibilities for taking these uses further. If the students had been given longer problem-solving tasks to work with, they could have been able to solve even more difficult problems. They could also have become more used to writing longer answers to the tasks, where they could give more explicit arguments for why their solutions were valid and perhaps also present proofs. In the new Swedish curriculum, two *abilities* are especially emphasised; ability to make argumentations and proofs, and ability to communicate. This type of technology and software could be of assistance in the learning process.

Students in the study showed significant progress in the instrumental genesis and also to some extent the documentational one. Nevertheless, a much more complicated process is required, and the results suggest that this may take a long time, maybe several years. It is difficult to insert technology as an organic part of the resources of a "document" (Guedet & Trouche, 2009) which represent whole work sessions or lessons in mathematics. However, even here a certain development was observed, and there were signs of a continuation of the process involving the TI-Nspire for both teachers and students, at a higher level.

Perhaps the most important result of this study is how TI-Nspire has been used in regular education in upper secondary courses. The various possibilities (Artigue & Bardini, 2009), of a technical, mathematical and conceptual nature, had the opportunity to appear in this longitudinal study taken over several months. It was interesting to be informed about the students' experiences and opinions of the technology. The technology was seen as contributing to speed and accuracy as well as providing a variety of forms of representation. It is also obvious that CAS represents a difficulty, especially for low-performing students, and simultaneously has an incredibly powerful potential in mathematics education. Only a few students are able to take full advantage of this potential, even though most of them can use CAS in a satisfactory way. Experiences from the use in the national tests were also positive. The barriers that existed for the use of laptops could be effectively eliminated, and this shows that it is possible to perform one of the sections of each national test with laptops as aids.

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BRIDGING DIAGNOSIS AND LEARNING OF ELEMENTARY ALGEBRA USING TECHNOLOGIES

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This paper presents research developed in the multidisciplinary PépiMep project (supported by the Ile de France region) which consists in transferring diagnosis and differentiation resources into the Sésamath platform, very much used by mathematics teachers in middle school in France. The research is based on the potentialities of the diagnosis software Pépite, which establishes an individual cognitive profile of the students in elementary algebra. We designed an interface to allow teachers to generate automatically exercises for differentiated instruction courses adapted to the learning needs of various groups of their classes.

Keywords: diagnosis and learning, differentiated instruction course in algebra, ICT, elementary algebra.

INTRODUCTION

Effectively helping students in the classroom with appropriate learning material is a difficult task for teachers. To make every student progress, they need detailed diagnosis about individual students' learning. But, simultaneously, teachers also need to manage the whole classroom by proposing differentiated activities adapted to groups of learners with close competences in algebra or who require the same teaching strategy. Our hypothesis is that software tools can help teachers to do that. This paper addresses theme 2, "students' learning with technologies", of the Working Group 15 and the question of the impact of using technologies on students' learning in the field of elementary algebra at the end of compulsory education in France (16 years).

Our research concerns the development and the use of online resources for diagnosis and differentiated learning. It takes place into the *Pépite* and *Lingot* projects, two multidisciplinary projects in Information and Communication Technology (ICT) (Delozanne et al., 2010). Based on a multidimensional analysis of algebraic activity (Grugeon, 1997), the *Pépite* software is a diagnosis tool which generates automatic multi-criteria assessments of students' competences in school algebra (Delozanne et al., 2008). Furthermore, it also shows the principal characteristics of students' activity in algebra which constitutes their cognitive profile.

This paper deals with the transfer and the integration of the diagnosis *Pépite* within the online databank *LaboMep*, developed by the French maths teachers association

Sésamath. The success of the *LaboMep* platform shows that such online resources may answer the teachers' needs. More precisely, from the potentialities of the diagnosis software *Pépite,* which establishes individual cognitive profiles of the students in elementary algebra, our goal is to design an interface to automatically index exercises for differentiated instruction courses in algebra adapted to the learning needs of various groups of a class. We deal with two research questions. On the mathematics education side: how can we automatically generate exercises of differentiated instruction courses proposed to students according to their diagnosis assessment about the learning objectives aimed at by the teacher? On the collaborative work side: how can the collaborative work between IT specialists and mathematics education researchers make possible the creation of an operational model to index a database of exercises that allows you to automatically produce differentiated instruction courses adapted to students' learning needs?

After clarifying some theoretical and methodological elements for the diagnosis, we will discuss the design of differentiated instruction courses. We will explain the principle of indexing exercises databases that allow for the automatic generation of instruction courses.

THEORETICAL AND METHODOLOGICAL ELEMENTS

Assessment

Diagnosis assessment is an important part of teachers' practice. But this term is used to refer to several types of assessment. Ketterlin-Geller and Yavonoff (2009) identify two practices within diagnosis assessment that use students' procedures and errors as their basis for analysis: analyses of students' answers to specialized tests and cognitive diagnosis assessments using standardized and psychometric models. In both cases, the objective is a local study of students' misconceptions. The diagnosis developed through the *Lingot* project aims to create an overall and multidimensional analysis of students' knowledge and abilities in algebra. It doesn't use psychometric models. It doesn't rely more on the conversation theory whose fundamental idea is that learning occurs through conversations between students which serve to make knowledge explicit (Scott, 2001). It is based on an epistemological study of elementary algebra from cognitive and anthropological approaches which enables to predict students' learning needs.

Assessment and epistemological references

Grugeon (1997) defined a model of algebraic competence at the end of compulsory education. It is the foundation of an epistemological reference to guide the design of an appropriate diagnosis (Artigue et al., 2001). This approach allows one to categorize tasks for a diagnosis test – problems of generalisation and proof, traditional arithmetical problems, problems where algebra appears as a modelling tool, algebraic and functional problems – and to structure the different aspects of the multidimensional analysis of students' activities in elementary algebra.

From an international synthesis of research related to the learning of algebra, Kieran (2007) proposed the GTG model of conceptualizing algebraic activities which differentiates three complementary aspects: (1) *Generative activities* involve the production of expressions or formulas or equations or identities (2) *Transformational activities* involve the usage of transformational rules (factorizing, expansion of products, rules for solving equations and inequalities, etc.) (3) *Global/meta-level activities* involve the mobilization and usage of the algebraic tool to solve different types of problems (modeling, generalization, proof). This model will be used to create student working groups composed of students who can work on tasks with the same learning goals.

Assessment and Anthropological Theory of the Didactic (ATD)

The ATD theory takes into account the institutional context of education. It proposes an epistemological model in which all human activity is to accomplish a task of some type of tasks with some technique. The technology of the technique is intended to provide justification for the technique and the praxeology's theory supposed to justify the technology itself. We postulate that assessment should locate the personal relationship of students with algebra in solving diagnosis tasks, and technological elements involved in their resolution compared to those expected, taking into account praxeologies to teach (curriculum, textbooks) and praxeologies taught (teaching practice) (Chevallard, 1999). The linking between epistemological references and institutional praxeologies in algebra allows one to identify learning needs often ignored by the institution and often implicit in the curriculum and textbooks (Bosch, Fonseca, & Gascon, 2004, Castela, 2008). For Chevallard, "assessment will focus, by practical necessity, for each student, on a sample of all types of tasks constituting referred praxeological organizations". Diagnosis tasks are thus characterized by a type of tasks, the complexity of algebraic objects involved, the level of involvement of tasks in the resolution.

Collaborative Work

An iterative process between educational researchers, computer scientists, teachers and trainers allowed to design and to test prototypes that implement the diagnosis in order to favor its evolution. There were four iterations to test the different versions of diagnosis (El-Kechaï et al., 2011). Our research approach is a bottom-up one informed by educational theory and field studies. In previous work (iteration 1), we started from a paper and pencil diagnosis tool grounded on mathematical educational research and empirical studies. Then (iteration 2), we automated it in a first prototype, also called *Pépite,* and tested by dozens of teachers and hundreds of students in different school settings (Delozanne et al., 2005). In more recent work (iteration 3), we implemented *Pépinière* that generalizes the first tool to create a framework for authoring similar diagnosis tools, offering configurable parameters and options (Delozanne et al., 2008). From 2010 (iteration 4), with the *PépiMep* project, we deployed the *Pépite* diagnosis tool on the *LaboMep* platform developed by *Sésamath*. For each iteration, we started with a didactic model and we defined a formal model for the implementation of the prototype. After, we tested prototypes with teachers or researchers (coding exercises, terms used in the interface to present the assessment, interface to index exercises) and, if necessary, we proposed an evolution of the prototype. Particularly, we tested the interface with teachers to adapt the terms used (type of tasks) with those of the curriculum and the mathematics textbooks (abilities).

FROM ASSESSMENT TO DIFFERENTIATED INSTRUCTION COURSES

We now present the principal elements taken into account to define the differentiated instruction courses related to the diagnosis assessment which supports the indexation of exercises.

Assessment and stereotypes

The diagnosis test included in the *Pépite* software implemented in *LaboMep* is composed of 10 diagnosis tasks (27 items), which cover the range of algebraic problems: exercises involving creating mathematical representations of problems in order to generalize, create a model, complete a proof, or write an appropriate equation (7 items); exercises covering techniques of algebraic calculation (8 items); or exercises in recognition (19 items). The diagnosis tasks may be multiple-choice or open-ended questions (Grugeon et al., 2012). Responses to the items of the diagnosis test are not only analyzed in terms of success or failure and mistakes. They are also coded according to properties and justifications used repeatedly, corresponding to institutionally recognized technologies, which highlight a coherent set of techniques (correct or incorrect) built within the institution. The main characteristics of a student's cognitive profile, considered relatively to its grade level, are automatically calculated by *Pépite* through a transversal analysis that codes the student's responses to the 10 diagnosis tasks.

The above model provides a description of the cognitive profile of each student. However, teachers need to use the diagnosis to form groups of students who require the same proposals of teaching to manage the whole classroom. We defined cognitive stereotypes in elementary algebra (Delozanne et al., 2010) as sets of equivalent profiles that can be considered close enough that students can work with the same learning goal tasks. The stereotypes model has three components: Usage of Algebra for solving problems (coded UA); flexibility in translating different types of representations (geometric figures, graphical representations, natural language) into algebraic expressions and vice versa (coded TA); ability and adaptability in the various uses of algebraic calculations (coded CA). For each of the three components, a scale with different technological levels has been identified, along with appropriate criteria for each level (Delozanne et al., 2005). For example, for the component CA, we distinguished three technological levels according to the types of manipulation and associated justifications: (CA level 1) expected technology taking into account the structure of expressions and their equivalence, (CA level 2) technology only supported on syntax rules, (CA level 3) technology without operational priority leading to concatenation rules and false linearity.

Stereotypes and differentiated instruction course

We postulate that stereotype definition in elementary algebra is an important step in comparison with categories (good, average, low) commonly used by teachers regarding the design and the implementation of differentiation strategies. Indeed, *PépiMep* automatically calculates groups of students who have close profiles in algebra (Grugeon et al., 2012). Figure 1 shows (on the left) the cognitive profile of a 9th grade student with (on the right) the groups of students (A strong, B or C weak).

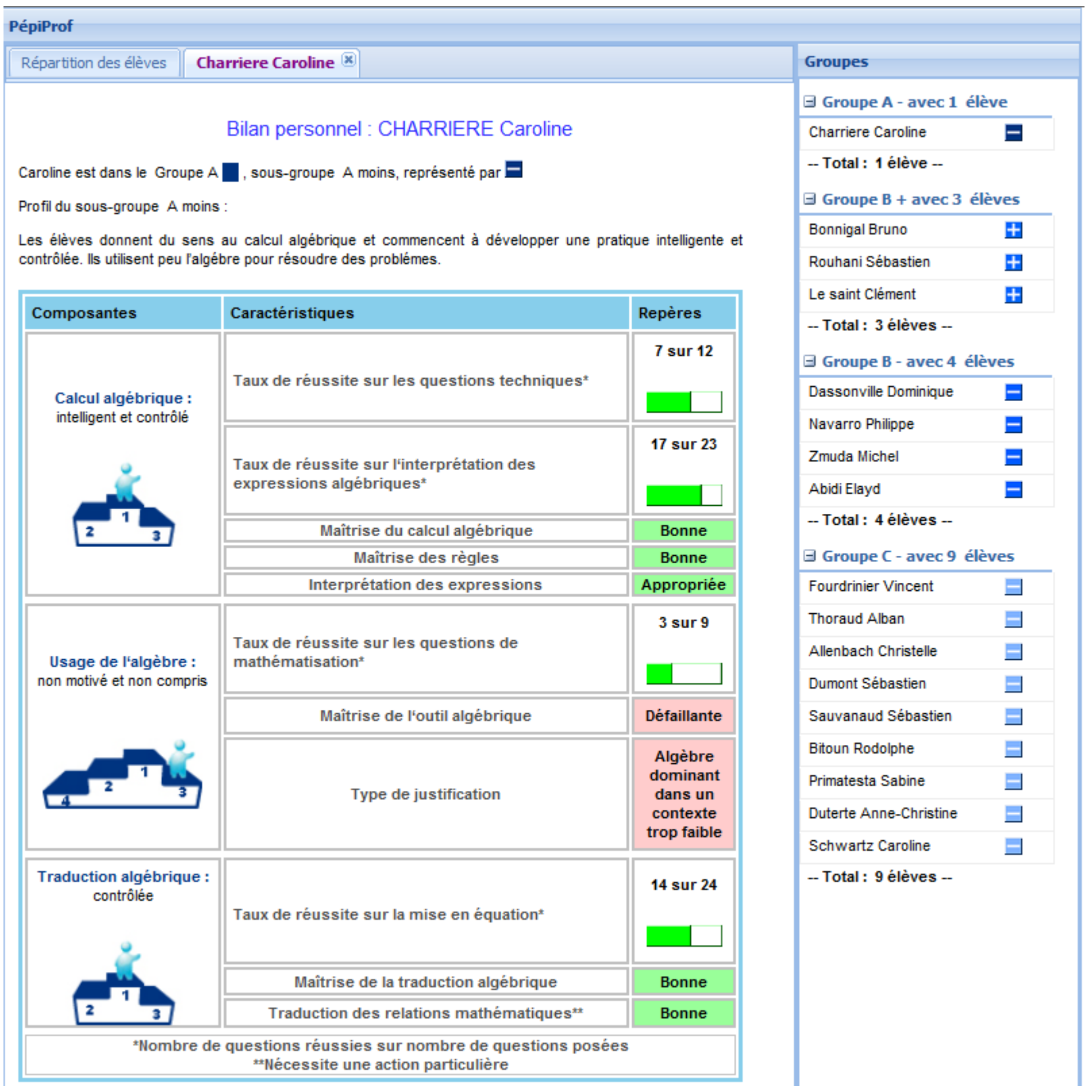


Figure 1: Personal cognitive profile of a 9th grade student

This model allows one to identify target learning needs often ignored by the institution. These are essential common learning issues to work on by all the students of a classroom: particularly, the need to produce general expressions to prove the equivalence of calculation programs, the dialectic numerical / algebraic, the double aspect, procedural / structural of an object, the equivalence of expressions (Pilet 2012). To define a differentiated instruction course, for a given mathematical topic, at a grade level, at a time of education, we identified issues of common learning for the class (presented above), i.e., the tasks involving types of tasks to work on. For a given purpose, we assigned tasks to each group, tasks that are associated with variables related to the technological level involved. An example of a differentiated instruction course is proposed in Grugeon (2012). This design also supports the indexation of the exercise database.

INDEXATION OF AN EXERCISE DATABASE

Our indexation is structured by ability. This choice can appear in opposition with the ATD theory briefly exposed previously, which organizes assessment like the personal relationship of students with algebra in solving diagnosis tasks, and technological elements involved in their resolution. This choice from the necessity to take into account the teachers' working context: they need exercises categorized by abilities in accordance with the curriculum and the mathematics textbooks. An ability refers to a kind of task to which is attached the targeted technology.

We developed an interface for the data capture relative to every exercise of the exercise database. For each exercise, our indexation takes into account: identification parameters (identifying in the database, title of the exercise), the school level for which the exercise is intended (7th grade, 8th grade, 9th grade and 10th grade), the targeted ability (detailed below), the mathematical domain concerned (literal calculation), the input and output objects (numbers, algebraic expressions…), the task complexity (detailed below), the input and output frame.

More precisely, the targeted ability is composed of three elements: relevant components (UA, TA or CA), main abilities for each component (see table 1), elementary abilities for each main ability (see table 2). The task complexity, as in a PISA test, is related to the level of proposed tasks described by Castela (2008): EL elementary, CS conceptual simple, MP multi-step, CX complex.

Component UA 0 – Conjecture that calculation programs or literal expressions are equal or not 1 – Produce a literal expression or formula for solving a problem 2 – Put in equation and solve a problem 3 – Demonstrate (calculation rules, properties, identities) or prove that calculation programs or literal expressions are equal or not 4 – Expressing a variable according to a formula in another
Component TA

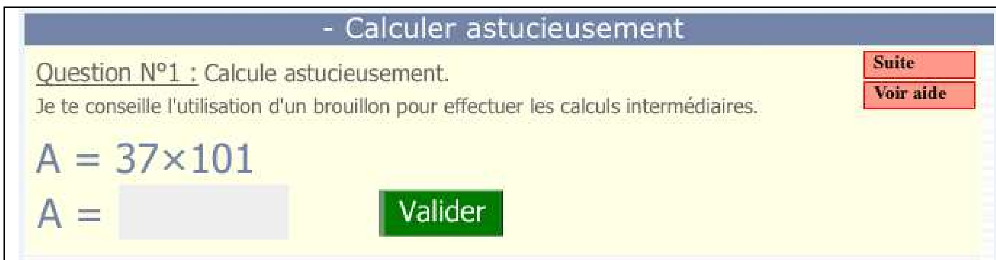
5 – Translate a literal expression, a calculation program…
6 – Graph a function (linear or affine), the solutions of inequality or of system
7 – Recognize an object
8 – Read on a graph
Component CA
9 – Calculate
10 – Test equality
11 – Reduce a simple polynomial expression
12 – Develop a simple polynomial expression
13 - Factorize a simple polynomial expression
14 – Know the remarkable identities
15 – Transform equalities
16 – Recognize the structure
17 – Choose the most suitable form of an expression
18 – Solve equality or inequality or system
19 – Determine the algebraic expression of a linear or affine function from data
20 – Identify a calculation error and correct

Table 1: Main abilities table

9 – **Calculate**	9.1calculate the value of a literal expression giving numerical values in the variables
	9.2 calculate a numerical expression using a rewrite
	9.3 calculate a numerical expression by using the remarkable identities
	9.4 calculate the result of a calculation program for a number
	9.5 calculate the value of a literal expression knowing a numerical relation linking variables
	9.6 calculate the image of a number by a function

Table 2: Elementary abilities for main ability 9 "calculate"

For example, exercise 578 (see figure 2) is relevant for component CA.



Calculate cleverly.

I recommend you the use of a draft to make the intermediate calculations.

Validate

Figure 2: Exercise 578

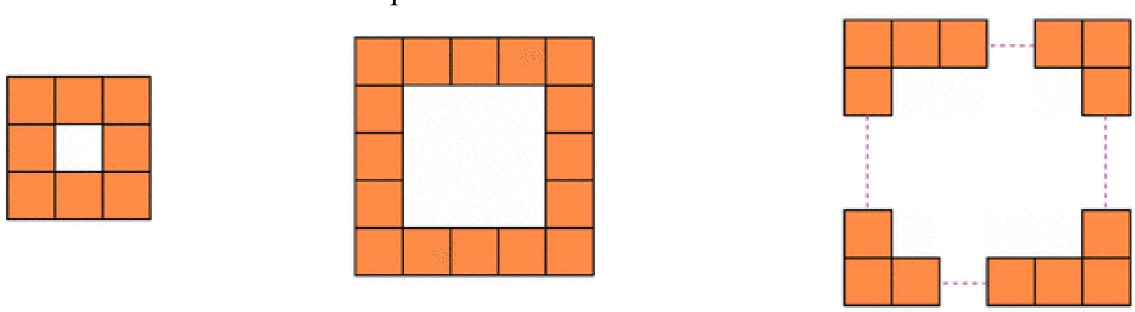
Three main abilities are worked on in this exercise: 9 calculate (9.2 calculate a numerical expression using a rewrite *(for example 101=100+1)*); 12 develop a simple polynomial expression (12.1 develop an expression by using the simple distributivity of multiplication on addition). In particular, the question of rewriting the expression convened in ability 9.2, is not often highlighted in class to carry out the calculation successfully.

EXAMPLE OF AUTOMATIC GENERATION OF A DIFFERENTIATED INSTRUCTION COURSE

The automatic generation of differentiated courses uses *LaboMep* exercises indexed by main abilities and elementary abilities. From the common learning objective selected by the teacher for all students in the class, a list of abilities to work on is proposed. In addition, the teacher proposes differentiated exercises for each group by changing the choice of expressions. We now present an instruction course having for objective: working on the role of algebra to solve generalization problems. This instruction course includes hidden abilities corresponding to learning needs ignored by the institution. More precisely, if generalization problems are discussed in classes, equivalence of calculation programs is very merely worked on.

This course leans on the lined square exercise (see figure 3), a generalization problem, which aims at proving that several calculation programs are equivalent. This problem consists in establishing algebraic expressions that allow one to calculate the number of square units colored with a figure built on the model below, whatever the number of square on the side of the white square is. Writing algebraic expressions involves operating priorities. It includes several steps: determine the number of squares colored for definite values of the number of squares on the side of the square, produce a mathematical expression, compare calculation programs.

We consider a white square surrounded with square units colored as on figures below. The objective is to calculate the number of square units colored.



1) If the white square has a side of 3 units, calculate the number of square units colored.
2) Same question with the white square with sides 4 units.
3) Same question with the white square with sides 8 units.
4) Same question with the white square with sides 100 units. Hint: indicate first the calculation process.
5) Write a formula which gives the number of square units colored according to the number of square units on the side of the white square.
6) Compare your formula with those found by your classmates. What can you say about these formulae?

Figure 3: The lined square exercise

In this exercise, algebraic expressions are objects with which you can make calculations replacing letters with numbers. How can we show that two calculation programs are equivalent? This is possible using calculation rules that guarantee the equivalence of calculation programs that translate expressions. Algebraic identities, such as simple distributivity, that students need to admit, are involved. The articulation between the procedural and structural aspects of an expression is merely worked on in the classroom. In this course, with the objective “prove that calculation programs are equivalent”, students in groups B and C (weak), worked with the lined square exercise which led to first degree expressions: N=4n+4; N=4(n+1); N=2(n+2)+2n; N=4(n+2)-4. The students in group A (strong) worked with a similar problem but with a different pattern which involves a second degree expression.

CONCLUSION [1] AND FUTURE WORK

We have described research on a diagnosis tool, developed in research laboratories, and transferred to *LaboMep*, an ICT platform mainly used by mathematics teachers at the secondary school level. The diagnosis tool allows the teacher to have, for every student, a very precise profile concerning their skills in elementary algebra. By using an indexation of the considered domain, the software automatically proposes, to groups of students identified as having close profiles, a differentiated instruction course adapted to their competences. This automation of the differentiated instruction course was made possible by the crossed successive enrichment of the underlying didactic model and the IT model. Using ICT leads to rethink the teaching resources not only for the students but also for the teachers, so two Ph. D. projects are in progress: Julia Pilet's thesis is interested in profits for the students while the thesis of Soraya Bedja studies the integration of the software in teaching practices.

Future research consists in honing the role played by abilities. What is the impact of the choice of abilities on the course automatically proposed by the software? What is the role played by abilities in the link with the activity of the teacher in his classroom? Can we specify an ontology in elementary algebra? Indeed, this question of the choice of abilities is at the heart of the problem of the transferability of our research in a wider mathematical domain, even in other mathematical domains.

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RESULTS ON THE FUNCTION CONCEPT OF LOWER ACHIEVING STUDENTS USING HANDHELD CAS-CALCULATORS IN A LONG-TERM STUDY

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Long-term studies of students using a handheld CAS-calculator encourage the hope that the gain of lower achieving students using digital tools is above average. The German CASI-project takes advantage of the three-tier school system to examine and evaluate the long-term use of handheld CAS-Calculators by lower achieving students in the 9th and 10th grade. The findings presented in this paper result from the pre-, post- and follow-up-test for the teaching unit on quadratic functions and equations which took place at the beginning, after approximately one third and near the end of 10th grade. The students had one year of experience using digital tools and already passed the tests on linear functions and equation systems. The theoretical framework for the test-design was based on the translation skills for functions.

INTRODUCTION

Functions have always been in the focus of research when the impact of digital tools on students' school achievements is evaluated. Especially, because

> the graphing calculator affords the user both the ability to create equations, tables, and graphs quickly and the facility to move among the representations rapidly (Hollar & Norwood, 1999)

there is special expectation of a handheld CAS-calculator to particularly foster the translation skills (Swan, 1982) between different representations of functions [1] in students. Indeed, Weigand and Bichler (2010) were able to observe an improvement regarding the transition between graph and equation of a function. Therefore this subject matter is an obvious starting point when focusing on the results of lower achieving students using digital tools.

O'Callaghan found

> that the CIA [2] students had a better knowledge of the individual components of modelling, interpreting, and translating as well as a better overall understanding of the function concept

supporting the idea of digital tools helping students to work on realistic problems involving functions. Schwarz and Hershkowitz (1999) claim

> that more students learning functions in the interactive environment than in a traditional environment had rich function concept images

and the result of a meta-study by Barzel (2012) states that conceptual knowledge in general can be fostered by using CAS. This is why there seems to be reason to hope, the students would be able do better when working on examination tasks involving situational descriptions of functions and the corresponding translation skills.

The German School System

The findings presented in this paper emerged from a project which took place in Germany and in order to judge the meaning of the results some knowledge about the German school system is necessary. There are four types of secondary schools in Germany which all start at 5th grade.

The students who show the highest academic abilities in primary school attend the Gymnasium. They graduate to the German Abitur in 12th grade [3], which is comparable to British A-levels and qualifies to study at a university.

The students showing moderate academic abilities in primary school attend the Realschule until 10th grade and then choose between switching to the Gymnasium or other higher schools to get the Abitur or a similar degree and starting an apprenticeship. If they do switch to the Gymnasium they graduate after 13 years of schooling since they have to start over in 10th grade.

The students who show low academic abilities in primary school attend the Hauptschule and most likely start an apprenticeship after 10th grade even though there are possibilities to switch schools and reach Abitur or similar degrees like the students who attended the Realschule. In some German states Hauptschule and Realschule are replaced by a new integrated type of school.

The fourth type of school is the Gesamtschule which incorporates all of the three other types. This is done by differentiation through special courses on various academic levels in each grade; some students learning at Gymnasium-level others at Hauptschule-level.

Studies concerning Students using handheld CAS-Calculators

Some empirical studies regarding the usage of handheld CAS-calculators have been undertaken in Germany. In particular the CALIMERO-Project in Lower Saxony and the M^3-Project in Bavaria have to be named. In Lower Saxony handheld CAS-calculators are employed at the Gymnasium from Grades 7 to 10 and up to the Abitur. The Bavarian study examined students at the end of the Gymnasium (Bruder & Ingelmann, 2009, Weigand & Bichler, 2010).

Since all major empirical studies on the usage of digital tools in mathematics education in Germany took place at the Gymnasium it is particularly interesting to note that while the overall test results of students using handheld CAS-calculators did at least not fall off in quality (Barzel, 2005) especially lower achieving students benefited from the use of handheld CAS-Calculators. The increase in school achievement found in this group of students was significantly greater than the average (Ingelmann & Bruder, 2007). Weigand and Bichler (2010) stated, that in the 10th grade particularly the weaker students my benefit from the use of digital tools.

International meta-studies even observe substantially better test results in classes using digital tools (Hembree & Dessart, 1986, Ellington, 2003). The results presented in the introduction give reason that the research on even lower-achieving students should be started with a closer look on their performance when solving tasks involving functions and translation skills.

THE CASI-PROJECT

The Project Computer Algebra Systems used in lower secondary schools (CASI-Project) examines the long-term use of handheld CAS-calculators at the German Realschule respectively courses of the same academic level at the Gesamtschule. It aims to support, test, and examine the use of digital tools by students taught in 9th and 10th grade. The focus lies on the design and evaluation of educational concepts for the use of handheld CAS-calculators regarding these by definition lower achieving students. Another part of the project concept is the aim to encourage and support diverse use of digital tools. The intent is to use the handheld CAS-calculator not only to calculate, but also to experiment, visualise, create algebraic models and control (Greefrath 2010).

The examined sample of students consists of 10 project classes at 5 different schools in North Rhine-Westphalia with 10 different project teachers who voluntarily signed up for the project without knowing which students they would be teaching except for their grade for obvious reasons. These 10 project classes were then equipped with ClassPad handheld CAS-Calculators. Aside from the loose planning of certain special content areas described below at project teacher meetings and the possibility to call for technical support at any time, the researchers did not influence the teachers' work. Additionally, the six parallel classes at the same schools serve as a control group.

Test Design

To achieve the goals specified in the paragraph above, several different research methods have been employed. During the 2 year duration of the project five content areas (linear functions and equation systems, Pythagoras theorem, circles and the number p, quadratic functions and equations, mathematic growth) were chosen to be of special interest and the educational concept of these is jointly planned and carried out by all project teachers and includes the definition of competencies the students are meant to achieve with digital tools, but also without any technological support at all. These study projects have been monitored using a pre-/post-test design as well as qualitative studies on mathematical test tasks and solution structure or strategies. Aside from mathematical tests, questionnaires on the students' opinions about digital tools and mathematics as well as journals of the teachers about the actual usage of the ClassPad during the lessons are evaluated.

The results presented in this paper follow from tests on the teaching unit on quadratic functions and focus on the quantitative side. These tests consist of two parts: While working on the first part of the test, students are not allowed to use any technology at all. The first task in this part is to solve two equations or equation systems to test for the algebraic abilities of the student. The second task involves drawing graphs of linear or quadratic functions and reading off zeroes or intersection points.

Question 1. A candle burns down. At this moment, it's height is 9 cm high. It shortens by one half cm per 10 minutes. Draw a graph into the given coordinate system that represents this situation.

Question 2. A stone drops from the top of a tower which is 12 m high. After one half second, it has dropped 1.25 m, after one second 5 m and after one and a half second 11.25 m. Draw a graph which assigns the elapsed time to the height of the falling stone over the ground.

Figure 1: Example questions for translation from situation to graph

After finishing this part, the students have to pass the first part to the teacher in order to be allowed to use their calculators to solve the following examination questions. In fact most of these tasks are solvable with little to no technological help and would by classified "tasks with optional or neutral benefit from digital tools" according to the scale introduced by Brown (2003). The questions have been designed to be parallel in matters of the involved translation skills after Swan (1982) – some examples for the translation between situation and graph from different tests being shown in fig. 1. As already pointed out in the introduction, emphasis was placed on the translation skills including situations. The translation skills taken into account for the tests' design are underlined in table 1.

From \ To	Situations	Tables of data	Graphs	Algebraic Expressions
Situations	---	measuring	sketching	descriptive modelling
Tables of data	reading	---	plotting	fitting
Graphs	interpreting	reading off	---	curve fitting
Algebraic Expressions	formula recognition	tabulating or computing	curve sketching	---

Table 1: Translation Skills (Swan 1982)

The lack of translations to algebraic equations takes the low age and therefore low experience in designing algebraic expressions into account. Also there might be some "curve fitting" and "fitting" involved due to the tabular design of the corresponding task (see fig. 3 for details), but the other direction is most likely the more important one.

The tests take place in the second year of the CASI-project. At that time the students attend the 10th grade and the chosen teaching unit takes place several weeks after the summer break. The pre-test occurs right before the subject changes to quadratic functions and with no repetition happening before. The post-test concludes the teaching unit and can optionally also be taken as written exam. The follow-up-test takes place about 3 month after the post-test without any kind of repetition.

RESULTS

The results presented in this chapter have to be sorted by the number of students involved in the analysis. This unfortunate premise is due to organizational problems with not-returning tests and decreasing numbers when working only with those students participating in all tests.

Therefore the results will be given in two sub-paragraphs. The first one will work with the data of all students who participated in all three tests to give the main impression and provide results in the most rigid form available. This first sub-paragraph will show an effect visible in the follow-up-test. To include the results of a substantially higher percentage of project and control group students and focus on the findings regarding the follow-up-test the second one will include the students who participated in the follow-up-test and at least one test of the other two. The combination of the results in this form proved to be the most accurate way to look at a larger number of students without flawing the results and also provide the more rigid view on the data.

Results of the students participating in all tests

The overall results of this group of students are presented in fig. 2. The analysis includes the data of 131 project students and 67 students of the control group.

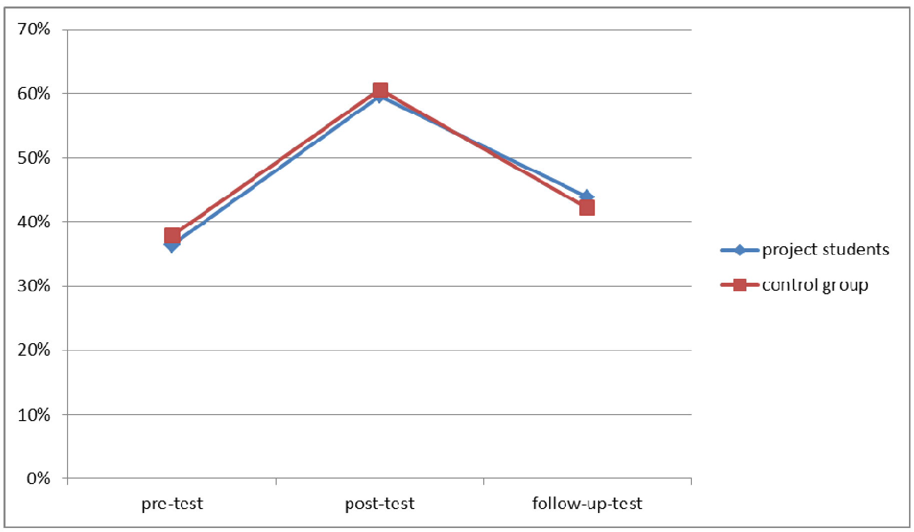

Figure 2: Test-results of the students participating in all three tests

The results show the usual peak at the post-test and then fall off until they reach a slightly higher level in the follow-up-test than in the pre-test. The small differences between the two groups are not significant, but the fact that the project students achieved a higher test-result only in the follow-up-test qualifies for a more detailed analysis of this special part. When analysing the examination questions on their own, mostly there are no significant differences between the project students and the control group. The few significant distinctions between these groups are summarised in the next paragraph.

There is a significant better result of the control group in the pre-test examination question on the translation from graph to situational description, t(89.3)=-2.779, p<.02, r=-0.226. In the post-test the control group has a significantly better outcome in an examination question where they had to read off values of graphs and insert modified versions of these graphs into the same coordinate system, t(169)=-2.297, p<.03, r=-0.161. The follow-up-test shows a significant advantage of the project students over the control group when solving tasks similar to the one shown in fig 3, t(196)=2.453, p<.02, r=0.171.

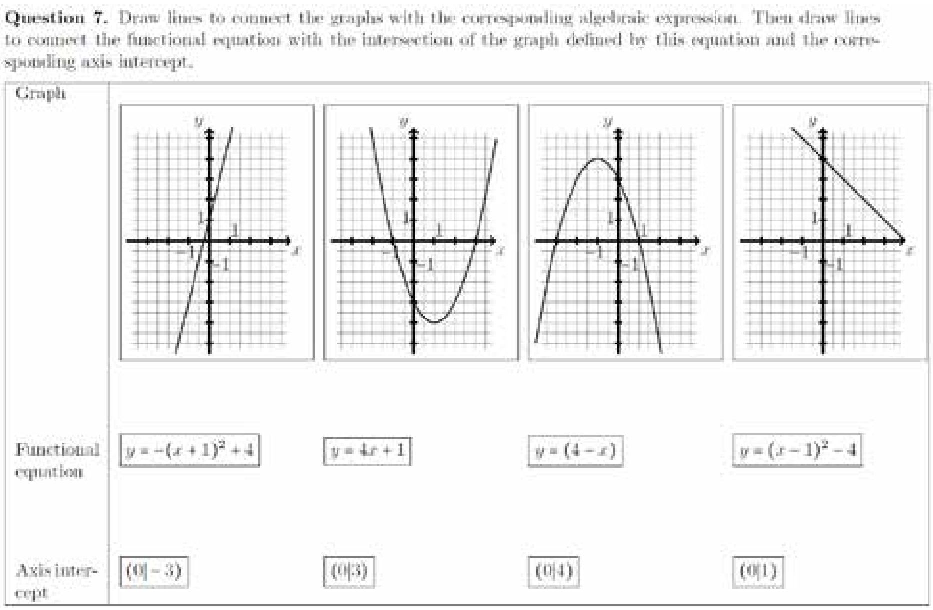

Figure 3: Example examination question from the follow-up-test

There are multiple translations involved in this examination question. Mainly they involve the translations between graphs and algebraic equations and algebraic equations and points on the graph which can be considered as a certain type of tables of data.

Results from the students participating in the follow-up-test

The results of the students who participated in both tests give ground to the assumption that a closer look on the results of the follow-up-test can lead to a better insight. So in this paragraph the students' total is defined by the participation in the follow-up-test and at least one of the other two tests in order to eliminate students for whom there is no data on their long-term progress. Admittedly, this way of looking at the data may be problematic when it comes to generalizations, but the strong tendencies found in the results of these larger set of students give some hint of explanation for the larger picture. Participation in either pre- or post-test and the follow-up-test still guarantees the students' participation over more than 3 month such that the students are familiar with project material and the way the parallel tasks are designed. Fig. 4 summarises the results in the three tests.

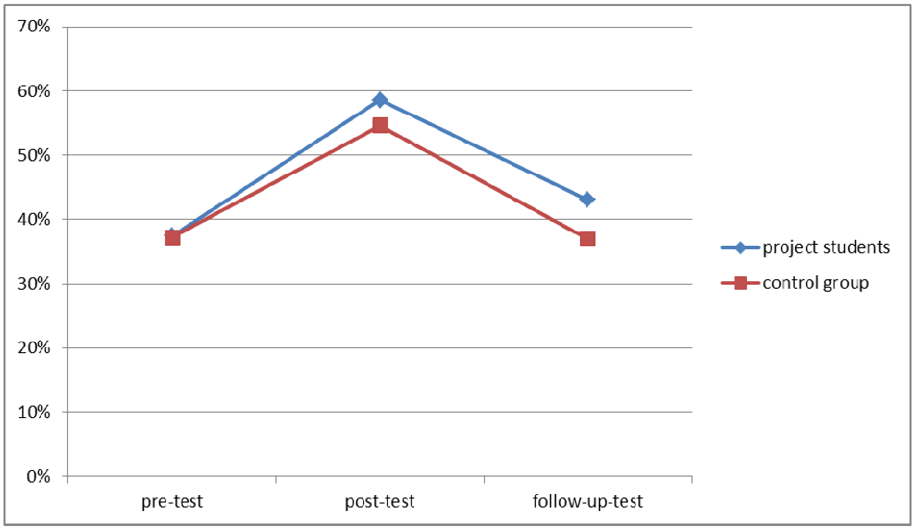

Figure 4: Results of the students participating in the follow-up-test

Both curves show the usual peak at the post-test's result, but there are more differences between the groups. The project students and the control group already differ significantly in the post-test and that gap gets bigger in the follow-up-test. The exact data is summarised in table 2.

	pre-test	post-test	follow-up-test
N (project stud.)	161	152	182
N (control group)	74	90	97
M (project stud.)	37%	59%	43%
M (control group)	37%	55%	37%
SD (project stud.)	15%	13%	13%
SD (control group)	16%	16%	15%
t	0.104	2.029	3.48
df	233	240	277
p	.917	.05	.001
r	0.007	0.137	0.204

Table 2: T-test results and effect size for the three tests

So there is no statistical difference at all in the pre-test, a significant difference with a small effect size in the post-test and a highly significant advantage of the project students over the control group with a small to medium effect size in the follow-up-test. A closer look at the examination questions yields the following findings [4].

The significant advantage of the control group when translating from the graph of a function to a situational description in the pre-test is still visible, $t(233)=2.294$, $p<.03$, $r=-0.155$, but equals out in the post-test and turns to a significantly better result of the project students in the follow-up-test, t(277)=2.11, p<.4, r=0.125. Additionally, there are still better results of the project students visible when solving examination questions of the type shown in fig. 3 with small effect sizes in the pre- and post-test and medium effect size in the follow-up-test. Table 3 contains the t-tests' results on these tasks.

	M (PS)	M (CG)	SD (PS)	SD (CG)	t	df	p	r
pre-test	59%	48%	34%	36%	2.294	233	.03	0.148
post-test	91%	84%	19%	23%	2.342	157	.02	0.157
follow-up-test	83%	64%	26%	33%	4.917	161	.001	0.301

Table 3: Results of the examination question of the type shown in fig. 2 [5].

The project students' results (M=45%, SD=32%) also significantly exceed the control group's results (M=35%, SD=30%) in the post-test examination questions on drawing functions and reading off values without any technological help, t(240)=2.699, p<.01, r=0.171. Furthermore there is a difference (project students: M=54%, SD=30%; control group: M=46%, SD=29%) in solving equations by hand, t(277)=2.073, p<.04, r=0.123.

CONCLUSION

Conducting a project on the use of handheld CAS-calculators at the German Realschule takes the research on digital tools to a class of students which has not been taken into account yet. It has been unclear whether students of this age and academic ability would be able to profit from technology at all or perhaps even be hindered by the necessity to learn the language of the computer in addition to everything else. The lessons during the CASI-project were conducted by interested but otherwise normal teachers and in spite of advice and planning of certain aspects for the special subject matters mentioned above in no way influenced by the researchers.

The results presented in this article regarding the students who participated in all three tests show that the test results under these circumstances do at least not fall off in comparison to the control group. This is valid for the tasks where all technical help was allowed as well as for the examination questions which have to be solved by hand only. The few significant differences regarding isolated examination questions cannot be observed in the corresponding parallel tasks and – with the exception of the task displayed in fig. 3 – do not show a tendency to be anything more than statistical artefacts.

Widening the pool of students by allowing students with only two written tests – as long as the follow-up-test is included – to enter the analysis leads to some additional findings. The project students achieve significantly higher results in the post- and follow-up-test with the effect even increasing towards the later test showing that mathematical knowledge is conserved better. This might partly be caused by the fact that the project students were able to use the handheld CAS-calculator to help solving the task shown in fig. 3 which is an examination question where there can really be a benefit from the use of digital tools . Even if this is the case it still shows the students' ability to autonomously use the digital tool for their benefit showing their progress in terms of instrumental genesis. (Guin & Trouche, 1999)

The results of the examination questions addressing translation skills which include translations with situational descriptions do not show any special advantage or disadvantage of either group of students. So there is no evidence to the assumption the richer conceptual images of the project students have an effect on these isolated tasks.

Additionally the abilities of the students to work on certain basic tasks without any technological help are at least conserved. This result was reached only by agreeing on certain competencies which are to be achieved and without the teachers being mandated by the project supervisors to train these skills with any special program. Students from the project group even had the tendency to be a bit better than the control group in these tasks.

A direct comparison of the effects observed in this project with the effects found by Ingelmann and Bruder (2007) as well as Weigand and Bichler (2010) examining students at the Gymnasium is difficult due to the different ages of the students and other ways the subject matters are addressed. The lack of improvement in examination questions addressing translation skills might be caused by the less abstract way of teaching functions at the Realschule. The main tendency of the results however is in line with the findings of these previous studies.

NOTES

1. See table 1 for a complete list of these translation skills as presented in (Swan, 1982).

2. CIA = Computer-Intensive Algebra

3. This has been subject to change quite often during the past ten years. Abitur after 12th grade is true for North Rhine-Westphalia at the moment, but this might also be about to change.

4. Only results with significance $p<.05$ are reported.

5. PS=project students, CG=control group.

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PUPILS' ROLE AND TYPES OF TASKS IN ONE-TO-ONE COMPUTING IN MATHEMATICS TEACHING

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The article presents results of data analysis of 11 mathematics lessons (29 episodes) in which netbooks were used within one-to-one computing. The research questions were: What are the types of tasks solved with the support of netbooks? What is the pupil's role in teaching in which netbooks are used (i.e., how does he/she participate in the solution to the task using the netbook)? The result of a qualitative analysis is a classification of tasks and of pupils' participation in the solution to tasks using netbooks (examples are provided). It was found out that traditional school types of tasks and a pupil as a user prevail. It ties with other research and points to the key role of developing teachers' technological pedagogical content knowledge.

Keywords: netbooks, one-to-one computing, pupils' participation, task types

INTRODUCTION

A relatively new type of ICT integration in the teaching of mathematics consists of so called digital classes in which each pupil has his/her own laptop with installed electronic textbooks and software for school and home work (one-to-one computing). Classrooms are equipped with a data projector, an interactive whiteboard and internet connection. There are many studies about one-to-one computing, however, only a few concern mathematics. For example, a study in a Grade 3 class has shown that the integration of laptops into the learning environment is a very complex process and that the key role is played by teachers (Neumajer, 2009). While the teachers welcomed the possibility of work with laptops at the beginning, they more or less impeded their more intensive use. Teachers mostly used laptops as a substitute for usual didactic means within a traditional frame of teaching. Similarly, Billington (2011) who investigated the teaching of mathematics at an upper secondary school concludes that laptops were mostly used within the curriculum – tasks could have been solved without them. Laptops strengthened the existing way of teaching rather than changed the teaching practice.

Freiman et al. (2011) describe a project in which Grade 7 and 8 pupils were equipped with laptops for use in several subjects. The authors conclude that "laptops in and of themselves may not automatically lead to better results on standardized tests, but rather create opportunities to enrich learning with more open-ended, constructive, collaborative, reflective, and cognitively complex learning tasks" (p. 136).

Vondrová & Jančařík (2012) analysed videorecordings of 15 primary mathematics lessons with netbooks and conclude that the episodes with netbooks concerned mostly revision and practice tasks and the teachers' explaining a new topic; tasks leading to the creation of knowledge by pupils were not seen. The “netbooks were utilised for activities easily accomplished without them (such as reading a text)” and “except for the applets in which pupils can work at their own pace and can often choose the level of difficulty, we did not witness any incidence of the teacher asking pupils to work on different tasks”.

One-to-one computing is a rare phenomenon in the Czech Republic. There are local projects with netbooks which usually aim to enhance pupils’ motivation, engagement in mathematics and knowledge, to promote individualized learning, to use more complex tasks which are difficult without ICT, etc. This paper aims to see, broadly speaking, whether this expectation has been fulfilled in one such project.

THEORETICAL BACKGROUND

It is generally accepted that the key elements influencing pupils’ learning of mathematics are tasks and the way the tasks are implemented in lessons – in this implementation, an active role of the pupil in developing knowledge is usually stressed. For example, the concept of *opportunity to learn,* seen as the single most important predictor of pupils’ achievement, is defined as the “circumstances that allow students to engage in and spend time on academic tasks such as working on problems, exploring situations and gathering data, listening to explanations, reading texts, or conjecturing and justifying” (Kilpatrick et al., 2001, p. 333). It includes “considerations of students’ entry knowledge, the nature and purpose of the tasks and activities, the likelihood of engagement, and so on” (Hiebert & Grouws, 2007, p. 379). Hiebert & Grouws (2007) point out that the teaching in which pupils are *struggling* with important mathematics leads to conceptual understanding. By struggling, they mean pupils expending “effort to make sense of mathematics, to figure something out that is not immediately apparent”.

The tasks and their implementation is a matter of concern in teaching with PCs, too:

> This is where the big issue is – the nature of the tasks and how they are presented to students. These should enable the student to experiment, investigate and draw conclusions. Students need to have access to the computer in the class, and the activities proposed should be rich enough and appropriate to promote learning. (Amado, 2011, p. 2150)

Thus, we are concerned with the way tasks are implemented in the digital class and what role pupils have in their solutions using netbooks – how engaged they are, if they struggle in the above sense or if they are invited to struggle at all.

There is a growing body of research focused on the types of mathematical tasks set in digital classes. One typology concerns the role technology plays in the solutions to the tasks (Böhm et al., 2004):

- Tasks where the use of PC and software is of little or no help; the solution is faster by-hand.

- Tasks that are solved faster or even trivialised by PC.
- Tasks testing the ability of using PC and software.
- Tasks starting from traditional ones that are extended to PC tasks (e.g., by including formal parameters or using realistic data).
- Tasks difficult, time consuming or impossible to solve without PC.

Another classification is introduced by Assude (2007): tasks of developing: instrumental knowledge and skills, mathematical knowledge and skills, relations between instrumental and mathematical knowledge and skills.

Zbiek et al. (2007) distinguish *technical* and *conceptual mathematical activity*. The former includes geometric construction and measurement, numerical computation, algebraic manipulation, graphing, translation between notation systems, solving equations, creating diagrams, etc. The latter involves understanding, communicating, and using mathematical connections, structures, and relationships, defining, conjecturing, generalizing, abstracting, etc. The authors stress that neither type of activity is more mathematically meaningful than the other.

Another point of view Zbiek et al. (2007) introduce is that of the initiator and maintainer of the activity. They distinguish *exploratory* and *expressive activity*. In the former, pupils are asked a question and given a procedure to carry out by the teacher; e.g., one type is "guided" exploration in which the pupils' goal is to produce a predetermined result chosen by the teacher. In the latter type of activity, pupils decide which procedures to use; they attempt to answer a question of their choosing with their choice of process. Zbiek et al. stress that "the nature of students' exploratory or expressive work depends on both the task and the activity" (p. 1181).

Finally, Zbiek et al. (2007) distinguish between the use of technology as *amplifier* and as *reorganiser*. The former accepts the goals of the current curriculum and works to achieve those goals better. The latter changes the goals of the curriculum by replacing some things, adding others, and reordering still others.

To sum up, PCs seem to provide a good opportunity to use rich activities in such a way that pupils struggle in the above sense; however, this opportunity is not often fulfilled. For example, the meta-study *The ICT Impact Report* (European Schoolnet, 2006) states: "Teachers' use of ICT for communication with and between pupils is still in its infancy. ICT is underexploited to create learning environments where students are more actively engaged in the creation of knowledge rather than just being passive consumers." Laborde et al. (2006), investigating the use of Cabri, conclude that teachers use it within a static geometric curriculum, i.e., traditional content and methods, and thus do not make use of educational powers of ICT.

Research has shown that also within ICT teaching, the key role of what is learned and how it is done is played by the teacher (e.g., Lozano & Trigueros, 2007; Fuglestad, 2005). Mishra & Koehler (2006) develop a model of the types of knowledge teachers need to teach with ICT. *Technology Knowledge* involves the skills required to operate technologies. *Technological Content Knowledge* is knowledge about the relationship between technology and content, i.e., how the content can be changed by the application of technology. *Technological Pedagogical Knowledge* is "knowledge of the existence, components, and capabilities of various technologies as they are used in teaching and learning settings, and conversely, knowing how teaching might change as the result of using particular technologies". The central part of the model is *Technological Pedagogical Content Knowledge* (TPCK) which "requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students' prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones" (p. 1029).

METHODOLOGY

In 2009, Grade 6 pupils in three classes in different schools were given one netbook each. The netbooks have been used in several subjects including Mathematics. The three teachers whose lessons are available to us were young, with several years of teaching experience. A publishing house that publishes, among other, e-textbooks provided them with training which concerned Technology Knowledge (use of netbooks, the interactive whiteboard, pupils' and teacher's software), Technological Content Knowledge (the use of the e-textbook, which includes multimedia, hyperlinks, e-tasks, etc., the use of dynamic geometry software, etc.) and partly Technological Pedagogical Knowledge (the way pupils can work in pairs, at the interactive whiteboard, how to create and use interactive exercises on the board and in netbooks). The teachers were rather passive during the training; they were not required to prepare digital material themselves. In other words, they were expected to develop their TPCK themselves while they were teaching. At the beginning of the project, their attitude towards ICT was positive and they believed in its potential benefits for pupils' learning.

We have revisited some of the data used for Vondrová and Jančařík's work (2012) and analysed them from the following perspectives:

1. What are the types of tasks which are solved with the support of netbooks?

2. What is the pupil's role when netbooks are used in the lessons? How does he/she participate in the solution to the task with the help of the netbook?

The data consist of videorecordings of 11 lessons, in which netbooks have been used, in Grade 6 and 7 in three schools. The videorecordings were made by us or the school provided us with them. The camera was typically static, at the back of the classroom recording the whole class and zooming in on the teacher and/or the whiteboard from time to time. The videorecordings come from different parts of the school year. At the time of our first observations, the teachers had had about half a year experience with using netbooks in their teaching. Thus, they already possessed necessary Technical Knowledge and their pupils had had an opportunity to familiarize themselves with netbooks and the software.

The analysis of videorecordings started by distinguishing episodes with netbooks, spanning 5 to 20 minutes (Tab. 1). By an episode, we mean a part of the lesson where one task was used. The episodes were coded in terms of the type of task used and the pupil's engagement in the solution to the task with the help of the netbook. We were only concerned with the part of the solution to the task which was done on the netbook, mostly disregarding the paper and pencil solution in the coding stage.

The coding was based on the above types of tasks but we also used techniques of grounded theory for coding the new emerging aspects of the tasks and their implementation. The coding stage resulted in a list of codes which were organised into two categories – pupils' engagement and types of tasks.

	Grade 6		Grade 7	
	No. of lessons	No. of episodes	No. of lessons	No. of episodes
Teacher X	3	3 / 5 / 3	1	4
Teacher Y	2	1 / 1	2	4 / 3
Teacher Z	2	3 / 1	1	1

Table 1: Number of lessons and episodes with netbooks

As an example, we will present the coding of three episodes from one lesson.

<u>Grade 6, February 2011. Topic: Practical use of volume and surface area of cuboids.</u>

<u>Episode 1:</u> Each pupil works on the internet with an interactive exercise to practise calculations of volumes and surface areas of cuboids. First, they recognise cuboids from among different solids which have sizes of edges given and record calculated volumes and surface areas in the box provided on the screen. The applet returns an immediate feedback whether the result is correct or not.

<u>Codes:</u> *Pupil as a user* (works according to instructions on the website), *Traditional school task* (distinguishing solids, calculating volume and surface area), *Technical mathematical activity* (numerical computations), *Netbook as amplifier*.

<u>Episode 2:</u> The teacher has sent the pupils a Smart Notebook presentation by wifi, the same presentation is open on the interactive whiteboard. The presentation shows some objects such as a house, a box, an aquarium and the pupils' task is to assign a box with dimensions $a \times b \times c$ to each object; that is, they should estimate measures of real objects and use units such as dm, cm and m correctly.

<u>Codes:</u> *Pupil as a user* (works according to instructions), *Traditional school task* (estimation of lengths, understanding units of measures), *Computer oriented tasks*

(namely, practising the dragging tool in Smart Notebook), *Technical mathematical activity* (estimation of dimensions, conversion of units), *Netbook as amplifier*.

Episode 3: Another task in the same presentation as used in Episode 2. The task is to calculate the height of water in an aquarium if we know its volume and dimensions of the bottom. The teacher asks the pupils to use the pen in Smart Notebook and colour edges of the bottom base in the pre-drawn cuboid. Next, they should use the blue pen to record the water level – lower than the upper base of the cuboid. Some pupils cannot do it and wait for the teacher to do it on the interactive whiteboard. The cuboid is then moved away and the pupils draw vertical edges according to the teacher's instructions and they can "see" a cuboid again: "a cuboid of water in the aquarium". The teacher refers to an analogical task solved earlier.

Codes: *Pupil as a user* (works according to the instruction), *Traditional school task* (calculating the height of the cuboid when the volume and measures of the base are given), *Computer oriented tasks* (namely, practising tools – coloured pens, deleting an object), partially *Mixed task* – the solution is supported by visualization (a picture of an aquarium – water "makes" a cuboid), *Technical mathematics activity* (calculation of volume), *Netbook as amplifier* (visualization of ordinary mathematics situation), *Exploratory activity* ("finding" a cuboid of water).

RESULTS

The first focus of the analysis was on the **types of tasks** which were assigned in the lessons and for which netbooks were used. Two aspects were followed: whether tasks were used for the development of knowledge and skills connected to mastering the netbook or mathematics, and whether the netbook with its software helped in the solution to tasks. Four types of tasks have been distinguished:

- *traditional school tasks* which can easily be solved without netbooks and for which netbooks only replaced paper-and-pencil techniques; netbooks were of no help or they only made the solution faster (13 episodes); the episodes included technical mathematical activities and 3 also conceptual activities; 1 episode included an exploratory activity.
- *computer oriented tasks* which developed computer skills through solving a mathematical task (8 episodes); all included technical mathematical activity and 3 conceptual activity, too; 1 episode was with an exploratory activity.
- *mixed tasks* for which the netbook and its software markedly simplified the solution or made it more illustrative, understandable (2 episodes); both consisted of a technical mathematical activity and exploratory activity, one included a conceptual activity, too.

To make the classification complete, we add one more type of tasks which did not occur in our data and which we, in view with Böhm et al. (2004), see as an ultimate goal of teaching with ICT:

- *computer tasks* which would not be possible without the support of the netbook; e.g., the algorithm is too complicated, the task is too numerically demanding, ICT mediates new insights, etc.

Thirteen episodes were not classified in terms of the task used. In 11, the pupils only read an assignment on the screen and did not use the netbook for the solution at all, in 1 episode they searched the internet for the word election and in another one they used the notebook for sending the teacher their homework by wifi. We did not witness any use of netbooks as reorganisers neither any expressive activity.

When solving tasks using netbooks, naturally the technical aspect plays a role. Thus it is quite difficult to distinguish which type of tasks of the proposed classification was the primary one, whether mathematically or technologically oriented. For example, in one of the episodes pupils look for the least common multiple of three different numbers. They use tables in Excel in which they are to generate multiples of the numbers. The teacher soon finds out that some pupils fill in the numbers manually and thus goes on to explain the solution in terms of technology – how to make the table with multiples in Excel automatically. The same problem with Excel appeared in the same class a month before this episode. A potentially *computer task* has become a *computer oriented task*.

The second focus of the analysis was on the type of **pupils' participation in the solution to the task with the help of netbooks**. The following classification has emerged in our data:

- *a pupil as a spectator* – pupils only observed what was going on on the screen of the netbook, they did not use any tools available on the netbooks (1 episode).
- *a pupil as a user* – pupils worked on the netbook according to the instructions of the teacher or the textbook, e.g., substituted numbers into the teacher's Smart Notebook presentation, used a mathematical applet for practising algorithms; we could say that the netbook substituted a worksheet with tasks to be completed but with a possibility of feedback (11 episodes); all episodes concerned technical mathematical activities and the netbook was used as an amplifier; one episode was classified as an exploratory activity and one as a conceptual activity.
- *a pupil as an active user* – the netbook was an important tool for pupils, they used its software to look for solutions to the task by, e.g., geometric constructions, computations, graphing, etc., or by looking for relationships, experimenting and making deductions; the netbook helped in the solution (4 episodes); the task was set by the teacher, who, through the choice of software, usually determined the solving methods, but the pupils could also make their own choices; all episodes were coded as amplifiers, 2 episodes as exploratory activities and 3 combined technical mathematical and conceptual activities.

Again, to make the classification complete, we suggest one more type of pupils' role which was not observed in our data:

- *a pupil as an independent and active user* – pupils use the netbook purposely to develop their knowledge and skills; if we paraphrase Amado (2011), the netbook is their partner in the learning process, "enabling [them] to achieve some knowledge that would otherwise be very difficult or even impossible".

In most episodes, the pupils' work was controlled by the teacher; the pupils were rather passive even in potentially computer tasks. Let us consider one episode in which the centre of the circumscribed circle of a triangle is looked for. The task is as follows: *Find a place for a feed depot for deer which has the same distance from three given feeding-racks which make a triangle*. Several pupils make some drawings in their paper exercise books and hypothesise that it will be a centre of gravity. One pupil makes drawings on the interactive whiteboard. The teacher says that it is not the solution and asks the pupil at the board to refute the hypothesis by using a figure in GeoGebra. The other pupils are asked to "try yourselves that it will not work". The teacher's instruction is not clear, only several pupils work with netbooks, the others look at the board. The hypothesis is refuted. The teacher goes on by giving hints towards the intersection of axes of sides. He asks one pupil to show this solution on a whiteboard using construction tools. The other pupils should make constructions into their exercise books. The situation potentially *a pupil as an active user* was classified as *a pupil as a user* as the teacher took a complete control of the situation. A potentially *mixed task* has become a *traditional school task*.

There were only 4 incidents of *a pupil as an active user*. For example, in one of them the teacher sent his own Smart Notebook presentation to the pupils in which a task was given: *There is an ancient temple with columns 60 m apart. We are to relocate the columns so that they are 45 m apart. Which ones will stay?* There is a picture of the temple and columns which can be moved by dragging. The teacher asks pupils to model the situation. The pupils experiment, relocate the columns by dragging, use the ruler on the screen to measure the distances. Not all the pupils find the solution. The teacher uses the problem to formulate the concept of the least common multiple. Smart Notebook was used for the visualisation of a mathematical situation and for experimentation and discovery of the solution. The teacher prepared a task and appropriate means of visualisation which determined the pupils' solving method.

CONCLUSIONS

The above four types of tasks can be combined with the four pupils' roles; e.g., we saw examples of computer oriented tasks in which pupils were users and active users. In a few episodes, types of tasks from the literature review were seen. Let us now take into account the *potential* of the four types of pupils' role. Netbooks as amplifiers as well as technical mathematical and conceptual activities could occur in all types of roles except for *the pupil as a spectator*. An expressive and exploratory activity as well as netbooks as reorganisers will not occur in *the pupil as a spectator* either as the netbook is not used for the solution at all. An exploratory activity can appear in *the pupil as a user* and *active user* while the expressive activity would tie with *the pupil as an active user* and *an independent and active user*. More data is needed in order to see whether this potential is fulfilled in practice.

The prevalence of traditional tasks and the pupil's role as a user or spectator in the episodes with netbooks tie with results of some other research focusing on ICT tools (Billington, 2011, Neumajer, 2009, European Schoolnet, 2006). It can be caused by the teacher's fear of sudden teaching situations (Laborde et al., 2006) or it can be related to the teacher's little experience with the ICT tools (6 to 12 months in our study). Providing teachers with technology and technological content knowledge is not enough. It is necessary that their TPCK is explicitly developed (see, e.g., Mishra & Koehler, 2006). The teachers go through stages. For example, when integrating Cabri in their teaching, the teachers begin with the use of software as a visual amplifier and end with creating tasks for pupils in which software plays a key role (Laborde, 2001). It might be interesting to see how the situation will develop for our teachers – pupils in the observed classes are in Grade 9 now and thus we plan to analyse data from Grades 8 and 9 to see whether there has been any progress.

Our study has its limitation in terms of data. More lessons or a series of lessons of one teacher would be desirable. The episodes were analysed in terms of an "average" pupil's participation. Borders between types of tasks and between types of pupils' participation in their implementation are somewhat blurred. Still, we believe that the study uncovered some aspects of real teaching with netbooks if teachers are not supported in professional training.

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USING DYNAMIC SOFTWARE TO FOSTER PROSPECTIVE TEACHERS' PROBLEM SOLVING INQUIRY

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Mathematical tasks are essential components that help prospective and practicing teachers to develop mathematics and didactic knowledge. What type of reasoning do problem solvers exhibit when they use a dynamic software to solve textbook tasks? We document the extent to which the use of the tool can offer prospective and in-service teachers the opportunity to construct and explore a task's dynamic model where visual, empirical, and geometric reasoning complement and enhance formal approaches. As a result of our research, it was evident that the use of the tool not only offers novel ways to think of the tasks, but also the nature of routine problems can be transformed into a series of nonroutine activities.

INTRODUCTION

There is an agreement within the mathematics education community that the use of technology plays an important role in the students' learning of mathematics. However, the extent to which teachers develop the experiences that allow them to recognize and incorporate the systematic use of the tools in their actual practice continues to be an important theme in teachers education programs. One way for prospective and inservice teachers to know the potential associated with the use of a particular tool is to get them involved in problem solving experiences that enhance the use of tools in solving and discussing routine problems found in regular textbooks (Santos-Trigo & Camacho-Machín, 2009). This report is part of a wider research project that aims to analyse and document the use of digital technology in teachers' education and students' learning. One essential phase during the development of the project was to characterize what ways of reasoning and problem solving strategies emerge when routine tasks such as those that appear in textbooks are approached through the use of digital tools. In this study, we focus on analyzing the problem solving sessions developed within a community formed by mathematicians, mathematics educators and prospective and inservice high school teachers which aimed to characterize and discuss ways of thinking and reasoning that the members of the community showed while using a dynamic software (Geogebra) to represent, explore, and solve a set of textbook problems. Thus, the research question that oriented our inquiry was:

- *What are the features of mathematical thinking and reasoning that distinguish and characterize a problem solving approach that relies on the use of dynamic software to deal with routine or textbook problems?*

CONCEPTUAL FRAMEWORK

To frame the study, we rely on two complementary approaches to explain ways for people to take decisions during the development of their activities. Specifically, we recognize that teachers take several important decisions during the preparation and development of their mathematical lessons. These involve the selection of problems, the introduction and discussions of concepts, ways to organize or structure learning activities, and ways to answer, evaluate, and orient students' comments and participation. How do teachers support and carry out choices and decisions related to the framing and development of a mathematical lesson? Schoenfeld (2011) proposes a framework to characterize and interpret ways in which people in different domains engage in, and develop practices associated with such domains or fields. "People's decisions making in well practiced, knowledge-intensive domains can be fully characterized as a function of their orientations, resources, and goals" (p. 182). Kahneman (2011) identifies two systems to account for or explain the decisions and choices that people make: "System 1 operates automatically and quickly, with little or no effort and no sense of voluntary control. System 2 allocates attention to the effortful mental activities that demand it, including complex computations. The operations of System 2 are often associated with the subjective experience of agency, choice, and concentration (p. 20). Hence, it becomes important to document the extent to which the opportunities and experiences that prospective teachers develop and encounter in their education influence the development of their practices.

How do teachers construct their orientations or beliefs, dispositions, values and resources to pursue their goals? What is the role of teachers' initial preparation and experience to achieve instructional goals that are consistent with mathematical practices? To delve into the teachers' preparation implies to recognize that there are multiple paths or programs and traditions to prepare prospective teachers around the world. In some cases, the faculty of education and the mathematics departments jointly share the responsibility to prepare teachers; other programs are part of a school or institutions (*e.g.*, normal schools) dedicated exclusively to the education of teachers. Both teaching models recognize the need and importance for teachers to develop the mathematical and didactic knowledge that can help them structure and implement proper conditions for students to learn the subject. However, the extent to which prospective and in-service teachers develop the mathematical sophistication needed to structure a sound mathematical lesson, to interpret students' ideas or comments and guide their learning has recently been questioned and generating an ongoing debate.

Even (2011) states that the assumption that "advanced mathematics studies would enhance teachers' knowledge of mathematics, which in turn will contributed to the quality of classroom instruction" needs to be reexamined in terms of what means for teachers to have adequate subject-matter knowledge to become an expert teacher and how a teacher can develop and use that knowledge in her teaching.

> Many practicing teachers, for different reasons, have not learned some of the content they are now required to teach, or they have not learned it in ways that enable them to teach what is now required. … Teachers need support if the goal of mathematical proficiency for all is to be reached. The demands this makes on teacher educators and the enterprise of teacher education are substantial, and often under-appreciated (Adler, et al., 2005, p. 361).

To shed light on the role of advanced mathematical knowledge in teachers' classroom decisions, Zazkis and Mamolo (2011) provide examples where teachers' awareness of that knowledge becomes useful to orient the development of a lesson. They use the construct *horizon knowledge* to refer to "teachers' advanced mathematical knowledge which allow them a "higher" stance and broader view of the horizon with respect to specific features of the subject itself (inner horizon) and with respect to the major disciplinary ideas and structures…occupying the word in which the object exist (outer horizon)" (p. 10). For instance, they describe a class event where Mrs. White asked her Grade 3 students to count the number of triangles formed by drawing segments from each vertex of a regular pentagon to the other vertices. When students provided their answers (32, 27), Mrs. White noticed that both were wrong and her recognition was based on using a mathematical result she had studied in her university course of algebra. Thus, Zazkis and Mamolo suggest that it is important for teachers to take advance mathematics courses in order to wide their mathematical horizon and to use that knowledge during the development of their teaching practices.

The ways in which prospective teachers study mathematics courses play a crucial role in developing resources and contents to be used in their teaching. That is, it is not sufficient for prospective teachers to take advance courses; but they also need to reflect on ways to connect mathematical results with other problems or situations. In this context, the systematic use of diverse digital tools can provide prospective teachers with the opportunity to enhance and extend problem-solving approaches that involve the use of paper and pencil. For instance, with the use of a dynamic software they can construct models of mathematical problems where objects or elements can be moved within the model to observe patterns of parameters that emerge as a result of moving those objects. In this context, it becomes important for prospective and practicing high school teachers to work and discuss ways in which the use of the tool helps them transform some routine problems into a set of activities that fosters mathematical reflection and connections between concepts. These problem-solving experiences are useful for teachers to explore, extend, and discuss mathematical concepts that later will shape their ways to frame and implement problem solving activities in their classrooms.

THE CONTEXT, PARTICIPANTS AND METHODS

Our research group formed by two mathematics educators, two mathematicians, and six high school prospective (three) and in-service teachers (three), has focused its agenda on analysing the types of reform that teachers' educational programs need to consider in order to incorporate the systematic use of digital technology to develop both mathematics and didactical knowledge. Our initial literature review allowed us to recognize that there is scant information on the characterization of ways of reasoning that subjects or learners can construct as a result of using a particular tool and how the use of several tools can help them enhance their problem solving approaches. That is, teachers need information regarding what types of representations and explorations of tasks or problems can be constructed with the use or a particular tools and the kind of arguments, including visual and empirical, that can be used to develop mathematical understanding and to support tasks solutions.

Our point of departure was to work on mathematics problems that appear in textbooks with the use of dynamic software and discuss them, within the group, features of mathematical reasoning that characterize and are consistent with this approach. Thus, we focus on contrasting the software approach that involves visualizing and supporting mathematical results through the use of empirical and geometric arguments with the analytic approach that relies on the use of algebra to identify mathematical relations. In this process, the software properties are helpful in visualizing the behaviour of particular parameters or relations without making explicit the algebraic model. We contend that while the use of the dynamic tool demands that problem solver think of the problem in terms of properties and their geometric meaning to construct a dynamic model; the analytic approach asks the problem solver to represent and explore the problem through an algebraic model.

The problem solving sessions and the unit of analysis. The group sessions were first developed through four three-hours meetings where we initially worked on a set of textbook problems and discussed, within the group, the role that the use of the software played in representing, exploring, and solving each problem. During each meeting one or two of the members of the group presented a task and possible ways to represent it through the dynamic software to the group. At this stage, the rest of the group got involved in open discussions that included asking for clarification issues, concept explanation or proposing other ways to approach the task. Later, we continued the group discussion via online conferences. In this report, **the unit of analysis** is the group's work exhibited while working on, and discussing the textbook problems. We do not intend to describe in detail the contribution of each group member to the solution; instead, we focus on what the group as a whole agreed and identified as important ideas associated with the problem solving process. To illustrate the common mathematical features that emerged during the solution process, we relied on a set of textbook tasks that appears at the end of unit that involves the study of perimeters and areas of triangles. It is important to mention that the initial task was to analyse the list of problems in terms of concepts involved and possible strategies needed to approach each problem.

We found that the format and wording of the textbook problems, in general, asked to find a particular answer to each problem and there was little opportunity for learners to think of multiple approaches or go beyond the asked solution. Here, we regrouped the problems and identified those that could be extended and connected with key mathematical ideas.

An Initial Prompt

Two textbook problems in which students are asked to find areas and perimeters of particular triangles were slightly changed and posed as:

a. Draw two triangles that share one side AB (base) and whose third vertex lies on a line that is parallel to segment AB. By observing the figure, do those triangles have the same area/perimeter (explain)? With the use of the software, observe what happens to the area and perimeter values of the triangles when the third vertex is moved along line L.

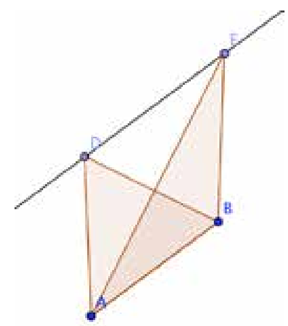


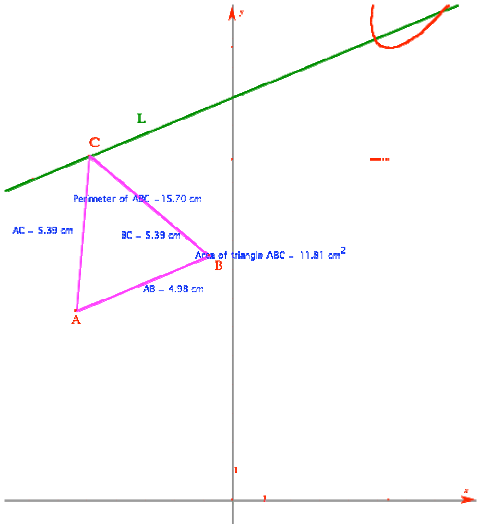
Figure 1: What happens to the triangle's area when point C is moved along line L?

b. Can you construct three triangles that share a common side (base) and also the same perimeter?

The group discussion of the tasks was framed around two problem-solving principles:

- All tasks are conceptualized as opportunities for learners to connect or extend initial statements.
- Solving the tasks involves looking for different ways to represent, explore and solve them and contrasting mathematical qualities associated with the solution process.

For example, during the process of dealing with the first task, the representation shown in Figure 2 was generated and it became a source of introducing several concepts into the discussion.



What happens the triangles' area and perimeters when point C is moved along line L? How can you graph the area and perimeter behaviours? Is it possible to identify a position where the triangle ABC reaches it minimum perimeter? How can we prove it?

Figure 3: Graphic representation of the perimeter and area of generated triangles.

During the discussion, it was recognized that the use of the dynamic software, became important to move objects within the representation and to quantify particular parameters (side, area, perimeter, angles, etc.) embedded in the representation and observe their behaviour. The use of the tool also allows the problem solver to graph particular functions or relations without defining the algebraic model explicitly. For example, the graph of the variation of perimeter as a function of the length of side AC and the corresponding perimeter as point C is moved along line L. Here, conjecture emerged:

> *From all triangles that are formed by moving point C along line L, the triangle with minimum perimeter is located when point C is the intersection of line L and the perpendicular bisector of AB. That is when triangle ABC is isosceles.*

Other concepts that were addressed while discussing this task include:

- The concept of height of a triangle and the calculation of the area.
- The concept of variation (graph of perimeter) and constant function.
- The concept of perpendicular bisector and its relation to the perimeter variation.
- The concept of infinity (how many triangles can be generated while moving point C along L?)
- The use of the Cartesian system and ways to support conjectures.

Comment: Thinking of this type of task in terms of paper and pencil approach often leads learners to use a formula (triangle area, in this case) to conclude that the area of two triangles that share the same base and height is the same. However, with the use of the tool, the dynamic model of the task offers the problem solver the opportunity to observe invariances or parameters behaviours that often become a set of conjectures that needs to be supported mathematically. The visual representation and empirical data associate with the parameters' behaviours were important to formulate conjectures that later needed to be proved or rejected. In addition, dragging points or objects within the configuration provided useful information to focus the group's attention on key elements or properties that help us construct arguments to support those conjectures.

The first task also provided the context to address problem b). The idea was to construct a family of triangles with a common base and the same fixed perimeter. With the use of the software, Figures 3 and 4 represent two ways to explore the problem:

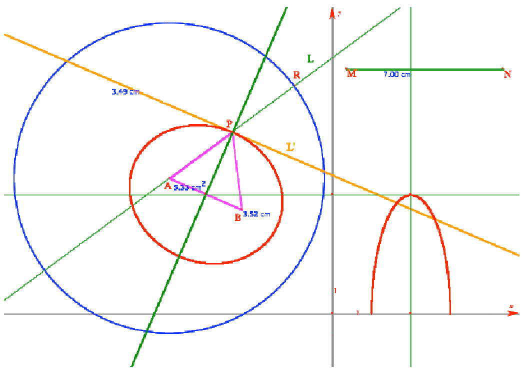


Figure 3: Drawing the condition through the perpendicular bisector

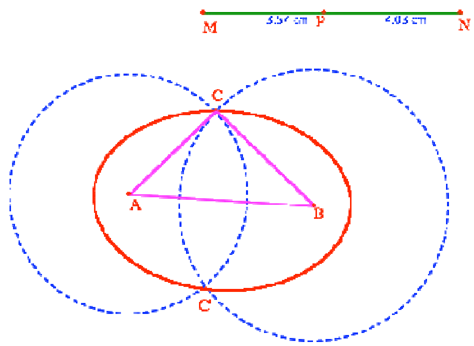


Figure 4: Another way to explore the problem

What is the locus of point C when point P is moved along segment MN?

In Figure 3, segment AB represents the common triangle side and segment MN the sum of the other sides of the triangle. That is, the perimeter of triangle ABC is the sum of segment AB and segment MN. Point A is the centre of a circle with radius segment MN and AR is a line passing by R (which is any point on that circle). L' is the perpendicular bisector of segment BR which intersects line AR at point P. The locus of point P when point R is moved along the circle determines the set of points that are the candidates to locate the vertex C to form a family of triangles with a fixed perimeter. Indeed, the locus is an ellipse since PB = PR (definition of perpendicular bisector) and AR is the radius of a circle. Figure 4 represents another way to explore the same problem: Segment AB is the common side and MN is the sum of the other two sides of the triangle. P is a point on segment MN. Two circles are drawn: One with centre point A and radius MP and other with centre at point B and radius PN. These circles intersect each other at points C and C'. The locus of point C when point P is moved along segment MN is an ellipse and each point on this locus determines and becomes the third vertex of triangle ABC with fixed perimeter. Thus, Figures 4 and 5 were two dynamic models used to represent the problem and the source to discuss:

- The relationship between the common side AB and the sum of the other sides (segment MN). That is, when can the triangle be drawn? (the triangle's inequality).

- Definition and properties of the locus (ellipse).
- The area variation of the family of generated triangles. Again, the intersection point of perpendicular bisector of segment AB and the perpendicular bisector of segment BR determines the vertex C where triangle ABC gets its maximum area (Figure 3).
- Connections that emerge while moving point B out of the circle (Figure 5). It is observed that locus becomes a hyperbola.

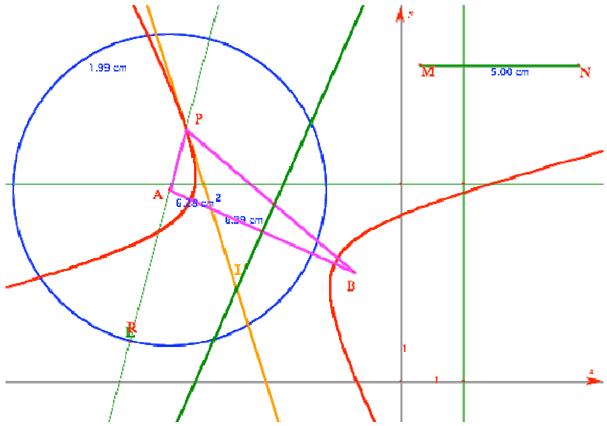


Figure 5: when point B is moved outside the circle, the locus is a hyperbola.

Comment: A main teachers' instructional goal is to encourage their students to extend and connect the contents they study with other ideas and situations. There is evidence that the task dynamic model exploration allows the problem solver to relate contents that in a traditional curriculum appear in different units. For example, the second task that asked for the construction of three triangles with the same perimeter became a platform to generate the conic sections that appear in an entire course of analytic geometry. In addition, the area behaviour of the family of triangle generated in figure 4 was analysed graphically to determine the triangle that reaches the maximum area. That is, the use of the tools offers problem solvers the opportunity to relate and connect concepts and themes that emerge as a result of moving objects within the representation or configuration of the task.

DISCUSSION AND REFLECTIONS

In retrospective, we recognize that the problem solving process involved while working on the tasks show several features of the type of reasoning and mathematical thinking that emerge as a result of approaching the tasks with the use of dynamic software. It was observed that there are several strategies and ways to think of and explore the problems that are associated with the use of the tool. Dragging objects within the model is a key strategy to identify and explore mathematical relations. The controlled movement of objects demands that problem solvers develop a set of heuristic that are crucial to explore and analyze the behaviors of objects relations. For example, drawing circles or extending segments to move lines on the plane or points along lines are strategies associated with the movement of the model. Assuming the problem as solved is a key heuristic that is enhanced with the use of technology. Here, by analyzing elements associated with the solution, problem solvers can obtain useful information to explore the problem dynamically. In addition, finding loci of particular objects not only becomes important to identify patterns of behavior of some parameters in the model; but also the same loci are tools to find mathematical relations. There is evidence that the use of computational technology (a dynamic software) can offer learners the possibility of transforming some routine problems, found in regular textbooks, into a set of tasks where they can exhibit and contrast different ways of reasoning about the problem that involve visual, empirical, and formal approaches. In this context, the use of dynamics tools plays an important role for conceptualizing the tasks as an opportunity for learners to engage into an inquiring or inquisitive process that goes beyond reporting a particular solution. For example, the dynamic models of the two tasks become a platform to explore not only different forms to represent emerging relations; but also ways to extend and connect the initial statement of the tasks. In this process, it is possible to generate graphic behaviour of particular relations (perimeter variation and locus of particular objects) without defining the algebraic model. In addition, concepts like perpendicular bisector and loci of points became relevant to explain and justify those relations. In general terms, the use of the tool offers problem solvers the opportunity to examine graphically relations that later can be explored and contrasted algebraically. In this context, the use of the tool complements or extends mathematical reflection that learners engage in algebraic approaches. Discussing the tasks and the use of dynamic software throughout the problem solving sessions provided us, as a research group, important information to think of possible routes for teachers to frame and implement problem solving activities in the classroom. Our current project goal is to include more high school teachers in our group and to work on their own textbook problems in order to discuss ways in which the use of technology can help them orient their teaching practices. By reflecting on main issues addressed in the initial research question, we learned that:

a) Many of the routine problems or exercises that appear in textbooks could be transformed, with the use of the tool, in a series of activities to engage learners in problem solving experiences. Thus, teachers do not need to think of a set of sophisticated problems to use the tool in problem solving approaches.

b) Problem solvers or learners can develop certain experience in the use of the tool by experimenting and discussing strategies that involve moving or dragging objects within the representation, finding loci of particular objects, quantifying attributes such as areas, lengths, perimeters, and by contrasting other learners' approaches to the problems.

c) The discussion of concepts and ways used to represent and explore the tasks became crucial to relate, integrate and extend the initial contents associated with the tasks. In this process, there is also an opportunity to review and extend some of the concepts or themes that appear while dealing with the tasks.

d) Working on the routine tasks with the use of the tools helped us design worksheets that can be used to orient teachers in the process of transforming the tasks into a platform to formulate conjectures and ways to explore and support them.

CONCLUDING REMARKS

In responding the research question posed initially, we argue about the importance for researchers and in-service teachers to work and discuss within a community what the use of the tool enhances in terms of ways of representing, exploring, solving and extending routine mathematical tasks. As a result, typical tasks found in textbooks offers a point of departure to construct dynamic models or tasks in which problem solvers can identify and explore not only different mathematical routes (contrasted with algebraic approaches) but also possible didactic ways to emphasize the coordinated use of empirical, visual, and graphic approaches.

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THRESHOLD CONSTRUCTS INSTRUMENTING TEACHERS' ORCHESTRATION OF AN INQUIRY WITH GEOGEBRA

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In this paper we investigate how to efficiently empower teachers to implement and orchestrate a mathematical learning activity supported by digital technologies. The particular learning activity in this study is intended to facilitate learners' transition from the Pythagorean Theorem to the distance formula and the equation of a circle. The activity comprises structured and guided inquiries involving laptops with GeoGebra and traditional resources. It has been tested with 38 upper secondary students and two mathematics teachers. Our results indicate that a singular discussion with the teachers, based on the researcher's prospective analysis of the activity with main focus on threshold constructs and self-regulating skills, suffices to support the teachers' implementation and orchestration of the activity.

INTRODUCTION

There is not only an abundance of digital technologies available in society, but also an abundance of research about learning mathematics with technologies. Still, research and current teaching practices do not seem to provide sufficient guidance on how to efficiently and systematically integrate digital technologies in mathematics education (Drijvers, 2012). However, numerous efforts in design-based research provide examples of singular good practices that serve as inspiration and proofs of existence that technologies can significantly enhance the teaching and learning of mathematics (Hegedus and Moreno-Armella, 2009; Sollervall and Milrad, 2012; Drijvers, 2012). The issue of scaling up innovations is being increasingly attended to by the mathematics education research community (e.g., Jo Boaler, plenary session at PME-NA 2012; Hegedus and Lesh, 2008). Several authors have recognized that research needs to address systemic issues at macro-level as well as how to efficiently implement classroom innovations at micro-level (e.g., Hegedus and Lesh, 2008).

In the current effort, we will address the implementation of a mathematical learning activity at micro-level, involving two teachers and 38 students in the beginning of their second year at the Natural Sciences Programme in upper secondary school.

Moving beyond teacher-driven improvement of their current practices, as addressed by Lesson Studies and Learning Studies (Lo, Marton, Pang, and Pong, 2004), we have in several research efforts applied a research-driven approach where researchers with complementing domains of expertise engage in the collaborative design of innovative mathematical learning activities supported by digital technologies

(Sollervall and Milrad, 2012). In order to fully exploit the opportunities for learning that are afforded by digital technologies, we have chosen to let the participating researchers with technological expertise be responsible for communicating the affordances of these technologies to the members of the research team. Rather than involving teachers at the level of technologies, we have designed prototypical activities that are presented to and adjusted by the teachers before implementation (Sollervall and Milrad, 2012). Because of the technical complexity in some of these prototypical activities, we have engaged expertise also in the implementation phase.

In the current research effort, we are making use of a stable, commonly used, and readily accessible interactive software, namely GeoGebra, that allows the teachers to be in charge of implementing the activity with their students. The three tasks in the activity are presented and discussed during a meeting on the day before implementation, with focus on mathematical constructs and possible teacher interventions that may be crucial for the students' successful completion of the tasks. The teachers propose additional possible obstacles that are discussed and addressed through alterations of the activity. Within the limited scope of this paper, it is not possible to address all aspects of the learning activity in detail. Instead, we will present the activity in a similar way that it was presented to the teachers, although this implies a superficial treatment of some of the theoretical underpinnings for the activity.

The activity blends students' constructions in GeoGebra and traditional work with pen and paper. The development of the activity is framed by the methodology of design-based research that allows us to attend not only to traditional outcome evaluation but also development of learning activities and prospective (a priori) analysis of hypothetical outcomes (Cobb, Confrey, diSessa, Lehrer and Schauble, 2003). The development phase involves negotiations of a preliminary activity with a prospective analysis based on hypothetical learning trajectories (HLT: Drijvers, 2003; Sollervall and Milrad, 2012). Another key aspect of design-based research is its cyclic character that allows adjusting and improving a preliminary activity (Drijvers, 2003). The version presented in this paper is in its second iteration. The first version was implemented with three first-year secondary students in May, 2012. The evaluation of this implementation resulted in minor alterations of the technical instructions and a decision to involve laptops instead of an interactive whiteboard. The latter decision was mainly due to an ambition to implement the second iteration of the activity in a whole class setting.

Furthermore, a restructuring was made by sequencing the first two tasks according to the involved mathematical processes – defining, representing, generalizing, and justifying – and the corresponding actions on the products of these processes, following the Processes and Actions framework (Zbiek, Heid, and Blume, 2012). These four processes, as well as other theoretical constructs, were used in our discussions with the teachers. In this paper, we investigate the issues addressed in these discussions and how they influence the implementation of the activity.

RESEARCH OBJECTIVES AND RESEARCH QUESTIONS

This paper addresses the implementation of an inquiry-based learning activity, intended to facilitate mathematical processes and actions related to upper secondary students' transition from the Pythagorean Theorem to the distance formula and the equation of a circle. Students engaging in the activity are challenged to create several threshold constructs – as external (physical) or internal (mental) constructs – that serve a crucial role in promoting their continued mathematical inquiry.

Our research questions are:

- How are threshold constructs and self-regulating skills addressed in the discussions between the researcher and the teachers?
- How do the problems that students encounter during the implemented activity relate to the threshold constructs and self-regulating skills?

In the next section, we discuss the notion of threshold construct as a local version of threshold concepts (Meyer and Land, 2005). We also address the notion of inquiry and the self-regulating skills that facilitate successfully completing an inquiry.

THEORETICAL AND METHODOLOGICAL CONSIDERATIONS

From a teacher perspective, it is essential to identify and address the threshold concepts that facilitate learning in a specific subject area. An example is the threshold concept 'limit' in Calculus. In general, threshold concepts can be seen as 'conceptual gateways' that lead to previously inaccessible ways of thinking about something (Meyer and Land, 2005). In this study, we will attend to threshold concepts that are localized to the learning trajectories for a specific activity focusing on learning coordinate geometry and specifically the distance formula. We refer to these local and specific threshold concepts as *threshold constructs* for the activity. In Figure 3 (left pane) the triangle PAB serves as a threshold construct (when the distance formula is not yet available) that together with the coordinates $(x,4)$ of the point B affords constructing the algebraic expressions x and $y - 4$, respectively, for the catheti of PAB. Such threshold constructs may be identified either in a prospective analysis as hypothetical constructs or actual constructs that during the implemented activity serve a crucial role in facilitating a student's learning trajectory. Since the threshold constructs are situated within a specific activity, they need to be empirically tested, updated and refined in an iterative research process that can be smoothly integrated within the methodology of design-based research.

The activity that will be presented in the next section is comprised of structured and guided inquiries (Herron, 1971) where the students have primary ownership and initiative. Inquiry-based learning challenges the students' self-regulation regarding cognition, motivation, behavior, and context, in corresponding phases of self-regulation: forethought, planning, and activation (cognition); monitoring (motivation); control (behavior); reaction and reflection (context) (Schunk, 2005).

In this paper, we will investigate if (and, if so, how) the teachers' interventions address scaffolding these phases of self-regulation or if they address other issues, with focus on the threshold constructs.

THE ACTIVITY AS PRESENTED TO THE TEACHERS

A meeting between the researcher and the two teachers (and a third teacher) took place in the afternoon on August 29, 2012, the day before implementation. Before the meeting, the teachers had received a preliminary printed version of the student instructions. These instructions consisted of a cover page with general information, seven pages with task instructions, and a brief two page manual to GeoGebra.

The first two tasks were presented as structured inquiries, while the third task was a less structured so called guided inquiry with no suggested sequence of steps for solving the task (Herron, 1971). The teachers were informed that self-regulated inquiries impose a substantial cognitive load on the students, which implies a need for the students to consolidate their experiences in order to process and retain what they have just learned (Kirschner, Sweller, and Clark, 2006). We decided to address this issue by allowing time (15 minutes) for the students to write individual reflections at the end of the activity. The teachers agreed to follow up on these reflections during the following lesson. This arrangement left 2.5 hours (8.30 am – 11.00 am) for working with the three tasks, including a short break.

As the activity involves interaction with laptops supporting the software GeoGebra that the students have only used on a few previous occasions, the issue of instrumental genesis (Verillion and Rabardel, 1995) was specifically addressed in the design process. We designed the first two tasks as staging activities involving not only mathematical learning objectives but also serving to familiarize the students with the character of the activity and specifically the involved technologies (Edelson, Gordin, and Pea, 1999). Furthermore, it was agreed to give the students sufficient time to explore, investigate and resolve problems by themselves, and that guidance should be given restrictively, although this could imply that they will not manage to complete all the tasks. It was agreed to be sufficient if the students complete two of the three tasks.

The activity was designed for students who are familiar with the Pythagorean Theorem, the square of a binomial, and the equation of a straight line, but with no previous experience of working with the distance formula. The general objective for the activity was to let students explore geometric relationships, pose hypotheses, confirm these hypotheses by producing algebraic proofs, and thus making connections between geometric and algebraic representations.

In the first task, the students are asked to find (other) points equidistant to two given points P and Q, that is, with the ratio 1:1 between the distances (Fig. 1).

This first task involves working only with GeoGebra following instructions that begin with setting **Perspectives** to **Basic Geometry**, thus giving the students a clear background to work on, showing no grid and no coordinate axes in order to stimulate them in attending only to the geometric relations. The students are asked to first place two points P and Q beside each other on the screen. Then they are asked to place a third point in the screen and were given the following information:

The condition in this task is that the new point should be located equally far away from the points P and Q:

The distance from the point to P should be equal to the distance from the point to Q.

One of the teachers objected that one of these sentences could be removed since they both give the same information. The researcher argued that the redundancy could help students who may be able to interpret one of the sentences but not the other. This position was further supported by arguing that the sentences are qualitatively different, as the first sentence addresses the condition from a procedural point of view while the second sentence objectifies the condition. It was decided to not change the phrasing of the sentences but to keep them as in the proposal.

Next, the students are informed that there are many points that satisfy the condition. They are instructed to place several such points on the screen (such a possible construction is illustrated in Fig. 1, left pane) and then answer the following question:

What geometric figure do you get from all the points that satisfy the condition?

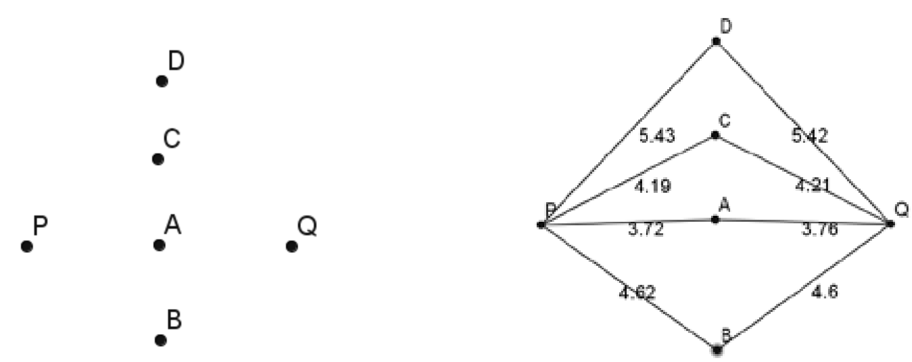

Figure 1: Constructions in GeoGebra according to the instructions for the first task.

On the next page, the students are invited to check the placement of their points, and if necessary adjust their placement, by having GeoGebra measure the distances from each point to P and Q, respectively (Fig. 1, right pane).

The second task addresses the same geometric 1:1 condition and the same question, but now with the points P and Q placed in a coordinate system. The students are instructed to keep their constructs from the first task, show **grid** and **axes**, and place the points at P = (0,4) and Q = (2,0) respectively. A construction according to the geometric 1:1 condition is illustrated in Figure 2 (right pane).

On the next page of instructions the students are asked to work with pen and paper to find the equation of the straight line.

Thus far, the tasks have called for the students to follow instructions. In relation to the Processes and Actions framework (Zbiek et al., 2012) they have been acting on definitions by interpreting and representing them, generalizing from a few points to all points, justifying geometric constructs by using GeoGebra, and acting on a generalization of a visual representation by representing it algebraically.

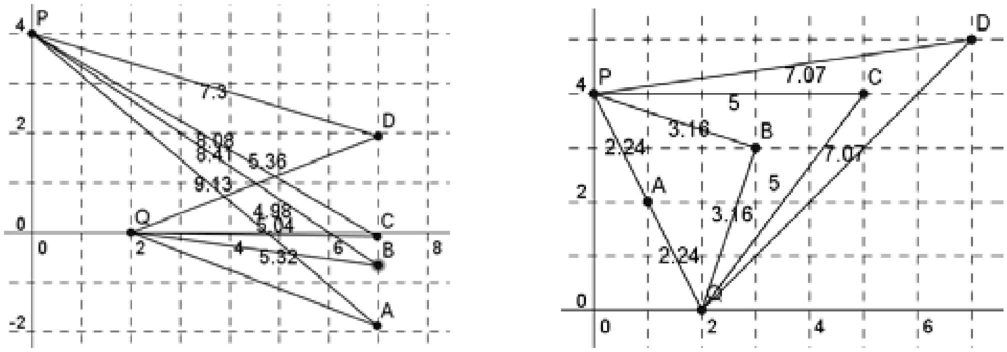

Figure 2: A possible initial setup (left pane) and a correct construction (right pane).

The continuation of the second task is more challenging. The students are now asked to *prove algebraically that all the points (x,y) that satisfy the condition lie on a straight line*. They are first instructed to remove all points except P, Q, A, and remove the labels on the segments AP and AQ. Next, they are asked to change the labels for P and Q so that they show **Name & Value**. Thereafter, they are asked to place the point A upwards and to the right of P and Q, and place a new point B directly to the right of P and directly below A (Fig. 3, left pane).

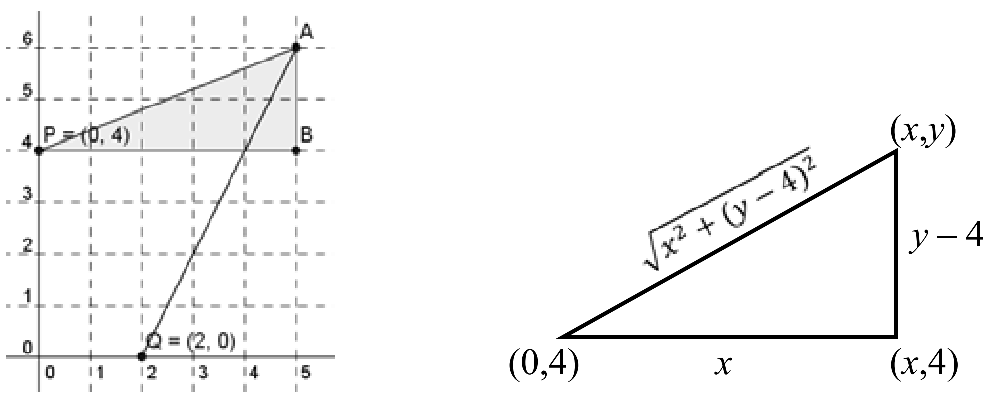

Figure 3: Continuation of task 2 in Geogebra (left pane) and on paper (right pane).

The students are then asked to draw the triangle PAB in GeoGebra (Fig. 3, left pane). Before turning to the next page of instructions, where they are instructed to work with pen and paper, the students are asked how the hypotenuse of the triangle relates to the task they are working on.

They are asked to copy the triangle PAB onto a piece of paper, including the coordinates for P and B. They are asked to set A = (x,y). Their first subtask is to find algebraic expressions for the catheti of the triangle, and the second subtask is to find an algebraic expression for the hypotenuse. In the first iteration, the students were instructed to first determine the coordinates for the point B, but in the discussions with the teachers it was decided to leave this construct as a challenge to the students. Next, the students were asked to find an algebraic expression for the distance between A and Q, without sequencing any steps in the instructions.

Finally, the students were instructed to set the lengths of AP and AQ equal to each other and simplify the equation. As expected, many students did not simplify carefully and made numerous algebraic errors. However, quite a few groups of students managed the first two tasks more or less on their own. Only one group of three students made a serious attempt on the third task, which we do not account for here. Instead, we proceed to highlight some of the presented results in relation to the research questions. We address both questions under each heading.

DISCRIMINATING AND GENERALIZING VISUALIZED POINTS

Already during the first iteration of the activity, it was confirmed that the reference points P and Q (compare Fig. 1) interfered with the interpretation of the geometric figure. The students' first guess was 'a rhombus'. During the second iteration, several groups initially answered 'a cross' or 'a triangle', even after they had constructed the line segments that allowed them to measure distances. Our interpretation is that the entire picture in Fig. 1 was prioritized before the condition, although the written condition was emphasized in the instructions by being italicized in a large font size. A comment by one of the teachers 'they are not used to working with figures as sets of points' can be seen as further explaining why the picture was favored in the students' work. Interventions by the teachers (and the researcher) involved suggestions to test each point against the condition by reading the condition for each specific point. When the students finally read 'the distance from P to P' they readily concluded that P does not belong to the figure. Furthermore, the teachers pointed out to several groups that the figure consists of all the points that satisfy the condition, not only the points they had placed on the screen. These incidents relate to discrimination and generalization (Bruner, 1966) as cognitive processes that the students need to apply in relation to a mathematical definition. Discriminating a geometric figure in a picture, generalize a geometric figure from a finite set of point, and acting on a formal definition, can all be regarded as threshold concepts in the cognitive domain, related to the self-regulating skills forethought, planning and activation.

CONTROLLING AND REFLECTING ON ALGEBRAIC TREATMENTS

As already mentioned, several students made incorrect treatments of the algebraic expressions in the final part of task 2, for example the classical mistakes of Case 1: replacing the square of a binomial with the sum of the squares of the two terms and Case 2: canceling square roots and squares term by term in the equation

$$\sqrt{(x-0)^2+(y-4)^2}=\sqrt{(x-2)^2+(y-0)^2}$$

Several of the student groups attempted to work very quickly with this subtask. When asked to check their expressions, they readily found their mistakes to Case 1 by writing the square as the product of two binomials. Case 2 was mainly handled by the groups themselves, when asked by the teacher to check their work. Our conclusion is that they knew the algebraic rules, but proceeded carelessly by working too quickly and not taking care in structuring and checking their written expressions. Hence, their algebraic mistakes can be explained by insufficient self-regulating skills related to control – as the mistakes are caused by the behavior of working carelessly – and reflection, which is particularly important in the inquiry context.

A SCAFFOLDED THRESHOLD CONSTRUCT BECOMES A CHALLENGE

During the first iteration, the students were instructed to determine the coordinates for the point B in Figure 3. As mentioned earlier, it was decided to remove this particular instruction in the second iteration and instead relying on the teachers to provide guidance to the students, if needed. Instead of providing embedded scaffolding for the particular threshold construct B = $(x,4)$, this part of the activity was transformed into a genuine challenge for the students. Some of the students managed this particular task by themselves, while those who got stuck were suggested by the teacher to attend to the coordinates of B. This proceeded smoothly, as the teachers were aware of this particular threshold construct.

In design-based research, particularly with technologies, the aim is often to work in the opposite direction and attempt to replace teacher scaffolding actions with scaffolds that are embedded in the activity, thus reducing the demands on the teacher and enhancing the possibilities for a teacher accepting to integrate the activity in her classroom (Wong, Looi, Boticki, and Sun, 2011). On the other hand, it is desirable to empower the students, show trust in their capabilities, and thus nurturing their collaborative and self-regulating skills (ibid.). Our approach of negotiating threshold constructs and issues of self-regulation with the teachers allows us to achieve a reasonable balance between teacher scaffolding actions, embedded scaffolds, and challenges intended to empower the students in their collaborative work.

CONCLUDING REMARKS

In a classroom environment where students mainly solve problems in a textbook, most mistakes can be easily adjusted by checking answers or asking peers or the teacher. In such an environment, individual students are not responsible for monitoring, control, and reflection on their work. Mistakes do not propagate and do not affect future work, so students are not stimulated to develop strategies for self-regulation. In comparison, even a minor mistake in an inquiry can have fatal effect on its continuation and may cause the students to fail in achieving the intended learning objectives. The open inquiry is sometimes, but certainly not always, the best format for an inquiry. By initially planning a structured inquiry we expose its inherent challenges regarding threshold constructs and demands for self-regulation.

Negotiating implementation of the activity with teachers creates awareness of these challenges and demands. This process empowers the teachers to take informed decisions about restructuring and possibly open up the inquiry by re-distributing scaffolds on the teacher and peers or embedding them in the activity. The negotiation phase also contributes to empower the teacher as orchestrator of the comprehensive activity and its scaffolds. In our opinion, this is an efficient way to implement research-designed activities in the authentic setting of the teacher's own classroom. From a research perspective, the negotiation phase is a good opportunity to bring teachers' knowledge of content in relation to students and teaching into the research process (KCS and KCT: Ball, Thames and Phelps, 2008).

It is apparent that GeoGebra works well in the upper secondary school classroom as a stable and user-friendly software with commands that align well with mathematical thinking and notation. Minor adjustments of settings are readily handled and specific commands are shared among the students. However, the pedagogical implementation of GeoGebra in the mathematics classroom has to be carefully considered so that it not only supports solving the tasks but also supports the students in achieving the mathematical learning objectives. In our activity, it would have been easier to have GeoGebra provide the equation of the straight line in the second task. Instead, we chose to let the students do this tedious work by hand, particularly since the learning objective was the distance formula and not the equation of the straight line.

Furthermore, it may be noted that the distance formula was not explicitly addressed in our activity, but rather emerged as a useful tool for solving a task in coordinate geometry. The teachers commented on this particular aspect by saying 'this is really a good way to work with coordinate geometry, instead of working with the dull standardized tasks in the textbook'. We wish that this may imply that the students who engaged in our activity have learned to appreciate the distance formula as a good thing to know and not only yet another formula to memorize.

In this study, we have identified threshold constructs that relate to geometric figures, algebraic treatment, and self-regulating skills. In future research efforts, we will continue to characterize the nature of the identified threshold constructs.

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GRAPHIC CALCULATOR USE IN PRIMARY SCHOOLS: AN EXAMPLE OF AN INSTRUMENTAL ACTION SCHEME

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This paper presents an empirically based case study design within a sociocultural theoretical framework. The research aimed to describe the implementation of a graphic calculator in a fifth grade primary school class when four students were engaged in mathematical activities performing tasks and challenges given by the researcher. The students' activities associated with the process of appropriation of a technological artifact played a prominent role in the data analysis. The distinction between artifact and instrument through the instrumental approach was in focus. The schemes of instrumented action were of special interest. The case study concerns students' discovery of the function key exponentiation with the inscribed symbol ^ at the key button, and an analysis of exponentiation by schemes of instrumented action.

Keywords: *graphic calculator, exponentiation, instrumented action scheme, appropriation*

INTRODUCTION

This case study is about students-initiated calculator activity. A student named Signe calls for attention about her findings concerning the exponentiation [1] function key on the keypad. The study is part of a larger research project where data was collected from task-based video interviews with students undertaken by the researcher. The focus of the reported research was implementation of an advanced graphic calculator (GC) in a 5th grade primary school class and characterization of its use. The choice of research strategy can be described as a case study design. The study is presented in a sociocultural theoretical framework where a GC is a sociocultural semiotic artifact. Artifact properties, such as the function keys on the keypad were variables considered in this study. The instrument-mediated activity approach was in focus. The theoretical framework referred to as instrumentation (Trouche, 2003) associated with research in technology educational environments is applied as the lens for data analysis. The students were accustomed to using a four-function calculator. Advanced GCs are usually not part of teaching math at the primary school level. The purpose of introducing the advanced GC was to investigate the longitudinal aspects of its appropriation by students. A task portfolio was designed for the project. The students' spontaneously initiated exploration activities of the exponentiation function sustained by the GC, was not part of the task portfolio or the project agenda.

THEORETICAL FRAMEWORK

The main theoretical perspective that frames our investigation is the instrumental approach associated with research in technology educational environments.

Instrument genesis is suitable to analyze processes described as appropriation by which an artifact becomes an instrument in the hands of a student, in terms of transforming an artifact into an instrument to become a cognitive aid for the student to solve a given task. The process of appropriation of an artifact to become an instrument, a tool for the competent user is further elaborated by (Trouche, 2003). The instrumental approach with the human/computer interaction and human use of tools constitutes two main processes of instrument mediated activity. The process directed from the artifact towards the human agent is named instrumentation. The tools convey, shape and transform the human agency, and the subject adapts to its constraints and affordances. The other process, instrumentalization, is directed from the human agent towards the artifact which includes stages such as familiarization with the instrument, mastering and adopting the instrument to one's own personal, specific needs. Rabardel (2002) drew on Wertsch's (1998) key construct of mediated activity. Students mediated calculator activity transferred to individuals-appropriate-technological-artifacts-in-cultural-practice constitutes the unit of analysis.

METHOLOGY

The choice of research strategy in this study can be described as a case study design on two levels. On the first level, there were 23 students in a fifth grade (students aged 11-12) primary school class. The first level formed a backdrop for the next level, the selection of four students. There were two boys and two girls placed into two small groups, one of each gender. The choice of the two boys and two girls was partly based on pre-tests, achievement tests from a pilot study involving a representative sample of the class, but also on the students' motivation as the two groups of friends wished to participate in this study that lasts over two years. We meant that the latter criterion was a good argument for the participants not losing interest during the period of data collection. The second level of the case study consisted of the two respective groups, and was a continuation and an elaboration of the study where the entire class constituted the background. In this study, each student had a handheld calculator at his or her disposal. The sophisticated icon-driven graphic calculator Casio fx9750G PLUS was introduced by the researcher including a 21-character x 8-line display. The mediated artifacts, among them the graphic calculator, were available and were included in their mathematical activities. The data collected and analyzed in this research study was from video-taped interviews when each of the four students were in work sections working individually, and when they were collaborating in small groups engaged in mathematical activities, challenges and tasks provided by the researcher.

ANALYSIS OF THE TRANSCRIPTION

The analysis of the transcript below is an example of students' appropriation of the artifact function key for exponentiation, a part of the graphic calculator, described by the instrument genesis process.

The built-in artifact exponentiation with its function key ^

The term artifact in the case of calculator is material, an example of a physical or touchable object but also a semiotic artifact. Less obviously, algorithms implemented on a computer or on a calculator, in a sociocultural perspective, are regarded as material artifacts. Algorithms are created and used in mathematics and can be programmed into computing devices. Through their materialization in written and spoken entities, algorithms do not exist without socio-historical invented signs. A graphic calculator is a complex technological digital artifact, which is constituted of a collection of artifacts. For example, if one considers the entire software package as a single artifact, a function for calculating the arithmetic mean or a one for solving quadratic equations are both built-in algorithms. Each of these examples is regarded as a single artifact. The physical artifacts such as the screen display, the keyboard or a single key on the keypad is each part of the artifact graphic calculator. The artifacts in focus in this case are the students' appropriation of the exponential application software within the GC and the associated exponentiation function key with the inscribed symbol ^ at the key button on the keyboard of the GC. The two artifacts can be considered as a consisting unit; a part of the software package which the agent is communicating with via the keystroke ^ for exponentiation in the context of a keystroke sequence 9 ^ 2 . The appropriation of the exponential function key means also appropriation of the entire artifact graphical calculator.

Transcript of students' exploration of the exponentiation key ^

The following excerpt is a transcript of a video clip of an interview of the students named Hilde, Kate, Signe and the researcher. The video clip is part of a comprehensive video material. The episode is initiated by Signe who enthusiastically requests the researcher's for attention about her findings concerning a new function key for exponentiation with the inscribed symbol ^ at the key button on the keyboard of her graphic calculator. Signe held out her hand with the graphic calculator towards researcher so he could take a closer look at the screen display.

The circumstances gave the researcher a golden opportunity to put aside the day's interview guide program and instead concentrate on Signe's initiative and encourage her to continue the exploration process by letting her lead the other students on the way. Furthermore, in the interview, the emphasis is on the students, on their own, to take the lead and not be guided or intervention by the researcher. In this context, concerning the role of the researcher as an observer, the researcher saw the necessity of balancing roles between being observer-as-participant and participant-as-observer avoiding unnecessary intrusion or interruption.

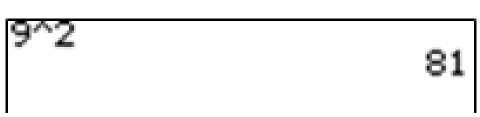

Figure 1: Signe's calculator display

112	Researcher:	Tell me Signe what do you have on your screen display?
113	Signe:	It is, eh, 9 (…) 2 up and it is equal to 81.
114	Researcher:	Could you show Tom and Hilde the calculator's screen display?
115	Signe:	Have you learned this yet? (Signe is speaking to Tom and Hilde while she is holding her graphic calculator in a position such that each of the students is able to look at the screen display.)
116	Tom:	I want to see it! What? Tell me, how did you do it? (Tom expresses astonishment at seeing the syntax shown at the screen.)
117	Hilde:	What does the number 2 mean? (Hilde gets a short glimpse of the screen display of Signe's graphic calculator.)
118	Tom:	I do not understand why it is 81 when you are taking 9 (…)
119	Signe:	Yes, because, right?, you take 9 times 9.
120	Tom:	Wow, I made it! (Tom bursts out a "wow" signalizing succeeded in typing in the keystroke sequence $9 \,^\wedge 2$ and calculated 81 that Signe just recently has shown him.)
121	Researcher:	What does the digit 2 mean?
122	Hilde:	Yes, times 2.
123	Signe:	No, 9 times 9. (Signe is making hand gestures to emphasize her answer through visualizing the arithmetic calculation by drawing in the air with her index finger the symbol 9, then the multiplication sign and another 9.)
124	Hilde:	How do I get the digit 2? Ok! (Signe is pointing with her index finger at the keypad and screen display to show Hilde the keys to be pressed that will generate $9 \,^\wedge 2$.)
125	Tom:	There we learned something. This is our secret.
126	Signe:	Yes, it is our secret.
127	Tom:	I'm not sure it is a very big secret. (Tom chuckles while talking)
128	Researcher:	What does the digit 2 mean?
129	Signe:	What the digit 2 is? It goes like two times up in the brain of the calculator.
130	Researcher:	Do you understand what Signe means?
131	Tom:	Yes, it goes twice.
132	Signe:	Two nines, it is the same as 9 times 9, only that we use the digit 2 to go little faster.

Signe's initiated exploration task is associated with the identification of the keystroke sequence displayed with the syntax $9 \,^\wedge 2$ and the students' experimental circumstances, the discovery of the mathematical identity $9 \,^\wedge 2 = 9 \cdot 9$. In (112), she has been able to make use of the function key for exponentiation on the keypad and she has managed to calculate $9 \,^\wedge 2 = 81$ appearing on her screen display. Researcher seized the opportunity by asking her to pronounce what is displayed on her screen. Signe answered in the next text line by reading aloud off the screen display and uttered the syntax "It is, eh, 9 (...) 2 up and it is equal to 81." Signe pronounced the digit 9 and the digit 2 and then "up", which could mean her visual perception that the digit 2 is located somewhat up in its position compared to digit 9. She uttered that the answer to the arithmetic calculation is 81. The reason for the researcher (114) for asking Signe to show Tom and Hilde her screen display, was, hopefully, to initiate a dialogue between the students about the new sign ^ and what the syntax $9 \wedge 2$ mathematically represents.

Making mathematical meaning of syntax for exponentiation

In (115), a dialogue is initiated by Signe's appeal to Tom and Hilde for attention by letting them see the syntax displayed on her calculator screen. Signe's utterance suggests she believed that the other two students did not have any experience of using this function key or that they didn't know about its existence and what it mathematically represents. She said this with an overwhelmingly proud smile to underscore her achievement. After Tom and Hilde cast a glance at Signe's screen, Tom appeared to be surprised (116) over the syntax configuration visualized on Signe's screen. Hilde's utterance (117) is associated with the visual appearance of the text string characters. Her focus is on the digit 2 and attention to the relative positions of the signs in the punctuation of the exponential notation (calculator syntax). Tom's utterance (118) is interpreted that he did not understand how Signe has managed to calculate 81 when she typed in 9. Tom is trying to make sense of Signe's calculator activities and his utterance concerned the mathematical content, the exponentiation. Signe chose to sit down next to Tom on the couch so they both could look at each others' calculator screens. Each of them was holding a graphic calculator in hand while Signe showed Tom the keystroke sequence and how to navigate and manoeuvre among the keystrokes to get the mathematical expression and the arithmetic calculation $9 \wedge 2 = 81$. In (119), Signe answered that it is the multiplication 9 times 9 that gives the answer 81, a result Tom most likely is familiar with. But Signe's answer may not be what he asked for, that was, the role of the digit 2 in the arithmetic calculation. Tom gave an outburst of a "wow" (120), after following guidance and instruction from Signe, signalizing that he succeeded in navigating and manoeuvring on the keypad and executed the arithmetic calculation by the use of function key for exponentiation. Researcher intervened three times (121, 128 and 130) in the students' dialogue with control questions to clarify Tom's and Hilde's perception of what the digit 2 in the calculator syntax mathematically to them represents. The questions were asked after Signe had given Tom and Hilde her verbal explanation and clarification by gesturing which consisted of keystroke sequence and the related mathematical content of the calculator syntax. Hilde's response (122) to the researcher's question (121) was that the digit 2 stands for the number of nines. However, Signe uttered (123) a definite "no" of what role the digit 2 is playing in the arithmetic calculation, and she corrected Hilde's statement by saying it means 9 times 9. Signe's utterance was simultaneously accompanied by hand gestures visualizing the arithmetic calculation. Hilde was trying to make sense of Signe's calculator activities and asked Signe about the role of the digit 2. Signe responded by showing Hilde the keystroke sequence, which constitutes $9 \wedge 2 = 81$. Hilde's response (124) after being guided by Signe was to outburst the "ok!" to confirm she had managed to type in the keystroke sequence for exponentiation. In a humorous tone (125-127), Tom and Signe proposed that they keep their findings as a secret from the other classmates. The discovery of the exponentiation button and the mathematical operator in the case of 9 raised to the power of 2 equals 81 is their little secret. The students were preoccupied with the fact that they, on their own, without the teacher's help, had learned about a new function key on the keypad on the calculator. The students had managed to give the syntax $9 \wedge 2$ mathematical meaning via the identity $9 \wedge 2 = 9 \cdot 9 = 81$. Their utterances like "there we learned something" and "this is our secret" are example of a personification of knowledge by making it their own, a characteristic feature of the process of appropriation as the process of "taking something that belongs to others and making it one's own" (Wertsch, 1998, p. 53). They reflected upon the knowledge they had acquired during the dialogue. In a joint activity, they established a shared focus of attention and a kind of working consensus of what they pay attention to in their mathematical activities. Our claim is that the students, in this context, possessed knowledge at a meta-level.

Finally, Signe (132) exhibited a pragmatic attitude to the use of the mathematical operation exponentiation. She probably connected prior knowledge about (repeated) multiplication and saw the opportunity to express the mathematical identity. Signe utilized, in this case, the advantage of exponentiation as a useful and a feasible calculation tool that is faster, more efficient and practical for an agent to perform, rather than performing calculations as repeated multiplication by the use of the calculator. According to (Artigue, 2002), Signe emphasized on the pragmatic value of technique by focusing on their productive potential like cost (time-consuming aspects), efficiency, and field of validity.

Asymmetry in the students' roles in the investigation

In the episode there was an asymmetry in the different roles of the three participating students which is crystallized in the events that unfolded through their respective calculator activities reflected in the dialogue. Signe undertook the role of a mentor and acted as a supervisor to Hilde and Tom. Signe advised the other two students about the function key ^ and the role it plays in the keystroke sequence, and regarding how they could navigate and manoeuvre between the keys on the keyboard to express the arithmetic calculation exponentiation in the case of $9 \wedge 2$. Signe was a mediator between Tom and Hilde, relaying her interpretation of how the displayed syntax should be understood and mathematically linking the operator exponentiation to the students' previously knowledge of the arithmetical operation (repeated) multiplication as 9 times 9. Signe answered the other students' questions. She explained and clarified misunderstandings like the digit 2 in the screen display syntax meant the multiplication 9 times 9 and not 2 times 9 as Hilde had presumed. Signe had the opportunity to guide the other two students and bring them into her knowledge, and which turned out to be correctly understood, her perception and understanding about how the graphic calculator operates with respect to calculating 9 to the power of 2 equals 81. On the other hand, Tom and Hilde were novice users and unfamiliar with the function key ^. Tom and Hilde were acting and participating in the role of learners that expressed their wishes towards Signe to be guided about the new function key and the associated syntax of the graphical calculator. They had questions about what the built-in application could perform and what the arithmetic operation represented mathematically with respect to how it operates on numbers. Hilde and Tom were asking Signe information about the new function key ^, its place and role in the displayed keystroke sequence and about syntax that enabled Signe to execute the arithmetic calculation. They sought assistance from Signe with their challenge of decoding the mathematical content of the syntax.

Anthropomorphic description of the calculator

It is interesting that Signe used a metaphor to explain and to make sense of the arithmetic calculation by attributing the artifact graphic calculator, which does not have human properties, human properties and qualities. This can for instance be seen in (129 and 132) where she uttered "it goes two times up in the brain of the calculator" and "the digit 2 to go little faster". Signe's explanation on how the calculator operates is associated with the human body as a reference where the brain is located within the head. The human brain can perform (mental) arithmetic operations on numbers. The characteristic feature can be compared and transferred to the calculator, which is equipped with a built-in brain with the objective to perform arithmetic calculations. Like in this case with Signe who ascribed the calculator human characteristics, empirical investigations give accounts of examples in which people provide anthropomorphic descriptions of phenomena related to computing devices (Morewedge, Preston, & Wegner, 2007). Signe's initiated exploration activity associated with the discovery of a new function key and identification of the keystroke sequence that appeared on the calculator syntax $9 \wedge 2$, which, to the students, corresponded to a new arithmetic calculation. The mathematical operation was exponentiation for positive integers in the case of the mathematical identity $9 \wedge 2 = 9 \cdot 9 = 81$. Based on the students' activity directed towards exploring the mathematical operation exponentiation sustained by the graphic calculator, the observations of the unfolding activity was analyzed and interpreted in light of the instrumental approach by the scheme of instrumented action.

Components of instrumented action scheme for the exploration of the mathematical operation exponentiation with the graphic calculator

(1) Knowing where to find the function key ^ for exponentiation on the keyboard of the graphic calculator.

(2) Knowing the existence of the command with the inscription ^ can be used to express exponentiation (as repeated multiplication)

(3) Remembering and being able to type in the keystroke sequence and corresponding displayed syntax $9 \wedge 2$ with the aim to perform the arithmetic calculation $9 \cdot 9 = 81$.

(4) Realizing that $9 \wedge 2$ can be substituted by the mathematical operation (repeated) multiplication, that is, recognizing the mathematical identity $9 \wedge 2 \overset{\text{identity}}{=} 9 \cdot 9$

(5) Being able to interpret the result $9 \wedge 2 = 81$

(6) Attempts to generalize by a change to a different value for the chosen base 9 and exponent 2.

The bullets (1), (2) and (3) concern aspects of observations of students' concrete gestures [2], restricted to techniques that the students carried out with the artifact graphic calculator. The technical skills, what the instrumental genesis of instrumented action requires of the student, is to be aware of its existence and the location of the function key exponentiation ^ on the keyboard of the artifact graphic calculator. Further, the student should be able to navigate and to manoeuvre among keys on the keyboard to type in the keystroke sequence. Navigating: by determining the position and course in the sense of orientation among the keypad to locate the keys to be used in the keystroke sequence to express potency $9 \wedge 2$. Manoeuvring: by managing the coordination and the steering of gestures that constitute the movement of the fingers on the keyboard in the agent's performance of keystroke sequence to execute the command exponentiation. Finally, by pressing the exe-button for the execution of the command exponentiation, the student is mobilizing the tool graphic calculator to perform the arithmetic calculation. The described technical gestures of the students' working with the graphic calculator consisting of a combination of keystrokes is an example of a usage scheme directly related to the artifact and oriented towards calculator management. According to (Trouche, 2003, p. 789), "We shall call gesture a student's elementary behaviour that may be observed, component of a scheme". The usage scheme (Rabardel, 2002), the operational manipulation of the tool, constitutes both by observable gestures and knowledge involved in the making of the gesture. The observable appearance of students' gestures on the keypad of the calculator is not to be considered in isolation, because the gesture requires knowledge from each student to carry out the task which is mobilizing, and utilization of the function key for exponentiation and recognition of the identity as repeated multiplication by the $9 \wedge 2 = 9 \cdot 9 = 81$. The enclosed transcript illustrates that Signe, Tom and Hilde not only were keen to know about the keystroke sequence to perform the arithmetic calculation, but they also were interested in the ulterior mathematical content to make meaningful use of the tool. My claim is that the mental process associated with the individual mental developing schemes of Signe, Tom and

Hilde, incorporates mathematical thinking when each of them were trying to attribute mathematical meaning to the technique of typing in the syntax. This can be seen as instrumented action scheme because of the conceptual aspects or conceptual components in the instrumental genesis. The scheme of instrumented action is oriented by the action itself. The students were not questioned, nor did they get into operationalizing the function key ^ with other bases than base 9 and other exponents than exponent 2.

What would Signe and the other students have said if for example the digit 2 in the syntax was replaced with the digit 3 or the digit 5, and whether they would have been able to calculate $9\wedge 3$ or $9\wedge 5$ and mathematically explain the arithmetic calculation? Based on Signe's statement in (132), "two nines, it is the same as 9 times 9, only that we use the digit 2 to go a little faster" and in (129) "two times up in the brain of the calculator.", indicates that Signe mathematically interpreted the exponent 2 in the exponentiation as the number of times the digit 9 shall be multiplied by itself. The screen displays not only the syntax, but also the corresponding answer 81 to the arithmetic problem, which gave Signe the opportunity to connect prior knowledge to the multiplication 9 times 9. The exponentiation corresponded to repeated multiplication and the mathematical identity $9\wedge 2 = 9\cdot 9 = 81$. This is an indication that Signe, for example, could have answered that $9\wedge 3$ was the same as $9\cdot 9\cdot 9$ and that the exponent digit 3 then told her "it goes three times up in the brain of the calculator." However, this is to draw the conclusion too far and remains a hypothesis. Bullet (6) is incorporated and implemented as a component in the instrument action scheme if the students, through a task design, were given the opportunity and challenged to use other bases and exponents and to make generalizations. The bullets (4), (5) and (6) require of Signe, Tom and Hilde to sharpen and extend the conception of multiplication for positive integers to repeated multiplication, and in a longitudinal aspect, relate it to the new syntax and the function type exponentiation a^b. The example illustrates the technical gesture involved in using the artifact graphic calculator interplay and interacts with mathematical thinking. Technical gestures and conceptual aspects are intertwined and co-developed. The three first bullets on the component list have a more technical character, while the three last on the list have more conceptual characteristics.

CONCLUSION

The notion of instrumental genesis is suitable to analyze the process of the transformation of an artifact into an instrument in the hands of a user. Instrumentation schemes distinguish between utilization schemes and instrument action schemes. The former schemes concern activity directly linked to the artifact and functions as building blocks for the more global integrated schemes of instrumented action related towards the artifact (calculator) with the aim of carrying out a given task. Mental schemes describe both technical and conceptual characteristics mediated by an artifact. In this case study we described an instrument action scheme concerning student-initiated exploration of the exponentiation key with the inscribed symbol ^ on the key button of the graphing calculator. The component of the instrument action scheme was the theoretical lens to analyze observations of the student's gestures (techniques) when they mobilised the calculator by navigating and manoeuvring among keystrokes to perform the task $9 \wedge 2 = 81$. The instrument action scheme also offers the researcher the opportunity to analyze the conceptual elements, the math concepts that are involved in students' activities when they are performing a task. This case study illustrates technical and conceptual aspects of the students' exploration activities concerning exponentiation, in terms of dialectical relationship, co-development and co-emergence.

NOTES

1. The mathematical operation exponentiation has not been introduced to the students at this academic level, and the associated function key was new to them.

2. According to Trouche (2003), a gesture is the observable part of usage scheme.

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DEVELOPING A GENERAL FRAMEWORK FOR INSTRUMENTAL ORCHESTRATION

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The need to understand teachers' actions as they teach mathematics in a computerized environment is greater than ever. A theoretical framework aimed at understanding teacher practice may help the research community guide practicing teachers in their struggle to integrate ICT into their teaching. The theory of instrumental orchestration is suggested as an appropriate lens for studying teacher practice. In the current study, instrumental orchestration types that have already been identified are examined in a variety of educational settings with the goal of further developing the theory from a critical point of view.

INTRODUCTION

Supporting mathematics teachers in their efforts to integrate technology into their daily practice remains a challenge for the mathematics education community. An essential step in meeting this challenge involves formulating a theory to describe teacher practice. Such a theory may then be used to inform and guide teachers in their efforts to integrate ICT successfully. Indeed, in 2004 Trouche introduced the notion of *Instrumental Orchestration* to describe teachers' need to support their students in the process of instrumental genesis. During the last eight years, Drijvers et al. and other researchers have further developed the notion of instrumental orchestration (Drijvers, Doorman, Boon, Reed & Gravemeijer, 2010; Drijvers, 2012; Tabach, 2011). Yet as technology changes, these already identified instrumental orchestration types need to be reexamined and possibly modified or extended accordingly.

The aim of the current study is to critically examine the instrumental orchestration types proposed by previous research. The analysis is based on classroom observations of 30 mathematics teachers who have faced the challenge and begun integrating technology into their mathematics teaching practice.

THEORETICAL BACKGROUND

The mathematics research community must develop ways to conceptually observe mathematics lessons in which teachers integrate technology as part of their everyday practice (Trouche & Drijvers, 2010). Pierce and Stacey (2010) proposed one such comprehensive framework, called *Mapping Pedagogical Opportunities*. According to this framework, ten pedagogical opportunities are clustered into three groups on a pedagogical map to reflect teacher practice: the *tasks* set for students; the *classroom interaction*; and the *specific subject* being taught. While this mapping enables researchers to characterize the teaching practice of different teachers, it provides little information for teachers who wish to incorporate technology into their lessons.

The powerful theory of instrumental genesis can be used to characterize aspects of student learning in a computerized environment. Vérillon and Rabardel (1995) proposed this theoretical construct based upon empirical findings and used it to describe how computerized tools become instruments for students and how diverse this process can be. That is, individuals and groups from the same classroom who solve a given task using the same tools can employ diverse strategies (Artigue 2002; Mariotti 2002). When students begin to use computerized tools, they construct a schema regarding what these tools can and/or should do for them. This schema is strongly related to their initial experiences and beliefs, the perceived nature and goals of their activities, their dialogue with peers, and the results of spontaneous explorations and serendipitous discoveries. This is especially true when the initiative to use, or not use, the tool is left to the students and their needs. Vérillon and Rabardel (1995) defined *instrumental genesis* as the process by which individuals create and change their perceptions of a tool while performing different tasks. Instrumental genesis is considered a bidirectional process in which both tool and user change. Trouche (2004) referred to these two aspects of the process as *instrumentalization* and *instrumentation*.

Whole-class discussions orchestrated by the teacher (Trouche 2004) can serve as an appropriate forum for talking about and sharing students' personal instrumental geneses for the purpose of further enhancing them. Trouche "introduced the term *instrumental orchestration* to point out the necessity (for a given institution – a teacher in her/his class, for example) of *external steering* of students' instrumental genesis" (2004, p. 296, emphasis in the original). Instrumental orchestration also has a socio-cultural aspect (Laborde 2003; Lagrange et al. 2003), since the technological medium serves as a boundary object between teacher and students, where "mutual negotiation and meaning-construction is the norm for both sides" (Hoyles et al. 2004, p. 321).

> Instrumented orchestration is defined by four components: a set of *individuals*; a set of *objectives* (related to the achievement of a type of task or the arrangement of a work-environment); a didactic *configuration* (that is to say a general structure for the plan of action); a *set of exploitations* of this configuration (Guin, Ruthven & Trouche, 2005, p. 208).

That is, while the didactical configuration refers to the arrangement of artefacts in the classroom, the exploitation mode includes "*decisions on the way a task is introduced and worked through, on the possible roles of the artefacts to be played, and on the schemes and techniques to be developed and established by the students* (Drijvers et al., 2010, p. 215)." The teacher prepares parts of his or her instrumental orchestration in advance, while other parts may emerge spontaneously during a lesson. That is, instrumental orchestrations have a time dimension that is related to didactical performance.

The theory of instrumental orchestration[1] does not suggest specific orchestrations. Nevertheless, several orchestration types have been identified based on empirical data from various studies (Table 1, left column including reference), so in this sense the categorization is not theoretically based. In all cases, open mathematical tools such as Dynamic Grapher or electronic spreadsheets were used in a computerized environment. For almost all orchestration types, the didactical configuration involves a whole-class setting in which the students sit facing one central screen.

	Didactical configuration	Didactical exploitation
technical-demo Drijvers et al., 2010	Whole-class setting, one central screen	The teacher explains the technical details for using the tool.
Explain-the-screen Drijvers et al., 2010	Whole-class setting, one central screen	The teacher's explanations go beyond techniques and involve mathematical content.
link-screen-board Drijvers et al., 2010	Whole-class setting, one central screen	The teacher connects representations on the screen to representations of the same mathematical objects that appear either in the book or on the board.
Discuss-the-screen Drijvers et al., 2010	Whole-class setting, one central screen	Whole-class discussion guided by the teacher, to enhance collective instrumental genesis.
Spot-and-show Drijvers et al., 2010	Whole-class setting, one central screen	The teacher brings up previous student work that he/she had stored and identified as relevant for further discussion.
Sherpa-at-work Trouche, 2004	Whole-class setting, one central screen	The technology is in the hands of a student, who brings it up to the whole class for discussion.
work-and-walk-by[2] Drijvers, 2012	Students work individually or in pairs with computers	The teacher walks among the working students, monitors their progress and provides guidance as the need arises.
not-use-tech Tabach, 2011	Whole-class setting, one central screen	The technology is available but the teacher chooses not to use it.

Table 1: Orchestration types identified

In the current study the instrumental orchestration types suggested by previous research are examined critically. The following research question was examined in the context of the classrooms of practicing mathematics teachers who integrate technology into their practice: To what extent are the categories of instrumental orchestration identified thus far sufficient to characterize the teaching practice of these mathematics teachers?

METHODS

Participants and data collection

The teaching practice of 30 mathematics teachers was observed and served as data for the current study. All participating teachers volunteered to be observed as they taught. The teachers varied in terms of their years of experience as mathematics teachers (from 3 to 22 years). Note that although the participants were experienced mathematics teachers, they had much less experience using technology in teaching. In fact, at the time the observation took place, the teachers had between six months and five years of experience in integrating technology. The participants also varied in the grade levels they teach: seventeen teach in elementary schools (Grades 3-6), eight teach in middle schools (Grades 7-9), four teach in secondary schools (Grades 10-12), and one teaches adults (see Table 2). Half of the teachers work in Hebrew-speaking schools and the other half in Arabic-speaking schools.

Grade Level	3	4	5	6	7	8	9	10	11	Adults
Number of teachers observed	2	5	7	3	1	1	6	2	2	1

Table 2: Number of teachers observed per grade level

Each teacher was observed for three to four lessons in the same class with the same students, over the course of one month, at a time set in advance with each teacher at his or her convenience. All observed lessons were recorded and transcribed verbatim. For most of the observations a camera was placed at the back of the classroom on a tripod and remained stable throughout the lessons. In a few cases, the researcher used two modes of recording the lesson. During whole-class discussions the camera was placed at the back of the classroom, but while the students worked the researcher followed the teacher as she circulated among the students in order to record the interactions between teacher and students. Understandably, most teachers were not willing to accept such an arrangement, so it was not used in all cases.

Data analysis

Each of the lessons was divided according to the teacher's actions. The researcher looked for the eight identified orchestration types, while at the same time keeping in mind the option of identifying new orchestration types. The process of identifying the orchestration types was not always trivial and was subject to the following methodological question: When can it be claimed that a particular orchestration type is a variant of another orchestration type? If the didactical configuration and the didactical exploitation of the teachers' actions are the same, we can identify this as the same orchestration type. Likewise, if the didactical configuration and the didactical exploitation differ, we can identify two different orchestration types. But what if only one of the characteristics has changed? In order to identify the orchestration as a variant, must we look for a fixed didactical configuration and allow only for variation in the didactical exploitation? Or is it didactical exploitation that is the core of the orchestration type, while variation in didactical configuration is less significant?

Trouche (2004), who proposed the *Sherpa-at-work* orchestration, pointed to the possibility that one student leads the work or several students consecutively lead the work. He analysed the situation as follows:

> This orchestration favours collective management of a part of the instrumentation and instrumentalization processes: what a student does with her/his calculator – the traces of her/his activity – are seen by all, allowing the comparison of different instrumented techniques and giving the teacher information about the instrumented actions schemes being built by the Sherpa-student (p. 298).

Tabach (2011) identified a variant of *Sherpa-at-work* that differs in its didactical configuration. Many screens were used and many students carried out the same action on their screens, but otherwise the essence of the didactical exploitation was the same in terms of the discussion that evolved. Tabach also pointed to a variant of *discuss-the-screen* orchestration, in which many screens were observed as a didactical configuration rather than one central screen. Nevertheless, the exploitation mode, which is the core of instrumental orchestration, remained the same.

The following analysis adopts this same approach. That is, an orchestration type is considered new if it differs both in its didactical configuration and in its didactical exploitation. In cases in which the didactical configuration differs but the exploitation mode remains the same, the instrumental orchestration is considered a variant of its parallel type.

FINDINGS

The following describes the cases of three teachers. These teachers were selected to offer a range of ways to integrate technology into teacher practice, with May at one end of the spectrum and Noam and Rona at the other end. In the case of Rona, an additional element, called *monitor-and-guide* was identified, as was a new orchestration type.

May

May is an experienced teacher who has been teaching mathematics in elementary school for the last 21 years. The observations took place in a sixth grade class comprising 27 students of mixed abilities. The three lessons took place in a computer laboratory, in which each student, or pair of students, sat in front of one computer and worked on various applets. The students worked at the computers for the entire duration of the three observed lessons, while the teacher circulated among them and provided assistance as needed. In this sense all of the lesson time was devoted to the *monitor-and-guide* orchestration type.

The researcher observing May's classroom followed the teacher with the video camera throughout the lessons, making it possible to further elaborate the orchestration types May used. Each lesson began with a variant of *technical-demo* orchestration, during which the teacher helped students enter the learning environment by providing passwords, checking for internet connections, and solving other technical problems. About 20% of the class time was devoted to this type of activity. The orchestration type used during the major part of the lessons (45% of class time) was a variant of *discuss-the-screen*. The teacher engaged in mathematical discussions with a student or a pair of students, at their request. In some cases, this orchestration type was followed up by a variant of *link-screen-board* or *not-use-tech*, mainly when May referred to students' notebook to clarify a mathematical point. These variants of the two orchestration types were used for about 10% of the class time. For the remainder of the time, the teacher walked around the classroom and monitored students' actions.

May's avoidance of any whole-class discussion during the observed lessons was puzzling, as the computer laboratory included a projector and a screen on which data could be projected. During an informal after-observation interview with May, she indicated that she did not know how to use the data projector and did not want to admit this lack of knowledge to her students.

Noam

Noam is an experienced teacher who has been teaching mathematics in elementary school for the last 17 years. The observations took place in a fifth-grade class comprising 30 students with mixed abilities. For the last five years Noam has been integrating technology into her teaching practice. The observations in Noam's classroom were by static camera only. Hence, we do not have a complete record of student-teacher interactions that were not part of the whole-class forum.

In the three observed lessons, a regular pattern emerged in Noam's orchestration actions. She began with *technical-demo* orchestration and then moved on to *explain-the-screen* orchestration. Next she alternated between *link-screen-board* and *discuss-the-screen*. She then returned to *technical-demo* or *explain-the-screen*, and she always finished with *monitor-and-guide* orchestration. In terms of time spent, about a third of the lesson time was devoted to *monitor-and-guide*, another third to *discuss-the-screen*, and the rest of the time was distributed almost equally among the other orchestration types.

Noam considers the computer to be a tool that helps her offer her students more diverse teaching and allows her to allocate her time to working with individual students who need more assistance. The learning management system allows her to monitor student work and identify students in need of further instruction.

Rona

Rona is an experienced teacher who has been teaching mathematics in elementary school for the last 11 years. The observations took place in a fifth-grade class comprising 30 students with mixed abilities. For the last three years Rona has been integrating technology in her teaching practice, encompassing the entire learning environment. The researcher observing Rona's classroom followed her around with the camera, so we have a record of the interactions between the teacher and her students.

The two double lessons (90 minutes each) that were observed took place in the students' regular classroom. Two student aides brought 20 laptops to the classroom on a wheeled cart and distributed them among the students. During this first phase of the lessons, which took about 9 minutes in each lesson (10% of the lesson time), the teacher led a whole-class discussion about the use of technology. This type of orchestration has not yet been identified. We named it *discuss-tech-without-it* to reflect the fact that possible uses of technology may be discussed even when the technology is not present.

The organization phase was followed by a similar lesson structure. The teacher used a mixture of *explain-the-screen* and *discuss-the-screen* orchestration in a way that did not allow the two types to be separated. Next she used *monitor-and-guide* orchestration while acting in one of three ways: answering technical problems - a variant of *technical-demo* orchestration; explaining the screen to a student or a pair of students - a variant of *explain-the-screen* orchestration; or monitoring students' progress via a learning management system that enabled her to monitor the individual progress of her students. In other words, *monitor-and-guide* orchestration may include an electronic element, in which the teacher may interact with students from a distance by sending messages rather than physically approaching students in need.

This sequence of orchestration types was repeated at least twice during the lesson. In addition, during the second lesson, the teacher used the *spot-and-show* orchestration type at the beginning of the lesson to clarify a homework problem that had been submitted to her via the learning management system.

DISCUSSION

The following question framed the current study: To what extent are the instrumental orchestration categories identified thus far sufficient to characterize the teaching practice of practicing mathematics teachers? The study examined the teaching practice of three experienced elementary school teachers (5th and 6th grades) via the instrumental orchestration lens. Notable differences were observed between the practice of May, a novice in integrating technology, and that of Noam and Rona, who were more experienced in using technology. A possible explanation for this notable difference between their practices lies in their technological pedagogical content knowledge (Mishra & Koehler, 2006). May's lack of technological knowledge limited her actions. It may possibly also have limited her ability to support her students' instrumental genesis process in particular, and their learning in general.

Yet May did use variants of the central orchestration types that involved interactions with students on an individual basis. Similar findings were reported by Drijvers (2012), who studied the practice of an experienced 12th grade teacher who was a novice in integrating technology. This teacher employed the *monitor-and-guide* orchestration type [there referred to as *work-and-walk-by*, as explained in endnote 2], which was further broken done according to the type of discussions taking place between teacher and students. Thus, variants of the orchestration types *explain-the-screen*, *discuss-the-screen*, *technical-demonstration* and *link-screen-board* were identified on an individual basis.

The practices of Noam and Rona were similar in terms of their clear pattern of lesson structure, as can be seen in the relatively stable sequence of instrumental orchestration actions in each of their lessons. Both employed a variety of orchestration types, which the two teachers sequenced in a similar manner. Drivers et al. (2010) reported the same sequences of orchestration types.

An additional orchestration type was identified in the practice of one of the observed teachers: *discuss-tech-without-it*. This new type emerged as a result of observing and analysing technological environments that have not been reported so far. The type refers to a special didactical configuration in which learning does not take place in a computer laboratory or with laptop computers owned exclusively by each student. Bringing laptops to the classroom on a wheeled cart enables the school to make use of any classroom as a potential host for mobile computers. Furthermore, the teacher's use of a learning management system demonstrates an electronic element in the *monitor-and-guide* orchestration.

Identifying a teacher's instrumental orchestration actions enables us to learn about his or her practice. We do not claim that *all* the instrumental orchestration types have already been identified. On the contrary, we hypothesize that as technology changes, new types of instrumental orchestration may begin to emerge. Some of these may be considered variants of already identified types, while others will be new. And yet, as the three cases discussed above demonstrate, identifying instrumental orchestration types offers a window into a teacher's classroom base practice. This window needs to be expanded by analyzing teachers' knowledge as well. The extent to which categorizing instrumental orchestration types can be used to inform the practice of novice mathematics teachers still remains to be studied.

NOTES

1. The notion of instrumental orchestration should not be confused with the notion of documentation genesis (Gueudet & Trouce, 2009). Although both notions stem from the instrumental approach and both focus on the teacher, documentation genesis includes the documentation work of the teacher outside of the class, while instrumental orchestration focuses mainly on teacher practices in the classroom. Still, the two notions do somewhat overlap, as documentation also refers to a *usage* component and orchestration also refers to a planning element.

2. The term *work-and-walk-by* was suggested by Drijvers (2012). However, in this study we refer to this as *monitor-and-guide*, which refers to the teacher's actions and hence is more appropriate here. I wish to thank Håkan Sollervall for this insightful suggestion.

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THE IMPACT OF THE INVOLVEMENT OF TEACHERS IN A RESEARCH ON RESOURCE QUALITY ON THEIR PRACTICES

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Since four years, a group of seven secondary school mathematics teachers and teacher educators has been involved in a research project dealing with the issue of dynamic geometry resource quality. The aim of this paper is to examine the impact of this involvement on their practices both as teachers and teacher educators. Based on the analysis of various resources produced by the group members before their participation in the research and nowadays, as well as on the group's auto-analysis of the evolution of their own practices, we could highlight a significant evolution in their way of using dynamic geometry in a classroom, as well as in their teacher training offer and content.

Research in mathematics education we carry out relies mostly on participatory methodologies, such as design-based research (DBRC 2003), which involve researchers and practicing teachers who are not mere experimental subjects, but they take an active part in all phases of the research. Like in other research projects, the focus is on the research itself and its outcomes and the issue of the teachers' professional development fostered by their participation to the research is not raised.

The question how the teachers' involvement into a research contributes to their professional development is central (Burns 2010). We report about an experience of a research group composed of three mathematics education researchers and seven secondary school math teachers (called DG-group in the sequel) working on dynamic geometry (DG) resource quality issues. We attempt to highlight the impact of the teachers' involvement into this research on their practices.

The paper is organised as follows. First, we briefly describe the research project on DG resource quality that gathered together researchers and teachers. This project will be referred to as I2G project. Next, we present the theoretical framework and the methodology we used to examine the impact of the teachers' involvement in the I2G project on their practices. Finally, we discuss the most significant findings and propose some concluding remarks.

I2G PROJECT

The I2G project, which ran between September 2008 and June 2012, was conducted in the framework of the Intergeo European project [1], aiming at developing a community of DG users all over Europe around an open web-based repository [2] specifically designed for sharing resources and practices related to the use of DG in mathematics teaching. In order to help platform users identify suitable resources regarding their instructional aim and context of use, as well as to allow the available resources to be improved, two main tools have been developed and implemented into the repository: a search engine based on mathematical notions and competencies ontology to help searching for relevant resources, and a resource quality review questionnaire helping the users analyse available resources and highlight aspects to be improved. Three mathematics education researchers and the DG-group were in charge of designing and testing the questionnaire. In the next section, we present the questionnaire and its design process, focusing on the roles of the researchers and the teachers involved in the I2G project.

Design of the questionnaire for DG resource analysis

The questionnaire, which is the main tool for the resource quality assessment in the repository (Fig. 1), was designed in a cyclical process consisting in the elaboration of its successive versions, followed by their tests and subsequent improvements.

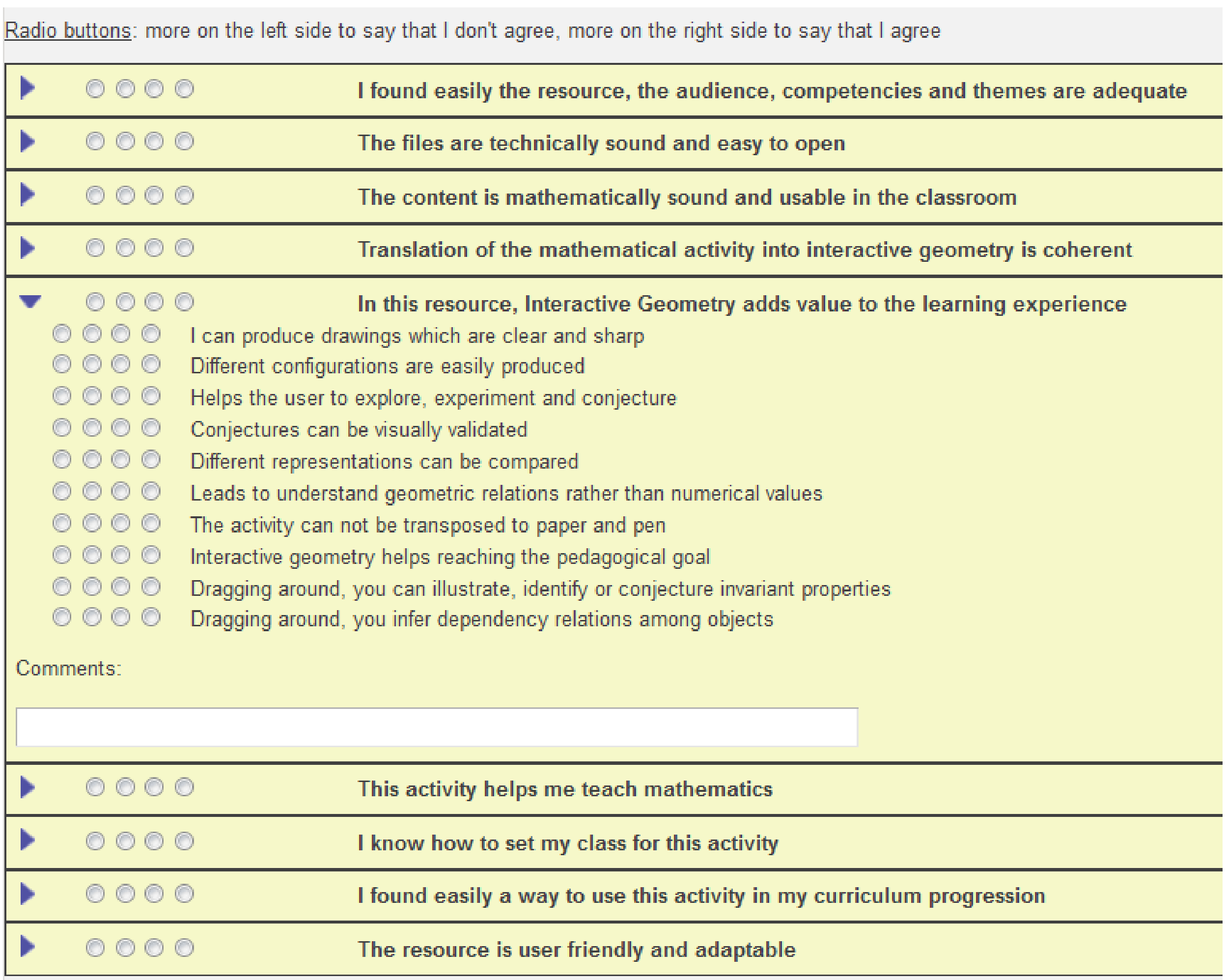


Figure 1: Online questionnaire for reviewing a DG resource in the i2geo repository

This methodology can be considered as a *design-based research*, in which “development and research take place through continuous cycles of design, enactment, analysis, and redesign” (DBRC 2003, p. 5), blending theory-driven design with empirical research. The first version of the questionnaire was designed by the researchers drawing on research results related to the use of DG in math teaching and learning. It proposed eight general questions related to the dimensions of a DG resource considered as critical with respect to its quality, such as technical aspect, mathematical content validity, instrumental aspect, and didactical and pedagogical implementation. Later, a ninth dimension related to the resource ergonomics has been added. Each of these questions can be developed into a set of more detailed criteria related to the corresponding dimension (Fig. 1). The theoretical considerations underpinning the choice of the dimensions and the definition of the criteria are exposed in some details in (Trgalová *et al*. 2011).

In order to make the questionnaire accessible to and usable by teachers, its elaboration has been done in a close collaboration with seven secondary mathematics teachers (DG-group), according to the schema in Fig. 2. Each version of the questionnaire was first reviewed by the DG-group who provided suggestions related to the relevance and the clarity of the quality criteria. An improved version was then tested with teachers in various contexts, such as pre-service or in-service teacher training programs or workshops (Jahn *et al*. 2009, Trgalová *et al*. 2011, Trgalová & Richard 2012). The questionnaire was then re-designed to take into account the outcomes of the tests.

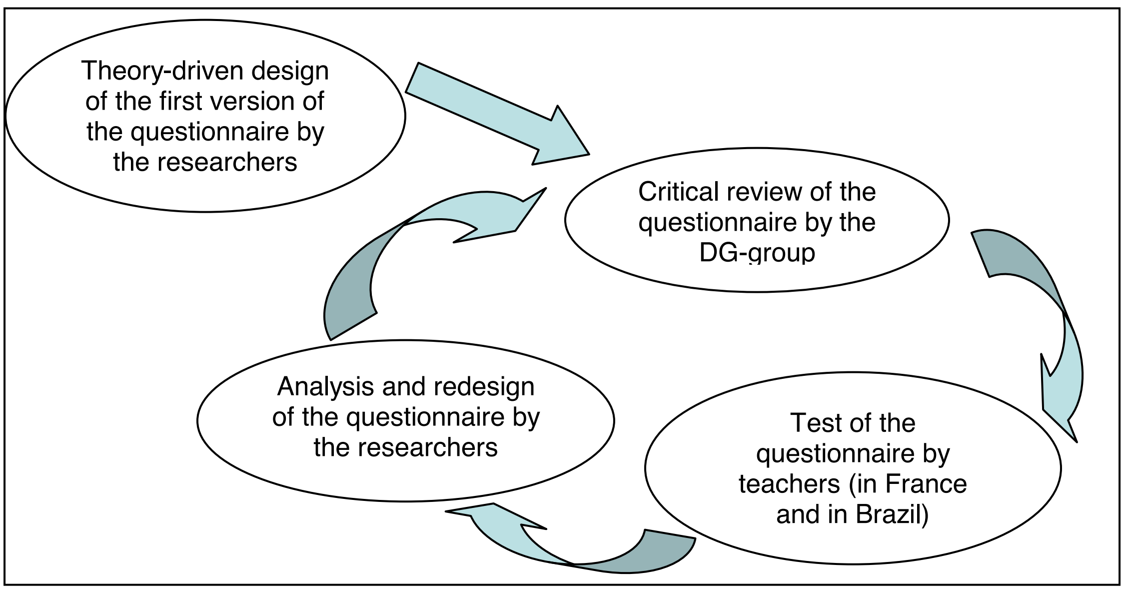


Figure 2: Schema of the resource quality design methodology

As it can be seen in the Figure 2, the teachers from the DG-group played an important role in the I2G research project and contributed to it significantly. During the last year of the project, we became naturally interested in what was the benefit of the teachers' involvement in the project in terms of their professional development. Below we outline the methodology and the theoretical background of this study

METHODOLOGY

The teachers from the DG-group started collaborating in 1996 within the Institute for Research in Mathematics Teaching in Lyon [3] as a group of "users and designers of DG resources" (Bourgeat *et al*., 2013). In 2008, the group volunteered to join the

I2G project and it collaborated until recently with math education researchers on the DG resource quality issues. Every year since 2003, the group offers training courses aiming at helping other math teachers master and integrate DG systems into their practices. All group members prepare together their courses during their regular meetings and produce various resources (documents, tools…), although only two or three of them, in turn, are in charge each time of the implementation of the courses.

For lack of possibility to set up a long-term observation of classroom and teacher training practices of the DG-group members, we gathered three kinds of data in order to identify possible evolutions of their practices. First we collected teacher training resources the group has produced since 2003. Second, recently the group was asked to reflect on changes in their own practices that the group members could observe since their involvement in the I2G project. This introspective activity, conducted during an informal interview with the teachers and summarized in the form of a workshop communication (Bourgeat *et al.*, 2013), yielded many interesting observations related mostly to the teaching practices, some of which are reported in this paper. Finally, we have analyzed some of the teachers' reviews and comments of resources in the Intergeo platform, which can also shed light on their practices with using DG in math teaching.

THEORETICAL LENS USED TO ANALYSE THE DATA

In the collected data we wished to observe indicators of the evolution of the teachers' practices with using DG in their teaching math or teacher training, as well as indicators of the changes in their sensitivity to teacher training issues. This concern led us to choose the following theoretical frameworks.

Instrumental approach and double instrumental genesis

Numerous research studies on the information and communication technologies (ICT) integration adopt the *instrumental approach* (Rabardel 2002) as a theoretical framework specifically designed for studying teaching and learning phenomena involving technology. The instrumental approach relies on a distinction between an *artefact*, a tool available to an individual, and an *instrument*, which is the result of a process of appropriation of the tool by the individual when s/he uses it in order to achieve a given task. The process of transforming an artefact into an instrument is called *instrumental genesis*. Some of these studies stress the complexity of technology integration, which requires a *double instrumental genesis* in teachers: a first genesis of an instrument for achieving mathematics tasks, and a second one of an instrument for achieving educational tasks (Acosta 2008). Haspekian (2011) evokes a *personal genesis* transforming a given tool into a *mathematical instrument*, and a *professional genesis* transforming it into a *didactical instrument*. According to Trouche (2004), the ICT integration requires from the teacher to be aware of the potentialities and constraints of artefacts, which is necessary to design suitable mathematical tasks. Moreover, the teacher has to be able to implement these tasks into the classroom and to foresee the spatial and temporal classroom management. The author introduces the term *instrumental orchestration* to refer to the didactical management of the artefact in a classroom. Drijvers *et al*. (2011) define the instrumental orchestration as

> "the intentional and systematic organisation and use of the various artefacts available in [a] computerised learning environment by the teacher in a given mathematical task situation, in order to guide students' instrumental genesis" (p. 1350).

These considerations will frame our analysis of the teacher training resources produced by the DG-group. We will look for elements in these resources showing whether the group is aware or not of the necessity of the double instrumental genesis in teachers wishing to integrate DG. We will also try to highlight the way the group orchestrates DG activities both in math classroom and in teacher training.

Potentialities of dynamic geometry

A dynamic geometry environment is computer-based software that allows the user to create geometrical figures and manipulate them into different shapes and positions by dragging their elements, mostly points. One of the distinctive features of DG is that when dragging, the geometrical properties of the figure defined in its construction are preserved. Three main modalities of dragging have been identified in the literature (Baccaglini-Frank & Mariotti 2010, Healy 2000, Laborde 2001, Arzarello, Olivero, Paola & Robutti 2002): (1) *dragging for verifying* consists in dragging to check the presence of the supposed (known) geometrical properties in the figure. According to Hölzl (2001), uses of DG are often limited to this modality, in the sense that students are expected to drag figures to confirm empirically the properties which are more or less given; (2) *dragging for conjecturing* consists in dragging to look for new properties of the figure through the perception of what remains invariant when dragging; (3) *dragging for validating/invalidating* consists in dragging to check whether the constructed figure preserves its geometrical properties when dragging.

In the analysis of the DG-group resources we will focus on the modalities of dragging in the activities it proposes and especially whether there are changes in the teachers' perceptions of the role of dragging.

DATA ANALYSIS

The analysis of the collected data shows a significant shift in the practices of the DG-group teachers in three main aspects. The first two are related mostly to their practices as mathematics teachers, the third one to their practices as teacher educators.

Modalities of dragging

Relying on the teachers' auto-analysis of their own practices before and after their involvement into the I2G project (Bourgeat *et al*., 2013), it appears that with their students, the teachers used DG mostly to obtain robust constructions aiming at highlighting invariants in geometric figures:

> "Yesterday, obtaining robust geometric constructions and highlighting invariants were the main goals assigned to students: they should construct a figure by using known properties […] which they could validate by the invariance of the figure when dragging."

Nowadays, they propose new types of activities in which they ask students "to explore figures in order to highlight invariants and/or conjecture new properties" (ibid.). The teachers seem to have acknowledged the importance of dragging for conjecturing in students learning and they thus propose various and richer tasks using different modalities of dragging.

The following comment [4], written by one of the DG-group members about a resource in the repository illustrating, with a robust construction, the equality of three ratios in a triangle with a parallel line to one of its sides, shows her awareness of the interest of soft constructions in geometry learning:

> "Several improvements are possible: 1. [The point] N can be set free, which will allow visualizing the difference between the case proportional-parallel and the cases where the ratios are not equal".

Although it is difficult to establish a direct link between the questionnaire and the evolution of the teachers' awareness of the DG contributions to the teaching and learning geometry, we can suppose that the numerous discussions about this issue, that eventually led to the definition of criteria related to the added value brought by DG to the math activity, are at the origin of this evolution.

Instrumental orchestration

Regarding the classroom management, the teachers confess to have struggled to combine phases of students' work on computers in a computer lab with collective phases of debate, which often needed to be postponed until the next session in an ordinary classroom, as they say (ibid.):

> "Before, the activities with ICT took place in a computer lab in the conditions that postponed the debate and the students-teacher interactions regarding their observations and manipulations in a digital resource."

Nowadays, the teachers orchestrate their ICT-based lessons in a more effective way: the use of a video projector allows articulating individual and collective phases. Indeed, the teachers say: "[Now] we observe the interactions in a genuine triangle "students – teacher – digital resource" (ibid.).

This shift can certainly be related to the pedagogical implementation of the resource, one of the nine aspects that we consider critical for determining the resource quality.

Awareness of the double instrumental genesis

The analysis of the teacher training resources produced by the DG-group reveals that the training programmes the group proposed before 2007, i.e. before its involvement into the I2G project, aimed mostly at helping trainees to master DG system tools. The training activities consisted in series of exercises to solve with DG chosen to illustrate the use of a particular DG tool. Figure 3 shows a typical training activity: the trainees would solve the exercise and indicate what DG tools they have used.

Exercise n°4:
a) Given a segment [AB], construct a square with [AB] as a side.
b) Given a segment [AB], construct a square with [AB] as a diagonal.
c) Construct a square with a side [AB] without using « parallel line » and « perpendicular line » tools.
N.B. For each question, verify that the construction remains stable.

Figure 3: Example of a teacher training activity proposed by DG-group in 2005

The focus of the teacher training programmes in this period was clearly on technical aspects of mastering a DG environment. In terms of instrumental genesis, the DG-group accompanied trainees' personal geneses of mathematical instruments.

Since 2009, the DG-group teacher training proposals show a significant shift towards considering didactical and pedagogical aspects of DG integration. Indeed, in a training programme proposed in 2009, the group announces the following objective: "The aim of this training programme is to accompany the teacher wishing to take her/his students to a computer lab". After a short phase of solving exercises aiming at getting acquainted with the main DG tools, the trainees are invited to reflect on activities suitable for the DG use, the goal being to bring forward the following aims: introduce a new mathematical concept, construct figures, and put students into a research activity with DG. After having solved a given exercise with DG, the trainees are asked to explore it in light of a possible implementation in a classroom: envisage possible adaptations, anticipate classroom management. Figure 4 shows a training resource, in which the two phases, solving an exercise and exploring it from didactical and pedagogical points of view, are present.

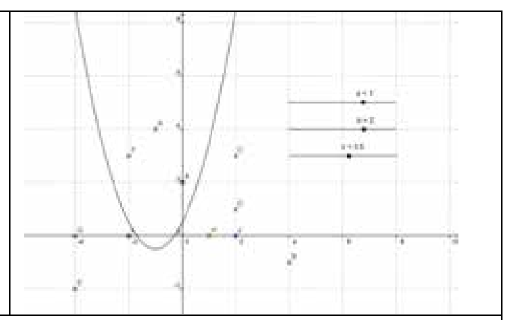
<table>
<tr><td>3. Studying polynomials
3.1. Relationship between graphical representation and expression of a first and second degree function
The trainees open the file « stage1-exo-fct.ggb »
They are given the document « stage1-exo-fct.odt »
Let the trainees solve the exercise.
Ask them to reflect on possible adaptations for a Grade 11 classroom for the next training day.</td><td></td></tr>
<tr><td colspan="2">2nd day: exploitation of the exercise 3.1 solved during the first day: Ask for possible adaptations. A specific attention should be paid to the method used by the students (successive trials without the properties). The need to review this point with them… how? How to manage this activity?</td></tr>
</table>

Figure 4: Excerpt of a training plan elaborated by the DG-group in 2011

The trainees are also led to create their own activities related to a math domain of their choice and adapted to the level of their class. They have to specify the teaching goals and envisage the classroom implementation of the activity.

The DG-group has developed specific resources to help the trainees with this task, such as a description sheet of a session using DG (Fig. 5) or a checklist with questions to ask before using ICT in a classroom, e.g., when to use ICT, do the ICT contributions favour students' learning, or how to integrate an ICT session into an ordinary teaching sequence.

DG software	
Class	
Mathematical topic	

TYPE OF OBJECTIVE OF THE SESSION

Represent a math object ❒	Research problem ❒	Discover a property ❒

OBJECTIVES OF THE SESSION

CONTRIBUTIONS OF THE DG

MODALITIES OF USE (COMPUTER LAB, VIDEO PROJECTOR, INTERNET…)

DEVELOPMENT OF THE SESSION

STUDENTS' PRODUCTIONS (IN A COMPUTER LAB)

Figure 5: DG session description sheet

These elements show that the DG-group has gotten awareness of the necessity to assist the trainees' professional instrumental geneses so that they can transform DG software not only into a mathematical instrument, but also into a didactical one. Moreover, the resources produced by the group entail signs of the influence of the quality questionnaire, namely considerations of several dimensions such as contributions of DG, didactical exploitation of DG potentialities or instrumental orchestration. This seems to confirm a highly positive impact of the DG-group involvement into a design of DG resource quality questionnaire.

CONCLUSION

In this contribution, we reported about a research on the issue of DG resource quality, conducted by a mixed group of math education researchers and in-service teachers. We attempted to show a positive impact of this collaboration on the teachers' practices both as math teachers and teacher educators. Regarding the use of

DG in their classes, we observe significant changes in the nature of tasks the teachers propose to their students: these are richer and more challenging, asking the students to explore figures and conjecture properties, rather than just verify supposed or known properties or validate robust constructions. The teachers are also able to envisage more productive instrumental orchestrations allowing a genuine integration of DG in their math classes. As teacher educators, the group seems to be now much more sensitive to didactical and pedagogical questions related to the DG integration than before. Initially, it focused mostly on technical aspects of mastering DG software, thus accompanying trainees' instrumental geneses yielding a mathematical instrument, whereas nowadays, its programmes include activities allowing the trainees to develop a didactical instrument as well.

NOTES

1. Interoperable Interactive Geometry for Europe, 2007-2010, http://i2geo.net/xwiki/bin/view/Main/About.

2. i2geo.net

3. Institut de Recherche sur l'Enseignement des Mathématiques (IREM). The IREMs gather together primary, secondary and university teachers to conduct research on problems in math teaching and learning at all school levels, to offer teacher training programmes based on research results, and produce and disseminate pedagogical resources.

4. http://i2geo.net/xwiki/bin/view/Coll_cdording/Egalitedes3rapportsavecunedroiteparalleleauncoteduntriangle?bc=&viewer=comments

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TESTS AND EXAMINATIONS IN A CAS-ENVIRONMENT – THE MEANING OF MENTAL, DIGITAL AND PAPER REPRESENTATIONS

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Digital technologies (DT) face new challenges in working with representations in mathematics classrooms and especially while using them in written tests and examinations. There are unexpected solution strategies of students which are not represented in an adequate – student expected – way, there are – beyond the mathematical competencies – tool competences necessary, and working with handheld technology causes the additional problem that the documentations of solutions have to be fixed on paper representations. Moreover, the student has to manage the interrelationship between her or his mental representations, the digital or tool representations and paper representations. How should the paper documentations look like to give the corrector the chance to track the work of the students? In the following authentic written exam, problems and authentic student solutions in a handheld computer algebra system (CAS) environment are analyzed. Finally criteria for documentations of paper solutions of exam problems are developed.

Keywords: digital technologies, examination, assessment, representation, documentation, computer algebra system (CAS), experimental investigation

REPRESENTATIONS AND DIGITAL TECHNOLOGIES

Digital technologies (DT) open new chances and new ways for the use of representations in mathematics education. DT especially simplify generating representations on the computer screen, they emphasize the experimental working by easily changing parameters, and they give access to the simple use of multiple representations. But there are also difficulties and obstacles while working with digital representations. The speed of the creation of screen representations may overcharge the mental abilities of the user, the interrelationship of different representations has to be developed by the user, and the question of the relationship between the traditional working with paper and pencil and with digital representations arises.

There is a long-standing and on-going debate about the meaning of digital representations in mathematics education (e. g. Guin, Ruthven & Trouche, 2005; Ainsworth, 2006; Ladel & Kortenkamp, 2009). Opportunities and problems are already discussed in the 1st ICMI study about DT: The Influence of Computers and Informatics on Mathematics and its Teaching (Churchhouse, 1986), it is an essential aspect of many ICMI activities in the last 25 years (Laborde & Sträßer, 2010), and it is still of essential meaning in the latest 17th ICMI study: Mathematics Education and Technology – Rethinking the Terrain (Hoyles & Lagrange, 2010).

There is not much known about the interrelationship between these different kinds of representation while working with digital technologies in a test or examination environment. There is a huge section (p. 81-284) "Learning and assessing mathematics with and through digital technologies" in the 17th ICMI Study, but it is about theoretical frameworks about learning with DT, changes in mathematical knowledge and practices resulting from access to DT, learning trajectories, automatic assessment and social learning, and it does not give answers to questions concerning the influence of DT on tests and examinations in the classroom. Especially, the use of DT in assessments and examinations is not explained in the frame of empirical investigations or in analyzing students' work in examinations.

THE LONG STANDING SYMBOLIC CALCULATOR M³-PROJECT

A long-term project (2005–2012) was started to test the use of symbolic calculators (SC [1]) in Bavarian grammar schools ("Gymnasien") in Germany in grades 10 to 12 (the M³-project [2]). The students are allowed to use SC in class, for their homework and in all tests and examinations (see Weigand & Bichler 2010a, 2010b).

The results of the evaluation of some tests in these classes showed that students in the project classes still have – also after one year of SC usage – difficulties in using SC and *(problem-) adequate representations* especially, as well as the *documentation of the solution* with paper and pencil. In a questionnaire at the end of the school year, nearly 40 % of the students in the project classes reported to have difficulties in using the SC, many students assigned these to "technical difficulties". But from interviews with and video studies from students, we learned that the real reasons for these difficulties are quite often on the mathematical and not the technical side. Students do not know how to use representations and how to get some information out of them.

In May 2010 the first final baccalaureate examination was given to the students at the end of school time in grade 12. In Bavaria, the baccalaureate is a state-wide examination with the same problems for all students, given by the ministry of education.

EXAMPLES FROM THE M³-PROJECT

In the following we concentrate on two main aspects of the above mentioned difficulties: how to choose the adequate representation in a problem solving process and how to document the solution of a problem. We will explain these aspects and difficulties in the frame of test and examination problems and authentic student solutions of these problems.

Adequate representations

A representation may be considered as *adequate*, if it represents situations or helps to solve problems the way one wants. In fact, representations always have to be considered in connection with possible and appropriate operations. The following example is taken from a test written in 10th grade.

Example 1: Given are f and g with $f(x) = \sin(x) + 1$ and $g(x) = 2^x$. How many intersection points have the graphs

a) in the interval [10; -10] and

b) for $x \in \mathbb{R}$? Give reasons.

Solving this equation on the symbolic level with the SC gives the result (Fig. 1):

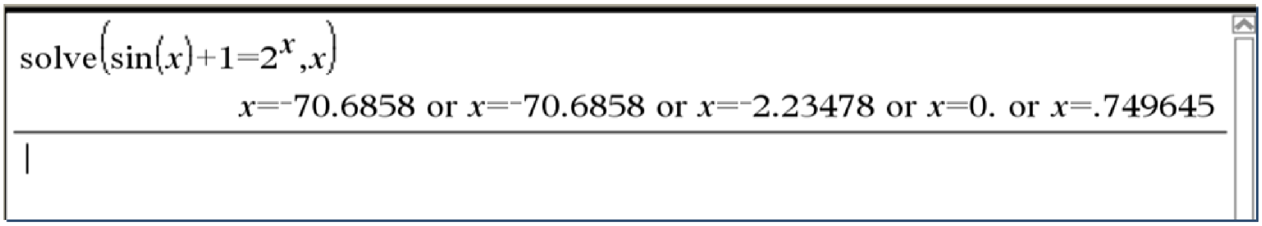

Figure 1: A warning sign appears in the display: "Some more solutions may exist".

It is quite difficult – for students (nearly) impossible – to interpret this display's numeric solution. Changing to the graphic screen and zooming into interesting sections is a good strategy. This gives the following graphs:

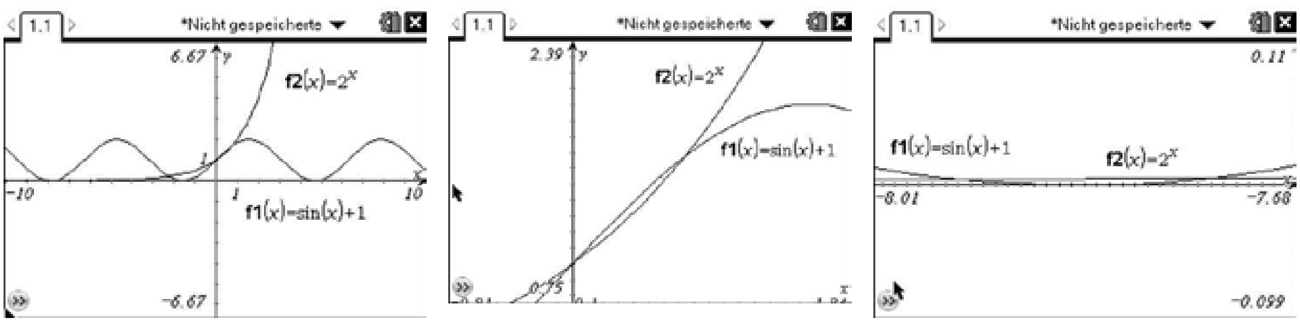

Figure 2: Screen shot of the functions with *$f(x) = \sin(x) + 1$* and *$g(x) = 2x$*.

Indicating the "interesting sections" and interpreting the graphs – for the area x < 0 – require advanced basic knowledge of the properties of the sine and the exponential function. The solution – an infinite number of intersection points – cannot be obtained from the calculator screens, it has to arise from the basic knowledge concerning the implied *mental representation* of the functions. 30 % of the students are able to solve the problem 1 a), but less than 5 % are able to solve the problem 1 b) with the general domain ID = $\mathbb{R}$.

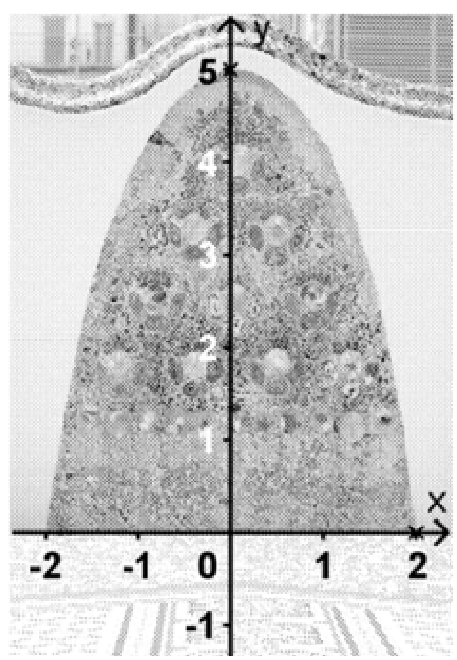

Figure 3: The art work of Gaudi

Unexpected digital representations

The following problem is from the final baccalaureate examination in Bavaria in May 2012. The text is given here only in a short form.

Example 2: A ceramic art in the Casa Batlló (see Fig. 3) has roughly the shape like a parabola.

a) *Give a model of the shape of the upper prim of the art work by using* $q(x) = ax^4 + bx^3 + cx^2 + dx + e$ *(For control:* $q(x) = -0.11x^4 - 0.81x^2 + 5$*)*

b) …

c) *The line g is parallel to the x-axis and divides the work of art into two sections. The area of the above part should be 71.5 % of the whole area.*

We concentrate only on the problem c). The situation shows the following screenshot (Fig. 4). You first have to calculate the integral of q from – 2 to 2 (Fig. 5).

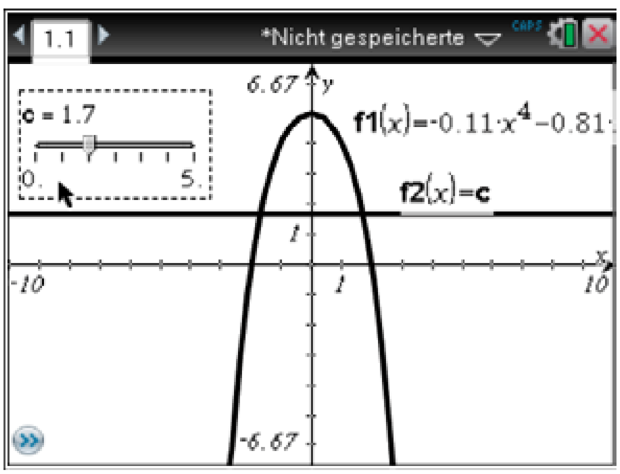

Figure 4: The graph G_q and the parallel line

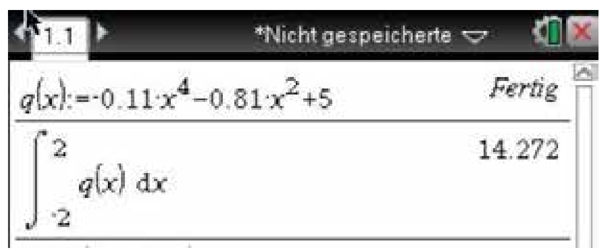

Figure 5: Calculation of the integral

If you solve the equation q(x) = c you get – with the handheld TI-Nspire – the following result (Fig. 6).

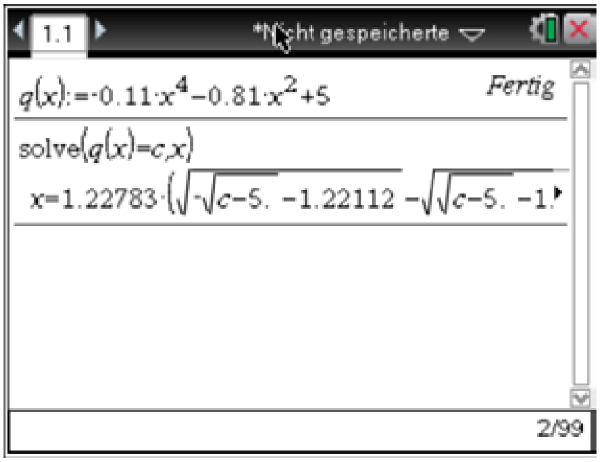

Figure 6: Solution with the TI-Nspire

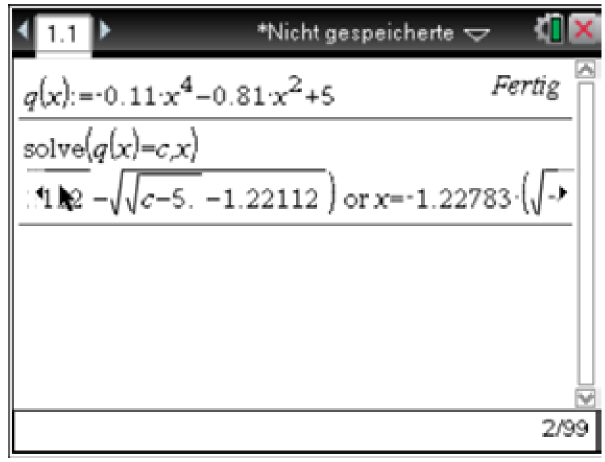

Figure 6: Solution with the TI-Nspire **Figure 7: Scrolling right**

The small triangle at the right side of the last line of the screenshot shows that the formula is not finished at the end of the screen. Scrolling to the right shows a surprising long line (Fig. 7). It is impossible – not only for students – to interpret this result. But also with a laptop representation (which was not available for the students in the exam), the result stays confusing (Fig. 8).

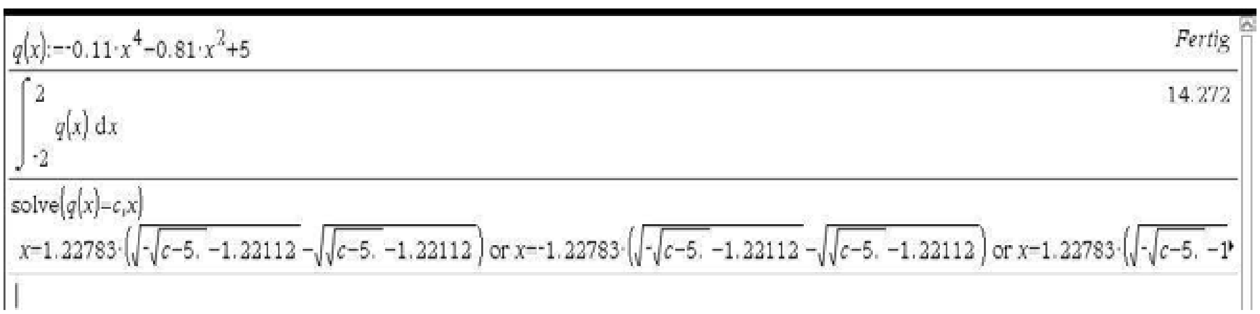

Figure 8: Solution with the TI-Nspire Notebook Version

Moreover, you see that the expression $\sqrt{c-5}$ in all solutions shows that there seems to be no real solution for c < 5, which is obviously wrong, because there exist two real solutions for $0 < c \leq 5$ (Fig. 4).

This example shows the problems that occur if the problem poser had another solution in mind and did not think of different solution strategies.

The documentation of solutions

Quite often – especially in written tests and examinations – the solution of a problem has to be documented on paper. This opens the question concerning the *adequate documentation form* of the solution on paper.

Example 3: Given is the function f with $f(x) = (x-2)^2 + 3$. Determine the equation of the tangent in the point P(1 / 4).

The following examples give some students' solutions and show different documentations. The underlining shows the use of the handheld device of the student during the problem solving process.

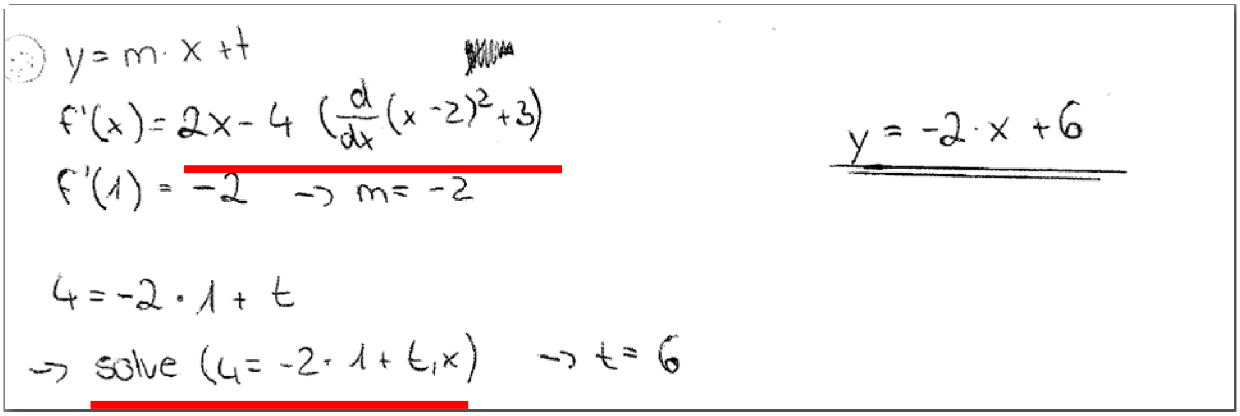

Figure 9a: A student solution quite similar to a paper and pencil solution

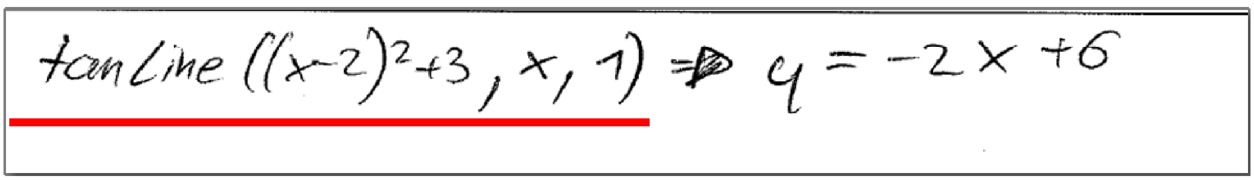

Figure 9b: A student solution using a handheld command

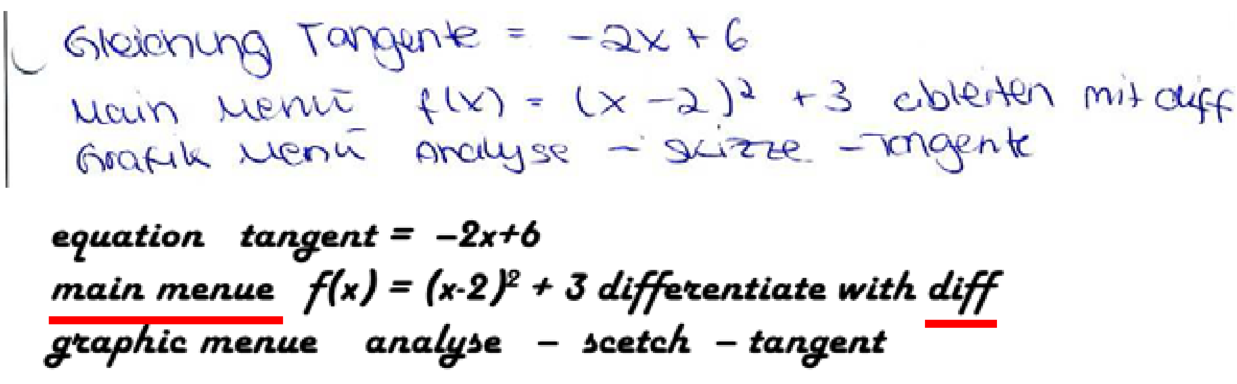

Figure 9c: A student solution and the English translation

You might be content with the solution 9a, the solution 9b might be accepted in a DT-environment, but it needs a special knowledge about DT-commands. But – for sure – you will not be content with the solution 9c!

Example 4. Give a sketch of some graphs of f_a with $f_a(x) = a \cdot x^3 - 3x + 1$, $a \in \mathbb{R}$. There are still some reasons why it might be useful or helpful to do a sketch of a graph (also) by hand, especially in relation with heuristic problem solving strategies. The question about the function and the expected accuracy of hand sketches arises. Fig. 10a and b show a screenshot and a student's hand draft of graphs of a family of functions. Even for heuristic reasons the hand draft does not fit to an expected accuracy and for sure it is not accepted as a solution of example 4.

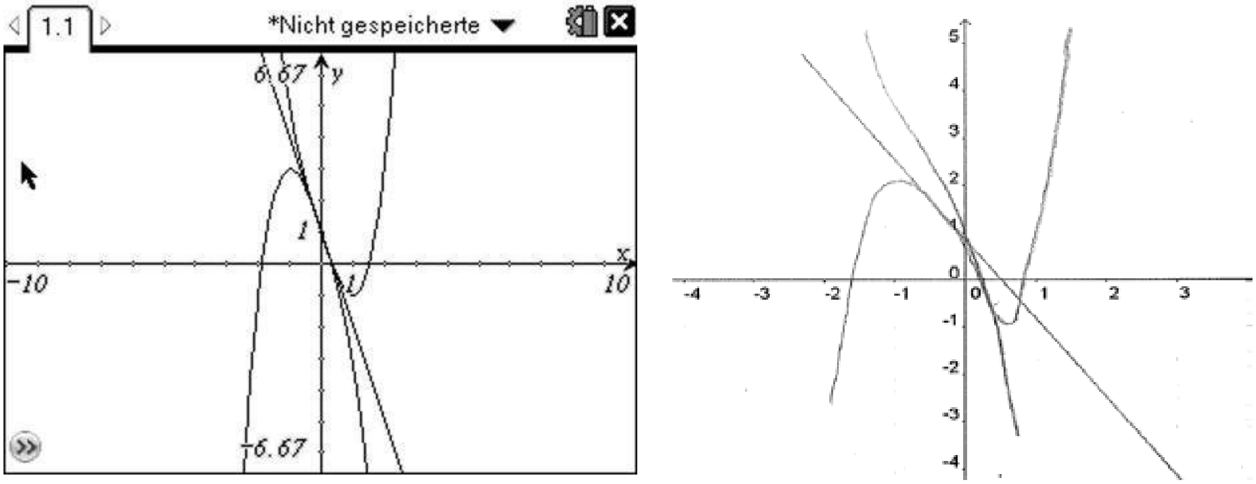

Figure 10: (a) Digital and (b) hand graphs of f_a with $f_a(x) = a{\cdot}x3 - 3x + 1$ for a = -1, 0, 1

EXAMPLES FROM THE FINAL BACCALAUREATE EXAMINATION

The following example is also from the final written baccalaureate examination in Bavaria in May 2012.

E. 5: Given is the family of functions defined in $\mathbb{R}$, f_a: $x \rightarrow \sin(ax)$ with $a \in \mathbb{R}\backslash\{0\}$.

a) Give two values of a, such that all zeros of f are integers.

b) Now a = 2. Calculate $\int_1^b f_2(x)\,dx$; b is the smallest zero of f_2 which is bigger than 1. Give a value $c \in \mathbb{R}$ with c > b, such that $\int_1^c f_2(x)\,dx = \int_1^b f_2(x)\,dx$. Give reasons for your answer using the graph of f_2.

With the use of DT the solution of example 5b might look as follows:

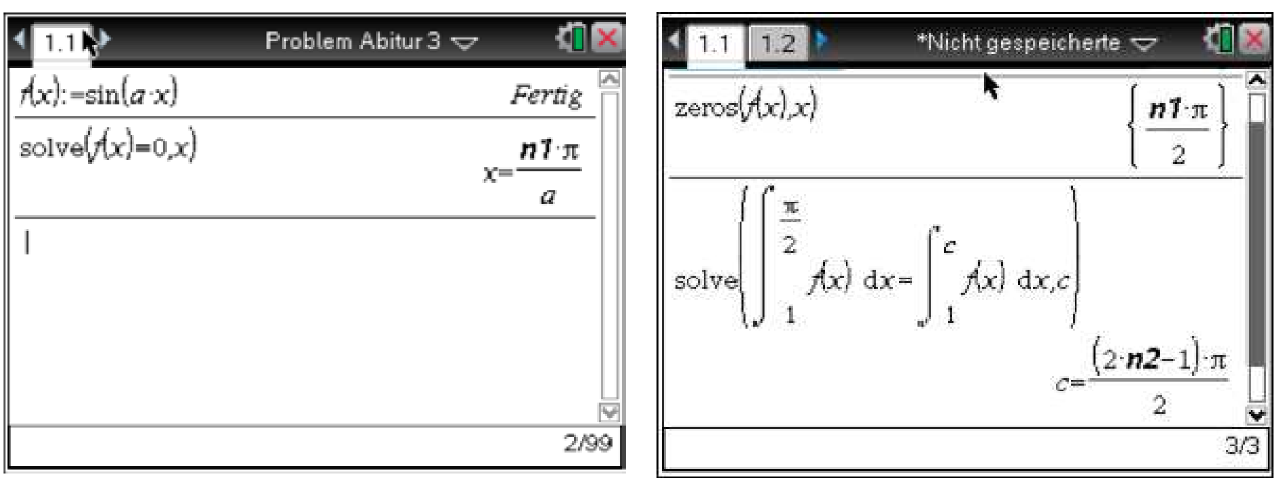

Figure 11: The DT-solution of (a) the example 5 a), and (b) the example 5 b).

These solutions (Fig. 11a and b) are prototypic for DT-solutions. The student has to find the problem solving idea and the starting equation, the calculator takes over the former hand calculations, and finally the student has to interpret the screen notations.

This example especially shows that it is (nearly) impossible to do this interpretation without a basic mathematical content knowledge about the given functions and concepts, e. g. the integral concept.

Working with DT it is always a good strategy to choose different representations of a problem. Fig. 12a and b show graphical representations of the problem which allow a solution of the problem without calculations.

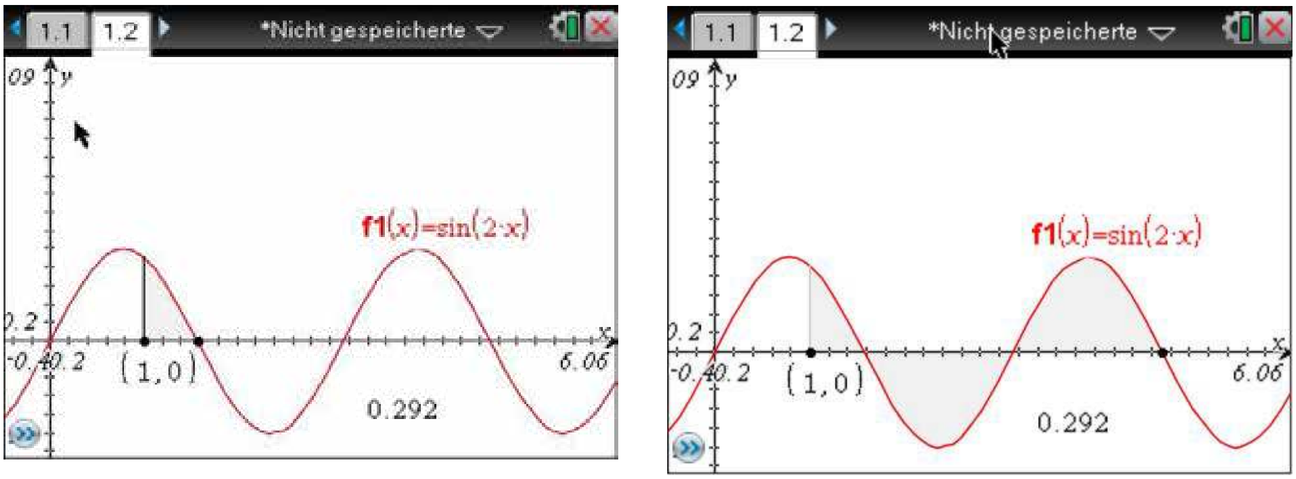

Figure 12: A graphical representation of the solution of (a) the example 5 a), and (b) the example 5 b).

Fig. 13 shows a student solution of example 5b) with a well-written reasoning concerning the solution. The problem for the teacher and the corrector of this examination is that the solution does not show whether and how the calculator was used.

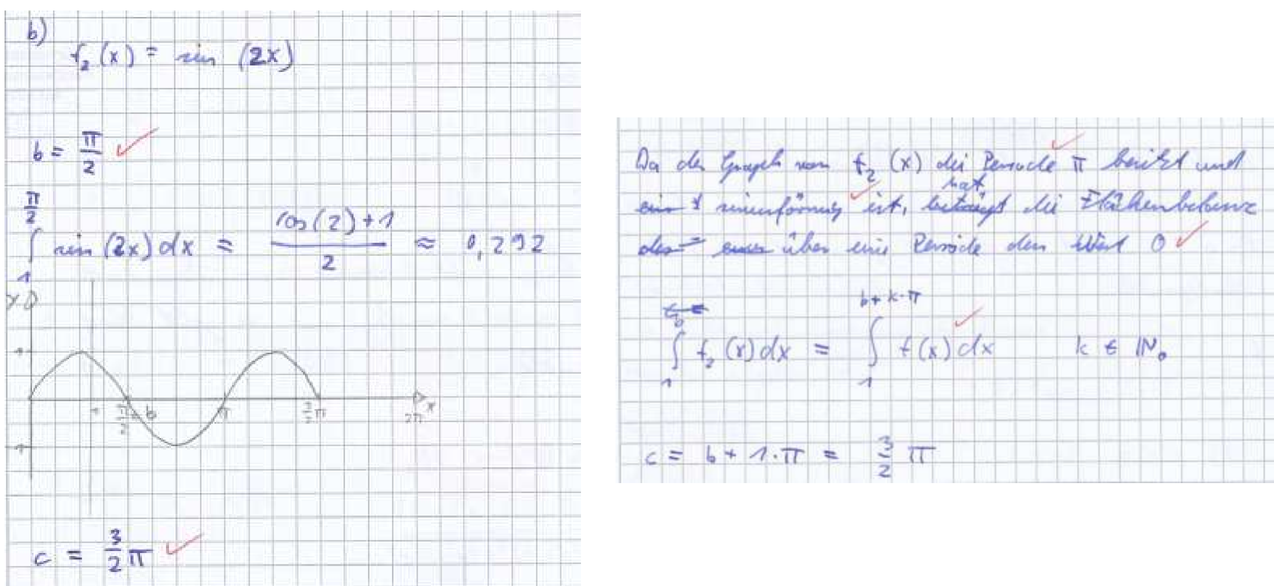

Figure 13. A – correct – student solution of the example 5b).

CRITERIA FOR DOCUMENTATIONS OF SOLUTIONS

Written examinations of students with DT ask for clear instructions for the documentation of written solutions. But there are no algorithmic rules or norms how to document a solution on paper. This opens the question concerning the *adequate documentation form* of the solution on paper.

In our project we started to develop *criteria* for (non-)correct, (non-)accepted documentations of solutions, e. g.

- It is not enough to only write down, what‘s on the screen!
- The solution has to be understandable „for others“, and it has to be seen when and where the SC was used.
- The solution describes the mathematical activities, it is not only a description in a special „calculator language“.
- The meaning of “keywords” (operators) in the problem definition has to be well-known to the student, e. g. “show”, “explain”, “determine”, “prove”, …

These criteria have to be discussed, evaluated and refined the next years.

FINAL REMARKS

This analysis shows the problem and difficulty of posing adequate test problems and the problem of the adequate students’ paper documentation of the solution of the problem. The ability or the competencies *to use SCs* adequately requires technical knowledge about the handling of SCs. Moreover, the knowledge of when to use which features and for which problems might be helpful.

The use of the adequate representation depends – of course – on the problem and the expected level of accuracy or strength. Furthermore, the problem solving process requires the knowledge about the *relationship* between *mathematical objects* or *concepts* and their *mental representations* and finally between the digital representation and the representations used in the documentation (paper and pencil representations). Despite the importance of different interactive multiple (digital) representations, the most important representation, which had to be developed, are the *mental representations*.

Concerning future developments there are some important research questions concerning the adequate use of representations in a digital examination environment:

- How are mental representations influenced while working with digital representations and vice versa?
- How should examination problems be posed to allow a greater variety of problem solving strategies while using DT?
- Are there general criteria for the documentation of problem solutions which cover a greater range of problems?
- Which documentation criteria are helpful for students?

NOTES

1. We used the TI Voyage 200, TI-Nspire and Casio ClassPad.

2. M^3: Model project new Media in Mathematics classrooms.

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MATHEMATICS, TECHNOLOGY INTERVENTIONS, AND PEDAGOGY – SEEING THE WOOD FROM THE TREES

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This research explores recent technological interventions in mathematics education and examines to what extent these make full use of the educational affordances of the technology and appropriate pedagogical approaches to facilitate learning. In an attempt to answer this question, a systematic literature review has been carried out, and a classification framework is presented that categorises the types of technology as well as the pedagogical foundations of the interventions in which those technologies are used. The potential of technology to fundamentally alter how mathematics is experienced is further investigated through the lens of the SAMR hierarchy (Puentedura, 2006), which identifies four levels of technology adoption: substitution, augmentation, modification and redefinition. The framework presented to describe the interventions thus ranges from enhancing traditional practice, to transforming teaching and learning through redefinition of how tasks and activities are planned and carried out. The development of such a framework will be beneficial for guiding teaching, increasing our understanding of learning in a technology rich environment, and improving mathematics education.

Keywords: Mathematics Education, Technology, Classification

The aim of this research is to gain some clarity regarding pedagogical approaches to technology interventions in post-primary mathematics education, as documented in recent literature. The objective is to increase the understanding of the kinds of teaching and learning of mathematics that technology has the potential to enhance. A long-term goal is to create, and test, a comprehensive 21st Century model of classroom practice for mathematics education.

Existing classifications of technology for mathematics education are investigated (Clarebout and Elen, 2006; Passey, 2012; Hoyles and Noss, 2003, 2009). The classifications by Hoyles and Noss provide the foundation for the technological aspect of the framework used in this study. In addition to grouping the interventions by technology, this research also classifies them according to learning theory, instructional approaches, and where they fall on the SAMR transformation hierarchy. The data emerging from the literature review is coded and stored in a spreadsheet pivot table. This allows the information to be arranged and visualised in diverse and meaningful ways that contribute to the extraction of a classification framework for technology interventions in mathematics education. From this framework, a set of guidelines relating to the shape that successful interventions might take is explored.

The emerging set of guidelines point to a holistic approach to technology interventions in mathematics education. Both the transformative and the computational capabilities of diverse technologies should be taken into account, providing for the investigation of challenging and interesting problems and the development of flexible and creative solving strategies. The assessment potential that technology offers needs to be utilised for successful integration. Innovation with regard to the working environment and class routine are seen as necessary in order to fully exploit the potential of technology in the teaching and learning of mathematics.

MobiMaths (Tangney, Weber, O'Hanlon et al., 2010), is an example of an intervention in keeping with the guidelines described. This particular intervention uses smartphone technology and conforms to the socially constructivist pedagogy of Bridge21 (Lawlor, Conneely, & Tangney, 2010) in order to create a learning experience that emphasises collaborative problem solving and contextualised learning. A set of activities pertinent to the curriculum are provided along with the technology in order to facilitate the integration of mobile learning in a pragmatic and meaningful way.

The poster presentation was made up of diagrams to illustrate the data from the literature review, as well as descriptions and illustrations of interventions deemed successful, such as MobiMaths. Further interventions were developed according to the set of guidelines and their implementations were also discussed.

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CONTINUING FORMATION AND THE USE OF COMPUTER RESOURCES

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Centro Universitário UNIVATES

***Keywords**: continuing education, computing resources, Math.*

The advent of technology always ends up causing revolution and now the computer is causing this effect in different sectors of the society and in education. Are these changes so significant? Is the school able to use this tool to the teaching and the learning? Are teachers prepared to make use of this resource in their classes? We believe that the inclusion of computational resources in pedagogical practice will only happen when the teacher experiences the process and when the technology represents an important means to improve teaching and learning.

Thus, the existence of spaces for teachers to exchange experiences, learning and teaching, is important in their professional development. Concerning the use of media in the classroom, we emphasize that it necessarily implies a review of practices developed so that they can propose real changes.

In this context, we addressed, in continuing education courses, the question of the use of computational resources in the teaching of mathematics and what is the impact on activities developed in the classroom by teachers. We sent an online questionnaire to mathematics teachers. Another questionnaire was sent personally to the municipal secretaries of education, with the objective of investigating the existence of computer labs in schools and how they conceive the offer of continuing education courses contemplating the use of computational resources in mathematics education. We also carried an interview with the Coordinator of Regional Education, in which we sought to investigate the incentive that offers to its teachers with respect to continuing education and the reality of the computer labs in schools.

From the data collected we decided to offer continuing education that problematizes the use of computational resources in mathematics classes.

With the objective of having active participation of teachers in the process of their education, we chose to base this work on the methodology of action research that seeks, starting from a preoccupation or a need in the classroom and from an identification of a specific problem, to formulate possible solutions to be implemented and tested. Participants develop in their pedagogical practice the activities problematized during the meetings and bring the results as well as their concerns and difficulties to be discussed in a large group.

NOTES

1. This work was supported by CAPES, the Brazilian government organization directed at the formation of human resources.

PROFESSIONAL COMPUTER ALGEBRA SYSTEMS IN UPPER SECONDARY MATHEMATICS

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We address the issue of use of professional computer algebra systems (CASs) in upper secondary mathematics as opposed to CASs that are designed with school educational purposes more explicitly in mind. We propose a didactical engineering project in which the epistemological analysis of CAS use sets out, through the dictum of C's: Assets of CAS are conceptual, computational and communicative, from the dialectics between CAS as an enhancement of mathematical modelling power and CAS as a virtual reality which itself needs to be modelled and interpreted by mathematics. Design, a priori and a posteriori analysis will be based in part on the theory of instrumental genesis and orchestration (Trouche, Drijvers) in part on ATD (Chevallard). Special emphasis will be given to the secondary-tertiary transition.

SECTIONS OF THE POSTER

The poster will be composed of

Illustrative Maple worksheets

Some sheets demonstrating the potentials and pitfalls of the use of CAS with professional power, such as

- A sheet demonstrating how excessive computational power may lead to shallow understanding and therefore deceptive instrumental genesis.
- A sheet exploring real numbers as (infinite) decimal expansions by exploiting CAS computational power subject to a dogma principle (arithmetic first principles).
- A sheet exploring continuity and fix points by exploiting the power of CAS visualization.

Outline of the epistemology of the impact of CAS use on mathematical content and routines

Vertices in the epistemological coverage will be

- Use towards authentic problems may lead to command of mathematics reducing to ability to choose appropriate CAS applications.
- Taxonomic nivellement of classical routines (i.e. `solve(f(x,y),x)` vs. `solve(f(x,y),y)`).
- Development of mathematical praxis with focus on mathematical core issues.

The epistemology will involve praxeological analysis in the sense of ATD (Chevallard, 1999) as well as mathematical content and concept analysis. For instance, the announced Maple sheet on decimal expansions can be described as a point praxeology involving direct CAS calls of decimal expansions encompassed by a local praxeology where the dogma condition, arithmetic first principles, determines the local praxeology technology as well as (part of) the point praxeology theory. How various incarnations of the full CAS program influence the scope of mathematical content and didactics will be a key issue.

Outline of implementation of designs in Danish secondary mathematics education through an action research framework for teachers.

This section will primarily be based on the theory of instrumental genesis and orchestration and a model description of a reflective practitioner community.

Outline of further steps in an a posteriori analysis

This section involves analysis of praxeologies (Chevallard, 1999) in the internal institutional setting of the classroom and the institutional setting determined by official regulations - including the didactical transposition in the transitions to higher education.

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A NEW INSTRUMENT TO DOCUMENT CHANGES IN TECHNOLOGICAL LEARNING ENVIRONMENTS FOR MATHEMATICAL ACTIVITIES DRAWN FROM HISTORY

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In Thuringia, one of the federal states of Germany, there was a fundamental shift in education policy last year. In 2014, every student will have to write a school-leaving exam using a computer algebra system (CAS). Thus, many schools have started working with such systems. Following the approach taken in Finnish studies, the expected changes in mathematical education can be documented by observing eight activities or features drawn from the history of mathematics.

THEORETICAL BACKGROUND

Looking at the history of mathematics, there are eight activities which were important for the improvement of mathematics. These activities were shown to lead very often to new mathematical results at different times and in different cultures over more than 5000 years. The activities are strongly connected to each other. In order to illustrate the various connections between all the activities, they can be arranged in an octagon (Fig. 1) (Zimmermann 2003). This model can be an element in a theoretical framework for the structuring learning environments, as well as in assessing the quality of mathematical education (Haapasalo & Eronen 2010). It is interesting to observe how these features translate into classroom activities and how students perceive these activities in their settled learning environment. Finnish studies have shown a shift in evaluating these activities in classrooms where digital tools such as CAS are used (Eronen & Haapasalo 2010).

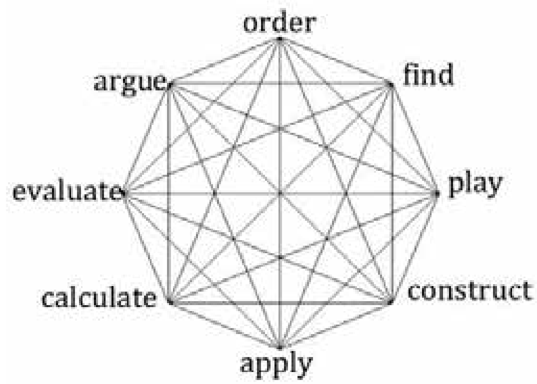
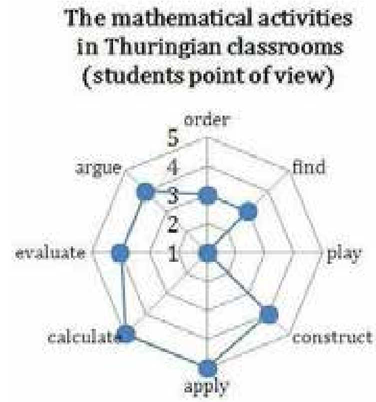


Figure 1: Octagon of math activities

METHODS

The Finnish questionnaire was translated and adapted for use in Thuringian schools. The web-based instrument consists of 24 statements ranked on a five point Likert scale. 523 students of grades 9 and 10 were asked how these activities appear in their mathematical education. The instrument was tested by a pilot group of 62 students and then reviewed by math teacher trainers in several rounds of discussions. At this point, only the median and the quartile were calculated to describe the data. To document the teacher's point of view, 21 teachers participated in the survey and answered the same questionnaire as the students.

RESULTS AND DISCUSSION



In students' view, argue, evaluate, calculate, apply and construct activities are more important in math lessons then order, find and play activities. In particular, play does not seem to exist in Thuringian math lessons, which is a pity when one considers their importance in the history of math. The range between upper and lower quartiles is low for all activities except argue, evaluate and construct, which indicates that students generally agree about the frequency of these activities in their classrooms.

The teachers' responses suggest that the eight activities are important in math lessons. Of particular importance were order, find, construct, evaluate and especially play. Furthermore, the teachers concur with the appearance of the eight activities in their classrooms. While the teachers understand that these activities are important for math learning, there seems to be a difference between the teachers and the students concerning their appearance. To explore the reasons for these differences, interviews will be done with the teachers.

The results of the students´ answers are interesting, but there are limitations to the validity of the research when students are asked just once a year. It cannot be assumed that students have a real overview of the whole year; they can be rather influenced by the most recent experiences in their math lessons. This has to be taken into account when interpreting the results. Despite that, it is expected that their views will develop over the next two years, mostly due to the use of modern technology such as CAS. Therefore, it will be interesting to observe how students will evaluate the appearances of the activities in the next two years following the introduction of CAS. This research will be conducted in the coming years, to investigate changes and to compare the results.

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USING ICT TO SUPPORT STUDENTS' LEARNING OF LINEAR FUNCTIONS

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In the poster a case study in a PhD project was presented, focusing on ICT and linear functions. The study investigates students' learning on the relation between algebraic and graphical representations of the linear function f(x)=mx+c, in particular the parameters. Changes of the scale on the axis also affect the graphical representation. How do the students experience this? Students' conceptions of how parameter values and changes of scale affect the visual representation in a dynamical environment need to be studied more. Students from upper secondary school in Sweden will work in pairs in a laboratory setting. The dynamic computer software GeoGebra will be used and the students will be engaged in two tasks. The students will be video recorded and their activities on the computer screen will be captured. The two tasks and the research design were presented in the poster in order to show examples of how to study exploratory learning in an ICT environment.

Keywords: Exploratory learning, ICT, Linear functions

THEORETICAL BACKGROUND

The role of the parameters is central in this study. In earlier research in connection to ICT one can find that parameters can show the students the differences literal symbols can play and that they can improve their symbol sense (Drijvers, 2003). Drijvers defines symbol sense as "the insight into and the structure of algebraic expression and formula" (*ibid*). "Variation of parameter value acts at a higher level than variation of an "ordinary" variable does; it affects the complete equation" (*ibid* p. 61). One aspect that according to Bardini and Stacey (2006) has not been studied much is students' work with the relation between the parameters *m* and *c* in the linear function *f(x)=mx+c* and their graphical representation. The parameters can be considered from different perspectives and *m* is more complex for the students (*ibid*). The *m* value can be seen analytic or visual (Zaslavsky, Sela, & Leron, 2002). From the analytic perspective the property does not depend on the representation, but in the visual perspective the slope depends on the scale on the axis (*ibid*).

AIM AND RESEARCH QUESTION

The aim of the study is to investigate students' conceptions of the linear function in a dynamic software environment.

- How do students' conceptions of the linear function change by exploring the parameters and in what way does the dynamical change of scale affect the students' conception of the linear function in a dynamic software environment?

METHOD

The research will be a case study of eight students, working in pairs, with Geogebra. The students will in the program be provided opportunities to manipulate the parameters and they will get an instant graphical feedback. The students will be asked to create fig. 1 (Magidson, 1992) and fig. 2 by writing the function with different parameter values in the algebraic field. A screen recording program and a video camera will capture the work and stimulated recall interviews will follow the tasks. At the end of the interviews the scale on the axis will be changed dynamically and the students will be asked to explain what happens with the *m*- and *c*-values and why.

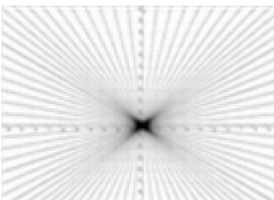

Fig. 1

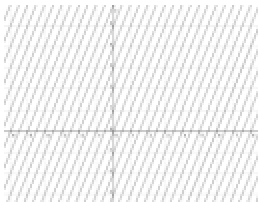

Fig. 2

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INFORMATION AND COMMUNICATION TECHNOLOGIES - AND MATHEMATICS TEACHING: RESEARCH CONDUCTED FROM ELEMENTARY TO HIGHER EDUCATION

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This paper presents results of studies related to the research lines: Mathematics Education and Educational Practices in Science Education and Technology. The studies have been developed from research on different fronts ranging from elementary to higher education. The study presented here involves a description of research conducted with undergraduates, masters and teachers, in order to promote reflections about the integration of ICT in mathematics education.

The proposed works involve reflection on the role of ICT in math pedagogic practice supported by technological resources of the Computer Laboratory and Distance Education to innovate teaching situations in the classroom. The studies of Bairral (2005), Borba and Villareal (2005), Maltempi (2008), Scheffer (2006) and Costa (2010), focus on the context of teaching mathematics among limitations associated with teachers´ education, especially in regard to information technology in teaching and the use of different possibilities that ICT offer, seeking to achieve alternatives to minimize this problem, pointing out the difficulties that undergraduates and teachers may have regarding digital inclusion in the school context.

This paper presents a brief discussion regarding inclusion of ICT in math teaching and highlights research studies that were developed and accompanied by the author, related to lines of research. The poster describes some examples, concluding with a brief reflection-discussion carried out up to now at different levels, and master dissertations that have turned more specifically to the study of practical proposals to be applied at school, considering the pressing needs of insertion of math teachers and their pedagogical practice in the digital context.

Scientific Initiation Research studies explored freeware to be used in teaching math. In these studies, Math Freeware and educational and interactive websites were selected and the quality and applicability of the programs to Elementary Education was investigated. Another area of studies turns to the creation of pedagogical proposals for discussion of math concepts, applied to digital classroom and appreciation of math argumentation of students and teachers conveyed in these learning environments. The studies investigated aspects of students' and teachers' Math Language and Argumentation, which considers the argumentative capacity manifested by verbal, non-verbal or written language, emphasizing the development of citizenship in the school context, i.e. in "math classroom", highlighting the importance of research oriented to math education with computer technologies.

Course Conclusion Papers include studies of the importance and use of ICT in math teaching from elementary to higher education, analyze different proposals with respect to how math contents are worked in digital classroom, besides considering the analysis of specific math concepts such as the exploratory study of representation and diagonal definition of a polygon.

Master Dissertations focused on the study of investigative practice proposals of Digital Inclusion, construction of Learning Objects, study of the importance and role of Distance Education, using platforms such as Moodle, with emphasis on initial and continuing education of teachers. The aim of these studies was to promote the intertwining of specific education with technology. The work was conducted with Wingeom and GeoGebra software that mediated resources to approach the Pyramids theme and Analytical Geometry. The results of Math Course undergraduates studies highlighted software benefits, possibilities and also difficulties to work with, although the benefits are great for the visualization process, discussion and reflection on geometric properties of polygons.

The resulting study proposals from researches that are presented in the poster include reflection on the role of ICT in teaching practice of math teachers and also of Distance Education in view of teaching and learning innovations in the classroom. Consequently, it is worth highlighting that the implementation of challenging educational practices and qualitatively significant from ICT is necessary for teachers in the current reality.

Working proposals resulting from these surveys cover reflection on the role of ICT and of Distance Education in pedagogical practice of mathematics teachers.

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG16: DIFFERENT THEORETICAL PERSPECTIVES AND APPROACHES IN RESEARCH IN MATHEMATICS EDUCATION

Ivy Kidron, Marianna Bosch, John Monaghan, Luis Radford

Keywords: *Inclusion in mathematical practices; iterative unpacking strategy; networking approaches; networking of methodologies; networking and epistemology; object-agents in the mathematics classroom; reference epistemological model; semiotic representation; teachers' identity development; theory and practice*.

The idea of the networking of theories has been investigated at CERME since the CERME 4 conference. In their introduction paper to the proceedings of the theory group at CERME 4, Artigue et al. (2005, p. 1242) wrote that "the central term that emerged from the working group was *networking*". They added that "as a research community, we need to be aware that discussion between researchers from different research communities is insufficient to achieve networking. Collaboration between teams using different theories with different underlying assumptions is called for in order to identify the issues and the questions". Networking became the topic of the working group at CERME 5 (Arzarello et al., 2007), CERME 6 (Prediger et al., 2009) and CERME 7 (Kidron et al., 2011). These working groups explored ways of handling the diversity of theories in order to better grasp the complexity of learning and teaching processes; and understand how theories can be connected or not in a manner that respects their underlying assumptions. These and other themes, like *the nature of mathematical objects* and the *semiosphere,* are a constant feature of this CERME working group, including CERME8. The *semiosphere* is a multi-cultural space of meaning-making processes and understandings generated by theories as they come to know and interact with each other (Radford, 2008). In the *semiosphere* a theory is considered as a dynamic interrelated triplet (P, M, Q) formed of theoretical principles (P), methodologies (M), and research questions (Q). Strategies for networking depend to an important extent on how "close" or "far" the networked theories are located in the *semiosphere*.

After a two stage peer review process, 12 papers and 3 posters were accepted for discussion in the Working Group. We revisit the manuscripts to raise themes that emerged from the conference discussion. We summarise these under three headings: (1) Methodology and theory; (2) Networking and epistemology; (3) Theory and practice.

DISCUSSION OF THE PAPERS WITH RESPECT TO: METHODOLOGY, EPISTEMOLOGY, THEORY AND PRACTICE

METHODOLOGY AND THEORY

Methodology was an explicit focus in the paper by Hickman & Monaghan on student teacher problem solving. A multi-media data collection artefact was used. Questions raised included:

- how an artefact can enable the networking of methodologies;
- where do “methods” and “methodology” lie in the Anthropological Theory of Didactics (ATD);
- what a methodology is and its relationship with “theory”;
- how to reconsider the work of past CERME “theory group” participants’ views on theories and methodologies?

The paper by Fetzer outlined theoretically and methodologically how object-agents can be integrated into a view on social interaction in classrooms. It employed Actor Network Theory (Latour) as a “background” theory. The focus was to understand (and trace) the role of “objects” in mathematical learning processes. Fetzer employed/networked three other theories: Goffman’s participation framework; Sack’s turn-taking system; and Toulmin’s theory of argumentation. A reason for employing these other theories was to develop an integrated theory-methodology for tracing the influence of objects.

The paper by Palmér focused on primary school mathematics teachers’ identity development. The two theories networked were Communities of Practice (CoP) and Skott’s “patterns of participation.” The paper analysed how the individual’s patterns of participation regarding teaching mathematics are influenced by and influence CoP. A conceptual framework is developed within a participatory perspective, making both the individual and the social possible as units of analysis. Both theories include the individual and the social but with different focus and different emphasis. The focus in the paper is the need for networking (coordination).

The paper by Barrera analysed students’ understanding of multiplication in a geometric context and their difficulties within a mathematics lesson requiring both mathematical frame changes and changes of register of semiotic representation. Barrera presented a combination of different theoretical approaches – Mathematical Work Space, Registers of Semiotic Representation and Semiotic Mediation Theory. Connecting the different theories allows the analyst to enrich the description of students’ work as well as increasing understanding of this work at both cognitive and epistemological levels.

The paper by Koichu demonstrated a new kind of strategy of networking. Specific connections between background and foreground theories are discussed. The dynamic relationship between theories is described: one theory may serve as an overarching framework in one case and as a source of conceptual tools (for elaborating on elements of another theory) in another case.

NETWORKING AND EPISTEMOLOGY

The role of epistemology in the networking of theories was an explicit focus in the paper by Ruiz-Munzón, Bosch and Gascón. It is addressed through the necessity for research to elaborate its own particular "vision" of the mathematical contents involved in research and to use this "vision" (conception or model) as a reference point from which to observe teaching and learning practices. This idea of a "reference epistemological model" (REM) is used for networking Chevallard's ATD and Radford's Theory of Knowledge Objectification (TKO). The paper deals with how each approach addresses the nature of algebraic thinking. It is an invitation for a dialogue between these two approaches that starts by presenting the point of view of the ATD, presenting its own REM about elementary algebra and the kind of questions addressed by this approach, in relation to the TKO.

The paper by Godino, Batanero, Contreras, Estepa, Lacasta and Wilhelmi analyzed two approaches to research in mathematics education: "Design-based research" (DBR) and "Didactic engineering" (DE), in order to study their possible networking. DE (closely linked to Brousseau's theory of didactical situations) focuses on epistemological questions; DBR does not adopt a specific theoretical framework, nor does it explicitly raise epistemological questions. The following question was discussed: is the epistemological focus only a question of "cultural and intellectual context" or is an epistemological reference necessary for each theoretical approach used in design based research in math education?

The paper by Janßen and Bikner-Ahsbahs offered insights about the processes through which algebraic structure sense develops. They described the specific contribution of two coordinated theories, Theory of Knowledge Objectification and Interest Dense Situation, focusing on the boundary between the theories and on the process of crossing it during the research that leads to a deeper insight into the development of structure sense. Questions about networking and the boundary between the theories were discussed.

Networking and epistemology was a focus as well in Castela's poster concerning ATD and CoP.

THEORY AND PRACTICE

The papers in this category gives substance to the hope of this Working Group, that exploring issues in networking theories is not a purely academic exercise; through this work we aspire to better understand the construction of mathematical knowledge towards the end of enriching learners' mathematical experiences. After networking and enriching our own theoretical frames, we need to consider questions how to apply our work.

The paper by Roos focused on CoP and inclusive pedagogies. The main aim is to understand the phenomenon of inclusion in mathematics and see how to problematize it. The connection between the theory of CoP and the framework of inclusion is done at the level of the principles of the theories since they both look at learning as a social phenomenon.

Verhoef, van Smaalen and Coenders reported on an empirical study exploring how six mathematics teachers investigated their teaching practices using a lesson study approach to determine characteristics of 'sensible mathematics'. Their study is essentially built on recent work of Tall and his colleagues on long-term mathematical thinking with regard to the approaches of Bruner and Freudenthal.

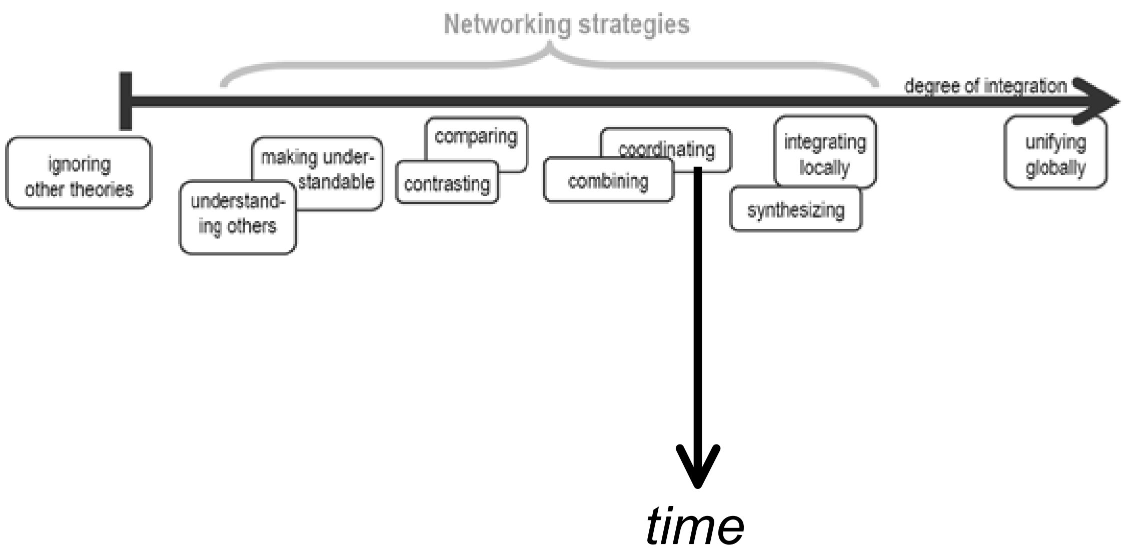
Two posters belong to this category of theory and practice: Czarnova's poster on Garcia & Piaget's triad and fairy tales and Godino's poster on instructional design tools based on the ontosemiotic approach.

CONCLUDING REMARKS AND THEMES FOR FURTHER CONSIDERATION

At CERME 7, we started work towards a *theory of networking theoretical approaches* in order to help the community develop in the direction of scientifically based multi-theoretical empirical research. At CERME 8, we continued the work but with a new focus on what a methodology is and the "irreducible link" between theory and methodology. Questions were raised about the place of methodology in theoretical frameworks and the relation of this issue to the methodology for networking theories as described in Radford's semiosphere (2008) and Artigue et al. (2012). The dynamic character of theories and how improving the methodology means improving the theory was discussed as well.

At CERME 8, the focus on networking and epistemology was stronger than in the previous working groups on theory. New issues compared to the last CERME included the impact of research approaches in the problem of professional development and teaching practices as well as in the formulation of research problems and the description of phenomena. In a sense, theoretical approaches need to be considered by what they enable researchers and practitioners to do, the questions raised, the regularities identified and described, that is, in a sense, the "results" obtained. .

The paper by Palmer raised a new issue, the dimension of "time" in networking strategies. If we consider the landscape of networking strategies proposed by Prediger, Bikner-Ahsbahs & Arzarello (2008) (see the figure below) and, for example, the strategy of "coordinating", then this coordination takes place in time. This temporal dimension is usually overlooked in consideration of networking and we look forward to taking this forward at the next CERME



We have experienced progress in our efforts on "networking theories" but at the same time our feeling is that, in the next working groups on theory, the stress on theories should be reintroduced and not only the focus on networking.

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ON THE GEOMETRICAL MEANINGS OF MULTIPLICATION: GEOMETRICAL WORK SPACE, SEMIOTIC MEDIATION AND STUDENTS' CHOSEN PATHS

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In this article, I briefly present an attempt to combine different theoretical approaches – Mathematical Work Space, Registers of Semiotic Representation and Semiotic Mediation Theory – in order to analyze students' paths within an experimental lesson connecting multiplication and some of its geometric meanings. I will also present reasons for combining these theories and I will illustrate how they have been used to analyze results from our experiments conducted in French high schools. Finally, I will conclude with some ideas and perspectives discussed in our working group and which I think could provide interesting perspectives for future research.

FROM INITIAL QUESTIONS TO A THEORETICAL COMBINATION FOR ANALYZING STUDENTS' CHOSEN PATHS

Geometry as a link between multiplication and its meanings for different sets of numbers

The fact that the notion of multiplication is closely associated with the idea of calculation can impede students from imagining a geometrical representation of the product. In the same way, the association between real numbers and the notion of magnitude can also get in the way when representing negative numbers, as well as when giving meaning to the product of two negative numbers. Thus, the multiplication of negative integers does not allow a geometric representation unless the "quantities" are treated in terms of orientation and direction (Argand, 1806). The extension of the operations' definition for complex numbers is linked to the representation of imaginary quantities by vectors. As a result, transformation is the only context in which multiplication and some of its geometric meanings can be connected. We are establishing a relationship between the different meanings of our mathematical object, with geometry as the glue holding them all together.

Geometric representations encourage the use of cognitive variables favoring the understanding of a mathematical object. The abstraction of arithmetic and algebraic concepts also stems from the fact that they are only represented through a symbolic diagram (Radford, 2003). So, it seems to us that it is always necessary to have an intermediary between a conception and access to its meanings, given that "there is not mathematical thinking without using semiotic representations" (Duval, 2008, p.1).

Consequently, we can formulate a key question related to the empirical context of our research: will students be able to establish connections between multiplication and geometry? We will see how we have integrated these three elements – multiplication, its meanings and geometric transformations – in an original experimental situation, which has been created to respond to this question.

Our main theoretical framework: The Geometrical Work Space

After analyzing the notion of Mathematical Work Space (MWS) (Kuzniak, 2011), we determined that this theoretical approach could suitably account for the complexity and richness of students' mathematical work. This notion assumes that a network has been created on two levels, one cognitive and the other epistemological. This network relies on a certain number of *geneses*, which can be *semiotic*, *instrumental* or *discursive* (cf. Figure 1). The analysis of this bilateral relation allowed us to expand our theoretical knowledge of mathematical work spaces and to determine the theory's flexibility. As we will see, this flexibility allows us to combine several theories in order to analyze students' chosen paths within an empirical mathematical working space.

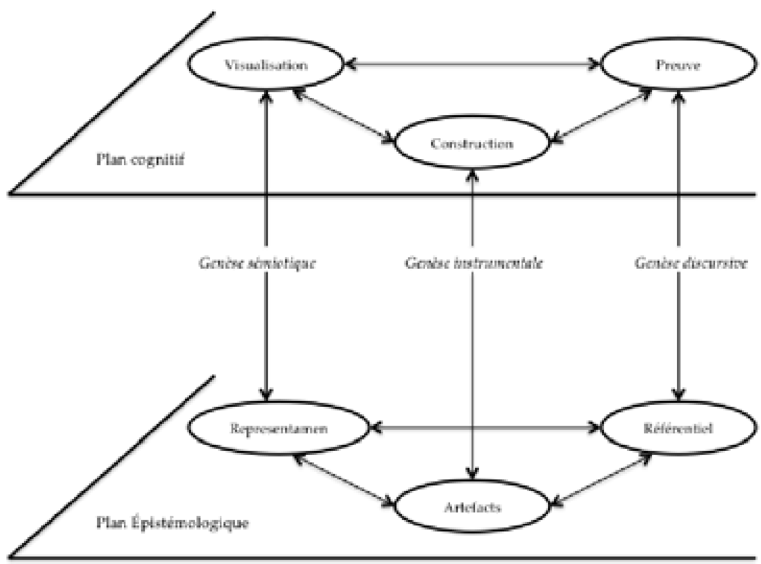


Figure 1: A genetic approach to the Geometrical Work Space

The starting point for the *geneses* linking the two levels of the MWS is traditionally placed on the epistemological level: for example, the visualization of an abstract mathematical object in a real or material space can be produced by the manipulation of artifacts in the construction of a figure. Still, the components on the epistemological level can be set in motion by needs on the cognitive level. A construction with artifacts can respond to a need for demonstration; the construction of a figure in a *paper-pencil* environment can be the result of a visualization allowing certain properties to assume a new configuration, or it can assemble the elements necessary for a proof. Thus, within these processes, called *geneses*, we can see not only the existence but also the permanent interactions between different registers of semiotic representation: we can make the transition from proof to construction through a change in register of representation (cognitive entrance); a geometric configuration, a sign or *representamen* (epistemological entrance) can prompt a visualization (cognitive action) making use of the properties and axioms (epistemolocial action) leading to a proof.

Finally, we have the makings of a hypothesis: being conscious of the metaphorical meaning of a mathematical object could allow a point of entry starting at the cognitive level of an MWS, which at the same time would encourage manipulating the components of the epistemological level:

> "Metaphors are not just rhetorical devices, but powerful cognitive tools that help us to build or grasp new concepts, as well as solving problems in efficient and friendly ways" (Soto-Andrade & Reyes-Santander, 2011, p. 2).

The combination of theories: Mathematical Work Space (MWS) and the role of sign-artifacts in a socially interactive space

All of the different studies dealing with semiotic notions present in the process of learning/teaching mathematics—whether they include information technology or not, whether or not they talk about registers of semiotic representation or pay special attention to the role of language and the understanding of mathematical objects—all of these positions "[revolve] around the relationship between mathematics and semiotics, concern questions of an epistemological, cognitive and sociocultural order" (Falcade, 2006, p. 3-4). However, within an MWS, the didactic question does not include explicit interactions between different individuals. Additionally, the MWS does not necessarily include other intermediaries, aside from the teacher, between the learners and the knowledge to be acquired or developed. It seems appropriate, then, to explicitly include *mediator intermediaries* where mathematics and semiotics can be found, where the different *geneses* occur, at the point where *semiotic mediation* and, when possible, *social mediation*, can facilitate access to research and the acquisition of meaning of mathematical objects. That said, given our special interest for semiotic genesis within a mathematical work space, the limitations of the existing semiotic approach as well as the theoretical definition of artifacts (Kuzniak, 2004) led us to look for other theoretical approaches dealing with semiotic mediation and the social construction of mathematical knowledge. We concentrated on Bartolini Bussi and Mariotti's (2008) work on Semiotic Mediation Theory, aspects of which we associated with Radford (2004) and Sfard's (2008) reflections on the social construction of mathematical knowledge and the complexity of the process of understanding a mathematical object. From a didactic point of view, Semiotic Mediation Theory includes elements such as the direct manipulation of tools, either in the form of concrete objects taken from the history of mathematics, or in the form of technological artifacts.

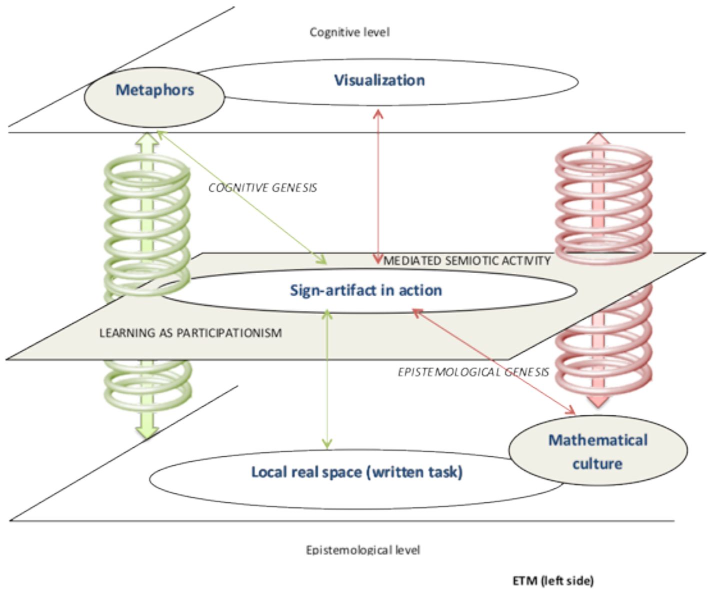


Figure 1: Diagram showing our theoretical combination. The dynamic arrangement of the MWS's components and of the epistemological and cognitive levels is due to the action of the sign-artifact in a context of semiotic mediation.

The theory also considers the precise organization of work in the classroom, where the relationships between the individual dimension, work in pairs and the collective dimension all play a role, and where oral and written activities complement one another. Finally, the theory also considers students' reading and interpretation of historical primary sources, aided by the teacher (Falcade, 2006).

The inclusion of historical and/or technological mediators as sign-artifacts on the one hand, and on the other hand the importance of collaborative work within the learning-teaching process, were the key elements that brought us to integrate Semiotic Mediation Theory and the Mathematical Work Space. Thus, we've included the MWS in a socio-constructivist learning process where the sociocultural and semiotic dimensions are included in the proximal development zone defined by Vygotsky (1934-1997).

AN OUTLINE OF OUR METHODOLOGY

Our desire to study the understanding of multiplication in a geometric context led us to design experimental course material. Observing several students' work on this non-traditional material allowed us to study their ways of solving problems in a mathematics lesson requiring changes of register of semiotic representation in a process of semiotic mediation.

Students in *Terminale S* (twelfth grade scientific track) were asked to solve a series of five questions suggesting a geometric approach to the multiplication of real and complex numbers (Appendix 1)[1].The activity was introduced in four *Terminale S* classes by their teachers. Thirty-four groups of two to four students worked on the activity for two hours in class. This session was integrated into the usual series of lessons by the teachers, who had just begun a chapter on complex numbers. At first, the students were instructed to make a geometric construction of the product of two real numbers in the plane, as proposed by Descartes in his *Geometry* (1637). Next, the students had to find a relationship between the points given on a plane and the multiplication of complex numbers. The final question of the series[2] called on students to think back on the entire activity. It played a fundamental role in the exploratory process, and its analysis allowed a first description of the paths followed by students moving between Descartes' multiplication and the understanding of the geometric meanings of multiplication for different sets of numbers. In order to describe the role of geometrization in students' approaches to multiplication, we studied their way of solving geometric construction problems involving the multiplication of real and complex numbers. Gradually, we've began forming a response to our research question (cf. Introducing the mathematical content of analysis: linking multiplication to geometry) "now transformed" and seeing it through the eyes of our main theoretical framework: are there interactions between the cognitive and epistemological levels of the MWS employed by students showing evidence of a geometric understanding of multiplication? Through this methodology, we determined students' chosen paths between Descartes' multiplication and the understanding of the geometric meanings of multiplication for different sets of numbers. In this article, we outline two paths so as to illustrate some of the experiment's results and the way we used our combination of theories to analyze students' mathematical work. In this work we analyzed the use of previously studied mathematical content and the way students employ it; interactions produced between the components of the MWS employed by students; the role played, as a *sign-artifact*, by the configuration of Thales' theorem corresponding to the geometric representation of Descartes' multiplication; the identification the *origin of geneses in a geometrical work space*. The mediating sign-artifact is therefore an essential element of our didactic proposals. As we will see, our artifact is a sign: a mathematical sign, a geometric representation and an icon of Thales' theorem. Recognized by students in the first question of the series as a tool to be employed in a proof, the sign must evolve throughout the collaborative lesson. The goal of its systematic use in the activities is the collective production of new signs, which correspond to new interpretations of the same artifact. We can associate this last point, especially concerning the evolution of signs and the way they influence discourse, with what Anna Sfard calls a "visual mediation." This is the place where "visual mediators have been defined as providers of the images with which discursants identify the object of their talk and coordinate their communication" (Sfard, 2008, p.147).We hope that our theoretical combination will allow us to study whether geometric representations, either given to or produced by the student, recognized as psychological tools or *sign-artifacts*, are capable of producing a precise mathematical object—in this case, multiplication.

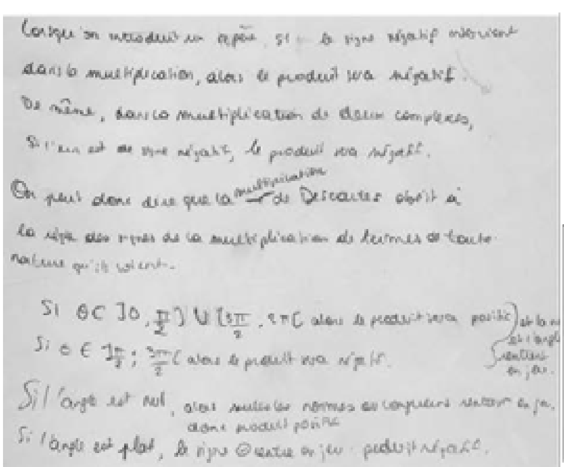
STUDENTS' PATHS: ANALYZING A FEW RESULTS THROUGH THE EYES OF OUR THEORETICAL COMBINATION

The individuals (groups) taking part in our experiment were initially classified by hand (i.e. we studied each sequence and each response to the final question, looking for elements of an answer that either corresponded to or differed from our initial determination) according to their responses to the last question of the series, leading to a first classification with three possible types of responses: Transformation (T); Proportionality and Thales' theorem (PTTh); complex (C), without explicit references to geometric transformations. The determination of students' paths was thus based on an analysis of the entire process leading them to their response to the last question.

Comparison between two groups showing different conclusions: Proportionality and Thales' theorem (PTTh) and Transformation (T).

The two groups were initially given two different classifications. We will examine the groups' differences beyond certain similarities in their responses to the final question. Group C1-I9 (T) bases its conclusion on an immediate visual connection between multiplication and the sign rule as seen in the Cartesian plane. Then the group extends this relation to Descartes' multiplication as well as any multiplication with any type of factor. The most striking observation we can make about their response, and which shows the close links between figural and discursive geneses, is the connection made between the sign rule and vectors' angles. The change of frames related to the sign rule is due to our activity, because the geometric manifestation of the sign rule does not appear in the French curriculum. In order to reinforce this idea, we emphasize that the group specifically identified the nature of the angles, especially the zero angle and the flat angle which allow a connection between real and complex numbers in this geometric configuration.

Group C3-I1 (PTTh) arrives at a conclusion that takes into account the different parts of the lesson. They show the properties of multiplication of real and complex numbers, justifying them with Thales' theorem.


Lorsqu'on introduit un repère, si le signe négatif intervient dans la multiplication, alors le produit sera négatif.
De même, dans la multiplication de deux complexes, si l'un est de signe négatif, le produit sera négatif.
On peut donc dire que la multiplication de Descartes obéit à la règle des signes de la multiplication de termes de toute nature qu'ils soient.
Si $\theta \in]0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi[$ alors le produit sera positif
Si $\theta \in]\frac{\pi}{2} ; \frac{3\pi}{2}[$ alors le produit sera négatif.
Si l'angle est nul, alors seules les normes des complexes entrent en jeu, donc produit positif
Si l'angle est plat, le signe ⊖ entre en jeu : produit négatif.

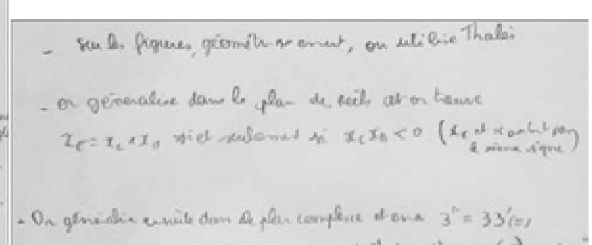

- sur la figure, géométriquement, on utilise Thalès
- on généralise dans le plan des réels et on trouve ... si $x_C x_B < 0$ (... même signe)
- On généralise ensuite dans le plan complexe ...

Figure 4: Left, pair C1-I9's conclusion (T). Right, pair C3-I1's conclusion (PTTh)

They present their geometric interpretation of multiplication of real and complex numbers as a generalization of Thales' theorem in the Cartesian plane and then in the complex plane. Can we say that the word "generalization" clearly accounts for a connection between the different aspects of the lesson? In a way, yes, because the lesson requires students to extend different sets of numbers.

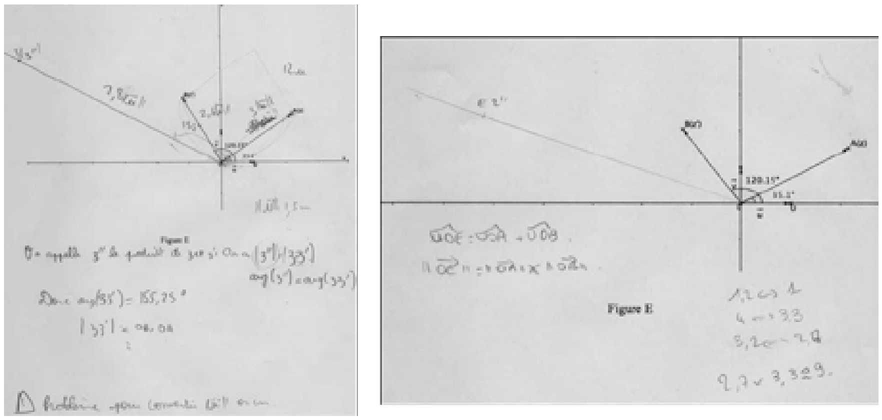


Figure 5: Left, C3-I1's answer to 4.b (PTTh). Right, C1-I9's response to 4.b (T)

For a more complete view, the figures above show the complete responses of both groups to question 4.b. It seems clear that the groups' algebraic knowledge of complex numbers guided their geometric construction, which shows no connection with the lesson's previous constructions. The algebraic properties of the multiplication of complex numbers are already part of students' *theoretical references* and they orient the students' construction, which is correct but isolated from the rest of the lesson. In their responses, nothing explicitly accounts for the visualization and geometric comprehension of complex numbers as a transformation in the plane. The entrance into the MWS for this question is epistemological, then, resulting in a *construction* with a ruler allowing the students to visualize the placement of the product of two complex numbers in the plane. In this lesson, the two groups take different approaches to question 2b. Here, only group C1-I9 (T) has used *transformations* in its response to one of the activity's first questions.

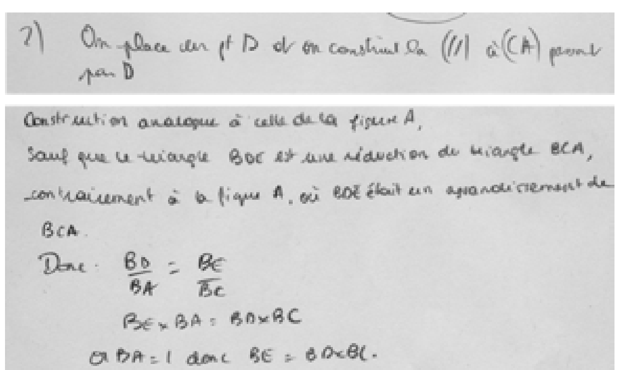


Figure 6: Above, group C3-I1's response to question 2.b. Below, response 2.b for group C1-I9.

In analyzing this response, it is quite interesting that the students refer to the *reduction* and *enlargement* of triangle BCA. The relationship of proportionality implied in Thales' Theorem and the geometric representation of Descartes' multiplication were not approached according to the segments-factors and the segment-product representing the proportionality. The visualization of similar triangles favors the immediate use of Thales' theorem because the students interpret this reduction-enlargement using their already available knowledge. These *paths* present their own specific characteristics. The "T" group mentions transformations quite early, in the second question, and their conclusion (final response) is very rich, relating the sign rule and geometry. The group using Thales produces a response describing multiplication for different sets of numbers but does not explicitly link the icon of Thales to a geometric representation of multiplication for different sets of numbers.

Synthesis of the analysis

Variable	**Analysis results**
Entry into the MWS	Mixed but largely epistemological.
Semiotic genesis/links between different registers of representation	Semiotic genesis of unknown origin, especially in the response given by C1-I9 to the second question. A semiotic genesis of cognitive origin may have occurred with the visualization of similar triangles, followed by the visualization of a transformation (reduction-enlargement) of these triangles through multiplication. A significant link between registers of representation was made during the association between the sign rule and the representation of the product of a positive and a negative number in the "affine" plane (C3-I1).
Semiotic mediation of the sign-artifact	The action and evolution of the *sign-artifact* were identified thanks to a specific explanation of the existence of a zero angle in Descartes' product (which was necessarily transposed into the "affine" plane). A link was therefore produced between the properties of the icon of Thales' theorem and the properties of multiplication of complex numbers.
Geometrical meaning of multiplication	Hypothetically, the product was interpreted as resulting from a transformation in the plane. This could have been a possible interpretation of the sign rule in terms of angles and by generalizing the meaning of any product to the product of two complex numbers.

Conclusion

We based our theoretical framework on a unified conception of cognitive and didactic elements. Several interests informed its development: the social dimension of learning processes; the study of semiotic mediation processes favoring the collaborative construction of a mathematical object; and the construction of meaning of mathematical objects. New lines of questioning emerge as a result of the diversity of concluding responses and the similarities and differences between the different students' chosen *paths*. They also demonstrate the difficulty of organizing the different *geneses* of the MWS. We realize that this is quite a complex cognitive activity since there is no direct conversion between one register of representation and another. This leads us to position the students' activity within a mathematical work space where the meaning of mathematical objects *emerges* as a result of a *cognitive genesis*. This *genesis* assumes the presence of complex semiotic interactions, such as those described by D'Amore and Fandino (2007) in order to describe the difficulties of moving between different representations. For example, the transposition of Descartes' product to a product operating directly on numbers, represented geometrically, positioned on a plane: this can only be the result of realizing that the activity is based on a *mathematical idea* (Lakoff & Nunez, 1997) that completely departs from the traditional knowledge of the mathematical object in question, i.e. the geometric meanings of multiplication. We are no longer working on the techniques of calculation or proof. Thus, because of the richness of the MWS introduced by the teacher as well as the diversity of students' personal MWSs, we must highlight the importance of our theoretical combination (MWS (Kuzniak, 2011) and TSM (Bartolini Bussi & Mariotti, 2008)). Through the lens of these theories, we have seen and analyzed students' ways of looking for the meanings of a mathematical object, dealing with a mathematical sign-artifact between an epistemological and a cognitive level, within a context of social interactions. This theoretical connection can still be improved if we look deeper, for instance, in Radford's semiotic approach (as suggested in our working group). Finally, the process of connecting theories must start with the analysis of their epistemological basis, which could help us understand the main focus of each one. Thus, I think we could better determine their complementarity, pertinence and even the most appropriate way – for example, in terms of chronology – to analyze students' mathematical work through them.

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[1] Link to the series of questions: https://docs.google.com/file/d/0B2PIBsYMh2gCS2EwQzYxSFlUSkk/edit?usp=sharing

[2] Last question: "Thinking about the work you have done today and in past mathematics lessons, what geometric meaning could you assign to multiplication?"

COUNTING ON OBJECTS IN MATHEMATICAL LEARNING PROCESSES. NETWORK THEORY AND NETWORKING THEORIES.

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Latour's actor network theory proposes a sociology of objects, accepting objects and things as participants in the course of action. Mathematics education has to deal with all sorts of objects, didactical tools and manipulatives, diagrams and signs. Mathematical learning appears to be closely connected to objects. Latour's approach is fascinating and irritating and provokes the research question, if and respectively how actor network theory can be a fruitful background theory to get a better understanding about the role objects play in mathematical learning processes. How is it possible to do research in mathematics education respecting Latour's perspective on social interaction? This paper outlines how a change of paradigm might be implemented through local integration of theories.

EXPOSITION: NETWORK THEORY AND NETWORKING THEORIES

On the one hand there is an empirical phenomenon: In the mathematics classroom all sorts of things, objects and visualisations are offered to improve mathematical learning and understanding. However, those objects prove to be resistant. They often neither function nor work the way teachers or learners expect or intend them to. On the other hand there is a theory: Introducing the Actor Network Theory (ANT) Latour reassembles the social. He develops a sociology of objects, accepting objects and things as participants in the course of action (Latour, 2005).

Studying Latour's sociological approach is fascinating and provoking. But is ANT a suitable background theory in the field of mathematics education? (How) is it possible to do *empirical* research in mathematics education referring to ANT? Even more precisely: (How) Can we investigate the role objects play in mathematical learning processes if we adapt Latour's perspective? What ways of acting can be empirically reconstructed, how do objects *act* in the mathematics classroom? This article performs the change in perspectives and focuses objects in the mathematics classroom through sociological lenses.

Latour himself does not suggest any methods of empirical analysis. Exploring appropriate heuristics means firstly to take additional approaches into consideration and secondly to connect them with ANT. In this paper, the networking of theories on the background level is the basis for the development of a local integration of theories (see Bikner-Ahsbahs & Prediger, 2010). Closely connected is the discussion and compilation of appropriate heuristics.

In this article, four approaches come into interaction: Latour's ANT, Goffman's Participation Framework, Sack's Turn-Taking system and Toulmin's Model of Argumentation (Latour, 2005; Goffman, 1981; Sacks, 1996; Toulmin, 2003). Footing on their networking, two methods of reconstructing the ways objects take effect in learning processes are introduced. The first analytic approach meets the sequential character of interactional processes: A systematic analysis of turn-partaking is implemented relying on own works (Fetzer, 2007, 2009, 2010, 2013), Goffman's participation framework (Goffman, 1981) and Sacks's turn-taking system (Sacks, 1996). The second analytic approach comes up with the lasting quality of objects: The functional orientated concept of argumentation analysis referring to Toulmin (2003) is adapted to trace objects' marks in mathematical learning processes. Aiming at the function of single actions, this analytic approach offers the opportunity to escape the sequence of interaction.

ACTOR NETWORK THEORY: AN 'ANT' IN THE RESEARCHER'S EAR

Latour's actor network theory ANT is a radical change of perspectives proposing a sociology of objects. He recommends a broader understanding of agency as well as action and extends the list of actors assembled as participants fundamentally.

"Any thing that does modify a state of affairs by making a difference is an actor." (Latour, 2005, p. 71). All actors, human or not, are "participants in the course of action" (ibid., p. 71). "Objects too have agency" (ibid., p. 63), and appear associable with one another, but only momentarily. They assemble as actor entities one moment and combine in new associations the next minute. Accepting objects as participants in the course of action, Latour gives in the idea of stable and pre-defined associations and actor-entities.

Looking through Latour's sociological lenses, not only the traditional understanding of agency has to be re-defined, but also the notion of action has to be re-thought. Objects participate in the course of action and take effect. But apparently their mode of action is different from the way human participants contribute to the social interaction.

Latour's approach fascinates and provokes. It puts not a bug, but an ANT in my mathematic researcher's ear. It triggers re-thinking of traditional ideas of the role of visualisations, manipulatives and other objects in mathematical learning processes. In this article, I introduce a local integration of theories in order to develop a new piece of synthesised theory on the role objects play in mathematical learning processes.

NETWORKING THEORIES:
TRACING OBJECTS IN MATHEMATICAL LEARNING PROCESSES

Tracing objects' marks in the course of mathematical learning processes referring to ANT means to substantiate the notion of action empirically. How do objects act or take effect in the course of action? This research issue entails the question, how objects' actions can be empirically traced and observed. What analytic methods prove to be suitable for reconstructing non-human actions? Below, two approaches to explore object's traces systematically are developed. Both are founded on micro-ethnographic research.

Micro-ethnographic approaches to classroom investigation like the method of interaction analysis help to reconstruct the development of interactional processes (see Fetzer, 2007; Krummheuer/Naujok, 1999). Interaction analysis is based on conversation analysis (Sacks, 1996; ten Have, 1999) and reveals, how the sequential organization of interaction is constituted. Accordingly interaction analysis is bound to be a sequential analysis. Every single *action* is interpreted extensively in the sequence of emergence. To investigate the aspect of *inter*-action, every single action is understood as a "turn" (Sacks, 1996) on a previous action. Turn-by-turn the emergence of the course of action is reconstructed. Traditionally, interaction analysis captures human actors as participants of an interactional process and investigates their actions. Own works on a micro-ethnographic approach to an object-orientated analysis of classroom interaction prove theoretically as well as empirically, that this interactionistic tool is a suitable and powerful basis for analysing the *networking* of all sorts of actors (Fetzer, 2009, 2010, 2013). Both approaches introduced below to trace objects' participating in the course of mathematical learning processes are methodologically based on an object-integrating analysis of classroom interaction referring to Fetzer (2009, 2010, and 2013).

Participation Framework

Goffman's "participation framework" (1981) provides "an essential background for interaction analysis" (ibid., p. 3). His approach offers the chance to distinguish between different forms of participating in "moments of talks" (ibid., p. 313). Right from the start Goffman tells hearing and speaking apart from the social slot in which these activities usually occur. "When a word is spoken, all those who happen to be in the perceptual range of the event will have some sort of participation status relative to it." (ibid., p. 3). Some might have the official status of participants. As "ratified participants" (ibid., p. 130) they may be listening or not be listening. Others might not be official participants, but still be following the encounter closely in the status of "eavesdropping" or "overhearing" (ibid., p. 132). Goffman discriminates ratified and non-ratified participants on a phenomenological basis. Besides, he introduces the status of "bystanders" (ibid., p. 132). Those not ratified participants find themselves in visual and aural range of the social encounter. The crucial aspect of this participative status is the fact, that their access to the moment of talk is perceptible by the official participants. Perceiving them as someone having the opportunity to follow the social encounter, the ratified participants assign them the status of bystanders. Thus, a bystander's role is determined in the *interactional* process.

Goffman's participation framework does pay no special attention to objects. Nevertheless, it proves to be a fruitful basis for investigating object-actors participating in the course of action empirically. Below, three connecting points are outlined. First, empirical research on the basis of ANT has to face the rapid change of networking and the unstable boundaries of associations. How is this flood of potential associations manageable? Who respectively what has to be considered as a participant in the course of action? Goffman takes a micro perspective and suggests focusing on "moments of talks" (ibid., p. 131) concerning the framework of participation. That combines well with an object-integrating analysis of classroom interaction. Investigating *moments of networking* in the sequence of emergence may capture the intermittent existences and permanent changes in assembling appropriately. Accordingly, ratified participants as well as bystanders are understood as participants at a certain moment of networking. Second, Goffman stresses, that not sound alone is at issue in social encounters, but also *other ways of perception* as sight or touch (ibid., p. 129f.). Opening the perceptual variety of interaction he clears the way for a wider range of observable actions and participating actors. Object's actions might rather be seen, felt or otherwise perceived than heard. Third, the introduction of the *bystander's role* is promising in the context of object's agency. Objects might be in the perceptual reach of ratified participants as potential actors. Assigned as bystanders, they might come into play, associate with other actors and take effect in the interactional process.

Turn-Taking System

Trying to differentiate empirically the way objects participate in learning processes remains unaccustomed. Nevertheless, own works on the development of an object-integrating analysis of classroom interaction (Fetzer, 2009, 2010, 2013) approve, that objects' contributions to learning processes become accountable in the process of interweaving. As soon as object-actors assemble with other actors they enter the course of action. Their traces render perceivable and can be captured by analysis. *Turn by turn* it can be reconstructed, how objects participate in the emergence of social reality. A closer look on the theoretical basis of the sequential organisation of conversation is a matter of consequence. Referring to Sacks's approach to conversation analysis, conversation is a coordinational problem (Sacks, 1996, Vl. II, p. 32). A basic challenge is to preserve "one party at a time" (ibid., p. 32), namely that any time there is at least one, but no more than one participant speaking. In matters of the "order of speakers" (ibid., p. 32; p. 521), it is decided on the next speaker or the next action, but not on the speakers or actions afterwards. Sacks refers to this sequential organisation of speaker change recurs as the "turn-taking system" (ibid., p. 524). He specifies several techniques of speaker-selection. Some of them correspond to the *next* speaker: "Current speaker selects next actor/speaker" (ibid., p. 524). Others are connected to the *current* speaker: "Next speaker may self-select himself." At this point Sacks's approach to conversation analysis may be connected to an interactionistic approach: Inter-action is based on mutual exchange. Actions are related to each other as turns. The current speaker may select the next actor. However, this turn-"distribution" (ibid., p. 533ff.) needs to be understood by the designated next speaker. Solely in this case he/she/it may either accept this distribution or refuse to pick up the offered turn. Eventually actors may simple take over the turn. Sacks does not specify in *turn-distribution* and *turn-partaking*. Indeed, this determination proves to be continuative and fruitful when combining conversational and interactionistic approaches in order to grasp objects' traces analytically.

Whenever participants in interactional processes change their status and become active actors, their current action can be interpreted as a turn on previous actions. The question arises, who or what provoked or initialised the change of participation status. Investigating the way objects participate in the course of action, especially a second aspect turns out to be crucial. Moments of networking and changes in participation status permit to reconstruct the *previous role* of the current active actor. Methodically, an "*analysis of turn-partaking*" is implemented (Fetzer, 2007, p. 126 ff.). This method of analysis stood the empirical test earlier in reconstructing actions, that are observable only indirectly (Fetzer, 2007, 2009). Turn-partaking and turn-distribution are linked very closely when investigating object-actors (Fetzer, 2013). In order to differentiate the notion of action in the context of objects' participating empirically, sequential analysis benefits from this strong connection. It is reconstructed backwards or indirectly how objects take effect in the course of learning. When a human actor is operating as a turn on an object, it can be deduced, that the object must have been kind of active before. It must have 'told' the human actor something. The object must have *made the offer to partake the next turn*. As a consequence, the human actor, namely a learning child, brings an active return to the previous objects action.

Empirical research on the basis of this analysis of turn-partaking confirmed, that several ways of taking over offered turns by object participants can be reconstructed. Sometimes human actors accept directly-offered turns. Sometimes humans pick up unspecific offers to take over the next turn (see Fetzer, 2013). In particular this method of analysis facilitates the reconstruction of the ways, in which objects take effect in social interactions and learning processes (see Fetzer, 2013).

Theory of Argumentation

Objects have a lasting quality. Following Latour, objects and things render more durable the constantly shifting interactions (Latour, 2005, p. 68). A book might be standing disregarded on a shelf for weeks, some-thing written on the board might remain unchanged a whole school morning, a bunch of manipulatives may lie on a desk untouched for minutes. Due to their (potential) durability, objects may take effect spreading place and time. Those moments of networking when objects take part actively might be temporarily delayed. Objects may overcome temporal bounds and limits. This fact brings up a methodological issue. Investigating objects' traces exclusively on the basis of a sequential analysis appears to be too short-handed. In addition, a second approach has to be implemented, that breaks up the narrow boundaries of sequential emergence. Referring to a sociological perspective on learning, mathematical learning processes emerge predominantly in (collective) argumentations (see Miller 1986; Krummheuer/Fetzer 2005). Consequently, investigating the role objects play in mathematical learning processes means to focus argumentative processes. In my research, I refer to the theory of argumentation and the "Toulmin Model" to capture objects' traces in mathematical learning processes analytically (Toulmin, 1958/2003).

Based on Toulmin's approach, arguments show a specific structure. The pattern of an argument has certain constituent elements, namely data, conclusion and warrant. These three functional categories are the core of an argument. The conclusion is the claim that needs to be established. When it is challenged, it has to be proven justifiable. The data is our personal knowledge, the facts we appeal to as a foundation for the claim. It is the ground we produce as support for the original assertion. It is the answer to the challenge: "What have you got to go on?" (ibid., p. 90). The shortest possible argumentation would be: Data *D* is the basis *so* the conclusion *C* can be established.

> "We already have, therefore, one distinction to start with: between claim or conclusion whose merits we are seeking to establish and the facts we appeal to as foundation for the claim – what I shall refer to as our data." (ibid., p. 90).

No amount of facts may establish any conclusion. There needs to be a connection of data and conclusion on another level.

> "Our task is no longer to strengthen the ground on which our argument is constructed, but is rather to show that, taking these data as a starting point, the step to the original claim or conclusion is an appropriate and legitimate one. At this point, therefore, what are needed are general, hypothetic statements, which can act as bridges, and authorize the sort of step to which our particular argument commits us." (ibid., p. 91).

These connecting links are warrants (ibid., p. 91ff.). They indicate the bearing on the conclusion on the data already produced and answer the question "How do you get there?" (p. 91). You can get from *D* to *C since* the warrant *W*.

Toulmin's analytical model is a methodological tool to reconstruct the *function* a certain action fulfils within the argument (see Kopperschmidt, 1989). It focuses on verbal as well as non-verbal actions. It is not restricted to analyse questions in dispute, but is open to all sorts of argumentative processes. Even implicit parts of an argument might be captured by analysis, as empirical research proves (Fetzer, 2007; Meyer, 2007; Schwarzkopf, 2000). Eventually, the Toulmin model connects well with the idea of tracing objects agency.

IMPACT ON RESEARCH AND PRACTICE: COUNTING ON OBJECTS IN THE MATHEMATICS CLASSROOM

Latour put an ANT in my researcher's ear, to re-think the way objects participate and take effect in the emergence of mathematical learning processes. Especially in primary mathematics education it is an expedient effort to change perspectives and explore approaches to "follow" the object-actors (Latour, 2005, p. 12, 156) and their traces. In this article the theoretical basis for empirical analysis on objects' participation in learning processes is outlined. Based on a micro-ethnographic approach to empirical research on mathematical learning processes, Latour's ANT, Goffman's Participation Framework, Sack's Turn-Taking system and Toulmin's Model of Argumentation are networked on the background level.

Actually, this local integration of theories proves to be a theoretical basis that renders empirical analysis well possible. Investigating how objects participate in the course of action leads to the development of a new piece of synthesised theory on the role objects play in mathematical learning processes. Several forms of object-participation can be reconstructed in the development of social learning processes. However, due to space restrictions, no examples of empirical analysis are given here. To learn more about the implementation of analysing objects traces see Fetzer (2013). Below, empirical results on the methodological level as well as on the theoretical level are abstracted. They outline the impact, this new approach might have on research and practice.

Methodological level

An object's opportunity to become an active participant in the course of action is strongly connected to human-actors interpretations and perceptions. It is only in moments of networking, that their acting becomes observable. When inter-acting with other actors, objects traces render perceivable. Accordingly, the basic idea of analysis is to grasp the acting of objects *indirectly*. The *analysis of turn-partaking* is the core of investigation. As soon as a human-actor takes over the turn offered by an object-actor, the current action (human) allows concluding on the previous action (object). In other words: The way students or teachers act as a turn on an object suggests how objects participate. What has the object 'told' them, when it was the object's turn?

The analysis of turn-partaking captures the *sequential* emergence of mathematical learning processes. However, one dominant feature of objects and things is their durability. As a consequence, the second methodological approach to investigate objects traces is the *Toulmin model* on argumentation. This method of analysis aims at the *function* single actions fulfil within an argument. Both tools of analysis prove to be empirically successful. They combine well in order to differentiate empirically the notion of 'action' in the context of objects.

Theoretical level

Objects take effect in the social learning process in a different way than students or teachers do. They may hold different *status of participation*. On the one hand, ratified actors might allocate them the role of *bystanders*. This is the case, if active actors perceive the object-actor as some-thing in the perceptual reach of the interactional process, but not directly involved in the course of action. On the other hand, object-actors may hold the role of *ratified participants*. In this participation status the other participants in the social learning process accept them as participants that take effect in the course of action.

The status of participation of an object is no stable allocation. An object-actor might be perceived as a bystander one moment and become a ratified participant the next minute. These changes of the participation-status from the bystander to ratified participant are triggered by other participants of the social encounter. The transition might be either initiated by human actors or by ratified active object-participants. Different conditions of emergence may be reconstructed concerning the change of participation-status:

- Objects become ratified participants, if the *process of problem solving stagnates* within the current ratified participants. Stretching the group of ratified participants and opening it to an object, which has held the bystander-status so far, often restarts the interactive learning process. These conditions for changing participative roles in case of stagnation emerge predominantly in group working phases, when teacher's interventions are minimal.
- Objects change their participative status, if *teachers* call students' attention to the turn an object offers.
- As soon as *objects* turn to be ratified participants, they often initialise the change of participative status of further objects. This might happen at the beginning of a problem solving process, if the task (an object-actor in the role of ratified participant) claims integrating manipulatives or visualisations (object-actors in the status of bystanders).

Empirical analysis provides a *differentiation of the notion of action* in the context of objects and offers answers to the following questions: How can actions be described, that are performed by object-actors? In other words: How do objects act, and what do things do in the course of action?

- As bystanders, *objects hold unspecific offers in readiness* to take over the next turn. Other participants might *pick up* this unspecific offer and get active in the interactive process.
- Taking the role of ratified participants, *objects partake turns*. At the same time they *make an offer to take over the next turn*. This specific offer addresses other participants. Its stimulative nature may vary from *volunteering* to

provoking to take over the next turn. Following, participants *accept* these volunteered or provoked offers to take over the next turn and get active.

- Objects *take part in the emergence of interactive processes*. They might contribute to the ongoing interaction especially if the problem solving process stagnates. Sometimes object-actors take over the role of a student's partner in interaction. In this case, they might not only contribute to the emergence, but *give the direction to the course of action*.
- As ratified participants *objects take over elements of an argument*. Playing the role of *data*, they clarify what there is to start from. In other cases they contribute to the conclusion. If objects take over the warrant, they legitimate the conclusion. It can be reconstructed, that objects particularly take over data and warrant. Accordingly, objects play a central role in mathematical argumentation and learning processes.

The ANT in my ear triggers a change of perspectives and shows objects through sociological lenses in a different light. Accepting objects as participants in the course of action and following the idea of objects having agency raises a scientific discussion and results in networking theories. This article shows that empirical research on the basis of Latour's approach is possible. It gives an example of how local integration of theories can lead to the development of a new piece of synthesized theory, a theory of objects participating in mathematical learning processes. Besides, the empirical results have impact on the practice of mathematics education. (For more detailed information on that aspect see Fetzer (2013)). I found out that objects may influence learning processes especially if they take over the role of ratified participants. As a consequence, our didactical efforts should aim on the following points: We should try to get objects out of the bystander-status. As mentioned above, conditions are good in group works, when students integrate objects as participants in a stagnating solving process. In other situations teachers might call students' attention to the turn an object offers. Once objects turned their status and became ratified participants, we should try to extend this participative constellation. And last but not least, objects play an important role in mathematical arguments. The empirical analyses revealed that objects participate in argumentative processes by taking over data and/or warrants. Thus they contribute to 'clearer' and 'deeper' argumentations. The more explicit and the more complex an argumentation is, the easier it becomes to follow the argumentative line and the more mathematics there is to learn.

Objects leave their traces in the emergence of social learning processes and take part in the course of action. We should count on objects in mathematical learning processes!

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DIDACTIC ENGINEERING AS DESIGN-BASED RESEARCH IN MATHEMATICS EDUCATION

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In this paper we analyze two approaches to research in mathematics education: "Design-based research" (DBR) and "Didactic engineering" (DE), in order to study their possible networking. The problem addressed in both approaches is the design and evaluation of educational interventions, providing research-based resources for improving the teaching and learning of mathematics. They also try to contrast existing theories, or characterize new educational phenomena. We conclude that DE could be seen as a particular case of DBR, linked to the "Theory of didactical situations", or that DBR is a generalization of DE that use other theoretical frameworks as foundations for designing teaching experiments.

Key words: networking theories, didactic engineering, didactic design, teaching experiment, didactic resources.

INTRODUCTION

The view of Didactic of Mathematics as a "science of design" is highlighted by several authors (Wittman, 1995; Hjalmarson & Lesh, 2008a; Lesh & Sriraman, 2010). Lesh and Siraman consider mathematics education as a science oriented to design processes and resources to improve teaching and learning of mathematics and reflect on the purpose of research in mathematics education:

> Should mathematics education researchers think of themselves as being applied educational psychologists, or applied cognitive psychologists, or applied social scientists? Should they think of themselves as being like scientists in physics or other "pure" sciences? Or, should they think of themselves as being more like engineers or other "design scientists" whose research draws on multiple practical and disciplinary perspectives— and whose work is driven by the need to solve real problems as much as by the need to advance relevant theories? (Lesh & Sriraman, 2010, p. 124).

Recent interest in Anglo-Saxon literature on design-based research (handbooks; special issues of high-impact journals) and on its role in mathematics education complements the traditional French literature on "Didactic engineering" (Artigue, 1989), which provided significant contributions from the 80's, but was virtually ignored in this literature. This suggests a certain isolation of the French didactic engineering regarding the research done in other countries with similar objectives.

The purpose of this paper is to establish some connections between both research approaches or paradigms by trying to answer the following questions:

- Can didactic engineering be included within the family of design-based research?

- What kind of synergies can be established between these research paradigms?

The problem of comparison and coordination of theoretical frameworks is a topic of relevance discussed by various authors and in discussion forums, such as the CERME Working Group ”Different Theoretical Approaches and Perspectives in Mathematics Education Research” (Prediger, Arzarello, Bosch, & L'enfant, 2008). Bikner-Ahsbahs and Prediger (2010) make a plea for exploiting theoretical diversity as a resource for richness and consider this diversity as a challenge and starting point for further theoretical development through networking theories.

In this paper we compare "Didactic engineering" (DE) and "Design-based research" (DBR), starting with the identification of their basic features and later carrying out a rational comparison of them. In the next two sections, a summary of the characteristics of DBR (Collins, Joseph, & Bielaczyc, 2004; Kelly, Lesh & Baek, 2008), and DE (Artigue, 1989; 2009, 2011) is carried out. Then we identify similarities, differences and possible complementarities in the problems addressed, the theoretical principles, methodology and intended results.

DESIGN-BASED RESEARCH MAIN FEATURES

Design-based research[1] (DBR) (Brown, 1992; Kelly, Lesh & Baek, 2008) is a family of methodological approaches for the study of learning in context. It uses the design and systematic analysis of instructional strategies and tools, trying to insure that instructional design and research be interdependent. It is assumed that educational research separated from practice cannot take into account the context influence on the complex nature of the results, or cannot adequately identify their constraints and conditioning factors: “We argue that design-based research can help create and extend knowledge about developing, enacting, and sustaining innovative learning environments” (DBRC, 2003, p. 5). This group of authors used the term design-based research methods to differentiate their approach from the classic experimental design in teaching, attributing five characteristics to this method:

1. The central goals of designing learning environments and developing theories or “prototheories” of learning are intertwined.
2. Development and research take place through continuous cycles of design, enactment, analysis, and redesign.
3. Research on designs must lead to sharable theories that help communicate relevant implications to practitioners and other educational designers.
4. Research must account for how designs function in authentic settings. It must not only document success or failure but also focus on interactions that refine

[1] It is also known as *design research* or *design experiments*.

our understanding of the learning issues involved.

5. The development of such accounts relies on methods that can document and connect processes of enactment to outcomes of interest. (DBRC, 2003, p.5).

As methodological paradigm, DBR, specifies how to conduct design studies, i.e. investigations of some length on educational interventions induced usually by a designed set of innovative curricular tasks and/or instructional technology. "Often, what gets designed is a whole 'learning environment' with tasks, materials, tools, notational systems, and other elements, including means for sequencing and scaffolding" (Reimann, 2011, p. 38). Usually there is no strict separation between development and theory test, but rather both are interconnected in a way reminiscent of "grounded theory" (Glaser & Strauss, 1967). However, in DBR there is no particular interest to avoid using previous theories; on the contrary it encourages theory building that incorporates elements beyond the observations. "Design experiments are conducted to develop theories, not merely to empirically tune 'what works'" (Cobb, Confrey, diSessa, Lehrer and Schauble 2003, p. 9). DiSessa and Cobb (2004) highlight the role of theory in what they call the "ontological innovation"— the invention of new scientific categories, specifically categories that do useful work in generating, selecting among, and assessing design alternatives.

Three phases are considered along a design experiment (Cobb and Gravemeijer, 2008): 1) Planning of the experiment, 2) Experimentation to support learning; 3) Retrospective analysis of the data generated along the experiment.

According to Collins, Joseph and Bielacyz (2004), design experiments, incorporate two critical elements to guide us in improving educational practice: the focus of design and the evaluation of critical elements. These experiments complement other empirical methods, such as ethnographic studies, clinical research, experimental or quasi-experimental studies to assess the effects of independent variables on dependent variables. But they also pose challenges that may be common to other research on education, such as:

- Difficulties arising from the complexity of real-world situations and their resistance to experimental control.
- Large amounts of data arising from the need to combine ethnographic and quantitative analysis.

DIDACTIC ENGINEERING MAIN FEATURES

The notion of "Didactic engineering" (DE) was introduced in the French Didactic of Mathematics in the early 80's to describe a research approach in mathematics education comparable to an engineer work. When carrying out a project the engineer relies on scientific knowledge from his/her domain; he/she agrees to submit the project to scientific scrutiny, while at the same time, is forced to solve issues more complex than those of science, and therefore to address problems that science cannot yet take over (Artigue, 1989, p. 283). Since its origin, didactic engineering was fundamentally linked to educational interventions (experiments) in classrooms, usually sequences of lessons; these experiences were guided by and tried to test some theoretical ideas (Artigue, 2011, p. 20). That is, DE is conceived as the design and evaluation of theoretically justified sequences of mathematics teaching, with the intention of trigger the emergence of some educational phenomena, and to develop teaching resources scientifically tested.

Considering the state of the French research in mathematics education at the beginning of the 80s, it is not surprising that the natural theoretical framework for didactic engineering was the Theory of didactical situations (TDS) (Brousseau, 1986; 1997). In fact, didactical engineering has been the privileged didactical research methodology in France. Some features of DE in its original sense are:

- It is based on classroom teaching interventions, i.e. on the design, implementation, monitoring and analyzing teaching sequences.
- The validation is essentially internal, based on the confrontation between a priori and a posteriori analysis (there is no external validation, based on comparison of performances in experimental and control groups).

Didactic engineering addresses case studies where the following phases are distinguished (Artigue, 1989): a) Preliminary analysis b) Design and analysis a priori of teaching situations; c) Experimentation; d) A posteriori analysis and evaluation. Perrin-Glorian (2011, p. 59) suggests that didactic engineering from the beginning "is more than a research methodology: it is also intended a didactic transposition viable in the ordinary teaching". That is, the view of didactic engineering as a product is as important as the method.

The evolution of didactic engineering was linked to the change of the Theory of didactical situations (TDS) itself, or the application of other theoretical models derived from the TDS, such as the Anthropological Theory of Didactics (Chevallard, 1992), or research about teachers practices. Perrin-Glorian (2011) distinguishes two types of didactic engineering according to the primary research objective: 1) didactic engineering for research aims to produce research results with experiments depending on the research question, without worrying about a possible wider dissemination of the scenarios used, 2) didactic engineering for development and training; here the short-term goal is the production of resources for teachers and for teacher training.

Although didactic engineering is not uniform, because of the changes that have been introduced, some sensibilities remain: epistemological sensitivity (expressed or not in terms of fundamental situation), emphasis on building tasks, concern for the organization of a *milieu* that offers a strong a-didactic potential, the key role played by the a priori analysis and the insight into the validation processes. As a result didactical engineering has become an object of fuzzy contour (Artigue, 2011, p. 23).

COMPARING AND RELATING BOTH METHODOLOGICAL APPROACHES

In this section we try to build a bridge between didactic engineering (DE) and design research in education. Already Hjalmarson and Lesh (2008b) compared design research with engineering:

> Our view of design in education research is based, in part, on the similarities and parallels to be drawn between education and engineering as fields which simultaneously seek to advance knowledge, impact human problems, and develop products for use in practice (Hjalmarson y Lesh, 2008a, p. 526).

For these authors engineering primarily involves the design and development of products that operate in systems, and includes the process of design and the tangible products of that design:

> The parallels between engineering design and education design begin with the nature of the systems where the products of design are used. The systems are not fixed even if they are often stable. The systems require innovation, respond to innovation (e.g., a curriculum, a piece of technology), and are changed by innovation (Halmarson y Lesh, 2008b, p. 107).

It is clear that the use in the field of education of design-based-research, and the comparison with engineering as design technology, is restricted to the "instructional design", which becomes synonymous with "didactical design". Didactical design in mathematics education, “Includes all types of ‘controlled intervention’ research into the processes of planning, delivering and evaluating concrete mathematics education. It also includes the problem of reproducibility of results from such interventions” (Winslow, 2009, p.2).

There is, however, a substantial difference between DBR and DE, which is closely linked to the methodological approach provided by the interpretive framework of a didactical base-theory, as the Theory of didactical situations (TDS). The close relationship of DE with the TSD provides explicit criteria, in the design, implementation and retrospective analysis, and an orientation to test and develop the theory itself. By contrast, although design-based research has similar goals, it does not adopt specific theoretical frameworks. DBR is, therefore, a family or category of educational research perspectives bonded for the interest or focus in the design, implementation and evaluation of educational interventions in naturalistic contexts, without explicit interest in epistemological questions. “A very noticeable aspect of the design research literature is the absence of discussion of epistemological issues. In the recent Handbook of Design Research Methods in Education (Kelly et al., 2008), for instance, the word ‘epistemology’ is not even used as an indexing term, and, while the word is not totally absent from the general DBR literature, there is no serious discussion of epistemological issues” (Walker, 2011, p.53). On the contrary, in DE epistemological questions are central because they are crucial in Theory of didactical situations.

While in DBR the focus is the design and search of instructional resources, DE goes further, trying to analyze the characteristics of these resources (and in general, everything that constitutes the *milieu*); the idea of a-didactic situation, where students are involved in solving a problem, without the direct guidance of the teacher, and the dialectic between didactic and a-didactic situations are particularly important.

Although both approaches are mainly qualitative (or mixed), an important difference is the establishment in DE of a priori hypotheses (before the experience design), while DBR tends to a qualitative posture, assuming that theories emerge from the data. Although the methodology stages are very similar, there is a greater influence of the previous theory in DE that seeks explicitly validation (even if internal) of the previous hypotheses, and the previous analyses of the design are detailed and complete.

In Table 1 we summarize the comparison between the DBR and the DE, considering the three key elements proposed by Radford (2008) as constituting a theory: paradigmatic issues, principles and methods. We also identify the results expected in its application.

Table 1. Comparison of DBR and DE (TSD)[2]

	DBR	DE (TSD)
Paradigmatic issues	- How to improve mathematics learning in realistic school contexts based on research results? - What instructional resources can be used to improve teaching and learning mathematics?	- What type of problem-situations give meaning to a specific mathematical knowledge? (Fundamental situations) - What features should have the milieu to achieve the student's independent learning of a specific knowledge? (Dialectic between a-didactic and didactic situations)
Theoretical assumptions	- The design is based on various interpretive frameworks - Theories emerge from the data	- The TDS guide the formulation of hypotheses about the design and the expected results - Data test the theory
Methodology	*Type:* Mixed (qualitative / quantitative) *Phases:* - Preparation of the experiment - Experimentation - Retrospective analysis	*Type:* Mixed, with positivist emphasis (nomothetic) *Phases:* (guided by the base-theory) - Preliminary phase - Design and a priori analysis - Experimentation - A posteriori analysis - Validation
Results	- Instructional resources	- Testing hypotheses derived from the

[2] We introduce the notation DE (TSD) to indicate the dependence of the "Didactic engineering" from the Theory of Didactic Situations. This will help to express possible generalizations of didactic engineering changing the base-theory used to support the instructional design.

	-Local emerging theories	base-theory -Fundamental situations -Instructional resources

The paradigmatic issues stated for DE (TSD) are somewhat similar to the DBR, although oriented and specified in the light of the underlying theory. The "fundamental situations" are models or representations of mathematical knowledge, i.e. characterizations of knowledge through the issues or problems for which that knowledge is an answer. The constructivist epistemological position about mathematical knowledge supporting DE, does not explicitly question an institutional relativity for that knowledge. This relativity was later highlighted by the Anthropological Theory of Didactic (ATD) (Chevallard, 1992, 1999) and the Onto-semiotic Approach (OSA) (Godino, 2002; Godino, Batanero and Font, 2007) assuming that the "raison d'être" of knowledge change in different institutions, or from its original construction in the history of mathematics, as regards its current use.

Consequently, we could view DBR as an extension of DE (TSD). While the former does not have a single or preferred theoretical framework, DE rests on a theory of intermediate level, the Theory of didactical situations which provides criteria to develop mathematical situations (search of fundamental situations for specific mathematical knowledge), and also for implementing and conducting the teaching situations in seeking students' autonomous learning. From the methodological point of view the preliminary study proposed in DE (TSD) may be a distinction, motivated and oriented by the base-theory towards the epistemological analysis of the mathematical knowledge to be taught.

FINAL REMARKS

As we have mentioned, the main problem of DBR is to develop instructional resources to improve the teaching and learning mathematics in naturalist school contexts, based on research. Since the research on educational interventions depends critically on the theoretical frameworks used to support the design, implementation and interpretation of results, different DBR will depend on the base-theory, or lack of theory. Therefore, we could conceive DBR as a family of methodologies or educational research approaches. Given that the aim is to develop a product based on research (curriculum, sequence of lessons, educational software, etc.), as Hjalmenson and Lesh propose, this type of research can be considered as a form of engineering inquiry. With a wider tradition, the French didactic engineering addresses a similar problem, and is supported by an explicit base-theory of intermediate level, the Theory of didactical situations. Therefore, we may consider it as an antecedent (in time) of the family of DBR, and a particular instance thereof.

It is also clear that instructional designs based on theoretical models different from the Theory of didactical situations are carried out and produce varieties of design-based research. These varieties share some paradigmatic issues, theoretical assumptions, methodologies and intended results, but may differ in others, as it is indicated in Figure 1 for didactic design based on the Anthropological Theory of Didactic (ATD), the Onto-Semiotic Approach (OSA), Realistic Mathematics Education (RME) (Freudenthal, 1991; Heuvel-Panhuizen, & Wijers, 2005), or, for example, the Lesson Study in Japan (JSL) (Fernandez & Yoshida, 2004). In this figure we also suggest to consider the expressions, DBR - Design-Based-Research, DE - Didactical Engineering, ID - Instructional Design and DD - Didactical Design, as synonymous, i.e. with equivalent meaning.

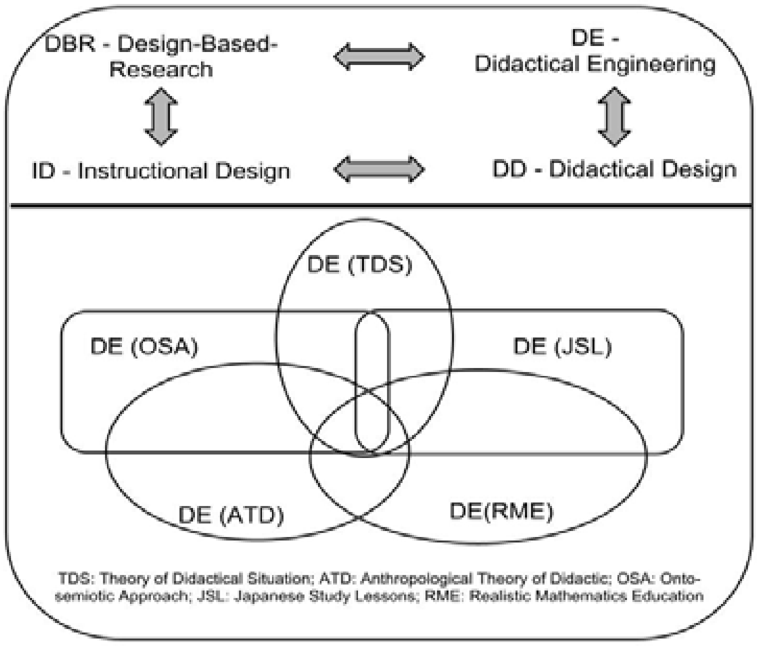

Figure 1. Varieties of design-based researches

All these types of didactic engineering (instructional design, didactical design, …) share similar issues. However, new analyses are needed to clarify in more detail that that carried out in this paper the methodological consequences that arise from a change in the base-theory supporting each particular engineering, and to explain the types of results that can be obtained in each case.

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NETWORKING METHODOLOGIES: ISSUES ARISING FROM A RESEARCH STUDY EMPLOYING A MULTI-MEDIA ARTEFACT

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This paper focuses on the methodological matters arising from using a multi-media artefact in an empirical study of student teacher collaborative mathematical problem solving. The use of this artefact in post-task stimulated recall interviews allowed two distinct methods of collecting data to be 'networked'. Issues raised by this 'networking of methodologies' include: how an artefact can enable the networking of methodologies; what a methodology is and its relationship with 'theory'; and a consideration of the work of past CERME 'theory group' participants' views on theories and methodologies.

Key-words: methodology, networking, think-aloud protocols, task-based interviews, stimulated recall.

INTRODUCTION

In the course of investigating student teacher collaborative mathematical problem solving we employed a specific multi-media artefact (the *Livescribe* pen, described later). The use of this artefact in post-task stimulated recall interviews suggested that two apparently distinct methods of collecting data on collaborative problem solving, think-aloud protocols (Ericsson & Simon, 1993) and task-based interviews (Goldin, 1997), could be 'networked'. In this paper we foreground methodological aspects of the investigation to explore issues related to networking theories.

This paper is structured as follows. We begin by describing think-aloud protocols and task-based interviews approaches. We then consider stimulated recall methodologies and *Livescribe* pens. We then outline our investigation of collaborative problem solving, to provide readers with the context of the research and a rationale for the approach we adopted. This is followed by an outline of past CERME 'theory group' participants, Radford (2008) and Artigue, Bosch & Gascón (2011), with particular regard to the place of methodology in theoretical frameworks. The paper ends with a discussion of issues raised and a consideration of the implications of our work for issues raised in the CERME 8 *call for papers*.

THINK-ALOUD PROTOCOLS AND TASK-BASED INTERVIEWS

Think-aloud protocols (T-AP) and task-based interviews (T-BI) can be regarded as methodological approaches for capturing data on domain specific problem solving. Both rely on participants' verbal reports and allow opportunities for participants to engage in 'live' reflection on their 'free' problem-solving performance in order to elicit reasoning. Both were developed in the last quarter of the 20th century and were relevant to research foci in expert-novice thinking, which was in vogue in this period. T-BI, but not T-AP, were developed within the mathematics education field. Neither was initially tied to a specific theoretical framework and both have been used by researchers espousing various constructivist and socio-cultural positions. Ericsson & Simon (1993) and Goldin (1997) are comprehensive expositions of each approach.

A key difference between the two approaches is that T-AP ask participants to 'think-aloud' during task performance whilst T-BI do not encourage this; T-AP afford reflection 'in the moment' by a self-directed participant whilst T-BI afford reflection 'after the act' via interviewer prompts on strategies undertaken. Ericsson & Simon (1993) note that there are different kinds of think-aloud verbalisations, from free association utterances (their Type I and Type II verbalisations, as discussed by Robertson (2001, p.13) involve 'direct verbalisation' and 'recoding of short term memory') to verbal descriptions of everything that the participant is conscious of whilst engaged in the task (Type III verbalisations); a risk with the latter type is that this will frustrate participants' problem solving. The exploration of learner thinking in T-BI has four stages and two goals:

> (a) posing the question ("free" problem solving) …[with] nondirective follow-up questions (e.g., "Can you tell me more about that?); (b) heuristic suggestions if the response is not spontaneous (e.g., "Can you show me by using some of these materials?"); (c) guided use of heuristic suggestions … (e.g., "Do you see a pattern in the cards?"); and (d) exploratory (metacognitive) questions (e.g., "Do you think you could explain how you thought about the problem?"). The clinician's goal is always to elicit (a) a complete, coherent verbal reason for the child's response and (b) a coherent external representation constructed by the child …(Goldin, 1997, p.45).

From a T-AP perspective such 'after the act' verbalisations are 'suspect' as they may provide data on what participants think they thought rather than on their thoughts-in-action; there is also a danger that the interviewer may co-produce knowledge (Hobson & Townsend, 2010). A T-AP approach, however, may not establish coherent verbal reasons for actions and much may remain unspoken, even if the interviewer encourages verbalisations of descriptions of everything in the participant's consciousness. There appears to be no scope for a hybrid form ('networking') of the two approaches[i] to capture data during problem solving since interrupting the think-aloud process with task-based interview style questions or establishing a protocol in which there is a strong emphasis on explaining and describing thinking, may impact upon free problem solving (that both approaches arguably require as a first step). Indeed, Ericsson and Simon (1998, p.180) state that "participants' efforts to describe and explain thinking can change the sequence of thoughts and lead to the intrusion of additional thoughts" thus potentially leading to "interfere[ence] with normal [mathematical] problem solving [processes] by either slowing…[them]…down or affecting the sequence of problem solving steps" (Robertson, 2001, p.13).

STIMULATED RECALL AND *LIVESCRIBE* PENS

Stimulated recall "is one subset of a range of introspective methods that represent a means of eliciting data about thought processes involved in carrying out a task or activity" (Gass and Mackey, 2000, p.1). A stimulated recall interview (SRI) typically involves the use of a stimulus such as an audio or video tape to "enable the participant to 'relive' the episode to the extent of being able to provide, in retrospect, an accurate verbalised account of…original thought processes" (Calderhead, 1981, p.212). Mason (2002, p.63) claims (and we agree) that SRI allows researchers to explore participants' "knowledge, views, understandings, interpretations, experiences and interaction"; there is thus a sense in which SRI and T-AP share ontological and epistemological principles (in the sense that Radford (2008) uses the term 'principles'). Given that a SRI is an 'after the act' replay, there is no reason why a SRI cannot be in two stages: (i) no interviewer prompting; (ii) interviewer prompting. The use of such a two stage SRI could then follow the protocols of both T-AP and T-BI.

Livescribe is the brand name of a digital pen with a built-in digital audio recorder. While it can be used as a regular pen on ordinary paper, when used with special proprietary paper, it records writing to be uploaded to a computer (which can then be played back/animated in real time) alongside the audio of sounds recorded at any point in the script. In addition to this, touching the pen to any point on a 'completed' page of notes/jottings also enables the instant replay of the precise sounds (for example, any conversation occurring at that point in the writing), thereby facilitating recall, should it be required, even without upload to a computer. This last point is very interesting in student playback of problem solving talk/writing as students can hear the talk associated with specific written symbols. *Livescribe* pens thus afford being used as a stimulus in SRI.

AN INVESTIGATION OF COLLABORATIVE PROBLEM SOLVING

Part-time postgraduate primary student teachers at York St John University took part in a pilot project (for the first author's PhD) exploring how digital audio recordings may provide opportunities to engage in closer consideration of, and reflection on, their mathematical problem solving performance. The research question was: how does thinking aloud, supported by digital audio recording, support student teachers' understanding of problem solving. Like most research questions, this one did not appear 'out of the blue', it arose from academic dialogue which tried to 'intellectualise' the first author's prior 'success' in engaging student teachers in 'real' problem solving and his prior experiments in using *podcasts* with his student teachers (*podcasts* were seen as a means to keep part-time students engaged in mathematics during periods of absence from the University). The practical idea 'behind' the research question is: if student teachers can critically reflect on their own problem solving, then this reflection may help them to design learning environments which support 'real' problem solving for their future pupils. The following (supplemented by a picture of an abacus) is one of the problems the students worked on in self-selected small groups:

Make as many three digit numbers as possible with 25 beads on one abacus.

An assumption behind this work was: student-student discourse during problem solving is important for mathematical development. We value Mercer's (1995) work in this area and consider his category 'exploratory talk' (where participants engage critically but constructively with each other's contributions) as important for mutual development. We also wanted a framework for problem solving and used that of Hošpesová and Novotná (2009), largely due to the connections that can be drawn between their categories and Mercer's (1995) work. A hybrid 'talk and problem solving framework' is detailed in Hickman (2011). This framework includes Mercer's (1995) three categories of talk (disputational, cumulative and explorative) but the explorative category is sub-divided into that in which relevant information is offered for joint consideration in: mathematical form; non-mathematical form.

Independent of this work the second author purchased *Livescribe* pens because he thought they might be useful for research. The first author 'tried these out' and we agreed that they were potentially useful for this research.

The research consisted of student teachers thinking aloud whilst engaging in group problem solving activities with *Livescribe* pens; at this point the interviewer employed a T-AP protocol. Just over a week later the students revisited (with the aid of the *Livescribe* pens) their work to identify potentially beneficially exploratory dialogue; at this point the interviewer employed a T-BI protocol. This 'revisitation' of their work included students using the hybrid framework to categorise particular responses.

It could be said that this 'revisitation' falls short of what some researchers regard as a SRI, for example, "the stimulated recall group…[speak]…their thoughts into a microphone as if talking to [themselves]" (Egi, 2008, p.226). However, the T-BI approach supported by the recorded material picked up by the *Livescribe* pens afforded the students the opportunity to identify themes, to talk to themselves and to reflect on the performance of their earlier 'selves'.

This pilot study was not without problems. Sometimes connections with mathematics recently engaged in were not noticed and this appeared to be linked to the presence of the digital audio recorder and the associated talk protocol, which, some students in the SRI claimed, impacted on their performance:

> Well, we knew that we *had* to discuss it in this way...

Some began speaking *before* they had fully considered the problem:

> I would have preferred to have had time on my own to look at it first and then come into it because...solutions started being talked through before I was at that point.

This indicates that, however important talk protocols may be, additional 'ground rules' are required before beginning a task of this kind. The same issues may have impacted upon the use of the *Livescribe* pens (i.e. making jottings because they felt they 'had' to) but the presence of the notes and the *Livescribe* pen often provided evidence of exploratory contributions that would not otherwise have been evident, 'making up for' and, ultimately, enhancing the quality of the original mathematical discussion. Participants were able to identify their exploratory comments more effectively within the *Livescribe* supported SRI than had been the case via their original T-AP (one potential cause of their less successful listening to each other's contributions being their level of concentration on their own verbal contributions and awareness of being recorded). For example, in the SRI (but not in the T-AP) of the *abacus problem*, participants noted that they had, in fact, been presented with a problem similar to one that had previously been encountered (indeed, the problem had been chosen for this reason). Therefore, the *Livescribe* supported SRI afforded students the opportunity to make connections, from their original contributions and working, that had not been explicitly identified in the original problem solving session. In the original recording, one participant cautiously observed:

> This is like one of the problems we did last week where after a certain number, you have to … you have … yeah…

Given the 'unfinished' nature of the verbalised thought, it is unsurprising that it was not effectively built upon (in Mercer's (1995) 'cumulative' fashion); it took the *Livescribe* SRI to make it clear to participants what had originally been propounded.

CERME PARTICIPANTS' VIEWS ON METHODOLOGY

In this section we outline our interpretation of the views of 'methodology' in Radford (2008) and in Artigue, Bosch & Gascón (2011). We select these papers because they deal with 'networking theories' and we critically value them.

Radford (2008, p.320) suggests that "a theory can be seen as a way of producing understandings and ways of action based on: … *basic principles* … a *methodology* … paradigmatic *research questions*". The principles (P) are a 'system' of unequally weighted 'elements' (views or statements) on pertinent constructs such as cognition, learning and social interaction; Radford would say that our statement above, "student-student discourse during problem solving is important for mathematical development", is a principle. Radford adds that "there is a *hierarchy* that organizes and prioritizes them" (ibid.). Our statement was explicit, behind our statement is an implicit view that development is cultural and is mediated by language.

Radford (ibid.) states that a 'methodology' (M), "includes techniques of data collection" and may go beyond 'positivistic' data collection. The word "includes" suggests that there is more that can be said about a methodology – we agree! Further to this a methodology must have *operability* (produce data to address research questions and distinguish between relevant and irrelevant data) and *coherence* (consistency with principles); relevant data is that in which there is coherence between the principles and the methodology of a theory.

Paradigmatic *research questions*, Q, are "templates or schemas that generate specific questions as new interpretations arise or as principles are deepened, expanded or modified" (ibid.). "Expanded or modified" suggests a 'state of flux' in the 'lives' of theories with Radford further stressing 'flexibility' and interrelations between P, M and Q. Both of these aspects endear us to Radford's approach but, despite the interrelationships, P, M and Q are distinct in Radford's exposition. We return to this in the next section and now turn to Artigue et al. (2011).

Artigue et al. (2011) is a novel but straightforward application of Chevallard's (via Mauss's) construct 'praxeology' to the phenomena of theorising. We refer the reader to Chevallard's CERME address (Chevallard, 2006) for an exposition of this construct but outline the terminology below.

A praxeology consists of four elements [T/τ/θ/Θ] in two pairs, "[T/τ] corresponds to the 'practice' … *types of problems* T that are approached and the *techniques* τ … [θ/Θ] forms the *technological-theoretical* discourse used to describe, justify and interpret [the practice]" (Artigue et al., 2011, p.2). This language underpins an epistemological model. An immediate insight from this perspective is that "talking about 'theories (as in the expression of 'networking theories') is the result of a metonymy used to point to the whole – *research praxeologies* – by only indicating one part, the theoretical block of praxeologies." (Artigue et al., 2011, p.2) It appears possible to replace the word 'technique' with 'methodology' in the case of research praxeologies and this certainly can be done in some research praxeologies but we feel that part of a methodology can be located in the technological component in our research.

An important adjunct in Artigue et al.'s (2011) consideration of research praxeologies is the construct (didactic) 'phenomenon', "empirical facts, regularities that arise through the study of research problems" (Artigue et al., 2011, p.3). Our informal interpretation of such phenomena is *what is important/striking in the research under consideration?* As an aside, we believe that this construct could be used to partition papers in recent CERME 'theories working group' into those with a (and those with no) central phenomenon; in the latter case there would appear to be a sense of 'networking for the sake of networking'.

Artigue et al. (2011) state "our approach is fully coherent with that developed by Radford (2008)". We too see commonalities in these two approaches but their distinct ontologies means that there can be no isomorphic mapping between them.

DISCUSSION

Our focus in this section remains on methodological issues. We address two issues: the role of *Livescribe* pens in 'networking methodologies'; whether T-AP and T-BI are simply methodological approaches for capturing data. In considering the second issue we explore the views of methodology in Radford (2008) and Artigue et al. (2011) with regard to our research.

We present an argument above that there is no scope for 'networking' the two approaches to capture data during problem solving. However, the *Livescribe* pen allows for an initial T-AP interview to be 'played back' in an SRI with a T-BI protocol. There is a real sense in which this artefact affords networking these interview approaches. With regard to Radford's (ibid.) view that a 'methodology' (M), "includes techniques of data collection and data-interpretation as supported by *P*" our use of both T-AP and T-BI could be seen as 'opportunistic' research which compromises basic principles (using the principles of T-AP in the initial interview and the principles of T-BI in the SRI). We, however, consider that the *Livescribe* pen allows the researcher to 'link' the initial T-AP activity and the subsequent T-BI activity whilst retaining the principles of both (but separately in the two stages of the activity). In terms of Goldin's (1997) four stage exploratory process T-AP are used (appropriately) in the free problem solving first stage and the remaining three stages (heuristic suggestions; guided use of heuristic suggestions and metacognitive questions) follow within the SRI. There is thus a sense in which the *Livescribe* pen does more than network interview approaches; it contributes to a 'stronger' T-BI protocol as heuristic suggestions, for example, may be more readily identifiable via a combination of 'jottings' and comments (most especially those that are, as discussed above, not fully verbalised) than through a written record or audio recording alone. Furthermore, it is arguable that, knowing that the *Livescribe* recording will preserve and actively connect these ordinarily disparate elements ensures that there is less necessity for an onerous verbalisation protocol, thus supporting the initial free problem solving.

We now consider whether T-AP and T-BI are simply methodological approaches for capturing data. We continue with a consideration of the Radford (2008) quote above but now focus on the word 'includes'. Radford is 'hedging' here. A methodology certainly includes data collection and analysis techniques, but many researchers, we feel would replace 'includes' by 'are'. We support Radford's 'includes' but (like Radford, we suspect) find it difficult to state (in the abstract) what a methodology holds beyond data collection and analysis techniques. We prefer to approach the abstract in this matter via the concrete: are T-AP and T-BI simply data collection and analysis techniques? Our view is 'no', they have evolved into quasi-theoretical approaches where an implicit (and flexible) 'theory' is intertwined with an explicit methodology. We have not explicitly researched their evolutions but both approaches were developed over a period long before the publication of Ericsson & Simon (1993) and Goldin (1997). We further suspect that each has two types of 'principles', explicit principles associated with data collection and analysis techniques and flexible implicit principles which may be appropriated by researchers of a variety of 'theoretical' approaches (e.g. constructivists and activity theorists amongst others). We hold that there is an historical 'irreducible bond' between theory and methodology in the various research uses of both T-AP and T-BI.

We now turn our attention to Artigue et al.'s (2011) research praxeologies but continue to consider the 'includes' vs 'are' matter above. In the language of Artigue et al. (2011), being in the 'are' camp would place methodology in the [T/τ] 'practice' pair of a praxeology. In the words ABG use, there are indications that they are in the 'are' camp; "Some of these phenomena enrich the initial theoretical framework to produce new interpretations and techniques or research methodologies" (p.3). But Artigue et al. (2011) was the first paper on this matter. In a more recent paper (Bosch, 2012, p.3) there is a suggestion that a part of the methodology can reside in the *technology* (θ) part of the *technological-theoretical* [θ/Θ] pair.

> Dans le cadre des PR, le niveau technologique permet de distinguer, dans le bloc théorique de description, interprétation et justification de la pratique, un ensemble d'éléments plus variables et mobiles que constituent le premier niveau de description des techniques, la « technologie », de la « théorie » formée par des principes et hypothèses plus stables qui font partie du « noyau dur » de la praxéologie et qu'il est plus difficile de faire évoluer sans que la construction complète en soit altérée. De cette manière, la prise en compte du niveau technologique des praxéologies de recherche permet de clarifier la relation, généralement assez obscure, entre une théorie scientifique et sa méthodologie de recherche.
>
> In the framework of research praxeologies, the technological level permits one to identify, in the 'theoretical block' of (or which contains) the interpretation and justification of the practice, a body of more variable and mobile elements which make up the first level of descriptions of techniques – the 'technology'. The 'theory' IS formed by the more stable principles and the hypotheses which are part of the 'hard core' of the praxeology and which are difficult to evolve without the complete construction being altered. Taking into account the technological level of research praxeologies allows a clarification of the relation, which is generally obscure, between a scientific theory and its research methodology.

This certainly 'makes sense' to our way of thinking, as a technique requires a rationale, a *technology* in the language of Artigue et al. (2011). We look forward to discussing this matter with the authors at CERME 8.

Summing up the considerations above, on what a methodology is, we focus on the concrete practice of researchers. 'Methodology' is a word to describe a part of this practice. The word should not be reified to hold mysterious qualities beyond concrete practices. In the concrete practice of some, but not all, researchers, a methodology 'is' a technique (resonances to Marx's 8th thesis on Feuerbach in this paragraph are intentional).

IMPLICATIONS

Our considerations in this paper impinge on the following themes in the 'call for papers':

- Examples of strategies for connecting theories [methodologies]
- Conditions for a productive dialogue between theorists [methodologies]
- Difficulties and strategies when gathering results from different frameworks
- The role of the empirical material (research data) in the networking and design of theories
- The interaction between contexts and theoretical approaches: the diversity of approaches towards context in different didactic cultures

Specifically this paper raises three issues for the on-going discussion of the CERME 'theory group'. First, it raises the issue of what a methodology is and its relationship with 'theory'; this is useful as methodology has been somewhat of a 'poor relation' in the past considerations of the CERME 'theory group'. Second, it suggests that an artefact, the *Livescribe* pen (together with SRI), can enable the networking of theories/methodologies (and possibly build a stronger theory-methodology in the process). Third, it encourages a reconsideration of the work of past CERME 'theory group' participants, Radford (2008) and Artigue et al. (2011), views on theories and methodologies; we say above "there can be no isomorphic mapping between them" but perhaps we can consider how these two meta-theoretical stances can be networked.

NOTES

[i] Individual researchers are, of course, free to do whatever they want in their research but, from the point of view of this paper, taking a bit of T-AP and bit of T-BI in a single research protocol would result in an approach that was neither T-AP or T-BI in principle.

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NETWORKING THEORIES IN A DESIGN STUDY ON THE DEVELOPMENT OF ALGEBRAIC STRUCTURE SENSE

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In this paper, we discuss how two different theoretical approaches are networked using a coordinating methodology with the aim to describe the development of algebraic structure sense in classroom interaction. From cycles of gathering, connecting, and structure-seeing in the social situation individual structure sense emerges in moments of objectification and subjectification, and can become visible in later instances of structure-seeing.

INTRODUCTION

The networking of theories has shown to be a helpful research frame for linking theories respecting their diversity as richness (Bikner-Ahsbahs & Prediger, 2010; Gellert, Barbé, & Espinoza, 2012). On the one hand it is a way of improving research by benefitting from the strengths of different theoretical approaches, on the other hand it may deliver an epistemological contribution to mathematics education through empirical studies on the networking of theories and through theoretical reflections on the matter and its meta-theoretical and methodological ideas (Arzarello, Bikner-Ahsbahs, & Sabena, 2009; Bikner-Ahsbahs et al., 2010; Kidron, Bikner-Ahsbahs, & Dreyfus, 2011). Radford (2008) postulates the semiosphere as a space for the networking of theories, where theories are distinguished by their identities and boundaries towards other theories. Theories can then be seen "as a way of producing understandings and ways of action based on the triple (P, M, Q)" (Radford, 2008, 320) where P is the set of basic principles, M in this pragmatic approach is a set of methodologies as a way of data collection and interpretation based on P and related to Q, while Q is a set of paradigmatic questions deeply related to P and M. However, methodology in our view also includes the meta-rules connected to P that imply specific ways of data collection, analysis, and interpretation. Concerning this definition, networking takes place between these three components. The degree of integration through networking may vary according to the *networking strategies* from understanding other theories and making understandable the own theory to coordinating or integrating theories locally and synthesizing them (Bikner-Ahsbahs & Prediger, 2010, 495 ff.). In this paper an example of empirical research exploring the strategy of coordinating is presented. It is part of a design study on the development of algebraic structure sense in grade 8. The specific contribution of two coordinated theories is pointed out focusing on the boundary between the theories and on the process of crossing it during the research that leads to a deepened insight into the development of structure sense.

RESEARCH BACKGROUND: ALGEBRAIC STRUCTURE SENSE

The research literature on algebra education has stated a demand for a better support of students' *algebraic structure sense* in many instances—most explicitly Linchevski and Livneh (1999), similarly Arcavi (2005). However, the only systematic attempt thus far to define such a sense has been undertaken by Hoch (2007, also see Hoch & Dreyfus, 2004). In her operational definition of algebraic structure sense for high school algebra (Hoch, 2007, 40) she refers to algebraic structures as a defined set of terms and expressions, which can be worked on using appropriate operations. On this basis, specific forms of algebraic structure sense are ascribed to a student, if he or she can recognize a familiar structure in its simplest form, deal with a compound term as a single entity and through an appropriate substitution recognize a familiar structure in a more complex form, and, most sophisticated, choose appropriate manipulations to make best use of a structure. Based on this definition, Hoch in a first step shows that many students lack the described competencies and then demonstrates that through individual training a sustaining improvement can be accomplished. Hypotheses are presented regarding the factors that have an effect on the individual's acquisition of algebraic structure sense. For everyday classroom instruction Hoch proposes to let the students work on tasks only vaguely described as "adapted to make them suitable for group work", the teacher could then help if necessary (Hoch, 2007, 133). However, both the construction and the implementation of the learning environments may be a very difficult task for many teachers, considered that almost half the students do not show structure sense in the tests, and that many teachers neither do (Hoch & Dreyfus, 2004, 55). Deeper insight into the development of algebraic structure sense is needed. The question thus is how students develop algebraic structure sense in everyday classroom interaction. The shift in attention from states to processes implies a shift in the theoretical understanding of algebraic structure sense: With a static definition referring to a hypothetical final product of algebra instruction no reliable insights will be gained about the processes through which algebraic structure sense develops. This leads to the approach of investigating the development of algebraic structure sense by means of networking two appropriate theories.

THEORIES CONSIDERED

To investigate structure sense as an ever developing attitude rather than as a spotty competence was inspired by the GCSt model that is part of Bikner-Ahsbahs' (2005) theory of interest-dense situations. It assumes seeing structures as the last of three collective epistemic actions performed to answer a mathematical question. Preceded by the *gathering* (G) of smaller units of knowledge and their successive *connecting* (C), *structure-seeing* (St) means the perception of mathematical structures, represented by regularities, rules, or exemplary solutions (Bikner-Ahsbahs, 2005, 202, our translation). When seeing structures, students refer to a unity of relations built on gathered and connected mathematical meanings. Theoretically, students may see new structures or known structures within new mathematical task contexts when producing mathematical meanings through gathering and connecting is saturated. This definition of structures allows for an open-minded view on algebraic structures as they are learned in school, defined rather by the possibilities of the learners than by the curriculum. The background theory for this approach is social constructivism linked with an interpretative view according to which people construct meanings about things through interpreting them in the situation. Thus, the epistemic actions are reconstructed through the interpretation of the students' utterances.

A basis for a deeper understanding of the connection between collective structure-seeing and the development of individual, enduring structure sense can be found in the *activity theory* of Leont'ev (1978). In this theory, the individual's *personality* develops in *activity* under the given cultural conditions. Activity in this theory does not mean some undirected motion but is characterized by its inherent *motive,* e.g. to solve an equation. On lower levels one can find *actions* directed to more specific *goals* and *operations* which are determined by the given *conditions*. This set of ideas has been made fruitful for mathematics education by Roth and Radford (2011; regarding algebraic structures see Radford, 2010). Their dialectic account describes mathematical meaning-making as *objectification*, i.e. the disclosure of an activity motive to the learner, and *subjectification*, i.e. the development of personality as the reordering of the subject's structure of motives in the very same process. Regarding methodology, an activity-theoretical account calls for a deep analysis of the societal, social, and individual processes. Roth and Radford concretize this by arguing for a semiotic approach.

Regarding the research question, the GCSt model may offer a clarification of the situational and social prerequisites of objectification/subjectification, where structure sense develops as a stable feature of the individual student's personality. A more detailed empirically based description of this interplay of the theories is the subject of this article. This first requires a consideration of methodology and method.

METHODOLOGY AND METHOD OF THE EMPIRICAL STUDY

The intended development of both theory and practice is precisely the kind of setting that *design research* has been developed and proven useful for over the last two decades (for an overview see van den Akker, Gravemeijer, McKenney, & Nieveen 2006). In a broad definition, the intent of this approach "is to investigate the possibilities for educational improvement by bringing about new forms of learning in order to study them" (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, 10). This design study took place between December 2011 and July 2012 in a grade 8 class of a comprehensive school in a socioeconomically deprived area of Bremen, Germany. Three factors led to this choice: First and most importantly, the teacher had to be interested in the reflection and improvement of her teaching. Second, a certain degree of heterogeneity was appreciated, for students' problems with algebraic structures are very likely to differ with school performance, however intense and directed the causality may be. The third argument was about the grade to investigate. Due to the given curriculum (Die Senatorin für Bildung und Wissenschaft, 2010) we could hope for a variety of algebraic structures that students can acquire a sense for in grade 8.

In this paper, learning processes regarding linear equations and linear functions are considered. The teacher was committed to patiently leave room for the learning process to take place, as this is regarded helpful not only in the theory of interest-dense situations, but also in Arcavi's (2005) considerations regarding symbol sense. In both cases the students were encouraged to make on-hand experiences in the initial phase of the unit. For the introduction to linear equations a variant of the matchbox algebra as presented in some German-language textbooks (first of all Affolter et al., 2003) was implemented. In this approach equations are materialized by matches (representing integers) and matchboxes (each holding the number of matches that is the solution of the equation, see fig. 1a). Already at this early point a focus was laid on the equations' structures. The idea was that there is a natural impetus to structure a pile of unsorted matches and boxes. Linear functions were introduced referring to the changing water level in a straight container steadily filled or emptied. The anchor representation of steady change with an initial water level was consciously chosen based on analyses of the experiences previously made in the unit on linear equations.

Qualitative analyses were conducted focusing on the cases where structure-seeing takes place. The goal was to clarify the connection between structure-seeing and objectification/subjectification. Assuming with Roth and Radford (2011, 25) that "subjects of activity … exhibit to each other whatever is required to pull off an event as that which it is", the analyses built on a comprehensive record of the classroom interaction and the expressed thinking of the students. It includes photographs as well as the written texts of the students, and, as the anchor of analysis, video footage. One camera provided an overview of the class, while two selected pairs of students were filmed by one video camera each to document their long-term learning process in depth. Each group included a boy and a girl, and a high and a low achiever according to the levels defined by the local board of education. Furthermore, the students were selected to represent a range in language proficiency as valuated by the teacher.

Networking strategies: an approach for linking two theories

The initial networking strategy was *combining*, i.e. juxtaposing the theories in order to better understand a specific data set from each perspective. This implies clarifying what structure-seeing means in the current epistemic process from the view of the GCSt model on the one hand, and on the other hand how the motive that constitutes objectification can be observed. In order to answer our research question the following *coordinating* analysis method is conducted linking the two theories' methodologies: a situation of structure-seeing is identified and the corresponding episode is transcribed. Then the epistemic process of this episode is analyzed by the use of the GCSt model. This way, the specific structure seen by the students is captured. Based on this, hypotheses about the motive of objectification and its relation to the development of structure sense are expressed and then validated by the successive data (methodical link). Before performing a developmental cycle of analysis, the a priori-status of the design theory will be described. It serves as a hypothetical model that illustrates the coordination of the two theories and will be the basis to further the design theory through data analysis.

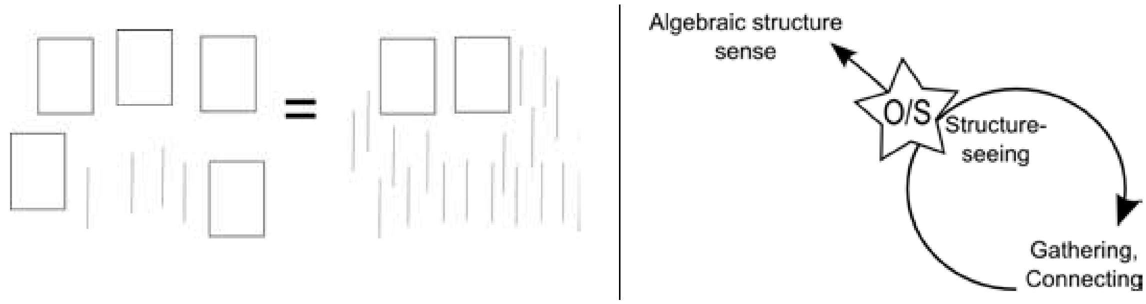


Fig. 1a (left): The matchbox equation the students worked on in the scene described below; Fig. 1b (right): The moment of objectification (O) as a special case of structure-seeing where algebraic structure sense develops through subjectification (S).

A HYPOTHETICAL MODEL AS A CASE OF COORDINATING

Coordinating theories is performed "when a conceptual framework is built by well fitting elements from different theories" (Bikner-Ahsbahs & Prediger, 2010, 491). This is especially used in the building of conceptual frameworks when complementary insights are approached. Hence, the networking strategy of coordinating is appropriate in the networking case presented in this paper.

In activity theory, one can assume that through processes of objectification the students individually change their view of the algebraic world. Subjectification, being the development of the student's personality in this very process, can thus describe the development of algebraic structure sense. Algebraic structure sense in these terms is a developing attitude towards specific algebraic structures, for example linear equations or linear functions. However, the theory of objectification remains unspecific about fostering conditions for objectification (and thus the development of algebraic structure sense). From the theoretical perspective of the GCSt model, objectification can be seen as an individual process emerging from instances of collective structure-seeing. This given, cycles of gathering, connecting, and structure-seeing are the soil for the development of algebraic structure sense. These processes may go on for some time until the specific moment of objectification. This moment constitutes a substantial change in quality: the students not only see the entity, they are also able to use it driven by the disclosure of the motive. The development of personality also passes through this change in that the structure of motives is reorganized towards the inclusion of the new motive that the students become aware of.

The connection between the models at the point of objectification deserves deeper consideration. Two states concerning a specific activity can be distinguished: in performing cycles of gathering, connecting, and structure-seeing, students look at the task with a tendency to perceive all kinds of structures in the situation. These are not necessarily mathematical and correct. After the point of objectification the students' attitudes have changed, they are able to use the specific structure within the activity and work it out. Before objectification the GCSt model is the foreground model to investigate epistemic processes that can lead to objectification. After the instance of objectification the GCSt model has taken the back seat and the dialectic of objectification and subjectification comes to the fore providing the frame to reconstruct the development of algebraic structure sense as a general attitude of the subject. The two models are complementary in the sense that they enable us to describe different aspects of one overarching process, with the moment of objectification being the tipping point. With this model (fig. 1b) in mind the development of structure sense before and after the instance of objectification can now be investigated through a coordinating analysis.

COORDINATING ANALYSIS AT THE MOMENT OF OBJECTIFICATION

The first scene presented here shows how the model presented above helps understand the crucial moment of objectification. In this case, it is supported by a metaphor that the teacher provides. Previously, the students have tried to keep "it equal, but more simple". However, the "it" is not clear. As part of the riddle only the mutual equality of the two sides must be provided, but the students additionally try to keep each side materially equal over time. In a first step they relax this rule by exchanging matches and boxes between the two sides, but notice that this does not bring any improvement of the situation. This material equality concept keeps the students stuck, and first the teacher is so with them, but then she gives a decisive hint:

1 /Teacher: okay so what can one do now on both sides (*taps the table on both sides*) so that it gets clearer you always have to do the same. (*H draws some dirt off the table, T looks at S, S looks at the table, still having three boxes in her hand*) (…) (*gets up and walks away, whispering in Sabine's ear*) ,take away something.

Transcription key:

w-e-l-l	speaking slowly
exact.	dropping the voice
exact'	raising the voice
exact-	voice kept in suspense
,exact	with a new onset
EXACT	with a loud voice
(.),(..),...	1, 2, ... sec pause
(7sec)	7 sec pause
/Karl:	interrupts the previous speaker
SX:	unidentified student

2 Sabine: (*looks at H, leaves one of the three boxes on the table and takes the two closer to herself*) Take away something

3 Herbert: A-h- ,man are you smart. (*S laughs, puts the boxes on the table again, slightly left of the others*) ,Mrs. Kahn is totally mean. ,take away something okay-

4 Sabine: (*looks at H, points at the right side*) so if we nine are five ,ah just take away four’ (*both students each take away four matches from the right side, H to the bottom right, S towards the equal sign*)

5 Herbert: One two three four I have.

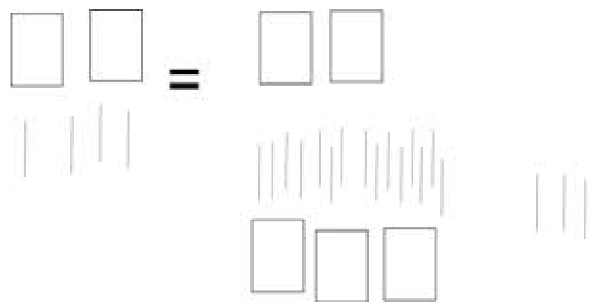
6 /Sabine: I have taken away here-

7 /Herbert: a-h-r. (*puts his matches back, S laughs*) ,are fifteen.

8 Sabine: Yes. ,and fifteen- (*points at the right side, looks at the left side*) (...) (*points at the five matchboxes on the left side one by one*) ,one two three four five- (*takes three of the boxes and puts them further to the middle*) ,divided by three- ,are five.

9 Herbert: Yes. (*looks at S, smiles*)

10 Sabine: And now we have (*puts the three boxes beneath the matches (see figure on the right), first points at the matches and then at everything around the equal sign*) this here and the three (*moves backwards with her chair*) we (*claps her hands*) HAVE IT. (*H looks at the table in front of him, S leans forward again*) ,look. ,because now you (plural) have (*first points at the 15 matches, then at the three boxes*) ,if these were in here (*looks at H*) ,right' (*points at the four matches put aside*) we still have four outside.



11 Herbert: Eheh'

12 Sabine: (*first points at the 15 matches, then at the three boxes*) And those are in there right. (*H looks at S, first grinning broadly, then with his mouth closed, then grinning again*) (4sec) ,do you think that works' (*puts the three boxes on the left side again*) (...)

13 Herbert: Well once more. (*puts his right hand on the right side*) ,there are fifteen.

14 /Sabine: (*takes the three boxes on the left side in her hand and shakes them while talking*) yes and her are also fifteen then because fifteen divided by three are five

15 /Herbert: (*joins in, holds up his right hand with all fingers up*) divided by three are five.

Sabine says "Yes" and calls the teacher (who is busy with other students). The students look at each other. "whoo" they shout, raise their arms and Sabine says: "Herbert we are good." Then the two students turn to a neighboring table and Sabine starts explaining her solution. The metaphor „take away" (1-4) enables the students to overcome the material equality concept and to rivet on solving the equation: having permission to "take away" they can ignore the parts materially identical on both sides (four matches and two boxes) (8-10). The remaining three boxes and 15 matches are connected by bringing them spatially together (10). The students see the divisibility by three and repeatedly check their result (10-14). Hence, the possibility of taking away helps seeing structures and experiencing objectification. The emotionality of the scene hints at both the leap the students do regarding the structure of linear equations and at the change in their personality regarding this specific situation.

In the second scene presented here we can see how the motive from the objectification described above is *conserved* in *key situations* expressed by the metonym "matches". It takes place five months after the first scene and more than three months after the last time the class worked on equations. The (higher track) students are asked to identify the moment in which the water level in two bathtubs is the same, when one of them starts at a higher level but drains at a higher speed. In the group the students and the teacher have argued that to find out the unknown, for which the terms of the functions deliver the same value, the terms have to be equal. This results in an equation on the blackboard, which is not named as such. Now the teacher asks:

/Teacher:	okay so what is that here now. (*briefly points at the blackboard, Rico puts his hand up, T briefly points at him*)
Rico:	(incomprehensible)
Teacher:	That are two terms yeah' (*many students talk, incomprehensibly*)
Sabine:	(*wags her right index in the air*) ISN'T THAT THE THING WITH THE cigarette things ,matches- (*T as well as Herbert and Ahmed look at her, the rest of the group looks at the blackboard*)
/Teacher:	p-s-t- ,o-h-
SX1:	equation'
Teacher:	(*looks at SX1, still points at the equation with her right hand*) An equation. exactly.
SX2:	A-h-

This is a case of structure-seeing in a new context. After a long period where equations did not play a role in the classroom, the word "matches" activates all the procedures that were developed based on the matchbox environment. Especially the repeated use of the metaphor "take away" in the following collective solution of the equation indicates the importance of the situation in which the earlier objectification took place (cf. Lakoff & Johnson, 1996). That means that in objectification knowledge can be conserved and made long-term-accessible in key situations. Structure-seeing activates this knowledge, not only for an individual, but available for the whole group by a metonym substituting the whole situation. This coordinating analysis expands the hypothetical model by the aspect of *conserving knowledge* and makes clear how previous processes of objectification/subjectification can again become part of remote discussions in the classroom.

REFLECTIONS ABOUT THE NETWORKING PROCESS

In this contribution two theories were networked by a coordinating methodology while investigating the impact of structure-seeing for the development of structure sense through objectification/subjectification. Objectification was characterized as emerging from instances of structure-seeing, thus forming a concept at the boundary of both theories. This leads to a temporal divide in the data: With the GCSt model the preparation of this moment can be investigated. Activity theory according to Radford and Roth provides the tools which capture how structure-seeing impinges the development of individual algebraic structure sense. Objectification seems to initiate a leap of quality: disclosing a motive it immediately enlarges the potential to act. The simultaneous emotional and valuing expressions indicate subjectification. Objectification seems to be experienced so deeply that it is conserved in the situation and can be activated in further situations, which are also prepared by gathering and connecting. Using the described coordinating analysis, objectification will be worked out more comprehensibly by further analyses leading towards local integration.

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NETWORKING THEORIES BY ITERATIVE UNPACKING

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An iterative unpacking strategy consists of sequencing empirically-based theoretical developments so that at each step of theorizing one theory serves as an overarching conceptual framework, in which another theory, either existing or emerging, is embedded in order to elaborate on the chosen element(s) of the overarching theory. The strategy is presented in this paper by means of reflections on how it was used in several empirical studies and by means of a non-example. The article concludes with a discussion of affordances and limitations of the strategy.

INTRODUCTION

A long-term program of our research group focuses on identifying and characterizing learning that possibly occurs when individuals or groups of individuals are engaged in problem solving and problem posing. In several of our studies, a typical data set consists of a series of recorded observations of how a learner or a small group of learners approaches an insight problem or tries to openly generalize a known theorem or pose a mathematical problem, which would be interesting to solve also to the posers. Many events, interactions and developments occur in such situations, so we, as many other researcher groups, repeatedly face the following research choices:

- Which of the observed developments are worthwhile enough in order to call them "learning"?
- The learners' successes and failures with problem-solving and problem-posing tasks are functions of many conditions and variables, some of which are out of our reach. So, which variables and conditions should become a focus of our attention, if we wish not only to describe the learners' actions and "learning," but also to explain what stipulated them?

These questions presume different theoretically-laden answers in different circumstances. Though individual studies in our group heavily rely on selected theories, their elements or their combinations, we are not adhered, as a group, to a particular theory or conceptual framework[1]. Thus, the need to develop certain strategies of calling into play several theories in one study as well as in a sequence of studies emerged for us. The goal of this paper is to introduce one of such strategies, which I would like to term *networking by iterative unpacking*.

ITERATIVE UNPACKING STRATEGY: AN INTRODUCTORY EXAMPLE

Let me introduce an iterative unpacking strategy by an example taken from a recent article by Simon et al. (2010). Simon and his colleagues presented a novel approach to studying learning through mathematical activities. The approach was developed for capturing subtle processes of transition the learners come through when progressing from one conceptual step of knowledge construction to a subsequent one. The scholars contrasted their approach with an approach developed by Thompson (1994) and Steffe (2003). That approach includes identifying sequences of developmental steps in students' mathematical actions and analysing them, in particular, in Piagetian terms of perturbation, accommodation and reflective abstraction. To further situate their approach in the literature, Simon et al. asserted:

> Our work also builds on the work of Hershkowitz, Schwartz, and Dreyfus (2001) who took on the challenge of explicating the formation of abstractions. Toward this end they described three 'epistemic actions' in the process of abstraction: 'construction, recognition, and building with.' They emphasize that construction is the key part of the process. *Our work can be seen as attempting to unpack construction* [italics is added] (p. 80)

Notably, Simon et al. refer to the Hershkowitz, Schwartz and Dreyfus's (2001) work in a dual way. On one hand, they refer to it as a theory, which unpacks a particular element of a previously developed theory. To this end and consistently with the Bikner-Ahsbahs and Prediger's (2006) terminology, the former theory can be seen as a foreground one, and the latter—as a background one. On the other hand, they use the Hershkowitz et al.'s (2001) work as a background theory or as an overarching framework, in which their own foreground theory is embedded. Simon et al. specifically point out which theoretical construct of the overarching framework they are going to unpack. They then perform the unpacking by developing an elaborated kit of tools for analysing the process of construction/transition in terms of seeing commonalities between the tasks in a sequence of tasks and anticipating solutions.

My point is that Simon et al. utilized two networking strategies, one of which had been explicated in the literature (e.g., Bikner-Ahsbahs & Prediger, 2006; Prediger, Bikner-Ahsbahs & Arzarello, 2008) and another was not. First, they utilized a *comparison strategy* by pointing out the differences between their approach and the Piagetian-oriented approach developed by Steffe and Thompson. The main difference was that Simon et al. suggested exploring learning without necessarily focusing on perturbation. Second, though Simon et al. did not use the “construction, recognition, and building with” language in their analysis, they considered it as an overarching conceptual framework for their own theorizing. To this end their contribution is in unpacking one of the key constructs of the Hershkovitz et al.'s (2001) theory. Therefore—*unpacking strategy*. Furthermore, the whole process of theorizing presented in the Simon et al.'s (2010) article includes the following chain of iterations: (1) from focusing on perturbation, accommodation and reflective abstraction to unpacking the formation of abstraction in terms of construction, recognition, and building with, and then (2) to unpacking construction in terms of seeing commonalities between the tasks and anticipating the solutions. Therefore—*iterative unpacking strategy*.

In sum, *iterative unpacking strategy* consists of sequencing empirically-based theoretical developments so that at each step of theorizing one theory serves as an overarching conceptual framework, in which another theory, either existing or emerging, is embedded in order to elaborate on the chosen elements(s) of the overarching theory. Note that different ways to embed an additional theory into an overarching theory may exist. For example, the way of unpacking "construction" offered by Kidron, Bikner-Ahsbahs and Dreyfus (2010) is very different from the way offered by Simon et al. (2010). It is also of note that, though an overarching theory and an embedded theory are, in a way, complementary, iterative unpacking does not necessarily imply that all the constructs of the overarching theory are to be preserved. In other words, the use of iterative unpacking strategy may influence also an overarching theoretical framework, as follows: unpacking a particular aspect of a theory may shed new light on the role of the rest of its aspects. For instance, the role of perturbation—one of the main concepts of the highest-level overarching theory in the above example—was reconsidered in the second iteration.

ADDITIONAL EXAMPLES

In this section an iterative unpacking strategy is illustrated by reflective accounts of two sequences of studies, in which I am involved during the last years. The first example concerns problem solving, and the second one—problem posing.

Example 1: iterative unpacking of problem solving

Problem solving as a research field is being attracting keen attention of the mathematics education community for more than 50 years. Foreground models of problem-solving are originated in seminal work of Polya (1945/1973) and developed in the eighties (e.g., Schoenfeld, 1985). Generally speaking, the models attempt to elaborate on how learners solve mathematical problems in terms of phases and attributes. For instance, a model offered by Carlson and Bloom (2005) postulates four problem-solving phases: orientation, planning, executing and checking, and operates with five problem-solving attributes: conceptual knowledge, heuristic knowledge, metacognition, control and affect.

First iteration

A heuristic knowledge component was chosen to be unpacked in the study reported in Koichu, Berman and Moore (2007). The study included a five-month teaching experiment in two Israeli 8th grade classes. The aim of the experiment was to test a particular approach to developing *heuristic literacy* in students. By *heuristic literacy* we meant an individual’s capacity to use the shared names of heuristic strategies in classroom discourse and to approach (not necessarily to solve!) mathematical problems by using a variety of heuristics. Changes in students' heuristic literacy were explored in three rounds of thinking-aloud interviews conducted at the beginning, in the middle and at the end of the experiment. The interviews were based on so-called *seemingly familiar problems*, that is, problems that looked similar to the problems previously offered in the students' mathematics classrooms, whose solutions, however, were essentially different. The following problem is an example:

> The sum of the digits of a two-digit number is 14. If you add 46 to this number the product of digits of the new number will be 6. Find the two-digit number.

Indeed, at first glance the problem has a solution by means of a system of equations, as many similarly looking problems approached by the students in the classroom have had. However, composing a system of equations appears ineffective at a second glance. Such problems were used as opportunities to elicit as many heuristics as possible from the students' thinking-aloud speech without discouraging the students from the beginning by facing unfamiliar problems. The interview protocols were segmented into content units and coded in terms of 10 pre-defined heuristic processes:

> (1) Planning, including (1a) Thinking forward, (1b) Thinking from the end to the beginning and (1c) Arguing by contradiction. (2) Self-evaluation, including (2a) Local self-evaluation and (2b) Thinking backward. (3) Activating a previous experience, including (3a) Recalling related problems and (3b) Recalling related theorems. (4) Selecting problem representation, including (4a) Denoting and labelling and (4b) Drawing a picture. (5) Exploring particular cases, including (5a) Examining extreme or boundary values and (5b) Partial induction. (6) Introducing an auxiliary element. (7) Exploring a particular datum. (8) Finding what is easy to find. (9) Exploration of symmetry. (10) Generalization.

Success or failure in solving the interview problem was obviously dependant on circumstances under which the problems were dealt with as well as the whole bunch of problem-solving attributes. Consequently, the rates of success were considered irrelevant to unpacking a heuristic component of problem solving. Instead, we introduced a notion of relative *heuristic richness* of solutions. We used the following comparison criterion: One solution was called *heuristically richer* than another if the number of different heuristic processes indicated in the first solution was greater by three or more than in the second solution. Given that 10 different heuristics were considered in our study, we considered the criterion "…three or more" very demanding, and, in turn, sufficiently robust. This criterion was applied to each student individually, for comparison of her or his solutions by pairs of corresponding problems given in the first, second and third interviews. We then developed an integrative measure of individual heuristic literacy development based on the number of the pairs of the corresponding problems, in which a solution to the second problem was heuristically richer than a solution to the first one.

The measure helped us to adequately account for some of the learning effects of the teaching experiment. One of the central findings was that positive changes in heuristic literacy occurred in most of the students, yet they were unequally distributed among the students, who were defined as "stronger" and "weaker" with respect to their achievements in SAT-M (Scholastic Aptitude Test - Mathematics) administered at the beginning of the experiment. In particular, those students, who were "weaker" at the beginning of the experiment, demonstrated more significant heuristic literacy development than their "stronger" peers. We explained this result by suggesting that the heuristic content of the teaching experiment was more novel and useful for the "weaker" students, whereas the "stronger" students might have possessed the strategies prior to the experiment. The novelty of this result was in the exposure of the role and the learnability of heuristic component of mathematical problem solving, which was identified in (relative) isolation from the rest of problem-solving components. In addition, this result enabled us to formulate some pedagogical implications.

Towards the second iteration

To recap, the study quoted above attempted to unpack the heuristic component of problem solving in terms of selected heuristic processes. Heuristic literacy was chosen as an object of learning. However, though the developed measure of heuristic literacy worked well for describing some of the learning effects of the experiment, it was too simplistic in order to capture how particular heuristics are called into play. This was particularly evident when we looked at the students' solutions containing comparable numbers of heuristics, which however differed in some other respects, such as the appearance of repetitions and cycles in the students' reasoning and the nature of their decisions when facing dead ends. Thus, we feel the need for further unpacking. Specifically, we are interested in unpacking a "heuristic richness" notion. To achieve this goal, we now experiment with three interrelated ideas.

Ovadia (Ph.D. in progress) studies how particular heuristics come into play depending on how the students perceive similarities and differences between problems that were discussed in a classroom and new ones. In particular, her study focuses on the ways by which the students learn to make connections between known and new mathematical problems at the level of common heuristics needed for solving these problems. To this end her study can also be seen as an attempt to unpack the process of seeing commonalities between the tasks pointed out by Simon et al. (2010) as one of the process underlying the process of construction.

Another study (Koichu, 2010) was conducted in order to better understand why those students who possess all necessary strategic and conceptual knowledge for solving given problems sometimes miss within-the-reach solutions. In this work, I consider alternative explanations of this well-documented phenomenon in terms of three theories developed within mathematics education research, point out the limitations of these explanations with respect to a particular data set and offer an explanation in terms of the Principle of Intellectual Parsimony. The Principle states that when solving a problem, one intends not to make more intellectual effort than the minimum needed. In other words, one makes more effort only when forced to do so by the evidence that the problem cannot be solved with less effort. The explanation is built on that *efforts* in problem solving can be of a different nature.

Finally, my interest in heuristic component of mathematical problem solving and Leron's interest in proving and applying cognitive science theories to mathematics education have fruitfully intersected in our common work "Proving as problem solving: the role of cognitive decoupling" (Koichu & Leron, in prep.). In this work, we re-analyse two thinking-aloud protocols from Koichu et al. (2007) study in terms of a conceptual framework representing the increasing importance attached to working memory capacity by researchers in cognitive psychology working on problem solving and decision making. The key notion of the framework is that of cognitive decoupling, i.e., human ability to form more than one mental model of the problem situation and attempt to hold them at the same time in working memory, all the time resisting the tendency for the models to be mixed and confused. Unpacking the heuristic component of problem solving in terms of cognitive decoupling seems us instrumental for better understanding the appearance of cycles in repeated problem-solving attempts and for the use of multiple representations.

Example 2: iterative unpacking of problem posing

Koichu and Kontorovich (2012) conducted a study, in which a group of pre-service mathematics teachers was asked to pose interesting mathematical problems based on a particularly rich problem-posing situation, the Billiard Task. Our goal was to identify those traits of problem-posing processes, which are involved in the posers' attempts to formulate *interesting* problems.

A coherent conceptual framework which would be sensitive to the subtleness of the problem-posing processes and simultaneously applicable to a broad range of problem-posing tasks is not yet established. However, the literature offers several conceptualizations of problem posing, which could be utilized in our study. We decided to adopt in our study a definition of problem posing by Stoyanova and Ellerton (1996) as an overarching conceptual framework. The definition states that problem posing is "the process by which, on the basis of mathematical experience, students construct personal interpretations of concrete situations and formulate them as meaningful mathematical problems" (p. 518). This definition required a great deal of unpacking, and, as will be evident shortly, we saw this fact as an opportunity rather than as a limitation. I present here our attempts to unpack two key components of the definition: the process of constructing personal interpretations of a given situation and the notion of a mathematically meaningful problem.

Iterative unpacking of the process of problem posing

Problem posing is a natural companion of problem solving, so we decided to unpack it in terms of attributes and stages, as it had been fruitfully done regarding problem solving. In line with our earlier research (Kontorovich & Koichu, 2009; Kontorovich et al., 2012), we focused on *mathematical knowledge base*, *problem-posing strategies* and *individual considerations of aptness*.

Mathematical knowledge base for posing problems includes the knowledge of mathematical definitions, facts, routine problem-solving procedures and relevant competencies of mathematical discourse and writing. In addition, it requires knowledge of mathematical problems that can serve as prototypes. Recalling and using a system of prototypical problems relies, in turn, on three components of the problem-solving ability, which were pointed out by English (1998): the ability to recognize the underlying structure of a problem and to detect corresponding structures in related problems, the ability to perceive mathematical situations in different ways and the ability to favour some problems over others in routine and non-routine situations. Notably, this conceptualization of mathematical knowledge base for problem posing can be seen as a result of iterative unpacking by itself.

Problem-posing strategies that emerged in our data (but had also been pointed out in prior studies) included constraint manipulation, symmetry, chaining, data-driven reasoning and hypothesis-driven reasoning. A part of our analysis was directed to unpacking the chaining strategy. *Considerations of aptness* are conceptualized as the poser's comprehensions of explicit and implicit requirements of a situation within which a problem is to be posed; they also reflect her or his assumptions about the relative importance of these requirements. Three types of considerations of aptness showed up in our data: aptness to the posers, aptness to the potential evaluators and aptness to the potential solvers of a posed problem. In the framework of Koichu and Kontorovich's study, considerations of aptness were indicated but not further unpacked. Their unpacking is one of the goals of a Ph.D. research of Kontorovich. Furthermore, one of the findings of the Koichu and Kontorovich's study was an identification and characterization of four problem-posing stages: warming-up, searching for an interesting mathematical phenomenon, hiding the problem-posing process in the problem formulation, and reviewing. Further refining and unpacking of these stages is another goal of a Ph.D. research of Kontorovich.

Interpreting a "mathematically meaningful problem" notion (a non-example)

For some time, we looked for a way of interpreting a "mathematically meaningful problem" notion among the existing ways of unpacking the closely related notions, such as "beautiful problem" and "interesting problem". However, the resulting interpretation has not been done by an iterative unpacking strategy. Thus, the chain of theoretical considerations presented below can be seen as a non-example of iterative unpacking strategy.

On one hand, the literature on aesthetic aspects of mathematics informed us that an agreement about what constitutes a beautiful problem is elusive, but offers quite stable lists of general characteristics of such problems and their solutions, such as clarity, mathematical deepness and complexity, cleverness, novelty and surprise. Second, we accepted the Crespo and Sinclair's (2008) argument that problems' descriptors such as "meaningful" belong to the rarefied discourse of mathematicians rather than that of learners. Crespo and Sinclair (2008) suggested that the learners' normative understanding of what qualifies as a worthwhile problem may develop around the notions of "mathematically interesting" or "tasty". Third, we decided to build on the Goldin's (2002) idea that general characteristics of problems, such as "meaningful," "beautiful" or "interesting," should be seen as instantiations of one's internal multiply-encoded cognitive/affective configurations, to which the holder attributes some kind of truth value, and not as "objective" qualifiers of the problems.

Consequently, we decided to treat in the study the descriptor "meaningful" in the manner that have been developed in past research for treating the descriptors “interesting” or “beautiful”. Namely, we operationally considered a posed problem mathematically meaningful (or interesting or beautiful) if it was evaluated as such by the poser of the problem, its readers or solvers. This decision suited our research needs, but could not be seen as unpacking, in the meaning specified in the rest of the examples. We rather bypassed delving in the cognitive and affective mechanisms underlying one's use of the descriptors and just explicated how the posed problems were operationally qualified in our data.

AFFORDANCES AND LIMITATIONS

In terms of Kuhn (1962/2012), the growing interest of the mathematics education community in networking theories might suggest that mathematics education as a research field is in transition from *pre-paradigm phase* to *normal science phase*. An iterative unpacking strategy discussed in this article is reminiscent of the accumulation-by-development strategy considered by Kuhn as the main developmental force of science during normal science periods. Indeed, the "further elaborating on…" discourse is typical for the periods of normal science, but not for the periods of paradigmatic shifts and scientific revolutions.

The presented strategy can also be seen as a particular case of the strategies of coordinating and combining (Prediger et al., 2008), the case that emphasizes accumulation of knowledge on local phenomena by establishing a specific connections between background and foreground theories. The specificity of the strategy is, in particular, in the dynamic relationship between the theories: one theory may serve as an overarching framework in one case, and as a source of conceptual tools for elaborating on elements of another theory in another case. These observations provide a background for pointing out some of the affordances and limitations of the strategy.

From a practicing researcher perspective, an iterative unpacking strategy can be instrumental for:

- situating a study in the literature and highlighting its theoretical contribution;
- wording research questions in terms of a particular conceptual framework without suppressing the possibility to further use additional conceptual frameworks in a coherent manner;

- justifying the chosen level of granularity in data analysis.

More generally speaking—here I follow the argument presented in Prediger et al. (2008)—an iterative unpacking strategy can be helpful for "better collective capitalization of research results, [adding] more coherence at the global level of the field,…, gaining a more applicable network of theories to improve teaching and learning and finally guiding design research" (p. 170).

As mentioned, one limitation of the iterative unpacking strategy is that it stops being important outside of the normal science periods (cf. Kuhn, 1962/2012). At the pre-paradigm periods, the strategies of ignoring, comparison or contrasting are typically in use. At revolutionary science periods, further elaborating on the elements of previously developed theories falls out of the mainstream. The use of the iterative unpacking strategy is limited also within the normal science periods. Briefly, all the conditions for the use of the strategies of coordinating and combining discussed in Prediger et al. (2008) apply.

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Endnote 1: The notions "theory" and "conceptual framework" are used interchangeably in this article.

CONNECTING THEORIES IN A CASE STUDY OF PRIMARY SCHOOL MATHEMATICS TEACHERS' PROFESSIONAL IDENTITY DEVELOPMENT

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This paper focuses on a connection between two theories used in a case study of the professional identity development of primary school mathematics teachers. The two theories connected are communities of practice and patterns of participation. The reason for the connection was the need for a framework that would make it possible to analyse both the individual and the social parts of professional identity development. In this paper, the connection is presented, illustrated briefly using empirical examples and evaluated.

Keywords: Identity, identity development, framework, networking strategies

INTRODUCTION

The focus of this paper is a framework used in a case study of novice primary school mathematics teachers' professional identity development. The teaching profession, with or without focus on mathematics teaching, is often described in terms of a changed profession which lacks continuity between teacher education and schools (Cuddapah & Clayton, 2011; Frykholm, 1999; Goodman, 1998). Graduating from teacher education and starting work as a teacher is described as a transfer or shift in professional identity where the interplay between the individual and the context is highlighted as a central part about which understanding should be developed (Bjerneby Häll, 2006; Cuddapah & Clayton 2011; Persson, 2009). The reason for the connection between theories in the framework used in the present study was the need for a framework making it possible to analyse this interplay between the individual and the context.

Eistenhart (1991) distinguish between three types of research frameworks (theoretical, practical and conceptual) whereof the here presented framework is a conceptual framework. Such a framework is built from different sources, e.g. previous research and literature that the researchers argue as being relevant and important when addressing the research problem.

EPISTEMOLOGICAL STANCES

In the study of novice primary school mathematics teachers' professional identity development, two notions in particular are focused on: identity and professional competence. Both of these notions are objectifications which, according to Sfard (2008), imply that, instead of talking about them as processes expressed as verbs, we talk about them as nouns as if they were physical objects. Objectification is not only a way of talking about the "same thing" it is also what creates the "things" we talk about and it is accomplished in two steps. The first step is reification which refers to how talk about processes and actions is converted into talk about objects. We observe actions but talk about them as objects (*he has got a lot of knowledge*). The second step in objectification is alienation where the objects are presented in an impersonal way as if they existed independently without the presence of specific people (*the level of knowledge in society is too low*). As objectified notions are originally based on actions they need to be de-objectified for it to be possible to study them. In the development of the framework in the present study, such a de-objectification of the objectified notions *identity* and *professional competence* needed to be done.

According to Lerman (2000), research into mathematics education has "been turn[ed] to social theories" (p. 20). He bases this on mathematics education research since the late 20th century, sees meaning, thinking and reasoning as products of social activities where learning, thinking and reasoning are seen as situated in social situations. The term situated refers to a set of theoretical perspectives and lines of research which conceptualise learning as changes in participation in socially organised activities and individuals' use of knowledge as an aspect of their participation in social practices (Borko, 2004). Whereas cognitive perspectives focus on knowledge that individuals acquire, situative perspectives focus on practices in which individuals have learned to participate (Peressini, Borko, Romagnano, Knuth and Willis, 2004). To participate means both to absorb and be absorbed in a community and Sfard (2006) describes this duality as "*individualization of the collective"* and "*collectivization of the individual"* (p. 158, italics in the original).

From a situative perspective, teacher learning is a process of increasing participation in the practice of teaching and, through that participation, becoming knowledgeable in and about teaching. To understand teacher learning, it must be studied within the multiple contexts within which teachers do their jobs, taking into account both the individual teachers and the social systems in which they are participants (Borko, 2004).

THE DEVELOPMENT OF THE CONCEPTUAL FRAMEWORK

A connection between theories depends on the structure of the theories involved and the goal of the connection (Radford, 2008). As mentioned, the goal of the connection presented in this paper was to better understand an empirical phenomenon by including both the individual and the social parts of identity development. Also, based on the epistemological stances, the theories was to be within a situative perspective with identity and professional competence as processes and not objectified objects.

The starting point in the search for a theory suitable to describe and understand the professional identity development of novice primary school mathematics teachers became theories focusing on identity and identity development. After exploring several theories, Wenger´s (1998) theory regarding communities of practice was found to be suitable and in line with the epistemological stances. However, his theory doesn't focus on mathematics education and/or teaching. There is no one definition of what constitutes professional competence for mathematics teachers but in research beliefs and/or mathematical knowledge for teaching is often focused on. However, traditional definitions of beliefs and mathematical knowledge for teaching are objectifications and the *situative* participatory perspective to be used in the here presented study was not in line with the acquisitionist perspective often used in research of beliefs and mathematical knowledge for teaching.

According to Skott (2010), the social turn that has been developed in research regarding other areas within mathematics education also needs to be developed within beliefs research. If beliefs research is to become social, the pre-reified and the pre-alienated processes of teachers' participation need to be focused on. Skott suggests a shift from focusing on objectified beliefs to focusing on patterns-of-participation as searching for patterns in how teachers participate in immediate situations and prior social practises. Skott's theory is in line with the epistemological stances of the study and focuses on those parts that were missing (mathematics education and/or teaching) in Wenger's theory and became the second starting point of the conceptual framework. Below first Wenger's theory will be presented followed by Skott's theory and after that the connection will be focused on.

Identity and identity development

According to Wenger (1998), identity formation is a complementary dual process in which one half is identification in communities of practice and the other half is the negotiation of the meaning in those communities of practice. A community of practice is a set of relationships between people, activities and the world; it is a shared learning history. The system of activity in a community of practice involves mutual engagement, joint enterprise and a shared repertoire. Mutual engagement is the relationships between the members, about them doing things together as well as negotiating the meaning within the community of practice. Joint enterprise regards the mutual accountability the members feel in relation to the community of practice and it is built by the mutual engagement. The shared repertoire in a community of practice regards its collective stories, artefacts, notions and actions as reifications of the mutual engagement. The shared repertoire proceeds from, and is a resource in, the negotiation of meaning within the community of practice.

An individual can participate in a community of practice through engagement, imagination and/or alignment (modes of belonging). These three ways of identifying and negotiating involve different approaches and different conditions and they do not require or exclude each other. Participation through engagement implies active involvement in a community of practice and requires the possibility to participate in its activities. Participation through imagination implies going beyond time and space in a physical sense and creating images of the world. Participation through alignment implies that the individual aligns in relation to the community of practice the individual wants to, or is forced to, be a participant of. Imagination and alignment make it possible to feel connected to people we have never met but whom, in some way; match our own patterns of action.

According to Wenger (1998), identity is a never ending negotiation and identification in communities of practice and, as such, identity is not an object but an endless becoming. Identity is temporary and identity development is constantly ongoing as a learning trajectory through a cluster of communities of practice. Our identities are shaped in the tension between our different memberships in various communities of practice and the identification and negotiation within these.

Professional competence

Pattern-of-participation research has been developed as a de-objectified participatory alternative to traditional research of beliefs (Skott 2010; Skott, Moeskær Larsen & Hellsten Østergaard, 2011) and mathematical knowledge for teaching research (Skott, in press). According to patterns-of-participation research a teacher participates in multiple simultaneous practices in the classroom and there are patterns in the ways in which the teacher participates in these practices. Researching patterns-of-participation is searching for patterns in teachers' participation in immediate situations and prior social practises, whereof some are mathematical. The aim in patterns-of-participation research is to understand patterns in how a teacher's interpretation of and contribution to immediate social situations relate dynamically to her prior engagement in a range of other social practices, not to her beliefs or mathematical knowledge as objectified mental constructs.

As such, patterns-of-participation are valid as the pre-reified and pre-alienated statements and/or actions previously objectified as mathematical knowledge for teaching or beliefs. Which statements and/or actions that were previously objectified as beliefs or mathematical knowledge for teaching respectively is no longer of interest. If the aim is to understand mathematics teachers and their mathematics teaching, their patterns-of -participation, their pre-reified and pre-alienated processes of participation, are of interest.

Connecting patterns of participation and identity development

The shift from beliefs and mathematical knowledge for teaching to patterns-of-participation makes a connection to identity and identity development according to Wenger (1998) possible. Wenger analyses identity development while Skott (2010) and Skott et al. (2011) analyse teaching situations. Both theories include the individual and the social but with different focus and different emphasis.

> In all of this, patterns from the teacher's prior engagement in social practices are enacted and re-enacted, moulded, fused and sometimes changed beyond recognition as they confront, merge with, transform, substitute, subsume, are absorbed by, exist in parallel with and further develop those that are related to the more immediate situation (Skott, Moeskær Larsen & Østergaard, 2011, p.33).

> […] it [identity] is produced as a lived experience of participation in specific communities. What narratives, categories, roles, and positions come to mean as an experience of participation is something that must be worked out in practice. An identity, then, is a layering of events of participation and reification by which our experience and its social interpretation inform each other (Wenger, 1998, p.151).

In the citation of Skott et al. above, the focus is on the immediate situation with "prior engagement in social practices" in the background. In the citation of Wenger, the relationship between situations and social practices is the opposite, with communities of practice in the foreground and the imprints must be "worked out in practice". What Skott et al. (2011) call social practices can be treated as communities of practice. A conceptual framework can be developed within a participatory perspective, making both the individual and the social possible as units of analysis where patterns-of-participation can be used to describe the-social-in-the-teacher-in-the-social (*individualization of the collective)* and communities of practices can be used to describe the-teacher-in-the-social-in-the-teacher (*collectivization of the individual)*. Patterns-of-participation offer a language with which to explain what is happening in situations while communities of practice offer a language with which to explain the emergence of the patterns-of-participation.

The participation of the teacher is double. At the same time as an individual participates in an immediate situation (the focus of Skott), she participates in several communities of practice (the focus of Wenger). Horn, Nolen, Ward and Campbell (2008) distinguish between a situation as an arena or as a setting. Arena refers to the "physically, economically, politically and organized spaces-in-time" (p. 63) whilst setting refers to "personally ordered and edited versions of the arena that arise as individuals interact in these contexts" (p. 63). If connected to the coordination between patterns-of-participation and communities of practice an individual's interpretation of an immediate situation (an arena) into a setting is based on his or hers nexus of memberships in communities of practices.

Wenger's theory becomes useful when analysing an individual's different participation in different communities of practice. Skott's theory becomes useful when analysing how such different memberships in communities of practice influences how the individual interprets and acts in immediate situations. Skott et al. (2011) write that it is the responsibility of the researcher to disentangle if and how a teacher's participation in past and present practices influences the classroom. Through combining patterns-of-participation and communities of practice such an analysis is possible. Based on analysis of the individual's participation in forms of engagement, imagination and/or alignment, interpretations can be made about communities of practice the individual seems to negotiate and/or identify with and how these memberships influence the merged patterns-of-participation.

According to Wenger (1998), identity development is an individual's learning trajectory through different communities of practice. That learning trajectory can be viewed through changes in the individual's patterns-of-participation in settings over time. To talk about professional [primary school teacher] identity based on participation in communities of practice means freezing the ongoing process. To talk about professional [primary school teacher] identity development based on learning trajectories in communities of practice means freezing the ongoing process several times over a prolonged time. By freezing and objectifying the ongoing process, one can talk about professional [primary school teacher] identity; however, it is important to remember that the identity is not an object within the individual but an objectification of an ongoing process.

USING THE CONCEPTUAL FRAMEWORK

In this section some short examples are given of how the conceptual framework may be used when analysing empirical material. The excerpts are from analysis made in the case study of novice primary school mathematics teachers' professional identity development. However, here the aim is not to analyse the identity development of these novice teachers but to show how the conceptual framework may be used when analysing empirical material.

The empirical material in the study was collected through self-recordings made by the respondents, observations and interviews. These varying empirical materials have different characteristics but are treated as, named by Aspers (2007), complete-empiricism, implying all the material constitutes a whole on which the analysis is based. Based on the conceptual framework research questions can be asked about the patterns-of-participation of one or several teachers, why these patterns-of-participation occurs, the modes of belonging in communities of practices of one or several individual and the influence of these memberships on patterns-of-participation.

The first example is an analysis of the mathematics teaching of Nina, one of the respondents, when she one day is teaching mathematics outdoors. In the example, the communities of practice are given names based on their shared repertoire so as to be distinguishable. As seen the analysis is not based solely on what is observed in the specific situation but on the complete-empiricism in the case of Nina.

> Three communities of practice are visible in Nina's patterns-of-participation in this teaching situation, the community of reform mathematics teaching, the community of teachers working in grade two and the community of teachers at Aston School. Her actions seem to be a merger of the shared repertoires in these three communities. Working with mathematics outdoors is part of the shared repertoire in the community

> of reform mathematics as well as the focus on students' communication. The emphasis on working fast to finish as many questions as possible is part of the shared repertoire in the community of teachers working in grade two. Finally, Nina's worries regarding if the students really learn and how she is to prove to her colleagues that they do is her participating in the community of teachers at Aston School.

Further different communities of practices can be analysed based on their mutual engagement, joint enterprise and shared repertoire. Similarly, the respondent's modes of belonging (engagement, imagination and/or alignment) in them can be analysed. The imprints of these modes of belonging can be analysed both with focus on the teacher as the social-in-the-teacher-in-the-social or on the communities of practice as the-teacher-in-the-social-in-the-teacher. Such analyses shed light on how the individual's patterns-of-participation regarding teaching mathematics are influenced by and influence communities of practice.

Below is first an example where the focus is on the teacher when analysing a pattern that occurs in several of the mathematics lessons taught by Barbro, another of the novice teachers in the study. In the next example the focus instead is on the community of practice.

> Barbro's mathematics teaching is strongly influenced by the community of teachers working in preparatory class. Her mode of belonging in this community of practice is engagement and also imagination as becoming "totally qualified" by writing her essay. The mutual engagement in the community is students learning Swedish. Even when Barbro is teaching mathematics her main focus is on the students learning Swedish and she gets positive feedback regarding this from other members in the community of teachers working in preparatory class.
>
> The mutual engagement in the community of teachers working in preparatory class is students learning Swedish. This joint enterprise is negotiated by the members in a harmonious way. Even so, there is a visible hierarchy between the members which seems to be based on their education. One part of the negotiated shared repertoire is not to speak English even when it could be a possible help for the children to understand the tasks in the mathematics text book.

Finally, analyses focusing on individuals, over a prolonged period of time make professional identity development visible as learning trajectories in nexus of communities of practice. The conceptual framework helps to answer questions regarding how and why, for example as below when analysing one changed pattern in Nina's mathematics teaching.

> The merger of the shared repertoire in the community of teachers working in grade two and the community of reform mathematics teaching has changed Nina's pattern regarding mathematics teaching. Last semester, before she became a member of the community of teachers working in grade two, she expressed the repeated pattern of the text book as something less good but now she stresses it as something good since "it is easy for the children to understand". The same expression is often heard by the other

members of the community of teachers working in grade two. In the community of teachers working in grade two, Nina is a member by engagement and she is active in the negotiation of its shared repertoire. Regarding the community of reform mathematics teaching, her new employment this semester is preventing her membership by engagement. As such, her mode of belonging is now in the form of imagination and it doesn't seem to influence her pattern regarding talking about and using (or not using) the text book in the same way as it did last semester.

EVALUATING THE CONNECTION IN THE FRAMEWORK

The connection between Wenger´s (1998) and Skott´s (2010) and Skott et al's (2011) theories implies a conceptual framework enclosing the process of professional identity development. The goal of the connection is to better understand an empirical phenomenon (identity development as a primary school mathematics teacher). Connections between theories could, according to Radford (2008), be made at the level of principles, methodologies, or questions or as a combination of these. In this study, the question constituted the need of the connection. The cores of Wenger's (1998) and Skott's theories may appear rather different at first sight. The two theories have different ranges. Skott's theory is new and under development, while Wenger's theory is better established. Wenger's starting point is learning and Skott's is beliefs. However, the two theories have a common starting point when looking at old phenomena (learning – beliefs) but from a (new) social perspective.

Wenger's (1998) starting point is to make a social theory of learning and Skott's (2010) is to turn beliefs research social. Social may, however, imply different things in various degrees. Wenger asks the question, "what if we […] placed learning in the context of our lived experience of participation in the world?" (p.3). Similarly, Skott suggests, "that a more participatory stance is adopted in research on the role of teachers for classroom practice" (p.1). The basic claim behind the notion of participation is that "patterned collective forms of distinctly human forms of doing are developmentally prior to the activities of the individual" (Sfard, 2006, pp.157). Further comparison shows that there are some connections between notions in the two theories. Wenger (1998) writes that knowledge always undergoes construction and transformation in use, and that things assumed to be natural categories, such as "bodies of knowledge", "learners" and "cultural transmission", require reconceptualisation as cultural, social products. This can be compared with Skott's (2010), Skott et al's (2011) and Skott's (in press) re-reification goal. What Wenger calls "cultural transmission" and "bodies of knowledge" respectively can be seen as similar to traditional reified beliefs or mathematical knowledge for teaching. Both Skott and Wenger work with re-reification focusing on processes. Certainly, a community of practice is a reification, but Wenger's theory is about the process of identity and identity development in and of those communities of practices.

Regarding methodology, neither Wenger (1998) nor Skott (2010) or Skott et al (2011) mentions it in their theories. However, based on the conceptual framework, inferences can be drawn about what kind of data should be collected. To be able to discover patterns-of-participation and communities of practice, an attended approach, e.g. ethnography, is needed. Regarding the basic principles both theories are within a participatory perspective focusing on processes involving both the individual and the social.

Österholm (2011) compares patterns-of-participation research with traditional beliefs research and writes that when researching patterns-of-participation, all prior experiences are of interest and that there seems to be a need for some form of separation between the different practices and different kinds of memberships. This separation is acquired by the connection between patterns-of-participation and communities of practice. Further, Österholm writes that within patterns-of-participation research, it seems to be difficult to talk about change since there is nothing to change. The connection between communities of practice and patterns-of-participation presented here makes visible the process of constant mutual change in patterns-of-participation and communities of practices.

In summary, if, using the words of Prediger, Bikner and Arzarello (2008), the conceptual framework implies coordination as built by well-fitting elements from different theories, being useful when the empirical elements of the theories complement each other and, coordinated thus, support a more complete analysis.

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INCLUSIVE MATHEMATICS FROM A SPECIAL EDUCATION PERSPECTIVE – HOW CAN IT BE INTERPRETED?

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The aim of this paper is to present a way of understanding the phenomenon of inclusion in mathematics. The theoretical framework consists of the connection between two theoretical perspectives and is tested in an empirical example of inclusive mathematics from the perspective of special education. The theory of communities of practice is used as an overall theoretical perspective along with a theoretical framework regarding inclusion. Sub codes were extracted from the empirical example to create a more fine-grained conceptual framework. The results show that the conceptual structure is beneficial for extracting a fine-grained conceptual tool in understanding and developing inclusion in mathematics.

Keywords: connecting theories, framework, inclusive mathematics, special education

INTRODUCTION

This paper discusses the connection of the theoretical perspectives in a research project focusing on the development of inclusive mathematics education, based on special education needs in mathematics. The connection is tested in an empirical example for understanding inclusion in mathematics from a pedagogue perspective.

The pedagogues that usually are involved in the situation of teaching students in special needs in mathematics are the mathematics teacher and the remedial teacher. The remedial teacher in mathematics needs to interpret the students' knowledge to be able to ensure that students' needs are met at all levels. On an overall epistemological level this can be made from a categorical perspective or from a relational perspective (Sjöberg, 2006). From a categorical perspective the problem is situated within the student. In contrast, the relational perspective investigates the environment surrounding the student and the influence it has regarding students' knowledge (Emanuelsson, Persson, & Rosenqvist, 2001). A relational perspective on mathematics difficulties stresses the need to consider how the teaching and learning activities in question affect the students' learning in mathematics (Dalvang & Lunde, 2006). The present project adheres to the relational view in striving to reach an understanding of inclusive mathematics. Specifically an inclusive perspective on mathematics education is adopted. In an inclusive perspective knowledge and mathematical understanding are viewed as cultural and social phenomenon.

Research (e.g., Ballard, 1999) regarding inclusion is a major field of research, which mainly looks at inclusion from the pedagogical perspective. However, little attention has been paid to the meaning of inclusion in mathematics and the identification of barriers or factors that appear critical in the students' learning of mathematics. Many Swedish schools use ability grouping in mathematics and the teachers envisions better goal achievements for students in special educational needs. Thus, research concludes that organisational differentiation does not give the positive impact on students' knowledge development the teachers expect (Boaler, 2008; Slavin, 1990). Educational differentiation and individualisation is a complex issue, which requires more investigation. Moreover, there is still much to learn regarding how different factors work and connect in the pursuing of an inclusive teaching of students in special needs in mathematics. On account of that, the present paper aims at discussing the connection of theories in the overall conceptual framework trying to frame inclusion in mathematics from a special education view with the perspective of pedagogues. The perspective is also inclusive, where inclusion is seen as a social process of participation in the mathematical practice. This requires identification of both the students' participation and the communities they have access to. To capture and put words to the offered communities in the practice and the students' participation, a conceptual framework is needed.

Eisenhart (1991) recognise three types of frameworks (theoretical, practical and conceptual). Since this framework is built from different sources, it is a conceptual framework.

THEORETICAL PERSPECTIVES

In the overall study two theoretical perspectives are used, a participatory perspective and an inclusive perspective. These two perspectives are used to capture the research questions of the study, namely: What can inclusion in mathematics be and how is it possible to develop inclusive mathematics education, based on special education needs in mathematics?

Conceptualizing inclusion

The concept of inclusion is complex and difficult to define (Brantlinger, 1997). Nonetheless, it is a well-used term in schools today, even in mathematics education. Historically, inclusion is a relatively new concept. It was first used during the early 1990s, before that, the term integration was used (Farrell, 2004). Then what is the difference between these two concepts from a school context perspective?

The concept of integration was developed towards the end of the 1960s, as a critique to the various institutions created for deviant groups in society. Within a school context this term was used to emphasize an assimilation process, children with special needs would be fitted into an existing school context (Nilholm, 2006). In the 1990s it was perceived that the concept of integration not fully covered the importance of participation and the term inclusion began to become more common (Rosenqvist, 2003). The concept inclusion aims at "that the school (the whole) will be organized based on the fact that children are different (the parts)" (Nilholm, 2006 p.14, own translation). The concept of inclusion refers to a continues process (Asp-

Onsjö, 2006) in schools. The introduction of the concept inclusion had an intention, a wish to change the perception regarding work with students in special needs (Nilholm, 2006), from exclusion to inclusion.

As previously mentioned, this project aims at investigating inclusion in mathematics from a participatory perspective. From this perspective inclusion does not just mean being in the classroom physically, it means to be included in the mathematical practice of the classroom, which can be anywhere, this form of inclusion has no physical condition, it is imaginary. Asp-Onsjö (2006) talks about *spatial, social* and *didactical* inclusion. Spatial inclusion basically refers to how much time a student is spending in the same room as his or her classmates. The social dimension of inclusions concerns the way in which students are participating in the social, interactive play with the others. Didactical inclusion refers to the ways in which student's participation relates to a teacher's teaching approach and the way in which the students engage with the teaching material, the explanations and the content that the teachers may supply for supporting the student's learning. In this study the content of learning is number sense, since it is the content covered in teaching observed. These three analytical categories will be used as an overall frame in developing a fine-grained explanatory framework. This framework aims at increasing our understanding of how students in special needs in mathematics are participating, develop their way of participating or become restricted from participating in the school mathematical practice.

Students in special education needs in mathematics in communities of practice

This investigation of inclusion in mathematics education is grounded in a social perspective on learning. The overall principle of this perspective is that learning is considered to be a function of participation (Wenger, 1998). Participation is to be seen as "a process of taking part and also to the relations with others that reflect this process" (Wenger, 1998, p. 55). Participation is an active process that involves the whole person and combines "doing, talking, thinking, feeling and belonging" (Wenger, 1998, p. 56). It "goes beyond direct engagement in specific activities with specific people" (Wenger, 1998, p. 57). The practice "exists because people are engaged in actions whose meanings they negotiate with one another" (Wenger, 1998, p. 73) and the practice reside in a community of individuals with *mutual engagement.* Members of a community of practice are practitioners who develop a *shared repertoire*, such as experiences, tools, artefacts, stories, concepts etc., the *joint enterprise* keeps the community of practice together. It is a collective process of negotiation of the participants in the process of pursuing it.

As previously mentioned, I will investigate inclusion from the perspective of pedagogues. In terms of participation this means that I look at how the teacher and the remedial teacher in mathematics allocate the problem of including students in special needs in mathematics to the mathematical practice. The prior presented learning theory that focuses on communities of practice (Wenger, 1998) is intended to be used. In this theory learning is seen as a process of social participation. The unit of analysis in this theory is the community of practice which is an informal community where people involved in the same social setting form the practice (Wenger, 1998). Although the social setting is important regarding special education needs in mathematics, the students´ conceptual understanding also has to be taken into consideration. In order to capture the students´ conceptual understanding I will follow Graven and Lermans (2003) interpretation of Wenger (1998). The reason for this choice is that they argue that the primary unit of analysis in Wenger´s theory is communities of practice, but for teacher learning it permits the primary unit of analysis to be "the teacher-in-the-learning-community-in-the teacher" (p.192). In the first phase of this project I will use Graven and Lermans (2003) unit of analysis, "the teacher-in-the-learning-community-in-the-teacher" (p.192). To reach the students (in the second phase of the project) I will modify Graven and Lermans unit of analysis into "*the student-in-the-learning-community-in-the-student"*. These units of analysis are coherent with Lermans (2000) unit of analysis from the social perspective, "person-in-practice-in-person" (p.38), where the person has an orientation toward the practice and the practice has become in the person. This will give access to the individual in the community as well as the community of practice.

THE CONNECTION OF THEORIES

When connecting theories it is crucial to know how the connection is made and what it is in the theories that make them work together. According to Wedege (2010) the connection can take place at different levels of the theories. In this project a connection is made between the theory of communities of practice (Wenger, 1998) and the theoretical framework of inclusion (Asp-Onsjö, 2006) at the level of principles of the theories, since they both look at learning as a social phenomenon. Hence, the theories have compatible cores in their view of learning. Although the complexity and size of the theoretical frameworks vary widely, they grasp the different aspects of the research question. Communities of practice are here seen as an overarching theory of learning. To this theory a connection is made to the three modes of inclusion that Asp-Onsjö (2006) presents.

Different strategies can be used connecting theories according to Prediger, Bikner-Ahsbahs, and Arzarello (2008). These strategies include having an understanding of different theories, to combine, coordinate or integrate them (Prediger et al., 2008). I coordinate communities of practice with inclusion, since they contain assumptions that are consistent. These assumptions are located in their social approach. The coordination creates a conceptual framework with well fitting elements that help in identifying both the students' and teachers' participation and the communities they have access to regarding learning in mathematics. The three modes of inclusion (Asp-Onsjö, 2006), which are deeply interconnected and constantly interacting with each other, puts words to and allows the development of a fine-grained framework regarding inclusion in mathematics. This framework may inform theory and therefore it has the potential to gain new contributions to the field through its explanatory power. The framework may also be able to contribute to the solution of the overall research questions: What can inclusion in mathematics be and how is it possible to develop inclusive mathematics education, based on special education needs in mathematics? This is done by identification of factors regarding inclusion in mathematics and their connection in the communities.

In this study the aim is to empirically investigate what inclusion in mathematics can be and how it can be developed by using necessary theoretical concepts. That is, what is and what is not inclusion in mathematics is an empirical question. In the overall study this is investigated through observations, group interviews and interviews with both teachers and students. In this paper the pedagogue perspective is in focus. The data used to capture inclusion in mathematics from their perspective is interviews. In order to identify what inclusion in mathematics can be from the interviews, the conceptual framework will be used as an overall frame.

METHOD

In the present project a large primary school, Oakdale School, located in a suburb of a medium-sized Swedish town is being observed. The on-going project is a longitudinal study with an ethnographic approach, which according to Aspers (2007) is a study where the researcher tries to understand a phenomenon through interpersonal methods. In this study the phenomenon of inclusion in mathematics is to be understood, hence the methods used are interpersonal. Interviews, discussions and lessons with teachers and the remedial teacher were observed and recorded during the first year of the project. 14 interviews, 24 observations at lessons and 3 group interviews were made. The construction of data is intended to continue for another year. In this article four interviews have been used in an empirical example. The interviews are with Ellie, Anna and Barbara. Ellie and Anna are primary teachers working at Oakdale School. They both teach mathematics in lower primary school. Two of the interviews are with Barbara. She is a remedial teacher in mathematics and her current assignment is as a remedial teacher with focus on mathematics.

In the interviews a qualitative, semi structured approach was used. The teachers were invited to elaborate on their view on students in special education needs and factors they consider crucial to for students participation in the mathematical practice. The interviews were recorded and transcribed in full. The analysis was made using coding of the data by labelling the empirical material. This type of coding is the ground for creating new theoretical categories (Aspers, 2007). In the coding of the data, sub codes were identified. Using these sub codes a few major codes arose. Subsequently, an analysis of how these major codes relate to the theoretical concepts was made. Using the concept of “communities of practice” (Wenger, 1998) the analysis started by identification of communities of mathematical practice from an inclusive perspective (step one). Step two in the analysis was using the framework of Asp-

Onsjö (2006) and make identifications of how pedagogues expressed their view of how to work with students in special education needs in mathematics. Digging deeper into the interviews fine-grained codes regarding inclusion and communities of practice in mathematics were identified. Then the analysis went between step one and two several times again.

RESULTS

Communities of mathematical practice at Oakdale Primary School

Analysing the interviews, three communities of mathematical practice were constructed. The first practice is the *community of inclusive mathematics (CIM)*, which is created from the fact that the research project started at Oakdale School and the teachers are invited to collaborate in this project. Barbara is a core member in this community, since she is the remedial teacher in mathematics, and is eager to develop inclusion in mathematics at the school. She wants to develop the teaching of mathematics for all students at the school, because it is "very easy to see the problem within the student instead of what it is in the teaching that does not benefit all [students]" (Barbara). Ellie and Anna are members of the community, since they are interested in developing their teaching of mathematics, and to "get more time to plan together" (Anna). The mutual engagement in this practice is the development of mathematics teaching for students in special education needs. Their shared repertoire is the talk about how to help the students understand, and their experiences of special education needs in mathematics.

The second practice visible in the data was the *community of mathematics classroom (CMC)*. This community is created in the classrooms and is thus two different communities of practice, one in Anna's classroom and another in Ellie's classroom. Ellie points out that it is important to "be involved" in the classroom activities, and Anna points out that it is "valuable that the students are present when the teacher presents the content". Barbara is a peripheral participant in these practices in her role as a remedial teacher and wishes to become more engaged, she wants to be "open about our roles in the class [room] "and "that we discuss together, what can I do". The mutual engagement in these practices is the mathematics learning for all students, that you work according the curriculum. The shared repertoire is the talk about the mathematics teaching, the curriculum and the use of teaching materials.

The third identified practice was the *community of special education needs in mathematics (CSENM)*. This practice is created by the fact that special education needs in mathematics exist at the school. It involves all mathematics teachers, though Barbara is a core member, since she is the only remedial teacher in mathematics at the school, "I shall serve from the first grade to the sixth grade". She points out that "I have been interested [in mathematics] and the others [remedial teachers at the school] not". Other remedial teachers in other nearby schools is part of this community, because Barbara points out that "we need to talk about how we do things

[…] talk about the subject and help each other". In this practice the students in special education needs are participants. They participate and influence the teaching, since "You ask them: How do you want it to be?" (Barbara). The mutual engagement is the students in special education needs development of mathematical knowledge. The shared repertoire consists of the artefacts involved in the teaching, such as materials, games and tasks. It is also the individual education plans and their content. All these three practices intercalate and influence each other, but there are differences of participants, mutual engagement and shared repertories in these practices that might influence the talk about inclusion.

Inclusion in the communities of practice

As mentioned earlier, in the screening of the data several sub codes regarding inclusion in mathematics were found. These were grouped into major codes and have been categorized into the three different communities of mathematical practice at Oakdale School. The three aspects of inclusion have also been taken into account in the categorisation. The role of mathematics is most visible within didactical inclusion, where the understanding of number sense becomes important. The codes in the matrix (Figure 2) are a fine-grained conceptual tool regarding inclusion in mathematics. Some of the codes occurred in more than one community of practice, but most of them only occurred in one of them.

Inclusion in the community of inclusive mathematics

In the community of inclusive mathematics practice (CIM) all the three aspects of inclusion were discussed. Within spatial inclusion the issue of being sensitive as a teacher regarding if the students in special education needs should be in the classroom or be in a small group with the remedial teacher were central to Barbara and Ellie. Ellie expressed that "they don't have to go away somewhere else if they don't like it". Barbara pointed out that "some [of the students] don't what to leave the class [room] so you have to think about how to help in the classroom". Acceptance was an issue within social inclusion referring to a permissive climate in the practice. Barbara pointed out that "it is an upbringing issue, you may not laugh at anyone". Barbara here connects "the knowledge process with the work with values". Ellie also emphasise this, referring to the classroom practice in saying "we have different needs […], [we] have taught the kids to accept that we work different, a permissive climate". The category referring to didactical inclusion in this community was teaching approaches. All the three teachers referred to this with impact. It referees to how to work with the students in special education needs in concrete. Anna talks about to "give everybody tasks that they can work with and understand" and Ellie says that one aspect of inclusion is to "give them questions that you know they will be able to answer". She emphasises that "you [the teacher] need to be aware of that you have the lesson at three different levels" referring to have several teaching approaches within the classroom. Here, Barbara points out "it is about how you ask the question" so that you include everybody.

Inclusion in the community of mathematics classroom

In the community of mathematics classroom spatial and didactical inclusion was visible. Within spatial inclusion Anna expressed her eager that all the students in the class attended the briefings in mathematics in the classroom, Ellie mentioned it, but was not so explicit. Looking at didactical inclusion three codes emerged: individualisation within the classroom; teaching approaches; and mathematical knowledge. Regarding the first, this refers to being able to "individualise tasks within the class" (Barbara) and to "think about to come to them [the students in special education needs] often" (Ellie). Talking about teaching approaches in this practice is practically the same as presented above in Inclusion in the community of mathematics classroom. The difference is that the students are more visible and engaged in this practice, "they support each other" (Ellie). Mathematical knowledge is here to be seen as knowledge about the mathematical content the current teaching includes, how to present it and to do different approaches to the content more specific. The content discussed is number sense, within the number range 1-1000. A central question here is "what are we going to teach them and who is responsible?" (Barbara), referring to the mathematical content.

Inclusion in the community of special education needs in mathematics

In the community of special education needs in mathematics all three aspects of inclusion are visible. Terms for teaching is within spatial inclusion and deals with issues regarding organisational aspects such as access to study rooms nearby and time to collaborate and be two pedagogues in the class at the same time. Within social inclusion Barbara emphasises students co-decision, "that you really ask, how do you want it?" that you [as a student] get to "be a part in the decisions". The didactical aspect of inclusion is mathematical knowledge, which refers to the mathematical knowledge of content the current teaching includes, as above. There is one distinction though, in this practice it is specifically how to deal with the knowledge that is difficult for students in special education needs. To be able to use different representations and have a "learning path that is truly systematic" (Barbara).

Inclusion Community		**Spatial inclusion**	**Social inclusion**	**Didactical inclusion**
CIM	Barbara	Teacher's sensitivity	Acceptance	Teaching approaches
	Ellie	Teacher's sensitivity	Acceptance	Teaching approaches
	Anna			Teaching approaches
CMC	Barbara			Individualisation
				Teaching approaches

				Mathematical knowledge
	Ellie	Attend the briefings		Individualisation
				Teaching approaches
				Mathematical knowledge
	Anna	Attend the briefings		Teaching approaches
				Mathematical knowledge
CSENM	Barbara	Terms for teaching	Student’s co-decision	Mathematical knowledge
	Ellie	Terms for teaching		Mathematical knowledge
	Anna	Terms for teaching		Mathematical knowledge

Figure 1. Inclusion in the communities of practice at Oakdale School

The matrix in Figure 1 sums up the result of the analysis, with major codes of the teachers’ expressions of how to work with students in special educational needs.

CONCLUDING REMARKS

In this paper the connection of the theoretical perspectives in the current research project has been presented and tested in an empirical example for understanding development of inclusive mathematics education based on special education needs in mathematics. The overall framework has been shown beneficial for extracting a more fine-grained conceptual tool in understanding and developing inclusion in mathematics. The empirical material has been instrumental in the development of the conceptual tool, and in the connection of theories. This is visible in the extracted codes in figure 2, where the three identified practices have served as a filter to identify inclusion at different levels at Oakdale School from a pedagogue perspective. Within didactical inclusion teaching approaches, mathematical knowledge and individualisation emerged with different emphasis within the three practices. The practices interact and influence each other, but inclusion looks a little different in the practices and this might influence the understanding and development of inclusive mathematics at Oakdale School. The identified codes have the potential to be generalised and it may be possible to identify other codes analysing the material in its entirety.

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COMPARING APPROACHES THROUGH A REFERENCE EPISTEMOLOGICAL MODEL: THE CASE OF SCHOOL ALGEBRA

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Luis Radford and Yves Chevallard, whose research programmes in mathematics education were awarded with the last Hans Freudenthal medals, have both given a prime place to the problem of teaching and learning elementary algebra. However, their approaches are far from being similar. Are they comparable? We are starting a dialogue considering how each approach, in a more or less explicit way, defines what algebra is and how it characterises 'algebraic thinking' or 'algebraic activities' with what we call a reference epistemological model *(REM) of elementary algebra. The dialogue starts by assuming the point of view of the Anthropological Theory of the Didactic, presenting our own REM, and the kind of questions addressed by this approach, in relation to the Theory of Knowledge Objectification developed by Radford.*

1. TWO WAYS OF APPROACHING SCHOOL ALGEBRA

For more than twenty years, Luis Radford's and Yves Chevallard's investigations have dealt with school algebra as a research domain. Nonetheless, the problems approached present very different formulations and scopes. They seem to deal with completely different worlds and there are very few mutual references, if any. It is clear that we are here considering two approaches that have been 'personalised' by the researchers mentioned but that, as the Hans Freudenthal award states, they represent two research programmes, involving several researchers from different countries. At the same time, the way each programme approaches the research problem of school algebra does not need to be exclusive. As in a case study, we are considering Luis Radford's approach as a representative of research dealing with 'algebraic thinking' (Radford 2002, 2008, 2012, Radford & Puig 2007). The other case, represented by Yves Chevallard's work, corresponds to the latest investigations carried out within the Anthropological Theory of the Didactic (ATD), mostly by our research team, around what we have called the 'process of algebraization' (Bolea et al. 1998, 2004; Ruiz-Munzón 2010; Ruiz-Munzón et al. 2012; Bosch 2012).

We will start by briefly formulating some of the main problematic questions or research problems addressed by each of the approaches, before pointing out some of their main differences and common points. The main questions addressed by the research approach proposed by Radford can be summarised around the notion of "algebraic thinking" and its characterisation: What are the relationships (filiations and ruptures) between numerical or arithmetical and algebraic forms of thinking? Could embodied forms of algebraic thinking observed in adolescents be accessible to young students? What is the evolution of the different components of algebraic thinking in young students? How do students interpret the meaning of algebraic symbolism? What characterises algebraic generalisations and what distinguishes them from arithmetic ones? These are only a few of the many questions that could be formulated in terms of the "iconicity" and "semiotic contraction" in the development of algebraic thinking, the "process of objectification", etc., in the sense these notions adopt in the Theory of Knowledge Objectification (TKO).

In the case of the ATD, the main question is the characterization of "*algebraized*" mathematical activities. Let us first remind that, in ATD, mathematical activities are described with the general model of "praxeologies", the inseparable union of *praxis* and *logos*, used to depict any human activity (Chevallard 2006). It is assumed that the algebraic character of mathematical activity is relative, a question of degree, and some indicators to measure the "*algebraization* degree" of a mathematical praxeology are defined and used to describe the process of *algebraization* of mathematical praxeologies. The questions that can then be posed are, for instance: What conditions are required for elementary algebra to normally exist as a modelling tool in an educational institution (for instance at lower secondary school, grades 7-10) so that the school mathematical organisations can be progressively *algrebraized*? In what sense can lower secondary school mathematical organisations be considered as poorly *algebraized*? What aspects of the *algebraization* process are difficult to introduce at school and what constraints hinder their introduction?

When comparing both approaches, the first obvious observation is that they question and problematize different aspects of the 'didactic reality' they wish to study: "algebraic thinking" *versus* the "process of *algebraization* of mathematical activities". It may seem that considering problems of such a different nature can make the dialogue between them rather difficult, at least if we stay at the level of the formulation of research problems. In fact, our postulate is that the differences in how problems are formulated are deeply dependent on the way of interpreting and describing algebra in each framework, that is, the *reference epistemological model of school algebra* used. Depending on how we define or consider the mathematical content involved in a didactic problem (school algebra, in our case), we will be able to formulate some research questions rather than others, to delimit the empirical unit of analysis considered, and to look for acceptable answers to these questions. Therefore, the distance between the two ways of approaching the problem of school algebra can be explained by the differences between the epistemological models assumed. We thus propose to start the dialogue between the approaches at this level.

2. WHAT ARE REFERENCE EPISTEMOLOGICAL MODELS?

When analysing any teaching or learning process of mathematical contents, questions arise related to the interpretation of the mathematics involved in it. For instance, what is elementary algebra or geometry, or statistics? How is it interpreted in a given educational institution? Why is it taught? How is it related to other contents? Etc. The different institutions interfering in the didactic processes propose more or less explicit answers to said questions. If researchers assume those answers uncritically, they run the risk of not dealing with the empirical facts observed in a sufficiently unbiased way. Therefore, the ATD proposes to elaborate what are called *reference epistemological models* (REM) for the different mathematical sectors or domains involved in teaching and learning processes (Bosch & Gascón 2005). In the ATD, those REM are formulated in terms of local and regional praxeologies and of sequences of linked praxeologies of increasing complexity.

It is important to insist on the fact that the epistemological models built by didactic research should be considered as *working hypotheses*. As such, they are always provisional and constantly need to be contrasted and revised. Even if they are given other names, other approaches in mathematics education use analogue theoretical constructs. For instance, the Theory of Didactic Situations proposes to describe mathematical bodies of knowledge in terms of fundamental situations (Brousseau 1997); the APOS theory (Dubinsky & McDonald 2002) uses the "genetic decomposition" of a concept; the Onto-Semiotic Approach (Godino, Batanero & Font 2006) talks about "systemic configurations"; the theory of "Abstraction in Context" (Dreyfus, Hershkowitz & Schwarz 2001) is concerned with "epistemic actions"; etc. In the case here considered, TKO is supported on a model of *algebraic thinking* proposing a specific way of interpreting and describing elementary algebra.

From the point of view of the Anthropological Theory of the Didactic, the kind of reference epistemological models considered have certain specific features. The empirical data taken into consideration do not only come from school mathematics, but also from the different institutions involved in the process of didactic transposition (the school and its environment, policy-makers, "scholar mathematicians", professionals, etc.). REM should not uncritically assume any of the viewpoints that are dominant in these institutions. In the ATD, epistemological models do not take into account the idiosyncrasy of the persons involved in the teaching and learning processes, or the specific conditions in which they take place. What they explicitly include are the concrete activities that can be considered as the *raison d'être* of the mathematical content involved in terms of problems to be solved or questions to be addressed, as well as the way it evolves to give rise to new problematic questions.

3. THE ATD REFERENCE EPISTEMOLOGICAL MODEL FOR SCHOOL ALGEBRA

With respect to school algebra, the ATD proposal is to interpret it as a *process of algebraization* of already existing mathematical praxeologies, considering it a *tool* to carry out a modelling activity that ends up affecting all sectors of mathematics. Therefore, algebra does not appear as "one more content" of compulsory mathematics, at the same level as the other mathematical praxeologies learnt at school (like arithmetic, statistics or geometry) but as a general modelling tool of *any* school mathematical praxeology, that is, as a tool to model previously mathematized systems. In this interpretation, algebra appears as a practical and theoretical tool, enhancing our power to solve problems, but also as the possibility of questioning, explaining and rearranging already existing bodies of knowledge.

This vision of algebra can provide an answer to the problem of the status and rationale of school algebra in current secondary education. On the one hand, algebra appears as a privileged tool to approach theoretical questions arising in different domains of school mathematics (especially arithmetic and geometry) that cannot be solved within these domains. A well-known example is the work with patterns or sequences where a building principle is given and one needs to make a prediction and then find the rule or general law that characterises it. This feature highlights another differential feature of algebra that is usually referred to as “universal arithmetic”: the possibility of using it to study relationships independently of the nature of the related objects, leading to “generalised” solutions of a whole type of problems, instead of a single answer to isolated problems, as is the case in arithmetic. Another essential aspect of the rationale of algebra is the need to organise mathematical tasks in *types* of problems and to introduce the idea of generalisation in the resolution process, a process making full use of letters as parameters.

In this perspective, the introduction of the algebraic tool at school needs to previously have a system to model, that is, a well-known praxeology that could act as a *milieu* (in the sense given in the Theory of Didactic Situations) and that is rich enough to generate, through its modelling, the different entities (algebraic expressions, equations, inequalities, formulae, etc.) essential to the subsequent functioning of the algebraic tool. In the model proposed, this initial system is the set of *calculation programmes* (CP). A CP is a sequence of arithmetic operations applied to an initial set of numbers or quantities that can be effectuated “step by step”- mostly orally and writing the partial results - and provides a final number of quantity as a result. The corpus of problems of classic elementary arithmetic (and also some geometrical ones) can all be solved through the verbal description of a CP and its execution: what was called a “rule” in the old arithmetic books. The starting point of the REM is therefore a compound of elementary arithmetical praxeologies with techniques based on the verbal description of CP and their effectuation “step by step”. However, working with CP soon presents some technical limitations and raises theoretical questions about, for instance, the reasons for obtaining a given result, justifying and interpreting it or the possible connections between different kinds of problems and techniques. All these questions lead to an enlargement of the initial system through successive modelling processes giving rise to *different stages of the “algebraization”* process that we will briefly summarize hereafter. A more detailed description can be found in (Ruiz-Munzón 2010; Ruiz-Munzón et al. 2012).

The *first stage* of the algebraization process starts when it is necessary to consider a CP not only as a process but as a whole, representing it in a “sufficiently material” way—for instance written or graphically—to manipulate it. This does not necessarily mean the use of letters to indicate the different numbers or quantities intervening in a CP (the “variables” or “arguments” of a CP). However, it requires making the global structure of the CP explicit and taking into account the hierarchy of arithmetic operations (“bracket rules”). This new practice generates the need of new techniques to *create and simplify algebraic expressions* and a new theoretical environment to justify these techniques. The notions of “algebraic expression”—as the symbolic model of a CP—and of “equivalence” between two CP appear. Following the standard terminology, we can say that this stage requires the operation of “simplifying” and “transposing”, but not the operation of “cancelling”.

The passage to the *second stage* of algebraization occurs when the identity between CP needs to be manipulated. In this stage, algebraic techniques include considering equations (of different degrees) as new mathematical objects, as well as the technical transformations needed to solve them. This case includes the resolution of equations with one unknown and one parameter, that is, the case where problems are modelled with CP with two arguments and the solutions are given as a relationship between the arguments involved. In the specific case where one of the numeric arguments takes on a concrete value, the problem is reduced to solving a one-variable equation. Nowadays, school algebra mainly remains in this last case (without necessarily having passed through the first one): solving one-variable equations of first and second degree and the word problems that can be modelled with these equations, without achieving the second stage of the algebraization process.

The *third stage* of the algebraization process appears when the number of arguments of the CP is not limited and the distinction between unknowns and parameters is eliminated. The new mathematical organisation obtained contains the work of production, transformation and interpretation of *formulae*. It is not much present at current secondary schools even if it appears under a weak form in other disciplines (like physics or chemistry). At least in Spain, the use of algebraic techniques to deal with formulae is hardly disseminated outside the study of the general “linear” and “quadratic” cases. However, they play an essential role in the transition from elementary algebra to functions and differential calculus, a transition that is nowadays quite weakened in school mathematics. Furthermore, secondary school mathematics does not usually include the systematic manipulation of the global structure of the problems approached, which can be reflected in the fact that letters used in algebraic expressions only play the role of unknowns (in equations) or variables (in functions), while parameters are rarely present. However, it can be argued (Chevallard & Bosch 2012) in which sense the omission of parameters—that is, the use of letter to designate “known” as well as “unknown” quantities—can limit the development of efficient modelling algebraic tools and constitutes a clear denaturalisation of the algebraic activity carried out at school.

Let us consider a short example to illustrate the three stages of the process:

> Take a problem of the sort: "Think of a number, multiply it by 4, add 10, divide the result by 2 and subtract the initial number", a process that we will represent by the CP: $P(n) = (4n + 10)/2 - n$. A problem where we know that $P(n) = 7$ can be solved in the first stage by first simplifying $P(n)$ and finding the equivalence $P(n) \equiv n + 5$, which gives the result $n = 2$. If the problem is $P(n) = 3n - 7$, the passage to the second stage seems more natural (even if we can always find complex techniques to solve it remaining in the first stage). If the CP is $P(n,a) = (4n + a)/2 - n$ ("Think of a number, multiply by 4, add another number, etc.") and the problem states that $P(n,a) = 2n - a$, the same type of techniques and theoretical environment enables to find a solution, which here appears as a relationship between n and a. The third stage corresponds to CP with more than 2 arguments, requiring new techniques to describe the relationships obtained, especially when we do not only work with linear equations.

This three-stage model of the algebraization process is a tool to analyse what kind of algebra is taught and learnt in the different educational systems, what elements are left out of the teaching process and what other elements could be integrated under specific conditions to be established. It is complemented with four general *indicators of the degree of algebraization* of a given mathematical content (or praxeology), as proposed by (Bolea, Bosch & Gascón 2001) to analyse the different possible constructions of the algebraic process by looking at the mathematical activities resulting from it. They correspond to: (1) the possibility to manipulate the global structure of the problems; (2) the need to "objectify" or "thematize" the mathematical techniques used and to question them; (3) the unification and ostensive reduction of praxeologies (their types of problems, techniques and theoretical discourses); (4) the emergence of new problems independent of the modelled systems.

The effort to explicitly state an epistemological reference model for elementary algebra has different purposes. It can first be used as a descriptive tool to analyse the kind of algebraic praxeologies that exist at school and to study the ecological effects (conditions provided and constraints imposed) of these praxeologies in other mathematical contents. It is also a productive tool when trying to connect investigations concerning school algebra carried out from different theoretical perspectives, as it helps specify the reference epistemological model of algebra more or less explicitly assumed by each research and compare the results provided by each one. For instance, one can consider what aspects of elementary algebra are not taught at school and inquire about the possible reasons of their absence, as well as the 'nature' and 'origin' of these reasons (Chevallard & Bosch 2012). Another interesting exploitation would be comparing different research works, as for instance the "structural approach" of the research strand on *Early algebra* or the "algebrafying" paradigm promoted by J. J. Kaput (2000) and the first stage of the algebraization process and its possible implementation in the classroom.

4. THE TKO AND THE PROBLEM OF SCHOOL ALGEBRA

Luis Radford's works present some answers to the questions addressed by the TKO to the problem of school algebra. First of all, *algebraic thinking* is characterized not by the use of symbolism but by its "analytical" character (Radford 2012b, pp. 16-17):

> I suggested, on both historical-epistemological and semiotic grounds, that algebraic thinking cannot be reduced to an activity mediated by notations. Although the modern alphanumeric symbolism constitutes a very powerful semiotic system, in no way can it characterize algebraic thinking. […] Algebraic thinking, I suggested, is rather characterized by the *analytic* manner in which it deals with indeterminate numbers—something where, as two fathers of algebra, Viète (1983) and Descartes (1954), explicitly stated, no difference is made between known and unknown numbers.

From this viewpoint, and based on different teaching experiences, algebraic thinking is postulated to emerge early among young students, even if this finding raises new difficulties related to the description of its evolution (Radford 2012b, p. 16):

> From a sensuous perspective on human cognition, it is not difficult to appreciate that 7–8-year-old students can effectively start thinking algebraically.

When considering algebraic symbolism and students' difficulties with *the interpretation of its meaning,* the operation with the unknown and the use of the first algebraic techniques to solve first degree equations is described in terms of the coordination of different systems of signs and the articulation of levels of abstraction. In this case, algebra is considered as a problem-solving tool:

> More precisely, our approach to algebra as a problem-solving tool means the development of an analytic technique based on a conceptually complex kind of mathematical thinking relying on the calculation of known and not-yet-known numbers or magnitudes that acquire a meaning as they are handled in the pursuit of the goal of the activity. [...] EISL [*elementary iconic symbolic language*] soft syntax allowed the students to accomplish the translation of elementary word-problems into an iconic statement and to suitably transform these statements in order to reach the solution. These iconic statements form an iconic text that, in the end, appears as a didactic device reducing the gap between the statement of the problem in natural language and a formal symbolic treatment of equations. The didactic goal was not to remove the gap (which is, I believe, an impossible task). The goal was to provide the students with an intermediary semiotic system from where to derive certain meanings to be used later in the semiotic system of symbolic algebra.

Algebraic generalisations are considered fundamental algebraic activities and should be defined in relation to *arithmetical* ones. More generally, the relationship between arithmetic and algebra must be clarified. Here, Radford assumes that the gap is located in the equations of the form $A\,x + B = C\,x + D$ where arithmetic methods (consisting in carrying out inverse operations) fail and the students should learn to operate on the unknown (Filloy & Rojano 1989), thus moving to *analytical* thinking:

> In order to operate on the unknown, or on indeterminate quantities in general (variables, parameters), one has to think analytically. That is, one has to consider the indeterminate quantities as if they were something known, as if they were specific numbers.

It is considered that, form a genetic point of view, arithmetic differs from algebra in this *analytical thinking* with indeterminate quantities, where unknown and known numbers are treated in the same way. An important consequence of this difference is that *algebraic formulae are deduced*, which is essential to distinguish algebraic generalisations from arithmetic ones. The production of a formula in the generalisation of patterns is not necessarily a sign of algebraic thinking (it can be the result of a guess, for instance). The use of algebraic symbolism is not a condition, neither necessary nor sufficient, to algebraic thinking and this conclusion opens new ways to the study of elementary forms of algebraic thinking in young students.

5. STARTING A DIALOGUE BETWEEN THE TKO AND THE ATD

Our proposal to initiate a dialogue between the TKO and the ATD approaches to school algebra is to start from some questions and assertions formulated within the ATD about how the TKO considers algebra, as a way to compare the *reference epistemological models* proposed by each approach. It is only the very beginning of the dialogue, since it has to be complemented with the reciprocal viewpoint: the vision of the ATD investigations from the TKO's own interpretation of algebra. In the ATD, the degree of algebraization is a characteristic assigned to mathematical praxeologies as a whole, not to the types of tasks or mathematical techniques considered separately. This point is certainly an important difference regarding the TKO approach. In our opinion, it is necessary to address it by looking more deeply into the general epistemological model *of mathematics* used by the TKO and the way it can be specified in the case of algebra. What is considered as "thinking" and "activity" (or "modes of reflexion" and "action") in the TKO could be related and contrasted to the notion of "praxeology". The ATD approach agrees with the TKO that a mathematical praxeology can be relatively algebraized without the need to explicitly use algebraic symbolism. A clear example is shown in Bolea, Bosch & Gascón (2001) related to "figurate numbers". Reciprocally, a mathematical praxeology can make extensive use of algebraic symbolism and still remain very poorly algebraized. There are, however, some issues where the points of view differ and need to be considered more deeply. The first one is related to the tasks dealing with equations of the form $Ax + B = D$. In the ATD model, these tasks only occupy a small part of the first stage of algebraization and, if simplification techniques are not required, they can even remain in the non-algebraic stage, needing only arithmetical operations to be solved. The same happens with the tasks dealing with equations of the form $Ax + B = Cx + D$ when the solutions obtained are numerical: they are only a small part of the second stage of algebraization. According to the three-stage model, other kinds of equations-or calculation programmes-should be considered in order to let algebraic techniques appear with all their functionality.The issue of the so-called "analytic character of algebra" that appears to be central in the TKO also plays an important role in the ATD characterization of algebra (see for instance Gascón 1993 and Chevallard 1989), related to the first indicator of the algebraization degree. The question arises about how this "analytic character" can be defined in the TKO when one moves beyond early algebra. Finally, one of the main functionalities of reference epistemological models is to let researchers get free from the institutional vision of the educational facts they are considering, as has been shown by Bosch (2012) in the case of the ATD approach to algebra. It is important, from this point of view, to consider what aspects of this institutional vision of the teaching and learning of algebra are questioned by the TKO's proposal and what others are assumed as valid ones. The dialogue between approaches requires specific work, like the one suggested in this paper, as well as specific tools such as the scrutiny of the epistemological models constructed and assumed by each frame.

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NETWORKING: THEORY AND TEACHING PRACTICE USING LESSON STUDY

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This paper describes the search of six Dutch teachers for the integration of theories to make sense of mathematics. The mathematics teachers investigated their teaching practices using lesson study. Two successive research lessons about the introduction of the derivative were jointly planned, implemented, and live observed. The teachers revised and re-teached the research lessons based on collaborative discussions at school and reflections at the university. The results of the study show that making sense of the derivative starts with encouraging students to communicate intuitively using own words. This is followed by the iconic development of visualizations and finally results in the use of symbols: operations with numbers and reasoning about operations with numbers.

INTRODUCTION

This study focusses on teachers' collaborative investigation to integrate theories of teaching and learning to make sense of mathematics. In 2008 the Dutch government recognized a stagnated progress in numeracy at scientific studies as a consequence of a lack of students' mastering of mathematical skills. This resulted in an increased attention for algorithms and correct calculations at secondary schools. The balance moved from a focus on Skemp's (1976) relational understanding to instrumental understanding of mathematical concepts.

Research at the University of Twente focusses on the effects of teacher design teams. In this context a number of mathematics teachers collaborated with the intention to improve mathematics education. The researcher (first author) invited six mathematics teachers to start a lesson study team. Teacher selection was based on good experiences the researcher had with the teachers during teacher trainee supervision. Lesson study is a professional development strategy in which teachers collaboratively investigate teaching and learning practices by means of live classroom observations and post-lesson discussions (Stepanek, Appel, Leong, Mangan, & Mitchell, 2007). We used lesson study to collaboratively investigate the integration of theories to make sense of mathematics.

Making sense of mathematics arises initially through coherent perception and action, and develops through coherent use of operations in arithmetic and algebra (Tall, 2012). The lesson study team's investigation of making sense of mathematics builds on Skemp's (1976) types of understanding and Tall's (2008) framework of long-term mathematical thinking in relation with Bruner's (1966) representations.

PROBLEM DEFINITION AND RESEARCH QUESTION

Lesson study is in Japan widely used and deeply rooted for over a century. Lesson study makes teaching approaches more practical and understandable to teachers through a deeper understanding of content and student thinking (Murata, 2011). In 2009 a four-year lesson study project was initiated at the University of Twente. The first project year focussed on the effects of lesson study on teachers' professional development. The results showed complexities with regard to culture differences with Japan (Verhoef & Tall, 2011). The second project year showed a positive effect of the use of GeoGebra in the context of the introduction of the derivative. This paper reports the third project year in which the search for the integration of theories to make sense of mathematics was central. Our research question is:

How to make sense of mathematics integrating theories in the context of the introduction of the derivative?

THEORETICAL FRAMEWORK

(a) A sensible approach to mathematics

A sensible approach to mathematics takes account of the structures of mathematics and of the increasing levels of sophistication as learning progresses from sense through perception, then through the relationships of operation and a developing sense of reason (Chin &Tall, 2012). This approach relates to Bruner's (1966) successive modes of representation. Bruner distinguished: (a) action based enactive representation, (b) image based iconic representation, and (c) symbolic representation including not only written and spoken language but also the symbolism of arithmetic and the language of logic. Enactive means gesture, movement of the body, and physical showing of ideas. Iconic, incorporating enactive, is visual and includes all forms of sensory recognition, touch and smell etc. In his framework Tall (2008) puts enactive and iconic together as conceptual embodiment. The enactive and iconic modes of human perception and action develop into the mental world of perceptual and mental thought experiment. Operational symbolism develops from embodied actions, such as counting and measuring, and encapsulates as symbols in arithmetic. The higher level of logic specified by Bruner is seen as a distinct level based on set-theoretic definitions and formal proof.

We suggest that, to make sense of mathematical thinking, the teacher should be aware of the changing needs of the student in new situations, to build on previous success and to realize that what worked before will need a new approach to make sense of the new situation. To do this we consider how the learner makes sense through perception based on fundamental conceptual embodiment and thought experiment, then through the coherent relationships in operational symbolism, and later in terms of reasoning based on definition and deduction. In school mathematics, reasoning develops in various forms, like the transition from the practical tangent $(f(x+\Delta x)-f(x))/\Delta x$ measuring the slope from x to $x+\Delta x$ to the theoretical tangent the ratio of the component of a tangent vector. In this paper we typify sense making of mathematical thinking in terms of *perception*, *operation* and *reasoning.* We distinguish the practical enactive and iconic representations, and the theoretical symbolic representation of the derivative. Perception means the dynamical look along the curve to *see* the changing gradient as the changing curves direction. Operation applies the changing practical slope and the relationship between the visual changing slope and symbolic computation of the slope that stabilizes on the derivative function. Reasoning develops practically (perception and operation) based on experiment, and theoretically based on definition and deduction.

(b) Lesson study as a strategy for professional teacher development

Lesson study can be typified as a *live* research lesson. The live research lesson creates a unique learning opportunity for teaching. Lewis, Perry and Murata (2006) describe three specific areas that develop through the lesson study process: (1) teachers' knowledge, (2) teachers' commitment, and (3) community and learning resources. While teaching is considered an independent and often isolated practice in many countries, lesson study brings teachers together to share goals, discuss ideas, and work collaboratively.

Murata (2011) reports the following five attention points. Firstly, lesson study is centred around teachers' interests. Teachers should perceive lesson study goals to be important and relevant for their own classroom practice. Secondly, lesson study is student focussed. The lesson study activities should direct teachers' attention to student learning and the relation between learning and teaching. Thirdly, lesson study has a research potential. Teachers share physical observation experiences and these provide research opportunities. Fourthly, lesson study is a reflective process. Teachers have to reflect on their teaching practice and subsequent student learning in an educational community. Fifthly, in lesson study teachers work interdependently and collaboratively. Isoda (2010) characterized the lesson study cycle process as consisting of the following collaborative elements: planning (preparation), doing (observation), and seeing (discussion and reflection). He advocated the use of scientific literature as a basis for deepening teaching strategies.

RESEARCH METHOD

Participants

Six mathematics teachers from different secondary schools participated in the lesson study team during the school year 2011-2012, see Table 1. The first three teachers participated in previous years. School management facilitated the teachers by giving them half a day weekly for participating in the lesson study project.

Table 1: Description of participants

	Work experience in 2010	Education and teaching experience
A	17 years	BSc math + MSc math education; lower level to upper level high school students
B	14 years	BSc math + BSc math education; mostly upper level high school students
C	one year	BSc engineering + MSc math education; mostly upper level high school students
D	26 years	MSc math + MSc math education; mathematics teacher team leader
E	19 years	BSc math + MSc math education; lower level to upper level high school students

F	one year	MSc math + BSc math education; a mathematics PhD

Besides the teachers, the lesson study team consisted of four staff members of the University of Twente: a mathematician, a mathematics teacher trainer, a PhD-candidate (second author) and the researcher (first author). The staff members had specific roles in the lesson study team.

Research instruments

The research instruments consisted of three lesson plans, field notes of student observations and written reports of the discussions at the teachers' school, and the plenary reflections at the university. The observers were participants of the lesson study team plus interested school colleagues.

Context of the study

The teachers revised the textbook with regard to the introduction of the derivative with a focus on sense making. They intended to pick up the textbook approach, with a focus on mastering differentiation rules, after the introduction.

Based on last year's experiences with lesson study, the teachers decided to use GeoGebra for sense making of the derivative. The teachers started a process of zooming in at a fixed point on the graph being aware of the rate of change using the visualization (Bruner's enactive representation). They wanted to introduce an icon to develop a link to the use of numbers (Bruner's symbolic representation).

The teachers worked in three pairs (P1, P2 and P3). In each pair, one teacher did not have any previous experience with lesson study. The pairs started successively teaching two lessons. Firstly, P1 started with the first research lesson. P2 continued the same day at another location. Secondly, P1 continued with the second research lesson next day while P2 continued later. The lessons were planned collaboratively, observed and discussed at the teacher's school. The first four lessons (from P1 and P2) were evaluated in a plenary meeting at the university. This resulted in a revision of the research lessons, and this was used in class by P3. This lesson was collaboratively discussed at the teacher's school and plenary evaluated at a university meeting.

Data collection, processing and analysis

P1's lesson plan was summarized. The other lesson plans were described in relation with P1's lesson plan. The field notes of the student observations were classified with regard to Chin and Tall's (2012) categorizations: perception, operation and reasoning. Remarkable (discussion and reflection) report statements were coded as practical (enactive, iconic) or theoretical (symbolic) based on Bruner's (1966) framework of representations. The classifications, codes and analysis were member checked with the teachers afterwards.

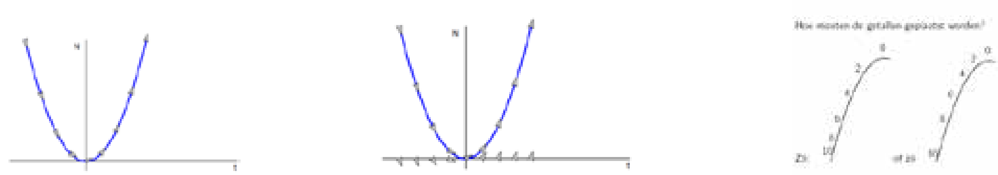
RESULTS

Lesson plans

Below, first the results of the lesson plans will be reported, followed by field notes of student observations, and finally elements from the discussions and plenary reflections. P1’s lesson plan emphasized student interaction. The teachers tried to make sense by activating students’ communication explicitly in their lesson plan (Figure 1).

The teacher introduces the increasing and decreasing graph in comparison with a jumping frog in the first lesson. The Power Point sheet shows the words: increasing/decreasing; monotonic; tangent; slope. The teacher gives each student pair one assignment. One student of each pair, sitting with backs against each other, receives an arbitrary graph on paper. The student describes the given graph in own words, the other student tries to draw the graph on his empty paper.

The teacher continues plenary with the graph of a parabola. He has drawn arrows on the graph (first two figures below). The teacher reminds the students of the computer game Angry Birds, making sense to the graph’s change in one point. The teacher shows the third figure below and asks ‘Do you know the right place of these numbers’?



The teacher continues the second lesson using numbers (slopes) illustrating a change each. He uses squares on his board and puts line segments in here with the comparable slopes. The students get an arbitrary graph on paper each. The teacher asks to put numbers – illustrating a change each - at some fixed points on the delivered graph.
The teacher ends plenary with the calculation of the slope of a straight line through two closed points on the graph, suggesting this gives one answer exactly: the change in one point on the graph.

Figure 1: Lesson plan of the first pair

P2’s lesson plan emphasizes operations with symbols. The teachers replace student interaction with worksheets. They want to reveal students thinking on paper as much as possible. The worksheet focusses on reasoning about numbers as rates of change. Figure 2 lists the core problem.

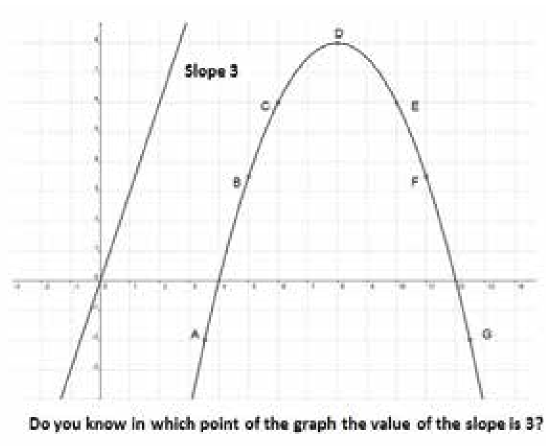


Figure 2: The slope in a point on the graph

The teachers use the problem in which students describe a graph in own words at the end of the lesson instead of in the beginning. The teachers don’t use arrows. They emphasize local straightness in a point nearby the top (more curved). They introduce the zooming in process as an analogy with a view at the earth from space. The teachers continue the procedure of zooming in according to two closed points using GeoGebra suggesting that the process of zooming in gives the same result (one slope). After that they calculate the derivative in different points on the graph and establish that the numbers are elements of one straight line. They end with the ratios Δy/Δx and (f(x+Δx) - f(x)/Δx).

P3's first lesson starts again with a student describing a graph in own words, and continues with the graph of the parabola with arrows, see the first graph in Figure 1.The students solve problems on a worksheet in pairs. The teachers ask the students to add numbers to the arrows on the graph. They continue with asking numbers as rates of change, see Figure 2. They end with asking the right numbers on the right places, see Figure 3.

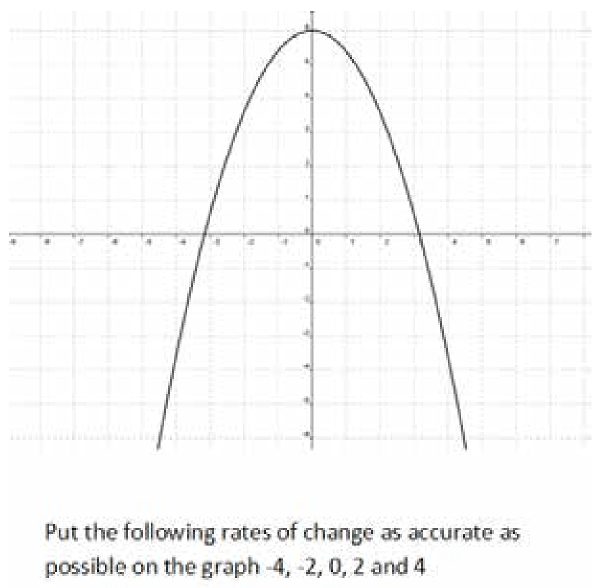


Figure 3: Right number on the right place

Field notes of student observations

Table 2 lists characteristic field notes of the student observations. The first column lists the successive pairs and the successive two lessons. The rest of the columns shows the classifications *perception*, *operation* and *reasoning*. The cells contain characteristic field notes of student observations per pair. The dotted line marks the reflective meeting at the university after P2's lessons.

Table 2: Characteristic field notes of student observations

	Perception Students see:	Operation Students:	Reasoning Students wonder whether:
P1-1	- the words 'increasing/ decreasing; monotonic; tangent; slope' standing on the board	- draw arrows (dove tails) on the graph	- does it involve the ratio on the coordinate system or the rate of increasing?
P1-2	- line segments in a square on the board indicating the change from picture to number	- fold paper and slide with it - calculate the difference quotient	- why numbers are important, if you are able to fold the change?
P2-1	- no data	- draw chords, not the tangent line in a point on the graph - try by calculating - calculate x- and y-coordinates	- make no comments
P2-2	- zooming in with GeoGebra - local straightness - a chord on a small interval - zooming out giving the tangent	- slide the lines, estimate and calculate - calculate the difference quotient	- make no comments
P3-1	- the words 'increasing/ decreasing; monotonic; tangent; slope' are given in advance	- sketch on the worksheet - slide a line with a ruler - draw chords instead of a tangent line - don't mention the word tangent	- do arrows on a straight line be the same ? - do the closer the arrows on the x-axis be, the flatter the graph? - does an arrow on the x-axis show the height of the graph?
P3-2	- zooming in with GeoGebra - local straightness - that Δx becomes smaller	- make a curve straight - slide the lines, estimate and calculate - formulate a straight line	- what happens if A coincides B? - what happens if Δx becomes zero? - does dy/dx not be defined, dx=0?

Perception: the perception moves to the visualization of symbolic representations during the lesson study process. The words 'increasing/ decreasing; monotonic; tangent; slope' in P1's first lesson stimulates students' communication. P1's students start to fold paper spontaneously when the teacher asks numbers – illustrating a change each - at some fixed points on the delivered graph (Figure 1, the last sentence of the last but one paragraph). P2 does not focus on perception in the first lesson.

Operation: the number of student activities increases during the lesson study process.

Reasoning: the intuitive reasoning becomes more important. P2 does not focus on reasoning at all. P1's students are trying to refine the icon 'arrow' (as a dove tail) to a line segment in a square. P2's students are impeded by the incorrect use of the 'tangent line method' - indicating some kind of smoothness of the graph -, learned from the physics teacher. They start with chords on a large interval (to prevent measurement errors as in physics). P3's students start to reason about the icon 'arrow'. They move to reasoning about Δx and dx without any transition. These students have to use symbols to be able to differentiate functions.

Reports from discussions and reflections

Table 3 shows the characteristic teacher comments from the discussions and the reflections based on the student observations. The first column lists the successive pairs and the successive two lessons. The rest of the columns shows the components in types of mathematical thinking: practical (enactive and iconic) and theoretical (symbolic subdivided in local or global). The cells contain characteristic comments from the reports. The dotted line marks the reflective meeting after P2's lessons.

Table 3: Teachers' reflections

	Practical		Theoretical	
	Enactive Reflection on action	Iconic Reflection on visualization	Symbolic (local) Reflection on symbols	Symbolic (global) Reflection on symbols
P1-1	- *no* words increasing/ decreasing; monotonic; tangent; slope in advance	- an arrow with direction refers to a move, preference an 'arrow' without any direction	- students not simply work with numbers after working with arrows	- differentiation globally should develops gradually
P1-2	- maximum and minimum are not a problem	- preference of an arrow without direction	- awareness of coaching to numbers	- support with calculations /difference quotient
P2-1	conflict with physics: derivative means chord	- the use of an icon is necessary	- change the context to a coordinate system	- calculations with a difference quotient
P2-2	- misconception that drawing a chord on a small interval when zooming in, gives a tangent line	- the use of an icon is necessary without giving a direction	- give equations of parabola and line and support students to reason about the slope	- calculations with a difference quotient
P3-1	- cut with a scissors, sew like a sewing machine does and emphasize two sides	- line segment with a dot halfway works best	- avoid stagnation in a chord, continue in coordinate system	- local discontinuous; connect increasing and monotonic increasing
P3-2	- zooming in on paper is not possible - calculation with small Δx takes time!	no data	- students think that a Δx of 0,0001 works exactly!	- there is a lack of numeracy

The teachers prefer to stimulate students' intuitive communication in own words, *not* giving the words 'increasing/ decreasing; monotonic; tangent; slope' in advance (enactive representation). They introduce new ideas like cutting with a scissors, and sewing like a sewing machine does and emphasizing two sides. The icon develops during the lesson study from a dove tail shaped arrow (suggesting a movement), via an arrow with a direction, to a line segment with a dot halfway – the picture of the graph magnified so highly that it looks like a straight line. For the students the transition to the use of numbers is impeded by a too large gap between perception (practical) and calculation (theoretical) by themselves. For the teachers the idea grows to introduce a coordinate system to insert both a line segment related to the slope value and a graph in relation with its equation. The teachers agree to stimulate communication with the students regarding three possibilities to calculate the slope at an interval: the interval [A-x, A+x], the interval [A, A+x] and the interval [A-x, x]. The discussions focus on a too large or a too small number. Another possibility is the use of counter examples like the relation with the graph of y=abs(x).

CONCLUSIONS AND DISCUSSION

The study shows that the use of an icon influences operational symbolism positively when the icon is chosen practically incorporating enaction and visualizing including sensory recognition, touch, smell etc. The 'dove tail icon' at the graph, seems to hide a line segment inside from the top to the bottom of the arrow, which may give rise to the idea that the concept of the derivative is inseparable from a difference quotient. Subsequently, the difference quotient gives rise to the differential quotient with which dividing by zero appears as an obstacle. The 'arrow icon' as a line segment and a v-sign on top, may give rise to the assumption that there is a continuous move because the direction is given and it resembles a vector used in physics. The 'line segment icon' with halfway a dot, gives rise to Skemp's (1976) relational understanding of the concept of a vector field as a basis for understanding differential equations in a later phase (Figure 4).

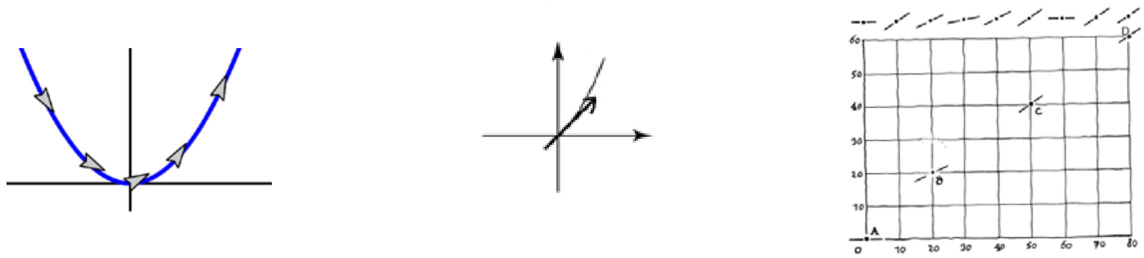


Figure 4: The development of an icon

The experiences with the development of an icon as being useful for sense making of mathematics were based on the student observation, discussion and reflection. Essentially, lesson study focused on classroom practices. The discussions after class and the plenary reflections contributed to teachers' own relational understanding of the derivative. The different theories encouraged the teachers to re-think and to refine the lessons in spite their textbook approach. Teachers' epistemological perspectives changed by the networking of theories (Oshimaa, Horinoa, Oshimab, Yamamotoc, Inagakid, Takenakae, Yamaguchif, Murayamaa, & Nakayamaf, 2006).

The Dutch textbooks are strongly influenced by Freudenthal's (1984) philosophy of mathematics as an activity, the principle of re-invention. Teachers' plenary reflections with regard to the 'line segment icon' with a dot halfway in a vector field gave rise to the derivative related to modelling of changing processes using differential equations. Being aware of this, the teachers made sense by thinking on the introduction of the derivative as a phenomenon. The teachers discussed the possibility to introduce the derivative on a curved area whereby student activities focused on describing the changing curved area. The teachers indicated students' instrumental understanding of a differential equation. They argued that students' tend to solve differential equations algorithmically and that students were unable to set up a differential equation, not being aware of the derivative as a rate of change necessarily for setting up a differential equation (Verhoef, Zwarteveen, Van Joolingen, & Pieters, 2013).

This study indicates that networking of theories - Skemp's (1976) types of understanding and Tall's (2012) framework of long-term mathematical thinking integrating by Bruner's (1966) representations – refines teaching practices. Lesson study stimulates the process of refining teaching practices.

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WORKERS' COMMUNITIES AS POTENTIAL INSTITUTIONS: A CONVERGENCE ISSUE TO ANTHROPOLOGICAL AND SOCIOCULTURAL COGNITIVE THEORIES

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This poster introduces a new presentation of the praxeological model within the anthropological theory of the didactic (ATD) and intends to show that this new model favours connections with cognitive theories inspired by the Activity Theory.

General context: the epistemological anthropology

What are the processes by which individual or local findings turn to be shared and recognized within an institution and then spread into other institutions? How do these social resources get transformed while circulating from their emergence institution to others? These are the issues that epistemology tackles.

The original notion of praxeology

ATD provides this epistemology with what is intended to be a general model for all human activities as well as for the resources to achieve these activities. This is the praxeological model [T, τ, θ, Θ] (Chevallard, 1999). In the case of mathematics, the common use of this model limits θ to theoretically proved mathematical results. Such a restricted conception fails to take into account the totality of what is known about mathematical techniques in the institutions where they are employed, whether in mathematics research, mathematics education or in any professional context.

A reorganisation of the praxeological model

Castela (2011) proposes the following presentation of the praxeological model of the resources institutionally available in a professional institution *Ip*:

$$\left[T^*, \tau^*, \begin{matrix} \theta^{*r}, \Theta^* \\ \theta^{p} \end{matrix} \right] \begin{matrix} \leftarrow I_r^* \begin{matrix} \swarrow Ir \\ \nwarrow Ip \end{matrix} \\ \leftarrow Ip \end{matrix}$$

This model distinguishes the category of research institutions (*Ir*) and supposes that one of them has produced a praxeology [T, τ, θ^r, Θ] (Πr). *Ir* specificity is that their social function includes developing praxeologies for other institutions, basing the legitimacy of the different components on a systematic process of validation which depends on the institutional paradigm. In *Ip*, some subjects carry out types of tasks deriving from T, Πr is known but the professional use has a transposition effect on its different components. It is assumed *a priori* that the transposed praxeology [T^*, τ^*, θ^{*r}, Θ^*] is submitted to a new legitimating process ($\leftarrow$) implicating both concerned institutions. Besides, the model considers that *Ip* contributes to the technological development, producing a practical work oriented technology θ^p, empirically validated within the professional activities. Yet the above representation does not make apparent the net of various sized institutions implicated in the praxeological institutional life, both in *Ir* and *Ip*. In the workplace, local institutions contribute to *Ip* cognitive development, by completing praxeologies coming from upper levels or by being the source of a new invention process initiated.

Possible connections with research inspired by the Theory of Activity

One French school of work psychology referring to the Theory of Activity is considered here, it has developed an interventionist approach, the Clinic of Activity (Clot & Kostulski, 2011). One major specificity of this school is to consider that the development process in working contexts produces socially shared resources, the "professional genre", variety of recommended ways of doing and ways of telling, which at the same time frames and sustains individual and collective activity. The genre retains the profession historical memory, it is also a living capital submitted to constant maintenance efforts to adapt to the changing working conditions. Producing and maintaining the professional genre is considered as a matter of the "transpersonal dimension of work" which, being socially oriented, goes past simple interpersonal interactions within local working communities. The Clinic of Activity group is commissioned to intervene on working contexts in crisis; its experience is that the elimination of the transpersonal is generally at the core of such crisis.

Interest of such a connection: asking new questions

These notions of genre and of transpersonal dimension of work address the epistemological anthropology with questions such as: under which conditions does a working collective turn into an institution producing and legitimating praxeologies renewing the genre? Under which conditions does this development turn itself towards the transpersonal perspective and contribute to the profession's cognitive productivity? Conversely these issues may be returned to the Communities of Inquiry Theory and other collaborative research perspectives, especially if their intention is to contribute to the development of the mathematics teacher profession.

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THE TRIAD OF PIAGET AND GARCIA, FAIRY TALES AND LEARNING TRAJECTORIES.

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The paper formulates a conjecture concerning the generality of learning trajectories based on the unusual similarity between observed elementary trajectories of learning in mathematics and learning progressions identified in a class of fairy tales. Both cases are explicated with the help of the Triad of Piaget and Garcia, (1989), clarifying their structural and interpretative similarity. The "learning triple" thus identified is conjectured to be one of the simplest progressions of learning, which can underline the general structure of learning trajectories in mathematics

RATIONALE

The central question of the presentation can be formulated as follows: *is there a "general strategy of human learning" that, already extant in fairy tales and their stories of heroes and heroines, could be drawn upon to inform learning in the mathematical sphere?* The presentation seeks to uncover the relationship between the Theory of Discovery method of teaching based on the work by Dewey, Piaget and Vygotsky and the structural approach to the analysis of fairy tales represented by Propp, Levi Strauss and Dundes. The methodology of comparison is the structural analysis of the small teaching sequences designed in the context of Discovery method with the structure of progressions of a hero's (or heroine's) struggle to succeed in the Heroic Quest class of fairy tales. The methodology of this investigation follows the doorsteps of the methodology used by Piaget and Garcia (1987) in formulating the Triad—that is a mechanism of thinking through which a schema of a mathematics concept can be constructed. The central idea of Piaget and Garcia's work is to demonstrate that "the mechanisms mediating transitions from one historical period [in the development of scientific concepts] to the next are analogous to those mediating the transitions from one psychogenetic stage to the next… That this dialectical Triad can be found in all domains and all levels of development seems to us to constitute the principal result of our comparative efforts." (*op. cit.* p. 28). A series of papers utilizing the Triad in explicating the construction of basic concepts of calculus such as chain rule or the limit had been published by a group RUMEC (e.g. Clark et al 1997; Cottril et al 1996). The presented approach is similar in that we are looking for a common mechanism to account for (1) learning progressions in mathematics identified during the Discovery method of teaching and (2) progressions of an adventure hero's (or heroine) struggle to succeed in a class of fairy tales. We demonstrate that the common mechanism which can account for both progressions is "a triple"—a contraction of the Triad. It is reminiscent of the term "trebling" in the scholarship of fairy tales, which refers to the pattern of three consecutive events occurring in a succession and structuring some fairy tales (Propp, 1975). The structural analysis of the trebling did not advance very much till the present day. Dundes (1961) analysis of Lithuanian fairy tales focuses on binary opposition between the triples rather than on their inner structure. Croft (2005) shows that the “triplicity” of “trebling” extends to the Russian culture at large, following the call of Levi-Strauss (1955) to extend the paradigmatic structural analysis of the myth to the world at large. This presentation proceeds along the path outlined by Levi-Strauss in relating the structure of fairy tales to “other aspects of culture” such as mathematics education.

METHODOLOGY

1. Discovery of the structural similarity between the mathematics classroom dialogs encountered during the teaching by the Discovery method and the adventure paths of heroes and heroines in the class of fairy tales.
2. Contraction of Piaget and Garcia’s Triad reduces it to the triple in the case there are only two different individual cases or manifestations.
3. Showing that Piaget and Garcia’s Triad correctly interprets the encountered sequence of triples in the mathematics classroom dialogs.
4. Showing that in the Fairy tales with Triples, triples can be explicated with the help of the Piaget and Garcia’s Triad.

CONCLUSIONS

Since the structure of the triples and their meaning in each of the two so different domains, mathematics education and wisdom of the folklore, is similar, there may be a structure underlying both, and this structure is, I am conjecturing, an element of the “general strategy of human learning”. Rates teaching sequence appears as an example of a Learning Trajectory construction based on the triples.

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INSTRUCTIONAL DESIGN TOOLS BASED ON THE ONTO-SEMIOTIC APPROACH TO MATHEMATICAL AND DIDACTICAL KNOWLEDGE

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Key words: instructional design, mathematics instruction, onto-semiotic approach

We present a set of diagrams in which we summarize the theoretical tools developed under the "Onto-semiotic approach" (Godino, Batanero and Font, 2007) (OSA) in mathematics education, from the perspective of the instructional design. We assume that the focus for the didactic-mathematical analysis should be the mathematical instruction processes, which involve the teacher and students working in a specific mathematical content, within an educational context, and with specific technological tools. We consider three analysis dimensions along the design and evaluation of a mathematics instruction process: phases, facets, and levels of analysis, for each of which the OSA provides specific theoretical tools. The potential of using a base-theory in design-based research (Kelly, Lesh & Baek, 2008) for mathematics education is also highlighted. The system of notions that constitute the mentioned base-theory can help not only to describe the processes and explain the educational phenomena, (goal of a scientific discipline), but also to develop instructional research-based resources (technological design component).

Within the OSA we have built a system of theoretical notions useful for the design and didactic analysis of mathematics instructional process (Godino, Contreras & Font, 2006). We distinguish three planes or dimensions of analysis: (a) The design phases dimension, that includes the preliminary, design, implementation and evaluation phases; (b) The facets dimension, consisting of the epistemic, ecological, cognitive, affective, interactional and mediational facets; and (c) the levels of analysis, where we consider the levels of practices (pragmatic meanings), configurations of mathematical objects and processes, configurations of objects and didactic processes, norms and didactical suitability.

As suggested by D'Amore and Godino (2006), the OSA approaches the didactic-mathematical problems from the epistemic and ecological dimensions. Firstly we problematize the nature of the institutional mathematical knowledge, in the same way that it is done by the Theory of Didactic Situations (Brousseau, 1997) and the Anthropological Theory of Didactics (Chevallard, 1992), two theories which are considered as the OSA starting points. The anthropological assumption about the nature of mathematical objects -conceived as emerging from mathematical practices-, is shared with the TAD, and is complemented by the notion of configuration of objects and processes, which allows a detailed epistemic and cognitive analysis of the mathematical knowledge involved in the planning and implementation of an instructional process. Moreover, the notions of didactic configuration and didactic trajectory (Godino, Contreras & Font, 2006), normative dimension (Godino, Font, Wilhelmi & Castro, 2009), didactical suitability (Godino, Batanero & Font, 2007) are new tools used in the detailed analysis of didactic objects and processes, the norms conditioning and enabling the teaching and learning processes, and to assess the relevance and adequacy of the various design components and decisions made.

Acknowledgment

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INTRODUCTION TO THE PAPERS AND POSTERS OF WG17: FROM A STUDY OF TEACHING PRACTICES TO ISSUES IN TEACHER EDUCATION

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INTRODUCTION

This working group addresses mathematics teachers and teaching, an area that has been given particular attention in mathematics education research for the last decades. Current research has started to consider theoretical and methodological frameworks that can capture the complex relationship between mathematics learning, teaching, teacher practices, and mathematics teacher education. This attempt was apparent in many papers submitted to the group and in the discussions that took place. Research in this area is exploratory and interventional with a main focus on teacher knowledge and its development while most studies adopt qualitative methodological approaches. Professional development and teacher education approaches are often based on collaboration and promote reflection.

Group 17 received 69 proposals (54 papers and 15 posters), which involved authors from 22 different nationalities. Each paper was reviewed by one of the group leaders and two authors. For most proposals we asked for some revisions. In the sessions of the working group during the conference, 44 papers and 11 posters were presented; 40 papers and 11 poster proposals are included in the conference proceedings.

Because of the large number of papers presented, the working group split into two subgroups (WG17A and WG17B). The group met as a whole for only the first part of session one and for the last session. All the papers were grouped into five thematic areas and distributed to the two subgroups. All the participants were informed in advance of the paper distribution in the two subgroups.

All participants of WG17 were expected to read papers previously to the session in which they were presented. In each session, three, four or five authors sketched the key ideas of their report (5 minutes each). One of the group leaders or participants then gave a prepared reaction to the set of papers (up to 10 minutes). In most cases, the reactor attempted to make links between the papers and suggested issues emerging from the papers that might form the basis for discussion.

In general, the work in the group seemed to be productive and most authors felt that they got useful comments to develop further their paper.

Thematic areas

We present the issues and ideas that emerged in reference to the five central themes.

Studying mathematics teaching

The complexity of studying mathematics teaching is overtly addressed in the six papers categorised in this thematic area. In some the focus is on detailed investigations of communication, for instance the process of questioning or orchestrating mathematical discussions in the classroom. Others discuss the design of an analytical tool to explore and characterise the complexity of mathematical classrooms, or teachers' actions during introductory activities to algebraic modelling. A broader perspective is taken for inquiring in the role played by the teacher in terms of patterns of participation or investigating the challenges teachers and students meet when introducing technology in teaching practice.

These articles all seek to conceptualise teacher-student interactions, even though the chosen theoretical perspectives differ. Several theoretical frames address the complexity of mathematical discourse where knowledge is viewed as co-constructed during teacher-students interactions in the classroom. A different approach is needed to conceptualise the role played by the teacher during teaching practice where the idea of "Patterns of Participation" enables the authors to analyse teaching practices as a dynamical process. The introduction of technology requires the elaboration of a framework where combination of Activity Theory and instrumental orchestration are central elements.

From a methodological perspective, all papers in this thematic area follow an interpretative paradigm, conducting research through case-studies focusing on in-service teachers and based on classroom observations and interviews.

Taking an overall perspective it is possible to identify the following three emerging issues: *Communication in mathematics classroom* is one of the main foci with emphasis on teachers' orchestration of mathematical discourse, on the practice of arguing and on good questioning. Another central issue emerging from these six papers concerns *mathematical content of teachers' teaching practice*, that is to what extent the mathematical content is brought to the fore during a lesson, and how this relates to the potentials and limitations of teachers' knowledge. The last issue addresses *reflection on and development of teachers' teaching practice* with emphasis on the dynamical nature of classroom interaction. These issues are discussed both from an in-service and pre-service teachers' perspective. Common to all these papers is recognition of the complexity and richness of teachers' teaching practice while developing theoretical and analytical tools as a means to conceptualise it further.

Resources for Teaching: Teacher knowledge and Teacher beliefs

A number of papers categorized in this thematic area were discussed in WG17A while others in WG17B.

The contributions discussed in WG17A (11 papers) were diverse, taking into account the teaching level they refer to (primary or secondary), the mathematical contents they attend to (e.g., fractions, functions, algebra, divisibility, and geometry), and the expertise of the teachers involved (from prospective to seasoned teachers). This diversity notwithstanding, all the contributions revolve around one or more of three key concepts: teacher knowledge, teacher beliefs, and teacher/teaching practice. For example, some papers focus on the link between teacher knowledge and teaching practice, others attend to the connection between beliefs and practice, while still others explore the interactions between beliefs and knowledge or even teachers' beliefs about the knowledge needed to teach effectively. In what follows, we discuss four central issues we see to collectively emerge from the contributions of this subgroup: (a) *Explorations situated in the context of teaching practice:* Almost all contributions discussed examine teachers' knowledge and/or beliefs as made explicit in some context of teaching practice. This can be the discussion of a real classroom episode, different real students' answers to a common problem, or working with and unpacking students' ideas. In doing so, collectively these contributions suggest that teacher knowledge and beliefs should be analyzed in the context of teaching practice. We consider this an important step that research has made over the past few decades, and one which has been fuelled by increasing realizations that knowledge and beliefs are inseparable from practice. (b) *Tendency to gain information about how different resources relate, more than a partial understanding of one of them:* Several of the contributions in this subgroup investigate interactions between teacher knowledge and practice or beliefs and practice; fewer are the investigations that explore interactions between knowledge and beliefs, and much scarcer are those that are situated at the nexus of all three components: knowledge, beliefs, and practice. Based on the promising findings of these contributions, we believe that working at the intersection of these components offers a very promising perspective for future research. (c) *Use of multiple theoretical and analytic frameworks and methodological approaches*: The contributions of this session capitalize on multiple theoretical and analytic frameworks, including, for instance, Shulman's framework, the *Mathematical Knowledge for Teaching*, the *Mathematical Pertinence of Teachers' Actions,* and the *Structure of the Milieu* frameworks—just to name a few related to investigating teacher knowledge. Given the complexity of the phenomenon under investigation—we believe that this diversity can help better understand the phenomenon at hand. However, as the number of these frameworks increases, we think that scholarly efforts should also be channeled in bringing together these frameworks, exploring if and how they talk to and complement each other, examining their affordances and limitations, and investigating how combinations of frameworks can prove helpful. Along the same lines, different methodological approaches pursued in the contributions of this session, ranging from qualitative case studies to quantitative experimental designs. (d) *Attention to the role of teacher preparation programs and in-service professional development programs*: Some of the contributions of this session report on research made in the context of teacher education programs for prospective or practicing teachers. We see a lot of value in this research strand and we encourage more systematic reflection on the role of teacher education programs in the development of teachers' beliefs, knowledge, and practice; at the same time, future works could invest more systematically in reflecting on how research results could inform the design of teaching programs.

The four issues just discussed are indicative of the multiple directions in which we see research evolving in the next decades. Collectively, these issues underline the multiple opportunities that lie before us as well as the various challenges that future research will face attempting to better unpack and understand the knowledge-beliefs-practice conundrum.

The contributions discussed in WG17B (10 papers) also adopt different perspectives on the way they use knowledge models. The importance to establish a general understanding on what meanings entail the content knowledge and how it affects the practice of teaching was highlighted both in the papers and in the group discussion. Moreover, models that allow conducting a more precise analysis of each knowledge component related to effective mathematics teaching were discussed. Mathematics knowledge for teaching (MKT) (Ball, Thames and Phelps 2008) as the mathematical knowledge needed to carry out the work of teaching mathematics was elaborated. It was stated that subdomains of the MKT model are not composed of mathematical knowledge only. As processes used in the creation of this knowledge were characterized (a) decompressing (working from a more compressed understanding of mathematics to a more unsophisticated form), (b) trimming (teachers present an advanced or sophisticated mathematical idea to students in a way that the fundamental nature of the topic is preserved but it is less rigorous), (c) bridging: making connections between mathematical topics or between mathematics and other subject areas.

From amongst the most recent conceptualizations of teachers' knowledge in the course of mathematics education, the MKT model and especially the development of the components of CCK and SMK, HCK (common content knowledge, specialized content knowledge and horizon content knowledge, correspondigly) were discussed. More precise clarification of the concepts and their deeper analysis was requested. Questions as: How could different types of knowledge (especially HCK) be promoted in pre- and in-service teacher education? Which research method should/could be use when analysing them? What is the evidence? Should the teachers be aware? More general questions like "what is the purpose of having a knowledge model?" were also addressed.

Teacher Education and Professional Development

Nine paper were discussed in this thematic area. The papers' topics comprise issues like curriculum development, the enhancement of mathematics teacher education practices through research, the integration of mathematics and science in teacher education, different approaches for professional development, or the sustainability of the impact of professional development programmes. In particular, some papers deal with teacher knowledge and students' work, some other highlight teacher education and classroom issues. Besides these empirical studies, some further papers deal exclusively with theoretical models and perspectives, and try to link theories or knowledge from other disciplines with mathematics teacher education.

The empirical studies' participants comprise both in-service and pre-service teachers, from primary and secondary level. The papers related to this thematic area use various theoretical frameworks, ranging from teacher knowledge theories, over theories of didactics, up to theories of learning. The methods used in this thematic area's papers are manifold; for example: questionnaires, document analyses, pre- and post-tests, teaching experiments, exploratory studies, or literature reviews.

A significant discussion in the working group concerns the question, whether and which theories are needed to analyse the impact of teachers' professional development and its sustainability. Another emerging issue is teacher educators' responsibilities regarding their research and its impact on improving mathematics teaching and mathematics teacher education. Yet another emerging question is how to become aware of unexpected impact of research results.

Teacher Collaboration

In the last decades, collaboration between teachers and academic researchers or research –practice oriented tasks has become the focus of research. In other words, in the study of mathematics teaching and teacher professional development, collaboration may be considered as a means for professional development, as a means to study teachers' innovative practices or even as a research focus. However, in WG17, collaboration appears in a very few papers (3) and never as an object of research.

Speaking about collaboration means to consider a joint work in order to provide mutual support and the achievement of goals, not necessarily the same for each member of the group. It is also assumed that the participants may have different roles, but necessarily contributing for a shared general goal that gives meaning to the collaborative work. Participants must be attentive to the needs of others and be open to negotiate understandings emerging from the collaborative effort. A confident environment is an essential condition for the development of collaboration. But this condition may raise some difficulties to the collaborative work between participants, in particular in the case of teachers and academic researchers, because they don't have, at the starting point, equal social status. What strategies, settings and content can we design to promote a collaboration between teachers and academic researchers in order to achieve a real collaborative work?

Using a collaboration setting for studying innovative practices seems, on the one hand a promising strategy, and on the other hand, a challenge for teachers and academic researchers. Successful situations have been presented in the papers. In a collaborative work, teachers and academic researchers were able to design a teaching intervention where innovation was included and studied. Collaborative practices were important for teachers during their teacher education and afterwards when they are all team teaching and collaborating with colleagues. But collaborative work is time consuming for all participants and needs time for reaching results. This is a challenging issue for us as to consider.

Teacher Reflection

Reflection was made the focus of the research in few papers. Three papers were categorized in this thematic area, but – while only two appear in the proceedings and are discussed here. Both papers focus on how to promote prospective teachers' reflection. One paper sees reflection to be promoted in situations of tensions and studied prospective teachers' reflective process on learning and teaching mathematics. In particular, emphasis was given on the nature of tasks that can promote reflection and a model of analysis with different dimensions of reflection was used to study prospective teachers' reflection. The other paper focus on how the prospective teachers reflect on the process of division with remainder through role-play. The paper describes how role- play has been used in the process of actualizing teacher –pupils-like interactions. The results indicate that prospective teachers through this reflective experience develop mathematical and pedagogical understandings. Issues that seem to emerge are: whether reflection requires always situations of uncertainty; how reflection is conceived in more social perspectives and what kind of research questions and methodologies such perspective will support.

Critical issues

We would like to close our introduction by including some critical issues that the participants of the two subgroups expressed during the closing session.

Critical issues addressed in WG17A

- We are working on multiple frameworks. What are the advantages and disadvantages of doing so?
- It is not clear to what extend our frameworks and results take into account the character of mathematics, especially for theoretical "imports". How can we achieve this?
- The role of the "Context" is not always sufficiently clear. How can we become better at it?
- The deficit model of teacher is still alive in our community. Are we aware of the consequences?

Critical issues addressed in WG17B

- What could be considered as a suitable model for teacher knowledge?
- What could be the purpose for developing new theoretical models or for modifying/revising the existing ones?
- How can teacher educators help students/teachers to develop different components of their knowledge? Are they different for prospective teachers´ education and in-service teachers?
- What are the ways of promoting teacher knowledge? Should be the teachers aware of different theoretical models?
- How can we analyse the influence (occurrence) of different types of knowledge?
- How can we study the mutual relation between teachers´ knowledge and practice?
- What is the evidence? Which research method should we use?

PAPERS

Studying Mathematics Teaching

Aizikovitsh-Udi , E, Clarke, D. & Star, J. *Good questions or good questioning: an essential issue for effective teaching.*

Badillo, E., Figueiras, L., Font, V. & Martinez, M. *Visualizing and comparing teachers' mathematical practices.*

Cusi, A. & Malara, N.A. *A theoretical construct to analyze the teacher's role during introductory activities to algebraic modeling.*

Drageset, O. G. *Using redirecting, progressing and focusing actions to characterize teachers' practice.*

Larsen, D. M. , Ostergaard, C. H. & Skott, J. *Patterns of participation-A framework for understanding the role of the teachers for classroom practice.*

Mali, A., Biza, I., Kaskadamis, M., Potari, D. & Sakonidis, Ch. *Integrating technology into teaching: new challenges for the classroom mathematical meaning construction.*

Resources for teaching: Teacher knowledge and Teacher beliefs

Berg, C. V. *Enchancing mathematics student teachers' content knowledge: Conversion between semiotic representations*.

Braich, P. & Mgombelo, J. *Exploring teacher perceptions of the development of resilience in a problem-solving mathematics classroom.*

Carreño, E, Ribeiro, M. & Climent, N. *Specialized and horizon content knowledge – Discussing prospective teachers knowledge on polygons.*

Carreño, E., Rojas, N., Montes, M. & Flores, P. *Mathematics Teacher's specialized knowledge. Reflections based on specific descriptors of knowledge.*

Carrillo, J., Climent, N. Contreras, L. C. & Muñoz-Catalán , M. C. *Determining specialized knowledge for mathematics teaching.*

Clivaz, S. *Teaching multidigit multiplication: combining multiple frameworks to analyse a class episode*.

Cooper, J. & Touitou, I. *Teachers' beliefs about knowledge for teaching- an indirect approach.*

Dreher, A. & Kuntze, S. *Pre-service and in-service teachers' views on the learning potential of tasks-does specific content knowledge matter?*

Ebbelind, A. *Disentangling a student teacher's participation during teacher education.*

Flores, E. Escudero, D. L. & Carrillo, J. *A theoretical review of specialized content knowledge.*

Jakobsen, A. Thames, M.H. & Ribeiro, M. *Delineating issues related to horizon* *content knowledge for mathematics teaching.*

Isik, C. , Ocal, T. & Kar, T.*Analysis of pre-service elementary teachers' pedagogical content knowledge in the context of problem posing.*

Klothou, A. & Sakonidis, H. *Primary teachers' assessment in mathematics: resources exploited in the pedagogical discourse.*

Kuhlemann, S. *Heuristic strategies prospective teachers use in analyzing students' work.*

Montes, M., Aguilar, A., Carrillo, J. & Muñoz-Catalán, M. C. *MTSK: From common and horizon knowledge of topics and structures.*

Pino-Fan, L. R. *Prospective teacher's specialized content knowledge of derivative.*

Raveh, I. & Koichu, B. *A reference framework for teaching the standard algorithms of the four basic arithmetic operations: from theoretical analysis to task design.*

Sayers, J. The influence of early childhood mathematical experiences on teachers' beliefs and practice.

Tomás Ferreira, R. A., Martinho, M. H. & Menezes, L. *The role of didactical knowledge in seizing teachable moments.*

Ubuz, B., Özdil, U, Çevirgen, A. E. *Pre-service teachers' knowledge and beliefs: their association to practice in the context of teaching function with analogies.*

Teacher Education and Professional development

Aslan-Tutak, F. & Ertas, G. *Practices to enhance preservice secondary teachers' specialized content knowledge*

Bas, S. M, Didis, G. A., Erbas, K, Cetinkaya, B., Cakiroglu, E. & Alacacı, C. *Teachers as investigators of students' writing work: Does this approach provide an opportunity for professional development?*

Hošpesová, A. & Tichá, M. *One possible way of training teachers for inquiry based education.*

Rasmussen, K & Winsløw, C. *Didactic codetermination in the creation of an integrated math and science teacher education: the case of mathematics and geography.*

Rowland, T. Turner, F. & Thwaites, A. *Developing mathematics teacher education practice as a consequence of research.*

Santos, F. L. & Domingos, A. *Mathematics in pre-service teacher education and the quality of learning: the proceptual divide.*

Watson, S. *Understanding professional development from the perspective of social learning theory.*

Zehetmeier, S. *What can we learn from other disciplines about the sustainable impact of professional development programmes?*

Zembat, I. Specialized content knowledge of mathematics teachers in UAE context.

Teacher Collaboration

Gunnarsdóttir, G. H. & Pálsdóttir, G. *New teachers' ideas on professional development.*

Guerreirio, A., Ponte, J. P. & Serrazina, L. *Mathematical communication: teachers' recognition of the singularity of students' knowledge*.

Semana, S. & Santos, L. *Teaching practices to enhance students' self-assessment in mathematics: Planning a focused intervention.*

Teacher Reflection

Helmerich, M. *Competence in reflecting- An answer to uncertainty in areas of tension in teaching and learning processes and teachers profession.*

Lajoie, C. & Maheux, J. F. *Richness and complexity of teaching division: Prospective elementary teachers' roleplaying on a division with remainder.*

POSTERS

Andersson, C &Vingsle, V. *Professional development program in formative assessment.*

Caseiro, A. *Statistical knowledge and teaching practices of elementary school teachers in the context of collaborative work.*

Ehrnlund, M. L Mathematics teachers' understanding and interpretation of their own learning and classroom practice.

Godino, J.D. & Pino-Fan, L. R. *The mathematical knowledge for teaching: A view from the Onto-Semiotic Approach to Mathematical Knowledge and Instruction.*

Liljekvist, Y & van Bommel, J. *Mathematical knowledge for teaching as a measure of coherence in instruction materials produced by teachers on the internet.*

McLean, J. A. *Pre-service elementary teachers' procedural knowledge of the greatest common factor and least common multiple.*

Sapkova, A. *The relationships between the traditional beliefs and practice of mathematics teachers and their students' achievements.*

Šunderlík, J &, Čeretková, S. *Usage of tasks during professional development of in-service mathematics teachers.*

Torres, E. G. *Professional mathematics teacher identity: An example with telesecundaria system teachers in Mexico.*

van Smaalen, D. *Teacher learning within the context of Lesson Study.*

Vanegas, Y. M. *Interaction suitability analysis with prospective mathematics teachers.*

GOOD QUESTIONS OR GOOD QUESTIONING: AN ESSENTIAL ISSUE FOR EFFECTIVE TEACHING

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It is the contention of this paper that it is not good questions that are essential for good teaching, rather, it is good questioning. We illustrate the significance of question-asking in mathematics classrooms by presenting the case studies of two teachers teaching the same topic to two different classes. Comparison of the two cases highlights important differences between "good questions" and "good teacher questioning practice." Our analysis suggests that good questions cannot be meaningfully considered or promoted independent of good questioning practice and that this distinction has significant implications for teacher education.

Key words: questioning practice; mathematics education; effective teaching

INTRODUCTION

Teacher questions are viewed as a critical teaching tool by many researchers and educators (e.g. Cunningham, 1987; Dillon, 1988; Ellis, 1993; Morgan & Saxton, 1991; and Martens, 1999). A question is an expression of inquiry that invites or calls for a reply. In a classroom, questions are used by teachers as instructional cues to assess student progress and to motivate student thinking. 'To question well is to teach well' (De Garmo (1911), p. 179 as cited in Wilen (1991), p. 5). It is clear that teacher questioning is universally viewed as a highly important instructional practice. Yet even this apparently unequivocal endorsement of the importance of teaching questioning conceals the essential distinction between "good teacher questioning" as instructional practice and the concept of a "good question" as an instrument of that practice (eg Benedict, Kaur & Clarke, 2007; Clarke & Sullivan, 1992). This distinction and the two constructs of "good question" and "good questioning" provide the theoretical and empirical focus of this paper.

The value of good teacher questioning is particularly endorsed in mathematics education. Yet teacher questioning can take many different forms and serve many commendable purposes. Therefore, within this general recommendation about question asking are many interesting and important questions about how teachers might implement question asking. The purpose of the analysis reported in this paper was to investigate some of the issues associated with the implementation of this important practice. The characteristics of a "good question" or of "good teacher questioning" as a practice can only be identified once the teachers' goals are known. A question intended to provide information on the current state of a student's understanding is likely to take a different form from a question intended to promote student self-regulation of learning and this would be different again from the sort of question that might stimulate engaging and productive whole class discussion.

Many researchers (Bingham, 2005; Black, 2001; Boyer & Piwek, 2010; Hufferd-Ackles et al. 2004, Moberg, 2008; Sigel & Kelley, 1986) have examined the culture of asking questions in class. Although there is consensus on the importance of question-asking, a variety of research has indicated that math teachers are not particularly good at asking questions (and/or at asking good questions). For example, studies have shown that teachers in an average class ask between 12 and 20 questions, yet approximately half of the questions are procedural questions regarding timetable, attendance, clarifying various technical issues, etc. Furthermore, most of the questions are closed questions for testing knowledge (straight recall). Only a very small percentage of questions encourage higher-order thinking. In addition, teachers tend not to allow students opportunities to think about questions; wait time has been found to average 1.2 seconds. Finally, 70% of the students' answers consist of three words and their duration is five seconds or less (eg Nystrand, 1997).

Curricula have been designed to help teachers improve their question posing, but based on the existing research it is not clear how successful these question-asking curricula have been at improving teachers' ability to ask good questions. It is this relationship - between curricula that aim to improve teachers' question asking and teachers' implementation of these curricula - that provided the focus of the analysis reported in this paper. More specifically, does teacher experience in using curricula explicitly focused on improving question asking lead to their regular and independent use of such advanced instructional strategies? Is good questioning a matter of using good questions or of questioning well? Are we talking about tools (good questions) or practice with those tools (good questioning)? The two case studies reported in this paper exemplify this distinction very clearly.

RESEARCH QUESTIONS

This study aims to provide evidence for the following research questions: What constitutes good teacher questioning and what contribution does the provision of “good questions” make to the enhancement of teacher questioning practice? That is, our research investigates the difference between good questions as an instructional tool and good questioning as instructional practice.

METHODS

We explored the question above through two case studies, each of which involved a teacher teaching the same content in a junior high school Algebra I class. The two teachers, Robert and Naomi (fictional names) were experienced algebra instructors (20 and 15 years of math teaching experience, respectively), who participated in a larger year-long project from which the present data are drawn. The data collection central to the analysis reported in this paper consisted of classroom observations and interviews with each of the teachers. We have analyzed two types of lessons taught by these two teachers. The first type was a regular lesson according to the curriculum adopted at the teachers' school, while the second type was a lesson provided to the teachers by the researchers, which was designed to promote the skill of question-asking. Both teachers participated in a one-week professional development institute in the summer prior to the data collection year. The institute focused on the use of the supplemental question-asking curriculum materials and was intended to facilitate the teachers' implementation of particular questioning strategies.

The advocated instructional approach focused on stimulating student reflection and discussion about “contrasting cases” of pairs of student responses to the same task. The questions provided to teachers emphasised the comparison of the students’ strategies, such as: “How do you know which answer is correct?” and “How similar or different are these two strategies?” and “When does the difference between using one or the other strategy matter?” A “discussion phases guide” offered teachers a three-phase approach to their questioning, providing sample questions that sought to *understand* the pair of student responses, *compare* the student responses and *make connections* between the two responses. Very specific sample questions were provided for each phase (as in the three examples above). In this study, the questions were intended to stimulate student (and teacher) reflection about two students’ strategies in solving a particular mathematical task. Potentially, the questions could scaffold student reflection on alternative mathematical methods (consistent with Holton and Clarke, 2006), and the guide provide a structure for teacher questioning.

RESULTS

The findings presented here examine the practices and the character of interactions in the classes of the two teachers. We use these two teachers' cases to help illuminate important and unexplored issues in the implementation of question asking: the teacher’s role in asking questions, the character of questions directed at students, and the way the questions are asked.

The Role of the Teacher

There were differences in the attitude of teachers to their role as “questioners.” One teacher, Naomi, defined her teaching style as "direct teaching." She delivered each topic stage by stage, being responsible for every stage (e.g. choosing every step in solving a problem). She provided many instructions during the process of solution: how to solve equations, perform calculations, work with models, when to contract, etc. Her teaching was very detailed and precise and was accompanied by oral and written explanations. Naomi solved each equation completely and did not skip stages in problem simplification, including detailed substitutions and making all necessary calculations. Such teaching as Naomi’s is well-described in literature about traditional teaching (Aizikovitsh-Udi & Star, 2011; Metz, 1978; Chazan, 2000). In contrast to Naomi, who saw herself as responsible for giving the suitable formula with explanations to the students, Robert preferred to allow the students to be the primary agents in finding solutions, while only assisting them to do so. Robert consistently applied methods that appear in recent educational literature as “innovative.” For instance, he encouraged reliance on intuition and gave a minimum of laws and rules. Also he did not function as an authority for deciding whether an answer is correct (Cazden, 1988). The practices of these two teachers embody the alternatives discussed by Lobato, Clarke and Ellis (2005) in their reformulation of “teacher telling” as the strategic alternation of initiation and elicitation.

Naomi

Naomi taught her class according to traditional patterns of teaching (Aizikovitsh-Udi & Star, 2011; Bauersfeld, 1988; Voigt, 1989). In this tradition, the place of the teacher in the teaching process is central. She gave ample explanations and instructions directly to the students while solving problems. In Naomi’s practice, the teacher served as a source of knowledge and was responsible for establishing correctness or incorrectness of answers. The teacher dominated the discourse in class, posing multiple questions but answering most of them herself. She did not encourage discussion. Most of the questions were focused on mathematical content and were intended to obtain information and evaluate answers rather than attempting to understand the individual student’s way of thinking. The students' answers to the teacher’s questions were very brief, and their own questions aimed at clarifying points they did not understand rather than furthering their investigations.

Robert

Robert taught his class in a method that was characterized by features of the contemporary “reform agenda” (Darling-Hammond, 1996; NCTM, 2000). While Naomi’s place in teaching was central, Robert’s place was central as well, but with different emphases: he guided the students, gave few rules and laws and allowed the students to choose different ways for solving a particular problem.

Robert did not answer his own questions, but waited for students to respond. In cases when the answer was slow to arrive, he repeated the question in different ways. He encouraged discussion in class, which can be seen from the encouragement he gave to students who replied, from his repeated questioning of students who could not reply, and from the fact that students asked investigative questions themselves. The purpose of the teacher’s questions was not only to evaluate knowledge of mathematical content but also to understand how the students think. Davis (1997) called this mode of listening to answers “interpretive listening.” In this practice, the teacher does not function as the authority for establishing the correctness or incorrectness of the solution or for correcting the solution, but directs students by means of questions to correct their mistakes. Such a pattern has been termed by Wood (1998) “the focused pattern” and contrasted with the more directive and convergent “funnelling pattern.” The two teachers offered a remarkable contrast: each employing instructional practices that have come to be identified with the stereotypes of “conventional” and “reform” teaching. In the following discussion, we juxtapose their actual practices in order to facilitate reflection on the role of questioning in both instructional models.

DISCUSSION

Although both teachers were considered good teachers and lesson structure was the same in both classes (an important point), the practice and “culture of question asking” of the two teachers were quite different and were shaped and applied differently. This has to do with the differences in the patterns of the teachers’ discourse. For instance, the pattern of Naomi’s discourse can best be described from the literature as the “funnel” pattern (where the teacher directs the students by means of questions toward the expected answer), while the pattern of Robert’s discourse is similar to the “focused” pattern (where the teachers leaves the responsibility for arriving at the solution to the students, while helping them to focus on the important aspects of the problem), as described by Wood (1998). Also the ways of listening of the two teachers were different, where Naomi predominantly exercised evaluative listening, while Robert applied both evaluative and "interpretive" listening (Davis, 1997). While most questions by both teachers were concerned with mathematical content, Naomi's questions were not directed personally to the students, that is, the purpose of the question was to receive a mathematical answer and not to emphasize the individual student (we have termed this type of question "technical"). By contrast, most of the questions in Robert’s classes required more explanation and argumentation than those in Naomi’s classes (we have termed this type of question "investigative").

In evaluating the teachers’ implementation of the practices advocated in the question-asking institute, we observed that both Naomi and Robert essentially preserved their teaching styles both in the supplemental question-asking portions of their classes and in the regular classes that adhered to their regular curriculum. Thus, in order to foster and encourage question asking in class, it appears that it is not enough to provide the questions to the teachers. The way in which the questions are asked, the timing and the number of times each question is asked have a central role in the culture of question-asking. In other words, even investigative questions can be asked in a technical way and consequently not give rise to any significant process of investigation. In particular, in relation to the professional development program that provided a key research site for this study, our results suggest that in order to change traditional teaching styles, it is not enough to give the teacher a small, narrowly focused exemplary learning unit (in this case, the supplemental questioning materials), even if, like Robert and Naomi, the teacher has previously taken a course in implementing related innovative teaching methods.

The initial evidence shows that teachers such as Robert, who have already assimilated some of the practices of the reform agenda into their teaching, are more likely to incorporate advanced instructional strategies into their practice than more conservative teachers, such as Naomi, whose pedagogical practice mirrors a personal commitment to stability and the inviolability (non-negotiability) of mathematical knowledge. However, the exposure of both teachers to the question asking teaching methodology was very brief, and it remains to be seen whether a more extensive exposure might or might not change a less innovative teacher's questioning strategies. Certainly, consistent with the existing literature on teacher change (eg. Clarke & Hollingsworth, 2002), neither the brief program nor the provision of the questioning material was sufficient to catalyse serious reconstruction by Naomi of her existing practices. And it may be that the efficacy of her existing practice was never seriously challenged by the institute. A new tool is less likely to be used, if the teacher's goals can be achieved successfully with existing, more familiar, tools.

Many in-service programs aim at enhancing teachers' teaching capabilities and expanding their repository of instructional strategies by emphasizing the connections between theory and practice. Indeed, making the connections between educational theories and practice in the classroom has been identified as essential (Zoller, Ben-Chaim, Ron, Pentimalli, & Borsese, 2000; Osborne, Erduran, & Simon, 2004). Certainly, the institute in which both teachers participated sought to establish or at least demonstrate this connection. However, it would appear from the two case studies that we have reported, that the motivation to change practice requires more than logical argument. We suggest, consistent with existing teacher change literature, that the inclination to change either resides in a teacher's existing willingness to experiment in the on-going improvement of her practice, or in a desire or need to change, arising from dissatisfaction with her ability to achieve her teaching goals. In the case of Naomi, neither condition was met. In the case of Robert, the institute and the materials provided, did not really constitute a change in practice, but rather an extension of existing practice and of an existing inclination to experiment and to innovate.

Questioning is so fundamental to a teacher's practice that a change in questioning methods may require a fundamental shift in teacher beliefs (similar to that documented by Tobin et al., 1994). Certainly, the two cases that we have reported suggest that this is the case. We have two distinct issues here: the nature of good question-asking by teachers, its form and its function; and, the means by which teachers can be led to change their practice in the direction of good question-asking. The two teachers that we have described, Naomi and Robert, exemplify two distinct questioning styles. Each style is in such widespread use that it is clear we are dealing with two sets of pedagogical beliefs and aspirations characteristic of two co-existing constituencies within the teaching community. The advocacy of one model of teacher questioning over the other (focusing over funnelling) is based on contemporary curricular aspirations to promote more than just facility with mathematical concepts and skills but rather more ambitious conceptions of "mathematical thinking" that integrate identifiable components, such as algebraic thinking, statistical thinking, critical thinking and metacognition[1].

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[1] Do these categories of "thinking" represent different <u>modes</u> of "thought" or are we categorising either the <u>context</u> in which thinking is undertaken (that is, mathematics) or the <u>type of objects</u> that are the subject of our reflection (mathematical objects)? Our success in promoting student

In summary, good questioning is central to good mathematics teaching and it is essential that teachers understand the importance of good question-asking skills in mathematics lessons. Good questioning involves the use of good questions as part of good questioning practice by teachers. What constitutes good questioning practice is directly reflective of the goals of the curriculum and the teacher and will differ significantly according to the cultural setting (Clarke, 2012). Teachers must create situations in which their own questions relate to the mathematical problem in hand, as well as modelling the skill of question asking for their students' benefit immediately and in the longer term (Holton & Clarke, 2006). To achieve this, it is necessary to plan teaching by choosing questioning tools that suit the student population, the teaching goals, the different needs and the teacher's own teaching style (and, we might add, the cultural milieu and traditions of practice that frame the classroom and constrain possibilities of change). The two case studies discussed in this paper demonstrate that the provision of good questions, even when these are presented as components of a coherent, structured questioning framework, was not enough to stimulate significant change in teacher questioning practice. For many teachers, the implementation of good questioning practices may involve profound change, not only in teacher practice, but also in teacher beliefs about the goals and purpose of mathematics teaching. Practical exemplars of good questioning practice must become a key resource in teacher education programs. Some interesting experiments have already been undertaken in this regard (Aizikovitsh-Udi, 2012).

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PRACTICES TO ENHANCE PRESERVICE SECONDARY TEACHERS' SPECIALIZED CONTENT KNOWLEDGE

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In this paper, the authors discuss practices to enhance preservice secondary mathematics teachers' subject matter knowledge specifically specialized content knowledge. The mathematical knowledge for teaching model (Ball et al., 2008) and Usiskin's perspective of teachers' mathematics (2001) were integrated to develop a content course for secondary mathematics teachers in Turkey. The focus of this paper is the nature of mathematical tasks used during the course. Those tasks are discussed from four approaches; unpacking concept definitions, applications and modeling, procedures and generalizations, and historical perspective of concepts. Pre- and post-test results and the participants' opinions showed that these practices were influential on their SCK.

Keywords: Preservice Secondary School Mathematics Teachers, Teachers' Content Knowledge, Specialized Content Knowledge, Mathematical Tasks, Content Course

INTRODUCTION

As the definition of teacher knowledge of Shulman (1986) consists of three types of subject matter knowledge (SMK), pedagogical content knowledge (PCK) and curriculum knowledge, SMK has an important role in this model. Brown and Borko (1992) asserted that preservice teachers' limited mathematical content knowledge is an obstacle for their training on pedagogical knowledge. Furthermore, according to the mathematical knowledge for teaching (MKT) model, developed from observations of elementary school teachers' classroom teaching, (Ball, 2000) there are six domains of teacher's content knowledge which can be categorized under Shulman's different types of knowledge (Ball et al., 2008). There are three domains under SMK: common content knowledge (CCK, mathematics knowledge not unique to teaching), specialized content knowledge (SCK, mathematics knowledge unique to teaching), and horizon content knowledge (knowledge of mathematics throughout the curriculum). Also, there are three domains under pedagogical content knowledge: knowledge of content and students (KCS, interaction of knowledge of mathematics and students' mathematical conceptions), knowledge of content and teaching (KCT, interaction of knowledge of mathematics and teaching methods), and knowledge of content and curriculum (interaction of knowledge of mathematics and mathematics curriculum). Among the three domains of SMK, specialized content knowledge stands out as teachers possessing deep understanding of the mathematics they will teach.

Furthermore, one of the most important features of this model is its emphasis on interweaving content and pedagogy for mathematics teaching (Ball, 2000). This model also describes kinds of practices which may be helpful to improve teachers' content knowledge. Ball (2000) stressed that it is crucial for teachers to experience mathematics from varied perspectives and in different contexts. Even though MKT provides a general framework for designing learning tasks for preservice teachers, it has limitations for secondary school mathematics teaching because it was developed as a result of research studies with elementary school teachers. In this paper, the authors provide discussion on using MKT framework and adapting it for secondary school teacher learning to design practices of a content course for secondary school teachers. Many examples given for the MKT model is from elementary school classroom. There are also examples of teacher education tasks for SCK (e.g. Suzuka et al, 2010) which may guide for task design but the examples are focused on elementary teaching. In our effort to adapt SCK concept of MKT model for secondary school teacher education, we used Usiskin's (2001) approach for type of mathematics that secondary school teachers need to experience. Usiskin (2001) provides the approach *teachers' mathematics* to develop mathematical practices to enhance secondary school teachers SMK. It should be noted that, Usiskin is not proposing a model for teacher knowledge research studies but provides an approach to guide practices to study mathematics with teachers. In this study of SCK tasks for secondary school preservice teachers, we use a blend of MKT model for teacher knowledge and approach of teachers' mathematics for teacher education practices.

Moreover, Usiskin and his colleagues (2003) explained three features of teachers' mathematics: concept analysis, problem analysis and mathematical connections. They discussed that even though these three kinds of mathematics experiences are essential for teachers, they are not addressed in a typical college mathematics course. Furthermore, he states that "Often the more mathematics courses a teacher takes, the wider the gap between the mathematics the teacher studies and the mathematics the teacher teaches" (p. 86). Among these three features of studying mathematics for secondary school teaching, concept analysis could be used for the SCK type of knowledge that a teacher needs to experience. In concept analysis, a secondary school teacher is expected to be able to discuss *alternate definitions*, *instances and applications*, *generalizations*, and *the history* of the concept. Furthermore, Suzuka and her colleagues (2009) stated that main features of a SCK task would be unpacking already existing knowledge and developing a flexible understanding of important concepts for mathematics that they would teach while those tasks allow teachers to build connections among representations or ideas. In other words, Usiskin's suggestion of concept analysis (in a sense relearning and deepening already existed knowledge for high school teachers) is very similar to features of SCK tasks as described by Suzuka and colleagues. Therefore, the concept analysis feature of the teachers' mathematics approach may be used to adapt SCK for secondary school mathematics teachers and design tasks for SCK. In this study MKT model and teachers' mathematics perspective for teachers' mathematics knowledge was integrated to answer the following research question: What kind of mathematical tasks enhance preservice secondary teachers' specialized content knowledge during a content course?

METHODS

This study took place at a public university in western Turkey. The target population of this research is secondary school preservice teachers in Turkey. There were 28 students enrolled in an elective content course for secondary school mathematics teachers. All of the students were asked to participate in the study voluntarily. Among them 16 of those students were chosen to investigate research question. It was not practical to choose all of the students for the study, so the participants were chosen to be representatives of the whole group. They were chosen according to their pre-test scores (three students from high, median, and low scores) and their grade level in the program (senior or junior). There were only four junior students and we chose them all, but there was missing data from one of them, so in total the data from 15 participants were used for this study.

Furthermore, it is important to portray students' mathematical background before discussing findings. They had been prepared in advanced mathematics topics such as group and ring theory, complex analysis, linear algebra in addition to calculus topics and differential equations. The curriculum for teacher education departments is provided by the Higher Education Institute (government) with some slight changes in it according to the university. There are some universities where students take mathematics courses from faculty members of college of education. There are also universities where students take advanced mathematics courses from faculty members of mathematics department. The study was conducted at a university of the latter case. When preservice teachers take many advanced level mathematics courses, it is believed that they will be ready to teach high school topics. Even though SMK is a prerequisite for PCK, it is important to address SMK as specialized content knowledge rather than just common content knowledge. In other words, preservice teachers will benefit greatly from a content course in which they will examine, relearn and reflect on mathematics topics that they will teach (Ball et al., 2008; Suzuka, et al., 2009; Usiskin, 2001). For this study we used *teachers' mathematics* practices (Usiskin et al., 2003) in the content course for secondary school preservice teachers. In this course, real numbers, complex numbers, equations, and functions (definition, exponential and logarithmic function) were studied.

The classroom environment for this course was community of learners such that future mathematics teachers explore mathematical ideas that they would be teaching. We would like to note that since participants studied mathematics as teachers, it was inevitable for them to learn SCK type of mathematics without thinking of teaching practice. Indeed, SCK was defined by Ball et al. (2008) as knowing mathematics for teaching. From this point of view, there were two main components of the content course: mathematical explorations and the context of teaching for mathematics. The mathematical explorations were addressed by mathematical tasks prepared by the researchers by adapting resources (e.g. Usiskin, et al., 2003). It should also be stressed that the authors worked with a mathematician for the development of mathematical tasks. Furthermore, the second component of the course, the context of teaching component, was addressed as participants collect and analyze student work for three topics (rational numbers, complex numbers, and logarithmic function) during the semester. The protocol for analyzing the student work was developed by the researcher for a previous study (Aslan-Tutak, 2009). For this paper, the focus will be the nature of mathematical tasks used in the course.

DATA SOURCE AND ANALYSIS

There were various data sources; mathematics knowledge pre-test, video records of the classroom instruction, two individual interviews during the semester, and mathematics knowledge post-test. All of the video records of the classroom instructions and individual interviews were transcribed.

The participants written mathematical work (pre-test and post-tests) were analyzed according to content; their correctness and their ability to provide valid mathematical proofs. The test was developed by the researchers. There were definition and proof questions in the pre- and post-tests such as definition of commensurability, proof of .99999... = 1 equality.

Individual interviews were used for two purposes; participants' perception of effective practices for their learning and their answers to given mathematics questions (e.g. the rule for converting infinite repeating decimals to rational numbers). We report both use of the interviews in this paper. There were two interviews conducted with 15 participants. The first interview was about in the middle of the semester while the second interview was at the end of the semester. Furthermore, the video-records of the classroom instructions were used to support other data sources as they were also used to study the practices of the course.

FINDINGS

The findings section will be organized according to the four types of SCK tasks did with for secondary school preservice teachers; *unpacking concept definitions*, *applications and modeling*, *procedures and generalizations*, and *historical perspective of concepts*. It should be noted that all of these types were used together during the course. In this paper, we tried to choose examples which describe best a particular task.

(a) Unpacking Concept Definitions

Concept analysis includes alternative definitions of mathematical concepts (Usiskin et al., 2003), how and why concepts arose mathematically would provide experiences to unpack definitions for concepts. In the MKT model, Ball et al. (2008) defined SCK as a skill unique to teaching and as it required a special work of unpacking of mathematics concepts. It is not necessary for others but it is a requirement for effective teaching. In the Turkish setting, preservice teachers' earlier experiences with secondary school mathematics concepts were limited to short textbook definitions to memorize, as participants stated in interviews. During their education they built further mathematics procedures or concepts without unpacking. However, these preservice teachers are asked to teach mathematics conceptually. So they had to look at these well-known concepts and definitions with the eyes of a teacher which requires to ask how and why questions. A primary purpose of the content course was to provide mathematical explorations to encourage preservice teachers to unpack the concepts that they thought they may know or understand superficially (Suzuka et al 2009).

All of the participants stated that they would not think of questioning already known mathematics concepts. In the individual interviews, most of the participants mentioned about the discussions of complex numbers and the geometric definition of number *i* as one of the most eye-opening experience for them.

From the video-records of the class discussions on complex numbers, we saw that all of the participants were aware of the rationale of the complex number i. They knew that when mathematicians were solving quadratic equations there were negative numbers inside square root. So they said that mathematicians came up with a method to represent -1 inside the square root. However pretest results showed that none of the participants were aware of geometric definition of number i before the content course. They defined the number i either as $i^2 = -1$ or as $i = \sqrt{-1}$ (Aslan-Tutak, 2012).

In order to explore number i, instruction started with discussion on the roots of an equation as its constant terms changes (Usiskin, et al., 2003). Then graphs of the equations were plotted on Cartesian plane (geometric representation of the equations) and their roots showed on a number line (geometric representation of real numbers). The number line was sufficient for the roots of first two equations but participants realized that they could not show the roots of third equation on number line. Then, the definition of the imaginary number *i* was discussed by rotation of 90^{o} perspective (Lakoff & Núñez, 2000; Trudgian, 2009; and Usiskin et al., 2003) which provides a geometrical understanding of the imaginary number. At that point, the participants realized the need of extending real numbers by discussing what it really means to have $i^2=-1$. During the classroom discussions and individual interviews they stated that none of them had known geometrical definition of the imaginary number *i* before this course. After the discussion on the geometric definition of complex number, *i* as a rotation of 90^{o} from real number line, they pointed out that understanding operations with complex numbers, and the connection between rectangular and polar form became easier. Furthermore, according to posttest results, six of the participants defined the imaginary number *i* as a rotation by 90^{o} (Aslan-Tutak, 2012).

> Participant 1: I consider the origin of the imaginary number i from a different perspective and I have learned from where it emerges [...] in the course everyone was able to aware and learn something that are substantially unknown and unnoticed before.

It is important to note that almost all of the participants were able to carry on simple to advanced calculations with complex numbers. However, being able to do some mathematical calculations is not enough for teaching mathematics meaningfully. Teachers' SMK is not just merely doing calculations (CCK) but also knowing the concept definitions in-depth.

(b) Applications and Modeling

During the content course, the students had the chance to investigate some mathematics concepts from application or modeling examples. For instance, participants discussed the alternative definitions of function and modeling activities or real life application word problems (e.g. water lily problem for exponential growth). When we analyzed the interview data, it was surprising to see participants' emphasis on modeling activities. Participants especially expressed that they did not use idea of function from modeling perspective; rather they solved computational problems by using functions.

> Participants 2: I didn't think of modeling for functions before. When I first looked at problems in the book I was very surprised. We never learned such modeling for functions.

First of all, it was interesting for authors to realize that preservice secondary school teachers' perception of the concept of function was limited with the definition of ordered pairs between two sets. They could not think of function as an operation to explain how variables changes according to each other.

> Instructor: What is function?
>
> Participant 3: a set which contains ordered pairs
>
> Participant 4: according to a certain rule, it is an operation which brings an element of a set to an element of another set.

By the end of the class hour, examining examples and classroom discussion yielded to the definition of a function as "If a v and a y can be related through an equation or graph, they are called 'variables': that is, one changes in value as the other changes in value. The two have what is known as a functional relationship; the variable whose change of value comes about as a result of the other variable's change of value is called a 'function' of that other variable" (Rao & Latha, 1995, p. 32). Later, two examples from Bremigan et al. (2011) were discussed whether if they were an example of a function. The purpose of the modeling discussion in this course was to make preservice teachers to relearn concept of function (a very crucial mathematical idea for high school mathematics) by studying alternative definitions.

As the concept of function was introduced from covariation perspective, during the following parts of instruction preservice teachers experienced modeling examples for exponential and logarithm functions. In addition to growth problems, participants were given a problem about oil spill and the cleaning procedure of such an environmental issue. The problem starts with some information about latest oil spill incidents in Gulf of Mexico and France. We discussed briefly about the enormous destruction of oil spill in even one day. So we decided that it is important to clean the oil in the fastest way and to know when all oil would be cleaned up. Then preservice teachers were given the problem: *If there was 8000 gallon oil due to a spill, and the crew can clean only 80% of oil for a week. So how much oil would remain after a week? And how long will it take to clean until there is 10 gallon of oil is left?* In this problem, one can start solving by using simple arithmetic calculations for percent. However especially for the second question, one should think of exponential function and its inverse logarithm function in order to find required time to clean the oil.

During the individual interviews, participants stated that they were surprised not being able to think of covariation definition of function, and not being introduced to modeling practices for functions until the content course.

(c) Procedures and Generalizations

Besides mathematical concepts, the procedures, rules and formulas are important elements in mathematics. The theorems including mathematical rules and formulas are deduced from both definitions and procedures. Deductions can be made from mathematical definitions (Usiskin et al., 2003), and the rules and formulas may seen as generalizations of procedures. Beyond knowing procedures and rules teachers need to know why a certain procedure works (SCK). Ball and her colleagues gave an example from elementary school years. Division of fraction procedure (invert and multiply) is commonly used but one may not necessarily need to know why the algorithm works. However, a teacher should know why it works, and this knowledge is defined as SCK. Therefore, in this content course, we also had mathematical tasks to explore underlying mathematical reasons of mathematical procedures and rules.

According to participants' responses for the first interviews, the most impressive experience in terms of unpacking procedures for them was the rule from the topic of rational numbers and their decimal representations. Furthermore, the pre-test included the question which asked the proof of the equality 0.999…=1. Seven participants tried to prove the equality and other eight did not answer the question. Among the seven answers, three of them just used the method of converting a repeating decimal into a rational number and four of them used different arguments. Also, at one point of the instruction on decimal representations there were discussions on the rationality of infinite repeating decimals. Students had homework from previous lesson and they were asked to give justifications of each step of arguments which are given in the textbook for the equality of 0.999…=1 (Usiskin, et al., 2003). During the classroom discussion, infinity concept was also discussed with this question; however, the focus of the classroom discussions was to study 0.999... as an infinitely repeating decimal. The purpose was to study the procedure of converting infinitely repeating decimals to rational numbers.

Individual interviews show that most of the participants (14 of all 15 participants) did not know why the procedure worked. The connection between the procedure (the method of converting a repeating decimal into a fraction) and the argumentation was provided in class by extending one of the arguments for the 0.999...=1 equality. The discussion was broken into proofs of three theorems for terminating, simple-periodic and delayed-periodic decimal representations respectively.

Moreover, participants' answers to mathematics questions asked in the individual interviews showed that for many of them, this was the first experience of elaborating the rule of converting an infinitely repeating decimal into a fraction. Until the content course they have known the rule but they didn't know why the methods work. After the course, according to post-test results, all of the participants could explain the rationale of the rule. This kind of knowledge which belongs to SCK is a special knowledge for teachers, knowing why a rule works but when teachers decide how to use this kind of knowledge in their practice that would be knowledge of content and teaching (KCT).

(d) Historical Perspective of Concepts

Another type of SCK tasks used in the content course is the historical perspective of concepts. It is important to note here that it is not the story part of mathematics history but the historical aspect of mathematics problems and concepts. Historical aspects of mathematics concepts were used to provide explorations to relearn mathematical ideas. Throughout the semester for each topic, there were discussions about historical perspectives in order to study why some rules work and how some mathematics concepts developed. Furthermore, almost all of the participants stated that addressing historical mathematics problems and how mathematicians handled them was a helpful approach for them to be able to see concept formation. Even though we cannot support their perceptions with pre- and post-test results, we take their feedback in consideration for developing tasks to enhance their learning in the content course. We will report about using history of logarithm because in individual interviews all of the participants stated that this was an approach that helped them to relearn logarithm.

First, logarithm was studied as the inverse of exponential function in the course. Then when we told preservice teachers that logarithm was developed before exponential function, they did not believe. Then the instructor introduced Napier and his problem which led to invention of logarithm function. Preservice teachers were given the problem and asked to work on Napier's problem (Panagiotou, 2011). Their work and Napier's solution yielded into the Napier's definition of logarithm. Later, there were discussions on Euler's definition of logarithm.

According to their feedback in the classroom and their responses during the second interview, we were able to conclude that even though this kind of knowledge of logarithm was available in resources, it was never introduced to preservice teachers, neither in high school nor in college level courses. They have been using logarithm since high school but none of them ever heard of Napier's problem or his definition of the logarithm function. Pre-service teachers were able to relearn logarithm by studying the historical perspective of the topic. They also reported that they thought this was a kind of knowledge that a teacher should have. Therefore, we may again conclude that this kind of practice, studying origins of mathematics concepts, allows preservice teachers to unpack and relearn some mathematics concepts.

DISCUSSION

In this paper, the purpose was to report findings regarding the type of tasks which can be used for content course to enhance preservice secondary teachers' SMK, specifically SCK. While studying the research question, we used the MKT model (Ball et al., 2008) as the theoretical framework, and teachers' mathematics (Usiskin et al., 2003) approach to construct the tasks for the content course. Learning tasks for secondary school preservice teachers should not be merely advance mathematical tasks. Rather the emphasis should be looking deeper in mathematics concepts, as unpacking and relearning them. Therefore, concept analysis practice as suggested by Usiskin and his colleagues (2003) was our primary approach to design learning tasks to enhance preservice secondary teachers' SCK. The SCK tasks developed for preservice secondary school teachers consisted of four types; unpacking concept definitions, applications and modeling, procedures and generalizations, and historical perspective of concepts. It should be noted that during the content course all four of these approaches were blended in a way to allow preservice teachers to study mathematics concepts. Since research questions were about the tasks, we reported findings according to these four types of SCK tasks. As, participants' knowledge from pre- and post-tests, and their perceptions from individual interviews and classroom discourse showed that all four of these elements of concept analysis helped participants to examine and study mathematical concepts.

Furthermore, all participants stated their purpose of relearning mathematics was to be more knowledgeable teachers. They were not engaging in those tasks not just as a student but as a future teacher. Even though there were no discussion on how to teach topics in this course, some findings suggested that preservice teachers were thinking of using what they learned for their teaching practice. So, we cannot propose that these practices affected only their SCK but not their KCT, because SMK is necessary but not sufficient condition for PCK. Here, the authors might conclude that as participants experienced learning mathematics for conceptual understanding, they valued teaching in such a way. One of the participants stated that the course was like a laboratory class for them to experience teaching for understanding. Further research may focus on including more topics to this study with a larger sample. The authors also plan to study effect of the content course on preservice teachers' belief about mathematics and teaching mathematics.

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VISUALIZING AND COMPARING TEACHERS' MATHEMATICAL PRACTICES

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We report the use of a tool to display and analyze essential elements of mathematical activity (definitions, properties, processes, etc.) arising during the development of a class. It has been applied to the study of the commonalities and differences among three classes conducted by three different teachers in the same institution, year and school level when they teach the bisector. The results allow us to infer some aspects about the mathematical knowledge of the teachers involved.

INTRODUCTION

Research on mathematical knowledge and the professional development of teachers has become increasingly important in recent years, and has revealed not only its complexity but also the limitations of the results (Sullivan and Wood, 2008). In particular, we need a research agenda that links theoretical outcomes with practice, conceptualizes teaching practice, and describes and discusses how mathematical knowledge for teaching is developed during a class.

Several authors assume the complexity of mathematical objects and that teachers' mathematical knowledge is large, intricate and evolves constantly. According to Davis and Renert (2013), instead of thinking of teachers' mathematical knowledge of concepts as a discrete set of basic knowledge in the heads of individuals, it is more productive to think of it as systems of changing instantiations (formal definitions, algorithms, metaphors, images, applications, gestures, etc..) emerging in practice and distributed among the whole community of professional teachers. Also, it is important to note that each of these instantiations has different conceptual value, and that it is desirable for teachers to know as many of these instantiations as possible in order to build a rich net of connections between them and acquire a more robust knowledge of the concept. We also assume that the complex nature of mathematical objects may be perceived in different institutions and different historical moments, different textbooks, or different methodological approaches. From this it follows that comparing the practice of different teachers working in the same institution, presenting the same mathematical concept at the same level and at the same time, will enrich our understanding of teachers' mathematical knowledge in practice. From this standpoint, the objectives of this research are as follows:

- To design a visualization tool that displays mathematical knowledge arising during a math class in terms of formal definitions, properties, examples and mathematical processes, among others.

- To use this tool to highlight the essential elements of the mathematical knowledge of the bisector used by the teachers during their practice.
- To account for the commonalities and differences of the mathematical activity of different teachers presenting the same mathematical content (bisector) in the same school year and the same institution.

TOOLS FOR VISUALIZING AND ANALYZING PRACTICE

In order to investigate mathematical practice we need tools specifically designed to address its complexity. To this end research in mathematics education has produced specific methodological tools and analytical frameworks. For example, the work of Rowland and colleagues (Rowland, Huckstep and Thwaites, 2005) and the knowledge quartet provides very specific conceptualization of practice. The works of Tomás Ferreira and Da Ponte (Tomás Ferreira, 2005; Martinho and Ponte, 2009) emphasizing the role of the teacher in the communication process, and the contributions from the Lesson Study methodology (Fernandez and Yoshida, 2004) include a collaborative model to enhance teachers to plan, implement, monitor, and reflect on math classes. All of these are important contributions that allow us to contextualize the interest of our research problem, namely to visualize the essential elements of the mathematical activity arising during classroom practice (definitions, properties, mathematical processes, etc.).

We observed the practice of three different teachers (hereinafter Laura, Antonia and Encarna) when they taught the perpendicular bisector in the final year of primary school (ages 11-12). The three classes were videotaped and transcribed for later analysis, looking for commonalities and differences in the mathematical activity. For the methodological design of the instrument we have decided to use the ontosemiotic approach for three reasons: First, it allows us to describe how mathematical objects emerge in the classroom and pay attention to the complexity associated with the object (*concept study* in the terminology of Davis (2013)). Secondly, because the ontosemiotical approach characterizes mathematical activity in terms of practices, objects and processes when primary and secondary curricula are also structured in terms of processes (reasoning, communication, modeling, etc.). Finally, because the ontosemiotic approach has already yielded visualization tools. Godino, Contreras and Font (2006) made an attempt to display primary mathematical objects during a class, and in this paper we enrich these attempts by incorporating also emergent processes.

For the analysis of transcripts we used one of the levels of the ontosemiotic approach (Godino, Batanero and Font (2007); Pochulu and Font (2011). It focuses on the primary objects and the mathematical processes involved in conducting practices, as well as those emerging from them. For the ontosemiotic approach (hereinafter OSA), mathematical activity is modeled in practices where primary objects (we will refer here to definitions, properties, construction procedures and problems) emerge. On the other hand, instead of giving a general definition of process, the ontosemiotic approach selects a list of processes that are considered important in mathematical activity, without claiming that such a list includes all the processes implicit in all mathematical activities (we will refer here to automation, institucionalization, argumentation, communication, modeling, and connection). Tasks or problems are considered primary objects because they are triggers of the mathematical activity.

An example to distinguish primary objects and processes is the construction of the bisector. For this construction, the student performs a sequence of actions, such as those underlying the use of the ruler and compass. In particular, students can use a construction procedure (algorithm) using only a ruler and a triangle with a 45 degrees angle (it is a procedure, which is considered a primary object type in EOS). With the repetition with other similar exercises the student engages in a process of automation.

TEACHING THE BISECTOR: MATHEMATICAL KNOWLEDGE INVOLVED

Detailed analysis of primary objects and mathematical processes illustrates relevant aspects of the structure and development of each of the three classes and permits us to distinguish many instantiations (using the terminology of Davis and Renert, 2013) of the bisector, as well as to establish relationships between them. Table 1 summarizes objects and processes that emerged from the practice of the three teachers, as well as the codes used to label them in Figures 1,2,and 3.

Table 1: Mathematical objects and processes emerging from the practice.

Mathematical objects	
Definition of perpendicular bisector: Teachers refer explicitly or implicitly to one definition of perpendicular bisector. It includes also definitions of related as segment or line.	
D_1:	Perpendicular line passing through the midpoint of the segment.
D_2:	Locus of all points equidistant from two given points.
D_2A:	Locus of all points equidistant from the ends of the given segment.
D_2B:	Line (boundary) which separates the plane into two regions, so that in a region all the points are nearer one of the two points than the other.
Properties: Any statement regarding the definition and the construction method of the perpendicular bisector, which can be true or false, but there is an attempt to justify it in class.	
P_1:	The point where the line intersects the segment is the midpoint.
P_2:	The obtained line is perpendicular to the given segment.
P_3:	Points on the boundary (D_2B) are aligned.

Construction procedure: Construction algorithm of the perpendicular bisector.	
Pr_1:	Euclid's procedure (Book I prop. X): given a *finite straight line*, describe an equilateral triangle on it (Prop. I) and bisect its angle (prop. IX).
Pr_2:	Perpendicular bisector as a locus: Given two points, find any other two points equidistant from them and connect these last two points with a line.
Pr_3:	Carpenter's procedure: Given a segment, measure its length, take its half and draw the perpendicular at the midpoint with the triangles 45 or 60 degrees, or the protractor.
Problem: tasks that incite mathematical activity, examples and counterexamples.	
EP:	Task based on paradigmatic examples
ENP:	Task based on non paradigmatic examples
CE:	Counterexamples

Mathematical processes		
Institutionalization: A definition, property or procedure is explicitly considered as valid, so from that moment on it is assumed to be known.		
Automation: Students are asked to repeat a certain procedure mechanical and individually.		
Communication: Oral or written statements on mathematical contents are expressed or understood. We explicitly exclude from this category mathematical arguments. Three subcategories have been included:		
EP: Teacher's lecture	DPA: Dialog among teacher and students	DA: Dialog among students
Argumentation: Existence of chains of mathematical arguments		
Modelling: At least one of the following phases of modelling (Blom, 2002) occurs: (a) starting point is a certain situation in the real world; (b) simplify, structuring and making the content precise; (c) objects, data, relations and conditions involved in it are translated into mathematics, and mathematical results derive, (d) retranslation into the real world.		

COMMONALITIES AND DIFFERENCES AMONG THE THREE CLASSES

In each of the three classes there emerge different definitions and construction procedures. In Laura’s class she defines the bisector as the perpendicular line passing through the midpoint of the given segment (D_1), and uses Euclid’s procedure (PR_1) for the construction. The carpenter’s procedure (PR_3), which is suggested by a student, is not institutionalized. In Antonia’s class, she institutionalizes the same definition (D_1), but a different procedure (PR_2: given two points, find two equidistant from them and connect them with a line). In Encarna’s class, she refers to the bisector as the boundary between two regions in the plane so that in a region all the points are nearer to one of the two points than the other (D_2B), and does not give any construction procedure.

In each of the three cases there appears one process that predominates, occupying approximately three quarters of the total time. In Antonia and Laura's classes it is the automation process, which takes about the last 75% of the class. In contrast, in the case of Encarna's the communication process predominates, taking up approximately three quarters of the total. We emphasize that in this third class modeling and argumentation processes appear, which have little or no presence in the other two.

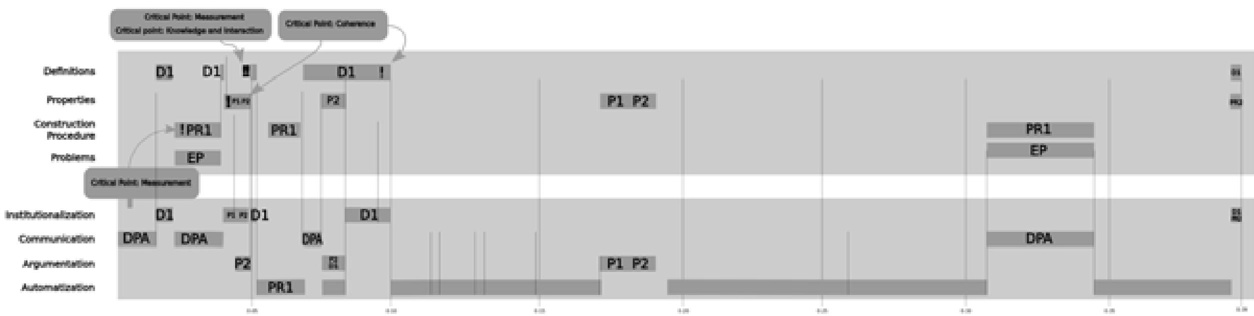

Figure 1. Graphic visualization of Laura's class. See Table 1 for label descriptions.

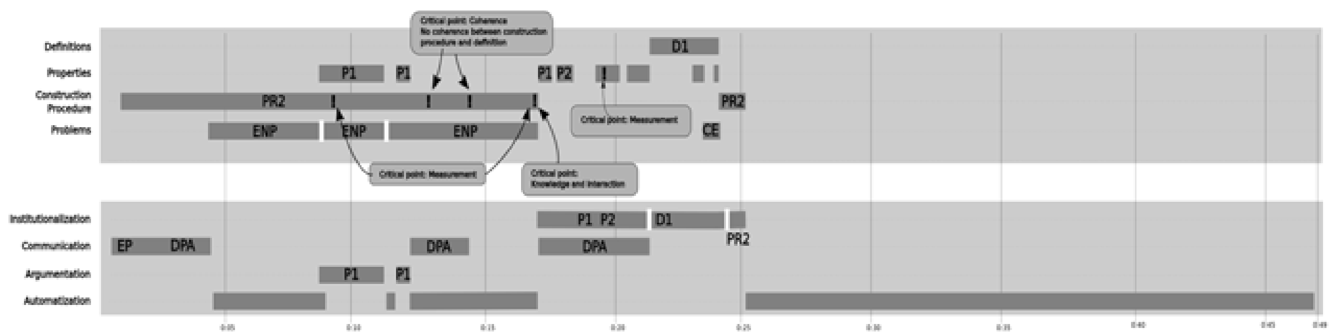

Figure 2. Graphic visualization of Antonia's class. See Table 1 for label descriptions.

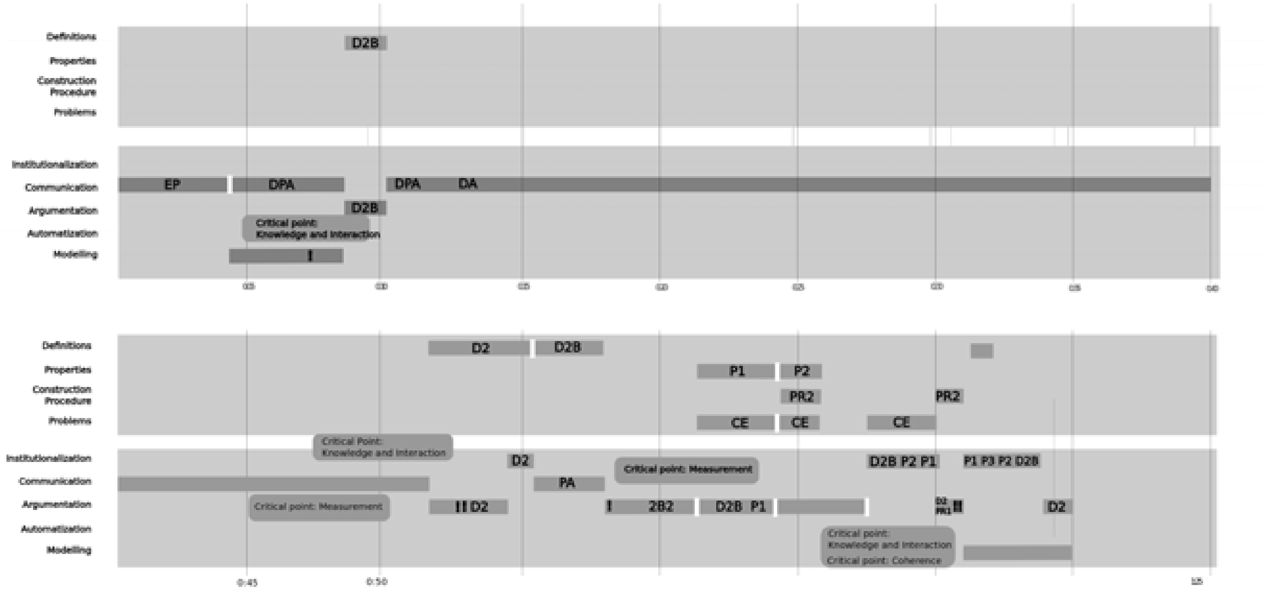

Figure 3. Graphic visualization of Encarna's class (significatively longer). See Table 1 for label descriptions.

A first analytical approach to the graphics shows also that mathematical activity is not uniformly developed during each class, but in all three we find slots of time in which the density of processes, definitions, procedures, etc. is higher. These periods of time occupy approximately from one quarter to one third of the total time of the class. However, in the case of Laura (Figure 1) there is an important accumulation phase during the first ten minutes, when most primary objects are presented. In the case of Antonia (Figure 2) this phase begins approximately in minute ten, while in Encarna's class (which is significantly longer) it appears at the end (Figure 3). We have analyzed in detail the way in which the three teachers connected primary objects and processes (considered instantiations of the bisector) as they emerged in these time slots.

CRITICAL POINTS IN TEACHING THE BISECTOR

The two following extracts from Laura's class illustrate some lack of consistency regarding the use of measure for constructing the bisector or proving its properties. In the first one, a student intends to find the midpoint of the segment using a graduated ruler, but the teacher makes explicit that direct measurement is not permitted for the construction.

Excerpt 1:

Teacher: So the bisector of the segment is nothing else than the straight line perpendicular to this segment that divides exactly it into two equal parts, right? What do you do to get the midpoint of that segment and split it into two equal halves? Say.

Student: I could put this on it -raising a ruler- and measure it.

Teacher: I could measure it with the rule but, would I obtain the same? Exactly? Exactly?

Student: With the compass.

Teacher: [nodding] With the compass (takes the chalkboard compass). The compass is the right tool with which the midpoint of the segment is going to be perfect.

However, in the second extract below, the same teacher uses direct measurement with the angle protractor to verify that the properties of the definition hold.

Excerpt 2:

Teacher: Therefore, one condition is that the line dividing the segment into two equal parts, the bisector of the segment, must be perpendicular. How can I know if these two lines are perpendicular? What do I have to do? Perpendicular (she points the four quadrants in the chalkboard)

Student: Measuring with the protractor.

Teacher: [nodding] Measuring with the protractor. (takes the chalkboard protractor)

Student: A right angle.

Teacher: and I have to obtain...

Students: A right angle, ninety.

Teacher: and I have to obtain four right angles. One, two, three, and four. If I put the protractor here (on the first quadrant)... Let's see. Note that I obtain exactly 90 degrees. OK? And If I put it this way I also obtain 90 degrees exactly. So I can say that the bisector of the segment is the line which is perpendicular to that segment and divides it into two perfectly equal parts. Exactly.

The selected dialogues above reveal as a fundamental aspect of professional knowledge related to the teaching of the perpendicular some reflection on the foundations of the mathematical activity. For the students, finding the midpoint of a segment leads naturally to a problem of direct measurement with the rule, while if the decision of the teacher is to follow the norms of Euclidean geometry, measuring with the ruler has no place in the constructing or proving properties of the bisector. This difference creates a professional problem that teachers can only manage with previous reflection on the validity of using approximated measurement in geometric constructions.

Such situations emerging from teaching practice on a specific concept are very important to connect the different instantations of it. The existence of this, and other similar situations lead us to define critical points, which require some interpretive analysis. A critical point is a manifestation of the difficulties that the teacher has to deal with the mathematical object (In This case the perpendicular bisector) due to its complexity. They are explicit in the form of errors, omissions, inaccuracies or lack of logical consistency in the teacher's speech. It is important to note that critical points relate to teachers' decisions and probably to their knowledge of the content, but are not necessarily a consequence of its lack. Seeking for those indicators in the transcription of the lessons, we have identified two critical points related to teachers' knowledge about the bisector. The first one has to do with measurement, as was illustrated above. The second one is related to the lack of consistency between the process of constructing the bisector and the definition in use. Both of them are reflected in Figures 2,3, and 4 with an exclamation mark.

The construction procedure (PR_2) used in Antonia's class considers the bisector as the locus of points equidistant from the two ends of a segment. However, she defines the bisector as the perpendicular through the midpoint of a given segment (D_1). This lack of coherence has important consequences for making sense of the bisector.

FINAL REMARKS

Having tools to visualize practice on a particular mathematical concept is important to decompose the intricate system of definitions, properties, processes, etc. (instantiations) that teachers use to approach this concept. Knowing not only which those instantiations are, but also how and when they arise in the course of a class, how they relate to each other, and the difficulties they involve is critical to understanding teachers' mathematical knowledge. Furthermore, it is assumed that mathematical knowledge for teaching is a complex system of instantiations distributed among all professionals.

We have applied a method based on the model proposed by the ontosemiotic approach to visualize three different mathematics classes on the bisector. The results highlight the use of different definitions, construction procedures and processes, as well as two critical points that inform about the difficulties for connecting the definitions, properties or procedures used. From our research derives the close relationship between the teaching of the perpendicular and the fundamental problem of using direct measurement for geometrical constructions, conjecturing or proving. To define the bisector as the perpendicular line through the midpoint of a given segment has been associated with an automation of its geometrical construction. On the other hand, defining the bisector as a locus has been associated with a broader communication process, and the emergence of modeling.

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TEACHERS AS INVESTIGATORS OF STUDENTS' WRITTEN WORK: DOES THIS APPROACH PROVIDE AN OPPORTUNITY FOR PROFESSIONAL DEVELOPMENT?

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The aim of the study was to examine how teachers' investigation of students' written works contributed to their professional development. The research was conducted at a public high school with the participation of six mathematics teachers and their students. The teachers have examined their students' written works as products of their solutions for some modeling problems for 5-week period. The preliminary analyses showed that teachers' collective examinations and interpretations of their students' written works have contributed to their professional development in terms of their subject matter knowledge and pedagogical content knowledge. Moreover, the collaborative learning environment in the study had positive effects on the affective domain regarding teachers' ways of knowing of students' thinking.

Keywords: Teacher education, professional development, student thinking

INTRODUCTION

This study reports on preliminary findings from a study carried out as a part of a larger research project about mathematical modeling where the primary purpose is to develop pre-service and in-service mathematics teachers' knowledge and skills about using modeling problems in teaching mathematics. The main three components of in-service teacher education dimension of this program were (a) planning a lesson in which a modeling problem was integrated, (b) implementing the problem in the classroom, and (c) investigating/assessing students' works in modeling problems. The focus of this study is on the third component.

More precisely, the purpose of this study was to examine how teachers' investigation of students' written work contribute to their professional development in terms of their subject matter knowledge and pedagogical content knowledge. The following research question guided this study: How does teachers' collective investigation of students' written works contribute to their professional development in terms of subject matter knowledge and knowledge of students' thinking as a sub-component of pedagogical content knowledge?

Teachers' Knowledge of Students' Thinking

It is widely accepted that teacher knowledge comprises three major dimensions: subject matter knowledge (SMK), pedagogical knowledge (PK), and pedagogical content knowledge (PCK) (e.g., see Shulman, 1986). In particular, teacher knowledge of students' mathematical thinking has attracted much interest. In this regard, teachers' knowledge of students' conceptions, difficulties and potential misunderstanding have been the subject of various studies as an important aspect of teachers' pedagogical content knowledge (e.g., Carpenter et al., 1996; Chamberlin, 2002; Gearhart & Saxe, 2004) as it could offer crucial contributions to teachers' professional growth. According to Gearhart and Saxe (2004), knowing what students know is crucial for effective classroom practice. When teachers attend to students' mathematical thinking, they can prepare the instruction with respect to their needs and level of understanding, emphasize important mathematical ideas, understand students' misconceptions and create learning environment to foster students' mathematical ideas (Kulm, Capraro, Capraro, Burghardt, & Ford, 2001 as cited in An, Kulm & Wu, 2004). All these studies highlighted that when teachers attend to and understand their students' thinking, both their instructional practice and students' achievement can benefit (Chamberlin, 2002)

However, although teachers' understanding of students' mathematical thinking is important for teachers to teach mathematics effectively, teachers usually had poor subject matter knowledge and their knowledge was not connected to students' thinking (e.g., Nathan & Koedinger, 2000; Bergqvist, 2005). Ball (1997) indicated that during classroom teaching, attending students' thinking was not easy because teachers often could not find an opportunity to interact with all of their students in their own classrooms. Additionally, teachers have difficulties in identifying students' ways of thinking as students' thinking can change under different circumstances or they cannot always express their thinking although they construct the idea in their mind (Ball, 1997; Chamberlin, 2002). Because of these reasons, it may be difficult for teachers to understand students' mathematical thinking.

All of these are particularly true when teachers attend to students' reasoning in modeling problems. One of the main features of these problems is that they allow students to think about the problem differently and solve it in different ways. Thus, teachers are confronted with different ways of student thinking when they attempt to understand students' reasoning while implementing the problem and assessing students' written works after the implementation. This is quite challenging for teachers (Doerr, 2007).

Professional Development Approaches That Focus on Student Thinking

In reviewing the works on the professional development of mathematics teachers, Sowder (2007) contented that one of the main goals of professional development is "developing an understanding of how students thinking about and learn mathematics" (p. 163). The studies that use students' ways of thinking on mathematical concepts as a tool for improving teachers' content knowledge and pedagogical content knowledge are designed based on the fact that attending students' mathematical thinking provides benefits for effective instruction and thus help support increasing students' achievement.

Ball (1997) suggests three approaches to improve teachers' attending to students' thinking. The first approach is "discussing cases of students' thinking" (p. 808). In this approach, teachers work together with written cases of episodes of students' thinking, and they investigate and find different interpretations of students' thinking. The second approach is "using of redesigned curriculum materials" (p. 808), and the third approach is "investigating artifacts of teaching and learning" (p. 811). The third approach is related to teachers' examination of "unnarrated" students' works and thoughts. Unlike the other two approaches, in the third approach, teachers engage in the artifacts obtained from real classroom settings, such as videotaped classroom lessons, students' written works and drawings on a paper or a chalkboard. Therefore, this approach is more realistic and provides teachers the opportunity to examine actual student products which are not interpreted by somebody else (Ball, 1997). The main idea of this approach can be seen in different professional development programs. For instance, *Cognitively Guided Instruction* (CGI) (Carpenter, Fennema, & Franke, 1996), *Multi-tier Program Development* (Koellner-Clark & Lesh, 2003), and *Integrating Mathematical Assessment* (IMA) (Gearhart & Saxe, 2004) can be considered as well-known examples of such programs. All of these programs acknowledged the importance of teachers' attending to and understanding students' mathematical thinking, although they used different approaches to develop it. For example, CGI was based on using the research-based knowledge of students' ways of thinking about a certain mathematics topic (e.g., addition and subtraction) to enhance teachers' instruction. As an alternative to CGI model, the professional development program of Koellner-Clark and Lesh (2003) situated in Models and Modeling Perspectives was grounded on providing opportunities for teachers to work on their students' works rather than providing research-based knowledge to teachers. According to this professional development approach, changes in teachers' knowledge and ultimately in their views about their teaching is possible only if they engage in situations where their existing knowledge is challenged and thus they experience some kinds of cognitive conflicts. The way of doing this includes activities of giving teachers tasks where they are required to interpret students' mathematical thinking (in modeling problems), and having teachers create conceptual tools (e.g., student thinking sheets, concept maps, etc.) to use in teaching practice (Koellner-Clark & Lesh, 2003). In this study, we adopted the *Multi-tier Program Development* (Koellner-Clark & Lesh, 2003) by having teachers engage in activities in which they investigate and think about students' modeling processes. Our main intention in this design was to foster teachers to think about their knowledge for teaching mathematics.

METHOD

Participants

This research was conducted at a public high school during spring semester of 2010-2011 school years. The school was selected because it was one of the distinguished schools with a well-organized workgroup of teachers in order to meet regularly across the semester. In addition, this school was willing to open their doors to carry out the study.

Six of the mathematics teachers in the school and their students participated in the study. The teachers had more than 8 years of experience and four of them had master degree in mathematics. On average, the teachers had strong mathematical knowledge and their students had quite high achievement level relative to other schools in the district.

Procedures

The teacher investigations lasted four weeks. Before the first week of the investigations, the introductory meeting was conducted with all teachers. In this meeting, the outline of the 5-weeks program was explained, the Student Thinking Sheet (STS) was introduced, and modeling problems that would be implemented in the classrooms were determined with the teachers.

During the 5-weeks period, each week a modeling problem was implemented in two classrooms at the same grade level by two teachers. Students worked on the problems in groups of 3 or 4 students during the two class periods (i.e., 90 minutes).

Before implementing a modeling problem in the classrooms, teachers were asked to solve the problem like a student and to create pre-implementation STS individually according to their predictions. After the modeling problem was implemented by two teachers in their classrooms, student works were collected and copies of them were provided to the teachers. Then, teachers were asked to examine the students' works in depth and to create individually post-implementation STS.

Next, teachers met for the follow-up meetings that lasted about 90 minutes. In these meetings, the teachers evaluated the classroom implementations. However, teachers who implemented the modeling problems shared their opinions about the tasks with other teachers and their experiences during implementation. These deliberations lasted approximately 15-20 minutes. Then, teachers interpreted their students' thinking strategies by the help of the STS. At this stage, teachers sequentially were asked to express verbally their individual written notes on the STS and to share with other teachers. Initially, the teachers, who conducted the classroom implementations, shared their thoughts. During this process, teachers showed examples of students' works while presenting their observations. In this way, teachers discussed their students thinking and collaboratively produced a shared STS which included students' fundamental thinking strategies.

Data Sources

Student Thinking Sheet (STS)

Student Thinking Sheet (STS) is a form designed to help teachers to think about and document students' mathematical thinking. It consists of a two-page document formatted as a table. The first page is divided into rows in which teachers are required to report different solution strategies used by their students in working on modeling problems. The table in the second page includes sections in which teachers are asked to report mathematical concepts, skills and process as well as students' errors and misconceptions for each different solution strategy.

Modeling Problems

For this study, five modeling problems were used to examine the interpretation of teachers' to understand their students' ways of thinking in certain mathematical situations. Modeling problems are non-routine tasks and they differ from the traditional textbook word problems. In each of these modeling problems, students interpret a complex real-world situation and formulate a mathematical description; therefore, what students produce has to go beyond short answers (Lesh & Doerr, 2003).

Weekly Meetings

The aim of the weekly meetings was to help teachers to examine and interpret students' different solution strategies, their misconceptions and errors. Each meeting lasted approximately 90 minutes and was audiotaped and videotaped. In these meetings, the researcher (the first author of the study) had a facilitator role and managed the group discussions. While teachers were sharing their thoughts about students' thinking strategies, the researcher posed some questions like "What do you think about the students' mathematical thinking underlying this strategy?", "What do you think about the effectiveness of this strategy to solve the problem?" in order to encourage teachers to think about students' ways of thinking more deeply.

Interviews

For this study, two different types of interviews were carried out. The first type of interviews was conducted with the teachers who implemented modeling problems in their classrooms. These interviews occurred before and after teachers implemented modeling problems in their classrooms and they lasted approximately 20 minutes. In pre-implementation interviews, teachers were asked about their predictions and expectations about students' ways of thinking; e.g., "What are your predictions about solutions students would have?" and "What are your predictions about students' difficulties and errors?" On the other hand, in post-implementation interviews, teachers were asked questions such as "Were there any ways of solutions which you found surprising? If any, what were they? Please explain them briefly" or "Which important mathematical ideas did students reach at the end of the process?"

The second type of interviews was conducted individually by all six teachers at the end of the semester and lasted approximately 40 minutes. The aim of the interviews was to inquire into how to examine students' thinking and work on STS and to see how attending weekly teaching meetings affect teachers' professional development. During the interviews, teachers were asked questions such as "Could you compare your interpretation of students' ways of solutions in the first week to the last week?"

or "How was this experience for you? Each interview was audio-taped and transcribed verbatim.

Data Analysis

The analysis of data was completed in two stages as pre-data analysis and in-depth data analysis. The pre-data analysis of the study has already been done and the in-depth analysis of data is in progress. For pre-data analysis, all written, audiotaped and videotaped data were examined carefully and organized by the researchers to prepare for analysis. Although students' written works produced during implementations were not the main source of data, they were used to provide background for the data analysis. Therefore, students' works were first examined by the researchers to determine the required data. The video recordings of the weekly teachers' meetings were carefully watched and notes were taken. Next, field notes were looked over; the transcripts of interviews and "Student Thinking Sheets" were reviewed. In this way, the initial codes were determined. In order to construct actual codes and present certain findings of the study, in-depth analysis of data will be conducted.

RESULTS

Pre-analysis of the data indicated that teachers' collective examinations and interpretations of their students' written works provided contributions to their professional development both in terms of subject matter knowledge and pedagogical content knowledge. Besides, there were some affective contributions of this type of collaborative learning environment to teachers' professional development. Despite the fact that finding out probable contributions of a professional development environment to teachers' knowledge requires a long-term study, the four-week study provided some clues about these contributions as follows.

Contributions to Subject Matter Knowledge

When teachers were trying to interpret students' different solution strategies from their written works, they were also trying to understand the mathematical aspects of these strategies. While they discussed the mathematical thinking underlying the strategy, they reveal their own mathematical knowledge and shared it with their colleagues. Especially when teachers detected some errors in students' solutions, they discussed what the mathematical sources of the error were. These discussions provided them with meaningful opportunities to reveal and improve their own subject matter knowledge. For example, the following excerpt shows that teachers discuss deeply on the mathematical concepts such as linear function, geometric series or slope etc. while they are investigating whether students' solution strategy is correct

Fevzi: Then, we accept the elements of the geometric series as linear. Is there such a thing?

Huseyin: No no. But actually, this ratio is the slope, isn't it? Can we think of this ratio as the slope?

Fevzi: It is just like the terms of a geometric series.

Huseyin: The ratios that I took their ratios in the geometric series.

Fevzi: The r's

Huseyin: Are the r's slope? Can we think of them as slope?

Fikret: Is there any linearity at there?

Fevzi: So, if x-axis [*represents*] is the number of bounces and y-axis is the height, linearity does not hold.

Huseyin: No, then it is not. Is it quadratic, then?

Handan: Yes, it becomes quadratic. When it is linear, it has to decrease at a constant rate, doesn't it? Isn't it a property of a linear function?

Contributions to Pedagogical Content Knowledge

This study focused on teachers' knowledge of their students' thinking as one main dimension of their PCK. Data indicated that this type of collaborative investigation of students' works had potential to improve teachers' understanding of their students' thinking. Teachers' interpretations of students' written responses to the modeling problems improved noticeably from the first meeting to the last one. For instance, in the first meeting, although there were many different solution approaches in students' work, almost all teachers were not able to see and detect the different solution strategies except for two or three strategies. In the interviews, they explained this situation that they didn't know exactly what the focus of the activity was. Especially, when a student used a strategy different from their expected solution, they tended to ignore this strategy. Some teachers explained the reason of this situation as they were accustomed to assessing students this way in their daily teaching practice. That is, student work often did not get credit if it did not follow teachers' "expected" way of solution. But as meetings proceeded, they tried more to examine and understand different solution approaches even if they were incorrect. Also, teachers could better see and interpret different solution strategies, mathematical ideas underlying these solutions and errors when they came across with them.

Additionally, the ways in which teachers interpreted and described students' solution approaches changed considerably from the first meeting to the last. For instance, at the beginning, they often looked only at what students did superficially. They did not tend to investigate what the mathematical thinking underlying the solution was, whether this solution was correct or not. As the following excerpt exemplifies, during the first investigation of students' written works teacher Handan just describe students' solution strategy superficially rather than making inference or providing detailed analysis.

Handan: Students obtained a value with trial and error and then they transformed it into a formula to support it. So, at least the students began with the trial and error method but then they formulated it.

As meetings proceeded, teachers focused more often on the underlying thinking processes rather than just looking at what students did. They considerably tried to understand their students' thoughts.

Contributions to Attitudes Towards Students and Collaboratively Working with a Colleague

Almost all teachers expressed the changes in their attitudes towards their students, especially towards those students who were not successful in standard tests and examinations. The following excerpt illustrated changes in a teacher's point of view towards their students.

> **Huseyin:** Now, I thought of that. These students shut down themselves after an hour. But, they solve whatever they want if they are provided with an appropriate question and the environment. They [*the project activities*] changed my thoughts about these students. Namely, yes, when you provide these children with an appropriate environment and offer them appropriate things, there is nothing they would not do. Now, it has increased my respect to these children.

Accordingly, teachers explained that through these close examination of students' work, they knew their students better and they appreciated their different ways of thinking. They also expressed their thoughts and feelings about the collaborative working with their colleagues. Similarly, the data from the whole group discussion in meetings showed the benefits of discussions and exchanging ideas for better interpreting their students' thinking and filling the gaps in their subject matter knowledge.

DISCUSSION

Our findings showed that interpreting students' different and sometimes unusual strategies on thought-revealing non-routine tasks were initially not so easy for teachers. This result is compatible with the findings reported by others (e.g., see Koellner-Clark & Lesh, 2003). However, as teachers analyzed students' work over time, they started to appreciate and understand students' solution strategies and also better interpret the mathematical ideas underlying these strategies. Like other professional development programs (e.g., Carpenter, Fennema, & Franke, 1996; Koellner-Clark & Lesh, 2003), these findings indicated that a learning environment in which teachers work collaboratively to analyze and interpret students' works on thought revealing non-routine tasks, provides crucial contributions to not only teachers' subject matter knowledge and pedagogical content knowledge but also to their attitude and beliefs towards teaching mathematics. On the other hand, collaborative work with colleagues on STSs and the nature of the modeling problems are two major factors that positively affect teachers' professional development. Nevertheless, when teachers did not solve the tasks themselves and/or come to the meetings without analyzing students' works carefully, they had difficulties in interpreting students' thinking. Therefore, as consistent with findings of Nathan and Koedinger (2000) and Bergqvist (2005), this study shows that such conditions can create barriers to provide adequate contributions to their professional developments while teachers investigate students' works. In addition, it is also observed that teachers need to spend adequate time to work on students' works for their professional development. In summary, this study contributes to our understanding of the literature on teacher knowledge and changes thereof in that investigations of students' works resulting from non-routine, thought-revealing activities could be a starting point for in-service teachers regarding what they know and think about their students' thinking and change their instructional practices.

NOTES

1. Work reported here is based upon a research project supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under grant number 110K250. Opinions expressed are those of the authors and do not necessarily represent TUBITAK's views.

2. The modeling problems used in this study were developed by the researchers working on this research project.

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ENHANCING MATHEMATICS STUDENT TEACHERS' CONTENT KNOWLEDGE: CONVERSION BETWEEN SEMIOTIC REPRESENTATIONS

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A developmental research project in mathematics teacher education is currently running at University of Agder. One of its aims is to enhance student teachers' content knowledge and to develop awareness of the specificity of mathematics as a subject-matter. Results from the research presented in this paper show the nature of the difficulties student teachers meet as they engage with mathematical tasks addressing the transition between semiotic representation registers. Implications for teacher education programs are discussed.

INTRODUCTION

How do we, as mathematics teacher educators, face the challenge of enhancing student teachers' content knowledge? And how do student teachers evaluate their own engagement with mathematical tasks addressing the idea of conversion between semiotic representations? In order to address these questions, I present results from a research project called "Inquiry-based mathematics teacher education: preparing for life-long learning", (the IBMTE project from now) and it is elaborated in collaboration with my colleague Barbro Grevholm. The ideas developed in our research project are rooted in our earlier research in teacher education (Grevholm, Berg, and Johnsen, 2006) where the idea of inquiry plays a central role. It was during spring 2011 that we initiated the IBMTE project and it is currently running at UiA. Based on this research project we elaborated the main aspects of an innovative course in teacher education currently running at the University of Agder (UiA), Norway. The aims of the research project are to strengthen mathematics teacher education at UiA by making explicit the link between theory (results from research in mathematics education) and teaching practice, and to facilitate the transition from being a student teacher to becoming an in-service teacher (Grevholm, 2003, 2010). In addition it aims at bringing the specificity of mathematics as a subject-matter to the fore by adapting and designing tasks as a means to enhance student teachers' awareness of the importance and the relevance of the use of semiotic representations (Duval, 1995, 2006).

In this article my focus is on student teachers' reflections on mathematical tasks which were designed as a means to develop both their awareness of the specificity of mathematics as a subject-matter, and to engage in reflecting on the transition between semiotic representations. Based on this approach the research questions addressed in this paper are: What is the nature of student teachers' reflections as they engaged with a task designed to encourage exploring the transition between different semiotic representations? And do student teachers recognise the importance of using different representations? The structure of the paper is as follows: First I present the rationale of the IBMTE project emphasising the role played by the ideas of "inquiry" and "transition between semiotic representations" in mathematics teacher education, considering a perspective both from student teachers' content knowledge and didactical knowledge (Durand-Guerrier & Winsløw, 2005). Then I turn to analysing student teachers' reflections after engaging with a mathematical task and discussing the relevance and meaning of the transition between semiotic representations. The data presented in this article are selected from student teachers' answers to a questionnaire and to an interview conducted with two students. I conclude by discussing possible development of and implications of this research for mathematics teacher education. Especially I argue for recognising the importance of developing student teachers' awareness of both the relevance of and transition between different semiotic representation registers.

THE MAIN ASPECTS OF THE IBMTE PROJECT

Rationale for the IBMTE project

Our research project emerges from the recognition that teachers and teaching play a central role in creating rich learning opportunities for pupils (Jaworski, 2006), and therefore we consider that it is important for student teachers to experience a rich and stimulating learning environment. From studies in teacher education it is well documented that the gap between theory and practice often creates a reality-chock for new educated teachers (Grevholm, 2003, 2010). Therefore our research project aims at designing a course that could enable new educated teachers to see how they could engage in a continuous learning process, as in-service teachers, and use tools like "inquiry cycle at three levels" as a means to engage in life-long learning while recognising the importance of results emerging from research in mathematics education. In Norway there is a demand for developing teacher education programs which are research based. However the consequences and implications of this claim are not clear. The IBMTE research project and the innovative course in mathematics teacher education emerging from this project are to be considered as an attempt to address this challenge. Finally by bringing the idea of *inquiry* to the fore, we introduce the dimension of reflection at three levels, as explained below, an aspect which we consider as central to the education of future mathematics teachers.

Inquiry cycle at three levels

The idea of inquiry is crucial in the IBMTE research project and it refers to a cyclical process consisting of asking questions, recognising problems, exploring, investigating, seeking answers and solutions and thereby engaging in an inquiry cycle (Berg, 2011b, 2011c, 2013; Jaworski, 2006, 2008). This process is understood at three levels: at the first level *inquiry in mathematics* as the student teachers engage in mathematical tasks; at the second level *inquiry in teaching mathematics* as student teachers engage in reflecting on and looking critically at how to organise and prepare teaching in order to enhance pupils' conceptual understanding of mathematics; and at the third level *inquiry in student teachers' own reflections* as they engage in looking critically at their own development as future mathematics teachers. Our aim is that student teachers recognise inquiry as being a useful tool and gradually move into considering "inquiry as stance" (Cochran-Smith & Lytle, 1999) or "inquiry as a way of being" in practice (Jaworski, 2006, 2008). We consider it important to offer student teachers' opportunities to engage in an inquiry cycle as a means to raise their awareness, as future teachers, of the specificity of mathematics as subject matter.

The specificity of mathematics as subject-matter

What is the specificity of mathematics as a subject-matter and what are the consequences of this specificity for learners of mathematics? On the contrary to physics, chemistry or biology where the objects of study are accessible either by perception or by instruments, mathematical objects like numbers, functions, geometrical shapes or vectors are not directly accessible. The only possibility to have access to these objects and to perform some operations on them is to introduce signs and semiotic representations (Duval, 1995, 2006). As a consequence learners need to use signs, diagrams, figures, and notations which stand for and represent the mathematical object at study. More generally Duval's approach emphasises the role played by visual representations in the learning of mathematics (Arcavi, 2003). For example, according to Duval (2006), the algebraic expression of a function, a table of the different values of the function, a text describing a situation, and a graph are four different semiotic representations of the mathematical idea of function (see Figure 1). Other representations are possible, as a correspondence between sets.

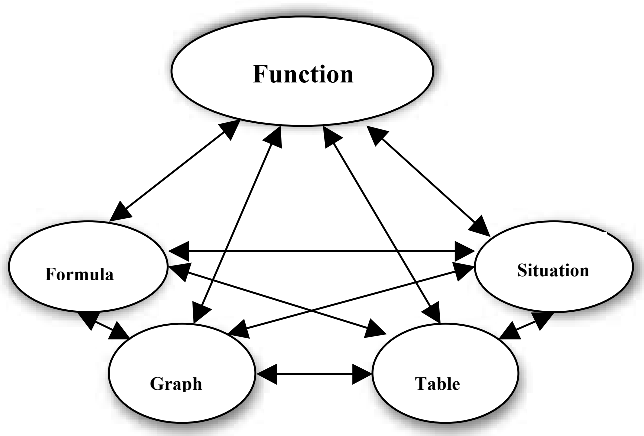

Figure 1: Distinguishing the mathematical object "function" from its semiotic representations (Berg, 2013)

Furthermore, Duval (2006) introduces the distinction between the ideas of *treatment* and *conversion* where the former is defined as "transformations of representations that happen within the same register" while the later refers to "transformations of representation that consist of changing a register without changing the objects being denoted" (p.111-112). The nature of treatments which can be carried out within a register is depending of the possibilities of semiotic transformations and these are specific to the used register. Conversion refers to "a representation transformation which is more complex than treatment because any change of register first requires recognition of the same represented object between two representations whose contents have very often nothing common" (p.112). For example the process of solving an equation belongs to treatment while the transition from the algebraic notation of a function to its graph brings the necessity of using two different semiotic representation registers and illustrates the notion of conversion. According to Duval (2006) the teaching and learning of mathematics witnesses a lack of recognition of the specificity of mathematics as a subject-matter and he argues that

> Changing representation register is the threshold of mathematical comprehension for learners at each stage of the curriculum. It depends on coordination of several representation registers and it is only in mathematics that such a register coordination is strongly needed. Is this basic requirement really taken into account? Too often, investigations focus on what the right representations are or what the most accessible register would be in order to make students truly understand and use some particular mathematical knowledge. With such concern of this type teaching goes no further than a surface level. … The true challenge of mathematics education is first to develop the ability to change representation register (p.128).

As teacher educators we need to take Duval's claim into account seriously and to avoid that student teachers engage in teaching mathematics at a surface level and without recognising its specificity. This acknowledgement brings with it clear demands to teacher education as to create opportunities for students to engage in activities where they are challenged by different semiotic representation registers and need to move between these. Furthermore I see Duval's distinction between treatment and conversion as related to the syntactic and semantic aspects of algebra. *Syntactic aspect* refers to the organisation and transformation of symbols following specific rules of manipulation, while the *semantic aspect* addresses the meaning endorsed by symbols and by expressions (Berg, 2009; Drouhard & Teppo, 2004; Puig & Rojano, 2004). Thereby the process of treatment is deeply related to the syntactic aspect of the expression at study (symbols manipulation), while the process of conversion is rooted in the ability of addressing the semantic aspect (developing awareness of the meaning of symbols, expressions and figures). I argue that bringing to the fore the specificity of mathematics as a subject-matter, as explained above, has deep implications for mathematics teachers' education. More specifically I consider that the design of activities aiming at developing student teachers' awareness of the importance of semiotic representation registers in the learning and teaching of mathematics is central and might be used as a means of enhancing their content knowledge and didactical knowledge (Durand-Guerrier & Winsløw, 2005).

Methodological considerations and research setting

The methodology adopted within the IBMTE project follows a developmental research approach. This implies that Barbro Grevholm and I, we engage in studying, documenting and researching student teachers' development and, at the same time, we recognise that our research activity contributes to that development (Goodchild, 2008). Concerning methods used in this research, the data was collected as part of the first teaching period during fall semester. The semester is organised as follows: students have first one week with teaching at the university and then three weeks with teaching practice in different schools, both primary and secondary, around Kristiansand. It was during the first teaching period that I decided to introduce a task related to the process of conversion between different semiotic representations as a means to address the specificity of mathematics as a subject-matter and to address Duval's claim concerning developing the ability to change representation register. Inspired by a task called "multiplied representations" (Swan, 2006), I designed an activity consisting of putting together cards representing the same mathematical idea, but using different representation registers (algebraic notation, sentence with words, geometric shape). I consider that this task offers a rich opportunity to discuss and compare, for example, the meaning of $3n^2$ compared to $(3n)^2$. Thereby student teachers need to consider not only the syntactic aspect of algebraic notation (calculation and manipulation of expressions) but also the semantic aspect, which corresponds to the meaning of these expressions (Berg, 2009; Drouhard & Teppo, 2004; Puig & Rojano, 2004). I present the following three corresponding cards as a complete and coherent set in Figure 2:

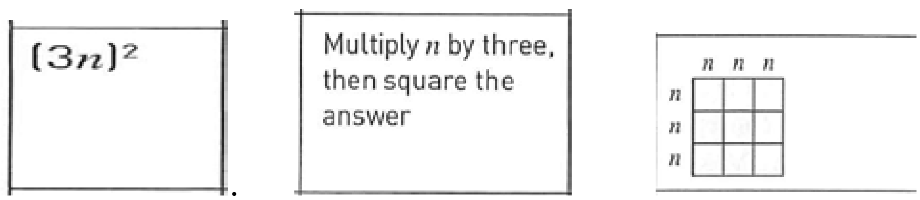

Figure 2: Example of the multiple representations task (Swan, 2006)

In addition I decided to give students some cards which were empty and where they had to draw the corresponding geometrical shape, to find the corresponding algebraic expression, or to formulate an adequate sentence with words in order to have a complete and coherent set of cards. This activity formed a basis from which I could introduce Duval's theory and emphasise results emerging from his research (importance of distinguishing a mathematical concept from its representation and developing the ability to move between several representations). The task was proposed to 52 student teachers and they were sitting in groups of three or four during the activity. I could follow how the task stimulated a lot of discussions as they tried to find corresponding cards and to make sets addressing the same mathematical idea. Right after the activity I distributed a questionnaire where I encouraged the student teachers to reflect on how they experienced this task and if they could see it as useful to introduce in their own teaching. In addition to students' answers to the questionnaire, I present excerpts from an interview conducted at the end of the semester with two student teachers who volunteered to reflect on their engagement with the task. The analysis of this data is presented in the following section.

Conversion between semiotic representations: Student teachers' reflections

Results of the analysis of data show that students' answers to the questionnaire can be grouped in three categories: those who are very positive to the task (38), those who think it was a nice task (12) and some few who find it not interesting (2). In the following I present results from the analysis according to these three categories, and results from the interview with two student teachers.

Analysis of student teachers' answers from the first group

Answers from that group are very positive and the students offer the following explanations

Student 1: I think this task was very funny because we get different perspectives within the same task. Get to look at the algebraic expressions in another way, what we actually did. It is easy to see the difference between $3n^2$ and $(3n)^2$ here. The cards with text help us about how to read these expressions. To draw was a nice exercise. It was not so difficult to find the cards which belong to each other, it was more challenging to draw [the corresponding geometrical shapes]. We could test if we understood what is happening

Student 19: Very exciting. Here we need basic understanding of what a sentence, a formula or a drawing means and represents. We get a larger overview over the connection [between these representations] and also the fact that these are the same even if they look different.

Student 15: I think it was a very interesting task because we really had to think hard to be able to draw the missing cards or to write the corresponding text. It was in fact challenging.

Results of the analysis indicate that the following aspects are in common in many student teachers' answers from the first group. Several of them refer to the task as "very funny", "very exciting" or "very interesting" and it seems that these students recognise the importance of engaging with different representation registers. In addition many students characterise the task as offering a very nice opportunity to explore the meaning of an algebraic expression. They use the expression "developing understanding" and my interpretation is that the task encourages them to inquire into the semantic aspect of algebra (Berg, 2009; Drouhard & Teppo, 2004; Puig & Rojano, 2004) as represented on the cards. Thereby, the focus is now on *the meaning* of the algebraic expression and not only on the computational aspect (syntactic aspect of algebra) of the expression and it seems that this element is new for these students. Furthermore, it seems that most of the student teachers have less difficulty in establishing a link between an algebraic expression and a text than linking these to a geometrical shape. However some students report on experiencing a challenge as they engaged in finding or drawing the corresponding geometrical shape. Analysis of students' answers from the second group brings further evidence concerning the nature of this challenge and its relation to the other representation registers.

Analysis of student teachers' answers from the second group

In this group the students were also positive to the task but it is possible to see in their answers the nature of the challenges they met

Student 2: I think it was ok to put together the corresponding cards, but when we had to write [the algebraic expression] or to draw [the geometrical shape] it became difficult. A lot of confusion.

Student 8: Interesting because we had to justify why we think it was correct, and sometimes we saw that it [the drawings] was not correct. It was unusual and difficult to see how the answer should be, especially when we had to draw [a geometrical shape].

Student 13: I think it was nice to see the connection between a figure, a text and an expression. It helps with understanding and one cannot make mistakes because one can see that the expression/text/figure do not fit together.

Student 42: I think the task was ok. It was challenging, especially because it is a while since I have been working with this.

Student 49: It was ok, but the problem is that my knowledge in mathematics is not quite good yet.

In this group it seems that the student teachers still agree about the usefulness of the task in offering an opportunity to see connections between different semiotic representations. At the same time they recognise the challenges they meet while engaging with it and report on "a lot of confusion", "challenging". One of them acknowledge the lack of having a robust knowledge in mathematics and my interpretation is that since the task does not address usual questions involving calculations and manipulations of expressions (treatment related to the syntactic aspect of algebra) students get confused and fail in developing understanding of the meaning of the different expressions (semantic aspect of algebra). Thereby they are not able to see that the same mathematical idea is illustrated through the different representations (conversion) and to group the cards in coherent sets. In the following I present a short excerpt of the interview with two students as it offers deeper insights into the reason why students found this task both interesting and difficult.

Analysis of excerpts from the interview

Student B: It was a nice task, but it was difficult.

Student A: Yes, and I realised very clearly what was difficult for me. Because I had difficulties with the geometrical shape, where you could see how big it [the expression] was. That part was a real challenge for me. I think it was difficult. For me algebra is so easy, I can see it at once, but this geometrical shape, it's all Greek to me. Then I could see clearly where my limits are….

Student B: Maybe we are at that level [draw a horizontal line with her hand], the level of formula and we do not manage to get down to the practical level.

Student A: …. yes, but it was also because I think, well I mean, for me this was unnecessary to have that kind of representation.

According to these two students the difficulties appear when they tried to establish a link between the proposed geometrical shape and the two other representations. It seems that, for these two students, a geometrical shape belongs to another level, “the practical level”, and stands in contrast to a more abstract level “the level of formula”. Furthermore they could not see the necessity of introducing this kind of representation and it provoked difficulties rather than bringing deeper understanding of the meaning of the algebraic expressions (semantic aspect). These insights are important for us, as teacher educators, as we aim at developing student teachers’ awareness of the importance of engaging with different representation registers.

Analysis of student teachers’ answers from the third group

Results of the analysis shows that there are only two students in this group

Student 21: Difficult, don’t remember basic knowledge in mathematics.

Student 44: Quite boring and too difficult.

It seems that these two students agree about the level of difficulty of the task, but unfortunately they do not offer deeper considerations about the nature of these difficulties. My interpretation is that the way the task is formulated is unusual for them and, due to a lack of mathematical content knowledge, they are not able to engage with it. It is also interesting that student 21 explains his answer by referring to the lack of “remembering basic knowledge in mathematics”. Here it seems that mathematical knowledge is understood as a process of memorising.

Discussion and conclusion

My aim with this article is to present how student teachers evaluate their engagement with a mathematical task which aim is to develop awareness of the process of conversion between semiotic representation registers (Duval, 1995, 2006). The analysis of student teachers’ answers both to a questionnaire and during an interview shows that most of the students recognise the task as interesting as well as challenging. More particularly they express difficulties at two levels: firstly in recognising geometrical shapes as useful representations, and secondly in deciding which geometrical shape corresponds to the two other representations. In addition, results seem to indicate that a conversion between an expression with words (a sentence) and an algebraic expression is understood as a process between entities at the same level of abstraction, while a conversion involving a geometrical shape introduces a level of different nature, “the practical level”. I consider these results as important since these bring to the fore a demand for us, as teacher educators, to illustrate and justify the use of several semiotic representation registers and to make explicit the important relation between geometrical shapes and other registers. I argue that in order to develop conceptual understanding of mathematics, it is necessary to offer student teachers rich learning opportunities where the specificity of mathematics as a subject-matter is brought to the fore and discussed. According to Duval (2006) recognising the importance of using and moving between representation registers constitutes the true challenge of teacher education and I consider that his claim needs to be seriously addressed by teacher educators. The research presented in this article is an attempt to face this challenge.

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EXPLORING TEACHER PERCEPTIONS OF THE DEVELOPMENT OF RESILIENCE IN A PROBLEM-SOLVING MATHEMATICS CLASSROOM

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This paper draws from an exploratory study of how mathematics teachers perceive the development of characteristics of resilience in mathematics students, as it is facilitated through a problem-solving teaching approach. Qualitative research approach was used to gain understanding of teachers perceptions of the development of resilience in a problem solving mathematics classroom. A key objective was to understand strategies which teachers used in their problem solving classroom, as well as their understanding of their role in developing resilience characteristics in students.

Keywords: Problem solving, resilience in mathematics, mathematics teaching

INTRODUCTION

In 2004, The Report of the Expert Panel on Student Success in Ontario, titled Leading Math Success (Ontario Ministry of Education, 2004), stated the following:

> Too often, society has accepted the stereotype that mathematics is for the few, not the many. The reality is that mathematics is deeply embedded in the modern workplace and in everyday life. It is time to dispel the myth that mathematics is for some and to demand mathematics success for all. (p. 9)

This statement alludes to a perpetuating dualistic belief about mathematics education which exists in society: you either can or cannot be a successful mathematics student, depending upon an innate ability which allows for less or more mathematical understanding. This false belief reinforces negative self-perceptions for students, hindering or completely ceasing their motivation to understand mathematics. This underlines the need for a shift in the understanding of teaching and learning mathematics, particularly with respect to strategies that encourage students to remain resilient in learning mathematics, regardless of the false belief. Educators are continuously attempting to understand teaching approaches for helping students who are “at risk.” The At Risk Working Group (2004) in part defined ‘students at risk’ as “students who are disengaged, with very poor attendance” (p. 16). These are the students who find no purpose in attending a mathematics classroom, and have probably internalized the notion that mathematics is not a subject which correlates with their skills and abilities. Ross, Hogaboam-Gray, and McDougall (2002) state that “one of the chief elements of mathematics education reform is teachers who make the development of student self-confidence in mathematics as important as achievement” (as cited in Ontario Ministry of Education, 2004, p. 26). Findings from Vatter (1992) suggest that at-risk students can be more successful if: “(1) school work is hands-on; (2) students’ feelings of worth and accomplishment are nurtured by the work itself; and (3) the work is tied to real work in the real world” (as cited in Ontario Ministry of Education, 2004, p. 38).

According to the National Council of Teachers of Mathematics (2000), “solving problems is not only a goal of learning mathematics but also a major means of doing so” (p. 52).The problem-solving approach could possibly help develop characteristics of resilience, since it is emphasized and promoted. Teachers need to understand and identify strategies which can help in building students’ mathematical understanding, as well as their resilience, so that they improve as mathematicians. This paper draws from a study that was aimed at exploring how mathematics teachers perceive the development of characteristics of resilience in students, as it is facilitated through a problem-solving teaching approach. The study was guided by the following questions: 1) To what extent are teachers aware of the development of characteristics of resilience within students in a problem-solving classroom? 2) How do teachers create conditions for students to succeed, through strategies and approaches, in a mathematics classroom, when solving problems? 3) What are the challenges for teachers, if any, in creating a problem-solving environment? and, 4) What successful factors/qualities have they seen develop in students, through implementing problem solving in classrooms?

THEORETICAL FRAMEWORK

The research was framed by the following ideas: Resilience, Problem Solving, Outcomes of Problem-Solving Approach and Development of Resilience.

Resilience

Although the question of resiliency in students within the mathematics classroom has not been widely researched, it is a topic of current interest in mathematics education. ‘Resilience’ describes “a set of qualities that foster a process of successful adaptation and transformation despite risk and adversity” (Benard, 1995, p. 2). The American Psychological Association (2013) states that resilience means being able to ‘bounce back’ from difficult experiences, and adapt well in the face of significant sources of stress. According to Borman and Overman (2004), characteristics underlying resilient children typically include high self-esteem, high self-efficacy, and autonomy. The development of resilience is seen as a process. Research focusing on adolescent resilience as a process “aims to understand the mechanisms or processes that act to modify the impact of a risk setting, and the developmental process by which young people successfully adapt” (Bond, Burns, Olsson, Sawyer, & Vella-Brodrick, 2003, p. 2).

Henderson and Milstein (1996) found that the capacity for resilience varies from one individual to another, and can grow or decline based upon “protective factors within the person that might prevent or mitigate the negative effects of stressful situations or conditions" (as cited in Borman & Overman, 2004, p. 178). Findings have been consistent in showing types of protective mechanisms that are important in the process of successful adaptation at three main levels: individual level, family level, and community level (Bond et. al, 2003). Borman and Overman (2004) conducted a study, in which they found that the most powerful model for promoting characteristics of resiliency was the *supportive school environment model*, which stressed fostering healthy social and personal adjustment of students. Three variables were the focus of this model: (a) safe and orderly environment, (b) positive teacher-student relations, and (c) support for parent involvement. It was comprehensively found that the most powerful school models for promoting resilience included "elements that actively shield children from adversity" (p.192). This illustrates the importance of identifying and regulating protective mechanisms, since these have shown associations with academic resilience.

Problem Solving

A focus on problem solving seems to be a key component of sound mathematics teaching and learning. There are various definitions of problem solving; these have been limited for the purpose of the paper. Lovin and Van de Walle (2006) define a *problem* by citing the definition of Hiebert et al. (1997): "any task or activity for which students have no prescribed or memorized rules or methods, nor is there a perception by students that there is a specific correct solution" (p. 11). Another definition used for the purpose of the study is given by Kantowski (1980):

> A task is said to be a problem if its solution requires that an individual combines previously known data in a way that is new. If he can immediately recognize measures that are needed to complete the task, it is a routine task (or a standard task or an exercise) for him. (as cited in Pehkonen, 2007, p. 1)

Within the problem-solving process, the lesson begins with the teachers posing a problem question or story, which contextualizes the learning, and then afterwards concepts and procedures are derived and understood by students (Small, 2008). Students should be engaging in tasks posed by the teacher, which allow them to engage in the mathematics that they are expected to learn through interactions, and struggle with the mathematics, by "using their ideas and their strategies" (Lovin & Van de Walle, 2006, p. ix).

Outcomes of Problem-Solving Approach and Development of Resilience

The idea of resilience being important to the study of mathematics is understood and better realized when educators value a holistic view of education, along with the constructivist understanding of problem based learning. Vatter (1992) suggests that at-risk students' success is attributed to a more hands-on, project-based approach to curriculum involving increased student choice, flexibility, and connections with students' everyday lives (as cited in Ministry of Education, 2004). By viewing mathematics education from constructivist problem based learning, the study of resilience can help in supplementing educators, to emphasize the need for students to be successful despite feeling that they are not a 'good' mathematics learner.

The Ontario Ministry of Education (2004) also encourages the idea that students need a certain amount of stress-producing conditions in order for learning to occur. Students must regularly be given the opportunity to struggle with mathematics problems. By denying them these experiences, or by providing excessive assistance to shelter them from what is perceived as mental pain, teachers and parents can end up 'crippling kids with kindness' (Chatterley & Peck, 1995, p. 39).

Glendis and Strassfeld (n.d.) understand that "students' mathematical development occurs within the social context of the classroom (Cobb, 1996), in an environment where the emotional experiences of students 'have the potential to influence teaching and learning processes' (Schultz & DeCuir, 2002)" (as cited in Glendis & Strassfeld, n.d., p. 4). Thus, it is important to learn strategies to help students regulate their emotions of low self-confidence and low self-esteem. Implementing problem solving in classrooms allows students to experience some stress and exit their comfort zone, thus, they push their existent learning into new situations. Therefore, educators need to explore and learn about strategies used in a problem solving classroom that help students in overcoming their emotions of inferiority towards learning mathematics.

METHODOLOGY

The study used a case study design where qualitative methods for data collection and analysis were used. Participants were selected using purposeful sampling (Creswell, 2008). Two teachers, who are well experienced in implementing the problem-solving approach in their classrooms, were selected. An interview guide with questions was the primary data collection instrument. Interviews were approximately one hour long; they were audio-recorded and transcribed. Teachers were asked open-ended questions. This allowed the teachers to discuss their experiences and share their teaching pedagogy and practice.

To analyze the interviews, an intrinsic case study analysis (Creswell, 2008) was used to separately understand each teacher's case. This put focus on the uniqueness of a single case, since a key objective of the study was to gain knowledge of specific strategies used by teachers; strategies could be similar or vary in ways, yet, have similar objectives and outcomes for teachers. Afterwards, a cross-cases analysis (Stake, 1995) was conducted to compare results from the two teachers by understanding commonalities and differences in teaching practice, as well as the understanding of developing resilience characteristics in students. These were examined by understanding common themes amongst the cases.

Description of the Cases

Mrs. A is currently teaching third grade. She described her personal journey, which lead her to implement "true, meaningful problem solving" in her classroom, over the last 5 years. Mrs. A had felt that the way she had been previously teaching was not the best approach, and there "had to be a better way to teach and develop true mathematical understanding in my students." As a result of feeling the need to learn a more suitable and successful approach for teaching mathematics, she turned to the Math Curriculum document, and was struck by the process skills discussed; specifically the statement that asserted "mathematical problem solving should be the mainstay of all math programs." She was beginning to understand that problem solving deserved more eminence; while teaching skills and concepts directly is also important, "problem solving should be folded into a teacher's approach in order to compliment the learning that is taking place." Mrs. B currently teaches grades 7 and 8. She has taught math as a core subject for 19 years. About 8 years ago she began to explore and teach math through the problem-solving approach. Mrs. B believes that mathematics needs to be taught in the real world context, and not "individually or singularly out of the blue," so students understand that it applies to the real world. She further states that "the more that you can bring it into their life or something they understand, is better, so that they can more grasp the concepts that you're doing."

FINDINGS

For the purpose of this paper, findings from the cross-case analysis of the two cases, named Mrs. A and Mrs. B, will be discussed in the following themes which emerged from analysis: Assessment providing relevance of mathematics curriculum in problem solving; Questioning; Successful instructional strategies in a problem solving environment; and Development of characteristics of resilience in problem solving.

Assessment Providing Relevance of Mathematics Curriculum in Problem Solving

The teachers described a crucial reason for implementing problem solving to be that they felt there was disconnect in the way that the mathematics curriculum was being taught to students, thus, creating a lack of relevance for them. A decisive link to be recognized is that in order to make mathematics learning more relevant for students, and to pose appropriate problem questions, assessment for learning is essential. Teachers must have an understanding of their students' level of mathematical knowledge in order to begin from a context which students can relate to, and sequence the lesson in a way that all students can make connections. Mrs. B discussed the use of diagnostic assessments as a form of assessment for learning. In order to choose an appropriate question which allows for multiple entry points to solve it, teachers must know their students' zone of proximal development. Using diagnostic assessments also allows teachers to anticipate the range of solutions which students might present. Mrs. A discussed how formative assessment, as a form of assessment for learning, is also advantageous for teachers. Since problem solving allows for more dialogue and communication, as students are solving the problem, teachers are able to help students in recognizing the relevance of their mathematics learning by questioning them, and further activating their prior knowledge and ability to make connections. Teachers are able to make more balanced, written and oral assessments of their own students as well.

Questioning

Questioning is a key strategy in facilitating the development of resilient characteristics in students, as teachers motivate students to remain persistent in solving problems. Asking students to explain their thinking, or why they chose the strategy were examples of the types of questions which could be asked. Mrs. B sometimes answers students' questions with a question, knowing that the students will be frustrated for some time, but she also assures them to keep struggling. This can be reassuring for students since they feel that the teacher understands that the students have the ability to meet the challenges presented to them. Findings by Gaye Williams (2007), an Australian researcher, also suggested that in some cases, giving students time to struggle and stress, and refraining from telling them how to solve problems will build more resilient children. Educators need to foster the characteristic of adolescents to meet challenges. Mrs. A asks open-ended questions such as: Why did you choose that strategy?; How does drawing the groups help you with your thinking?

These types of questions further the thinking of the students, and sometimes they realize that their strategy may not be on the right track, or the most efficient. While doing so, it is important for Mrs. A to ensure that students understand that there is no "right" way to solve a problem, since everyone has a different level of understanding, and some may prefer a methodical approach, whereas another may have a more creative approach to solving a problem. She reiterates that "no child is told they are not doing it right." Mrs. A gives the example of a student who came in her classroom almost afraid of math, because she had weak memory skills. She slowly began to understand that the teacher was not looking for the "right" way to solve problems. Math began to make sense to her, and something as simple as decomposing numbers into parts that were easier to manage, and then putting them back together made working with larger numbers so much easier for her (e.g. 63 +15 + 21 became 60 + 10+ 20 = 90 and then 3 + 5 + 1 = 9 and then 90 + 9 = 99). This opened up her mathematical thinking and allowed her to think differently about math. She began to volunteer lots of answers and show guests her most recent strategies, and how she solved certain problems.

Successful Instructional Strategies in a Problem-Solving Environment

During the time when students are struggling, strategies that teachers use to facilitate a problem solving classroom can be useful. These include group work and *gallery walks*, where students go around and observe solutions of peers with sticky notes; they put any questions on one colour of sticky notes, and comments on another colour. Mrs. A has also discussed how she challenges her stronger students to try different problems or work with more difficult numbers. The *congress*, or extended discussion that occurs once all students have had time to work on the problem, is also a successful strategy in the problem-solving approach used by Mrs. A and Mrs. B. Students gain comfort and confidence in their learning as they explain their solutions to the class, and are able to justify and defend their understanding to their peers by being questioned or asked to reaffirm their thinking. As this occurs, simultaneously the rest of the class is able to learn from their peers and value the diversity in thinking within their class. In relation, Mrs. A also discussed how students who are weaker in their mathematical understanding may not participate in the mathematical conversations, but they will still be exposed to ideas and skills. Gradually, all students can gain confidence in themselves as they realize that others are thinking along the same terrain; those who have approached the problem in either a very distinct or less efficient, sophisticated way are valued members of the class.

There are various ways to differentiate learning for students, so that they feel safe to struggle and take risks. Mrs. A states that children often deal with their math frustrations by shutting down, asking for help at every step, or simply putting anything down. Her method is to ensure that a safe math environment has been set up, as mentioned previously, and there is lots of teacher modelling, especially when it is time to share each others' mathematical thinking during a math congress. Encouraging the use of manipulatives is also useful, in order to create models of the problems in order to make sense of the problems. Using parellel tasks is also helpful. This is where the same question is posed but different sets of numbers are used, so that students can still maintain similar discussions later. Mrs. A and Mrs. B have both witnessed positive development of qualities in students, and become aware of how the problem-solving approach creates a sense of community in the classroom. This, in turn, helps students become more confident in their learning, as they feel a sense of belonging. Mrs. A gave the example of a comment she had heard: "I want to try the strategy that _______ used next time I do a problem." This type of comment can be powerful for students, since they feel as if they are a contributing member of their class. Mrs. B also spoke to this point by discussing one of the positive outcomes of the learning community: students are able to learn from peers that they might not normally interact with. Within the relationship dynamics of a classroom, this is influential for students who may feel that they are not noticed or may have low self-esteem as a learner. Tools, such as the *bansho*, have also shown to be a source of pride for the entire class as well. The Japanese term *bansho* means board writing (Kubota-Zarivnij & Kestell, 2010), and is used in a very systematic manner, in order to keep record of all parts of the problem solving lesson and show various approaches used by students to solve a problem.

Students learn to use their peers as a source of help, if they feel that they are failing individually; this is acceptable and advocated in a safe classroom. Students are able to use the strategy of *tailgaiting* or doing a *spywalk*, where they are allowed to move around the room, looking at or questioning how others are solving the problem. It is not about children taking their paper around to classmates and copying their work, but rather a way to stimulate the thinking of those who are not sure where to begin or are more resistent to get started. This can help students become more resilient since they learn to ask peers for help in their learning, and not feel isolated. Also, they are able to clear misconceptions that they may have, through extended mathematical talk with others, and being able to fuse the ideas or suggestions of others with their own. This relates to Margaret Glendis and Strassfelds' (n.d.) findings from their action based research, in which it was observed that students had begun to rely on one another when they worked through their 'crucial points,' and felt a sense of pride in their work as a group. Another finding, stated by Mrs. A, was that the problem-solving approach strengthened friendships between students, as the feeling of inclusiveness developed in the classroom. These relationships extend beyond the subject, and into other curriculum areas.

Development of Characteristics of Resilience in Problem Solving

Mrs. A and Mrs. B discussed the importance of allowing students to solve problems in their own ways, and build self-confidence in them. This also concurs with Small (2008), who believes that allowing students to approach problems in their own way builds confidence and maximizes the potential for further understanding. When students are able to succeed in solving problems through their chosen approach, then they feel ownership of their learning. If they struggle during the process of finding a solution, teachers are able to help accordingly. Strategies, such as spy walks, tail gaiting, gallery walks, group work, extended discussion or congress, and guided questioning all help in acting as protective mechanisms against failure. Allowing students to struggle, and helping them understand that it is a natural part of the problem-solving process is important.

Another significant type of protective mechanism which Mrs. A and Mrs. B advocated, was the development of a safe community in the classroom. This is done through the various strategies mentioned above. Using peers as resources around the classroom, and discussing with them to clear misconceptions; these are qualities of resilient mathematics learners. Mrs. A and Mrs. B have identified that these are achieved through extended discussion, and time given to discuss with peers. Creating a supportive atmosphere is the key to creating a safe classroom in which students can progress and become resilient in overcoming the struggles which they face.

Being a resilient mathematician describes a learner who has "a set of qualities that foster a process of successful adaptation and transformation despite risk and adversity" (Benard, 1995, p. 2) that they may face in the process of working on problem solving in a mathematics classroom. The perceptions of Mrs. A and Mrs. B, in discussing their awareness of the development of resilience, shows students want to be successful mathematics learners. Although some students may come into the classroom believing that they are not good at math, or they dislike math, Mrs. A and Mrs. B have observed positive outcomes in teaching through the problem-solving approach. It can be assumed that all humans want to feel successful and good about themselves, therefore, naturally, they would be open to developing characteristics of resilience. This coincides with the understanding of Benard (1995), as it is explained that "we are all born with an innate capacity for resilience, by which we are able to develop social competence, problem-solving skills, a critical consciousness, autonomy, and a sense of purpose" (p. 2).

The cases of Mrs. A and Mrs. B have shown that students are able to succeed and become resilient mathematics learners. This is enabled through structured strategies for problem solving which are used to create an environment in which students can develop characteristics of resilience. This is very important in the teaching of mathematics, since the subject has been stereotypically known to only be understood by some students, not all.

CONCLUSION

Findings of this study highlight the importance of the problem-solving approach in teaching mathematics, and the development of resilient characteristics in students. This study extends the conceptualization of resilience within mathematics education, claiming that the definition is based upon the context and types of risks which are under study. For the purpose of this study, the risk was students failing in understanding mathematics, and believing that they are not good at mathematics. The conceptualization of resilience remains open to interpretation, since there is no unified theory or definition. Findings are useful for educators who are interested in supporting students to gain specific characteristics of a resilient mathematics learner, which include: (a) developing self-confidence, (b) regulating emotions of mathematical anxiety and struggling to succeed, (c) remaining persistent by struggling in solving problems, (d) using resources around the classroom, (e) and clearing misconceptions and incorrect solutions. The findings from the case studies show that the strategies that teachers use when teaching problem solving promote the above characteristics for students.

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SPECIALIZED AND HORIZON CONTENT KNOWLEDGE – DISCUSSING PROSPECTIVE TEACHERS KNOWLEDGE ON POLYGONS

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The specialized and horizon content knowledge (SCK and HCK), two of the MKT (Mathematical Knowledge for Teaching) sub-domains, are the focus of this paper. We focus our attention on these two sub-domains of teachers' knowledge due to the delimitations problems concerning the content of SCK and the need to discuss the nature of HCK with greater precision. We approach this problem from a case study focusing on the knowledge of the concept of polygon that has a prospective high school math teacher. Such knowledge is discussed from descriptors referred to SCK.

Keywords: Mathematical knowledge for teaching (MKT), specialized content knowledge, horizon content knowledge, polygons, prospective secondary teachers.

INTRODUCTION

The teaching-learning process is influenced by a large set of factors (e.g., beliefs, goals, class/school size). One of the most influential factors concerns teachers' knowledge (e.g., Nye, Konstantopoulos & Hedges, 2004). Such knowledge influences, inter alia, focus, the pursued goals, and, consequently, what and how students are intended to learn, as well as the view of mathematics they are intended to experience and acquire. Thus, it is of fundamental importance to call attention to teachers' knowledge at all moments, being one of the more crucial initial teachers training (Llinares & Krainer, 2006).

Although there is almost unanimous acceptance of the influence of teachers' knowledge on students' learning, what is comprised in such knowledge, its type, content and specificities (or not) can be understood differently under different perspectives (e.g. Ball, Thames & Phelps, 2008; Rowland, Huckstep & Thwaites, 2005).

From amongst the most recent and influential conceptualizations on teachers' knowledge in the course of our recent work, we opted for the Mathematical Knowledge for Teaching (MKT) model (Ball et al., 2008). This option was made taking in consideration the very specific orientation to the teacher's mathematical knowledge of MKT, placing emphasis on the mathematical reasoning involved in the work of teaching. Such a choice was also connected to our perception of the sub-domains of the MKT as a relevant starting point for designing tasks for the mathematical preparation of teachers, and for doing research on what inputs to teachers training show effects on students and practices. Intending to contribute to an advance on the understanding of the MKT and on the content of its sub-domains, we began research focusing on discussion of the content of some of the sub-domains. That broader research includes work focusing on teachers practices from kindergarten till Higher Education and is leading us to question some particular aspects of the MKT conceptualization concerning the similarities and differences in the content of some of the sub-domains (Carrillo, Climent, Contreras & Muñoz-Catalán, 2012). In this paper we discuss the limits of SCK and HCK nature from mathematical knowledge, concerning polygons, revealed by Marta, a prospective high school math teacher. Such a discussion also aims to call attention to the need of a deeper specification of the content of these sub-domains.

SPECIALIZED CONTENT KNOWLEDGE (SCK) AND HORIZON CONTENT KNOWLEDGE (HCK)

One of the core aspects that the MKT brings to light concerns the specificities of teacher's knowledge for teaching, when compared with the knowledge involved in other professions in which the same topic (mathematics) is, also, used.

Having the genesis on Shulman's (1986) work, the MKT conceptualization considers the two domains of teachers' subject matter knowledge (SMK) and pedagogical content knowledge both divided in three sub-domains (see Ball et al., 2008). Our focus here concerns only the sub-domains included in SMK. The MKT conceptualization considers that Common Content Knowledge (CCK), the knowledge that allows one to know how to do mathematics for oneself, although necessary, is neither sufficient nor specific for the teaching of math. Thus, included in the content domain is a specialized Mathematical Knowledge for Teaching (SCK), which differs from the knowledge needed in settings other than teaching and also another kind of knowledge, related with how things can be perceived interconnected along schooling (HCK) – but not related with the content being approached at the moment.

The SCK is perceived as *the mathematical knowledge and skills unique to teaching* and thus, it only makes sense to be part of teachers' knowledge and not necessarily part of the knowledge needed outside teaching. Such knowledge implies *the use of decompressed mathematical knowledge* (Ball et al., 2008), in such a way that teachers are in a position that allows them to understand and explain (with and for understanding), amongst others things, the mathematical reasons embedded in students' errors or alternative reasoning; generate or reformulate examples that reveals key mathematical ideas; or identify the critical characteristics of a given definition or when proposing one. SCK is, in this sense, seen as the knowledge required by the teacher who genuinely wishes their students to understand what they do, and not merely mechanically run through a set of given procedures – this is not limited to procedures but has a broader meaning that includes the necessary concepts and how they interrelate and influence each other (Ribeiro & Carrillo, 2011).

On the other hand, HCK can be understood as an awareness of how the current mathematical topic fits into the overall scheme of the students' mathematical education, how the various topics relate to the others, and the way in which the learning of a particular topic may relate with others as one moves up the school. (We have to remark that HCK does not concern the knowledge immediately related with the content being approached at the moment -Jakobsen, Thames and Ribeiro, 2012). It is thus a kind of knowledge that is neither common nor specialized and that is not about curriculum progression. It is knowledge about having a sense of the larger mathematical environment of the discipline (Ball & Bass, 2009), allowing one to perceive different types of connections between and amongst different topics or even in the space of one topic at different school levels or in one level (e.g., Fernández, Figueiras, Deulofeu & Martínez, 2011). Such knowledge (even in these complementary perspectives) can enable teachers to make judgments on what is mathematically important and worthwhile to pursue – not being part of the curriculum they are supposed to teach – at a particular moment/school level. Such knowledge can also be perceived as an advanced mathematical knowledge from an elementary mathematical stand point (Montes, Aguilar, Carrillo & Muñoz-Catalán, 2012), meaning that advanced mathematics for teachers need to be demonstratively related to the teaching that takes place in school.

In our research group, and what concerns specifically the work focusing specifically on teachers' knowledge, we went across several stages: firstly, we aimed to identify, from observed classroom practice, the knowledge teachers show evidence of deploying at every specific moment (e.g., Ribeiro & Carrillo, 2011); the knowledge prospective teachers evidenced when answering a questionnaire/solving a task on fractions or polygons (e.g., Ribeiro & Jakobsen, 2012; Carreño & Climent, 2010) or even when interpreting and giving sense to students nonstandard productions (e.g., Jakobsen & Ribeiro, 2013). Such an approach leads us to a complementary focus of attention, in which we discuss and reflect upon the content of the MKT sub-domains and the conceptualization itself (e.g., Carrillo et al., 2012; Montes et al., 2012).

Although there is a certain consensus that there are no well-defined borders around MKT domains, such undefined borders make it difficult to differentiate what is included in each one of the domains (e.g., when analyzing a classroom practice, a questionnaire or an interview).

Specifically, if we are looking at what could be considered for inclusion in the SCK, we face some problems in achieving such a unique identification because it might be impossible to say, for certain, if some aspect of knowledge is specifically related with SCK or could be included in the CCK or even, according to with some perspectives (as connections), included in the HCK. These difficulties may be related to the fact that all these sub-domains have a straight relationship with the SCK (it influences how teachers perceive all the teaching-learning process and, thus, all the others domains) and thus with the specificities of teachers' knowledge when compared with other professionals who use mathematics as a tool. Such difficulties are perceived as one of the limitations of MKT and, if the aim is to call attention to the specificities of teachers' knowledge, it should be possible (at all times) to distinguish the nature (and content) of all its sub-domains.

These limitations of the MKT lead to a possible (re)conceptualization of teachers' knowledge (e.g., Carrillo et al., 2012), having as a starting point the problematic of the specificities of teachers' knowledge and if such specificities are better perceived as integrated in the previously mentioned sub-domains and not only in one (or some) of them. In such a process of deepening and clarifying the nature and content of the mathematical knowledge of the mathematics teachers, we felt the need to characterize each one of the involved MKT sub-domains, which lead to an enlargement and more refined delimitation of each of the components of teachers' knowledge. A discussion of its similarities and differences is central in this process.

METHOD AND FINDINGS

In this paper we use Marta's case as a means to address the problems of delimitation of SCK and the link with HCK, and at the same time, question the nature of these two sub domains. Thus, the specialized knowledge revealed by Marta when solving a questionnaire on the concept of polygon as an object of teaching and learning, is what allows us to discuss the focus of this paper.

Marta is one of the 7 prospective high school math teachers (PMT) who are part of a larger study aimed at characterizing the specialized knowledge that PMT reveal on the subject of "polygons". She is 22 and answered the questionnaire when she was a student in the course named Practical Teaching 1 when she started the last year of Initial Training. This group of students is the class group of the matter referred to an academic year at a private university in Peru. The academic mathematics courses results were generally low, but Marta's attitude towards reflection and debate on their own knowledge, was one of the reasons to consider her part of the study. In addition, Marta was the one with more closeness to the teaching-learning situations (from peers) because she had some teaching experience, which claimed she had some contact with students and their reasoning and arguments and thus, one could think that her mathematical content knowledge was linked to SCK and HCK and not simply to CCK.

We decided to address the concept of polygon because it is a subject that is studied from elementary through high school and it becomes conflicted when it comes to using it in differentiation of figures that are or are not polygons, or in the study of polyhedra. The analysis of the questionnaire was made from desirable descriptors of mathematical knowledge for teaching of the subject of polygons. Some of these descriptors were elaborated from the literature review (research on the knowledge of students and teachers and difficulties with the subject – e.g., Tall and Vinner, 1981; Fischbein, 1993, Fujita and Jones, 2007; Battista, 2007) and others emerged during the analysis process.

Here we propose a discussion and reflection on the nature and content of the SCK and HCK, in conceptualizing MKT, from three SCK descriptors used in the case of Marta. That reveals some difficulties in determining the boundaries of SCK, regarding HCK. We have chosen, in turn, a questionnaire item for illustrating such descriptors.

In the analysis of each answer, we need to identify the subdomain that addressed each question, respectively. One of the possible ways to overcome difficulties in determining the boarders of the SCK, in concrete related with HCK, was to equate and question ourselves (and, explicitly, other experts) as to whether a certain specific aspect of knowledge on polygons would, or would not, be shared with other professions – such as mathematicians. Although we are not absolutely certain of the limit of the mathematical knowledge of a mathematician on this topic, it was assumed that a mathematician could know a definition of a polygon without linking it to the basic necessary concepts (sub-concepts) that are fundamental in order to be able to elaborate the notion of such a geometric being with and for understanding. We considered that such knowledge related to linking the sub-concepts to elaborate a geometric notion is necessary and somehow exclusive for developing the tasks of teaching (e.g., choosing and developing useable definitions; selecting representations for particular purposes). One other option had to do with the fact of considering (after some discussions with persons with some mathematical training) that in any other profession would be necessary (involved) to note the critical characteristics of a polygon, being that such knowledge is essential for teaching mathematics while elaborating/exploring a definition of polygon. Such centrality has to do with, amongst other things, the fact that these critical features correspond to the ones students' are supposed to note while differentiating polygons from non-polygons.

From the previous ideas, assuming the sub-concepts and critical characteristics as core elements in a definition of a geometrical concept, we have elaborated some indicators aimed at describing prospective secondary teachers' SCK. We have selected three of them to discuss here (our choice is due to the fact that these are the ones more present in the prospective teachers' answers and, on the other side, because they allow for illustration of the difficulties in defining SCK borders, the focus of our paper).

> ***SCK1:*** *Know to identify the concepts and sub-concepts involved in a geometrical concept.*

Understanding a sub-concept as a specific concept that allows the elaboration of a more general concept, it is assumed that a teacher must have knowledge that allows for its identification. Such identification allows for the elaboration of a deeper sense and understanding of the concept itself, with an ampler sense and not associated to prototypes, and thus also allowing for detection of students' conceptual errors and misconceptions, being useful while elaborating a verbal definition of a geometrical object. While discussing some aspects related with the sub-concepts, Blanco & Contreras (2002, p. 112) mentioned that, in order to represent (draw) the orthocenter of an obtuse triangle it is necessary to know the concept of orthocenter and high of a triangle; but in order to be able to represent this last concept of high of the triangle, it is essential to have an understanding of the sub-concepts of perpendicularity from a point, of a line perpendicular to each other and of vertex opposite to a side of a triangle. Such knowledge is essentially mathematical knowledge but specific to the task of teaching, which justifies its inclusion in SCK. On the other hand, identifying the sub-concepts involved in a certain geometrical concept implies the knowledge of a net of relationships between concepts (which is also perceived as HCK in Fernández et al., 2011) and differentiating them between more or less fundamental (basic) concepts (in the sense of Ma, 1999) and others in which such sub-concepts form a part. This idea of mathematical structure is one of the reasons that makes us doubt if such knowledge, which was initially, in an obvious way, included in SCK, should be included in the HCK – as defined also by Ball and colleagues.

In one of the questions, several different (pseudo)definitions of polygon were given and the prospective teacher had to comment on them and discuss the sub-concepts involved. (There was a clarification in the question of what meant a sub-concept – necessary concepts know in order to be able to understand the concept of polygon.)

Three such (pseudo)definitions were: (i) *it is a geometrical shape with sides and different angles. They can be concave, convex, regular and irregular*; (ii) *a polygon is a geometrical figure with equal sides and angles*; (iii) *it is the interior of a geometrical closed figure made by the lines that join three or more dots and the dots should not cross*.

When analyzing the given definitions, Marta points as sub-concepts only those that are involved explicitly in a given definition. When commenting on the first two (pseudo)definitions she considers as sub-concepts: geometrical figure; geometrical shape; angle; classification of angles; measurement of a line (as an element of a polygon). But in the third she introduces a nuance considering "interior region" as a sub-concept associated to a critical characteristic to differentiate between what is and is not a polygon (such an idea is reinforced when she proposes a four-sided polygon, with the sides crossed, as a way of identifying the error in pseudo definition, as it would define two regions (iii)).

> ***SCK2****: Know to identify the critical features (characteristics) of a definition or given notion of a geometrical object.*

Perceiving as critical features (characteristics) the ones that are necessary and sufficient to define a concept, we assume this corresponds to the knowledge the teachers must possess to distinguish the general characteristics from the specific or accessory ones. As an example, from Marta's answers in the first given (pseudo)definition (mentioned previously in (i)), when discussing the misunderstood features (they can be concave, convex regular and irregular) she mentions that, "*here the student is confusing classification by its angles and by its sides with the definition*". In such, she reveals that in the definition of polygon there is no need explicitly to classify and she thus describes such a characteristic as accessory rather than critical. Putting together this evidenced knowledge and the previously mentioned in SCK1 concerning Marta's knowledge (a polygon is closed and has only one interior region), it lead us to wonder if knowing these defining characteristics of the concept of polygon can be considered as common content knowledge, not being exclusive from the teachers' knowledge space.

> ***SCK3:*** *Know to identify the critical characteristics when proposing a definition of a geometrical object.*

The elaboration of a geometric definition is influenced by the choice of the fundamental concepts to focus on and the critical characteristics that allow differentiating two different objects. In Marta's case, when asked to give her own definition, she includes as sub-concepts and critical characteristics some of the ones she discussed previously in the students' (pseudo)definition. For Marta, a polygon is:

> "*a geometrical flat closed figure with a polygonal region and its border is the union of three or more non-collinear dots*".

Although Marta considers a set of critical characteristics that may give us an idea that she has a specific knowledge that can be associated with the SCK on what a polygon is (flat and closed figure; interior of a polygonal line; with the minimum of three points), she mainly reveals herself to have knowledge related with the structure of a mathematical definition. Such a nuance is essential when analyzing teachers' SCK and it can be better discussed by analyzing Marta's examples, and non-examples of polygons in light of her given definition[1]. When asked to represent polygons and non-polygons, in light of the definitions she presents, we could expect her to draw convex and concave or even complex polygons (with sides crossed) or simply closed flat forms as a circumference, but she draws the following representations:

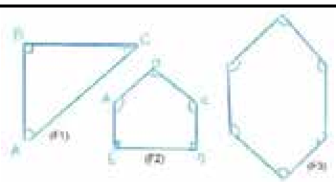
Examples of polygons	Examples of non-polygons
	

Figure 1: Examples and non-examples of polygons drawn by Marta

Putting the puzzle together (both the critical characteristics she presents and the representations she draws), it can be seen that the concept image she has contributes more information to her definition of polygons. Such an image leaves aside the critical characteristics of a polygon (in her own notion/knowledge of it), which lead

[1] These examples, and non-examples also, are obviously related to the example space she possesses and we, as teacher trainers, are supposed to allow teachers and prospective teachers to develop.

to a deficient/imprecise and incoherent definition with the mental image she has of the geometrical object.[2] Such discussion lead us, again, to equate if, when elaborating a definition, the critical characteristics are not more closely related with knowing what a mathematical definition is than with an SCK.

CONCLUSIONS AND FUTURE WORK

We have used a prospective secondary teacher's answers to one questionnaire for giving examples about some questions on the nature and content of teachers' SCK and HCK. Some of these questions concern the prospective teachers' knowledge on the role of the sub-concepts in the definition (and thus, what a definition is and the different possible approaches in defining a certain geometrical object). Examples of such questions: Is the identification of the sub-concepts and the acknowledgement of the critical characteristics associated to polygons part of SCK or is it part of CCK (in the sense that any person with some mathematical training is supposed to have a knowledge on these sub-concepts and its role in the definition)? Is the identification of the sub-concepts an SCK or a specific knowledge related with connections and thus in the space of HCK (in the sense of the HCK conceptualization presented by Fernández et al., 2011)? Is the fact of considering the critical characteristics when proposing a definition (of a polygon) related to the nature of mathematics itself?

The previous questions are part of the examples that lead us to discuss and reflect on the limitations (and difficulties) in differentiating some aspects of the MKT sub-domains in terms of its content and nature and in particular on which aspects of the mathematical knowledge are specific for teachers to teach mathematics with and for understanding. Aligned with this problem, our research group has started working specifically on the space of the knowledge a teacher has/must have in order to perform their work of teaching, leaving aside a discussion on whether such knowledge is shared with other professions or not. Carrillo et al. (2012) present a theoretical discussion of such an approach and its impact on the conceptualization of teachers' knowledge. Further research is needed in order to clarify the content of each of the domains. Such identification and discussion would allow for the creation of a body of knowledge allowing a deeper and more fruitful focus of attention on the more problematic and promising aspects of teachers' training. It would allow us to focus our attention where it is most needed, as well as reinforce the need for a specialized knowledge for the teaching of mathematics.

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[2] Although the differences between conceptual image and conceptual definition in the same object have been previously discussed (e.g., Tall & Vinner, 1981) we consider that in order to be able to reflect on the critical characteristics it is essential to have an awareness of the characteristics that limit the concept definition, leading to a harmonization of image and definition.

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MATHEMATICS TEACHER'S SPECIALIZED KNOWLEDGE. REFLECTIONS BASED ON SPECIFIC DESCRIPTORS OF KNOWLEDGE

Emma Carreño; Nielka Rojas; Miguel Montes; Pablo Flores[1]

We present some reflections on mathematical knowledge for teaching arising from the descriptors proposed by Sosa (2010) and linked to the model of mathematics teachers' knowledge constructed by Ball, Thames and *Phelps (2008). The former leads us to reflect on the model and to adopt the proposal of Mathematics Teacher's Specialized Knowledge so as to make progress in describing the mathematical knowledge brought into play when teaching. This suggests the need to scrutinise the model carefully and to refine its characterisation. To do so, we embark on a search for evidence which allows the incorporation, integration and interconnection of aspects of knowledge apparently unrelated in the model of Mathematical Knowledge for Teaching (MKT).*

Key words: Mathematical knowledge for teaching, mathematics teachers' specialised knowledge.

INTRODUCTION

In this paper we discuss the theory underlying the different domains which comprise the construct of mathematical knowledge for teaching developed by Ball, Hill, and Bass (2005) and Ball, Thames, and Phelps (2008), a model widely used in Mathematics Teaching research for the purposes of teacher training. The model devised by these authors allows teachers' knowledge to be studied from observations of classroom practice in primary education. Building on the work of Shulman (1986), it focuses on mathematical content through the categories of content knowledge and pedagogical content knowledge.

[1] This article represents part of the studies into teachers' knowledge by the SIDM group (from the Spanish 'Research Seminar into Mathematics Education') based at the University of Huelva, Spain. It comprises the following researchers: José Carrillo (coordinator), Nuria Climent, M. Cinta Muñoz-Catalán, Luis C. Contreras, Miguel A. Montes, Álvaro Aguilar, Dinazar I. Escudero, Eric Flores and Enrique Carmona (University of Huelva), Pablo Flores, Nielka Rojas and Elisabeth Ramos (University of Granada, Spain), C. Miguel Ribeiro, Rute Monteiro and C. Susana dos Santos (University of the Algarve, Portugal), Leticia Sosa and José L. Huitrado (University of Zacatecas, Mexico), and Emma Carreño (University of Piura, Peru).

We discuss the model of MKT in terms of the general descriptors of mathematical knowledge defined by the research group headed by José Carrillo at the University of Huelva, Spain. Amongst the results of this line of research are the studies by Sosa and Carrillo (2010) and Sosa (2010), which report on an interpretative study into the MKT displayed by two teachers working on the topic of matrices at Spanish baccalaureate level. For more specificity of mathematical knowledge for teaching, Sosa identifies and builds indicators or descriptors of knowledge or specific skills of the teacher while teaching (p.54). In order to specify more precisely mathematical knowledge for teaching, Sosa identifies and draws up a set of descriptors and indicators of the specific knowledge and skills deployed by teachers in their work (p.54). These studies suggest that further work on the features demarking the subdomains of mathematical knowledge would be beneficial. Sosa's study, dealing with general aspects of mathematical knowledge, is a good starting point.

This paper aims to discuss the knowledge domains described by Sosa (2010) and to scrutinise the model in order to develop its characterisation. The need to illustrate a new system of organising teachers' mathematical knowledge, drawing on MKT and being made operative by means of practice-based descriptors, leads to a new organisation which lays emphasis on the specific features of this knowledge in relation to teaching.

Below we discuss aspects of MKT in terms of Sosa's indicators, and consider how the new perspective might deal with certain problems arising, with the aim of refining the model so as to better understand the knowledge teachers display as they go about the work of teaching.

THE RELATION BETWEEN MKT AND KNOWLEDGE DESCRIPTORS

Mathematical knowledge for teaching is understood by Ball and her collaborators as the specific knowledge required for teaching mathematics. This model builds on the work of Shulman and collaborators several decades previously. Furthermore, as an emergent contribution of the qualitative analysis of the classroom practice of various practising and trainee teachers in the United States, it confirms the specialisation of the knowledge required for teaching mathematics, distinguishes components of this knowledge (in terms of domains, subdomains and descriptors), and includes the subdomain of specialised content knowledge, which is fundamental to the work of teaching.

With this in mind, the group based at the University of Huelva aims to explore the MKT model so as to better understand and identify its components. Hence, in 2010, the group compiled a list of 100 general descriptors (Sosa, 2010) for mathematical knowledge which enable elements to be differentiated and features of the observed knowledge to be established in the practice of teaching. These descriptors correlate with the components defined by Ball et al. (2008), and are carefully phrased to capture each particular characteristic pertaining to the various knowledge domains, such that items representing evidence of aspects of knowledge in the MKT model can be built up, integrated and inter-related. Below, we describe how the MKT knowledge subdomains relate to some of the descriptors proposed by Sosa (2011) and discuss several issues arising.

The subdomain *Common Content Knowledge* (CCK) is based on the notion of encapsulating the mathematics that anybody making use of the subject might know, such as might be the case in using definitions, rules, properties and theorems associated with a specific topic (*CCK1*), using mathematical notation, and understanding the importance of an item (*CCK2* and *CCK3*), and knowing how to apply mathematics and do demonstrations (*CCK4* and *CCK5*). Complementing the foregoing, we would add that is a kind of knowledge that teachers need, and although other professions might draw on it, too, it forms an integral part of what makes mathematics teachers specialists.

The subdomain *Specialised Content Knowledge* (SCK) refers to a deeper and more thoroughgoing knowledge of mathematical content, and includes understanding the significance of concepts (*SCK1*), knowing the unseen steps behind procedures (*SCK2*), intuiting the root of pupils' mathematical errors (*SCK4*).

The descriptors corresponding to *Horizon Content Knowledge* (HCK) refer to associations between concepts, relationships between general and specific content (*HM1* and *HM2*), and awareness of interdisciplinary applications (*HM3*). In the case of descriptors referring to understanding how one item relates to another that comes before or after it in the curriculum, the knowledge that is brought into play concerns curricular issues and the sequencing of the subject, as the teacher has to know the contents both previous and subsequent to any particular item being taught.

Pedagogical Content Knowledge (PCK) combines knowledge of teaching with knowledge of mathematics (Ball et al., 2008, p. 401). This subdomain concerns aspects such as those embodied in descriptors *PCK30* and *PCK31*, respectively: "*Knowing how to introduce a new concept by relating it to concepts studied previously*," and "*Knowing different ways of introducing a mathematical topic through some information or brief historical background about it; or knowing how to contextualise a topic through a brief anecdote or historical background*." These descriptors illustrate that PCK is a subdomain which implies learning mathematics with meaning. Indeed, sometimes descriptors tend to be associated with features of learning mathematics such as "Knowing which exercises to leave the pupils for homework."

With respect to *Knowledge of Content and Students* (KCS), there are two sets of descriptors, those referring to general pedagogical knowledge, and those referring to knowledge about the students' interaction with mathematics. The first of these sets represents knowledge that mathematics teachers probably need, and hence the descriptor (*KCS2*), "*Understanding the needs and difficulties of students with mathematics*." The second group is based on understanding how pupils assimilate and apply material, as can be seen in the descriptor (*KCS3*), "*Anticipate the misunderstandings that might arise with specific items being studied in class*." The descriptors in this subdomain enable us to distinguish when knowledge involving aspects of learning mathematics is being deployed, which is an appropriate starting point for describing this subdomain.

Finally, regarding *Knowledge of the Curriculum* (KC), Sosa (2010) sets out three descriptors describing the organisation of content in textbooks (*KC1*), the prior and subsequent treatment of an item (*KC2*), and content deriving from teachers' institutional environment (*KC1).* These descriptors also teachers' critical responses to the established objectives and standards to be noted.

TOWARDS THE SPECIFICATION OF MATHEMATICS TEACHERS' KNOWLEDGE

Taking into consideration the conditions for distinguishing the subdomains of MKT and attempting to find those defining features which specify mathematics teachers' mathematical knowledge, we find it necessary to place mathematics at the hub and focus attention on the knowledge that is significant only to mathematics teachers. This position has given rise to the MTSK model advocated by Carrillo, Climent, Contreras, and Muñoz-Catalán (2013), in which specialisation receives an alternative focus centred on mathematics teachers' specialised knowledge (MTSK), which abandons the notion of mathematical knowledge for teaching to centre on knowledge of significance only to mathematics teachers.

In the following section we summarise several key aspects of devising descriptors for the six subdomains of the MTSK model. We begin with the three subdomains concerned with mathematical knowledge (MK), which all concern the way teachers' understand mathematics.

Knowledge of Topics (KoT). The fundamental concern of this subdomain is the idea of "knowing" a topic, and this would need to be reflected in the corresponding descriptors. However, it is important to consider everything this knowledge implies, from rules, procedures and calculation methods associated with the concept to the different meanings of a topic in itself (e.g., la derivative as the gradient of a curve or as the limit of finite increments). Equally important is to consider the different phenomena associated with mathematical concepts (Rico, 1997). These are some of the considerations that the group which developed the MTSK model took into account when drawing up the description of KoT. Nevertheless, a thoroughgoing review of the various considerations involved in this subdomain is still necessary. For this subdomain, for the topic of matrices considered in Sosa (2010), we could write a descriptor such as the following, *Knowing why the elements of a specific matrix in a problem are laid out in a particular way*.

Knowledge of the Structure of Mathematics (KSM). In the MKT model, the characterisation of mathematical horizon knowledge tends to make us think of connections as the defining element of this subdomain (drawing on Fernández (2011) and Martínez, Giné, Fernández, Figueiras, and Deulofeu (2011)). However, knowledge of structure represents understanding the connections with elements that are prior and subsequent to the item being studied (Montes, Aguilar, Carrillo, & Muñoz-Catalán, 2013). It is worth noting that such shifts forward and backwards in time are not so much curricular as mathematical. An example is knowing the connection between the integration and measurement, even though they are not concepts which occur in the same year. Hence, we regard the descriptors in this subdomain as being related to the topic being studied. Thus, one such descriptor might be, *knowing the relationship between matrix algebra and geometry,* which gives us information about the knowledge of a topic which could be covered in one year, with matrix geometry, which could be from another topic in the same, or other, year.

Knowledge of Practices of Mathematics (KPM). This subdomain, which corresponds to the idea developed by Ball (1990), refers to ways of dealing with mathematics (Carrillo et al., 2013; Montes et al., 2013). In order to provide a description of knowledge about mathematics it is necessary to give preeminence to aspects of mathematical reasoning. For example, the descriptors should incorporate concepts such as definition, demonstration and argumentation, which reflect understanding of what constitutes a definition, or when a demonstration has been completed, or when a particular line of reasoning is valid. Depending on the topic in hand, this subdomain will reflect certain aspects of mathematical procedure, but these will not be concepts inherent in the concept itself. Hence, for multiplying matrices again, a KPM descriptor such as *Knowing that definitions and properties have limits* could be phrased as *Knowing why any two matrices cannot be multiplied*, which would already form part of KoT.

In the following section we will consider the subdomains corresponding to pedagogical content knowledge.

Knowledge of Features of Learning Mathematics (KFLM). It is worth reminding ourselves that understanding the manner in which pupils learn mathematics is not in itself purely mathematical, although it clearly involves mathematics, but it is very closely linked to the work of teaching. In our opinion, teachers' understanding of their pupils' learning is influenced by they way they understand learning, and hence there will be differences between those teachers who endeavour that mathematics be learnt mechanically or meaningfully (Skemp, 1978), and those who do not, or who consider only one of the modes. Thus, the descriptors for this subdomain will totally depend on the topic under consideration, as the learning related to fractions, for example, need not necessarily share the same features as the limit, although theories such as those advanced by Sfard (1991), APOS (Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1996), or any other theory of learning could prove useful in devising descriptors for every learning state. Such a descriptor from this subdomain, referring once again to the multiplication of matrices, could be phrased as *Knowing that students tend to use the commutative property for multiplying matrices*.

Knowledge of Mathematics Teaching (KMT). The knowledge described in this subdomain is that which allows teachers to take the complex series of decisions that constitutes the task of teaching, such as making the choice of an appropriate textbook, selecting a representation for a particular concept, or finding specific resource material for dealing with a topic. Hence, this subdomain requires descriptors which reflect teachers' decision making processes for carrying out a lesson. A typical descriptor would be, *Initiates the teaching of matrix algebra using non-square matrices of limited size (as with* *the scalar product of vectors)*, which draws on knowledge that is considered in other subdomains, but which also has its own existence.

Knowledge of Mathematics Learning Standards (KMLS). In this case, we consider the descriptors proposed by Sosa (2010) for curricular knowledge a good starting point, although we think it necessary to add descriptors relating to teachers' institutional context, such as aspects of knowledge deriving from professional associations (such as the NCTM), journals or research groups, beyond the confines of the prescriptions of educational authorities. As such, it would be necessary to explore which of these teachers considered conventional sources of information. For example, *being aware of professional papers dealing with student problems with matrix algebra* would be accepted as a descriptor as here research literature constitutes a standard source for developing one's educational knowledge.

A FINAL REFLECTION

As suggested above, the MTSK model represents a change of perspective in the MKT model, given that it considers all specialised aspects making up mathematics teachers' knowledge, including both the teaching profession and the object of teaching, in this case, mathematics. The specialisation of mathematics teachers' knowledge means going deeply into the idea of pedagogical knowledge and distancing on self from it, in order to achieve a 'mathematisation' of the model, which goes from considering the specificity of pedagogical knowledge referring to teaching and learning mathematical content, to focusing on the mathematical pedagogical knowledge which defines the profession of mathematics teacher.

Given that the MTSK model is under construction, we think that one of the most important tasks to see through in future studies is a more precise description of the model, creating descriptors for the subdomains which enable us achieve a better understanding of the nature of teachers' knowledge in line with the above. In like fashion, access protocols to the distinct components of teachers' knowledge are needed to thus resolve the controversies surrounding them, as in our opinion certain subdomains, such as mathematical horizon knowledge in the MKT model, despite being generally accepted as a dimension of teachers' knowledge, has limited accessibility for describing what is observed.

NOTES

[1] Each descriptor with its respective acronym and definition can be found in Sosa (2010, pp. 63-70).

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DETERMINING SPECIALISED KNOWLEDGE FOR MATHEMATICS TEACHING

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Drawing on the work of Deborah Ball and collaborators in the field of Mathematical Knowledge for Teaching (MKT), we draw attention to several areas of difficulty in applying this framework to actual samples of mathematics lessons, due to a tendency for the subdomains that make up the model to overlap. Tackling these shortcomings by viewing all mathematics teachers' knowledge as specialized has led us to reinterpret and rename these subdomains in what can be considered a reformulation of MKT.

Keywords*: mathematical knowledge for teaching, teachers' specialized knowledge, mathematics teachers' knowledge.*

INTRODUCTION

One of the benefits of research into teachers' knowledge – in our case relating to mathematics teaching – is ascertaining desirable elements and characteristics that can be taken as reference points when working with teachers. From the many such characterisations of knowledge developed in the last two decades (Bromme, 1994; Rowland, Turner, Thwaites, & Huckstep, 2009), the theory of *Mathematical Knowledge for Teaching* (Ball, Thames, & Phelps, 2008) has proved to be especially powerful in describing the knowledge required by teachers in their practice, underlining its ties with mathematics while at the same time considering other elements involved in the teaching process (e.g. the pupils and their learning, and the curriculum) and the connections between them. What is more, *Mathematical Knowledge for Teaching* (MKT) has pioneered consideration of mathematical knowledge from the point of view of teaching, including knowledge of the structure of the subject, the rules governing how it works, and careful thought about the contents and their relations. In this respect, it seems to us that the purpose of MKT is that it should be an analytical tool for studying teachers' knowledge, as opposed to a model of such knowledge itself. On the other hand, we are aware that the authors' description of teachers' knowledge is partial, omitting other equally important

[1] This article represents part of the studies into teachers' knowledge by the SIDM group (from the Spanish 'Research Seminar into Mathematics Education') based at the University of Huelva, Spain. It comprises the following researchers: José Carrillo (coordinator), Nuria Climent, M. Cinta Muñoz-Catalán, Luis C. Contreras, Miguel A. Montes, Álvaro Aguilar, Dinazar I. Escudero, Eric Flores and Enrique Carmona (University of Huelva), Pablo Flores, Nielka Rojas and Elisabeth Ramos (University of Granada, Spain), C. Miguel Ribeiro, Rute Monteiro and C. Susana dos Santos (University of the Algarve, Portugal), Leticia Sosa and José L. Huitrado (University of Zacatecas, Mexico), Diana Vasco (State Technological University of Quevedo, Ecuador) and Emma Carreño (University of Piura, Peru).

dimensions, such as teachers' beliefs and knowledge not specifically related to mathematical issues such as class management.

MKT has meant a significant advance in attempts to characterise mathematics teachers' knowledge domains. The most significant contributions are probably the differentiation of the subdomains *specialized content knowledge* (SCK), *common content knowledge* (CCK), and *horizon content knowledge* (HCK), within Shulman's (1986, 1987) classic *content knowledge*, in addition to the subdomains *knowledge of content and students* (KCS) and *knowledge of content and teaching* (KCT), within *pedagogical content knowledge*. However, as the authors themselves recognise, the new subdomains do not in practice mean an exhaustive classification of a teacher's knowledge, and often it is difficult to know whether an excerpt of classroom practice is unambiguously illustrative of one of these subdomains or rather the intersection of two or more of them. What is more, the very differentiation between SCK and CCK leaves the system open to the possibility that all teachers' knowledge is to some extent specialised.

It is precisely such dificulties in applying MKT to our studies that we wish to highlight in this paper, at the same time suggesting a reformulation of the framework from a perspective that does indeed regard all teachers' knowledge as specialised.

OUR EXPERIENCE WITH MKT

In our work on mathematics teachers' knowledge using the MKT framework (Sosa, & Carrillo, 2010; Figueiras, Ribeiro, Carrillo, Fernández, & Deulofeu, 2011; Ribeiro, & Carrillo, 2011a, b; Climent, Romero, Carrillo, Muñoz-Catalán, & Contreras, in press), we have identified various difficulties which have led us to raise questions about the model. The chief shortcomings were recognised by Ball and associates in their 2008 work, and concern *specialized content knowledge* (SCK) and *common content knowledge* (CCK [1]). Essentially, there are two related problems [2]:

1) The difficulty in deciding where CCK ends and SCK begins, as a result of the very definition of CCK. In brief, CCK is defined as that knowledge held by anybody educated to the corresponding level under analysis (Ball *et al*., 2008). In this way, although the model of MKT is based on observation, in order to decide whether the knowledge underpinning a teacher's action during a teaching episode corresponds to CCK or not, we need to compare it with the hypothetical knowledge of someone at a hypothetical level of education, without knowing anything about the educated person's practice or their typical knowledge, but instead, a compendium of desirable knowledge drawn from various curricula. Hence, for example, it is not clear whether CCK or SCK is invoked when explaining why the same denominator is needed for adding or subtracting fractions (but not in the case of multiplication or finding the quotient, although this might lead us to an alternative algorithm). Deciding whether such knowledge is typical of a well-educated individual involves a large degree of speculation. It therefore occurs to us more reasonable to define

CCK intrinsically, that is, referring exclusively to mathematical knowledge itself, without reference to other professions or qualifications.

2) The difficulty in demarking SCK from HCK, and SCK from KCS, again as a result of the definition of SCK. SCK is understood as a way of thinking about mathematics which occurs only when considered as something to be taught. However, it is sometimes difficult to determine whether this reflection refers to the relations between the item to be taught and others (HCK) or to the learning of the item (KCS). In this case, we can consider the example of the commutative property in relation to different objects. First, we will consider this property in relation to the addition and multiplication of natural numbers. Although both operations fulfill this property for this particular numerical set, from the point of view of their meaning, we can say that addition is semantically commutative, but not multiplication in general (adding or uniting 2 elements and then 3 is the same whichever the order; considering 3 groups of 2 elements is not the same, however, as considering 2 groups of 3 elements). This subtle difference affects how each case is perceived, and relates to how each is learnt. Now we will consider the property in relation to multiplying matrices. In this case, commutability does not generally occur, except in the case of square matrices in which the operation can be done either way (although these matrices do not fulfill the commutative property either). This fact differentiates the multiplication of matrices from that of numbers, and knowledge of this difference implies associating both contexts, which we would argue forms part of HCK. Additionally, it provides a mathematical explanation for a common student error in multiplying matrices, which associates it with KCS.

We have found, then, problems in the demarcation of the subdomains, in which respect we concur with the impression of other authors (Silverman, & Thompson, 2008). There is a need, we feel, to define the subdomains in a slightly different way, more appropriate, we would say, to teachers' knowledge regarding teaching mathematics. At the same time, we have tried to see to what point SCK permeates, or is included within, other subdomains, thus emphasising the valuable contribution it has made to the MKT model.

It is important at this point to give due recognition to the development of the MKT model by Ball and her collaborators, although at the same time the difficulties noted above lead us to think that it would be more appropriate to alter the focus of teachers' knowledge so that, on the one hand, it can be better understood, and on the other, its contents can be better discerned.

The difficulties in demarcation suggest the need to look more closely at MK (ie, 'mathematical knowledge' [3], that is the left hand side of the MKT model) and to progress towards defining and delimiting CCK, SCK and HCK. This is the work which we have undertaken and which we would like to present in this paper, along with the reformulation of the subdomains pertaining to *pedagogical content* *knowledge* (PCK, the right hand side of the MKT model). Throughout, we have been guided by two premises. First, we have not limited ourselves to merely observing episodes of classroom practice, but have proposed a sound theoretical model, which can be subsequently tested in practice (observations), especially longitudinal studies combining classroom observation and shared reflection. Secondly, we have remained open to the possible restructuring of the MK domain, and the potential for new or different subdomains, and even the possibility of the subdomains of PCK being affected. Moreover, this model is designed to reflect teachers' beliefs about mathematics and its teaching and learning. Although the role of these beliefs in the model is not the focus of this paper, the fact of its inclusion marks a divergence from the MKT model.

SPECIALISATION AS A GENERAL FEATURE OF MATHEMATICS TEACHER'S KNOWLEDGE: MATHEMATICS TEACHER'S SPECIALIZED KNOWLEDGE (MTSK)

What interests us as researchers and trainers within the area of Mathematics Education is the extension of teachers' professional knowledge linked to mathematics as the focus of the teaching-learning process, the recognition of which was one of the chief contributions of the work of Shulman (1987). For their part, the research team headed by Ball (Ball *et al*, 2008; Ball, & Bass, 2009) outline the mathematical knowledge within the specialised area, and it is precisely this mathematical character which causes problems when it is applied to *pedagogical content knowledge*.

We attempt to focus the specialisation of mathematics teacher's knowledge from another perspective. Instead of talking about 'specialised content knowledge' (as a part of teacher's knowledge), we talk about 'mathematics teacher's specialised knowledge' (*MTSK*). We try to distance ourselves from the idea of mathematical knowledge for teaching and think of mathematics teacher's knowledge that makes sense only to them (in which, therefore, the specialised nature defines all knowledge under consideration).

The specialisation of *MTSK* should allow it to be differentiated from general pedagogical knowledge (knowledge of pedagogy and general psychology, which also forms part of mathematics teacher's professional knowledge), from the specialised knowledge of teachers of other disciplines, and the specialised knowledge of other mathematics professionals. In other words, it is specialised in respect of mathematics teaching.

We have reconsidered the content of this knowledge from this perspective, basing ourselves on the domains of MKT and on our beliefs concerning what we consider desirable as the content of mathematics teacher's specialised knowledge in each subdomain. The outcome is that we propose to eliminate the reference to 'common content knowledge' from the domain of mathematical knowledge (given that our interest lies only with knowledge in relation to mathematics teachers; for example, we believe that teachers should possess not only the knowledge of *how* but the knowledge of *why*, and the students too, see Flores, Escudero, & Carrillo, 2013). As a result, ‘specialised content knowledge’ ceases to be necessary and ‘horizon content knowledge’ broadens its scope (resulting in two related subdomains). With regard to ‘pedagogical content knowledge’, we have renamed and reinterpreted KCS, KCT and KCC, recallibrating them to what we believe is their content.

We present this new proposal in more detail below and in a visual display (Fig. 1).

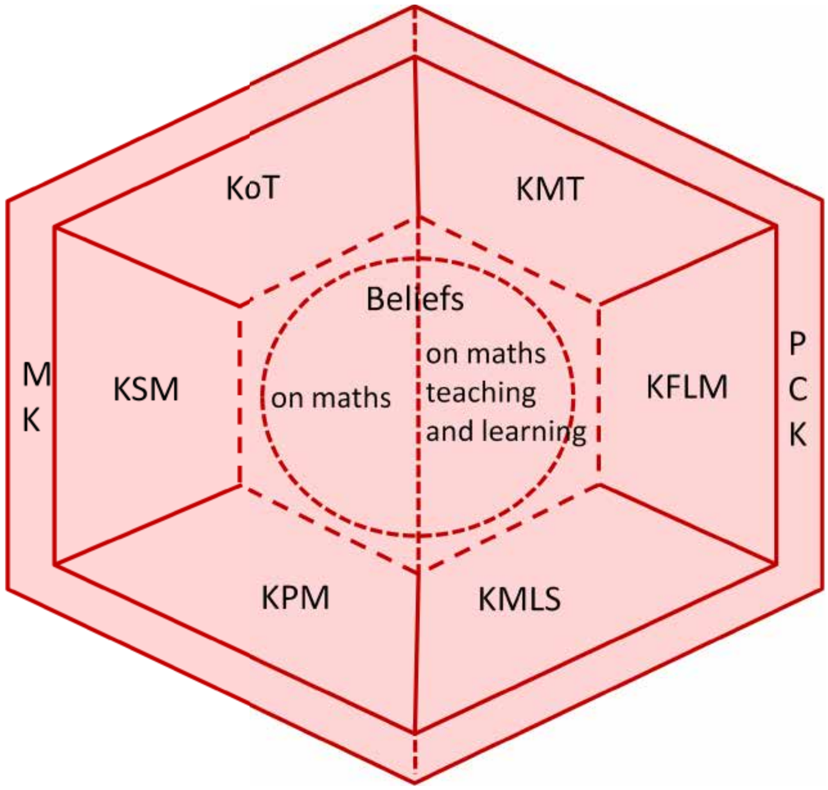

Fig. 1: Chart of *MTSK*

Elements of *MTSK* refering to Mathematical Knowledge (MK)

a) Knowledge of topics (*KoT*)

This includes the knowledge of mathematical concepts and procedures along with the corresponding theoretical foundations. We can say that all knowledge considered desirable for a pupil to be in possession of at any particular level [4] would form part of the teacher’s CCK at this level, including a certain degree of formalisation or vision of the content from a somewhat higher viewpoint (for example, knowing that the property of commutability represents a more technical explanation of the fact that the order of addends in an addition sum does not affect the result).

b) Knowledge of the structure of mathematics (*KSM*)

Building on Ball and Bass’ (2009) description of horizon knowledge, we consider two elements of MK relating to the structure of the discipline (this subdomain) and the ways of proceeding in mathematics [5] (the next subdomain).

The first of these elements, knowledge of the structure of the discipline, includes knowledge of the main ideas and structures, such as knowledge of properties and notions relating to specific items being tackled at any moment, or the knowledge of connections between current topics and previous and forthcoming items. It implies seeing the content in perspective, basic mathematics from an advanced point of view, and advanced mathematics from a basic point of view. Also included is the idea of increasing complexity, as explained in Montes, Aguilar, Carrillo, & Muñoz-Catalán (2013, in this volume).

c) Knowledge of the practice of mathematics (*KPM*)

The second of these elements refers to ways of proceeding in mathematics. It includes knowledge of ways of knowing and creating or producing in Mathematics (syntactic knowledge), aspects of mathematical communication, reasoning and testing, knowing how to define and use definitions, establishing relations (between concepts, properties etc), correspondences and equivalences, selecting representations, arguing, generalising and exploring. Knowledge about relations or connections between concepts, pertaining to *knowledge of the structure of mathematics,* should be distinguished here from knowledge about how such relations are established.

Defined in this way, MK extends over the full range of mathematical knowledge, covering the whole universe of mathematics, comprising concepts and procedures, structuring ideas, connections between concepts, the reason for, or origin of, procedures, means of testing and any form of proceeding in mathematics, along with mathematical language and its precision. The denomination *KoT* emphasises that the subdomain is defined in purely mathematical terms, and this we think makes it clearer that *knowledge of topics* and *knowledge of the structure of mathematics* form a complex system. At the same time, this way of defining *KoT* avoids the somewhat mechanical slant which the definition of CCK was prone to, as sometimes was the knowledge *of* mathematics (in the sense of knowledge *of* and *about* mathematics, Ball, 1990).

Returning briefly to one of the examples above, knowing that the product of matrices is not commutable pertains to *KoT*; knowing that in this sense it is different from the multiplying natural numbers would pertain to *knowledge of the structure* (as it means taking a basic viewpoint to the multiplication of matrices, like multiplying numbers) and knowing that the pupils believe that the product of matrices is commutative because they extrapolate this property from multiplying numbers (which they learn at school) would form a part of *pedagogical content knowledge* (as we will now explain). If we take the point of view of *specialized content knowledge* within the MKT model, reflection about specialisation in both contexts is a reflection about the specific content of the act of teaching, for which reason we consider it as SCK, which, as we have noted above, results in overlap with *horizon knowledge* and *knowledge of content and students*.

Elements of *MTSK* refering to Pedagocical Content Knowledge (PCK)

d) Knowledge of Features of Learning Mathematics (*KFLM*)

KFLM derives from the teacher's need to understand how pupils think when faced with mathematical activities and tasks, the same as KCS in Ball's model. It is important that the teacher is aware that the pupils may have problems with a particular topic. This awareness is fed by the teacher's general knowledge of the topic and by their familiarity with the pupils. This subdomain encompasses a range of knowledge, including (and not, we believe, explicitly included in KCS) theories or models of how students learn mathematics (for example, the process which takes pupils from action to schema according to the APOS perspective –Asiala, Brown, DeVries, Dubinsky, Mathews, & Thomas, 1996). It is not a question of knowing these theories or perspectives, but rather their significance, that is to say, what these theories contribute to describing the process of learning mathematics. *KFLM* is not mathematical knowledge, although the teacher needs to have a background in mathematics in order to understand it and put it to use. KCS refers to content and students, while *KFLM* is concerned with how mathematics is learned, that is, with identifying the features of mathematics learning.

e) Knowledge of Mathematics Teaching (*KMT*)

KMT is not mathematical knowledge either, though it does require it. It is the kind of knowledge which allows the teacher to choose a particular representation or certain material for learning a concept or mathematical procedure, and which allows them to select examples or choose a textbook, in much the same kind of way as Ball's KCT. We would underline here (encapsulated in the name of the subdomain) the integration of mathematics and teaching, in that it is not a question of mathematical knowledge on the one hand and teaching knowledge on the other; pedagogic knowledge is not included here in the context of mathematical activities, but rather only that in which the mathematical content constrains the teaching. In *KMT* we locate knowledge of resources from the point of view of their mathematical content or the knowledge of approaching a structured series of examples to help pupils understand the meaning of a mathematical item.

f) Knowledge of Mathematics Learning Standards (*KMLS*)

KMLS concerns knowledge of curricular specifications, the progression from one year to the next, conventionalised materials for support, minimum standards and forms of evaluation, in the same way as KCC does in Ball's model. However, *KMLS* seeks to extend knowledge of learning objectives and standards beyond those deriving from the institutional context of the teacher. We include objectives and measures of performance developed by external bodies such as examining boards, professional associations and researchers, thus adding an element of assessment and evaluation drawn from the appropriate educational agencies.

FINAL COMMENTS

With *MTSK* we have intended to focus solely on mathematics teacher's specific knowledge with respect to teaching the subject, eliminating any reference to a common core of knowledge shared with others who make use of mathematics. *Knowledge of topics* and *Knowledge about mathematics* are shared by all mathematicians, as is to a certain extent *Knowledge of the structure of mathematics*, though not to the degree of familiarity required by teachers. Conversely, *KMT*, *KFLM* and *KMLS* are exclusive to teachers.

With MKT the focus was on the class as a whole, including pedagogical concerns (KCT, KCS) to an extent that the framework might be applicable to other disciplines, but in so doing it shifted away from mathematics and its core essence. In contrast, rather than consider the class as a whole, we aim to consider mathematics as the hub of *MTSK*, around which this new framework offers different ways of viewing the mathematics which the teacher knows and uses. We refer not only to mathematics in itself, but to reflections about mathematics that a teacher establishes by interacting with it in their daily practice, out of which aspects of mathematics pedagogy inevitably arise (*KMLS*, *KMT*). MKT concerns the educational circumstances constraining the teacher: recognising the causes of error, using powerful examples, identifying incorrect definitions in textbooks, etc. In contrast, *MTSK*, by virtue of being designed to encapsulate teacher's specialised kowledge, focuses its attention on mathematical content and, with greater precision, on the different ways of fully engaging with mathematical content when teaching.

In this paper we have endeavoured to discuss the defining features of the subdomains comprising MKT and suggest an alternative model based around it. We propose that this model (and others) be conceived of, and brought into play, as a kind of researcher's kit which helps them to avoid a prescriptivism which might impede understanding the phenomenon under scrutiny. Studying *MTSK* using this kit will enable our knowledge of its categories and subdomains to be reinforced.

At the same time, it would be interesting to pursue another line of research attempting to situate the model within a theoretical framework, in which, amongst other things, we would have to explicitly present our grounded position regarding how we understand teaching and learning mathematics, teacher training (both initial and in-service), our mathematics beliefs, the utility of models and other analytical tools, the purpose of our research, and the role of the subjects being studied/participating in that research, especially in relation to the researchers and the carrying out of the research itself.

In our work group we are drawing up research projects into mathematics teachers' knowledge in terms of different topics and different kinds (some structuring, such as the notion of infinity, and others more local, such as the concept of a polygon), and focusing on different dimensions of *MTSK*. Some of these projects involve experienced teachers (occasionally in the context of professional development), others with novice teachers, and others with students' teachers. Different educational phases are also involved, and likewise different stages of the teaching-learning process (exemplification, the introduction of concepts or procedures, designing tasks, and making decisions in class). Our aim across the board is to explore the limits and potential shortcomings of our proposal for *MTSK*, and to refine it further.

It stripes us that a better specification of desirable professional knowledge for a mathematics teacher from research is expecially important in contexts of professional development, particularly in collaborative situations, where the group itself is at liberty to decide what to study and reflect on (in terms of professional practice, for example), *MTSK* being one such possibility. It is not a question of having available a model which can be gradually assimilated, so much as having this model available as a point of departure for shared reflections forming the platform on which to design the group's collaborative work (Carrillo & Climent, 2011).

Acknowledgements

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NOTES

1. We use the acronymns coined by Ball and collaborators themselves; in addition to those above, KCT for Knowledge of Content and Teaching, KCS for Knowledge of Content and Students, and likewise HCK for Horizon Content Knowledge and KCC for Knowledge of Content and Curriculum.

2. For further explanation, as well as examples of the difficulty in applying MKT, see Flores, Escudero, & Carrillo (2013, this volume).

3. Mathematical knowledge is understood as Shulman's subject matter knowledge. We use MK instead of SMK to avoid confusion with SCK or SMK in reference to specialised content or mathematical knowledge, respectively.

4. From a conception of school mathematics in which the pupils also learn the 'whys' of procedures and the reasons for certain concepts (for example, why fractions with the same denominator are needed to add fractions, but not to multiply them).

5. Ball, & Bass (2009, p.6) mention four elements constituting HCK: "a sense of the mathematical environment surrounding the current "location" in instruction; major disciplinary ideas and structures; key mathematical practices; and core mathematical values and sensibilities". In our case, we have considered the first three elements, given that values and sensibilities means introducing a different kind of element from the rest of the components of the model.

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TEACHING MULTIDIGIT MULTIPLICATION: COMBINING MULTIPLE FRAMEWORKS TO ANALYSE A CLASS EPISODE

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This paper provides an analysis of a teaching episode of the multidigit multiplication algorithm, with a focus on the influence of teacher's mathematical knowledge on his teaching. The theoretical framework uses Mathematical Knowledge for Teaching, mathematical pertinence of the teacher and structuration of the milieu in a downward and upward a priori analysis and an a posteriori analysis. This analysis shows a development of different didactical situations and some links between mathematical knowledge and pertinence. In the conclusion, the contribution of the two traditions originated frameworks is briefly addressed.

Keywords: *Mathematical knowledge, teacher, elementary teaching, multiplication algorithm, pertinence, structuration of the milieu*

This paper originated in a doctoral research project (Clivaz, 2011) that aimed to describe the influence of the mathematical knowledge of primary school teachers on their management of school mathematical tasks. The origin of that question partly came from U.S. mathematics education research and partly from the French *didactique des mathématiques*. Ball's categories of Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008) were used to describe the teacher's knowledge, while I described the effect on teaching through mathematical pertinence of the teacher (Bloch, 2009). I analysed the teacher's knowledge and the effect of this knowledge in ordinary classroom situations with the model of structuration of the milieu (Margolinas, Coulange, & Bessot, 2005) to take into account the complexity of the teacher's activity.

After a brief explanation of these three frameworks, their interaction will be shown through an episode about the teaching of the algorithm of multidigit multiplication. Finally, I will discuss the interaction of these frameworks for analysing the teacher's knowledge and teaching.

FRAMEWORK

CATEGORIES OF MATHEMATICAL KNOWLEDGE FOR TEACHING

Refining Shulman's (1986, 1987) categories of teacher knowledge for mathematics, Ball, Thames, and Phelps (2008) provide a practice-based division of Mathematical Knowledge for Teaching (MKT).

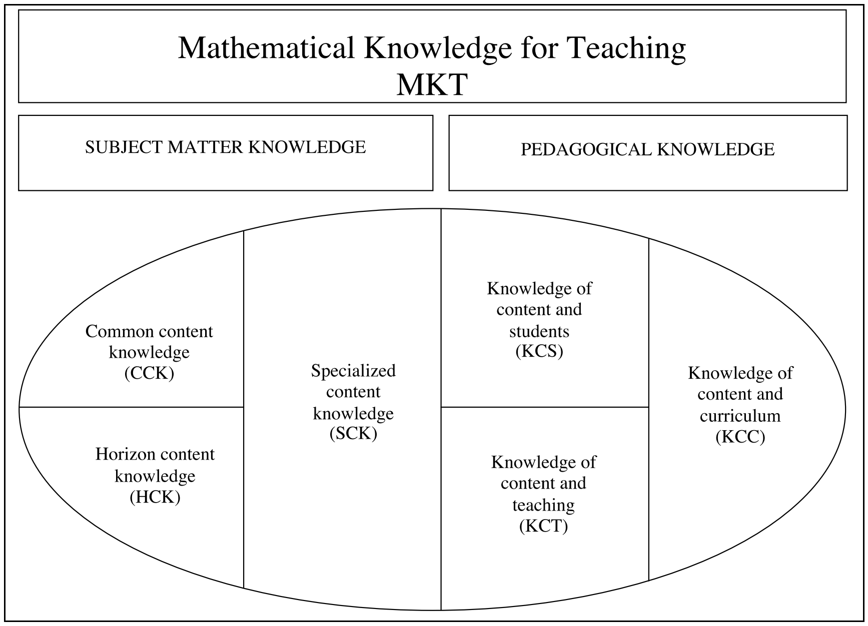

Figure 1: Domains of mathematical knowledge for teaching (Ball et al., 2008, p. 403)

One of the special features of this categorization is the existence of a *Specialized Content Knowledge* (SCK), defined as

> the mathematical knowledge and skill unique to teaching. In looking for patterns in student errors or in sizing up whether a nonstandard approach would work in general, […] teachers have to do a kind of mathematical work that others do not. […] This work involves an uncanny kind of unpacking of mathematics that is not needed–or even desirable–in settings other than teaching" (Ball et al., 2008, p. 400).

One of the examples of the use of MKT Ball and her colleagues often provide is the teaching of the multiplication of whole numbers algorithm (Ball, Hill, & Bass, 2005, pp. 17-21). I studied examples of this teaching with four teachers in the Lausanne region of Switzerland (Clivaz, 2011), and the case I will analyse is about this teaching.

MATHEMATICAL PERTINENCE OF TEACHER'S ACTIONS

In order to detect the effects of a teacher's mathematical knowledge, Bloch (2009) suggests considering the *mathematical pertinence of teacher's actions*. An action is pertinent if it allows the student to grasp the functionality of mathematical object, with enouncement of mathematical properties, mathematical arguments for the validity of procedures, or for the nature of mathematical objects[1]. Bloch gives three criteria for this pertinence, the first being the "ability to interact with the students on mathematical aspects of the situation and to encourage their activity by the means of interventions and feedback on their mathematical production[2]" (p. 32).

STRUCTURE OF THE MILIEU

To describe the teacher's activity, Margolinas (Margolinas, 2002; Margolinas et al., 2005) has developed a model of the teacher's *milieu*[3], based on Brousseau (1997), which she also uses as a model of the teacher's activity.

+3	*Values and conceptions about learning and teaching* Educational project: educational values, conceptions of learning, conceptions of teaching
+2	*The global didactic project* The global didactic project, of which the planned sequence of lessons is a part: notions to study and knowledge to acquire
+1	*The local didactic project* The specific didactic project in the planned sequence of lessons: objectives, organization of work
0	*Didactic action* Interactions with pupils, decisions during action
-1	*Observation of pupils' activity* Perception of pupils activity, regulation of pupils' work

Figure 2: Levels of a teacher's activity (Margolinas et al., 2005, p. 207)

At every level, the teacher has to deal not only with the current level, but at least also with the levels directly before and after the current level. This tension makes a linear interpretation of teacher's work inaccurate (Margolinas et al., 2005, p. 208). In fact, a more complete model can be considered, including the student (E, for *élève*), the teacher (P, for *professeur*), and the milieu (M). Each milieu M_i is constituted at each level i by the lower E_{i-1}, $P_{i-1,}$ and M_{i-1} component and the situation S_i is made at each level i by E_i, $P_{i,}$ and M_i. This can be written as $S_i = (M_i ; E_i ; P_i)$ and $M_i = S_{i-1}$, or more visually represented in an onion diagram (Figure 3) or in a table where the teacher's levels range from +3 to -1 and the student's levels range from -3 to +1.

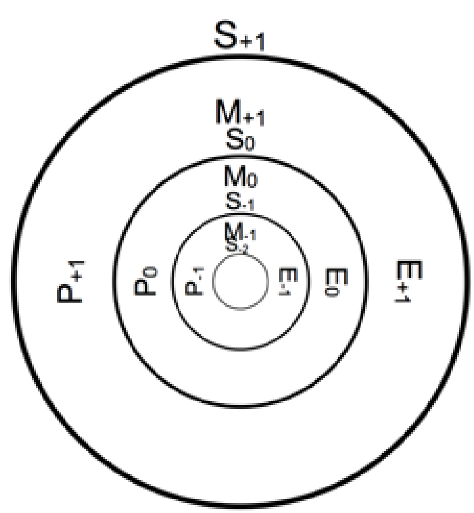

Figure 3: Structure of the milieu, level -1 to +1

Therefore, S_0, the didactical situation, can be determined either from the teacher's point of view, by a *downward analysis*, or from the student's perspective, by an *upward analysis*. This latter may conduct to one or more didactical situations S_0 which may not be the same as the situation determined by the downward analysis. Margolinas (2004) calls this a *didactic bifurcation*.

The downward analysis uses mainly the audio-recorded interview we had with each teacher before the lessons about multidigit multiplication. The upward analysis is *a priori* "in the sense that it doesn't depend on experimental or observational facts[4]" (Margolinas, 1994, p. 30). Both are then tools to analyse the classroom observations in an *a posteriori* analysis.

THE CASE OF DOMINIQUE

The episode, 27 minutes in duration, that was analysed is taken from the series of seven lessons Dominique, a Grade 4[5] teacher, devoted to this algorithm in his class. The analysed episode features the moment where Dominique plans to show the multidigit multiplication algorithm. The video recording was transcribed and coded with Transana software (Fassnacht & Woods, 2002-2011) according to the categories of mathematical knowledge for teaching, mathematical pertinence, and the levels of the teacher's activity. This extract can be situated in the series by means of a *synopsis* (Schneuwly, Dolz, & Ronveaux, 2006) and a *macrostructure* (Dolz & Toulou, 2008).

The *a priori* downward analysis will be presented in the following sections of this paper. Due to space constraints, the *a priori* upward and *a posteriori* analyses cannot be developed here. We will very briefly describe the situations S_0 these *a priori* analyses reveal and, in the *a posteriori* analysis, we will suggest reasons for some highlighted troubles, in terms of types of Mathematical Knowledge for Teaching (MKT) and mathematical pertinence.

A PRIORI ANALYSIS

Downward analysis

Based on the interview with the teacher, the downward analysis goes from level +3 to level -1 and determines *a priori* the didactical situation S_0 from the teacher's point of view. The topics Dominique addresses are various, so I focus only on the question about the type of algorithm, the MKT linked to that question, and the consequence about the determination of S_0 didactical situation.

Dominique thinks that pupils should understand what they do in math (level +3). He also feels this way about the algorithms, but he views algorithms as tools for problem solving (+3). So, for the series of lessons about multidigit multiplication, the main goal is that the students are able to carry out the algorithm and use it efficiently (+2). The type of algorithm is not important if it is efficient for the students (+2). Dominique knows that there are several kinds of algorithms for multidigit multiplication, and he plans to show two of them: the "table algorithm" (Figure 4, left) and the algorithm *en colonnes*[6] (EC) (Figure 4, right). This way of showing more than one algorithm is consistent with the official regional instructions (DFJ, 2006) and textbooks (+3) (Danalet, Dumas, Studer, & Villars-Kneubühler, 1999). At the end of the chapter of the textbook about this topic, Dominique will ask students to only retain and use the EC algorithm, with the justification that this is the algorithm everybody learned at school, and it is more efficient than the table algorithm (+2).

For the lesson, Dominique plans to show first the table algorithm on a two-digit by two-digit multiplication question and then to show the EC algorithm on the same example. He knows that the two algorithms give the same results and that the partial results can be compared line by line (SCK, +2), and he plans to show that (+1). However, he doesn't mention any other link between the table algorithm and EC – in general (+2), when planning the lesson (+1), or when envisioning teaching the lesson (+1). In addition, he does not observe the students using the two algorithms on the same multiplication question (-1). To make the line-by-line comparison possible, he plans on asking the students to "put the tens below": "It's not very logical, but it allows the student to have the two (lines) in front of each other"[7]. He never mentions any other reason or justification for this step and never mentions the possibility (and the effects) on the inversion of the two factors (SCK, +2).

Dominique thinks that the only problems pupils will face in the EC algorithm are multiplication facts and the second line zero (KCS, +2). He foresees that he will observe many errors about this zero (KCS, -1). So he plans to repeat the *zero rule*: "when one works with tens, a zero must be added"[8].

For Dominique, multiplication is shortcut for addition (SCK, +2). He never plans to mention any link with area when explaining the table or EC algorithm (+1) even if he asked one area problem to introduce the necessity for building an algorithm (+2). Challenged about his representation of multiplication, he never gives any other representation, and when asked about the link between multiplication and area, he answers that the area has to be computed with multiplication (CCK, +2).

These elements contribute to determine the didactical situation S_0 from the teacher's perspective. It can be summarized in four points:

1. Show the table algorithm for the example 12×17, requiring writing the units first for 17.
2. Show the EC algorithm on the example 12×17. Write the EC algorithm next to the table. After the first line, highlight the fact that the results of both algorithms' first lines are the same.
3. Write the zero at the right place in the second line, because "when one works with tens, a zero must be added". Carry out the second line and highlight the fact that the results of both algorithms' second lines are the same.
4. Finish by adding the two lines.

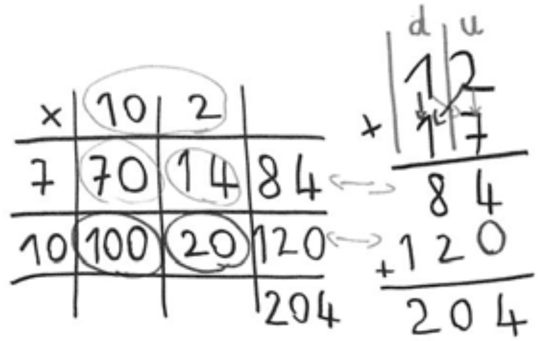

Figure 4: Two algorithms for the multiplication 12x17, written by Dominique on a poster.

Upward analysis

The elaborate upward analysis (Clivaz, 2012) starting from M_{-3} material milieu, shows that the student can deal with M_{-3} in different ways about the parallelism between the table and the EC: one row with one row (same correspondence as the teacher), partial product to partial product, just copy the results or do the two algorithms independently. He/she can also apply the *zero rule* in three ways: he/she can write a zero before beginning the second row with no further interrogation, he/she can link this zero to each zero in the table's second row, and he/she can literally apply the teacher's explanation, adding a zero each time he/she works with a ten. The combination of these two dimensions conducts to twelve situations S_{-2}, from which six seem consistent and lead to six S_0 didactical situations.

A POSTERIORI ANALYSIS

The detailed *a posteriori* analysis of the 27-minute episode (Clivaz, 2011, pp. 194-204; 2012) compares the *a priori* analysis with the video and the transcript of the actual lesson. It shows that most of the students considered the two algorithms independently or in a line-by-line correspondence, and wrote the zero without interrogating. These students were in two of the S_0 situations that the upward analysis determined. However, one student, Armand, repeatedly asked questions about why the teacher didn't add a zero *each time* he used tens. He was in another S_0 situation. He also asked several times, "Is it 1×1 or 10×10 ?[9]". However Dominique kept *his* S_0 situation and was not able to even understand Armand's interrogations.

MKT and pertinence

The proliferation of didactical bifurcations and the inability of the teacher to notice that the Armand's S_0 radically differed from his have their origin in the teacher's choices made at the +3 to 0 levels. Additionally, these choices may be understood as a consequence of the teacher's MKT.

The first choice was to use the table algorithm and particularly the correspondence between the lines' sum in the table and in the EC's lines, but with no explicit correspondence between each partial product. Dominique's MKT about the table were accurate, as revealed in the interview, but they were not pertinent, since they didn't allow him to interact with the students on the mathematical parts of the situation (pertinence's first criteria according to Bloch, 2009). The reason for this discrepancy between knowledge and pertinence was the knowledge of multiplication itself. For Dominique, multiplication was only a shortcut for repeated addition. He never considered it as a Cartesian product or as the area of a rectangle. Therefore, to Dominique, EC and table algorithm were two ways to perform multiplication; they were not linked to multiplication itself and they were just linked to each other because they gave the same result.

The second choice was the "recipe" for *zero rule*. This rule is problematic in many ways: use of additive words (add a zero), lack of link with place value, and above all, fallacy if literally applied. Regarding these two choices, Dominique had a working *Common Content Knowledge*, but he couldn't unpack it and couldn't use the corresponding *Specialized Content Knowledge* to explain why a zero appears when one multiply by tens.

CONCLUSION

This episode analysis used the structure of the milieu to highlight and to analyse links between mathematical knowledge for teaching (MKT), pertinence, and teaching choices of the teacher. It showed that not only Common Content Knowledge is necessary to apply pedagogical MKT, but also "that each of these common tasks of teaching involves *mathematical* reasoning as much as it does pedagogical thinking" (Ball et al., 2005, p. 21). It is one illustration of the way one U.S. mathematics education framework and elements of the *Theory of didactical situations* (Brousseau, 1997) can interact to analyse a math teaching issue. The original question was about mathematical knowledge of the teacher, but the finesse of the structuration of the milieu was necessary to show the multiplicity of the various didactical situations. The distinction of specialized content knowledge among MKT was needed to analyse the causes of the phenomena when the structuration of the milieu and the pertinence were crucial to capture the movement of didactical situations beyond the static character (Ball et al., 2008, p. 403) of Ball's categories, to see not only mathematical knowledge for teaching but mathematical knowledge *in* teaching (Rowland & Ruthven, 2011).

The combination, in the sense of Prediger, Bikner-Ahsbahs, and Arzarello (2008), of the frameworks from two cultural backgrounds allowed to "get a multi-faceted insight into the empirical phenomenon in view" (p. 173). It was more vastly used in other parts of the doctoral research (Clivaz, 2011) and gave some interesting results, for example about the correlation between MKT and pertinence. For that purpose, we invite the interested reader to read the full dissertation.

These two frameworks also raise one more general issue. The question of mathematical knowledge of the teacher is widely studied in the English-speaking mathematics education community, but, according to the National Mathematics Advisory Panel (2008), it was often studied with the quantitative point of view of the influence on students' test outcomes. It is far less disputed[10] in the French speaking *didactique des mathématiques*, even though the developed model could offer tools to discuss the influence of MKT on teaching. I hope that studies using frameworks originated in the two contexts will be developed to connect different theoretical approaches as promoted in the ERME spirit.

NOTES

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[1] « Une intervention mathématique est pertinente si elle rend compte dans une certaine mesure de la fonctionnalité de l'objet mathématique visé ; ou, s'agissant d'enseignement, si elle permet au moins de progresser dans l'appréhension de cette fonctionnalité, avec des énoncés de propriétés mathématiques contextualisées ou non, des arguments appropriés sur la validité de procédures ou sur la nature des objets mathématiques. » (Bloch, 2009, p. 32)

[2] « […] capacité à interagir avec les élèves sur des éléments mathématiques de la situation et à encourager l'activité des élèves par des interventions et des retours sur leur production mathématique. » (Bloch, 2009, p. 33)

[3] Milieu is the usual translation for Brousseau's French term "milieu", but, in French, it refers not only to the sociological milieu but it is also used in biology or in Piaget's work. A more accurate translation would be "environment".

[4] « dans le sens qu'elle ne dépend pas des faits d'expérience ou d'observation », my translation.

[5] 9-10 year-old students.

[6] Literally "in columns" but the accurate English name would be "long multiplication".

[7] « Mettre les dizaines en dessous. C'est pas très logique, mais ça permet d'avoir les deux en face. »

[8] « Quand on travaille avec les dizaines, on ajoute un zéro. »

[9] « c'est 1×1 ou 10×10 ? »

[10] With the notable exception of Quebec and the presence of a Working Group on the topic in EMF congress (Clivaz, Proulx, Sangaré, & Kuzniak, 2012).

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TEACHERS' BELIEFS ABOUT KNOWLEDGE FOR TEACHING – AN INDIRECT APPROACH

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In this paper we analyze teachers' beliefs about the knowledge needed for teaching elementary school mathematics. Eliciting such beliefs is important for designing and evaluating teacher education. We find indications of these beliefs in anonymous feedback questionnaires the teachers submitted in an in-service professional development course. This indirect approach avoids discrepancies between teachers' declared (conscious) beliefs and tacit beliefs that actually influence their learning. We found that beliefs changed during the course, at first favouring pedagogical content knowledge (PCK) as a learning goal, and shifting toward subject matter content knowledge (SMCK). The significance of this research is not only its findings but also its method, which avoids some issues inherent in traditional methods.

Keywords: Teacher beliefs, knowledge for teaching, mathematics education, professional development, mixed methods

INTRODUCTION

Teachers' beliefs are an important theoretical construct which has been receiving much attention in recent years. Teachers' beliefs about mathematics and about teaching and learning mathematics have an impact on their in-the-moment teaching decisions (Schoenfeld, 2010); however in this paper we are interested in teachers as *learners*. Teachers' beliefs about the knowledge they need for teaching influence what they learn and how they learn it. "*... teachers may be guided by their beliefs about teaching knowledge ... Such beliefs may lead them to question the value of information presented...*" (Fives & Buehl, 2008, p. 135). This idea is also supported by the theory of Adult Learning (Knowles, 1990), which states that adults tend to have a task-centred orientation to learning, i.e. are motivated to learn to the extent that they believe the learning will be instrumental in their professional practice.

There are at least two ways in which teachers' beliefs about knowledge for teaching might be considered when designing in-service professional development (PD[1]): taking their beliefs as given and aligning the PD teaching goals with them, or conversely, considering their beliefs as something that may change as a result of the PD and making beliefs an explicit teaching goal, thus aligning them with the intended PD content. In either approach, a reliable tool for revealing teachers' beliefs about knowledge for teaching is required.

The most direct method for revealing teachers' beliefs is asking the teachers about them, either through questionnaires or through interviews. A problem with such an approach is related to what Toerner (2002) calls *membership degree* attributes of beliefs, and more specifically, levels of *consciousness* and levels of *activation* of beliefs. Belief systems are complex; people may hold a variety of beliefs, more or less conscious, possibly conflicting, which are activated in specific situations. It is questionable whether beliefs that are activated when answering a questionnaire (necessarily conscious beliefs), or even during an interview, will be consistent with "situated" beliefs activated while teaching in a classroom or while learning (or refraining from learning) in a PD course. Beliefs activated in a "lab" environment may reflect an idealized version of teachers' beliefs, disregarding the complexity of natural situations, where unconscious or tacit beliefs may influence behaviour. Group interviews have been used to address this issue (Fauskange, 2012). A group discussion is more likely to elicit beliefs in all their complexity (Bryman, 2004). However, group dynamics are quite complex in their own right. It may be difficult to discern individuals' beliefs from a group interview, and furthermore, a group interview may actually influence the individual's beliefs - a fact which may serve the PD goals, but impairs the method's validity as a research tool.

In this paper we use an indirect approach to eliciting teachers' beliefs about the knowledge they need for teaching. We examine anonymous feedback forms, where the teachers reflect on more and less successful PD activities, and explain *why they were more or less successful in their opinion*. We will argue, after presenting our methodology, that this indirect approach to beliefs addresses the theoretical and methodological difficulties listed above; however, we justify this claim only from a theoretical perspective. Justifying this claim empirically is a goal for future research.

We aim to answer the following research questions:

- What beliefs about the nature of mathematical knowledge needed for teaching are tacitly implied in the teachers' feedback questionnaires?
- How did these beliefs change during the one-year PD course?

RESEARCH SETTING

In our research project we observed a mathematics professional development course for in-service elementary school teachers. A unique aspect of this PD was the fact that it was conceived by a professor from the mathematics department of a leading university in Israel, and was taught by graduate research students, primarily mathematics Ph.D. students. In this setting, the teacher-students and the mathematician-instructors had quite different initial beliefs about the knowledge that should be taught in the PD. Roughly speaking, interviews with the instructors revealed a belief that the teachers' knowledge of mathematics should be deepened and broadened, whereas teacher expectation questionnaires revealed beliefs to the effect that what they need most is knowledge related directly to the teaching of mathematics, primarily teaching strategies and ready-made classroom activities. The course consisted of 10 3-hour sessions. *Learning episodes* typically began with a mathematical problem related to the teachers' grade-level content (i.e. multiplication properties, representations of fractions, etc.). Sometimes the problems were designed to challenge the teachers' understanding of these topics. The problems often led to open discussions where the teachers could raise pedagogical concerns. More details can be found in (Cooper & Arcavi, 2012).

In this paper we focus on one group of 19 elementary school teachers (grade 3), taught by two graduate research students in tandem[2]. The teachers were all general teachers, who teach a variety of elementary school subjects in addition to mathematics. In addition to expectation questionnaires, which revealed the teachers' *prospective* views, the teachers were invited to submit anonymous feedback questionnaires after each of the PD sessions. These questionnaires comprise the data for our research. We present here the questions, along with authentic sample answers:

1. Select an activity from today's PD which you consider particularly successful. Explain in what ways it was successful. *"...[because the activity] provided a look at what goes on in the children's heads."*[3]

2. Select an activity from today's PD which you consider less successful. Explain in what ways it was less successful. *"...[because the activity is] not suitable for my classroom."*

3. If you have any additional comments, write them here. *"...[the PD] is becoming more accurately aligned with needs from the field."*

Throughout the 10 PD sessions, a total of 69 feedback forms were submitted, comprising approximately 40% response rate. In the general research design, the main purpose of these questionnaires was as a means to identify more and less productive learning episodes as perceived by the teachers. However, in this paper we are not interested in the selected episodes themselves, only in the *reasons* the teachers brought to justify their selections. Our assumption is that this will elicit teachers' beliefs (possibly tacit) about the knowledge they need, and that eliciting these beliefs indirectly may alleviate many of the problems inherent in traditional methods. In referring to teaching episodes (and not to themselves), the teachers are more likely to reveal genuine beliefs. Furthermore, these beliefs are activated in the context of learning mathematics, which is exactly the situation in which these beliefs are relevant to mathematics education and research. Another theoretical issue that our approach addresses is related to the differences between teachers' prospective and retrospective views (Roesken, 2011). By gathering information throughout the PD, we are filling the gap between prospective views, as indicated at the outset, and retrospective views formed as a result of the PD.

METHODOLOGY

In this research we mixed qualitative and quantitative methods. We analyzed the teachers' feedback in a qualitative manner – coding and categorizing the data – and proceed to analyze the coded data quantitatively to find patterns and trends.

Once the feedback forms were transcribed and fed into a technological tool (Atlas.ti), we segmented the data into phrases (*quotations* in the tool's terminology), and coded each data segment according to the knowledge for teaching that it implied. This is a crucial point: Teachers did not *tell* us what they believe, but rather they *made use* of their beliefs, perhaps tacitly, by selecting outstanding episodes, and revealed these beliefs indirectly by explaining their selections.

This initial coding was fine grained, resulting in 81 different codes. For example, the phrase "*... [the activity] provided a look at what goes on in the children's heads*" was coded as "*understanding students' thinking*", since this utterance, as a reason for selecting a particularly successful activity, indicates that for this teacher, understanding students' thinking is important. At this early stage, codes were **not** mapped to any particular categories. The fine-grained coding was intended to permit a variety of different categorizations. Although we had a theory-based coding scheme in mind (MKT, described below), we could later adopt a completely different categorization scheme very efficiently, without re-coding.

In explaining why a PD activity was particularly (un)successful, the teachers are implicitly answering the question "what kind of mathematical knowledge should be taught in the PD"? The theoretical framework of mathematical knowledge for teaching (MKT), as put forth by Hill, Ball, & Schilling (2008), was a natural framework for categorizing responses, since we expected the data to reflect the teachers' beliefs about the types of mathematical knowledge that they need in order to teach, and therefore hoped would be taught in the PD. This framework, inspired by Shulman (1986), differentiates between subject matter content knowledge (SMCK) and pedagogical content knowledge (PCK), and further refines each of these categories. SMCK is sub-categorized into common content knowledge (CCK - mathematical knowledge that is used in teaching in ways similar to the ways in which it is used in other occupations that use mathematics), specialized content knowledge (SCK – mathematical knowledge that specifically serves teachers when engaging in teaching tasks), and horizon knowledge (HK - an awareness of how mathematical topics are related over the span of mathematics included in the curriculum). PCK is sub-categorized into knowledge of content and teaching (KCT – knowledge that combines knowing about teaching and knowing about mathematics), knowledge of content and students (KCS – content knowledge intertwined with knowledge of how students think about, know, or learn this particular content) and knowledge of curriculum (KC – a familiarity with text books and other teaching resources).

Codes were categorized into one of the six categories of MKT. For example, the code *understanding students' thinking* was categorized as KCS. Some teachers were more verbose than others, and indicated a variety of reasons for selecting activities. In such cases more than one code was assigned to a single quotation, but each code was mapped to one MKT category at most. Some codes (for example *I enjoyed the activity*) were not mapped to any MKT category.

All the coding was carried out by the two researchers, working together. Some quotations were difficult to code and to categorize, due to ambiguity in the teachers' words. These and all other points of disagreement were debated until unanimous agreement was achieved. As a rule, when teachers indicated satisfaction or dissatisfaction with the subject matter, we tended to assume they were referring to

SCK, since the **common** content (CCK) for lower elementary school (the mathematical content that the children should acquire) is quite straightforward.

In our analysis we checked the prominence of the various categories, and looked for trends over time.

ANALYSIS

We begin with some descriptive statistics. In 67 out of the 69 forms the teachers indicated a particularly **successful activity**. A total of 32 different activities were selected as particularly successful, and were indicated a total of 80 times (clearly, some teachers listed more than one successful activity in a single form). Nearly all of the teachers explained their selection. These explanations generated 27 codes in 82 quotations, for example: the quotation "*... use this method to uncover student errors*" was coded as *discovering student errors*. In 24 out of the 69 forms a particularly **unsuccessful activity** was indicated. A total of 12 different activities were selected as particularly unsuccessful and were indicated a total of 24 times. The teachers' explanations for their choice generated 16 codes in 23 quotations, for example: the quotation "*the movie clip was tiresome*" was coded as *tiresome*.

28 out of the 69 forms included **general comments**, which generated 38 codes in 62 quotations. Teachers' general comments, e.g. "*please teach us fractions*", "*I like the PD because it provides tools I can use*", were analyzed separately from teachers' reasons for choosing particular activities. Although general comments do elicit beliefs about knowledge for teaching, we felt that this data suffers from many of the methodological problems we described in the introduction, since they are not situated in learning activities. We analyzed these data, and found that the picture they reveal is similar to the one drawn by the other data, but this analysis is not included here.

The number of quotations associated with each category is summarized in table 1:

SMCK			PCK		
Common Content Knowledge	Specialized Content Knowledge	Horizon Knowledge	Knowledge of Content and Teaching	Knowledge of Content and Students	Knowledge of Curriculum
2	18	0	32	10	0

Table 1: Number of quotations assigned to each knowledge category

We see that reasons related to PCK (42 quotations) were provided twice as many times as reasons related to SMCK (20 quotations). Within SMCK, specialized knowledge is more prominent than common knowledge, and within PCK, KCT is more prominent than KCS. We note that HK and KC were not indicated at all.

We now take a temporal view, presented in figure 1. In order to compare PD sessions, where varying numbers of teachers submitted feedback questionnaires, we normalized the quotation count by dividing by the number of submitted forms. At the beginning of the PD aspects of PCK were prominent, and this prominence tended to decrease as the PD progressed. Specifically - the average number of PCK-related comments per feedback in the first three sessions was 3 times greater than the average in the other session. On the other hand, the prominence of SMCK started out low and tended to increase over time.

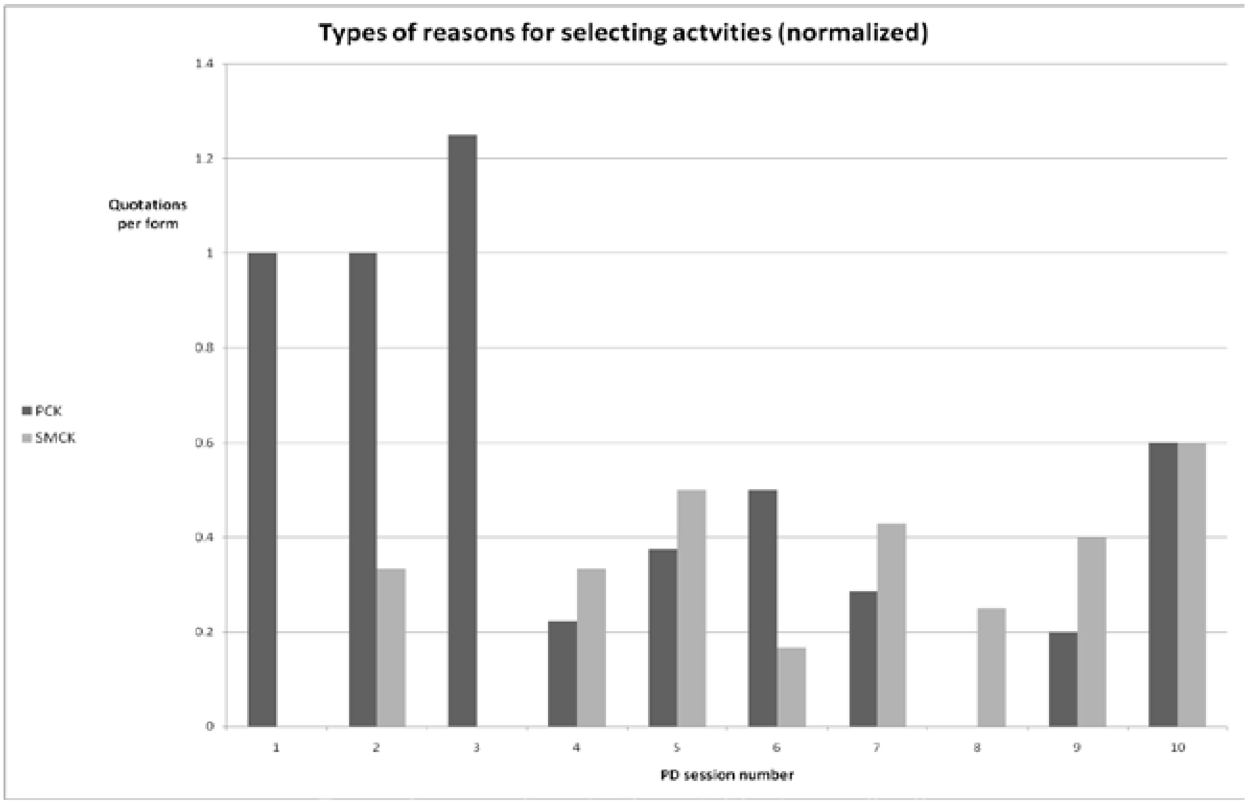

Figure1: SMCK/PCK reasons for selecting activities (normalized)

We see in these findings an indication of a shift in the teachers' beliefs. In this shift, the teachers are moving closer to beliefs that are consistent with the instructors' goals for the PD, focusing on SMCK. We cannot say with certainty what caused this shift, but we speculate that it is connected to the nature of the content that the instructors brought to the course. Initially, the teachers did not believe there was anything new for them to learn about grade 3 mathematics, but the mathematicians managed to bring a new depth to this content, as described in (Cooper & Arcavi, 2012) and in (Cooper & Pinto, 2012). In a broad sense of the term, this can be considered an indication of the teachers *learning* in the PD.

METHODOLOGY AND ANALYSIS – A DIFFERENT CATEGORIZATION

Our initial analysis provided some interesting results, but we were not totally satisfied with the coding scheme. Ten of our codes, representing 33 quotations (30% of the total), did not map to any MKT category, suggesting that there may have been an important message in the data which we were missing. We decided to look for an alternate categorization scheme, based on the data. Upon re-reading the feedback forms, we realized that the teachers' reasons for selecting particular activities said something not only about *what* they would like to learn, but also about the role of PD in their eyes, and how it should relate to their teaching. At one end we found quotations implying that the PD should contribute to the teachers' practice in a direct way, mainly by providing classroom activities and teaching tips, for example: "*...[the activity] teaches how to teach in the classroom*". We call this category *classroom focus*. At the other end we found indication of a loose connection, where the teachers' comments were aimed directly at what occurred in the PD, with little or no reference to teaching practices, for example: "*I really enjoyed that activity*". We call this category *PD focus*. In between were quotations that indicated a connection between the PD and teaching practice, but not by contributing directly to classroom teaching, for example: "*...[the activity] got me thinking about ... how to teach for understanding*". This intermediate attitude seems to see the teacher as having a role in incorporating knowledge learned in the PD, and adapting it for her classroom practice. We call this category *teaching focus*. In this new categorization, <u>all</u> codes were assigned a category, suggesting that this categorization has better grounding in the data. This new category scheme is very different from the first. Its categories cut through the various MKT categories, for example, a focus on *classroom* teaching is related to SMCK in some quotations (e.g. "*content was not appropriate for my classroom*") and to PCK in others (e.g. "*...ways for dealing with students' difficulties*"). Furthermore, in this new scheme some codes which appeared indistinguishable in terms of MKT were seen to be quite different. For example, *ways for dealing with students' difficulties* is a case of *classroom focus*, whereas the apparently similar *understanding students' thinking* is a case of *teaching focus*. Using the MKT categories, they both mapped to KCS.

We see (table 2) that the quotations are distributed among all three categories, with more prominence for *PD focus*, but again the temporal picture is more interesting (figure 2). The focus on *classroom* practices started high and tended to gradually decline over the course of the PD, whereas the focus on the *PD* itself started out low and increased quite steadily. The prominence of the *teaching* category changed very little over the course of the PD. It appears that teachers started out with beliefs indicating that the PD should contribute to teaching practices in a direct manner, and gradually accepted the possibility of less direct contributions, even to the extent where they consider the PD on its own terms, setting aside the question of how it will contribute to their teaching. This shift in beliefs, like the shift described in the previous section, can be attributed to the nature of the content that the instructors brought to the course. If the teachers accept goals of deepening their understanding of the elementary content as worthwhile, it is natural that they should not see this as contributing *directly* to their teaching practices, since this is not content they will bring to their own classrooms.

Classroom focus	Teaching focus	PD focus
33	20	43

Table 2: What teachers focus on in their reasons for selecting activities

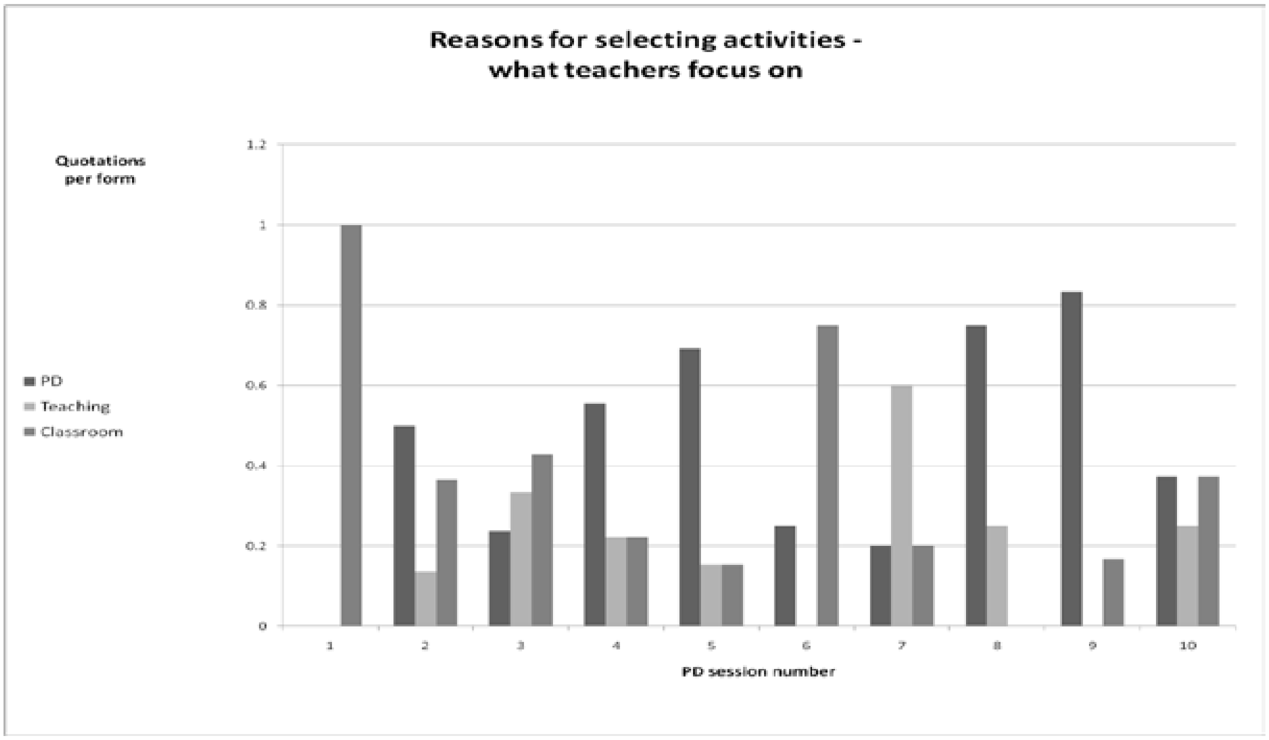

Figure 2: Reasons for selecting activities - normalized

DISCUSSION

We have shown trends in the teachers' attitude towards the PD, both in the knowledge it should focus on, and in the nature of its role as a means for improving teaching practice. The teachers moved from beliefs whereby the PD should enhance mainly PCK in a manner that can be utilized in classrooms, to beliefs whereby the PD may also enhance SMCK, and that the PD need not focus on classroom teaching directly. Although the change is modest, this is encouraging news for teacher educators, since it is usually difficult to show evidence of changes in teacher beliefs as a result of PD. But beyond these results, we would like to reflect on the research *method*.

Validity and reliability of the methodology

Although we believe that teachers *have* beliefs that transcend particular situations, our methodology is grounded in a particular situation - the PD. It is not only the teachers' beliefs that changed; the teacher educators' goals and the nature of the activities evolved in parallel. The change that we have shown is in the beliefs of the teachers *as they interact with the activities and the instructors*[4]. In this context issues of sustainability are not pertinent[5]. The scope of the teachers' beliefs about knowledge for teaching is the duration of the PD. This is acceptable, since our interest in these beliefs is mainly how they affect the teachers' attitudes toward learning in the PD.

The reliability of our method would appear to be dependent on the particular activities in the PD. Were the activities varied enough to elicit a broad spectrum of beliefs about knowledge for teaching? We believe the answer is yes, though space constraints do not permit us to describe these activities. However, we wish to claim that such a description would have been of limited relevance. When a teacher says that an activity was successful because it *suggested ways for dealing with students' difficulties*, we are concerned mainly with the implication that knowing how to deal with student difficulties is important knowledge for this teacher. The question of reliability is thus whether each PD session provided activities that were rich and varied enough to elicit the teachers' beliefs reliably. Consider, for example, what would happen if all the activities in a particular session were to focus exclusively on SMCK. There would be no opportunities for the teachers to select a successful activity based on its PCK nature. This is a serious concern, which is addressed as follows: If there is a lack of activities which focus on PCK, teachers will still be able to indicate their beliefs about the importance of PCK through *unsuccessful* activities ("*this activity was unsuccessful because it did* ***not*** *address PCK*"). If the feedback questionnaires had asked only about successful activities, reliability would have relied strongly on the variety of the PD activities. The coupling of the two questions greatly reduces the dependence on the choice of activities.

CONCLUSION AND IMPLICATIONS

This paper has both practical and methodological implications. On the practical side, it highlights an unusual PD course, conceived by a mathematics professor and taught by mathematics Ph.D. students, where there is an explicit focus on SMCK and on teachers' attitudes to mathematics. We have shown a change in teachers' beliefs during the course, and there is reason to believe that this change is a result of the PD. This suggests that the PD is worthy of the careful scrutiny it is receiving in the first author's Ph.D. dissertation. One of the issues under investigation is how these teacher educators, with their university approach to mathematics, managed to influence the teachers' beliefs. On the methodological side, we presented an indirect approach to eliciting teachers' beliefs about knowledge for teaching. This method was shown to be sensitive enough to reveal a change in teachers' beliefs. Although the change was modest, we have more faith in these results than we would have had in results obtained directly, for all the reasons listed in the introduction to this paper. We believe our method is less susceptible to teachers' tendency to gratify researchers, and can reveal tacit beliefs which may even contradict their declared beliefs. Whether or not this method really is an improvement over traditional methods is an important question left for future research.

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NOTES

[1] PD is used throughout the paper for "professional development".

[2] One of the instructors was a mathematics Ph.D. student, the other was a computer science M.Sc. student.

[3] The scope of this paper does not permit us to describe the activities to which the teachers referred. Data of this nature will be presented at the conference.

[4] The nature of this interaction is a central research question in the first author's Ph.D. dissertation.

[5] The PD's impact on teaching practices is being investigated in the first author's Ph.D. dissertation.

A THEORETICAL CONSTRUCT TO ANALYZE THE TEACHER'S ROLE DURING INTRODUCTORY ACTIVITIES TO ALGEBRAIC MODELLING

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Abstract

In this work we will introduce a theoretical construct that we have elaborated as a tool for the analysis and the interpretation of the teachers' actions during class activities which are aimed at fostering an aware learning of the use of algebraic language as a thinking tool. Through the analysis of an excerpt of a class discussion concerning introductory activities to algebraic modelling, we will show how this construct could provide "transparent" indicators to highlight the effectiveness of the teachers' actions during class interaction.

1. INTRODUCTION

The idea of an early approach to algebra, with a strong focus on generational activities (Kieran 1996) to help students overcome the well-known difficulties they usually face in the study of the formal aspects of algebra, is widespread and consolidated (Kaput&Al. 2007, Cai&Knut 2011).

Starting from the 90s, these ideas are developed by research together with a new vision of the teaching of arithmetic, characterised by a focus on relational aspects and meta-level activities, aimed at making students control the properties subtended to arithmetical equalities in order to create a connection between arithmetic and algebra (Kieran 1996, Lincevski 1995). Althoug in the first decade of 2000 many research studies are devoted to the implementation of these activities at school (also at the primary level), only few of them consider the role played by the teacher together with the problem of teacher education (Carpenter&Franke 2001, Blanton&Kaput 2001).

Our research studies, that can be conceived within this frame, are devoted to the planning of innovative didactical paths in arithmetic and algebra (grades 4-8) to be implemented through a socio-constructive approach in a strict cooperation with the teachers (Malara&Navarra 2003). The possibility to cooperate with different teachers (while in the first period we only collaborated with teacher-researcher, in these last ten years many other motivated and experienced teachers were involved) enabled us to highlight two main gaps: (1) a gap between theachers' declared conceptions and the hidden ones displayed by their behaviours in the classes; and (2) a gap between the theoretical assumptions they shared with researchers and their actual practice (Malara 2003). These results suggested us to focus not only on class experimentations but also on teacher education activities. In tune with Mason (1998) and Jaworski (2003) ideas, we designed and implemented specific tools and methods aimed at fostering teachers' development of different levels of awareness through the activation of joint critical-reflection practices (Cusi, Malara&Navarra 2011).

Our actual research objective is to design a specific methodological tool aimed at guiding teachers in the fundamental process of a-posteriori reflection on their own practice. Our idea is to refer to a theoretical construct, that has been defined as a result of one of our studies (Cusi&Malara 2009, Cusi 2012), as both a diagnostic tool in the analysis of class processes and a tool for teachers' self-reflection on their own teaching: the construct of "*model of aware and effective attitudes and behaviours*" (in the following M-$_{AE}$AB). Although it was aimed at identifying the specific features of a teacher who is able to make his/her students develop fundamental competences in the use of algebraic language as a tool for thinking, we believe that this construct could represent an effective "theoretical lens" for the analysis of the role played by the teacher, even during different algebraic activities, such as those aimed at the introduction of algebraic modelling.

In the following, we will introduce the M-$_{AE}$AB construct in the theoretical frame within which it has been developed and, through the analysis of a class excerpt, we will show how it could be refered to as a tool for analyzing the role of the teacher during introductory activities to algebraic modelling, highlighting its effectiveness in providing "transparent" indicators to describe the teachers' actions.

2. THE M-$_{AE}$AB CONSTRUCT FOR THE ANALYSIS OF THE TEACHER'S ROLE

The theoretical frame within which the M-$_{AE}$AB construct has been developed is constituted by two threesomes of components. The *first threesome* refers to the theoretical components we identified for the analysis of the development of thinking processes through algebraic language: (a) the model of didactic of algebra as a thinking tool proposed by Arzarello&Al. (2001), who, in particular, highlight the essential role played by the activation of conceptual frames and appropriate changes from a frame to another for a correct interpretation of the algebraic expressions which are progressively constructed; (b) the idea of anticipating thought developed by Boero (2001), who introduces it as a key-element in the "game" transformation-interpretation, which is typical of the processes of construction of reasoning through algebraic language; (c) the theoretical analysis proposed by Duval (2006), who identifies in the coordination between different representation registers a critical aspect in the development of learning in mathematics. Thanks to previous studies (Cusi 2009) we were able to highlight that an effective use of algebraic language as a thinking tool requires the management of three main key-components: (a) the appropriate application of conceptual frames and coordination between different frames; (b) the application of appropriate anticipating thoughts; and (c) the coordination between algebraic and verbal registers (on both translational and interpretative levels).

The *second threesome* of components is related to our theoretical framework of approach to the study of the teaching-learning processes and of the role played by the teacher. The first component is Vygotskian: we, in particular, refer to Vygotsky's stress (1978) on the importance of a teaching aimed at expanding students' zone of proximal development in order to stimulate, thanks to their interaction with the teacher or with more expert classmates, the activation of internal learning processes associated to a higher level of mental development. The second component draws its inspiration from the work carried out by Leont'ev (1978), who stresses the importance of making students increase their awareness about the meaning of the processes they activate during class activities in order to foster their learning. Our third component is the cognitive apprenticeship model introduced by Collins & Al. (1989), which draws its inspiration from an idea of learning as an "aware" apprenticeship and pursues the objective of "making thinking visible", through the activation of teaching methods which give students the opportunity of observing, discovering or even inventing the experts' strategies in the same context in which they are worked out. We, in particular, refer to two sets of typical methods of cognitive apprenticeship: (a) modeling, coaching and scaffolding, aimed at helping students acquire skills through processes of observation and guided practice; (b) articulation and reflection, related to metacognitive objectives and aimed at helping students achieve a conscious control of their own problem-solving strategies.

We believe that the 'games' of coordination between different linguistic registers and of interaction between the syntactical level, the interpretative level and the level of activation of anticipating thoughts, which can be automatically set up by an expert, *should be "made visible" to novices* in order to make them acquire and understand their meaning. Therefore our hypothesis is that, in order to help students progressively develop the competences and awareness necessary to carry out advanced tasks through an effective use of algebraic language, it is necessary that the teacher, during class interaction, adopts and makes visible specific attitudes and behaviours. *In this way, his/her students could be guided to the acquisition of the same attitudes and behaviours, which corresponds to an effective management of the three key-components we have previously introduced.*

This is the reason why we decided to refer to the expression "*teacher as a model of aware and effective attitudes and behaviours*" to highlight the approach of a teacher who consciously behave with the constant objective of "making thinking visible", in order to make his/her students focus not only on syntactical aspects but also on the effective strategies and on the meta-reflections on the actions which are performed.

In order to identify the peculiar characters of a teacher able to adopt this kind of approach in the class, during previous studies (Cusi&Malara 2009; Cusi 2012) we analyzed the audio-recordings of whole class activities and the subsequent students' small-groups activities, with the aim of highlighting: (1) *the role* played by the teacher during class activities as a "stimulus" to foster an approach to algebra as a tool for thinking, and, (2) *the links between the types of approach proposed by the teacher and the types of approach chosen by students* during small-groups activities, with particular reference to the meta-reflections they propose. It is important to stress that the teachers who carried out the activities we analysed (activities focused, as we stated before, on the construction of proofs by means of algebraic language) have been involved in a shared process of both didactical innovation and professional development conceived within the theoretical framework we presented before and were therefore aware of the importance of acting with the constant aim of "making thinking visible".

Thanks to our analysis of class discussions and students' small groups activities, we were able: (1) on the one hand, to highlight how unsuitable teacher's choices can lead to a missed acquisition of competences and awareness by students; (2) on the other hand, to identify the specific characteristics of a teacher who is able to act in order to foster students' acquisition of the key-competences in the use of algebra as a thinking tool and their development of an awareness of the meaning of the activated processes. The distinguishing features of an effective approach were, therefore, identified highlighting not only the teacher's capability of making his/her fundamental thought processes "visible" to students, but also the development of new competences the students display when they autonomously face the same kind of problems. These features, which can be placed in some fundamental behavioural categories of the cognitive apprenticeship, characterize the $M_{AE}AB$ construct.

The following table summarizes the characters of a teacher who poses him/herself as a $M_{AE}AB$, relating them to our theoretical framework of reference: in the first column the main roles of a $M_{AE}AB$ are recapped; in the other columns the associated teacher's actions (second column) and the reference to our theoretical framework (third column) are highlighted.

Roles played by a teacher who acts as a $M\text{-}_{AE}AB$	**Corresponding actions of the teacher**	**Reference our theoretical framework**
(a) *Investigating subject* and *constituent part of the class* in the research work being activated	He/she tries to stimulate in his/her students and attitude of research towards the problem being studied	It is in tune with *Vygotsky*'s ideas of learning as a social process, according to which the interaction with adults or with more expert peers enables students activate internal learning processes which help them achieve a higher level of mental development.
(b) *Practical/ Strategic guide*	He/she shares (rather than transmit) with his/her students the adopted strategies and the knowledge to be locally activated.	It refers to the *modeling category of cognitive apprenticeship*: it requires that an expert performs a task externalizing the internal processes in order to make students observe and build a conceptual model of the processes that are required to accomplish it.
(c) *"Activator" of processes of generalization, modelling, interpretation and anticipation*	He/she provokes and stimulates the construction of the key-competences for the development of thought processes by means of algebraic language	The teacher has to play the role of both: - activator of *interpretative processes* (fostering a correct identification of the *conceptual frames* that have to be chosen to correctly interpret and transform algebraic expressions and a good *coordination between different frames*) and - activator of correct *anticipating thoughts*. At the same time it refers to the *categories* of *modeling*, *coaching* (which consists of offering to students hints and feedback while they carry out a task) and *scaffolding* (it refers to the supports the teacher provides to help students carry out a task).

(d) *Guide in fostering a harmonized balance between the syntactical and the semantic level*	He/She helps his/her students control the meaning and the syntactical correctness of the algebraic expressions they construct and, at the same time, the reasons underlying the correctness of the transformations they perform.	It requires to foster a good coordination between the verbal and the algebraic *register*, through the activation of *correct conversions* and *treatments* (Duval 2006). This role also refers to the *articulation category* of cognitive apprenticeship, which involves the methods applied to make students articulate their knowledge, way of reasoning and problem-solving processes.
(e) *Reflective guide*	He/She stimulates reflections on the effective approaches carried out during class activities in order to make students identify effective practical/strategic models from which they can drawn their inspiration.	Through this role, the teacher highlights those processes which can be associated to an effective activation of the *three key-competences* in the use of algebraic language as a thinking tool. It refers, in particular, to the *reflection category* of cognitive apprenticeship, which involves enabling students to compare their own problem-solving processes with those of an expert or of another student, so that they ultimately could be able to compare them with an *internal cognitive model of expertise*.
(f) *"Activator" of both reflective attitudes and meta-cognitive acts*	He/She stimulates and provokes meta-level attitudes, with a particular focus on the control of the global sense of processes.	The focus is on the *control of the processes associated to a real acquisition of the key-competences* in the use of algebraic language as a thinking tool. Again the reflective practices that the teacher aims at provoking are focused on the *articulation* of the activated processes in order to evaluate their appropriateness. Playing this role, which also refers to the *reflection category*, fosters, in tune with the ideas developed by *Leont'ev*, students' development of a real awareness of the meaning of both the class activities and the learning processes themselves.

The first three roles (a, b, c) that a teacher should perform in the class require him/her to carry out the activities posing him/herself not as a "mere expert" who proposes effective approaches, but as a *learner* who faces problems with the main aim of making the hidden thinking visible, highlighting the objectives, the meaning of the strategies and the interpretation of results. The other three distinctive characteristics of the profile of a teacher as a $\mathrm{M_{AE}AB}$ (d, e, f) refer to a different role played by the teacher: he/she must also be a point of reference for students to help them clarify salient aspects at different levels, with an explicit connection to the knowledge they have already developed.

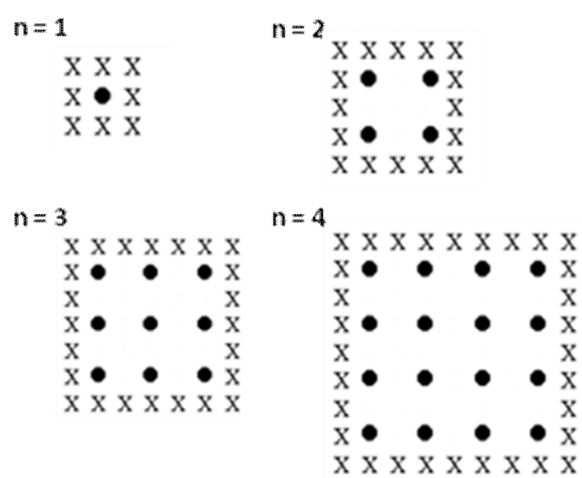
The previous table highlights that the algebraic and the social/methodological dimensions of our theoretical framework result to be complementary in combining to each other to foster an analysis of the teaching practices that goes beyond the discourse of the teacher in such a manner as to highlight the underlying intentions that link directly to the mathematics at stake. For this reason, it is important to stress that our aim is not to give an exhaustive definition of "effective teaching" as an absolute. Nevertheless, through the distinctive features of the $\mathrm{M_{AE}AB}$ we aim at identifying those teaching practices that can directly influence the quality of student learning in the specific context of the teaching of algebra. At the same time, the construct enables the identification of those attitudes and behaviour that can negatively influence students' learning. As we stated before, in fact, our studies allowed us to contrast the positive effects of this kind of approach with the effects of an approach which is not in tune with the $M_{AE}AB$ construct, which can provoke students' development of a sort of pseudo-structural approach to the use of algebraic language as a thinking tool (Cusi 2012).

3. THE ANALYSIS OF A CASE

In this paragraph we analyze an excerpt of a class discussion focused on an introductory activity to algebraic modeling, referring to the $M_{AE}AB$ construct as an a-priori tool for analyzing teaching.

The excerpt refers to the initial part of a discussion conducted in a first class of lower secondary school (grade 6). The teacher, who is really motivated and interested in engaging with projects aimed at fostering curricular innovation, has collaborated with us in different experimentations, sharing with us the aims, the methodology and the meaning of the approach subtended to the activities we promote. Since the following discussion constitutes a good example of conduction of class activities aimed at the introduction of algebraic modelling, we chose to focus on this excerpt in order to test the effectiveness of the $M_{AE}AB$ construct in providing "transparent" indicators to describe effective teachers' interventions.

The specific activity was proposed at the end of an introductory path to the algebraic modelling of figural sequences. The problem situation was adapted from the Pisa task usually named as "the apple trees". The characterizing feature of this task is the combination of the figural and verbal registers with the aim of fostering generalization and the algebraic formalization of the relationship between the number of apple trees and the number of conifers in the different possible configurations. In order to simplify the problem situation and to help students in its exploration and in making the identified relationships explicit, tables were introduced together with the requirement of specific argumentations. Due to space limitations we do not present the original worksheet, but only the proposed patterns and the first questions.

	Below you can find the patterns which represent the disposition of apple trees and conifers in relation to the number (n) of the rows of apple trees. 1) After having carefully observed the patterns, what can you say about the disposition of apple trees and conifers in the different cases? 2) Try to reproduce, through a drawing, the disposition of apple trees and conifers when n=5. Motivate your answer. 3) Explain how you can find the number of apple trees if you know the number of rows.

In the class discussion that we propose, the teacher tries to guide her students to the exploration of the number of conifers and apple trees in the different patterns.

The left column of the following table contains the excerpt of the first part of the discussion (T stands for the teacher, while the other alphabetical letters stand for the different students who take part in the discussion; due to space limitations we will skip some interventions which are not fundamental in the development of the discussion). In the right column we propose an analysis of the teacher's interventions with reference to the $M_{AE}AB$ construct.

<table>
<tr><th>Class discussion excerpt</th><th>Analysis of T' interventions through the $M_{AE}AB$ construct</th></tr>
<tr><td colspan="2">The class exploration starts with T's request of reproducing the patterns on the workbook while she is doing the same at the blackboard.</td></tr>
<tr><td>1. T: What did you check, while I was drawing on the blackboard, to exactly reproduce the disposition of apple trees and conifers?</td><td>T poses herself as an investigating subject, stimulating an attitude of research towards the problem. Moreover she simulates an attitude of sharing.</td></tr>
<tr><td colspan="2">2. M: (I checked) how many conifers there are on each side.
... Other pupils intervene.
6. K: In the first drawing there are 9 conifers.</td></tr>
<tr><td>7. T : In the first drawing there are 9 conifers. How did you determine the correct number of conifers, K?</td><td>T poses herself as a reflective guide. When K looks at the total number of the conifers and makes a mistake, T does not express any judgment. On the contrary, she intervenes to turn K's attention to the counting strategies he adopted in order to prompt a correct attitude of inquiry and to foster a self-correction.</td></tr>
<tr><td colspan="2">8. K: I did 3... 3… I got wrong.</td></tr>
<tr><td>9. T: Try to explain that.</td><td>T poses herself as an activator of metacognitive acts: she fosters an attitude of enquiry, encouraging K so that he can be able to make his thoughts explicit.</td></tr>
<tr><td colspan="2">10. K: They are 8. I considered 3 at the beginning, on the first side, then I added 2, then 2 on the other side, and then 1.</td></tr>
<tr><td>11. T: Eight. Good, K! And how many apple trees are there instead, G?</td><td>T encourages again the students and poses herself as a practical-strategic guide, making them focus on the first configuration and re-directing the inquiry towards the identification of the interrelation between the number of conifers and the number of apple trees.</td></tr>
<tr><td colspan="2">12. G: One!</td></tr>
<tr><td>13. T: Let's explore the other representations as well. How many conifers are there in the second representation?</td><td>T poses herself as a participant, constituent part of the class group, and as a strategic guide, drawing students attention toward the second configuration.</td></tr>
<tr><td colspan="2">14. GF: 8 multiplied by 2. Two is the number of the rows. Therefore 16.</td></tr>
<tr><td>15. T: A said that he would have wanted to know how many apple trees are exactly in the drawing.</td><td>T poses herself as an activator of reflective attitudes, trying to focus students' attention to a comparison between the different cases. Focusing on the objective of the discussion, she is also trying to activate correct</td></tr>
</table>

<table>
<tr><td></td><td>anticipations, with the aim of making them highlight a correlation between the number of conifers and the number of apple trees.</td></tr>
<tr><td colspan="2">16. GP: In this one there are 4 (apple trees)
17. M: I noticed that the number of rows is equal to the number of apple trees in the rows.</td></tr>
<tr><td>18. T: What would you say about M’s observation?</td><td>T does not judge M’s observation and ask the other students to examine it, posing herself as a reflective guide, with the aim of both stimulating reflections on the different approaches proposed and making them explicit.</td></tr>
<tr><td colspan="2">19. A: It’s right. When n=2 there are two apple trees in every row.
20. G: So, in order to calculate the number of apple trees in the enclosure we should multiply the number of the rows by the number of trees in every row.
21. K: I didn’t understand anything.</td></tr>
<tr><td>22. T: The observations actually overlapped.</td><td>When K declares his doubts, T poses herself again as a participant, stimulating the class in order that the different proposed observations could be better made explicit. In this way she fosters the sharing of knowledge and poses herself as an activator of both reflective attitudes and metacognitive acts.</td></tr>
<tr><td colspan="2">23. G: I meant to say that in this case, K, in order to calculate the number of apple trees you must take the number of apple trees in every row and multiply it by the number of rows. Therefore two multiplied by two.
24. M: That is you must multiply the number of rows by itself because the number of apple trees is equal to the number of rows.</td></tr>
<tr><td>25. T: So let’s see if I am able to understand. What do “the number of rows” and “the number of apple trees in every row” mean?</td><td>Instead of evaluating G and M’s observations, T poses herself as a reflective guide, asking students’ to clarify the meaning of some terms, with the aim of “making their thinking visible”. At the same time she poses herself as an activator of interpretative processes, trying to stimulate correct conversions from the verbal to the symbolic register.</td></tr>
<tr><td colspan="2">26. A: The rows are those (he points at the drawing) ... that is the number of rows, how many rows there are. The number of apple trees is how many apple trees there are in every row.
27. K: I have understood!</td></tr>
<tr><td>28. T: I have understood now. Thanks, M. So how can we write this number 4 which stands for the number of apple trees?</td><td>T stimulates and provokes the construction of key-competences for the development of thought processes by means of algebraic language, posing herself as an activator of interpretative processes, gradually stimulating the activation of correct conversions from the verbal to the symbolic register.</td></tr>
<tr><td colspan="2">29. Group of students: 2 multiplied by 2!</td></tr>
<tr><td colspan="2">The discussion continues with the analysis of the number of the conifers in every pattern and the following identification of the symbolic expressions which represent the relation between the number of apple trees and the number of conifers in every configuration and the number of rows. Lastly it ends with a naive study of an inequality in order to determine in what cases the number of the apple trees exceeds the number of the conifers.</td></tr>
</table>

4. FINAL REMARKS

The analysis we conducted testifies that the $M_{AE}AB$ could represent an effective diagnostical tool in the analysis of the quality of the teacher's management of introductory activities to algebraic modeling.Through the theoretical lenses we adopted, in fact, it was possible to highlight an effective action of the teacher, characterised by a specific focus on the strategies aimed at making students control their thinking processes and develop an awareness about the meaning of the performed activities. In this discussion the teacher's interventions associated to the roles strictly connected to the algebraic dimension of our construct are less frequent than the roles associated to the meta-cognitive dimension. For this reason, the construct could also enable to highlight that, while during global-meta-level activities (Kieran 1996) there is a need of a good balance between the algebraic dimension and the meta-cognitive dimension, introductory activities to algebra requires a major focus on those roles which can better help students develop a deep awareness of the meaning of the processes they are involved in.

We believe that the $M_{AE}AB$ construct could be also a useful tool to promote teachers' reflection on their own practice. In tune with Mason's idea of teaching as "educating awareness" (1998), we think that making the teachers analyse their class processes through specific theoretical lenses could provoke what Mason defines "shifts of attention", which play an essential role in fostering the development of new awareness and hence in determining an effective teaching. We believe indeed that these activities could allow teachers to perform their first "guided" reflective practices, receiving and afterwards interiorizing the necessary stimulus for the construction of their own models for reflection, to which they can refer everytime they have to analyse their practice. In the future we intend to test this hypothesis referring to the $M_{AE}AB$ construct in the work with both pre-service and in-service teachers, proposing it to them as a tool for self-analysis.

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USING REDIRECTING, PROGRESSING AND FOCUSING ACTIONS TO CHARACTERIZE TEACHERS' PRACTICE

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'Redirecting, progressing and focusing actions' is a framework for describing how teachers use students' comments to work with mathematics. In this article, the framework is presented using examples from two teachers' practices. Then the article demonstrates how this framework can be used to characterize the two practices and differences between them.

Keywords: Communication – Teachers role – Orchestrating discourse

INTRODUCTION

There is a need for more detailed understanding of communication in practice generally, and how teachers' use or not use students' comments to work with mathematics. This article presents an example of how the framework of 'redirecting, progressing and focusing actions' (Drageset, 2012) can be used to characterize teachers' practice. This is done by looking into the amount of different types of teacher comments in two teachers' practices and exemplifies how this might be interpreted.

COMMUNICATION

Researchers often finds that classroom discourses are dominated by teacher talk, in a discourse pattern where the teacher initiates the questions, the students respond to them, and the teacher evaluates the responses (Franke, Kazemi, & Battey, 2007). This pattern is often labeled as IRE (initiation-response-evaluation). Cazden (2001) describes IRE as 'the default option – doing what the system is set to do 'naturally' unless someone makes a deliberate change' (Cazden, 2001, p. 31). In this pattern, the students are normally engaged in a procedure-bound discourse, such as calculating answers and memorizing procedures, and with little emphasis on 'students explaining their thinking, working publicly through an incorrect idea, making a conjecture, or coming to consensus about a mathematical idea' (Franke et al., 2007, p. 231).

During what Stein, Engle, Smith and Hughes (2008) call the first generation with respect to mathematical discussions in the classroom, focus was on the use of cognitively demanding tasks, encouragement of productive interactions, and letting the students feel that their contributions were listened to and valued. Little attention was directed towards how teachers can guide the class towards worthwhile mathematics, and many teachers had the impression that guidance should be avoided (Stein et al., 2008). The result could be that the students took turns sharing their solution strategies without any filtering or highlighting.

However, even though an increased level of discourse is positively related to student learning we know that just getting students to talk is not enough (Franke et al., 2007). Merely making your thinking available to others is insufficient because too much is normally unsaid. The manner in which we make our thoughts available seems to be crucial (Kieran, 2002). Consequently, details matter, or in the words of (Franke et al., 2007, p. 232): 'One of the most powerful pedagogical moves a teacher can make is one that supports making detail explicit in mathematical talk, in both explanations given and questions asked'

The second generation practice 're-asserts the critical role of the teacher in guiding mathematical discussions' (Stein et al., 2008, p. 320). The hallmark is that the teacher actively uses students' ideas and work to lead them toward more powerful, efficient and accurate mathematical thinking. Ball uses the term 'show and tell' as an example of the same:

> 'For the lesson to be more than a drawn out "show and tell" of the different methods requires the composition of a mathematical discussion that takes up and uses the individual contributions … making available one child's thinking for the rest of the class to work on.' (Ball, 2001, p. 20)

Ball here emphasizes an active use of students' contributions. However, even though there is increasing agreement that students' contributions must play an important role in classroom communication there is a need to understand how this can be achieved. Carpenter, Fennema, Franke, Levi and Empson (1999) suggest using a careful selection and sequencing of student strategies. Stein et al (2008) suggest a similar strategy as part of a model that specifies five key practices in order for a teacher to use student responses more effectively in discussions; anticipating likely student responses, monitoring, selecting responses to be presented, sequencing the presentation, and making connections.

This model may move attention away from learning mathematical content independently of student thinking. Instead, attention is directed towards how students' thinking about mathematical content can be used to create reflection and learning. Such a strategy will also give the teacher regular access to students' ideas and the details that support them. This is essential knowledge for teaching and learning in mathematics (Franke et al., 2007).

Fraivillig, Murphy and Fuson (1999) and Cengiz, Kline and Grant (2011) report studies of how teachers actively use the students' ideas to lead them towards more powerful, efficient and accurate mathematical thinking and in which situations this occurs. Fraivillig et al (1999) present a framework called 'Advancing children's thinking' (ACT) based on an in-depth analysis of one skilful first grade teacher. The framework has three components: eliciting children's solution methods, supporting children's conceptual understanding, and extending children's mathematical thinking. While the eliciting and supporting components focus on the assessment and facilitation of mathematics with which the students are familiar, the extending component is focused on the further development of the students' thinking. Each of these components is defined by several categories of instructional techniques, for example 'encourage elaboration', 'remind student of conceptually similar situations' and 'demonstrate teacher-selected solution methods'.

Alrø and Skovsmose (2002) introduce the notion of inquiry co-operation as a particular form of student-teacher interaction when exploring a landscape of investigation. As part of the inquiry-cooperation model they identify eight communicative features: Getting in contact, locating, identifying, advocating, thinking aloud, reformulating, challenging and evaluating. These features were present both in the student-student interaction and in the teacher-student interaction.

Several scholars have described a phenomenon where the teacher dominates the solution process and in different ways reduces the complexity for the students. Brousseau (1997) describes that teachers sometimes provides more and more information to help students when they fail repeatedly. The result is that the teacher gradually takes responsibility for the essential part of the work. When the target knowledge disappears completely, Brousseau (1997) describes it as the Topaze effect. A similar way for teachers to reduce complexity for students is described by Lithner (2008) using the term guided algorithmic reasoning. In guided algorithmic reasoning 'all strategy choices that are problematic for the reasoner are made by a guide, who provides no predictive argumentation' (Lithner, 2008, p. 264) and the remaining routine transformations are executed without verificative argumentation. Predictive arguments are related to why the chosen strategy will solve the task, while verificative arguments are related to why the strategy solved the task. A third concept for this phenomenon is funneling (Wood, 1998). A teacher's questions funnel the conversation when the teacher does most of the intellectual work and 'the student's thinking is focused on trying to figure out the response the teacher wants instead of thinking mathematically himself' (Wood, 1998, p. 172).

Several studies have developed tools for characterizing teaching practices, such as Wood's (1998) funneling and focusing and Brendefur and Frykholm's (2000) four levels of communication. While these concepts have explanatory power in the study of entire practices, the limitation lies in the lack of detail. Other studies, such as 'Advancing children's thinking' (Fraivillig et al., 1999), its further development by Cengiz et al (2011), the inquiry co-operation model described by Alrø and Skovsmose (2002) are different, as these studies characterize elements found in teaching without describing an entire practice. These are concepts that enable us to describe single teacher comments at a level of detail which is not possible using more general concepts such as funneling and focusing. Detailed descriptions are critical for researchers to be able to describe and analyze teachers' communication in more detail. It is also crucial for professional development as teachers have little use for general advice. Further development of detailed frameworks is needed in order to create concepts to describe and understand how single comments might contribute to the mathematical discourse.

PURPOSE OF THE ARTICLE

The purpose of this article is to illustrate how the framework of redirecting, progressing and focusing actions can be used to characterize and interpret teachers' communication in practice.

REDIRECTING, PROGRESSING AND FOCUSING ACTIONS

The redirecting, progressing and focusing actions framework was developed through several stages. The data comes from five teachers practices at upper primary (grade 5-7, students aged 10-13). All their mathematics teaching for one week was filmed from the start of the topic of fractions, typically four or five lessons. The five teachers participated in a larger survey (Drageset, 2009, 2010) and were selected for further study based on a selection of diverse profiles from the survey. Altogether this meant that the data was approximately 2000 teacher comments. In this case a teacher comment is defined to be a response to a student comment.

In the first step of the development, each teacher comment of several excerpts from five teachers was characterized with respect to how teachers use or not use student comments to work with mathematics. Similar comments were collected in groups that formed initial categories with a preliminary definition. The definitions were inspected with each comment added and adjusted whenever necessary, and categories were sometimes divided or merged as a result of this. When the categories seemed to have stabilized, all the rest of the data were coded. During this coding the definitions were adjusted whenever necessary, and also at this stage some new were created, some were merged and others divided. During the end of this work the categories were organized in three superordinate groups, the redirecting, progressing and focusing actions. This method has similarities with grounded theory (Charmaz, 2006; Glaser, 1978; Glaser & Strauss, 1967). However, this is done without following the original emphasis on discovery, detachment of theory, and the step-by-step procedure. For further information about the development, see Drageset (2012).

TWO CASES COMPARED

This article will use the practices of Anne and Linda, two of the original five teachers, as an example. During the filming of Anne and Linda, their practices were considered to be quite different. Their practices were described in several ways, both intuitively and using existing concepts and coding schemes. For example, concepts from 'The knowledge quartet' (Rowland & Turner, 2009; Rowland, Turner, Thwaites, & Huckstep, 2009) and the 'Content knowledge for teaching' framework (Ball, Thames, & Phelps, 2008) and the coding scheme for measuring the quality of mathematics in instruction (LMT, 2006) were used. But all these failed to have any explanatory power related to the perceived differences. When these initial attempts were unsuccessful, it was decided to instead try the approach of characterizing single comments and creating categories of similar comments. This turned out to be a more productive approach, resulting in the framework of 'Redirecting, progressing and focusing actions' (Drageset, 2012).

ACTION	CATEGORY	ANNE	LINDA
Redirecting		**6 %**	**22 %**
	Put aside	2 %	8 %
	Advising a new strategy	2 %	5 %
	Correcting question	2 %	9 %
Progressing		**58 %**	**53 %**
	Demonstration	3 %	7 %
	Simplification	4 %	17 %
	Closed progress details	45 %	26 %
	Open progress initiatives	5 %	2 %
Focusing		**36 %**	**25 %**
	Enlighten detail	10 %	7 %
	Justification	2 %	1 %
	Apply on similar problems	0 %	0 %
	Requesting assessment from other students	2 %	0 %
	Notice	20 %	13 %
	Recap	3 %	4 %

Figure 1: Amount of each category and summarized for each action. The percentages are based on number of comments. When 'Put aside' is 2% for Anne it means that 2% of all of Anne's comments were of this type.

The three redirecting actions of 'put aside', 'advising a new strategy' and 'correcting questions' are all comments with the common feature of trying to change the students approach. Linda uses redirecting actions far more than Anne (see figure 1). In fact, more than one fifth of all Linda's comments were about changing the students approach without trying to understand the reasons behind the answers given. Especially the categories of 'correcting questions' and 'put aside' are used far more by Linda than by Anne. This is consistent with the intuitive observation that this teacher gives fast, direct and sometimes harsh feedback to the students. One example of Linda's direct feedback is this one (an example of 'put aside'):

> Teacher: No, but we are not talking about pizza now, we are talking about… pure calculation with numbers, what is it that really happens?

This comment works redirecting because the student approach is rejected and a new direction is suggested. It is also fairly direct, not giving any form of support to the student's suggestion. And even with less direct feedbacks like 'I think you answer on something different than I ask for' (Linda) such a large amount of redirecting actions will necessarily characterize the practice.

Anne seemed to avoid such confrontations by rarely using redirecting actions. Instead, Anne used focusing actions considerably more than Linda. By using focusing actions instead of redirecting actions, she appreciates the student's approach and might also understand more of the reasons behind it. However, the dominating type of focusing actions in Anne's practice is 'notice'. This is comments that stops or even interrupts the students and points out what the teacher finds important. It is a teacher-led focusing action, as this example from Anne's practice illustrates (the teacher just asked what the student did to expand the fraction):

Student: Expanded by four

Teacher: You expanded by four, so that you could have a common denominator. Yes.

The core of 'notice' is that the teacher points out and emphasizes some information that occurs during a dialogue. Sometimes 'notice' comments acts supporting for the student, and sometimes it is a way to add information necessary for the rest of the students to understand.

While Anne's practice has a considerably larger amount of focusing actions than Linda's practice, they are similar in that 'notice' is the dominating type of focusing actions and that the other major type is 'enlighten details'. One example of the use of 'enlighten details' from Linda's practice is this one (the task is to find out how many fifths nine of fifteen chips are):

Student: The answer is three

Teacher: The answer is three. Explain to me. Here are three. Here are three fifteenths… no fifths, is it? Three, six, nine, twelve, fifteen, there are fifteen chips. What are three fifths then?

In this example the teacher is not satisfied with only an oral answer but requests the student to find three fifths of the chips also. This is about requesting students to enlighten reasons behind the answer and to make thinking explicit. According to Franke et al. (2007), making detail explicit is one of the most powerful pedagogical moves a teacher can make. It is worth mentioning that in both practices comments requesting students to enlighten details are almost only used as a response to correct answers.

Progressing actions are dominating both practices, but there are differences on which type of actions are used most frequently. The dominating type of progressing actions in Anne's practice is 'closed progress details'. This category is formed by comments where the teacher asks for one detail at a time, moving along one step at a time. Instead of asking about the final answer, the teacher splits it up into several smaller tasks and asks for answers to each of these. One aim of this strategy might be to ensure that every student is able to follow the line of thought by following them through every important step. The result is that the teacher takes control of the process and probably reduces the complexity of the task for the students, as that they do not need to see the whole picture. These questions typically have only one correct or desired response, which is quite often easy to find. This type of comments is dominating the entire practice as 45 % of all Anne's comments are 'closed progress details'. This illustrates Anne's tendency to split up tasks, control the process and request the students to answer rather simple step-by-step questions. The following example of 'closed progress details' above is from Anne's practice. The task is to add 1/2, 2/5 and 1/10. The teacher writes '1/2+2/5+1/10=', and then this follows:

Teacher: What is the common denominator?

Student: Ten

Teacher: Ten. And then you did what here? (points out 1/2)

Student: Multiplied by five

Teacher: Multiplied by five, above and below. And here? (points out 2/5)

Student: Two

Teacher: Multiplied by two, above and below. And here? (points at 1/10)

Student: Nothing.

Teacher: Nothing. Okay, and then you got?

'Closed progress details' are also the most frequent category in Linda's practice, but the use is much less frequent. On the other hand, Linda uses simplification considerably more than Anne. The comments that form the simplification category are typically comments where the teacher simplifies the task by adding information, changing the task, giving hints or telling the student how to solve the task. And it is characteristic for Linda's practice that she adds information or changes tasks in order to make the student give the wanted response. One striking example comes when the teacher asks how much two fifths and three fifths are. There are ten orbs at the blackboard, and the teacher has told that two orbs is one fifth.

Student: Ten.

Teacher: *Two* fifths and *three* fifths, how many fifths is that? (emphasizes *two* and *three*)

Student: Ten.

Teacher: If you have two fifths here (holds up two fingers) and three fifths there (holds up three fingers on the other hand), how many fingers do you see?

Here, the teacher first emphasizes the numerators as a response to the student answering ten. When this does not help the teacher asks the student to count fingers so that the correct and wanted response 'five' will be said. The large amount of 'simplification', 'closed progress details', 'notice' and redirecting actions is characteristic for Linda's practice as the teacher gives clear and direct feedbacks of both incorrect (redirecting actions), correct (notice) approaches and when the student fails to progress (simplification and closed progress details).

The dominating amount of 'closed progress details' combined with a large amount of 'notice' is characteristic for Anne's practice as the teacher controls the process by dividing up tasks and pointing out what is important. Also, the students are quite frequently asked to explain how or what (enlighten detail).

By looking at some specific types of student comments more information is available. In Anne's practice there are very few incorrect answers from students (7 %), which might be explained with the large amount of closed progress details that reduces complexity. In Linda's practice the amount of incorrect student responses are larger (23%) and these are mainly followed up by redirecting actions. This indicates that the Linda is not interested in the reasons or thinking behind incorrect answers, but instead tries to change the students approach to something more productive. It might also mean that Linda opens more up for student suggestions than Anne does. Looking at student explanations changes this picture slightly. There are a larger amount of student explanations in Anne's practice (15% of the student comments) than in Linda's practice (9%). Also, these are more often followed up by focusing actions (mainly notice) by Anne than by Linda.

CONCLUSION

The 'Redirecting, progressing and focusing actions' framework adds to concepts that can describe and characterize comments and practices in detail. This article has illustrated how these concepts can be used to characterize practices and understand differences based on a simple counting of the different types of comments. Further details are accessible by inspecting how teachers respond to different types of student comments. This is just briefly exemplified for two rather intuitive types of student comments, incorrect answers and explanations. It is also possible to inspect qualitative differences within each category, for example how teachers ask students to enlighten details, how teachers use closed progress details in different ways, or even how different kinds of student explanations are followed up by the teachers. One might also be able to find patterns of comments frequently used by teachers, for example when repeated use of redirecting actions has no effect on the progress towards a solution this might lead to the use of simplification.

The example from the practices of Linda and Anne illustrates how concepts from the 'Redirecting, progressing and focusing' framework can be used to characterize and interpret teachers' communication in practice. For example, a practice dominated by the use of closed progress details is a practice where the teacher takes control of the process by doing all the important strategic choices and leaving the calculation to the students. This reduces the complexity and probably also has an effect on what opportunities the students are given to learn mathematics. Another practice might be dominated by teacher comments that request the students to enlighten details. This means that the reasons, thinking and arguments behind answers are being made explicit regularly, which according to Franke et al. (2007) is one of the most powerful pedagogical moves a teacher can make. In this way, concepts from the 'Redirecting, progressing and focusing' framework can be used to study qualities in these practices.

But there is one main limitation to the approach in this article as only the amount of each category is studied and not how different types of teacher comments work together in sequences. To progress it is important to study how different types of teacher actions, such as for example closed progress details and enlighten details, can interact productively in a mathematics discourse in the classroom. One such example is to study how each type of teacher comment affects the students' comments, approach or behavior. Then it also becomes possible to study qualities of different teacher actions.

The power of research frameworks lies in the concepts created. Further research is needed to find out the explanatory power of the concepts developed in the 'Redirecting, progressing and focusing' framework when describing, interpreting or characterizing entire practices or shorter discussions (sequences of comments) in mathematics classrooms.

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PRE-SERVICE AND IN-SERVICE TEACHERS' VIEWS ON THE LEARNING POTENTIAL OF TASKS – DOES SPECIFIC CONTENT KNOWLEDGE MATTER?

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This study examines views of pre-service and in-service mathematics teachers on the learning potential of tasks and interrelations of such views with relevant content knowledge. Focusing on the role of representations for learning and the content domain of fractions, the paper hence aims at connecting different sub-aspects of professional teacher knowledge. The results indicate that the learning potential of problems focusing on a conversion of representations is hardly acknowledged in comparison to tasks requiring only a calculation on a numerical-symbolical representational level and giving a rather unhelpful pictorial representation. However, there is a tendency that teachers with higher content knowledge scores rate the learning potential of the first type tasks comparatively higher.

INTRODUCTION

Epistemological views related to tasks are expected to play a key role when mathematics teachers select or create problems for the classroom. Hence, professional knowledge related to overarching aspects of pedagogical content knowledge (PCK) – such as the idea of using multiple representations – should also be examined on the level of task-related views. However, other components of professional knowledge might have an influence on task-related views, such as domain-specific content knowledge (CK), but unfortunately there are still very few studies making such links. Consequently, this paper focuses on views of pre- and in-service teachers regarding the use of representations in tasks in the content domain of fractions, for which the teachers have also been assessed in a CK test.

The results indicate that there is task-specific variation in the views about the learning potential of the tasks presented to the teachers. Nevertheless, the analysis yielded two types of tasks, in line with the theoretical design of the corresponding questionnaire unit: tasks making use of the learning potential of changing between representations (1) and tasks with rather unhelpful pictorial representations (2). In-service academic-track secondary teachers rated the learning potential of type 1 tasks higher than pre-service teachers. Moreover, teachers with higher CK scores tended to acknowledge the learning potential of those tasks comparatively more.

The following second section gives a brief overview of the theoretical background, which leads to the research interest of this study presented in the third section. We will then describe the methods and design in the fourth section, present results in the fifth section, and conclude with a discussion in the sixth section.

THEORETICAL BACKGROUND

National standards in many countries emphasize the importance of dealing with multiple representations for mathematical learning. In the case of the German standards for the mathematics classroom "using mathematical representations" is stated as one out of six core aspects of mathematical competency - in particular, recognizing interrelations between different representations and changing between them is stressed explicitly (KMK, 2004). There are very good reasons for such an emphasis of multiple representations in the mathematics classroom: Representations play a major role in all kinds of mathematical activities, since the perception of mathematical objects is dependent on representations (Duval, 2006). We take the notion *representation* to mean something which stands for something else – in this case for an 'invisible' mathematical object (cf. Goldin & Shteingold, 2001). In particular, *pictorial* representations are illustrations, diagrams or sketches. Since usually a single representation makes visible only some aspects of the corresponding object, multiple representations complementing each other are needed to develop an appropriate concept image (cf. Tall, 1988). Hence, the ability to recognize a mathematical object behind its different representations, to use them flexibly, and in particular to change between them is a key for successful mathematical thinking and problem solving (i.e. Lesh, Post, & Behr, 1987; Duval, 2006). Consequently, reflecting and discussing interrelations and conversions between different representations should be part of the mathematics classroom in order to foster the students' ability to use multiple representations flexibly. In particular, tasks focusing on conversions from one mode of representation to another (and back), which promote insight into their interrelations, can make an important contribution to students' understanding (Duval, 2006). Therefore, the awareness of the importance of dealing with multiple representations and also of the difficulties which come with them for learners, as well as knowledge about how to foster their abilities in making use of multiple representations are important aspects of PCK.

Against this background the question arises as to what extent teachers are aware of the learning potential of tasks focusing on conversions of representations. The results of a prior study about pre-service teachers' views on pictorial representations in tasks indicate that many pre-service teachers tended to overemphasize the motivational aspect of pictorial representations and hardly saw the learning potential of such pictorial representations which enable students to take an additional approach to mathematical concepts. (Dreher & Kuntze, 2012; cf. also Dreher, 2012). In this prior study, the teachers were asked about their views regarding the pictorial representations. A follow-up question was how these views may impact on the learning potential of a task as a whole in the eyes of the teachers, i.e. how they see the potential of the learning opportunity which is set by the task. In addition, there has been a need for including data from in-service teachers in order to get insight into the role of experience e.g. in organizing contents for the classroom. Both of these follow-up research interests have been taken up in the study reported here.

Views about the learning potential of tasks are considered to be individual, conviction-like and in the first place restricted to the particular case of the task considered. However, when looking at types of tasks and investigating the views of teachers related to such types, the data may give insight into whether teachers are able to ‘see the difference’, i.e. to realize opportunities for conceptual learning associated with the types of tasks. In this sense, task-related views also reflect components of PCK on a level beyond the specific case of a particular task.

As a theoretical background for analyses on these two layers, this study uses a multi-layer model of professional knowledge (cf. Figure 1), which combines the spectrum between knowledge and beliefs (e.g. Pajares, 1992), the professional knowledge domains by Shulman (1986; cf. also Ball, Thames & Phelps, 2008) – with levels of globality respectively situatedness (cf. Törner, 2002; Kuntze, 2012). As professional knowledge is often structured episodically and as there might be inconsistencies e.g. between global beliefs and situation-specific views, it appears as necessary to consider these levels of situatedness respectively globality separately and to examine their interrelatedness an approach which is facilitated by the model (Kuntze, 2012).

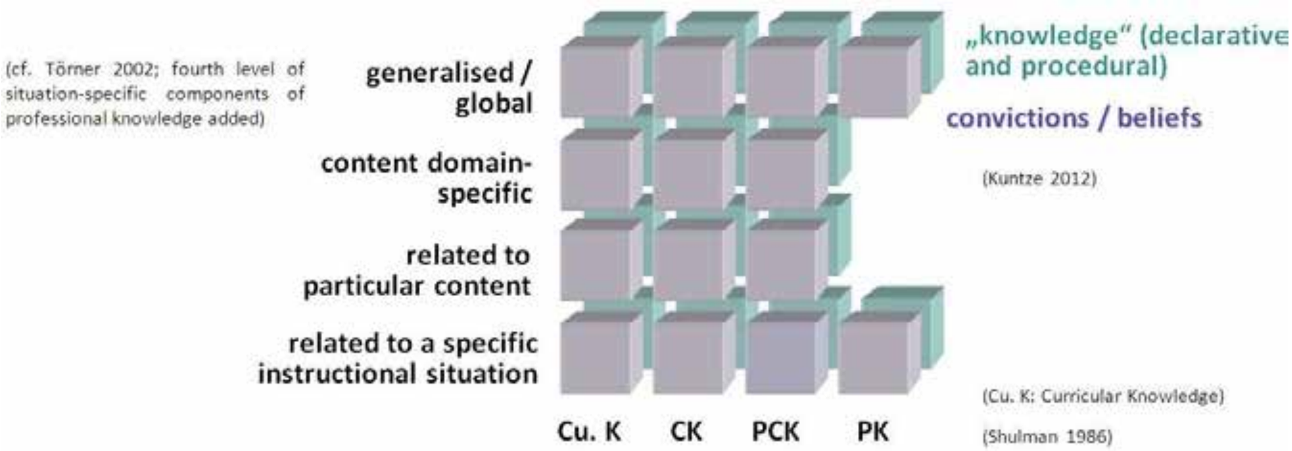


Figure 1: multi-layer model of professional knowledge (Kuntze, 2012)

Task-related views can – according to this model – be described as basically content-specific convictions in the domain of PCK (Kuntze, 2011). As argued above, there is the possibility of going up one level of globality and making a bridge to the knowledge side, if the data affords looking empirically at types of tasks.

In the case of the present study, the tasks are situated in the content domain of fractions with an emphasis on representations of fractions and on operating with fractions. The PCK component mentioned above is hence bound to this content domain. As low CK mastery has repeatedly been reported especially for pre-service teachers (e.g. Toluk-Uçar, 2009) this aspect calls for including content domain-specific CK. This affords answering the question whether CK is sufficient for PCK related to the use of representations in the task types and/or which role CK plays for

PCK in that content domain. Exploring such links between content-specific convictions (related to tasks) and relevant CK could provide useful information for designing effective professional development activities.

RESEARCH INTEREST

Given the significance of professional teacher knowledge related to the idea of using multiple representations (cf. Kuntze et al. 2011) in particular related to the design of learning opportunities provided by problems, the previous section highlights that task-related views and corresponding PCK and CK components are in the center of interest. Moreover, empirical insight into relationships between these components of professional knowledge is needed. In our study, we hence concentrate on the following research questions:

- Which task-specific views relevant for the use of representations do mathematics teachers hold? In particular, how do they evaluate the learning potential of types of problems which make use of multiple representations in different ways?
- Are there differences in the views between groups of teachers with different qualification levels?
- Is CK interrelated with such task-specific views?

DESIGN AND METHODS

For answering these research questions, a questionnaire was administered to 219 pre-service teachers (183 female, 26 male, 10 without data) and 83 in-service teachers, of which 58 were teaching at academic track secondary schools (23 female, 32 male, 3 without data) and 25 at secondary schools for lower attaining students (15 female, 10 male). The pre-service teachers were on average 20.7 years (SD = 2.5) old and at the beginning of their first semester of teacher education. The teachers at academic track secondary schools resp. at secondary schools for lower attaining students were on average 41.5 (SD = 12.3) resp. 39.9 (SD = 11.3) years old and had been teaching mathematics since 13.6 (SD =12.3) respectively since 10.8 (SD = 9.5) years.

Corresponding to the first two research questions for this study, the participants were asked to evaluate the learning potential of six fraction problems by means of multiple-choice items. A sample item is: "The way in which representations are used in this problem aids students' understanding." The teachers could express their approval or disagreement concerning these items on a four-point Likert scale. They were told that the problems were designed for an exercise about fractions in school year six. Three of these tasks are about carrying out a conversion of representations, whereas solving the other three tasks means just calculating an addition or a multiplication of fractions on a numerical-symbolical representational level. The pictorial representations which are given in the problems of the second type are

Make up a situation or a word problem which is suitable for the calculation $3 \div \frac{1}{4}$ and then use it to solve the calculation.	Do you know what $\frac{1}{2} \cdot \frac{1}{4}$ is? You can use the pictures below to help:  

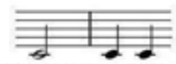



Figure 2: Samples for tasks of type 1 (left) and of type 2 (right)

rather not helpful for the solution, since they can't illustrate the operation needed to carry out the calculation. Some of them may even be misleading. Samples for both kinds of tasks are shown in Figure 2. These two types of tasks were chosen, because from the perspective of our previous research (c.f. Dreher & Kuntze, 2012; Dreher, 2012) they represent two rather opposite ways of using multiple representations i.e. focusing mainly on affective aspects respectively focusing rather on content-specific criteria.

In order to find answers to our third research question, a further section of the questionnaire was included in the analyses: A test on specific CK about dealing flexibly with multiple representations for fractions and their operations. As in the sample item shown in Figure 3, given (incorrect) conversions between different forms of representations had to be checked and corrected or a conversion had to be carried out.

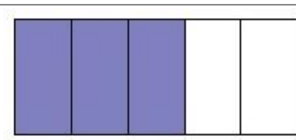
Please change the diagram, if necessary, so that $\frac{3}{5}$ of $\frac{1}{4}$ is shaded. Otherwise just tick the box on the right-hand side.		

Figure 3: Sample item of the CK test

RESULTS

We start with the results concerning the first two research questions, namely the teachers' evaluation of the learning potential of the six tasks given in the questionnaire. The design of this questionnaire section could be confirmed by a factor analysis: For each task there is a single reliable four-item scale (Cronbach's α range from 0.72 to 0.87) about its learning potential with respect to its use of representations. Figure 4 shows the means and standard errors of these six scales for all three subsamples. The value 1 means strong disagreement, whereas the value 4 stands for strong approval. Looking for differences between the means of the subsamples yields mainly that the in-service teachers at academic track secondary schools have evaluated the learning potential of the first two type 1 problems much higher than the other participants.

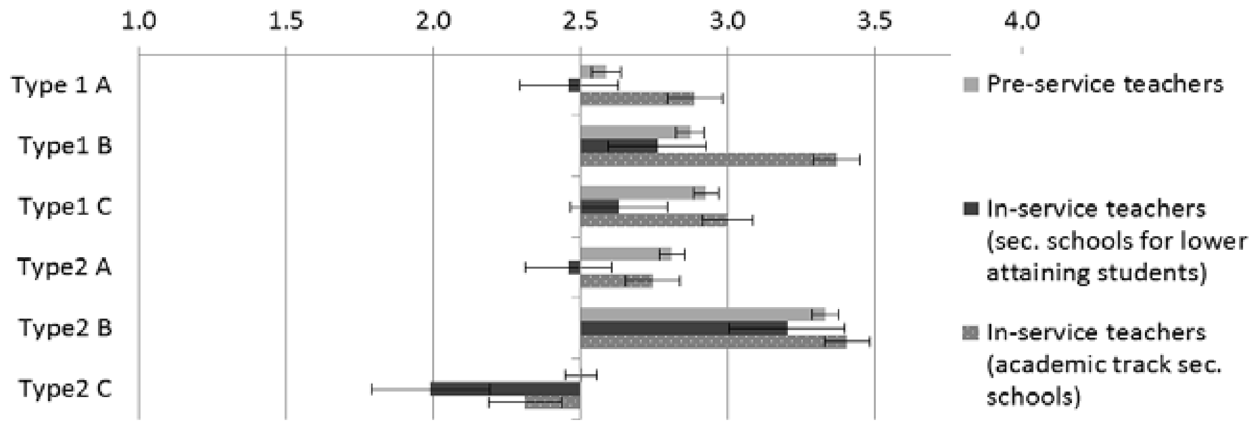


Figure 4: Evaluations of the task-specific learning potential

Comparing the evaluations of the different tasks creates the impression that the views, which were expressed here, are very task-specific. The uniqueness of each problem seems to be predominant over their classification into two types according to their use of representations: The tasks with the most positive and the most negative ratings both belong to the second type (calculation tasks with rather unhelpful pictorial representations). Nevertheless, the theoretical classification of the tasks underlying their creation can be reconstructed empirically from the teachers' evaluations of their learning potential:

Component	Type1_A	Type1_B	Type1_C	Type2_C	Type2_A	Type2_B
1	.719	.696	.659			
2				.786	.628	.610
Extraction Method: Principal Component Analysis. Rotation Method: Varimax with Kaiser Normalization.						
a. Rotation converged in 3 iterations.						

Figure 5: Factor analysis (Rotated Component Matrix; 48.9 % of variance explained)

Carrying out a factor analysis with the six scales about the learning potentials of the six problems yields two "meta-scales" linked to the two types of tasks (cf. Figure 5), where both scales are reliable with $\alpha = 0.79$. Having now scales corresponding to the learning potential of two types of tasks which make use of multiple representations in different ways, it is worthwhile comparing the aggregated evaluations of the subsamples once again. The means (and standard errors) in Figure 6 show an

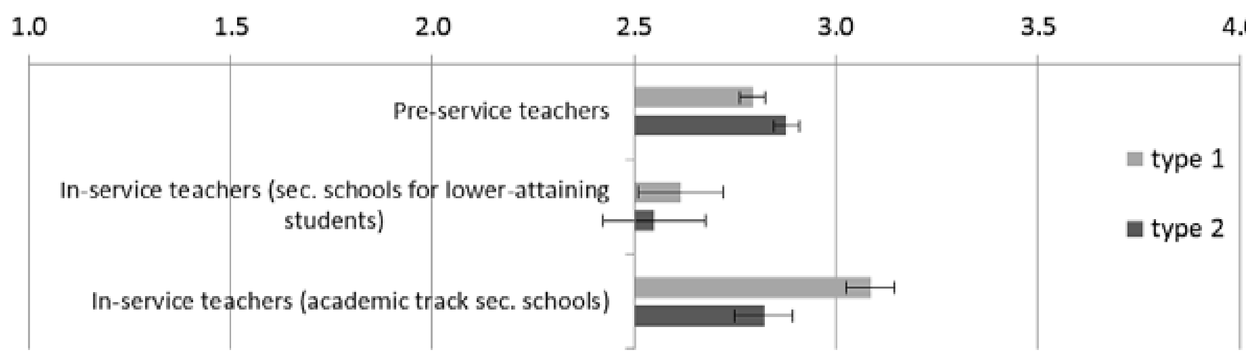


Figure 6: Evaluations of the learning potential regarding the two types of tasks

interesting result concerning the subsamples of this study: While the pre-service teachers' rating of the learning potential is higher for type 2 tasks than for type 1 tasks (T=2.121, df =218, p<.05, d=0.18), the pattern might be reversed for the in-service teachers at secondary schools for lower-attaining students (not significant) and is completely reversed for the in-service teachers at academic track secondary schools (T=3.015, df=57, p<.01 d=0.53). Focusing on the views about the learning potential regarding tasks of the first type, a comparison between the subsamples yields that the in-service teachers at academic track secondary schools have given higher ratings than the pre-service teachers (T=4.221, df=275, p<.001, d=0.63) and than their colleagues at secondary schools for lower attaining students (T=4.113, df=81, p<.001, d=0.98). Comparing the sub-samples regarding their view about type 2 tasks on the other hand shows that the pre-service teachers have assigned a higher learning potential than the in-service teachers at secondary school for lower-attaining students (T=2.487, df=26.9, p<.05, d=0.68).

These results give rise to the third research question of this study: Is specific CK interrelated with such views concerning the learning potential of types of tasks as a part of domain-specific PCK? Comparing the evaluations of those participants having a score of at least 50% in the test about specific CK to the rest of the sample might give some insight. Figure 7 shows that the participants with at least 50% CK

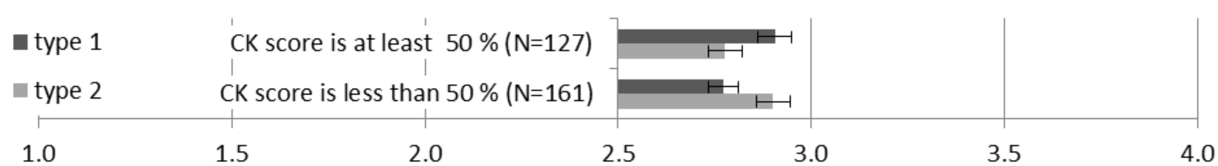


Figure 7: Evaluation of the types of tasks according to CK score

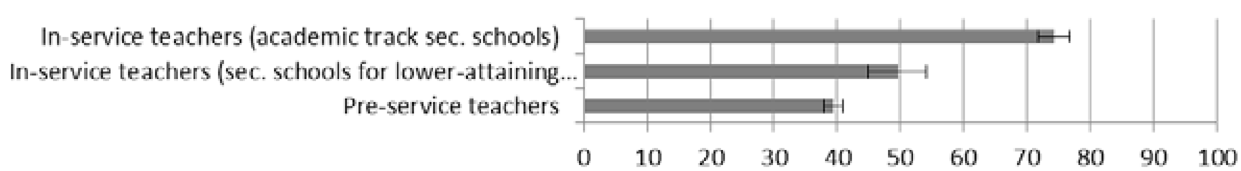


Figure 8: Content-specific CK scores (in percent)

score have assigned a higher learning potential to type 1 tasks than to type 2 tasks (T=2.413, df=126, p<.05, d=0.27), whereas the evaluations of the subsample of teachers with a lower CK score shows the reversed pattern (T=2.564, df=160, p<.05, d=0.29). The rather low effect sizes are not the only reason why this result should be interpreted with care: Another indication for scepticism can be found in the results concerning the CK scores shown in Figure 8. The mean scores of the three subsamples in the test on specific CK are very distinct: The in-service teachers at secondary schools for lower-attaining students have on average scored higher than the pre-service teachers (T= 2.196, df=229, p<.05, d=0.47), but lower than their colleagues at academic track secondary schools (T=5.111, df=80, p<.001, d=1.23).

Thus, the division into groups according to CK scores as done above leads to a rather uneven distribution of the subsamples. This calls for considering the groups separately: as CK may not be the only aspect in which pre-service teachers differ from in-service teachers, it doesn't necessarily have to be the specific CK that is decisive for the distinct task-specific views of the two groups. Consequently, Figure 9 shows scatter plots according to subsamples.

Subsample 1: Pre-service teachers

Subsample 2: In-service teachers at academic track secondary schools

Subsample 3: In-service teachers at secondary schools for lower-attaining stud.

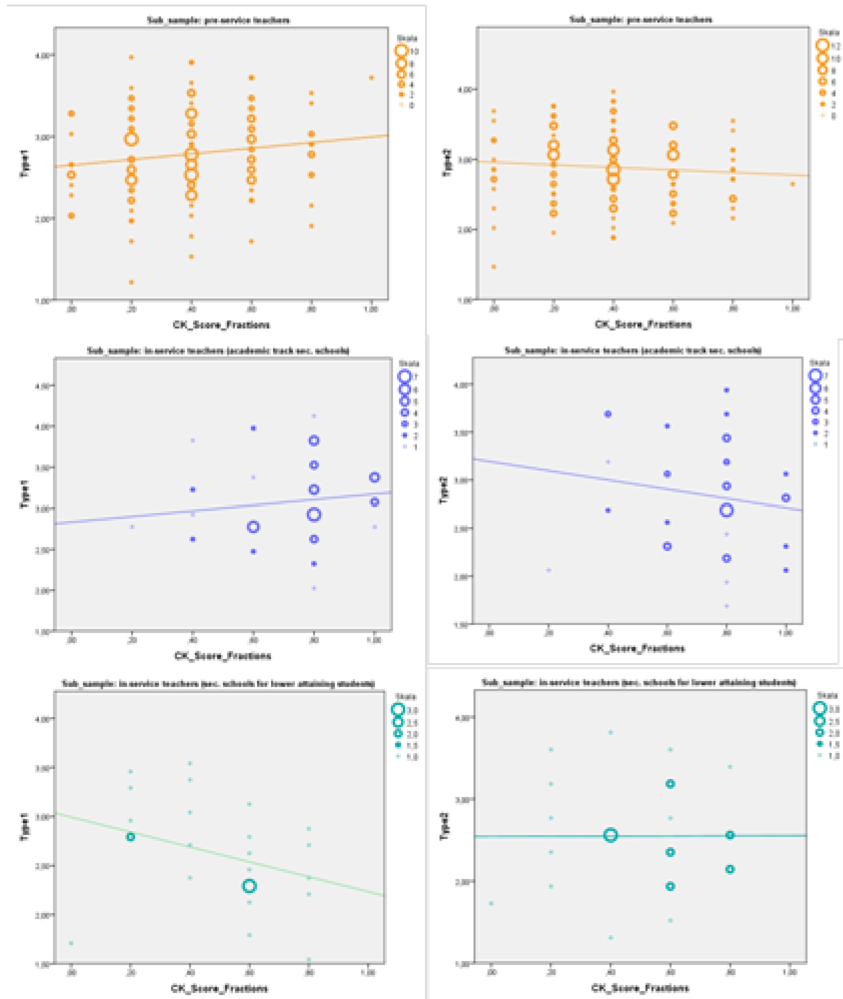


Figure 9: Content-specific CK scores and task-specific views according to subsamples

Regarding the pre-service teachers and the in-service teachers for academic track secondary schools in the sample, a relationship of the task-specific views with CK is visible to some extent, whereas it cannot be found with respect to teachers working at secondary schools for lower-attaining students.

DISCUSSION AND CONCLUSIONS

The interplay between situatedness and globality is a challenge for teachers in their everyday work: e.g. specific and individual classroom situations with learners, snapshots of learning processes seen through the lens of interactions in the classroom on the one hand and the rather non-individual contents, the relatively 'stable' and 'global' mathematical knowledge catalogue on the other hand may be seen in contrast to each other, and the teacher's role is to bridge this gap in order to support the students' (individual) learning. This contrast is in a way 'mirrored' in the teachers' professional knowledge. Consequently, research about professional knowledge on different levels of globality respectively situatedness can help to describe interdependencies between these two poles.

The present study aims at explaining task-specific views through more general characteristics of the tasks which are linked to the way they make use of representations. Hence, the overarching idea of using multiple representations is reflected in the tasks (to different degrees). Against this background, one of the major results of this study is that the overarching idea can explain task-related views even to the extent that they form scales according to task types.

As knowledge about using multiple representations in teaching and learning situations can be considered as PCK, this empirical structure makes the analysis of task-specific views to an indicator of domain-specific PCK. The results suggest that the subsamples differ with respect to this aspect of PCK.

However, even if these subsamples also differ with respect to their content-specific CK, CK differences are not sufficient for explaining differences in task-related views, as the study shows for the example of using multiple representations in the content domain of fractions. Beyond a base of CK, domain-specific PCK appears as a professional knowledge component of its own right.

A key follow-up question concerns the further exploration of such PCK and its structure. Corresponding evidence may come from an analysis of other questionnaire sections, which is currently being carried out.

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DISENTANGLING A STUDENT TEACHER'S PARTICIPATION DURING TEACHER EDUCATION

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The effect of teacher education is of international interest, at the same time expectations on newly educated teachers increase. Deep understanding of what it means to become a primary school mathematics teacher is necessary and this constitutes the focus in the research project. The aim of this paper is to illustrate how two conceptual frameworks, System Functional Linguistics and Patterns of Participation, have been used in the study. The first has been used as a methodological tool and the second as an analytical tool. The use of these will be illustrated by the case of Lisa, a student teacher. The results show that System Functional Linguistics successfully disentangles the heritage of Lisa's past and present practices, and facilitates interpretations through Patterns of Participation.

Keywords: identity development, participation, student teacher, teacher education

INTRODUCTION

The role of teacher education in the educational system is highlighted and discussed more and more, both in Sweden and internationally. Expectations on teachers seem to increase, and the increased demands raise questions about the quality of teacher education and how its various parts affect student teachers' knowledge and identity development (Hejzlar, 2008). To develop as a teacher is a long process that evolves gradually. By understanding that process leading to the formation of a teacher can we better understand what it means to become an upper primary school mathematics teacher.

The content of this paper derives from a study focusing on student teachers' identity development. The study aims to disentangle these students' participation in different practises, analyse and understand how different practices influences students' identity development. It is about understanding and contributing to knowledge in the process of becoming an upper primary school mathematics teacher. The point of reference is that student teachers' participate in different situations that reflect aspects of what mathematics is, how to learn and teach mathematics and what it means to know mathematics. The focus in the study is on how student teacher's participation, according to these questions, change during teacher education in relation to external and internal influence. Internal influences may be different parts of teacher education and possible external influences may be family, media, social structures or relationships.

Historically, in Sweden, there has been some changes within the educational system regarding teacher education between 1968 when the program started to 2012. One of these changes concerns 1988, when teachers went from a special upper primary school teacher education to a general primary school teacher education (Hejzlar, 2008). In 2011 a new upper primary school teacher degree was reinstated. The main reason for this was that the former teacher education had neglected the specific content regarded to teach children aged ten to twelve.

Phillip (2007) points out that traditional research on teacher development generally has focused on the individual. However, in recent years the focus has been changing towards more social theories. According to Morgan (2010) taking a social perspective to analyse and understand teacher development and identity can bring greater understanding of processes within teacher development. Above all, a social theory provides us with other information about teachers' identity development and change than, for example, traditional research on beliefs.

The aim of this paper is to illustrate how two conceptual frameworks, System Functional Linguistics and Patterns of Participation, have been used in the study. The case of Lisa, a student teacher at teacher education, will be used as illustration.

METHODOLOGICAL AND THEORETICAL FRAMING

Two conceptual frameworks are used as methodological and analytic tools. First to address and unfold situated communication System Functional Linguistics (henceforth SFL) has been used (Halliday, 2004). It is strictly considered as an methodological part in the analysis when disentangling students participation. Also, it is a well establiched concept when adressing and unfolding communication that are situated (Björklund Boistrup, 2010). However, SFL does not empisise any content or situation, it mainly focus on, in this case, how the student teacher addresses the content or situation. Therefore there is need for a framework that focuses on the content and situations that has been addressed. Doing this, the conceptual framework Patterns of participation (henceforth PoP) will be used (Skott, in press).

Coordinating theories

As stated above, the purpose of this paper is to present the analysis of a case study using as analytical approach concepts from two different conceptual frameworks. I hold that I am coordinating two different theories. Prediger, Bikner-Ahsbahs, and Arzarello (2008) make a distinction between coordinating and combining theories. They define coordinating as a term for bringing theories together that contains interpretations of notions that are compatible, whereas combining is when theories are only juxtaposed.

In both perspectives the concepts: learning, identity development and evaluations are seen as situated within context. They do also explicitly focus on the construction of context as something that arises within situations. In this sense they are alike, but they complement each other in one imported way. SFL addresses and PoP describes shifted participation in situations, evaluations and communication. The outlined approach uses therefore SFL to unfold situated communication to reveal traces of context (Morgan, 2006). These traces of context are then interpreted through the conceptual framework of PoP (Skott, in press).

System Functional Linguistic

SFL draws upon the notion that a text is not something predefined, rather, it is something that is constructed while participating with others. It regards a text as being handled in three different functions (the ideational function (1), the interpersonal function (2), and textual meta-function (3)), so-called meta-functions, simultaneously. Morgan (2006) says that this unfolding into meta-functions does not create descriptions about mathematical situations but it serves as a crucial window when following processes. It provides means that can be interpreted through a compatible framework.

I will present these meta-functions while addressing the following sample from the transcript of Lisa's first interview. The approach described below is illustrated in Table 1.

> I do not really know what I want to be, and I still do not really know. Then I was at a school and had practice, so it was there the idea came. But as I said I do not still really know [...] I thought that teacher education would be a little easier than this. You have of course heard from various people, that everyone goes through and no one will fail. That was why I chose it too. That is, if you have nothing to do, you can always be a teacher.

The first meta-functions concern people's expressed experience, the ideational function (Halliday, 2004). It focuses on the construct of reality by linguistic means. It concerns statements that address the interpretation of one self in different contexts. An example of this is when Lisa expresses her scepticism towards the choice of being a part of teacher education. "I do not really know what I want to be, and I still do not really know."

Ideational Function	**Interpersonal Function**	**Textual function**
We **address experience** in relation to things, events, situations, thoughts and feelings. It is interested in the verbs that are expressed and the entities that it refers to. Firstly you have those processes that involve **physical actions**. There is an actor (doer) that does something. Secondly you have those processes that are **mental** like thinking, wanting, to know etcetera. The senser is addressing a phenomenon.	Concerns **past and present experiences**, focus on the **addressed interaction.** *"Then I was at a school and had practice."* The subject is "I" and finite verb "was". The addressed interaction is "a school". When using a finite verb the evaluations can signal: 1. Whether the proposition is valid for the present, past or future time. 2. Whether the proposition is about positive or negative validity.	Finally we have to **consider the context and the language that surrounds us.** We have to construct our message so that it fits into a bigger conversation. "You have of course heard from various people, that everyone goes through and no one will fail." Here the theme is connecting to a meta-discussion that addresses coherence. Coherence is concerned a mental phenomenon that cannot be identified.

Table 1: The methodological framing

While using language and express our experiences we at the same time focus our participation against someone or something. This is the second function and is called the interpersonal meta-function. Lisa continues, "Then I was at a school and had practice, so it was there the idea came." This is more active than the ideational meta-function; it is interpersonal by mean that it is both interactive and personal (Halliday, 2004). This means that a text is about something and is addressing someone. This is done within situations and calls the textual meta-function (the third function). It concerns the process of construing the coherence of a text. Lisa continues. "You have of course heard from various people, that every one goes through and no one will fail. That was why I chose it too. That is, if you have nothing to do, you can always be a teacher."

Patterns of Participation

Lerman (2002) suggests that research on teacher development would go from focusing on acquisition to focus on participation. To learn is to participate and become another. This means to pay more attention to the interpersonal, namely the personal experience in a social setting. This perspective sees student teachers' identity formation and learning as a result of shifted participation in educational situations and acknowledges that all activities are situated (Lave, 2000; Skott, 2010).

Through students' participation, there are several aspects of patterns that are related to different levels: individual, social, institutional, political and cultural. It is the shifting participation in these levels and aspects that PoP want to describe by focusing the pre-reified processes that are said to precede mental construction (Skott, in press).

Teacher education includes several contexts of actions that are separated and connected in different ways and intends to reproduce and alter students' participation in situations (Dreier, 2000; Lave, 2000). According to Dreier (2000), a context of action is an arranged situation in a specific location with specific content and specific participants. This notion, context of actions, is not used within the PoP framework but will be used in this article to emphasis that it is the actions that makes the context occur. This notion is consistent with Skott's (2010) use of context as something that arises in situations.

Participation requires participants, that is, individuals who by participating develop their individual participation. The individual is part of a situation and a part of the context of actions overall repertoire. Since participation is situated in specific location individuals participate in a specific way. A prospective teacher brings multiple ways of participating into the educations different contexts of actions. Some of them derive from mathematics and others do not. The different contexts of actions that prospective teachers during teacher education involves in will somehow transform their way of participating in relation to what mathematics is, how to learn and teach mathematics and what it means to know mathematics.

In PoP student teachers' individual mathematic skills or beliefs are not central, but the process said to precede it. This process is phrasing, in participatory terms, the shifted movement when relating to these different aspects of teaching and learning mathematics. PoP is described as an alternative approach to research about beliefs, teacher change and identity development (Skott, 2010).

METHODS CONCERNING THE STUDY

The study, from which the content of this paper derives from, is a theory driven multi-sited ethnographic study. Theory driven because theories that emphasize the social are guiding the choices made during the ongoing project (Walford, 2009). Multi-sited because the mode of construction is not a single site, instead the mode of construction is a process that takes place in multiple sites (Pierides, 2010). In the study four student teachers are followed through their teacher education. They are observed and interviewed before, during and after different context of actions like courses, lectures, seminars, internships, study groups and examination work.

The focus in observations and interviews are on learning and teaching mathematics and the interviews have been transcribed word by word. The assumption is that student teachers cannot evaluate their participation in past and present context of actions without simultaneously identify their self and relate to different people in different ways. When this occurs they present specific content while in communication addresses others through an evaluation.

In the analysis one case will be focused on, the case of Lisa. The generated data intended to give examples of Lisa's participation in different past and present context of actions and comes from semi-structured interviews (Kvale, Brinkmann & Torhell, 2009) with voice recording.

When analysing the data first SFL has been used. Every clause in the transcripts has been connected with the specific meta-function. Then the unfolded communication has been coded, analysed and organized in a Static – Dynamic Analysis (Aspers, 2007). Then the aspect found in the transcripts are summarized and compared. The next step in the analysis was to apply the framework of PoP with the intention of phrasing Lisa's participation in participatory terms to describe the shifting movement when she relates to different aspects of teaching and learning mathematics.

THE CASE OF LISA

Lisa, who is in her 20s, started teacher education directly after high school. According to Lisa she comes from a family history of not entering University. Her family is proud that she wants to continue studying after high school. Lisa is included in this study because of her interest in mathematics and because she intends to write at least one master thesis, 15 credits, at advanced level in mathematics education. This means that she will have at least 45 credits in mathematics education after her graduation. The school where she carried out her first internship is located in a small town. The supervisor teaches Mathematics and Natural Science. In this example I choose for space reasons only to present one aspect, among others, that Lisa emphasised in her initial interview. This aspect is related to competitions as inspiration for motivation in learning.

The first interview was directly conducted in the beginning of her teacher education and focused on past classrooms experiences and current understanding about teaching and learning mathematics. The second interview was made at the end of the first five-week internship period, this after seven months at teacher education and concerned Lisa's experience gained during this period in relation to what mathematics is, how to learn and teach mathematics and what it means to know mathematics.

Lisa highlights in the first interview her own experiences from her own schooling by pointing out that different competitive elements increased her motivation, "this to get it a little more challenging." The example is collected from her experience when learning multiplication, which was the beginning of her interest in mathematics that continues today: "I have studied to level D and it has been great fun to study mathematics.". Mathematics is one of the topics ranked highest, and she looks forward to teaching mathematics after her teacher education is completed. In this interview, Lisa relates to how these experiences occurred. She mentioned one very good teacher that taught in a playful manner in which the element of competition is presented as important: "we had a lot of competitions". Competitions helped Lisa become more interested in the subject and "get better results because you are more engaged". The competitive element made "you get interested".

In the second interview, Lisa's involvement with the supervisors on the internship makes her have a general discussion about what and how a teacher should be. She highlights that: "[…] teachers should be committed; being knowledgeable is very important, the desire to develop students, to be close to students and I think it is important to have a sense of humour and the ability to see the students. It is possible to make the long list even longer.". All these diverse criteria were fulfilled, according to Lisa, in the mathematics teacher whom she had met in recent weeks.

In the second interview, the experience she gained from being part of the classroom is that the mathematics teacher engaged in teaching, according to Lisa, closely to the pupils' thoughts. "The teaching is at their level, it is very playful all the time, experiments and interaction with the students, they contend all the time.". Participation and the competitive element are something she highlights in this interview in relation to the above elements: "where the competitive instinct may be awakened and so.". This is highlighted as something positive, that the mathematics teacher according to her awakes the competitive spirit in the students.

One additional interview was conducted a month before Lisa's first internship with her future supervisor and it concerned school culture, experience about teaching and learning mathematics and prospective teachers' expected participation during internship. While interviewing the supervisor, before the practice period began, the mathematics teacher stressed the students' individual competing against themselves as central to how the students would be able to see their own individual learning.

> I teach multiplication every third week. We do it both on Mondays and Fridays on time. It may seem a bit stressful and messy but they like it very much because it is easy to measure against? oneself. I always tell them that you have to beat yourself and no one else. Then we do the same thing on Friday and compare. There is no one that I can show that they have not developed and I think that is pretty cool.

This is a deliberate strategy by the teacher, to increase their motivation and confidence. "When I say I have not done anything, it is you who have done this. I think that is fantastic when they realize that I have actually taught myself this.". The key features described above are illustrated in Table 2.

Table 2: Aspects and key features in Lisa's evaluation

Aspect	Evolution		
	Interview 1	Interview 2	Remarks
Competitions as central for learning mathematics.	Ideational – connected to past experience Increasing motivation Challenging Good performance through commitment Beginning of Lisa's interest in mathematics Playful Better results	Interpersonal – connected to present experience Playful Interaction Good performance through commitment Competitive instinct Competitive spirit Recurrent contending in competitive element	The mathematics supervisor at internship emphasise students' individual competing against themselves as central to how the students would be able to see their own individual learning.

Interpreting the case from a participatory stance

Lisa re-engages initially, in the first interview, in a pattern that emphasises the element of competition as central for creating engagement and motivation. The competitive element makes teaching, according to her, more challenging and it was seminal to her interest when participating in past mathematical contexts of actions. This is her own re-enactment in relation to the situations she herself experienced through the schooling she participated in, her earlier experience that is part of her current patterns. Through participation in various contexts of actions the aspect described will either be enacted or re-enacted, moulded, fused, and sometimes changed beyond recognition as they confront, merge with, transform, substitute etcetera (Skott, in press).

Lisa's representation emerged through participation in a specific teacher's context of action. This teacher is, according to Lisa, the person who kinder her interest in mathematics through participation in various competitions. Participation in these competitions enables you to relate your own skills towards yourself and to others. In Lisa's first two interviews this is a key element for learning mathematics. Her prior interest in mathematics is unsupported by the subject itself. It is the engagement that arose through participation, in competitive teaching that is central. Good performance is achieved by commitment and this occurred, in Lisa's case, through participation in competitive elements.

The difference between the representations in Lisa's first and second interview is that the first one is to a great extent ideational, because she cannot relate to anything else but personal experience from her previous life. This observation is relevant when discussing past and present participation and how Lisa re-enacts to these practices. It is only when following up on the competitive aspect, later in interview one, that she relates it to the teacher's physical actions from her past experience. In the second interview she relates the aspect entirely to the interpersonal, i.e. the experience she gained from participating in the mathematics teacher's classroom during the practise period. Her prior engagement and experience from historical mathematical classrooms has been absorbed, merged and/or changed into another representation. Lisa's identity has changed through shifted participation.

However, she discusses how a teacher should be, without relating to social relations, and then intertwines this argumentation to include the mathematics teacher whose practice she participates in. Lisa describes an active classroom where participation in the mathematical situation is crucial. What she describes as central in this participation is the competitive instinct that the teacher creates in students. The teacher is said to enact and re-enact this competitive instinct recurrently, which gives good working pace and makes the teacher's instruction varied.

Lisa's representation regarding 'competitions as inspiration for motivation in learning' has been both complemented and constructed through her participation in relation to the first interview. Competitions during interview one is motivating, challenging, playful and engaging. After participating in the mathematics classroom, competitions form a natural part of teaching. My interpretation is that the repertoire of words she uses to describe the aspect expanded while she participated in the supervisor's, mathematics teacher's context of action and has deepened her reasoning about the importance of competitive elements.

After the analyses of Lisa's two interviews were conducted, an iteration of the supervisor's interview was done using the categories that had emerged from the initial analysis. Regarding the element of competition, there was a long section, about four minutes, where the mathematics teacher devoted himself to describe his teaching. He emphasised competitions as a developmental force when teaching multiplication, that it is in this phase that students clearly view their progress. What the supervisor describes as a three-week interval, Lisa describes as something frequent, a natural part of teaching. Whether it is every third week or a natural part of teaching, it can be interpreted as Lisa's participation in this mathematical classroom has re-enacted her past experience and that her representation about competitions as inspiration for motivation in learning in the mathematics classroom has been complemented and strengthened.

SUMMARY

The article is based on a participatory perspective on students' identity development and is derived from the overall interest to better understand what it means to become a primary school mathematics teacher. To follow this process of becoming makes it possible to take into account both internal and external influences from past and present practices. Both the internal and external are a natural part of all situations which student teachers participate in. The participation in these past and present practices has in this paper been highlighted or more explicitly one chosen aspect has been presented and how this aspect has evolved during the first part of teacher education.

To follow student teachers in teacher education is therefore based on patterns that occur not only as institutional context of actions, but because of the length of the education and it's complexity it evolves from diverse social practices. The analytical tools showed potential in disentangling Lisa's participation in these past and present complex practices and enabled interpretations through PoP. Especially, when describing the difference in Lisa's two interviews concerning the aspect, competition is central for motivation in learning mathematics.

Lisa's evaluation is not seen as an expression or an individual imprint that she believes in, but as an identity expression that arising from the situation. Nor does her statements sees as expressions of a position but as a continuous variable motion through teacher education. SFL has helped to demonstrate this movement by the assumption that texts have different functions, i.e. addressing different objects and situations. In relation to different aspects and levels it is possible to disentangle and compare patterns and discuss if they have been enacted or re-enacted, moulded, fused, changed, merged, transform etcetera.

By using the specific conceptual framework PoP it has been possible to interpret different patterns and present one aspect in participatory terms. I have also presented how this aspect during practice at a school has an effect on Lisa's identity development. The data construction and data analysis made it possible to interpret her shifted participation during the first year of education. The analytical tools have contributed to highlighting how Lisa's representation of competitions as inspiration for motivation in learning has changed.

SFL showed to have unexpected potential in relation to the ethnografic approach. As mentioned earlier the mode of construction is not a single site, but a process that takes place in multiple sites. When addressing and unfolding situated communication it became, in many cases, very clear which sites or phenomenon to further explore. In Lisa's case for example the teacher from past experience and the present supervisors' classroom. SFL made it very explicit where to further conduct research to understand Lisa's participation in past and present practises.

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A THEORETICAL REVIEW OF SPECIALISED CONTENT KNOWLEDGE

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This work is a bibliographical review of Specialised Content Knowledge from the model of Mathematical Knowledge for Teaching. It offers a discussion of the most frequent definitions for this subdomain. We work with two examples of specific tasks which, according to the authors, require specialised knowledge on the part of the teacher. We identify the essential characteristics of the mathematical knowledge involved in these tasks and contrast these with the features commonly employed to identify specialised knowledge. We conclude with a discussion of the nature of specialised knowledge, which serves as the starting point for several papers by our research group to be presented to this working group.

Keywords: Specialised Content Knowledge, Mathematical Knowledge for Teaching.

INTRODUCTION

What does a teacher need to know to teach mathematics? What mathematical knowledge does the teacher require to teach a specific topic? How and where can teachers use this knowledge in practice? Questions like these have prompted numerous research projects aimed at studying the *ideal* knowledge and skills which a mathematics teacher should possess. In particular, a research group based at the University of Michigan has spent several years working on a scheme which allows them to categorise the typology of mathematical knowledge observed in, and required by, education: Mathematical Knowledge for Teaching (MKT).

MKT refines the map originally put forward by Shulman (1986) into subdomains, thus: the superordinate 'Subject Matter Knowledge' domain is subdivided into 'Common Content Knowledge' (CCK), 'Specialized Content Knowledge' (SCK), and 'Horizon Content Knowledge' (HCK); 'Pedagogical Content Knowledge' is in its turn subdivided into 'Knowledge of Content and Students', (KCS), ''Knowledge of Content and Teaching' (KCT) and 'Knowledge of Content and Curriculum' (KCC). This breaking down of domains into more finely defined sub-categories owes as much to lesson

[1] This article represents part of the studies into teachers' knowledge by the SIDM group (from the Spanish 'Research Seminar into Mathematics Education') based at the University of Huelva, Spain. It comprises the following researchers: José Carrillo (coordinator), Nuria Climent, M. Cinta Muñoz-Catalán, Luis C. Contreras, Miguel A. Montes, Álvaro Aguilar, Dinazar I. Escudero, Eric Flores and Enrique Carmona (University of Huelva), Pablo Flores, Nielka Rojas and Elisabeth Ramos (University of Granada, Spain), C. Miguel Ribeiro, Rute Monteiro and C. Susana dos Santos (University of the Algarve, Portugal), Leticia Sosa and José L. Huitrado (University of Zacatecas, Mexico), and Emma Carreño (University of Piura, Peru).

observation as to reflection on what sort of knowledge teachers should have, and the demands, in terms of mathematical reasoning, intuition, understanding and skill, the profession places upon them (Ball, Thames, & Phelps, 2008).

The aim of this short research is to take a closer look at SCK, and map out the advances made in the field, its nature and the difficulties that arise when it is applied to systematising teachers' mathematical knowledge.

DEFINITION OF SCK

One of the main contributions of MKT, according to its authors, is the identification of knowledge in terms that are purely mathematical and specific to the profession, SCK. This has been largely well received by the research community in that it specifies the teacher's knowledge. However, there are also some drawbacks which make it difficult to observe and analyse.

In this section we present the results of a wide-ranging literature review on what, from the point of view of the definition, is understood by SCK, drawing on the work of various authors who have used MKT in their research, whether seeking a better understanding of this subdomain or aiming to develop it in some way. Our intention is to give an overview of a collection of studies, looking specifically at how the definition has been adapted and developed over time.

Our starting point is the work of Ball *et al* (2008) in which the definition of SCK is predicated on notions of the profession and Common Content Knowledge, a practice followed by many subsequent authors (Hill *et al*, 2008; Delaney, Ball, Hill, Schilling, & Zopf, 2008; Hill, Ball, & Schilling, 2008; Krauss, Baumert, & Blum, 2008; Knapp, Bomer, & Moore, 2008; Carreño, & Climent, 2009; Suzuka *et al*, 2009; Kazemi *et al*, 2009; Markworth, Goodwin, & Glisson, 2009; Rivas, Godino, & Konic, 2009; Godino, 2009; Van, 2009; Godino, Gonzato, & Fernández, 2010; Sosa, & Carrillo, 2010; Ribeiro, Monteiro, & Carrillo, 2010; Castro, Godino, & Rivas, 2011; Herbst, & Kosko, 2012; Rivas, Godino, & Castro, 2012). However, none of these definitions specifies the nature of the knowledge in itself, but rather they all evoke external agencies.

Definitions alluding to professional demands, tend to make reference to the mathematical knowledge and skills unique to education, and which are generally not used in other contexts. Education requires knowledge beyond the pupils' mode of thinking. This implies a particular way of unpacking mathematical knowledge which is not necessary (or even desirable) in other professions (Ball *et al*, 2008).

Great emphasis is placed on the insistence that this kind of knowledge pertains exclusively to the ambit of mathematics teaching, and is not required in other professions. Nevertheless, one might justly ask how it is that we know that a certain kind of knowledge is not required in other professions. Is it necessary perhaps to check what kind of mathematical knowledge is used in each profession?

Fortunately, an indication of how this task might be undertaken is offered by such definitions themselves. The use of the term 'skills' indicates what, until this point, has been lacking in determining the nature of the knowledge involved in SCK. In other words, the definitions of SCK tend to be phrased in terms of what having this knowledge enables one to do: "responding to students' 'why' questions, […] choosing and developing useable definitions, modifying tasks to be either easier or harder" (Ball *et al*, 2008, p. 400), to mention just a few.

Drawing on the work of Rivas *et al* (2012), amongst the skills which can be attributed to this kind of knowledge are selecting and designing class activities, and making representations and giving explanations of curricular items. Suzuka *et al* (2009) emphasise that one skill demanded by SCK is that of interpreting mathematical productions, both those generated by students and those to be found in materials.

From the above, then, it follows that SCK is defined as unique to teachers in that the tasks it allocates to them are indeed specific to mathematics teachers. Nevertheless, it seems to us that there remains the question of whether the mathematical knowledge which allows these tasks to be successfully performed is shared by other professions.

The other tendency which is frequently deployed when defining SCK is comparison with CCK. CCK is defined as the knowledge required in order to solve such tasks as are given to pupils. Other definitions describe it as the knowledge held by a well-educated adult at the educational level in question. Markworth *et al* (2009) symbolically define SCK as "content knowledge needed for the teaching of mathematics, *beyond* the common content knowledge needed by others" (pp. 69). Hence it is knowledge that the pupil may not necessarily learn. Are we to understand 'beyond' in this context as a deeper or amplified kind of CCK? And what if the educational intention was to extend and amplify the knowledge of a topic, with the result that these defining features now formed part of CCK? Is this form of knowing content separate from the way mathematicians usually know mathematics or is some kind of intention required, and hence knowledge of teaching/learning to be so? What benefits are there to separating out mathematical knowledge in this way?

Defining SCK in this way raises the difficulty of clearly demarking what can be considered common knowledge from specialised knowledge. The point at which one shades into the other depends on various factors ranging from general considerations (educational level, the school system) to more specific ones (the teacher's particular intentions).

EXAMPLES OF SCK

By way of illustrating what SCK refers to, in what contexts it is used, and how it is applied, we collated various examples from the literature. These include classroom sequences, or episodes, in particular those in which the teacher has to deal with difficult or unexpected circumstances, which show how the teacher interacts with mathematics.

In this section we offer a full analysis of two of the most representative examples that have been employed to illustrate SCK. The examples are reported in the literature as specific educational tasks. So as to identify as explicitly as possible the purely mathematical features required to solve these tasks, we will go through each task step by step, unpacking the information.

In the first example (Figure 1), a subtraction problem is given along with a typical algorithm for solving it and two potential errors that pupils might make. The mathematical knowledge involved in the analysis of procedures leading to the detection of these errors (one of a teacher's specific task) is identified as SCK.

Subtraction by regrouping

The subtraction is presented:

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline \end{array}$$

Most readers will know an algorithm to produce the answer 139, such as the following:

$$\begin{array}{r} \overset{2}{\cancel{3}}\overset{9}{\cancel{0}}7 \\ -\ 168 \\ \hline 139 \end{array}$$

Many third graders struggle with the subtraction algorithm, often making errors. One common error is the following:

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 261 \end{array}$$

Consider another error that teachers may confront when teaching this subtraction problem:

$$\begin{array}{r} 307 \\ -\ 168 \\ \hline 169 \end{array}$$

Figure 1: Subtraction by regrouping. (Ball et al., 2008, pp. 396-397)

The authors offer the following commentary:

> …in the subtraction example […], recognizing a wrong answer is common content knowledge (CCK), whereas sizing up the nature of an error, especially an unfamiliar error, typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of specialized content knowledge (SCK). In contrast, familiarity with common errors

> and deciding which of several errors students are most likely to make are examples of knowledge of content and students (KCS). (pp. 401)

This example offers a description of SCK which leads us to wonder about the terms employed: What does 'sizing up the nature of an error' refer to? What is meant by an 'unfamiliar error'? What does it mean to have 'nimbleness in thinking about numbers' or 'flexible thinking about meaning'? What can we say about going beyond the answer to the subtraction? Indeed, we could ask ourselves what can be identified as purely mathematical here?

To answer this last question, let us go through the mathematical arguments leading to the identification and characterisation of the origin of each of the errors set out in the example.

Regarding the first, the algorithm is misapplied such that the smaller number is subtracted from the larger one in each of the three columns, resulting in error as the pupil fails to grasp the importance of the relationship between the top and bottom rows in the subtraction, as Ball *et al* (ibid) make clear. The way of thinking which leads to this error is the understanding of subtraction as the 'distance' between two numbers: the fact that '1' appears at the foot of the column with '7' and '8', '6' at the foot of the column with '0' and '6', and '2' at the foot of the column with '3' and '1' strongly suggests that this was the operation applied to these numbers.

In the second example, the source that Ball *et al* (ibid) suggest for the error is a failure to recognise the positional values of the numbers at the moment of regrouping. To this we can add the consideration that according to the algorithm, zero is associated with an absence of value, so that it cannot *lend*, forcing it to borrow from the '3'. The knowledge which leads to this interpretation is the *statements* about 0 and the use of the subtraction algorithm.

In both examples, ample understanding is required about the mechanics of subtraction by regrouping works, and particularly the expanded notation of numbers.

According to the reasoning of the authors, we can say that the kind of knowledge identified above (the use and justification of the subtraction algorithm, numerical notation, subtraction as the 'distance' between two numbers, the statement about zero) would fall within SCK, given that it refers to knowledge put to use by the teacher in providing or evaluating mathematical explanations of how such errors occur, in addition to recognising and analysing them. However, we consider that the mathematical elements brought to light in the detailed analysis of the task do not provide sufficient evidence to guarantee that such knowledge is exclusive to mathematics teachers. What is more, all the knowledge involved could be categorised as Common Content Knowledge, depending on the researcher's beliefs regarding how the pupil should understand the topic in question.

This example allows us to see one of the most repeated difficulties in the literature, the lack of a clear distinction between CCK and SCK.

Another example which is a particular challenge for mathematics teachers consists in coming up with a story which represents the division of fractions, such as $1\frac{3}{4} \div \frac{1}{2}$. We will focus on the second part of the activity suggested by Suzuka *et al* (2009), in which the teacher has to analyse a story which apparently contains errors in the set up, but which has the correct answer (Figure 2).

Story Problem

I have two pizzas. My friend eats one quarter of one of the pizzas and then I split what's left evenly between two of my other friends, Dan and Heather, and each person gets three and a half pieces of pizza.

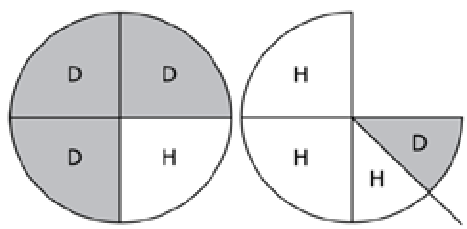

Figure 2: Incorrect story problem to represent $1\frac{3}{4} \div \frac{1}{2}$. (Suzuka et al., 2009, p. 11)

As mentioned above, one of the skills considered representative of SCK is that of interpreting mathematical productions – whether by pupils, other teachers or written material – something that this example embodies.

Let us repeat the exercise of going over the task making use of purely mathematical arguments to identify and describe the specific knowledge required to solve it. For this example we set ourselves the task of identifying the knowledge which enables one to know, on the one hand, that the result is correct, and on the other, that the setup of the story is incorrect.

Understanding why the result is correct requires being able to see the relation between the operation that the task intends to represent $\left(1\frac{3}{4} \div \frac{1}{2}\right)$ and the operation actually represented by the story problem $(7 \div 2)$. In the first of these, given that the divisor is a fraction, it is not possible to devise a natural context based on the partitive sense of division. The metamorphosis into the second operation attempts to express precisely this sense of division, but in this formulation it is not $1\frac{3}{4}$ pizzas that are divided but 7 slices of pizza (the size of each of which is $\frac{1}{4}$). This number of slices is the numerator of the improper fraction deriving from the mixed number: $1\frac{3}{4}=\frac{7}{4}$ The dividend in the second operation is the numerator of the corresponding fraction, once this has been transformed into an equivalent so that the denominators of both are the same: $\frac{1}{2}=\frac{2}{4}$ That is:

$$1\frac{3}{4}\div\frac{1}{2}=\frac{7}{4}\div\frac{2}{4}=\frac{7\times 4}{2\times 4}=\frac{7}{2}$$

It should be noted that we are not suggesting that this was the mental process by which the story problem was devised, rather we are setting out the mathematical arguments which allow one to analyse and account for the equivalence between the operations and for the answer being the same in both cases.

Hence, the knowledge involved in this first part is: knowing that the quotient of two fractions is equal to the quotient of any two equivalent fractions; knowing an algorithm for dividing fractions; knowing the multiplicative inverse property of numbers.

In order to understand why the problem posed in the story is incorrect, the required mathematical knowledge concerns the use of the meanings of division $a\div b$ as quantifier (How many times does b go into a?) and as sharing out (How many does each b get if we share out a?). The meaning inherent in the story problem cannot be extrapolated to the operation to be represented.

As with the example of subtraction, there is no purely mathematical knowledge here that can be seen as exclusive to mathematics teachers, or to which the pupil cannot have access.

CONCLUSION

In this paper we have aimed to scrutinise elements of knowledge pertaining to SCK. We began with an analysis of definitions and looked closely at examples which, according to the authors, involved specialised knowledge. We kept in mind throughout the notion of SCK as purely mathematical knowledge, whether viewed as an accumulation of special knowledge or as a special way of regarding content.

The definitions review we carried out leads us to conclude that these always employ elements which are extrinsic to specialised knowledge, such as making reference to other professions or the notion of going beyond CCK. In the analysis of the examples, we found that the specific tasks called for knowledge of meanings, properties and definitions of the mathematical topics involved. Nevertheless, it is natural to ask whether this knowledge can be considered specialised with respect to mathematics teachers. We would maintain that the answer is ‘no’. We cannot see any way of viewing the topic concerned that is particularly special, nor can we perceive any specialist mathematical knowledge which is habitually inaccessible to pupils or other professionals. What is evident is a specific use of this knowledge.

We are not trying to say that mathematics teachers' specialised knowledge does not exist; however, the data suggest that it might not be exclusive to the mathematical domain. We believe that it is impossible to think about this kind of knowledge without bringing to mind knowledge about mathematics teaching, such as ways of constructing the subject, the development of complexity within topics, and the features of learning mathematical content, amongst others. Specialised knowledge takes into account more aspects than meanings, properties and definitions.

The above conclusions form part of a series of considerations and reflections which together have led the research group SIDM in the Department of Mathematics Teaching at the University of Huelva (Spain) to work towards the development of a model which focuses on the study of what is specialised in terms of the results of an interaction between types of knowledge of and about mathematics, the structure, the teaching, the characteristics and standards of mathematics education, as well as connections to beliefs about mathematics (and its teaching/learning), mathematical knowledge always occupying the central focus. This work is the first of a series of papers (Carreño, Rojas, Montes, & Flores, 2012; Carrillo, Climent, Contreras, & Muñóz-Catalán, 2012; Montes, Aguilar, Carrillo, & Muñóz-Catalán, 2012) to be presented in this volume with the aim of offering a full picture of our advances as a research group (Carrillo *et al*, 2012, in this volume).

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THE ROLE OF DIDACTICAL KNOWLEDGE IN SEIZING TEACHABLE MOMENTS[1]

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Drawing on a short analysis of a classroom episode, we reflect on the teacher's actions and their relationship to his/her didactical knowledge, namely in its dimensions of knowledge of mathematics and knowledge of instructional processes. Focusing on these dimensions, we discuss the answers of some future and practicing teachers to a written assignment based on that episode. Anchored in the notion of didactical knowledge, we raise some issues regarding teacher education programs and their adequacy to comply with current demands of mathematics teaching.

TEACHERS' DIDACTICAL KNOWLEDGE AND CLASSROOM EPISODES

Portugal's recent mathematics curriculum for basic education (grades 1 to 9, pupils aged 6 to 14) (Ministério da Educação (ME), 2007) stresses three transversal skills – problem solving, reasoning, and communication – which are seen of crucial importance towards achieving the curriculum overarching learning goals. However, we believe that the recommended changes in the dynamics of the mathematics classroom are the crucial features which put the biggest challenges to teachers. Indeed, teachers and their students are called to play very active roles within mathematically rich environments.

In our work with future and practicing mathematics teachers, we pay special attention to issues of classroom communication, stressing the teacher's role in the process (Bishop & Goffree, 1986; Menezes, 2004; Tomás Ferreira, 2005; Martinho & Ponte, 2009; Ruthven, Hofmann & Mercer, 2011). The analysis and discussion of short classroom episodes – written vignettes of lesson snapshots – is a way that has been found to be useful in helping teachers recognize situations which illustrate challenges that they find when engaging students in meaningful mathematical discourse (Ruthven et al., 2011; Tomás Ferreira, Menezes & Martinho, 2012). Having a sound didactical knowledge seems to be of utmost importance to attain such goal.

Though acknowledging other interpretations for the idea of didactical knowledge (Ponte, 2012), we follow Ponte's (1999) perspective in which it is directly related to aspects of practice and is "essentially oriented towards the action" (p. 61). The notion of didactical knowledge encompasses four inter-related dimensions:

(1) *knowledge of the content* that is to be taught, including connections amongst mathematical concepts and connections with other areas and their reasoning, argumentation, and validation forms;

(2) *knowledge of the curriculum*, its goals and objectives, and its horizontal and vertical articulation/alignment;

(3) *knowledge of the students*, their learning processes, interests, and most frequent needs and difficulties, as well as knowledge of social and cultural factors that may influence students' performance at school; and

(4) *knowledge of the instructional process*, namely the planning and teaching of lessons, and the assessment of teachers' own practices. (Ponte, 1999, p. 61 [italics added])

This notion also involves knowledge of the contexts (e.g., school, community) and knowledge of self as a teacher (Ponte, 2012). Didactical knowledge is dynamic in nature since "the experiences and situations of practice the teacher encounters in the classroom contribute to its development and constant reformulation" (Tomás Ferreira et al., 2012, p. 283; see also Ponte & Santos, 1998).

The study reported in this paper emerged from our practice as mathematics educators. We start by analyzing a classroom episode, discussing some aspects of the teacher's didactical knowledge that support her core actions. We then present the analyses of that episode made by prospective and practicing teachers, discussing aspects of their didactical knowledge regarding the domains of *knowledge of mathematics* and *knowledge of instructional processes*. Finally, we share some thoughts about teachers' didactical knowledge and raise issues about teacher education programs and their adequacy to comply with current demands of mathematics teaching.

A Classroom Episode

In the episode *Rita and Prime Numbers* (Figure 1; Boavida, 2001, adapted from Prince, 1998), the teacher starts by proposing a closed task – to list all prime numbers up to 50 – which has a low level of cognitive demand for her students (the students in the episode correspond to Portuguese 7th graders, aged around 12). Yet, by building on a student's (Rita) comment, the teacher raised the task's cognitive demand, engaging the students in complex thinking processes, such as conjecturing, refuting, arguing, and proving. In addition, they have the opportunity to discuss aspects of basic logics (such as implications, reciprocals, examples, and counterexamples).

Rita's teacher asked her class to **find all prime numbers up to 50**. After some time, Rita noticed that the prime numbers larger than 5 she had identified so far ended in 1, 3, 7, or 9. She called her teacher to show her this finding. The teacher asked Rita to work with her partner in order to find the best way to share her finding to the class during the collective discussion of the work. Rita listed on the board all prime numbers smaller than 50 and she read what she had written in her notebook:

Rita: The prime numbers except 2 and 5 end in 1, 3, 7, or 9.

The teacher then asked the class to analyse if the same thing happened with other prime numbers. The students started checking several cases of prime numbers, some of which much larger than 100, and they did not find any prime that would not end in 1, 3, 7, or 9. Shortly, they were strongly convinced that what Rita had found was true for all prime numbers, regardless of having been checked, because all prime numbers that they would check always ended in one of those digits. At this time, the teacher wrote on the board:

Rita's conjecture: ***All prime numbers, except 2 and 5, end in 1, 3, 7 or 9.***

She made sure the students remembered the meaning of conjecture and she challenged them to find a process that would allow them to be sure if the conjecture were, indeed, valid for all prime numbers and why that was so. The students tried to respond to the challenge and, in this process, they reinforced their conviction that the conjecture was true; yet, their work did not progress. Then, working with the whole class, the teacher wrote on the board the numbers from 0 to 9, circling 1, 3, 7, and 9. Almost immediately the students offered several suggestions:

Maria: Teacher, cross out numbers 0 and 5. A prime number larger than 5 cannot end in 0 or 5.

Teacher: Why?

Maria: If it ends in 0 or 5, it is a multiple of 5 and, therefore, it is not prime.
Daniel: You also have to cross out 2, 4, 6, and 8. If it is larger than 2 and is a prime number, it cannot be even!
Bernardo: Of course not. 2 divides...
Teacher: So what?
Bernardo: A prime number can only have two factors.
Rosa: Yeah. If it ends in 2, 4, 6, or 8, that is because it is even and even numbers are multiples of 2.
Rita: We only have 1, 3, 7, and 9 left. Therefore, all prime numbers except 2 and 5 end in the way I found. We're now sure of it.
Inês: **But the opposite is not true.** For example, 21 ends in 1 and it is not prime.
Teacher: Why isn't 21 prime?
Several students: Because 3 divides 21; 3 times 7 is 21.
Bernardo: It has divisors that are different from 1 and 21.
Teacher: So, check if this is true, or not, this that I am going to write on the board: all numbers ending in 1, 3, 7, or 9 are prime.

One could hear several voices saying "it's not true". They mentioned 21, 27, 33... (…)

Figure 1. Episode *Rita and Prime Numbers*

Which instructional actions are central in the unfolding of this episode? The teacher refrained from validating Rita's idea; instead, she gave Rita and her colleague the opportunity to present their finding to the class. By building on Rita's input, the teacher extended the original task, asking the class to check whether Rita's idea would work for other prime numbers. The class naturally accepted its truthfulness as students were unable to find a way to contradict Rita's finding. The writing of "Rita's conjecture: All prime numbers, except 2 and 5, end in 1, 3, 7, or 9" on the board seems to have been deliberate – the teacher knew that the proof of a conjecture and the role of examples in that process were at stake, and that the term "conjecture" could be unfamiliar to some students. In addition, she turned Rita's conjecture more explicit to the whole class by clarifying its scope.

The collective discussion that was initiated engaged students in reflecting on the meaning of conjecture and of proving or refuting a conjecture. This was not an easy task for the students who could only see their ideas reinforced by finding more and more examples which, nevertheless, proved nothing. After letting the students struggle with this, the teacher wrote on the board all ten digits and circled those corresponding to the units of a prime number – her intention seems to have been to list all possibilities for ending a natural number while highlighting those related to Rita's conjecture. Drawing on their knowledge of divisibility criteria, the students eliminated the non-circled digits and the teacher's questions ("Why?"; "So what?") ensured that they justified all their options. By building again on a student's comment (that the opposite of Rita's conjecture was not true), the teacher involved the students in working with counterexamples, which started to emerge after she wrote the new conjecture on the board. In sum, the teacher's actions raised the cognitive level of the initial task and helped engaging the students in significant mathematical activity.

We can identify some aspects of the teacher's didactical knowledge in the episode *Rita and Prime Numbers*. For example, the teacher listened to her students in a responsive manner (Empson & Jacobs, 2008), valuing all contributions as worthy discussing collectively, regardless of their correctness or rigorousness in language. By giving the students the responsibility for proving or refuting the two conjectures presented in the episode, the teacher orchestrated a collective discussion in a productive way (Stein, Engle, Smith & Hughes, 2008), pushing for a shared understanding of conjecture and for explanation and justification of all assertions.

We believe the teacher's actions were anchored in her *mathematical knowledge*, which allowed her to recognize a teachable moment triggered by Rita's finding, and in her *instructional knowledge*, which allowed her to seize the situation and build instruction upon Rita's idea, encouraging her students to do mathematics (Tomás Ferreira et al., 2012). The teacher transformed a task of procedures without connections (Stein & Smith, 1998) into a task with much higher cognitive demand, involving processes of proof. The new task and the fruitful discussion around it pushed the students to engage in rich mathematical activity.

DATA GATHERING

Our practice as teacher educators reflects our belief that is it important to have (prospective) teachers discussing aspects of the teacher's role regarding the management of mathematical communication in the classroom. For that purpose, we frequently resort to the analysis of classroom episodes such as the one presented before. The data we present and discuss next is based on the analysis of the episode *Rita and Prime Numbers* guided by the following questions: (1) *How do you think the teacher should lead the classroom discourse after the last interventions of the students?* and (2) *Do you believe Rita's conjecture is proved? If so, why? If not, why?* At the end of 2011/12, a group of 12 prospective teachers, enrolled in a 2-year master's teacher certification program, was asked to complete a short written, individual, in-class assignment which included the analysis of the episode *Rita and Prime Numbers* and accounted for 10% of their final grade of a mathematics education course. There was great variation in the answers obtained not only regarding the mathematics underneath the episode but also in terms of the didactical choices that were thought to be adequate to give continuation to the episode.

Feeling the need to see practicing teachers' reactions, we asked a group of eight teachers, enrolled in a professional development course, to analyse the same episode, using the same guiding questions; yet, the assignment did not count explicitly for assessment purposes. The two cohorts of participants worked in different universities located in large urban areas in northern Portugal; in both contexts, participants had been involved in reflecting and discussing several issues of communication, especially the teacher's role in managing meaningful classroom discourse (National Council of Teachers of Mathematics (NCTM), 1991), and the challenges faced when orchestrating productive mathematical discussions (Stein et al., 2008). The participants' written productions were analysed in the light of Ponte's (1999) notion of didactical knowledge, focusing on the dimensions of mathematical and instructional knowledge.

RESULTS

In this section, we present and analyse some of the data collected from both groups of participants, resorting to our translation of the participants' work because it is originally written in Portuguese. We chose the work of three prospective and two practicing teachers as it illustrates the respective cohort productions. We structure our discussion by each of the two questions that guided the analysis of the episode.

How Should The Episode Continue?

Júlio held a bachelor degree in mathematics from the same institution he was seeking teacher certification. In his response to the first question there was evidence that he acknowledged the existence of two implications in the episode, one being the reciprocal of the other. Focusing on sense making and knowledge building, he emphasized the need of recognizing and distinguishing reciprocal implications, and understanding the role of examples and counterexamples in proofs and refutations:

> Based on the students' answer[s], the teacher should tell them that they had shown the assertion was false, through a whole-class discussion, making them understand that it is enough to give an example that does not verify the assertion for this to be invalid. Then, she should ask the students to relate Rita's conjecture to the latter one, questioning them about their difference[s] and truthfulness, in order to conclude the task.

Júlio's sensitiveness towards the important issue of developing mathematical reasoning, particularly formulating, testing, and proving (or disproving) conjectures, in the teaching and learning process seemed to be clear.

Carlos was a colleague of Júlio's, with a similar academic background. Unlike Júlio, Carlos did not evidence much understanding about the episode, due to an incorrect interpretation of the episode or to weaknesses in his didactical knowledge. His suggestion to continue the episode began with some considerations about Rita's finding, which evidence fragilities in his mathematical knowledge:

> The way Rita phrased the conjecture seems to indicate that all prime numbers are all odd numbers except those that end in 5. During the lesson, it became clear that this is not true since 21, 27, 33 are odd numbers ending in 1, 7, and 3, and they are not prime.

This prospective teacher did not seem to realize that the discussion at the end of the episode was about the reciprocal of Rita's conjecture and that the examples provided by the students (21, 27, and 33) were, indeed, counterexamples for the reverse of Rita's conjecture, not counterexamples for the conjecture itself. Besides a poor understanding of the mathematical situation underlying the episode, Carlos's suggestions to continue the episode missed some important points emphasized in current curricular orientations:

> After the students said that it was not true, that all prime numbers end in 1, 3, 7, or 9, the teacher should ask them for explanations. Some mention examples that do not verify the conjecture; yet, the teacher should ask for more examples and have them discussing the

> reason why they are not prime [numbers]. Afterwards, [the teacher] could build on the fact that 9 is not prime since the conjecture said that all numbers ending in 9 were prime.

Carlos did not assign an appropriate value to having students understanding the meaning of conjectures, reciprocals, examples, counterexamples, proofs, refutations, etc. (at the level of 7th graders), nor to having a moment in the lesson to summarize the ideas that emerged during class discussion. In addition, it was not clear why, according to Carlos, the teacher should deal with the number 9 in a *special way*. Data suggested that Carlos had a poor mathematical understanding of the episode.

Joana had earned a bachelor degree in applied mathematics and computing several years before enrolling in a teacher certification program. She worked in the field of applied mathematics and had a very short teaching experience. Her knowledge of mathematics exhibited several weaknesses, which seemed to account for inadequate instructional decisions. She misunderstood Rita's conjecture and its reciprocal; hence, not surprisingly, her suggestions to continue the episode seemed to be senseless:

> The teacher should have let the students reach the conclusion that 'all numbers ending in 1, 3, 7, 9' are not prime and she should not have written on the board and telling the conclusion. Maybe saying the students should conclude or even writing only the sentence 'all numbers ending in 1, 3, 7, or 9 are prime; do you agree?' because, by saying 'So, see if it is true' she is implicitly telling the students that something is wrong.

Joana did not interpret the teacher's intentions when writing on the board the two implications as a means to help students understanding what was at stake and to help them differentiating the two situations; instead, she saw the teacher's actions as intending to offer the students with clues for what would be correct or incorrect.

Cláudia was a certified teacher with more than ten years of experience teaching 7th to 9th graders (ages 12 to 14). In her response to the first question, she stressed critical features for continuing the episode: she believed that "at the end, the teacher would probably let the students prove that not all numbers ending in those [units] digits are prime [numbers]", giving the students accountability and ownership for drawing an important conclusion based on their own (counter)examples; in addition, she reinforced the need for a moment of synthesis – "in the end, it is important that they make a synthesis" – making explicit the main ideas that had been discussed.

Lina's teaching experience was similar to Cláudia's but she worked with 5th and 6th graders instead. Lina might have not understood what was being discussed at the end of the episode. She believed that "after the last interventions […] the teacher should ask the students to reformulate the conjecture". Her response to the second question (which we discuss later) sheds more light into her thinking.

Is The Conjecture Proved or Not? Why?

Júlio understood that Rita's conjecture was true and realized the process that was used to prove it collectively: "the conjecture is proved, since the students know that all the numbers end on some digit between 0 and 9, and using divisibility criteria, they managed to exclude the even digits and the 5, remaining 1, 3, 7, and 9”. The data suggest that Júlio valued the whole-class discussion and students’ (prior) knowledge as means to help them proving Rita’s conjecture: “In this way, and using their own knowledge, the students proved Rita’s conjecture, through discussion and exchange of ideas”. Thus, Júlio seemed to have pulled adequate aspects of instructional knowledge to the analysis of the episode *Rita and Prime Numbers*.

Carlos believed that Rita’s “conjecture may lead to two interpretations, the first being that all prime numbers are all [numbers] that end in 1, 3, 7, and 9, except 2 and 5, and, on the other hand, that all prime numbers except 2 and 5 end in 1, 3, 7, or 9”. Although this latter interpretation was, in fact, Rita’s conjecture, it was the first interpretation that Carlos believed to be at the core of the episode. He did not think that Rita’s conjecture was proved during the lesson: “Rita’s conjecture was not proved since prime numbers except 2 and 5 end in 1, 3, 7, [or] 9. What was proved was that the numbers ending in 1, 3, 7, and 9 aren’t always prime”. Resorting to the two possible interpretations of Rita’s conjecture he had identified, Carlos stressed that “in [this] lesson, the only thing that was proved was that the first interpretation is not valid”. Data seemed to suggest that Carlos did not recognize a process of proof (of whatever conjecture he would consider) in the teacher’s and students’ joint work.

Joana did not understand Rita’s conjecture *per se* and, in fact, it seemed that she had no understanding of what a conjecture is, nor what might be entailed in proving (or refuting) such an assertion:

> Rita’s conjecture was proved and it was incorrect, since the students checked for a large array of numbers, even bigger than 100, thus establishing a degree of certainty in their answers and even finding numbers like, for example ‘21’ which though ending in 1 is not prime since 3 divides 21.

Joana confused the two reverse implications involved in the episode; yet, she seemed to value the testing process as important to strengthen one’s conviction.

Cláudia believed that Rita’s conjecture was proved during the lesson; yet her response may indicate how easily teachers do what they say should not be done! In fact, she referred that “…although the student [Rita] had said ‘We’re now sure of it’, she [the teacher] should show that there were prime numbers ending in 1, 3, 7, and 9”. Cláudia showed concern for illustrating the core idea that was being discussed (Rita’s conjecture) with concrete examples, which she seemed to believe that would help in reassuring the validity of the conjecture or in better understanding the conjecture. Though understandable, such a concern and subsequent actions may actually induce students into an erroneous conception of proof, namely mistaking proof for exemplification using particular cases.

Lina did not seem to be sure about whether Rita’s conjecture had or had not been proved; in fact, after stating that the conjecture had been proved, she changed her opinion supported in a mathematically incorrect argument. Like Carlos, Lina brought up the number 9 into the scene, suggesting that she also had an unclear understanding of the conjecture that was collectively proved during the lesson:

> It seems that Rita's conjecture is proved because valid arguments were used allowing to conclude that the assertion is valid for all prime numbers except 2 and 5. However, the number 9 is not a prime number. There is at least one exception that was not considered; thus, the conjecture is not proved. Proof, in mathematics, entails demonstration.

On one hand, Lina believed that the conjecture had been proved but, on the other hand, the *proof* that was made during the lesson was not enough! Data suggested that Lina held a rigid and formal perspective of proof and made contradicting assertions since a logical chain of valid arguments no longer seemed to be at the core of a proof.

CONCLUDING REMARKS

The participants' difficulties in analyzing the episode *Rita and Prime Numbers* seemed to be anchored in a poor knowledge of the mathematics involved. The differences in academic background of the prospective teachers may have accounted for the differences in the mathematical knowledge they evidenced. However, caution must be exercised. Carlos, whose background and grade point average was similar to Júlio's, also showed gaps in his mathematical knowledge. Joana, unlike her colleagues, did have some teaching experience; yet, her knowledge of instructional processes in the classroom emerged as much weaker than that of Júlio or Carlos.

An inadequate understanding of proof (in Cláudia's case) or a very rigid and formal conception of proof (in Lina's case) may also have been at the origin of the difficulties found whilst analysing the episode. It was possible to find, in both groups of participants, illustrations of misunderstandings regarding the role of examples and counterexamples in proving or refuting assertions; yet, only in the participant practicing teachers did we find clear evidence of closed conceptions about proof, which may have hindered them from recognizing a process of proof in the episode.

The gathered data suggest that a poor knowledge of mathematics on the (future) teachers' part seems to be associated with a weakened instructional knowledge. This supports the claim that adequate instructional decisions can hardly be made when teachers do not have a deep understanding of the underlying mathematics of teaching situations (Kahan, Cooper & Bethea, 2003). In particular, the orchestration of productive mathematical discussions and the systematization of (new) knowledge, two complex communicative actions (Menezes, Canavarro & Oliveira, 2012; Stein et al., 2008) and essential aspects of the teacher's role within the current (Portuguese) curricular orientations, cannot be appropriately approached if the teacher's knowledge of the mathematics underneath the teaching situation is not sound (Martinho & Ponte, 2009; Ponte, 2012; Tomás Ferreira et al., 2012).

Teachers need solid mathematical and instructional knowledge, in Ponte's (1999) sense, to be able to build on teachable moments such as the one triggered by Rita's contribution and, more generally, respond adequately to the many demands of classroom teaching. Yet, the current typical organization of teacher education may be failing to develop (future) teachers' didactical knowledge, including mathematical knowledge, despite a significant emphasis on content courses, especially in many teacher certification programs. In addition, (prospective) teachers may not be developing adequate conceptions of proof, aligned with current recommendations for school mathematics, which go much beyond processes such as two-column proofs.

Despite a heavy content load in the participants' academic background and despite all the emphasis put in the teacher's role in managing meaningful classroom mathematical communication in the two contexts in which the participants worked, we were surprised to see how much difficulty they had in making sense of the episode and in reflecting upon it in the light of current curriculum orientations. We believe that the discussion of concrete situations, based upon classroom episodes as the one presented in this paper, may contribute to teachers' increasing consciousness about their conceptions and practices, helping them in recognizing teachable moments and in building on them, seizing the opportunities that emerge during classroom interaction and taking the most of them. But this is obviously insufficient.

We had no opportunity to interview the participants in this study in order to have a better grasp of their mathematical and instructional knowledge. Our data provides only a limited and short glimpse of what might be happening. Further research is needed to address more deeply how (future) teachers manifest their didactical knowledge and how teacher education and professional development programs may help them in developing that kind of professional knowledge.

NOTES

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MATHEMATICAL COMMUNICATION: TEACHERS' RECOGNITION OF THE SINGULARITY OF STUDENTS' KNOWLEDGE

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This paper discusses the role of collaborative work in the development of social interactions in the classroom and how teachers value such interactions in the development of the modes of communication, as well as in the interaction patterns centered on student' individual knowledge. Data are collected in the context of collaborative work involving a researcher and three teachers participating in this study. The development of the interaction among the students themselves and between them and the teachers, along with the recognition of the students' singular mathematical knowledge determined the adoption of reflexive and instructive modes of communication and also the extraction and discussion patterns in mathematical communication.

Key words: mathematical communication; modes of communication; interaction patterns; teaching practices; collaborative work.

INTRODUCTION

Mathematical communication is as much part of mathematics classes as mathematics itself. Such communication takes different forms according to the conceptions of the nature of mathematics, its teaching and learning and the role of communication in teachers' professional practices. Teachers' positions concerning the nature of mathematics as a way of understanding society are translated into forms of mathematical communication as the foundation of the mathematics teaching and learning process, considering the existence of several strategies and singular forms of mathematical knowledge in the classroom.

Mathematical communication as the foundation of the teaching and learning process is based on the recognition of the social interactions in the classroom between the teacher, the students and mathematical knowledge (Sierpinska, 1998) and considers discourse as a social practice (Godino & Llinares, 2000), where the role played by all actors in the negotiation of mathematical meanings is of the utmost importance. In this perspective, mathematical communication is a social process in which participants interact, exchanging information, influencing each other, taking up the attitude of the other and, simultaneously, expressing and asserting his or her singularity (Belchior, 2003; Mead, 1992). Teaching becomes the organization of an interactive and reflexive process, where the teacher is continuously engaged in updated and differentiated activities with students (Cruz & Martinón, 1998) and where students become aware of their cognitive and affective processes, mobilizing them for learning (Ponte, Brocardo & Oliveira, 2003).

This paper looks into the role of collaborative work in the teachers' recognition of the interactions among the students themselves and between them and the teacher, as well as the way in which these social interactions are valued in the recognizing the singularity of the students' mathematical knowledge. We focus on the relation between the development of social interactions among students between and them and the teacher, and the recognition of students' mathematical knowledge. We also seek to understand how the valorization of the students' mathematical knowledge is related to the modes of communication and the interaction patterns centered on reflecting and sharing of knowledge among all the actors in the mathematics classroom.

RESEARCH DESIGN, ACTORS AND COLLABORATIVE WORK

Access to empirical data concerning classroom practices, conjugating action and meaning, resulted in the adoption of the case study format (Stake, 1994) for this study, which relied on participant observation and inquiry, with the development of a collaborative work, grounded on the reflection and questioning about communication practices in the classroom. Three teachers of the 1st cycle of basic education − Alexandra, Carolina e Laura (pseudonyms) – took part in this study for a period of over two years (December 2006 – February 2009), in the course of which there were twelve meetings of collaborative work (collaborative meetings), fourteen classroom observations by each teacher (participant observation) and two interviews and two individual meetings by each teacher (inquiry) for over two years (December 2006 – February 2009) (for more details see Guerreiro, 2011).

These teachers had a significant professional experience, with a minimum of twelve years of teaching in this cycle, and were highly motivated to work in collaboration with a view to strengthen their professional knowledge about mathematical communication in the classroom. The definition of common interests and goals resulted from the need to ensure a level of intentionality of the collaborative work with regard to their professional development (Boavida & Ponte, 2002) and to the construction and reconstruction of the classroom practices, giving way reflective constancy and successful changes in the partnership that existed between the first author of this paper (from now on referred to as "the researcher"), and the participating teachers (Ruthven & Goodchild, 2008).

The axes of collaborative work were trust, as referred to by Boavida & Ponte (2002), and the valorization of the critical analysis of the researcher and his colleagues, who acted as external observers, as mentioned by Saraiva & Ponte (2005). However, the teachers' initial response after the first classes were conducted and discussed was of anxiety in relation to the quality of their performance and that of their students', evidencing insecurity about their knowledge and professional performance. This anxiety waned as the collaborative work went on and was replaced by a significant level of comfort and complicity among actors.

Self-knowledge about teaching practices through the viewing of classes and critical reflection fostered by the researcher led to changes that were acknowledged by teachers and within the scope of the students' group work and were visible in the presentation of mathematical tasks, in the acceptance of error as an educational resource, in the promotion of communicative interactions between the students themselves and between them and the teacher, as well as the recognition of the students' personal and individual mathematical ideas and strategies.

This attitude towards the mathematical ideas of the students and of others justified the teachers' growing disposition to further their training in order to expand their professional and mathematical knowledge. The changes that the teachers made resulted from the recognition of the students' singular knowledge. They also indicate an effective valorization of learning as teachers look upon the students as autonomous learners within the context of communicative sharing marked by the negotiation of mathematical meanings.

MATHEMATICAL COMMUNICATION PRACTICES IN THE CLASSROOM

Mathematical communication practices in the classroom are structured into several modes of communication, in accordance with the teacher and the students' role in the classroom discourse. Brendefur e Frykholm (2000) classified them as unidirectional, contributive, reflexive and instructive communication. The unidirectional and contributive modes of communication are associated with the teacher's absolute, or almost absolute, power over the classroom discourse. Reflexive and instructive modes of communication show the importance of classroom discourse as an object of reflection and sharing of knowledge between the teacher and the students, which takes on a metacognitive and reflexive dimension.

The nature of the interactions between the teacher and the students is expressed by interaction patterns that swing between the control of the students' thought on behalf of the teacher and the sharing of mathematical ideas and strategies through reflexive discussions about the processes of construction of mathematical knowledge. The existence of reciting, funneling and focusing patterns (Godino & Llinares, 2000, Wood, 1994, 1998) results from the centering the students' thought and knowledge on the teacher's knowledge by means of routine processes of reproduction of mathematical knowledge. Extraction and discussion patterns (Godino & Llinares, 2000, Wood, 1994, 1998) reveal the students' individual contribution in the construction of mathematical knowledge, through a collaborative dialogue between the teacher and the students.

Modes of communication

The usual communication practices of the teachers seem to result from the conception of the roles of both teacher and student as doers of a structured and finished mathematics, based on the resolution of mathematical tasks. In this perspective, mathematical communication is characterized by the dominant role of the teacher in the classroom discourse as the holder of knowledge to be passed down and the role of the student as keen listener, which is what defines unidirectional and contributive modes of mathematical communication, through the inclusion of examples, solution strategies and correct solutions of exercises and mathematics problems.

The teachers felt relatively uncomfortable in countering these practices of educational and communicative control, and were unable to conceal a significant effort not to condition the autonomous work of the students and to accept alternative or diverging solutions, giving some scope for the comparison and sharing of different solutions to the mathematical tasks, even if confusing, incomplete or incorrect. The growing participation of students in the classroom discourse, through the valorization of the moments of communicative interaction within the group or the class, led to an important autonomy on the part of the students and to the reflection in the action, by the teachers, incorporating in the discourse those ideas and difficulties expressed by the students or felt by the teachers, thus promoting the reflexive and instructive modes of communication:

> [We have to] give them time and tasks where they interact with each other; to give them time to interact, to understand, to discuss strategies, first within the group, without having to keep an eye on them all the time, guiding them, [or] leading them to the right answer; to give them time and some room.
>
> [February_2009 _ interview _ Alexandra]

The way in which the teachers learned to increasingly value the personal knowledge of each student resulted in a significant reinforcement of the sharing of knowledge among all actors and in the recognition of the value of the knowledge held by the students and teachers alike. Such a change increased the students' critical awareness and sense of responsibility in the autonomy with which they validated the solutions of mathematical tasks, as a feature of reflexive communication. This strengthened the sharing of mathematical ideas and strategies, instead of explicating the solution of the tasks.

The instructive dimension of communication derived from the debates about the reasonability of the results, from the reformulation of incorrect solutions, from the process of knowledge construction generated by solution of the mathematical tasks and from the recognition of the students' learning. Carolina's students began to compare their strategies to reach a mathematical solution regularly, thus showing a greater autonomy in the comprehension of alternative processes:

Jessica (reading the question paper): − Antonio needed a fence seventy-six meters long to enclose his backyard… (she does not finish reading the question)

Dennis: − We wrote a division statement where we divided the figure by two (he writes the traditional division algorithm on the blackboard): seventy six divided by two, to see how much is half of it (the student makes the calculation resorting to his notes). And then we divided thirty-eight by two to estimate the half of thirty-eight to get the answer (he makes a new calculation). And then to see if it works out correctly we (he consults his notes) calculated nineteen times four (he writes an addition sentence with equal parcels). And it makes seventy-six. Therefore, our equation is correct.

Beatriz: − Why didn't you just divide it by four?

Teacher: − Your colleague is asking you a question!

Dennis: − I wanted… wanted half of this.

Beatriz: − Half? But if the square has four sides, I think that instead of dividing seventy-six by two, well, you could simply divide it by four, since the square has four sides.

Dennis: − We did it this way...

Beatriz: − You did it step by step?

Dennis: − We divided it by two, and then, after getting those two, we would find out the other two.

Beatriz: − Yes, I know. You did it step by step, didn't you?

Dennis: − Yes.

Beatriz: − It's just that you could have done it all in one. Instead of thirty-eight divided by two, you could have gone straight on to divide it by four. But OK, you did it step by step, which is all right, too.

Dennis: − More questions?

[March _ 2008 _ class _ 3rd year _ Carolina]

These occurrences are characterized by a metacognitive dimension of reflection about the action itself, assigning value and integrating the students' knowledge. The teachers' assumption of the specific and personal knowledge of each student gave birth to the development of a reflexive and instructive communication, as well as the increase in the students' autonomy in the teaching and learning process, besides the students' recognition of the mathematical knowledge of the other students.

Interaction patterns

The permanent validation of the students' solution in the course of such an autonomous work seems to result from the uniformity of strategies and solutions expunged of errors and fitting the teachers' knowledge. This attitude on the part of the teachers derives from a learning process centered on the solving of mathematical tasks, without imprecision or ambiguities, within a short time span. This drives towards the right answer avoiding errors owes much to the initial interaction practices between teachers and students according to the reciting, funneling and focusing interaction patterns based on the teacher's knowledge:

> Sometimes, as I have my own strategy, when they divert from it and are confused, maybe, and have some difficulty in explaining how they managed it, when I explain, I explain it my way. (…) when they are moving away from it, instead of listening through to the end, we cut it short to guide them. (…) Maybe not to waste time and not to let them create wrong concepts.
>
> [January _ 2008 _ meeting _ Alexandra]

Accepting and understanding the error as a learning resource, conjugated with the valorization of the interactions between students and the decentering of the teachers in relation to the blackboard in the classroom, resulted in the valorization of the students' individual and personal knowledge, and triggered the appearance of extraction and discussion patterns based on the recognition of the students' mathematical thinking. The existence of these patterns reveals the valorization of the students' individual thinking on the part of the learning community.

The extraction pattern results from the valorization of solutions that were different and potentially incomprehensible to other students and the teacher herself, giving way to a questioning pattern where one intends to recognize the validity of ideas, solving strategies or solutions presented by students. In the above problem that consisted in determining the length of the side of a square from its perimeter, one group of students chose to go on dividing by two successively in order to establish the fourth part of the initial figure, as illustrated above. Carolina tries to understand the boy's thinking using and extraction pattern:

> Dennis: − Does anyone else want to ask a question? Yes, please?
>
> Teacher: − I do. I heard what you explained to Bia. Bia ended up explaining, ended up saying, getting your explanation, but I want to know the reason why it crossed your minds to divide seventy-six by two.
>
> Dennis: − To find out how much the half was.
>
> Teacher: − One question: the half of what?
>
> Dennis: − Half of seventy-six.
>
> Teacher: − Why?
>
> Dennis: − To find out the half...
>
> Teacher: − And that half of seventy-six is what? What is it half of?
>
> Dennis: − Where, Ms.?
>
> Teacher: − Mind you, I'm not saying it is wrong, Dennis. Did I tell it was wrong?

Dennis: − No.

Teacher: − OK. I just want to know… to figure out… to try to understand what you were thinking.

[March 2008 _ class _ 3rd year _ Carolina]

Questioning seems inconclusive, which warns us about the difficulty to know the singularity of the knowledge of the other, even through a questioning based on an extractive patterns. Such difficulty seems to have roused a moment of uncertainty in the student in relation to the correction of his solution, which made the negotiation of mathematical meanings problematic. Likewise, the discussion pattern manifests itself in the way the teacher helps to publicize and explain the different types of reasoning, solving strategies or solutions present by the students in group/class. Laura contributes to the explanation of the students through a set of questions so as to guarantee that the students' solutions are validated and accepted by the whole class. In the next classroom episode, the algorithmic solution of an addition statement containing many parcels was set against a strategy of mental calculation involving notes to support it.

Teresa: – We did the following: we added six and five, which is B plus E, six plus five, which makes eleven. And then we added four and four, which makes eight; this makes nineteen.

Miguel: – We started with the larger numbers.

Teresa: – Then we added four, which makes twenty-three. And then three plus three, this makes six, twenty-three, twenty-nine, and then one. This makes thirty.

Teacher: – Doubts?

Students: – No.

[November _ 2008 _ class _ 3rd year _ Laura]

Faced with different solutions, Laura takes on the role of inquirer aiming to clarify procedures, conjugating extraction of knowledge with the sharing and discussion of solving strategies among the students:

Teacher: – So how did you proceed so as to avoid getting lost?

Miguel: – We ticked the numbers that he had already added.

Teresa: – We added E and B, which makes eleven, and we drew a dotted line, an X to know (she writes on the board as she speaks).

Miguel: – Right.

Teresa: – We then added G, F and C (she writes on the board as she speaks).

Teacher: – Because those are worth what?

Teresa: – Four.

Teacher: – Each.

Miguel: – And we started with the large ones.

Teresa: – We added G and F, that makes eight, plus… (she follows the letters with the piece of chalk) twelve (she writes "twelve").

Teacher: – And now how much do you have?

Teresa: – We already have twenty-three. Then we went on to add H and A. They are three each, it makes six. Twenty-three plus six makes twenty-nine (she writes the partial results). And then we drew another dotted line and added another one …

Teacher: – Which was the line that was missing.

Teresa: – And we got thirty in the end…

Miguel: – … of the news stories.

[November 2008 _ class _ 3rd year]

The revelation of these patterns results from the teachers accepting the existence of singular mathematical strategies on the part of the students, instead of the normalization of mathematical knowledge grounded on the teacher's knowledge. The teachers' recognition that it is possible to conduct a positive exploitation of the error and that students have a personal knowledge of their own resulted in a sharing, among all, of the singular knowledge of every student, based on patterns of discussion and extraction.

MATHEMATICAL COMMUNICATION AS VALORIZATION OF STUDENTS' KNOWLEDGE

Collaborative work took on a significant role in the development of teachers' self-knowledge of their practices of mathematical communication. These were broadened by viewing classroom episodes and by the critical questioning of the researcher. This collaboration resulted in a change in teaching practices based on the valorization of the interactions between students and these and the teacher, in the recognition of the error as a learning resource and in the recognition of the singularity of the students' ideas, strategies and mathematical knowledge.

The progressive valorization of the individual student's mathematical knowledge induced a change in the classroom mathematical communication practices, encouraging the teachers to broaden their mathematical knowledge, as a way to contribute to the understanding of students' mathematical knowledge.

The path from the initial mathematical communication practices, grounded on teachers' knowledge, and the later mathematical communication practices was largely based on the valorization by teachers of the interactions among students and between students and teacher as well as of the singularity of students' knowledge. This led to the strengthening of the students' capacity to communicate their mathematical ideas and to interpret and understand the ideas of the others. The deepening of social interactions in the classroom promoted the existence of reflexive and instructive modes of communication, generating a sharing of responsibilities between the teacher and the students in the recognition of the mathematical knowledge constructed in the classroom. The teacher's acceptance of the students' original mathematical ideas and strategies, though confusing, incomplete or incorrect as a valid learning resource triggered processes of knowledge comparison fed by discussion and extraction patterns centered on students' knowledge.

In this way, mathematical communication practices were far beyond the reproduction of the teachers' mathematical knowledge and assumed an intentional action in the valorization of singular mathematical knowledge constructed by each student. The existence of reflexive and instructive modes of communication and of patterns of extraction and discussion evidenced students' mathematical knowledge, generating a significant level of autonomy in the construction and validation of mathematical solutions. The recognition of the existence of diverse mathematical ideas and strategies on the part of students and teachers promoted mutual respect for the opinions, mathematical ideas and strategies of students.

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NEW TEACHERS´ IDEAS ON PROFESSIONAL DEVELOPMENT

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It is a challenge for teacher educators to constantly evaluate and reflect on the studies they are organizing for future teachers. The aim of this paper is to find out whether newly graduated teachers find that an emphasis on learning communities in mathematics teacher education has been useful during their studies and now in their work as teachers. We also want to know what further support they would wish for in their professional development. Five teachers, who are working in compulsory schools and graduated in 2010–11 with 80 ECTS specialization in mathematics, were interviewed. Our conclusion is that the collaborative practices in their studies were important for them both during the studies and now when they are all team teaching and collaborating with colleagues. All five teachers work in schools where no other teacher has the same degree of specialization in mathematics and are given leading roles in their schools. They feel they need support from others outside the school and seek varied possibilities to develop professionally outside their school. This is a challenge for those responsible for teacher education.

Keywords: learning communities, professional development, teacher education

INTRODUCTION

Mathematics teacher education in Iceland has been changing and developing during the last decades. From 1971 teacher education for teachers in compulsory school (grades 1–10) has been a three year B. Ed. Degree (180 ECTS). The structure has varied but student teachers have always specialized in one or two subjects. From 2007–2011 the B. Ed. degree consisted of 80 ECTS in pedagogy and didactics, 80 ECTS in specialization and 20 ECTS in studies of their own choice. From July 2011 teachers have to have a master's degree in order to qualify as teachers.

The authors of this paper have taught different mathematics education courses for more than 20 years and have taken part in developing the studies in cooperation with colleagues. We have chosen to introduce our students to various ways of collaborating and building learning communities during their studies. They have been introduced to lesson study (Lewis, 2002), they prepare their teaching practice in groups, they discuss course readings in groups and write reflective diaries together, prepare for oral exams in groups and work on various other group assignments (Gunnarsdóttir & Pálsdóttir, 2010; Gunnarsdóttir & Pálsdóttir, 2011). This is based on the belief that the creation of learning communities in teacher education gives the students good learning opportunities for developing a professional language and a collaboration competency. Our aim is also to introduce professional learning strategies to our students they can use when they enter the teaching profession.

In the spring 2010 and 2011 approximately 24 students graduated with a B. Ed degree and 80 ECTS specialization in mathematics and mathematics education. We found it interesting to find out whether students from this group found the emphasis on learning communities in their mathematics teacher education useful during their studies and now when they have entered the teaching profession. We also wanted to know how they were thinking about their professional development and what further support they felt they would like to get.

PROFESSIONAL DEVELOPMENT AND LEARNING COMMUNITIES

According to the OECD –Teaching and Learning International Survey (TALIS) 22,4 % of teachers in lower secondary school in ICELAND took no part in professional development activities during the last 18 months and 48,5% took part in activities that lasted for less the 11 days. When asked what prevented them from more participation 47,2% of those who give a special reason said that there was nothing on offer that suited them (Ólafsson & Björnsson, 2009).

A study on the formal professional development possibilities given to lower secondary school teachers in Iceland during the period of 2005-2010 shows that the opportunities given are limited and do not meet features that characterize effective professional development (Desimone, 2009). That applies specially to duration and coherence which are considered very important features along with, focus on content, active learning and collective participation. The organization and funding of professional development in Iceland does not seem to allow for continuation and progression (Gunnarsdóttir, submitted).

Teachers have many opportunities for professional development both within formal professional development settings such as courses and in-service days and informal settings such as common planning and discussion of lessons, self-reflection and reading of professional journals (Borko, 2004; Desimone, 2009; Wei, Darling-Hammond, Andree, Richardson, & Orphanos, 2009). Several researchers have tried to point out some principles for effective professional development by synthesizing results from various research and development projects (Borko, 2004; Desimone, 2009; Loucks-Horsley, Stiles, Mundry, Hewson, & Love, 2010; Wei, et al., 2009)

Wei et al. (2009) define effective professional development as development that leads to improved knowledge and instruction by the teachers and improved student learning. They draw on research from both the US and elsewhere that links student learning to teacher development. Darling- Hammond, Wei and their colleagues put forward four main principles for designing professional development:

- Professional development should be intensive, on-going, and connected to practice.
- Professional development should focus on student learning and address the teaching of specific curriculum content.
- Professional development should align with school improvement priorities and goals.
- Professional development should build strong working relationships among teachers. (Darling-Hammond, Wei, Andree, Richardson, & Orphanos, 2009)

They also indicate that other factors like school-based coaching and mentoring and induction programs for new teachers are important and likely to increase the effectiveness of teachers. They also point out that intensive professional development rooted in practice is most likely to change teaching practices and lead to increased student learning.

Loucks-Horsley and her colleagues have for more than a decade worked on professional development and in the third edition of their book *Designing Professional Development for Teachers of Science and Mathematics* published in 2010 they review new developments in the knowledge base for professional development and use it to enrich the basic principles of effective professional development they presented in their earlier work.

According to Loucks-Horsley et al. (2010) effective professional development is designed to address students' learning goals and needs. It is driven by images of effective classroom learning and teaching and gives teachers opportunities to develop both their content and pedagogical content knowledge and inquire into their practice. It is research based and implies active learning for teachers in learning communities with their colleagues and other experts. It is a lifelong process, linked to other parts of the school system and should be continuously under evaluation.

Professional learning communities seem to play an important role in supporting teachers in continuously improving their teaching and sustaining their professional learning (Fernandez, 2002; Loucks-Horsley, et al., 2010). Lesson study is referred to as an example of a professional development strategy that has many of the aspects that characterize effective professional development. Lesson study enhances teachers´ knowledge and quality teaching, it develops leadership capacity and the building of professional learning communities (Loucks-Horsley, et al., 2010).

According to Desimone (2009) there is a consensus among researchers on the main critical features of professional development that can be linked with changes in teachers practice and knowledge and to some degree in student learning. She points out five main features. These are focus on content, active learning, coherence, duration and collective participation. According to Desimone there is strong evidence that focus on content and how students learn that content in professional development can be linked to teacher development and to some extent to student learning. Active learning where teachers engage in various activities like observations, reviewing of student work and discussions is also an important feature. Teachers also need to feel a coherence between their beliefs, knowledge and their experiences in professional development and reforms and policies at all levels. Collective participation and duration are also very important features. Teachers need time to work with, reflect on and try out new ideas and they need to do this in a learning community with others dealing with the same issues. The critical features Desimone points out seem to capture the core in principles for effective professional development both Darling-Hammond et al. (2009) and Loucks-Horsley et al. (2010) present.

They also have much in common with what Borko, Koellner, Jacobs & Seago (2011) claim to be the shared view of many teacher educators on professional development. According to this view professional development for teachers should be a collective endeavour, it should be about the work of teaching and the learning opportunities should be situated within the teachers practice.

METHOD AND DATA

In this study we interview five teachers from the group that graduated as mathematics teachers in 2010 and 2011 from the University of Iceland, School of Education. They were chosen from the group we knew were teaching in compulsory schools which is less than half of the whole group. They were chosen because we knew how to reach them and they were willing to participate with short notice.

The interviews were semi-structured (Bryman, 2004). Both researchers conducted the first three interviews but in the last two only one researcher was present. The themes discussed in the interviews were experiences especially regarding lesson study and learning communities in the interviewees´ pre service education and their ideas on how to develop professionally now when they are professionals. All interviews were recorded, transcribed and analysed. We read and coded the interviews and from the codes some themes emerged (Flick, 2006). The themes, linking professional development to practice, opportunities for collaboration with other teachers, and better opportunities for professional development are evident in the data. In writing up the cases we summarize what the teachers said in relation to these themes. We looked at each teacher as a case and tried to capture more in-depth information about their conceptions of learning communities and their actual situation now.

The case of Eva

Eva finished her teacher education in 2010 and has since been teaching in a private primary school. She remembers taking part in planning teaching in cooperation with fellow students which now she considers a useful experience and a good preparation for the team teaching in her school. She thinks that in teacher education it is necessary for students both to learn to work alone and in collaboration with others. She learnt a lot from her teaching practice when a practice teacher gave her critical comments on her teaching and discussed it with her.

In her school Eva has the role of a math leader as she is the only teacher who has mathematics as a specialization. She has been giving courses in her own school and other schools working according to the same ideology..

> I have given lectures within our schools about mathematics teaching and the use of teaching materials. We do not use textbooks for the children only as support to us and teachers ask what do we do when you don’t have a book ... what do you if a child cannot do subtraction?

She finds it important that teachers can have discussions and get support in their school. They should have easy access to information about new ideas and materials on mathematics. At the same time they should also have opportunity to attend courses. She has participated in some short courses which she found very useful for her professional development.

> I don´t think it matters if I take courses inside my school or outside, but in-service days where you learn something new, where something new is happening and just give a course, for those who wish to attend.

Eva expresses the view that teachers have to extend their knowledge and be prepared to search for new knowledge otherwise they will not be able to develop their teaching.

The case of Freyja

Freyja finished her studies in 2011. From her teacher education she remembers the lesson study process as a time of cooperation and discussions. She found it rewarding to share ideas and knowledge when preparing and planning a lesson together with others and having to come to an agreement at the end of that process. She also found it most rewarding to observe a fellow teacher student teach the lesson

> …and how because we are two different persons doing the same thing, it turns out different even though we have the same plan, then there are always some aspects that change, both the class and the teacher. I also find it just interesting to come and sit down and see how another class reacts to the presentation and how the teacher is...

Collaborative planning and teaching during teaching practice Freyja considers a good preparation for the job as a teacher. She feels that she has learnt that the teacher's personality, the group of students and the situation are all important factors in teaching and that it is impossible to plan teaching exactly the same way everywhere. She also finds teachers´ collaboration very important and vital for students´ wellbeing.

> We need to decide what to do, also cooperation with other teachers in the school is important. For example in the upper grades where we are working together, because I am only teaching two subjects and then there are other teachers teaching and we are teaching the same students.

Freyja has been trying out many of the ideas she got during her studies but she still has many questions and is looking for new experiences.

> ...and I left the studies here with a lot of questions. Even though I felt I had learnt a lot there were many questions I would like to get answers to, so maybe it is just the interest of the individual, whether there is an interest to continue educating yourself, keep nurturing what you have been doing or I don't know.

Freyja is seeking knowledge and advice when she meets new challenges both from colleagues and her former teacher educators. She finds it difficult to meet the different needs of students and wants to get new ideas to cope more effectively and to change emphasis in her teaching towards more hands-on projects. She thinks it is important to have professional development activities like short courses/workshops within and outside the school and be able to have access to the instructors later for questions and discussions.

Freyja finds it important that mathematics teachers have a variety of opportunities for support and professional development and to her the possibilities lie in teamwork at her school and in courses for mathematics teachers.

The case of Helga

Helga finished her teacher education in 2010 and has been teaching both in grade 3 and grades 8-10. She remembers clearly the lesson study process:

> ...I remember that we prepared the teaching all together, decided on the goals and planned a lesson from A-Z with the reactions to, that if this happens, try to find the ifs too. Look, then we went out and taught it and then the others observed and one taught and then we summarized and could go back home and work on that and try to improve the lesson, fix the goals, or as we did it then I remember that we divided the tasks, we had some group work, then there was one that observed the teacher while the other observed the students. So yes this is what I remember most from that.

Helga says that through the lesson study process she learnt to think more about the structure of a lesson and that the discussions were deeper and more open because the group discussing was jointly responsible for the planning of the lesson.

> I found that we thought much more about the structure of the lesson than when we were planning other lessons and then this process afterwards and we taught twice, ... I found that made our reflections deeper, because we reflected twice.... you get more things to discuss because everyone was more aware of the their role in the lesson... because most of the time we were three in the lesson even though we had taken on different roles and so, then we talked about it afterwards, it was still different, you could criticize the lesson without criticizing the one who taught, because we were working on it together.

When asked about the courses in mathematics education she attended during her studies she mentions the experience from lesson study and the work on reflective diaries on readings in groups as valuable tasks.

Helga has taken part in a professional development activity concerning literacy instruction. It was a process of two years where all the teachers at the school formed a learning community and met regularly both within the school, with teachers from other schools and the course-leaders. She is interested in trying out a similar program in mathematics or to participate in a lesson study with teachers from other schools. She is the only mathematics teacher in grades 8-10 and would find it challenging to plan with other mathematics teachers even though she works in a team in her school with teachers of other subjects and feels she can discuss her ideas about mathematics teaching with them. She finds it important for teachers in small schools to have the opportunity to meet other teachers and finds meetings with mathematics teachers in others schools in her area important. Such meetings are organized by the school authorities a couple of times each school year.

> ...it gives you another point of view, and ideas and support and of course the people you are with in school, you have talked to them and they have the same kind of thought and then you need someone else to get out of that...

Helga has just started a course in mathematics teaching for teachers in her school community that will run the whole school-year.

> But I also find it very good to have the possibility of, like this course in mathematics where you start with a course and then you can continue just to get some support so you do not forget the ideas when you are back out there.

Helga finds it challenging to try something new and feels she needs support to teach more in the way she would like to teach. She remembers an international math conference she attended during her studies and found that very interesting. She is also considering further studies.

The case of Lára

Lára graduated in 2010 and has been teaching in grades 7-10 in a school in rural Iceland. She remembers doing a lot of group-work in her studies. She found it useful to share ideas and have discussions as a teacher student and experiences it also as a teacher.

> I found it useful to hear from others what ideas they had and I hope they got some ideas from me

She feels that in her school teachers share ideas, tell each other about good experiences, collaborate and help each other out.

> I like it a lot and I feel that here in my school people work together and it is very easy to ask someone and everyone gives you advice and tells you what has worked well before.

Lára means that her students are the driving force in her professional development.

> Just how different students are and that things somehow are never the same, no lesson, no group or just students, they change from year to year and even from one month to another

She feels that because of how different students can be and how many unexpected things can happen in teaching she is always challenged to develop her teaching. She believes that with more experience she will become more competent as a teacher

She is the only teacher in her school that has specialized in mathematics. That means that she has a leading role and chooses teaching materials for the school. She feels that it is a big responsibility and finds it hard to have to argue for her decisions for teachers, parents and school authorities. Now she would like to have support from outside, from teachers from other schools and some specialists through meetings and courses. In her studies Lára wishes that there had been more emphasis on mathematics teaching for young children and teaching materials for all grades.

Lára claims that she needs more opportunities to meet and discuss with other mathematics teachers. She has good support in her school but not in developing her mathematics teaching and she needs support to work on and develop her own views and opinions.

The case of Margrét

Margrét finished her studies in 2010 and has been teaching in two different schools. She mentions books and articles that she read during her studies and how she can use ideas from them to guide her in her teaching. She is now using some of these resources for building up her competence in using questions in teaching.

Margrét thinks that lesson study rests on interesting ideas that could be useful for her and she finds it helpful to share the classroom with another teacher and to be able to discuss at the end of the day the teaching experience.

> I think that it can be good to use lesson study. … I think we get some of the good things from it in team teaching, especially if you are honest and critical in the discussion when evaluating the teaching.

What Margrét misses from her teacher education is more knowledge about students with special needs and feels that she needs more tools and concrete learning materials to meet their needs. She would like to have an opportunity to join a course or get support to increase her knowledge on that matter.

In Margrét´s school the teachers are organized in teams. She feels that it gives her flexibility and support in her professional development. She also mentions that the workload can be spread because the teachers trust each other. Margrét tells that she is working in a new school where the teachers are developing new practices and are curious and inspired to develop their teaching.

> .. and in this school, I don´t know if it is because it is a new school, very different from the old school where I taught the first year, …everyone is so ready and willing to develop professionally.

Margrét feels that she is in an environment where many things are happening that will help her in her professional development and that she still has many issues and ideas from her teacher education to work on and develop further.

DISCUSSION

In this research study five different teachers tell about what they found useful from their experiences from taking part in learning communities during their initial teacher education and their ideas about professional development after their first years of teaching. Here we draw attention to what they all mention and discuss those issues.

In the interviews all the teachers show interest in their own professional development and have some ideas about their needs and possibilities. What they mention as important factors for them is in line with what Darling-Hammond and her colleagues (2009) put forward as four main principles for professional development. They see their professional development as closely linked to their practice and are striving to meet the needs of their students. They want to form some working relationships with teachers both in and outside their schools. Because of their specialization they all have leading roles in mathematics in their schools and feel that they need outside support to develop and meet the challenges of being math leaders. As young teachers they feel that collaboration and team teaching gives them the support they need to meet the daily challenges of teaching. Experiences from the collaborative practices in teacher education prepared them for this part of the work but the in-depth discussions about mathematical content and practices are missing. Opportunities for participating in learning communities with teachers from other schools as for example in lesson study could meet these needs as Loucks-Horsley et al., (2010) point out.

The fact that these new teachers are all given leading roles in their schools shows that they meet some expectations from the school environment and some support from their school principals. This also reflects that schools need professional leaders with good specialized knowledge and what these new teachers bring with them to the schools is valued and in line with the schools' goals for development and improvement. That should be a good basis for their professional development and it is an important principle for effective professional development as Darling-Hammond et al. (2009) draw attention to.

All the teachers are aware of the need for further professional development. They mention courses, in-service days, conferences and meetings with other mathematics teachers as a support for their development. Professional development activities within the schools do not meet their needs to develop as mathematics teachers and mathematics leaders in their school. This applies both to teachers in schools in urban and rural areas. From the interviews it is evident that the teachers would like to have had opportunities to attend courses, workshops and other professional development activities. Because of the size of the population it is difficult to fulfil their wishes and it is important to seek new ways to create a forum for teachers to meet in reality or in virtual space. This is a challenge for us as teacher educators.

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COMPETENCE IN REFLECTING – AN ANSWER TO UNCERTAINTY IN AREAS OF TENSION IN TEACHING AND LEARNING PROCESSES AND TEACHERS PROFESSION

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Teaching and learning processes and the teacher's profession are afflicted with tensions of conflicting goals. After giving an outline on these areas of tension, the competence in reflecting will be suggested as an answer to the question on how to deal with tensions. Reflection support teachers and learners in bearing the often unsolvable tensions. The article then suggests facilitating actions in university education by integrating reflection tasks in teacher's education. Examples of students' reflections and first analysis results in changes of student's beliefs are given.

Keywords: reflection, tension, teaching, learning, process

THE SITUATION OF TEACHERS IN LEARNING PROCESSES

In Mathematics teaching and learning processes (prospective) teachers often experience uncertainty in dealing with conflicting situations. They feel lost in having to deal with contradicting interests. Prospective teacher students tend to search for „simple answers" - not for vagueness in decisions on how to set up and manage the learning process, to get the „one solution" on deciding how to teach, trying to step on one side of the contradicting poles of areas of tensions. This tendency is quite understandable as teaching and learning processes challenge teachers on a wide range of diversity: heterogeneity of primary school students shows in social and cultural background, literacy, language, interests, gender, skills, effort and achievement ability, and many more. In addition to those already demanding aspects (prospective) teachers are challenged to fulfill high expectations of school administration in their teaching and pedagogical actions. Modern teaching methods and a strong orientation towards the learning-outcome and competences leave (prospective) teachers often helpless.

The areas of tension are already coining the university studies and some of them are just lying in the nature of mathematics. Mathematics is presented strongly product-orientated as an elaborated theory which leaves no space for individual ways of thinking. Moreover it is actually just the power of mathematics being a product with clearly outlined rules and structures. But learning processes start at a totally different point. Students activate their pre-knowledge and try to connect to the mathematical concepts, what hardly turns out to happen in a straight way and nor getting to the intended rules. Teachers have to and should support this process, but often struggle in that area of tension. Both sides are important, having an individual conceptualization and the conventions, and it seems difficult to set up a balanced learning environment.

The answers from first year students at University of Siegen who were asked in a survey 2010 about their opinions and experiences with mathematics and their expectations of their studies reveal an image of mathematics and learning processes which can be characterized by putative conflicting interests and traits of mathematics, which tend to become even more diffuse during university studies. From these opinions of students the areas of tension could be drawn: students, enrolled for mathematics educational studies, come with a strongly closed view on mathematics and the teaching and learning processes due to self-experienced classroom action. Their view is affected by conventions and rules, strongly regulated learning activities, which leads to internal tensions if the (prospective) teachers simultaneously reach out for more real life applications and comprehension orientated activities, clearness and more openness in teaching mathematics. The survey was used to extract some main areas of tension from students in mathematics training. The extracted areas of tension correspond well to my observations in university lectures and in mentoring teacher students in their first practical experiences at school.

These areas of internal tension, which afflict educational studies students and teachers, could be grouped in five classes (see Helmerich, 2012), given here by stating the extreme poles of tensions:

- form and content
- openness and closeness
- rigour and intuition
- product and process
- singularity and regularity

These concepts are the result of a survey among first year students in year 2010. The students were asked to describe their view on mathematics and teaching and learning processes with their own words. Basis for the clustering of the students' answers was the work of Krauthausen and Scherer (2007) on the foundations of teaching mathematics as "at first contradicting extremes" (translated by MH). The extreme poles mentioned by Krauthausen and Scherer (2007), like application orientation versus structure orientation, student orientated versus subject orientated teaching, individual ways of solving problems versus conventions, open tasks versus closed-ended problems link quite well to the areas of tension identified of the Siegen research group and mirror the results of the prospective teachers' survey. I focused rather on the mathematics-related aspects of the "contradicting extremes" of Krauthausen and Scherer instead of the more general pedagogical ones.

Although we know that teachers encounter conflicting situations, and their professional knowledge and necessary competences are extensively discussed and investigated, studies like TEDS-M still describe areas of tension as a liability for teachers, but not the great potential of activating these tensions in the learning processes as dialectic concepts. Helsper names it "constitutive professional antinomies of teaching" (Helsper, 1996, 2004) (translated by MH), "which entangle the uncertainty of representative interpretation and the simultaneous aspects distance and proximity.” (Baumert & Kunter, 2006, p. 471) (translated by MH).

> “Taken the antinomious structure of teaching seriously, teachers necessarily will have to make decisions about their actions in teaching, which cannot be in accordance to both conflicting claims of validity at the same time. This situation will only turn out to be bearable and productive, if there is a working agreement of their free will” (Baumert & Kunter, 2006, p. 471) (translated by MH).

Among many areas of tension the five mentioned above are vital in mathematics education of our students – relating to their image of mathematics and their issues in learning mathematics – and seem to be characteristic for the nature of mathematics. Many other areas of tension could be brought into consideration, but most of them are rather on a pedagogical level and apply entirely only to classroom interaction but not to specific learning processes in mathematics.

REFLECTION IN AREAS OF TENSION

The experience in teacher education is that many university concepts do not spring into action in school teaching. One reason might be the strong beliefs built up over years during school which could hardly be affected by university teaching and breaks through under pressure in school teaching situations. So we set up a couple of actions in university teaching to form at least some awareness of the students beliefs, make them reflect on their own beliefs and attitudes towards mathematics, teaching mathematics and the learning process of mathematics.

Teaching mathematics does only work in areas of tension, teachers have to make decisions, have to hold on these conflicts lay out in the nature of mathematics, in the clash of own standards and environmental restrictions. In order to stand these tensions, and to achieve “mathemacy” (Skovsmose, 1998, p. 199), which is “more than an ability to calculate” (Skovsmose, 1998, p. 199), reflection competence is required.

Before showing examples for these reflection tasks and what students made of it, the term of reflection, as it is used here, is explained. The American Heritage Dictionary [TAHD] gives the following definition on reflection:

> “1. The act of reflecting or the state of being reflected. (…)
> 3. a. Mental concentration; careful consideration.
> b. A thought or an opinion resulting from such consideration.” (TAHD, 2009)

This definition points to important aspects of reflection: consideration of your own actions and thoughts and come to an opinion on that. To reflect your own standards on learning processes, the reasons for your decisions and the origin for your belief systems (see Törner & Pehkonnen, 1995) is necessary to get over the uncertainty and learn to deal with it, accept it as a necessary state in which learning and teaching will happen.

> "Reflection means reconsidering what is taking place. Reflection is a mental, a conscious or a theoretical activity. In my terms: It is a critical activity and a process of grasping basic processes of social development. If reflexive modernization has mathematics as constituent, then reflections with respect to mathematics become of particular importance." (Skovsmose, 1998, p. 200)

Reflection on opinions of prospective teachers towards mathematics and learning processes can oppose to uncertainty in areas of tension. The own positioning between the poles of areas of tension and knowing the reasons and justifications for the positioning must be learned and trained already at university.

The problem arising at this point is, that reflection in university's education could only be a "reflection on action" (Schön, 1983), or even less it might be only a reflection on *possible* actions, since the reflection project presented below stimulate reflection on very general issues concerning teaching and learning processes. The aim of encouraging teacher students to reflect during the university studies is to make them aware of their attitudes *on* teaching, learning and learning environments, and how this could affect their performance *in* classroom action. The reflected knowledge about one's beliefs and preferences, about potentials and threads in areas of tension, strengthen the teachers in their later reflection processes in action. If teachers accept the tensions in teaching situations as a fact and furthermore that teaching actions are not bind to one of the far ends of the tension's spectrum, teachers will feel less pressure and less uncertainty in such situations. If the prospective teachers learn by reflection during their education about how to take benefit from tensions as dialectical concepts, they will be able to get a broader view on learning processes. Solutions and questions of pupils could be judged and answered more open-minded, considering the areas of tension offer different views on learning processes. Those areas of tension could be even made explicit and integrated as opportunities for reflection in classroom interaction. This leads to more communication about mathematical ideas, pupil's conception of mathematics and a reflected knowledge and image of mathematics.

A REFLECTION PROJECT IN TEACHERS EDUCATION

This section illustrates how reflections and reflection competence could be encouraged and facilitated in university education for prospective teachers. Examples of reflection essays from students are used in a case study on reflection competence. In this paper a first attempt of a theory orientated analysis of these essays is given by using Skovsmose's dimensions of reflection (see Skovsmose, 1996, 1998) as a framework for analyzing the statements of the students. Further research on the development of reflection competence in teachers training by writing reflection essays will follow. So far the aim of this project was to get students to a reflected position of their image of mathematics and how to learn and teach mathematics at school.

In the beginning of the project accompanying a lecture, occasions of reflecting were integrated, to start off with just describing opinions on material presented in my lessons, and to continue with writing tasks throughout the whole semester, provoking processes of reflection and enforce my students to take a firm stand on their beliefs, decisions and opinions in teaching and learning processes. The university course on “Learning Mathematics as Construction Processes” for third year students seemed to be the right place for starting with the reflection project. About 200 of the 350 enrolled students took part in the take-home reflection assignments which came in addition to the tutorial work and homework, included some kind of reflection tasks already as for example to discuss advantages and difficulties of different teaching settings or mathematics problems.

With the reflection assignments, shortly called „E-Reflex“, the teacher students are encouraged to reflect their own learning process and to comment on the presented contents, as reflecting is crucial for reasonable and sustainable learning und understanding. In this way of reflecting the teacher students got the possibility to express their observations and thoughts in order to become aware of their own learning process and issues. By working on the reflection assignments the mathematical and didactical content knowledge is increased and competence in reflecting is trained and elaborated. Moreover those E-Reflex texts gave me a good feedback on the learning progress of my students and enabled me to respond in further lectures. The student texts had to be uploaded to a learning platform “Moodle”, were revised by myself and handed back with attached short comments. With these comments the students got a feedback on depth and width of their thoughts.

As a guideline for setting up the reflection tasks and an analysis and evaluation tool I adapted the theory of Skovsmose (1996, 1998), who differentiates several dimensions of reflection in mathematics:

- mathematics-oriented reflection
- model-oriented reflection
- context-oriented reflection and
- lifeworld-oriented reflection.

Usually these dimensions are used by Skovsmose to describe reflection processes of mathematical actions. For the E-Reflex project the aspects were applied for reflection on learning and teaching processes of mathematics. Mathematics-oriented reflection was transferred to didactics-oriented reflection, covering questions on the coherent usage of specific didactical theories, stringent lines of argumentation and reasonable usage of didactical terms and concepts for describing phenomena in teaching or learning processes. The model-oriented reflection applies to considerations about the ideas of teaching and learning processes and their adequacy. Furthermore it questions whether a didactical theory or concept covers a certain idea or process, or whether an alternative conceptualization could be brought up. Context-oriented reflection gives insight to thoughts on learning environments and surrounding requirements, whereas the lifeworld-orented reflections should connect didactical and pedagogical thinking and action with individual beliefs and discussion of the relevance of didactical theories for distinct actions. The reflection tasks try to cover these dimensions and link them to specific mathematical problems or classroom action considerations.

Besides giving a framework for the formulation of the reflection tasks, these dimensions of reflection are later deployed for the analysis of reflections of the students in mathematical educational studies. Since the reflection tasks are not focused on one possible answer, one need to have an analysis tool for differentiated evaluation and feedback processes.

The first E-Reflex task covered the didactical concept of learning by discovery, which is based on a constructivist position. The students should reflect on the question whether mathematical discoveries of primary or secondary school students are possible at all, and if so, to give a practical example, how discoveries could be made in certain teaching settings or mathematical problems. In addition, it had to be discussed, if there are specific mathematical concepts and contents which could not be learned by discovery. Last but not least the teacher students should state their preferred mathematical content for interrogative-developing teaching method and teacher centered classroom action, and the reason for their decision.

In this first assignment the students had to reflect on possibilities for discoveries in mathematics, taking a closer look on mathematics in primary school in a mathematical-oriented way, but also with respect to pedagogical concepts and beliefs on how teaching might be. Student Kerstin wrote:

> "(…) In my opinion, almost everything can be learned by kid's own discoveries, assumed that one provides specific learning materials. Even concepts could be established by working on tutorial sheets. But I think it is good for kids, if you bring some variety to teaching and don't withdraw yourself all together. 'Let the kids just do their own thing' is not a healthy attitude in discovery learning. (…)" (translated student's text)

In order to reflect on the given question Kerstin has to make her own concept of teaching explicit, moreover she combines her experience and belief of good teaching with lifeworld attitudes and take up a position 'pro discovery learning'. Most of the student's texts argued for discovery learning as an important principle in teaching, but some students expressed their tendency to go for teacher centered, guided learning methods if it comes to secondary school and putative more abstract and conventional mathematics.

The second E-Reflex assignment challenged the students to justify mathematics in school. With Heymann's claim that mathematical instruction should provide a general education (Heymann, 1996) and Winter's call for important basic experiences with mathematics (grasp mathematics as a certain way of looking at our world, experience mathematics as a structured, well-formed theory and as a certain way of thinking providing problem solving abilities (see (Winter, 1995)) the students got two positions during lecture to potentially set up their arguments. But the reflection task aimed on their own opinion and their individual justification for teaching mathematics in school, too. It was motivated to make their point of view explicit by giving practical examples or choosing only distinct contents out of the wide range of mathematical knowledge.

Most students argued with the importance of mathematics for every-day life, applying a dimension of lifeworld-oriented reflection. A typical sequence is found in Eva's text:

> "(…) Personally I think, that mathematics is sort of basic competence for social life in our culture (in the jungle of course applies something else). But in our industrialized world mathematics is essential. Regardless whether it is shopping, work or leisure we will be confronted with mathematical topics. Some kids say things like: 'I wanna be a hair dresser, gardener or baker, what do I have to know maths for?' But you can easily find concrete examples for mathematical topics even in these jobs. (…)" (translated student's text)

The third E-Reflex problem was a rather mathematical-oriented task. The students were confronted with a supposedly 'illogical spot' of mathematics. The students then had to understand, what is going on in this problem and complete a dialogue of two school kids talking about this problem and trying to figure out, how it works. By using the dialogue setting, the students had to think about the answers to the given problem, but also about difficulties in understanding the mathematical concept underlying the problem, so they had to think about both parts in the conversation. This method and its positive learning effects are discussed more deeply in Wille (2011). After writing the dialogue it was mandatory to reflect on this kind of task: How did it feel to work on this assignment? What could be learned by doing this? Is this method a possible starting point for learning processes at school?

Working on this task the students experienced the constraints and opportunities of putting yourself in another's position. Many students found themselves struggling with the problem and so having difficulties to take over the explaining part in the dialogue. But they all acknowledged the high potential for learning and deeper understanding of mathematics throughout this reflecting way. Since the dialogues turned out rather lengthy and would have to be shown in reasonable large sections to grasp the line of argumentation, examples could not be displayed in this article.

In the last E-Reflex task, the students were asked to describe their idea of heterogeneity in classroom action and how they would deal with this diverse situation in teaching and why in this way. To carry it further on, a preference for teaching in a homogeneous (assuming this would be possible to form) or a heterogeneous class should be given. Insisting to take up a stance on the dialectics of learning processes is important to really reach the students in their core beliefs and get a reflection process started on this. A final remark should be made to outline possible actions of differentiation in learning processes and to discuss, if – assuming again that homogeneous learning groups could be put together – differentiation is actually necessary.

In the last task I have noticed a huge step towards a reflection competence in the student's texts. Almost all reflections turned out longer than the obligatory two pages and showed the effort of lay out the reflection in different dimensions. The student got more confident in positioning themselves in areas of conflict demonstrated in sophisticated and deliberative lines of argumentation. Exemplarily the text of Diana shows the progress in belief changes:

> "(…) If I had been asked several weeks ago, whether I prefer a homogeneous over a heterogeneous learning group for teaching, I would have probably said yes. Because one cannot imagine anything better for a teacher than teaching kids who are all on the same knowledge level and learning the same content at the same pace and time. (…) The teacher would be able to plan teaching precisely and would cover all anticipated content. (…) I want to be a teacher actually just because of the variety and individuality of the students. Of course I want to teach a heterogeneous learning group. (…) The individuality of learners has to be exploited for a good learning environment." (translated student's text)

CONCLUSIONS AND FURTHER RESEARCH WORK

The reflection tasks were used in the course to enable prospective teachers to overcome their uncertainty in areas of tension by reflecting on mathematics teaching and learning issues, their own point of view and last but not least their options in dealing with conflicting situations.

This first review of the student's reflections is encouraging for future reflecting projects in university's teacher education since a development from uncertainty in teaching actions towards a reflected way of taking position in the range of possible actions in areas of tension is noticeable.

The reflection impetus of the assignments could be improved by stressing out even more the importance to have a look at both sides of the tension spectrum. Not only the positioning on one pole of the areas of tension, but the experience of opportunities and threads of the far ends and the ability to find your place in many situations somewhere in the middle, not being able to make an easy decision for one side but instead having to bear with the tension, is a desirable goal in future reflection projects. This will help students to establish a reflected way of knowing in which situation the decision for one pole is appropiate and in which situation you have to stand the tension.

In further research it is profitable to take a closer look on the student's texts and carve out types of reflection characters to achieve a deeper insight of persistence or approaches to changes in belief systems and thereby a better understanding of the effectiveness of university teaching on students.

This reflection projekt is linked to the broader research topic of the Siegen group, searching for objectives and educational goals and a curriculum for mathematics teachers (see Lengnink, 2011; Helmerich, 2011). With those objectives the competences in mathematical repertoire (content knowledge in mathematics), attitude on and to mathematical content and the observable performance are combined to an education framework. In this framework reflection links repertoire, attitude and performance together by defining the relation between the triplet: education manifests in a reflected attitude on a mathematical repertoire which implies performances in teaching and learning processes.

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ONE POSSIBLE WAY OF TRAINING TEACHERS FOR INQUIRY BASED EDUCATION

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The paper describes the results of a research study aimed at preparation of pre-service primary school teachers for inquiry based mathematics education (IBME). The teaching experiment involved pre-service teachers' work using a variety of techniques (with strictly formulated questions and open problems) in various environments suitable for this approach. We concentrated on pre-service teacher content knowledge and ability to apply the gained knowledge in problem solving.

Keywords: inquiry based mathematics education, pre-service teacher training, subject matter knowledge, knowledge base for teaching

INTRODUCTION

We have been focusing on various issues connected to the process of professionalization of teachers for many years. Contemporary we tried to find the ways of improvement of teachers' knowledge base for teaching (Scherer, Steinbring, 2004) through introducing substantial learning environments (Tichá & Hošpesová, 2011). The issue is closely connected to the question of how to introduce these environments into reality of teaching at schools. This lead us to inquiry based mathematics education (IBME), to the role of the teacher as understood in thus conceived education, and to the requirements on pre-service teacher training. This contribution is concerned mainly with the quality of pre-service teachers' content knowledge, and their ability to apply the gained knowledge in solving process of problems leading to IMBE (requiring discovery, exploratory approach).

INQUIRY BASED MATHEMATICS EDUCATION

IBME in contemporary education raises great interest (for example Canavarro, 2011). It transmits procedures known from real scientific research to everyday school work with pupils. It puts emphasis on independent discovery (a self-discovery), on proper justifying, and on links with everyday reality (i.e. on a practical desirability).

> "Inquiry based mathematics education refers to an education which does not present mathematics to pupils and students as a ready-built structure to appropriate. Rather it offers them the opportunity to experience how mathematical knowledge is developed through personal and collective attempts at answering questions emerging in a diversity of fields, from observation of nature as well as the mathematics field itself, ..." (Artique, Baptist, Dillon, Harlen and Léna 2011, p. 10).

The concept of this approach is far from new. Let us recollect here for example the concept of *guided rediscovery,* anchored in the concept of genetic style of teaching governed mainly by psychological reasons and the concept of guided discovery. It built on characteristics of genetic style of teaching of Brunner (1966), Wittmann (1974), Freudenthal (1973).

> "H. Freudenthal expressed his opinion on guided discovery and genetic style of teaching on a number of occasions. He characterizes the genetic principle as "rediscovery", i.e. such teaching of mathematics, in which discoveries stand for what they really are, i.e. for discoveries. Discovery takes place even in a modest pupils' rediscovery by non-prescribed procedure... Freudenthal adds the attribute guided to the concept of "rediscovery". Although this attribute should be self-evident, it is not useless because of left-wing didacticians who refuse to understand and interpret rediscovery as a process in which pupils should discover everything on their own." (Vyšín, 1976, 584-585).

As we want to focus on creation of future teachers´ knowledge base for teaching in IBME, it is worthwhile to recollect at this point also Brousseau's concept of a didactical situation (explanation in endnote 1) and *a*-didactical situation (Brousseau, 1997). "In *a*-didactical situation the educators enable the student(s) to acquire new knowledge in the learning processes without any explicit intervention from them" (Brousseau in Novotná, Hošpesová, 2012, p. 282). This does not necessarily imply that the student (solver) must discover the new knowledge on his own, but IBME can become a specific type of *a*-didactical situation. This makes us ask what the educator's role in situations of discovery is. Brousseau distinguishes several phases in an *a*-didactical situation that can be used in IBME:

> "Situation of action – its result is an anticipated (implicit) model, strategy, initial tactic
>
> Situation of formulation – its result is a clear formulation of conditions under which the situation will function
>
> Situation of validation – its result is verification of functionality (or non-functionality) of the model" (Brousseau in Novotná, Hošpesová, 2012, p. 282)

The educator's intervention in these phases will be of different nature. Let us presume that a discovery is initiated by the educator using a problem that opens opportunities for collaborative orientation. We can expect the educator to intervene in the first phase only if the pupils/students are not active, if they do not look for solutions. In contrast the educator's role in the second and the third phases is crucial. This role is far from traditional: the educator does not explain, produce illustrative examples and exercises. He poses questions, asks for clearer explanations, for assessment of validity of argumentation (e.g. questions: Why do you think so? Could you explain it more clearly? How do you mean this? Are you sure it is so?). In mathematics, this is of specific meaning, as the nature of mathematical knowledge is also specific.

In our study the students were in dual role. They solved the problem (the role of the student) and at the same time, they are supposed to think as the teachers and analyze the didactic potential of the solved problem (the role of educator). The aim of this paper is to find out how this dual role influenced the process of problem solving.

TEACHING EXPERIMENT, ITS PARTICIPANTS, METHODOLOGY

Participants

The teaching experiment was carried out with two groups of participants. We used similar framework steps for both groups.

The first group consisted of pre-service primary school teachers in the second half of their studies attending the course of Didactics of Mathematics (in total all 63 registered students). The students completed in their previous study several courses of mathematics, which focused on the theoretical basis for teaching mathematics for primary school level. Partly we solved problems in learning environments suitable for implementation of IBME. Students analyzed their mathematical content, discuss the possibility of putting them into practice. We tried it show how the traditional subject matter could be enriched through IBME.

The other group consisted of students who had already finished their course of Didactics of Mathematics but wanted to amplify their knowledge by attending an optional subject with the aim of connecting the knowledge gained in the seminars of mathematics and didactics of mathematics in isolated topics and to gain a full, comprehensive view of primary mathematics education. They do not focus specifically on IBME.

Teaching experiment and research tool

Our approach will be illustrated on the problem "Who hits 50". The assignment of this problem was inspired by a paper of Scherer and Steinbring (2004). The problem was posed as follows (the original German wording of the problem is in the endnote 2):

> The following rules for calculations hold in the scheme in Fig. 1:
>
> - You can choose arbitrarily the start number (initial - given in an oval) and the number you are adding (addend - given in the circle).
> - You gradually get the numbers in the other four squares by adding the "addend".
> - You get the "target number" by adding all the numbers in the five squares in a row.
> - Which natural numbers do you have to choose as "initial" and "addend" to get the "target" number 50?

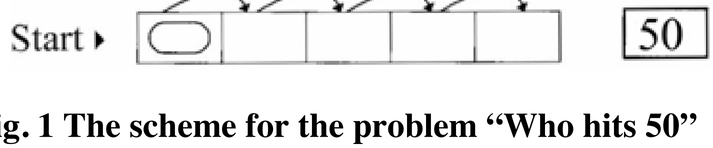


Fig. 1 The scheme for the problem "Who hits 50"

The students were introduced to the environment and assigned the problem. However, they were not told that the problem focused on searching for patterns, coherences and relations and their properties. They were expected to find that out on their own.

The procedure:

1. The participants solved an unknown problem/set of problems in order to get familiar with the environment.
2. a) They were asked questions and assigned problems in the form: “What will happen if the target number is 60, 90? What will happen if there are 6 boxes?” etc.

   b) The students were asked to think over the aim of the assigned problem. They were asked to pose other tasks and questions.
3. This group work was followed by a discussion in which various aspects of the problem were analyzed (mathematical content, number of solutions, possible obstacles for the pupils, different models and representations, possible solving methods, possible extensions of the problem).

In the background of the experiment was our conviction that our students should first get hands-on experience with discovery in these environments, i.e. try to solve problems similar to those they will assign to their prospective pupils (number walls, race to twenty, etc.), which should be followed by a discussion on the didactical aspects.

The students in both groups were first asked to solve the assigned problem, answer the question.

The students in **the first group** then worked in groups of four on modifications of the assigned problem. They were looking for answers to questions aiming at developing their subject matter knowledge and were solving the assigned tasks:

- What relations could you observe in the scheme?
- Propose other target numbers and find several solutions.
- Solve the problem with a different scheme: e.g. a different number of squares; a different target number; 6 squares and target number 60.
- Articulate conclusions on the relationships between the target number and the shape of the scheme.

In **the other group** the students worked in pairs. The work in seminar concentrated mainly on students´ ability to elaborate didactically the subject matter. The students were asked to think over the following questions (and look at the situation in a different perspective):

- What task was given? Why? (Which concept, solving method is being developed?)
- Why did we assign this problem? What is the benefit of this problem? What may become a source of difficulty, problems?

Later, in a whole-class discussion, they came to the conclusion that this was about regularities and dependencies, the students were asked to:

- Think of other questions and tasks leading to discovery of regularities.
- Assess whether this was a suitable and stimulating task.

In the end students in both groups were asked to pose other questions and problems in this environment.

Data collection and analysis

When analyzing the students' work we were inspired by the method of grounded theory. We classified the students' written productions and sorted them with respect to their characteristic features. We gradually identified and formulated emergent phenomena, and interpreted them.

FINDINGS AND DISCUSSION

We realized several phenomena. The students differed considerably in:

- ways how to find the solution,
- ability to describe what they saw and which representations they were using,
- which regularities and dependencies they uncovered,
- what type of problems they posed.

How did the students proceed when looking for the solution?

All students managed to find several solutions to the problem. The students were not satisfied with one solution of the problem; they tried to find all solutions. But the most frequently used strategy was the trial and error strategy. The initial numbers were most often chosen at random and some of the groups of students did not try to proceed in a systemic way. This was the case of looking for the "first" as well as the following solutions (that is pairs of starting and adding numbers). In consequence, some groups of students were not sure in the concluding discussion on the number of solutions whether they managed to find all of them. Moreover, they could not tell how to make sure that all the solutions be found. Trial and error strategy is the natural way of solving this type of problems unless the solver knows the procedure and is undoubtedly appropriate in case of primary school pupils (although Scherer & Steinbring (2004, p 68) showed that some of German four-graders were able to proceed in a systematic way).

Students also did not attempt to apply algebra. The exception was a group in which two students who originally studied mathematics worked (see their solution in endnote 3). The question is whether their approach was not caused by the student tendency to solve the problem in a way understandable to their future pupils. Rather, it seems that they were not able to use the knowledge they have acquired in previous studies, although in the seminar they regularly worked like this.

What representations did the students use?

Previous findings are related to the modes of representation, which students used. Despite the attention paid to different modes of representation, translation among them, and transformation within them (Lesh et al., 1987) the students probably neglected and underestimated the role of iconic and pictorial representations in concept construction and tend to overuse verbal representation. They for example wrote:

> The sum of the start and the final numbers is the same number as the sum of numbers in the 2nd and the 4th squares. In case of the target number 50 it is 20. This means that the number in the middle is always the same number. In this case 10. And as there are five squares, the "target number" must be divisible by five. If there are 6 boxes, the "target number" must be divisible by 6.

Visual representation (Fig. 3 for target number 90, Fig. 4 for 6 boxes) accompanying the verbal description of the calculation was used only rarely.

> We divide the final number by the number of boxes (squares) (5) and get the middle number. The sum of the first and the fifth numbers and of the second and the fourth numbers is twice as large as the middle number.

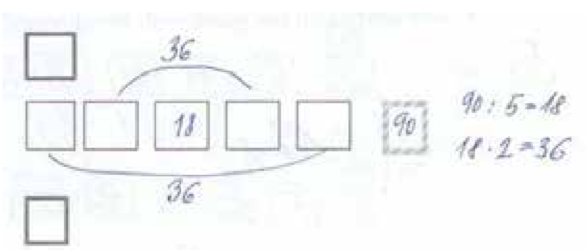


Fig. 3 Visual representation accompanying the verbal description

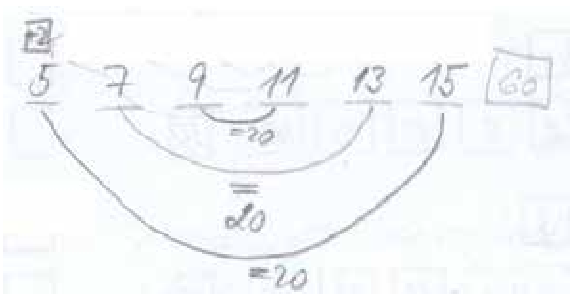


Fig. 4 Visual representation for schema with 6 boxes

In several cases, the visual representation helped us grasp how the students were reasoning (Fig. 5a, b). As important we consider the students´ efforts to seek patterns and coherence. For example Griffith claims that mathematics may be characterized as the search for structures and patterns that bring order and simplicity to our universe. Moreover, it is the discovered patterns and coherence that give mathematics its power (Griffith, 2000).

> An odd number of numbers means that the middle number is always the same and the sum of the outside numbers is double the middle number. An even number of numbers means that the two numbers in the middle are three times smaller than the final number.

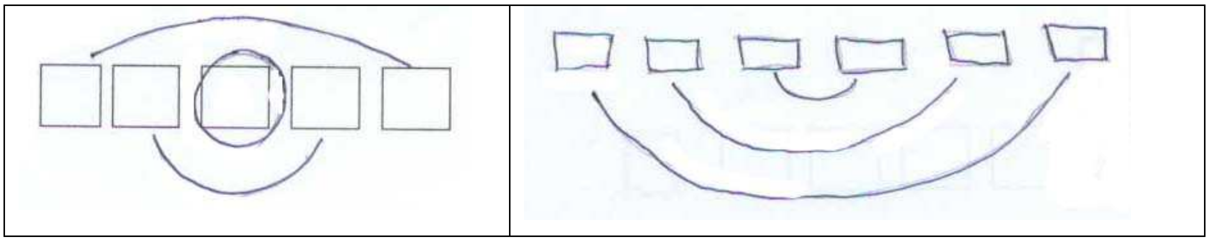


Fig. 5 Visualisations of statements mentioned above

The joint discussion made most students realize the “usefulness of making illustrations”. However, only some of them were able to grasp the table that followed these visualizations (Fig. 6).

x – 2a	x – a	x	x + a	x + 2a

Fig. 6 Visualisation through table

What regularities and coherence did the students notice?

The students usually described their calculation; they stated what they had been doing and how they had proceeded. They mostly recognized that the final number must be divisible by the number of squares but rarely did they verify or justify this conclusion. Their explanations are not always easy to understand. For example

> 50/5 = 10, 10 must be in the middle because we divide 50 into five parts (10, 10, 10, 10, 10) and then the 10 in the middle is left while I always subtract something from the tens on the one side and then I have to add something to the tens on the other side.

> 50/5 = 10 ... to the third. We divide the final number by the number of boxes, the result of which will be in the middle box. We subtract a selected number in the upper box to the left and add it to the right.

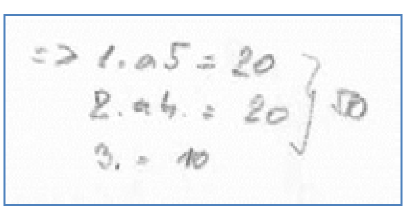


Fig. 7 Students’ solution

> We can see that there is number 10 in the 3rd box. We calculated 10 as 50/5 = 10.

These observations also included some interesting remarks that were consequently discussed in the joint reflection. For example:

> The sum of the pair of outside numbers is 20.
>
> The addend and the number in the second square are together 10.
>
> If the addend is larger by 1, the initial number is smaller by 2. (This was observed in cases that the students' procedure of looking for other solutions was systematic.)
>
> The greater the initial number is, is the smaller the addend is.

Posed questions and problems

The students' interest in problem posing was raised in case of both groups by the following tasks: Why did we assign this problem to the children, what were our intentions? What kind of problem would you recommend as suitable for assignment to children in the given situation? Why? With what intention?

Let us present a few questions posed by the students:

> Which least number can there be in the middle?
>
> What would happen if the middle number were 8?
>
> What would change?
>
> What least number can the target number be?
>
> Can the start number be zero? Can the addend be zero?
>
> What will happen if the target number is odd?

In general, the students tried to pose such questions and problems that could be answered or solved unequivocally.

The students in the second group also considered how to broaden the solution in case that the problem was not limited only to natural numbers. In other words how this environment could be used for motivation of negative or even rational numbers.

CONCLUSIONS

In this study, we dealt with the problem how to prepare future teachers for IBME. We assumed that the solution of appropriate problems and connecting it with didactic questions shows the students how important their knowledge of mathematics is and how they can use it in school practice. Our starting point was in accordance with other authors, for example Lamon stated: "... facilitating teacher understanding using the same questions and activities that may be used with children is one way to help teachers build the comfort and confidence they need to begin talking to children about complex ideas." (Lamon, 2006, xiv)

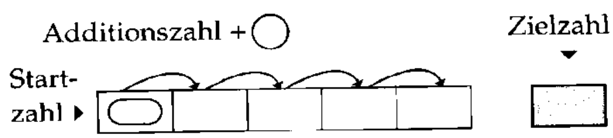
One of the key problems was the students' failure to realize the connection of what they learned before in mathematics courses to the needs of school practice. This fact was in our study manifested by students' preference of arithmetic when solving given problem. Algebra as a tool for solution was used only by those students who were proficient enough, and who are persuaded that they are able to work quickly and without errors. Other students tended to stereotypical procedures: they regarded the use of visual aids, illustrative examples as manifestation of lower level of knowledge. It seems that they perceived as substantial only the questions of WHAT? and HOW? to teach. The question WHY? did not interest them so much. Such belief is in our opinion unsuitable, inadequate, especially in case of primary school teachers.

We formulated several questions for continuation of this research: how to change this opinion, how to develop the ability to visualize, explain using visualization, how to show the students that thorough study of mathematics is important for teachers; how to change students' understanding of the role of reasoning in education (not only mathematical) and the related generalization, looking for similarities and differences, discovery of regularities, patterns and coherences. It is also the question of making the students aware of the dangers of badly-founded statements (hypotheses, proclamations) and of when it is the right time for drawing conclusions.

In our previous research (Tichá, Hošpesová, 2006) we realized that joint discussion and reflection of problems and their solutions can be a fruitful way. IBME offers, in our opinion, good grounds for such activities.

END NOTES

1. A system in which the teacher, student(s), milieu and restrictions necessary for creation of a piece of mathematical knowledge interact "to teach somebody something". The educator "organizes a plan of action which illuminates his/her intention to modify some knowledge or bring about its creation in another actor, a student, for example, and which permits him/her to express himself/herself in actions" (Brousseau & Sarrazy, 2002).
2. The scheme of the problem "Wer trifft die 50" in the original (Scherer, Steinbring, 2004, p. 65).



3. The students formulated and solved an equation:

$$x_1 + x_1 + a + (x_1 + a) + a + (x_1 + a) + a + a + (x_1 + a) + a + a + a = 50$$
$$5x_1 + 10a = 50$$
$$5(x_1 + 2a) = 50$$
$$x_1 + 2a = 10$$

and systematically proceeded in its solution:
10, 10 + 0, 10 + 0, 10 + 0, 10 + 0 = 50
8, 8 + 1, 9 + 1, 10 + 1, 11 + 1 = 50
6, 6 + 2, 8 + 2, 10 + 2, 12 + 2 = 50 , …

4. Acknowledgement: This research was partially supported by the RVO 67985840.

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ANALYSIS OF PRE-SERVICE ELEMENTARY TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE IN THE CONTEXT OF PROBLEM POSING

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In this study, it was aimed to study the level of pre-service elementary teachers' proficiencies in determining 5th grade students' problem posing errors about addition with fractions. This study was realized with 36 senior class pre-service elementary teachers in an eastern university during spring term in 2011-2012 academic year. Error Determination Test, which has six problem statements about addition with fractions, was applied to pre-service elementary teachers. Participants were asked to compare addition operations with the given problem statements and to clarify errors if there were any. Research findings indicate that participants had problems in determining errors and made different errors in their explanations about the students' errors.

Keywords: *Pre-service elementary teachers, problem posing, fractions, addition operation with fractions.*

INTRODUCTION

Problem posing takes increasing attention in recent years. Main reasons that lie behind this attention are establishing relationship between mathematical concepts, operations and daily life (Abu-Elwan, 2002; Dickerson, 1999; Knott, 2010), transitions between representations (English, 1998; Işık, Işık & Kar, 2011) and it contributes to these issues. This study draws attention to the analysis of pre-service teachers' pedagogical content knowledge which includes being aware of students' misconceptions and also errors. Particularly, the following aim was addressed: studying pre-service elementary teachers' proficiencies in determining errors in problem statements about addition with fractions.

THEORETICAL FRAMEWORK

There are some knowledge categories that professional teachers should have. One of these categories as determined by Shulman (1987) is pedagogical content knowledge. It is specifically about illustrations, explanations, and examples used in making a subject more comprehensible to learners. In detail, it is about knowing functional representations and illustrations of content and concepts, knowing what the issues that makes learning content easier or harder, knowing students' misconceptions and errors, knowing analogies, symbols, examples or explanations that helps overcoming misconceptions and understanding concepts, and lastly knowing different age group and level students' thoughts, perceptions, and previous knowledge about concepts (Shulman, 1987).

Problem posing is about generating new problems and reformulating problems from the given problems or situations (Duncker, 1945). As English (1998) indicated that students could improve defining symbolic mathematical expressions ability and relating them with daily life issues ability through problem posing. Besides, Crespo (2003) stated that problems posed by teachers give students chances to learn mathematics. Işık and Kar (2012) determined elementary school mathematics teachers realized more problem posing activities about fractions sub learning domain than other ones under numbers learning domain. Additionally, all of the teachers, who gave more place to problem posing activities about fraction sub learning domain, expressed that problem posing contributes students in relating conceptual understanding to symbolic expressions in daily life.

Moreover, Hill, Rowan and Ball (2005) stated that teachers' mathematical knowledge should give chance to students to explain and teachers to analyze their students' answers. On the other hand, there are limited studies studying teachers' and pre-service teachers' pedagogical content knowledge in operations with fractions. Toluk-Uçar (2009) found out that pre-service elementary teachers thought fractions represent pieces instead of amount and as well the solutions of the problem they posed necessitates addition in natural numbers instead of addition in fractions. The problem a participant posed for the operation $\frac{1}{3}+\frac{1}{2}=$ was "My mum gave me 1 of her 3 apples and my brother gave me 1 of his 2 apples. How many apples did I have altogether?'' and it exemplifies this situation. Ticha and Hošpesová (2009) asked pre-service teachers to pose a problem from $\frac{1}{4}\text{x}\frac{2}{3}$ operation; afterwards, from the problems participants posed, they asked them to evaluate three of them. The researcher determined that pre-service teachers ignored the conceptual dimension of the operation, could not relate the given operation with the daily life issues, and some of them posed problems that necessitate multiplication instead of addition and students indicated that it was easy to formulate same type of problems but it was difficult to formulate problems of a growing difficulty. Işık (2011) concentrated on conceptual analysis of the problems about multiplication and division in fractions posed by pre-service elementary mathematics teachers. The results of study showed that pre-service teachers had difficulties in the conceptual dimension of fractions and operations in fractions. On the other hand, in the literature there is not any study about teachers' and pre-service teachers' pedagogical content knowledge in determining the errors in the problems posed about addition with fractions by students.

Problem posing is effective in both clarifying students' mathematical skills thoroughly and giving chance to assessing what students did (Whiten, 2004). In addition to this, problem posing informs teachers about students' skills, attitudes, and conceptual learning about a situation (Işık & Kar, 2012). As mentioned before, one of the categories of pedagogical content knowledge is being aware of students' misconceptions and errors. From this aspect, when pre-service teachers are encountered with the students' errors in the problems posed, pre-service teachers' awareness could be enhanced.

The success of problem posing activities was based on the guidance of teachers to students about true problem posing and exploring. When the successes of instructional process and student were affected from teachers' knowledge (Dooren, Verschaffel & Onghena, 2002; Fennema & Franke, 2006; Shulman, 1987) issue is considered, it is significant to analyze teachers' and pre-service teachers' pedagogical content knowledge under different dimensions. Besides, as Crespo (2003) mentioned mathematical problem posing is one of the difficulties in learning mathematics, and as well it isn't clear when and how the pre-service elementary teachers could learn about this issue. In this regard, this study is realized with pre-service teachers and it was aimed to study pre-service elementary teachers' proficiencies in determining errors in problem statements about addition with fractions. Therefore, this study would contribute possible planning processes in teacher education.

METHOD

This study was realized with 36 senior class pre-service elementary teachers in a public university in eastern part of Turkey during spring term in 2011-2012 academic year. These participants took Basic Mathematics I and II (freshman year course) as well as Teaching Mathematics I-II (junior year course) during their instructional process. In addition, they had also chance to observe and practice in class instructional activities at schools. Pre-service teachers were coded with pseudo names like PT1, PT2,…, PT36.

Error Determination Test (EDT), which contains six problem statements about addition with fractions, was applied to pre-service teachers. Items in the test were given in Table 1 below.

Table 1: Problem Statements in EDT

	Problem Statements
1.	I ate $\frac{1}{3}$ of the oranges my mother bought. My brother ate $\frac{1}{2}$ pieces. How many oranges are left?
2.	We bought a cake for my birthday. My sister and I ate $\frac{1}{2}$ of the cake together. $\frac{1}{3}$ of the rest of the cake was eaten by my mother and father. Accordingly, what is the fraction of the rest of the cake?
3.	First day, Ali's classmates planted $\frac{1}{3}$ trees to the school garden. In the second day, they planted $\frac{1}{2}$. Hereunder, how many trees did Ali's classmates plant totally?
4.	Süleyman at first picked $\frac{1}{2}$ of the roses and then picked $\frac{3}{4}$ of the roses. How many roses did Süleyman have totally?

5.	Ahmet joined the game with $\frac{1}{2}$ of the marbles and Mehmet joined with $\frac{1}{3}$ of the marbles. So, what is the amount of marbles from the total did Ahmet and Mehmet join the game with?
6.	Ali participated in a penalty game two times. In the first game, he made one goal from two kicks, and in the second game, he made three goals from four kicks. Therefore, what is the fraction of goals did Ali do at the end of two games?

First four problem statements in Table 1 were selected from problem statements posed by 5th grade students from $\frac{1}{3}+\frac{1}{2}=\blacksquare$ and $\frac{1}{2}+\frac{3}{4}=\blacksquare$ operations. In the first item, the sum is a proper fraction, and in the second item, the sum is a mixed fraction. Işık and Kar (2012) asked 210 7th grade students to pose problems from the five items given about addition with proper and mixed fractions. Afterwards, the researchers analyzed the problems posed by the participant students and determined error types. In the first four problem statements in EDT, there are six error types determined by Işık and Kar (2012); *expressing the added second fraction over the remainder of whole (E1), failure in expressing the operation in the question root (E_2), attributing natural number meaning to the result of the operation (E_3), confusion about units (E_4), attributing natural number meaning to the added fractions (E_5)* and *failure in establishing part-whole relation (E_6).* Fifth and sixth problem statements were added to EDT by the researchers. In the literature, these types of problem statements were utilized by different researchers (Chick & Baker, 2005; Newton, 2008; Ward & Thomas, 2007) for determining students' conceptual knowledge about fractions. Explanations about the errors in problem statements were presented in the findings part.

In the implementation process of EDT, participants were told that *fifth grade students were asked to pose resolvable problems about addition with fractions based on only the given operations. All of the problem statements in the test were posed by students.* There is an explanation like *compare given addition operation with problem statement, and express error types if there is any;* in this process for participants not to lose their motivation and for them not to analyze with prejudice, fifth and sixth problem statements were also said as problem statements posed by students. EDT was applied to pre-service teachers in one class hour. Answers of pre-service teachers were analyzed with content analysis method.

Two different researchers analyzed pre-service teachers' answers about each item in the EDT concurrently and independent of each other. Based on the analysis of each item in EDT, the consistence was found as; 87,5%, 98,43%, 96,88%, 100%, 98,43% and 90,62%, respectively. In the comparison process, answers that were not appropriate to determined error types were presented under *other* category. This category contains some statements that do not express errors instead problem statements were written with little changes in the sequence of words (ex. *what is the fraction of money collected?* Instead it was written like *what is the fraction of money* *collected from the class did our teacher spend?*), and expressions were not open enough to make error analysis.

FINDINGS

Conceptual Analysis of Problem Statements

In the first problem statement, it was mentioned that $\frac{1}{2}$ *of the pieces* were eaten by brother. With the *piece* word in a fraction form, it was tried to express quantities like in the natural number form. Therefore, there is an E_5 error type in the problem statement. Besides, *how many oranges are there left?* question has a subtraction meaning. From this aspect there is an E_2 error type in the problem statement.

In the second problem statement in EDT, addend $\frac{1}{3}$ fraction is expressed via rest of the cake. However, this necessitates $\frac{1}{2} \times \frac{1}{3}$ operation. On that sense there is an E_1 error type in the problem statement. In addition, in the problem statement rest of the cake is asked instead of the amount of that was eaten; so, it does not meet addition meaning. Consequently, there is also an E_2 error type in the problem statement.

Third problem statement in the EDT involves $\frac{1}{3}$ *trees* and *how many trees were planted totally?* question, with these expressions it was tried to give addend fraction and the result of the operation which were fraction the natural number meaning. From this aspect, there are E_3 and E_5 error types in the problem statement. Besides, with $\frac{1}{2}$ expression it is not clear if $\frac{1}{2}$ indicates the amount of trees or $\frac{1}{2}$ represents the area of school garden. On that sense, fraction does not represent an appropriate unit and there is an E_4 error type in the problem statement.

In the fourth problem statement of EDT it is mentioned that at first $\frac{1}{2}$ of the roses in the garden were collected. In this situation rest of the roses was the half of the total roses. On the other hand, student gave place to *later on* $\frac{3}{4}$ *of them were collected* expression. This situation is not logical from the point of part-whole relation. From this aspect there is an E_6 error type in the problem statement. In the question it is asked like how many roses did Süleyman have totally?; with this expression the result of the operation which was a fraction was given a natural number meaning. Therefore, there is an E_3 error type in the problem statement.

In the fifth problem statement in EDT, there is not any information that shows if the addend fractions were taken from the same whole or not. On that sense, although it looks like formal of problem statement meets the operation, in a conceptual sense it is not possible to add fractions from different wholes. Besides, although two wholes are similar the total of two fractions is not the result of the problem statement. In the sixth problem statement in EDT it was mentioned that at the end of the game only four kicks of the six kicks were turned into goal. On the other hand, the result of $\frac{1}{2} + \frac{3}{4}$

operation which was tried to be posed a problem based on this doesn't meet this result. Problem statement necessitates $\frac{1}{6}+\frac{3}{6}$ operation which accepts the total amount of kicks as a whole instead of $\frac{1}{2}+\frac{3}{4}$ operation. From this aspect, problem statement and the operation which was tried to be posed a problem based on this are not consistent.

Pre-service Teachers' Proficiencies in Determination of Error Types in Problems Posed

The distribution of what error types pre-service elementary teachers found in the first four items in EDT is presented in the following Table 2.

Table 2: The Distribution of Error Types Found in the First Four Items in EDT

Items	E_1	E_2	E_3	E_4	E_5	E_6	Errorless	Blank	Other
1		27(75)			18(50)		3(8,3)	0(0)	2(5,5)
2	18(50)	23(63,9)					5(13,9)	1(2,8)	1(2,8)
3			13(36,1)	8(22,2)	16(44,4)		8(22,2)	0(0)	2(5,5)
4			11(30,6)			3(8,3)	4(11,1)	3(8,3)	2(5,5)

*Values in table were calculated based on 36 participants and presented with frequency (percentage).

According to Table 2, in the first problem statement one fourth of the participants could not determine E_2 error type and half of them could not determine E_5 error type. From these proportions, it could be said that to determine E_5 error type is harder for participants. 13,9 % of the participants mentioned that there was not any error in the second problem statement. Moreover, half of them did not determine E_1 error type, almost 36 % of them could not determine E_2 error type. From these proportions, it is harder to determine E_1 error type. About the third problem statement 22,2 % of the participants mentioned there was not any error. Besides, 44,4 % of the participants could determine E_5 error type, meanwhile the proportions of determination of E_3 and E_4 error types are less. Specifically to determine E_4 error type is harder for participants. In the fourth problem statement, 11,1 % of the participants stated that there was not any error in the problem statement. Almost 70 % of them could not determine E_3 error type, and 92 % of them could not determine E_6 error type. From these proportions, it could be said it is harder to determine E_6 error type.

Besides, about the fourth problem statement some of the participants made different error types in their explanations about the errors in this statement. Five of the 11 participants who determined E_3 error type stated that the question should be like *what is the fraction of roses collected?*. PT23's explanation about this issue was;

> PT 23: It is a wrong way to ask how many roses did Süleyman have. If it is asked like what is the fraction of roses collected, it would be a true problem.

In the PT23's explanation the result must be fraction, so, it was emphasized with *what is the fraction of* expression. On the other hand, part-whole relation was ignored; due to this, in the question *what is the fraction of* was used. This expression is not appropriate to logical aspect. Six of the 11 participants who determined E_3 error type stated that they did not know the beginning amount, so, they could not answer the issue that was asked in the question. PT4's explanation about this issue was that;

PT4: In here, the total number of roses in the garden is not mentioned, so, it is not possible to calculate how many roses were collected. Whereas if it is known given operation could be calculated.

Students were asked only to pose a resolvable problem with just $\frac{1}{2}+\frac{3}{4}$ operation. In the given application directive it was emphasized that pre-service teachers should analyze problems posed through this aspect. Therefore, explanations about the necessity of knowing the beginning amount are not appropriate approach when the given operation is considered. In addition, when it is thought if the beginning amount is known, in a logical sense this would not be an appropriate problem. Due to the result of the given operation being a mixed fraction, expressing addend fractions as a part of the whole would cause violation of part-whole relation.

The distribution of pre-service teachers' answers to fifth and sixth problem statements in EDT was presented in Table 3 below.

Table 3: The Distribution of Answers Given to Fifth and Sixth Items in EDT

	Errorless	**With Error**	**Blank**	**Other**
Fifth Problem	9(25)	24(66,7)	2(5,5)	1(2,8)
Sixth Problem	21(58,3)	6(16,7)	7(19,5)	2(5,5)

*Values in table were calculated based on 36 participants and presented with frequency (percentage).

According to Table 3, 25% of the participants stated that there was not any error in the fifth problem statement. On the other hand, two thirds of the participants mentioned that problem statement was not appropriate to given operation. 20 of the 24 participants, who mentioned the problem statement had an error, said that if the reference fractions indicating the number of marbles were equal or not was not known, so, these two fractions could not be added. PT27's explanation about this issue was as follows;

PT27: There is an error. Are the numbers of Ahmet's and Mehmet's marbles equal? This is not known, so, it is not possible to find the true answer with just addition.

In addition, four participants found the error in the way of how the question was asked. Participants mentioned that problem should be asked like *what is the fraction of total marbles that they joined to the game?*. PT5's expression about this issue was;

PT5: Problem should continue like: Accordingly, what is the fraction of total marbles that Ahmet and Mehmet joined the game with?

According to Table 3, more than the half of participants (58,3 %) stated that there was not an error in the sixth problem statement. On the other hand, six participants determined the error in the problem statement, but they could not give a conceptual explanation to the reason of this error. Four of the participants said that there is not appropriate relationship between the result as $\frac{5}{4}$ to the operation and the result of the solution of the problem posed. Four of the participants mentioned that the result of the problem posed based on the given operation was not consistent with problem statement. PT1 mentioned about this issue as;

PT1: The result of given operation found as 5/4. The number of kicks that were done as goal was higher than the whole, so, the problem had an error.

Moreover, two participants saw the reason of the error as different number of kicks in each penalty kicking game. PT3 made an explanation about this issue as;

PT3: It was played two times. There were two different wholes, so, the sum could not give how many kicks were turned into goal.

RESULT AND DISCUSSION

In this study, pre-service teachers' proficiencies in determining the errors that 5th grade students done in problems posed about addition with fractions were studied. It was found that determining E_1 and E_2 error types in the problem statements was higher than other error types. In the focus of these errors, it could be said that there was problem of transferring fractions and addition to the problem statements. It could be said that these errors were related with formal aspect of the operation, but other error types were related with conceptual aspect of the fractions. Therefore, increased determination E_1 and E_2 error types could be thought that analyses were generally done on the formal aspect. It was found that pre-service teachers had more difficulty in attributing a natural number meaning to the result of the operation in the first four problem statements of EDT, not stating fractions with appropriate units and not associating fraction in the reference amount with whole. Specifically, not associating part-whole relation was the least found error type done by pre-service teachers. It could be mentioned that participants not being able to determine the sum as a mixed fraction caused them living more difficulty. Moreover, some participants made new errors in their explanations about their errors. These findings supported by different studies' results (Işık, 2011; Redmond & Utley, 2007; Rizvi, 2004; Toluk-Uçar, 2009; Zembat, 2007) that pre-service teachers had difficulties in problem posing about operations with factions.

25 % of pre-service teachers indicated that there was not an error in the fifth problem statement and only 55,6 % of them could make a conceptual explanation about the reason of the error. These findings were supported by Newton (2008) as pre-service teachers had difficulties in determining the impossibility of addition with fractions defined in different wholes. Only six pre-service teachers determined the error in the sixth problem statement in EDT. On the other hand, none of them could make a conceptual explanation to error. These findings were similar to the results of other studies realized by different researchers (Chick & Baker, 2005; Ward & Thomas, 2007).

Crespo (2003) mentioned that pre-service teachers posed problems without thinking mathematical and pedagogical aspects and did not study the resolvableness of them. Based on the findings of this study, they supported what Crespo mentioned, and besides they indicated that these skills like addition with fractions should be enhanced. Findings of this study were gathered from six problem statements about addition with fractions topic. This could be seen as a limitation of the study. In the future, this study could be extended with problem statements involving different fractional number and possible other error types and these studies could be done with pre-service teachers or in-service teachers. By qualitative studies, from participants' frames of mind that include the reasons behind different error types from some pre-service teachers' explanations it could be revealed. Experimental studies could be realized for removing the difficulties by considering the results of these like studies in the future.

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DELINEATING ISSUES RELATED TO HORIZON CONTENT KNOWLEDGE FOR MATHEMATICS TEACHING

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The conceptualisation of mathematical knowledge for teaching (MKT) has recently received much attention. Scholars have provided examples, studied effects, and debated importance. However, from among the MKT domains, horizon content knowledge (HCK) has received less attention. In particular, the nature of the knowledge as it is related to teaching is unclear. We argue for efforts to clarify definitions and to test and refine those definitions with the use of realistic and vetted examples of professional work. To advance this agenda, we provide a working definition of HCK and use it to discuss a vignette involving irrational numbers.

INTRODUCTION

Teachers' content knowledge is of current interest, both the nature of such knowledge and ways to improve it. Among proposed conceptualisations, one that emerged from trying to understand and describe the nature and form of teaching and its mathematical demands is mathematical knowledge for teaching (MKT) (e.g., Ball, Thames, & Phelps, 2008). Grounded in Shulman's (1987) notions of subject matter knowledge (SMK) and pedagogical content knowledge (PCK), these researchers define MKT to be mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students and propose a refinement of SMK and PCK into sub-domains. Of these, horizon content knowledge (HCK) is less-developed.

The problem of defining HCK stems from an overabundance of metaphors and from inadequate clarity and consensus, especially regarding HCK's relation to teaching. Thus, deeper discussion of what HCK comprises (what it is and what it is not) is needed. Ball and Bass (2009) situate their conception of HCK within their practice-based theory of MKT. They describe HCK as "a kind of mathematical 'peripheral vision' needed in teaching, a view of the larger mathematical landscape that teaching requires" (p. 1). They provide a compelling foray into the ideas, but their provocative proposal leaves much to further development.

Starting with Ball and Bass' ideas, Zazkis and Mamolo (2011) use Husserl's work to propose a conception of "knowledge at the mathematical horizon." They use Husserl's notion to analyse ways in which topics from undergraduate mathematics provide inner and outer horizons of school mathematics. Their paper prompted two commentary papers. Foster (2011) discusses what he calls “peripheral mathematical knowledge” to refer to mathematics that matters for teaching but is out of the view of the learner. Figueiras, Ribeiro, Carrillo, Fernández and Deulofeu (2011) point out that the language for HCK needs to be consistent with basic assumptions of the nature and role of teacher content knowledge. They argue for locating the meaning of HCK in the work of teaching instead of conceptualising HCK as advanced knowledge that is then applied to teaching. They write, “Our critique of Zazkis and Mamolo’s paper is much more in terms of their assumptions about the nature of the mathematical knowledge that elementary and secondary teachers need, rather than in terms of their conceptualization of knowledge at the mathematical horizon” (p. 26). The point they seem to be making is about whether the knowledge from an advanced course would have the bearing on practice that Zazkis and Mamolo claim.

A different view of knowledge in relation to teaching is evident in the work of Vale, McAndrew, and Krishnan (2011). They use the phrase “connecting with the horizon” to characterize the knowledge implicated by teachers comments that learning more advanced mathematics in a professional learning program helped them see more connections and structure both among representations related to a topic and among different topics. They report that, as teachers saw connections between topics they teach and more advanced topics, it helped them see connections and more general structure inside the mathematics they teach.

In these studies, several images, phrases, and issues recur, often in ways that reveal unresolved issues. For example, some scholars identify with the language from Felix Klein’s book title, elementary mathematics from an advanced standpoint, or higher perspective, while others suggest inverting it to be advanced mathematics from an elementary perspective. As another example, references are sometimes to students’ horizons and other times to teachers’ horizons. In describing horizon knowledge issues also arise regarding distinctions between HCK and knowledge of the curriculum and its trajectory. Some scholars are concerned with the importance of requiring undergraduate mathematics courses, while others are concerned with the treatment of that content and the way in which it is framed and named.

We suggest that these scholars are engaged in a difficult process of developing a clear definition of HCK around which consensus could be built and that examples are central to this process. Explicit definitions, good examples, and disciplined analyses are crucial to making progress. In this paper, we offer our current “working definition” of HCK and then use it to examine a candidate example of HCK. We then use this examination to reconsider some of the issues central to HCK. Although situated in empirical work, our primary goal is conceptual — to surface, delineate, and clarify key issues for advancing work on HCK.

THEORETICAL BACKGROUND

Researchers have found that teachers' mathematical knowledge and experience, broadly construed, are not consistently associated with greater student learning. Instead, the mathematical knowledge associated with achievement gains is specifically related to the work of teaching and to the mathematical tasks that constitute that work. It is this evidence that led Ball and her colleagues to develop a conceptualization of mathematical knowledge for teaching, where the "for teaching" expression conveys a practice-based characterization of teacher content knowledge.

In addition to common knowledge of the subject (SMK), Shulman (1987) defines PCK as knowledge that is an amalgam of knowing the subject with knowing how students engage with the subject and knowing effective ways of representing the subject and rendering it for learning. Ball et al. (2008) subdivide both SMK and PCK. PCK contains: i) knowledge of content and students (KCS); ii) knowledge of content and teaching (KCT); and iii) knowledge about content and curriculum (KCC). SMK contains: i) common content knowledge (CCK), mathematical knowledge that is involved in teaching but not unique to the teaching profession; ii) specialized content knowledge (SCK), mathematical knowledge that is unique to teaching and not used in professions outside teaching (namely, knowledge that allows the teacher to engage in tasks specialized to teaching, such as analyzing patterns of errors or readily solving problems in multiple ways); and iii) horizon content knowledge (HCK), knowledge about mathematics outside the curriculum. An important contribution of the work of Ball and her colleagues is that all of these domains are defined in relation to the work of teaching: MKT is knowledge that serves as a resource for addressing the mathematical demands of teaching. It is this knowledge "for teaching," with clear links to the demands of specific tasks of teaching, has been shown to have positive effects on student achievement (Baumert et al. 2010; Hill, Rowan, & Ball, 2005; Kersting, Givvin, Sotelo, & Stigler, 2011; Rockoff, Jacob, Kane, & Staiger, 2008). In this discussion, it is also worth noting that, because MKT is intimately linked to teaching, it is different for different school levels and topics: MKT for kindergarten differs from MKT for upper elementary differs from MKT for secondary, and MKT for geometry differs for MKT for number and operation differs from MKT for algebra.

HCK is one of the sub-domains of such a practice-based mathematical knowledge for teaching. It involves a sense of how mathematics at play in instruction is related to a larger mathematical landscape (Ball & Bass, 2009). HCK is thus perceived as implicated by the proximal demands of teaching but not directly related to the curriculum (mathematical content) that has to be taught at a particular point in instruction. Importantly, even though it is about mathematical knowledge removed from the content being taught and learned at a particular level, HCK needs to be demonstratively related to the teaching that takes place in school.

In collaboration with Ball and Bass' research group at the University of Michigan, we developed the following working definition.

> Horizon Content Knowledge (HCK) is an orientation to and familiarity with the discipline (or disciplines) that contribute to the teaching of the school subject at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory. HCK includes explicit knowledge of the ways of and tools for knowing in the discipline, the kinds of knowledge and their warrants, and where ideas come from and how "truth" or validity is established. HCK also includes awareness of core disciplinary orientations and values, and of major structures of the discipline. HCK enables teachers to "hear" students, to make judgments about the importance of particular ideas or questions, and to treat the discipline with integrity, all resources for balancing the fundamental task of connecting learners to a vast and highly developed field.

HCK is neither common nor specialized, and it is not about a curriculum progression, but more about having a sense of the larger mathematical environment of the discipline being taught. In that sense, when discussing HCK, it is not sufficient to simply consider knowledge about advanced mathematics or knowledge about different topics that may arise in students' future studies. HCK also includes, but not to the exclusion of other things, knowledge that would allow teachers to make additional sense of what students are saying and to act with an awareness of connections to topics that students' may or may not meet in the future.

HCK is distinct from specialized content knowledge (SCK) because SCK is immediately about the content being taught and HCK is not. SCK is about unpacked, elaborated, explicit versions of the content being taught, in ways that are useful to teachers as they teach. Beyond what students directly need to learn, it includes knowledge about representations, explanations, language, and features of these that increase teachers' capacity to teach them. The distinctive character of SCK is evident in Foster's (2011) discussion of peripheral mathematical knowledge. As he writes when discussing knowledge that he found useful as a teacher (such as knowing that both $x^2 + 17x + 30$ and $2x^2 + 17x + 30$ factorise with integer coefficients and how to generate other such pairs), "that whereas the process of *coming to know* these things may be of great value for learners, *knowing them* may not be" (p. 25).

The notion of SCK as distinctively mathematical knowledge directly related to the content being taught but that is specialized to the work of teaching has a certain parallel to the notion of an inner horizon as described by Zazkis and Mamolo (2011, p. 9) as not at the focus of attention, but also intended. If the last part were amended to "also present" or "also relevant to teaching," the parallel would be quite strong. It is this issue of relevance to teaching that seems to distinguish Zazkis and Mamolo's examples from those of Foster's. When we read Zazkis and Mamolo's example of knowing that the number of triangles in a pentagon with all diagonals drawn is a multiple of five, we can follow the logic that there might be a situation in which this might be relevant as a teacher, but it does not pass a kind of reality test for professional knowledge. At a practical level, in an extended discussion in the professional community, we are not convinced that this would be seen as having much utility, whereas many of Foster's examples seem likely to stand up to such a test. At the heart of this is the basic definitional character of MKT — that it be professional knowledge *for teaching*.

The other distinction to make is between HCK and KCC. We argue that it is important to distinguish these and that doing so reveals a central issue, that HCK is distinctively relevant to the conversation about "advanced" mathematics courses and the role of mathematicians in the education of teachers, whereas KCC is much more about an understanding of school mathematics and particular approaches to organizing the school curriculum. Unfortunately, Ball et al.'s description of HCK as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" confounds the issue. Their statement was meant to be about one example of HCK and the meaning of "related" was not meant to be about the curricular development of the content, but about the other kinds of relatedness that might exist among topics (personal communication). In this unfortunate wording, we see potential problems with the term "horizon." The idea of a curricular horizon as being about a curricular trajectory is distinct from what we mean by HCK. The statement also raises the issue of how remote something needs to be in order to be about the horizon. Our current proposal is that HCK is not the content being taught and not about curricular development of that content. Next we discuss the empirical context of our work and then present a vignette from secondary school teaching and use it to discuss the constitution of HCK.

METHODS

To establish a practice-based conception of mathematical knowledge at the horizon, we have grounded our analyses in teaching and practice-based reasoning about teaching, such as that occurring in the context of focus-group interviews, professional development programs, and collaborative investigation of teaching. The data consist of records of such practice from the United States, Norway, and Portugal. From this data, we selected and developed candidate, teaching vignettes — critical situations perceived as effective teaching and supporting professional deliberation, discussion, and discernment. We then analyze these vignettes in relation to our working definition of HCK and vet them with other members of the mathematics and mathematics education community.

Although we focus on practice, it is important to note that the object of study is not a particular teacher or classroom, but tasks entailed in teaching and an analysis of their mathematical demands (Ball & Bass, 2003). With a focus on idealized tasks of teaching and the demand they create for horizon knowledge, we developed vignettes from selected episodes gathered in different contexts. The vignettes are consistent with professional practice, which allows us to use them to examine and test notions of HCK. The vignettes provide a reference point for discussing, reflecting upon, and further analyzing both the work of teaching and our findings about practice-based HCK. Given limited space, we discuss a single example, and then use it to illustrate and reflect on the proposed definition.

Vignette of HCK in practice

Mr. Lee's class has been discussing different types of numbers. While his students have a firm knowledge of whole and rational numbers, they have now been introduced to irrational numbers and are given examples of such numbers (like $\sqrt{2}$ and π, listed on the board). Based on what is on the board, Mr. Lee asks his student whether they can think of any other rational or irrational numbers they have learned about in the past. A student, Jay, suggests $2\sqrt{2}$. Writing it on the board, Mr. Lee asks, "And is $2\sqrt{2}$ rational or irrational?" To his surprise, Jay says it is rational. Jay continues and another student Ben responds.

Jay: If you have a rational number and an irrational number and you multiply them, the product will be a rational, and you will still have a fraction.

Ben: I don't think so, because when we multiplied a rational with an integer, we still got a rational — I think the same… the product of an irrational and a rational will be irrational.

Jay goes to the board to explain his thinking:

Jay: Look…, say you have a rational *a/b* and multiply it by the irrational *v*, you get *av/b*, which is rational, see?

Ben: No, that can't be…. If it's rational… that is only possible if *a/b* is zero, and that was not the case.

Jay: What? … how come?

Ben: Oh, now I understand why you're saying that it's rational… you were missing something… well…, if the product is rational, then *v* is rational too, and it can't be because we said at the beginning that it was irrational...

There are several issues here that teaching needs to handle. First of all, a teacher would need to decide if the argument provided by Ben is correct. Second, after understanding that Ben's solution is correct, a teacher would need to decide whether it is worth pursing, in particular when the argument used by Ben is unrelated to the learning goals of the lesson and is outside Mr. Lee's secondary curriculum. Asking students to explain their ideas to their peers may be risky when the mathematics is in an area that is less familiar to the teacher. These situations, corresponding to improvisations (Ribeiro, Monteiro, & Carrillo, 2009), lead to contingency moments (Rowland, Huckstep, & Thwaites, 2005) in which the teacher has to put in practice all of his or her intuitive knowledge. Even when teachers are aware of some of the possible implications in following, or not, a certain path, they face dilemmas that can profit from a familiarity with mathematics beyond the scope of what is being taught.

A teacher should notice that Jay's first argument is wrong. In the vignette, Mr. Lee does not intervene and lets the two students continue, possibly because Ben is providing an argument and is trying to generalize from previous experience — an integer multiplied by a rational number yielded a rational number — so the teacher lets the students continue without intervening. However, in responding to the (wrong) argument Jay presented at the board, Ben is using a kind of proof-by-contradiction argument, content that is outside of the curriculum Mr. Lee intends to teach. When Ben states that av/b being rational implies $a/b = 0$, he starts by assuming that a/b is not zero. Then $(a/b)(v) = (av)/b = p/q$ is a rational number, and because a/b is not zero, $v = (pb)/(aq)$, meaning v is a rational number too, but that contradicts the assumption. So, a/b cannot be different from zero if av/b is a rational number. Having knowledge about proof by contradiction and the characteristics of irrational numbers would help a teacher to understand Ben's argument and to handle the teaching of this matter with integrity. However, having learned about irrational numbers and proof by contradiction in advanced university courses in mathematics is not a warrant that a teacher can make sense of students' arguments in a classroom. What is needed instead is a treatment that accounts for how these topics may arise in teaching.

As stipulated in the definition, HCK includes an understanding of disciplinary ways of knowing and of establishing validity. Being familiar with proof by contradiction and other important ways of building mathematical arguments would help a teacher hear student reasoning, in the many different and emerging forms it takes, whether at the primary or secondary level. Such knowledge would allow a teacher to appreciate and understand how proof by contradiction can be related and imbedded in students' comments and reasoning in many topics. Proof (and in particular proof by contradiction) is a topic more extensively taught and used in university mathematics courses, yet it is rarely related to what is done in teaching (and not all advanced topics have such a relation — direct or indirect).

The decisions faced by Mr. Lee require subject matter knowledge not part of the curriculum he is teaching. Hence, it is not obviously part of SCK or CCK, which involve representations, explanations, and unpacked knowledge of the content being taught. Knowledge of proof by contradiction and being able to use this knowledge in teaching, can help a teacher hear students, see beyond the topics being taught, and make judgments about what to do in the unfolding dynamics of instruction.

SUMMARY AND REFLECTING COMMENTS

We have used this example to examine a view of advanced mathematical knowledge related to teaching and how it is different from advanced mathematical knowledge typically taught to prospective teachers. Knowledge of advanced content is in itself not a warrant that a teacher can make sense of student thinking in instruction.

Returning to our working definition of HCK, teachers need to have an "orientation to" and "familiarity with the discipline." It would be helpful for teachers who are working with specific content to know how the discipline handles this content at different stages in its development. A topic can be related to other content, outside the immediate curriculum, with different aims and not directly related to the content being taught. Proof by contradiction is often taught in the context of number theory, yet familiarity with it can have a bearing on the teaching of a wide range of topics. Thus, teachers need knowledge that supports connecting the notion of a proof technique, and the mathematical steps of such a proof, with what students may say. This lead us back to our working definition, which says that HCK "contributes to the teaching at hand, providing teachers with a sense for how the content being taught is situated in and connected to the broader disciplinary territory."

Even though the reasoning of the students may be considered to be outside the curriculum, such as in the vignette above, a teacher may value the way students make use of specific proof techniques in their arguments, without necessarily trying to make the logic behind the proof an important part of the discussion. Instead, a teacher may recognize a proof technique, its validity, and the intuitive way it convinces students. Such knowledge is an important resource for teaching.

Our view is that teachers need a treatment of "advanced" mathematics tailored to the orienting and navigating demands of the teaching in which they engage. Experiences with proof in general, and proof by contradiction in particular, in the context of teaching, provide a teacher with resources for hearing the mathematical ideas behind Ben's argument — ideas related to major structures and developments in the discipline, another part of our working definition of HCK. A teacher needs to make decisions about how to handle discussions that occur in the classroom, and to do so in a way that has integrity as students learn additional mathematics. Knowing mathematics at the horizon gives a teacher awareness of potentialities of situations and suggests possibilities for dealing with the mathematical content being taught at a given level. In order to do so, a teacher does not need to know everything about proof by contradiction, but needs to have a sense of what it is and its potential. That would allow students (at least in theory) to further understand and make sense of other topics, both directly and indirectly related.

Developing tasks related to advanced content, yet situated in artifacts from teaching — such as student work, a task from a textbook, or a dialog among students — might motivate prospective teachers to study mathematics, provide a focus for learning that mathematics, and develop a sense of when and how to use such knowledge in teaching. Similarly, developing instruments to measure HCK might help advance our understanding of HCK by forcing greater clarity about what it is we are trying to measure. Instruments could be similar to many of the recent multiple-choice instruments developed to measure other domains of mathematical knowledge for teaching (e.g., Hill, Schilling, & Ball, 2004; Tatto et. al, 2008), or they could be developed using interviews or observational techniques. Indeed, measuring and validating the models underlying such concepts would be important steps in developing an understanding that could inform policy and practice.

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PRIMARY TEACHERS' ASSESSMENT IN MATHEMATICS: RESOURCES EXPLOITED IN THE PEDAGOGICAL DISCOURSE

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Teachers' pedagogical discourse concerning mathematics education and, particularly, the resources they draw upon within it allow the study of how they make sense of and evaluate pupils' work in the mathematics classroom. The article focuses on the resources utilised within five primary school teachers' pedagogical discourse articulated in the context of a semi-structured interview on mathematics education issues. The analysis of the data indicated the exploitation of not necessarily compatible resources which might lead to surface or inequitable evaluations.

Key words: assessment, mathematics, resources, pedagogical discourse, teachers.

THEORETICAL CONSIDERATIONS

Assessment constitutes a critical factor of the educational process, which, in combination with others, shape teaching and learning. The relative research in Mathematics Education had been rather limited until early 1990s, focusing mainly on the development of tools for measuring mathematical knowledge (Gipps, 1999). The recognition of the determining role of assessment in cultural reproduction and social stratification, particularly in mathematics, with its profound consequences for future educational and professional success of both students and teachers, led to the adoption of a sociological perspective for its study. Within this perspective, assessment practices are viewed as social in nature: reflective of often implicit assumptions about knowledge and what counts as valued knowledge; about the relationship between learning, teaching and assessment; about teachers' evaluative practices and their understandings of students' achievements and curriculum demands (Wyatt-Smith & Gunn, 2009).

Contemporary views on assessment attribute special significance to the ways teachers make sense of and engage in the assessment process (e.g. Morgan, 2009). The investigation of these ways has been related to the notion of the pedagogical discourse, which offers a framework for studying the linguistic activity developed by a teacher with respect to teaching and learning mathematics that allows the identification of the meanings attributed by him/ her to the assessment process (Morgan et al, 2002). The pedagogical discourse is constituted through processes of 'recontextualisation' of knowledge and practices. These processes are determined by the agents responsible for them, the processes of selection, simplification, condensation, repositioning and refocusing through which this discourse is enacted and the resources exploited (Bernstein, 2000). Resources are accumulated knowledge structures people hold in their heads and "draw upon when they produce or interpret texts – including their knowledge of language, representations of the natural and social worlds they inhabit, values, beliefs, assumptions and so on" (Fairclough 1989, p. 24).

Within the above perspective, teachers construct meanings when reading a mathematical text dependent on the features themselves discern in the text. These meanings are determined by the discourses enacted for the reading of the text for assessment purposes, designated by the resources and the positionings exploited (the latter concerns different relationships to students and authorities and different orientations towards texts and assessment tasks).

Morgan (1998) examined how teachers assess students' texts (verbal and nonverbal behaviour) and found that they tended to draw on resources from different and often contradictory discourses. In a later study, Morgan & Watson (2002), looking at teachers' assessment activity both in the classroom and with respect to written tasks, confirmed the previous finding, highlighting particularly the interpretive nature of the assessment activity due to the different resources individual teachers bring to the assessment task. These resources may include (a) teachers' personal knowledge of mathematics and school mathematics, including their personal "mathematical history", (b) their beliefs (conceived as dynamic rather than as static constructions) about the nature of mathematics and the ways these are related to assessment, (c) their expectations about how mathematical knowledge can be communicated, (d) their experience and expectations of students and classrooms, (e) their experience, impressions and expectations for individual students, and (g) their cultural background and linguistic skills as decisive parameters for the assessment process, related to downgrading students' mathematical achievements (Watson, 1999; Morgan & Watson, 2002).

Morgan's and her colleagues' studies also showed that, when assessing students, teachers are positioned in certain, distinct and occasionally contradictory ways and this can lead to different evaluations of the same text (Morgan et al, 2002). For example, a teacher might be positioned as an examiner using personal criteria or criteria determined by exterior factors or as an advocate, who seeks opportunities to afford a grade to the student. Different positionings are likely to lead to the employment of different resources during the assessment process and, consequently, to the realization of different actions and judgments from different teachers or from a teacher at different times and in different circumstances. However, the issue teachers' positions as they evaluate students' work is not addressed here and it is only mentioned in order to provide a complete and comprehensive account of the perspective employed.

The sociological perspective for examining how teachers make sense and evaluate pupils' mathematics productions briefly outlined above has not yet attracted much research attention. Only sporadically one comes across studies barely related to this perspective. For example, examining mathematics classroom assessment interactions from a semiotic point of view, Björklund Boistrup (2010), in accordance with Morgan's and her colleagues' studies, concluded that these interactions are part of different discourses, which steer the individual towards what is valued and who has the authority to act, but also provide possibilities for active involvement, dependent, however, on the interplay of these discourses.

Also, Wyatt-Smith & Gunn (2009), focusing generally on assessment within a sociocultural perspective, view it as critical inquiry in need to be considered in relation to four main interrelated and interdependent lenses: (i) conceptions of knowledge, including its nature and the related capabilities to be assessed; (ii) conceptions about the alignment of assessment, learning and teaching and how teachers enact them in practice; (iii) teacher judgment practices, especially related to standards, moderation opportunities, requirements and expectations of quality performance and (iv) discipline-specific literacy demands required to participate in and contribute to knowledge. There are certainly similarities between these lenses and the discourse resources exploited by Morgan and her colleagues, both notions enabling particular characteristics of enacted assessment related to the suite of conceptions, values and assumptions at play to come to the fore.

The present study focuses on the resources designated in primary teachers' pedagogical discourse as a means to examine the meanings they attribute to the assessment process in mathematics. These resources constitute what Wenger (1998) calls a "repertoire", the elements of which shape the ways they think and act when assessing pupils' mathematical texts and thus their relevant assessment practice (the term 'practice' is used as suggested by Wenger, that is, to involve the whole person, acting and knowing at once). Gaining insight into the content and structure of this repertoire might be seen as important in understanding how classroom assessment occurs.

THE STUDY

The present study constitutes part of a research project examining primary teachers' pedagogical discourse about basic components of mathematics education developed in various contexts (e.g., in classroom, in informal exchanges, in interview settings and so on). The aim of the project was to study the resources exploited and the positionings adopted by the teachers within this discourse, seeking to understand how they make sense of and put in practice assessment in mathematics.

The focus here is exclusively on the resources emerging in the pedagogical discourse of five primary teachers evolving in the context of a semi-structured interview about mathematics education issues and, in particular, on how these resources frame the ways in which they understand and evaluate students' mathematical texts. The interview consisted of three parts. The first part included questions related to the nature of mathematics as well as to learning, teaching and assessing in mathematics, while the second part focused on teaching and assessment practices in mathematics. Both these parts intended to provide opportunities for the pedagogical discourse to unfold, allowing for the resources exploited by the teachers to emerge. The third part of the interview sought information related to the teachers' mathematics background and professional profile. The pedagogical discourse developed around the questions of all three parts was completed over three meetings, each of which lasted for more than two hours.

All five teachers (4 females and 1 male) were teaching in public primary schools in the north eastern part of Greece and their professional profile varied in terms of teaching experience (5 to 19 years), their University degrees (one or more) and their professional activity (none to high participation to conferences, research programs and short training courses). The teachers were each the subject of a corresponding case study carried out in the context of the main research project, which included classroom observations as well as informal discussions and interviews, like the one under consideration here.

A combination of Grounded Theory and Content Analysis techniques was used for the analysis of the teachers' transcribed discourse (Thornberg & Charmaz, 2012; Berg, 2004), while the categorization/ scheme suggested by Morgan and Watson (2002) was used to identify the content and the structure of the repertoire of resources of each teacher. More specifically, a three stage analysis was followed: (a) careful reading of the data and detection of extracts relevant to each category of the scheme (b) coding of the extracts within each category and grouping them in sub-groups with relevant meaning and (c) repetition of the previous stage within each of the emerging sub-groups for the formation of the third level resources. For reliability reasons, the whole process was realized simultaneously by the two researchers. Finally, the systemic network was used to present the results of the analysis, as it allows a united depiction of the outcome in repeated levels of complexity, rendering thus transparent the internal relations of the data (Bliss et al, 1987).

RESULTS

The data analysis led to the construction of a systemic network for each teacher with regard to the resources s/he was drawing on within her/his pedagogical discourse and are informative of the ways s/he is making sense of and evaluates pupils' mathematical texts. The five individual networks were then combined to form an overall network of resources that depicts the content and the structure of the individual as well as the collective repertoire of resources identified. Due to space limit, only part of this overall systemic network is presented below. In particular, Figure 1 concerns the resource "Beliefs about the nature of mathematics and their relation to assessment" (see Morgan and Watson, 2002 above) which dominated all five discourses and especially its branch that focuses on the teachers' alleged assessment practices in mathematics, which is closely related to how they conceptualize and assess pupils mathematical productions.

Below an overview of the resources utilized and their interconnections is first provided and then particular features of these resources are discussed in relation to how teachers view and practice assessment, substantiated by data extracts.

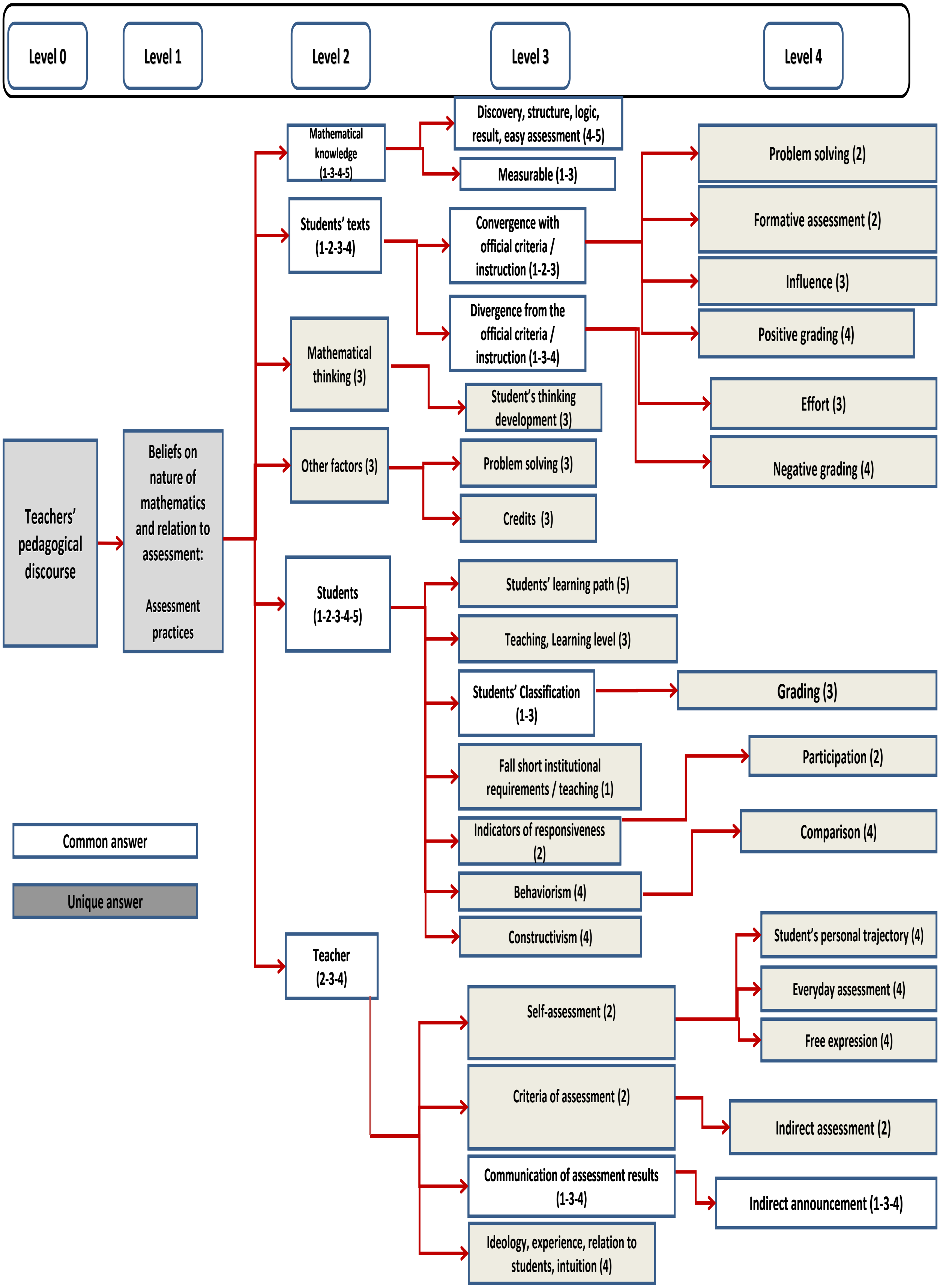

Figure 1. Teachers' alleged assessment practices as a sub-branch of the resource "Beliefs about the nature of mathematics and their relation to assessment" [1=Nikitas, 2=Chrisa, 3=Anastasia, 4=Antonia, 5=Antiopi]

Overall, the analysis showed that the teachers of the sample appear to approach the task of assessing pupils' mathematical texts with a set of predispositions and experiences. The resources they employed in their pedagogical discourse concerned mainly their beliefs about the nature of mathematics and how these are related to assessment, as well as their expectations about how mathematical knowledge can be communicated. In addition, these resources were more collective than personal, were frequently related to the official discourse of assessment and were often incompatible to one another across time and contexts.

The view that mathematical knowledge can be measured characterized two of the teachers' discourses (Fig. 1, levels 2 & 3 – mathematical knowledge).

> I would say, it is more measurable in mathematics (the knowledge)… because, perhaps, we have to deal with more standardised things… What I mean….when we do, for example, simple additions, one succeeds and another does not. While the concept, what you define as love, can be discussed, you just need to convince me. But you can't convince me that 1+1 makes 9, it is a little difficult to convince me. Perhaps this is the reason why assessing in mathematics is different…the results are more expected.
>
> (Nikitas, 16 years of teaching experience, BA in Education & BSc in Mathematics, limited professional activity).

The teachers tended to judge students' mathematical texts on the basis of the degree these satisfy or not the official criteria of assessment, as substantiated, however, within the classroom (Fig. 1, levels 2 & 3 – students' texts). In this context, the assessment process allows the comparative classification of each student according to his/her mathematical behaviour and its divergence from the officially defined, as well as from the one designated as desirable in each class (Fig.1, levels 2 & 3 – students).

> My tests are always in the "spirit" of the textbook, perhaps a little bit different with respect to the way questions are phrased or to the difficulty level. This means you might find much easier but also much harder exercises compared to the ones in the textbook. Not unrealistic (exercises)… neither I do something extreme, nor I look for genius students in the class…. This is not my aim!
>
> (Chrisa, 17 years of teaching experience, BA in Education, high professional activity).
>
> For example, I will assess a student… I will give him two problems and if he solves them with no difficulty and somebody else solves half of it…this is for sure an indicator of assessing differently each one of them…interest and effort is what counts
>
> (Anastasia, 19 years of teaching experience, moderate professional activity)

Teachers' pedagogical discourse often drew on resources related to learning behavior rather than to specific achievements, when reporting on individual pupils; they tended to avoid describing specific features of their texts. Although they valued oral language in assessing, the teachers were not consistent with this view. For most, evaluation was predominately based on a combination of students' written performance, memory and effort made. Furthermore, referred to the importance of knowing the students they were about to assess, as this knowledge provided a framework within which they could interpret students' mathematical texts instead of drawing on informal, loose criteria (Fig. 1, levels 2 & 3 – teacher).

Few teachers articulated a pedagogical discourse that presented some internal coherence. All of them, however, tended to employ discourses which came from different fields, thus allowing conflicts and contradictions to emerge within their pedagogical discourse, which can be summarized as follows: students' mathematics knowledge is not imprinted in the texts they produce, is seen as if it can be measured on the basis of the compatibility of certain features it bears with official ones, which, however are disputed with respect to their effectiveness and the necessity in the classroom, thus questioning the traditional function of the assessment process to classify students according to their mathematics performance (Fig.1, across and within levels).

> The safest index for my students' assessment is their willingness to engage with mathematics. That is, if I see a student, even if he is not good, to want to begin the lesson, to come to the board, to be asked, for me is the safest index
>
> (Antiopi, 10 years of teaching experience, BA in Education, high professional activity).
>
> I am not interested in all these! That is, when I am in good terms with my students, the difficult part is to allocate grades to them and many times I offer higher grades, because I am interested in their relationship with mathematics … It is better for a student to be not always excellent but to be given "10/10", in order to escape home oppression… I also rely on the teacher's guide that describes the objectives …But in mathematics, during my teaching … How do I know what do the students know? … by assigning some assessment tests. I do this often, very often! It might be some short exercises. It might be a problem to solve and is usually something intended for the average student, so they are not blocked. Short tests, small exercises …
>
> (Antonia, 5 years of experience, BA in Education and BSc in Mathematics, substantial professional activity)

Based on the preceding findings, the pedagogical discourses developed by the five teachers appear to share features but also present tensions as well as contradictions and inconsistencies with respect to how the assessment process of pupils' mathematics is understood and allegedly exercised. These discrepancies are due to the different set of resources utilized across time and contexts and emerge not only between teachers but also in a single teacher.

Generally speaking, the teachers' pedagogical discourse tended to draw on resources that come mainly from two opposing discourses (traditional/performance oriented and alternative/ competence oriented respectively) (Broadfoot & Pollard, 2000 drawing on Bernstein), to adopt official criteria, re-contextualised according to each teacher's personal biography in mathematics education and shows overall moderate internal coherence.

DISCUSSION AND CONCLUDING REMARKS

Teachers' assessments are inevitably influenced by a number of factors, neither necessarily relevant to the students' mathematical achievements nor indispensably compatible to the official evaluation criteria. This influences their official, summative assessments of individual students. The results of the study presented here support this and could be attributed to the fact that in the Greek educational system, teachers are not yet required to proceed to assessing students using reliable and clearly defined criteria (officially or personally) and produce judgments for which they will be accountable.

The results also suggest that different teachers can interpret the same text or similar students' texts in many different ways, either because of judging different characteristics as significant or because of attributing different value to similar features. These different approaches might arise in informal classroom assessment contexts as well as in official circumstances. Due to the highly interpretive nature of every assessment activity in mathematics, it is almost impossible for a teacher to avoid such discrepancies. This raises important questions related to issues of equity in mathematics education. More specifically, it brings to the fore the possibility that some groups of students, who do not share the same social, cultural and linguistic background with their teacher-assessor, might be provided with fewer opportunities to access the resources s/he draws in assessing. As a consequence, it is harder for them to produce texts bearing the features expected by the teacher, in order to be highly valued. This way, there is a strong possibility that these groups of students will be systematically disadvantaged because of their failure to exhibit the mathematical behaviour expected and valued by their teacher (Morgan & Watson, 2002).

It is clear that assessment depends heavily on teachers' interpretations of specific students' textual productions. These interpretations might not be compatible with students' intentions, depending on the personal resources exploited by the teacher when assessing. The pedagogical discourses articulated in the context of this study indicate teaching practice as the basic regulating factor in the assessment process, which re-contextualises the official discourse, in order to become functional in practice. This process finally leads to the formulation of a limited number of assessment criteria, mostly inflexible and not necessarily compatible to one another. Thus, assessment is downgraded to a simple "confirmatory" act of what students do rather than of what they are able to do in mathematics, based on criteria which are not always compatible to the official ones.

The matters discussed above illustrate the interpretive nature of assessment in mathematics and provide some insight into the possible sources of different evaluations, which are likely to disadvantage certain groups of students. Such an insight may enable teachers to engage in critical reflection on their everyday professional practice. The recognition of the incompleteness of their awareness of students' behavior, of the potential for alternative interpretations of this behavior and its situated and temporary nature may increase the sources of evidence on which teachers base their judgements. Overall, may help them become aware of the social and cultural forces in effect, their own beliefs and practices included, when success and failure in school mathematics is at stake. Providing experiences and opportunities of reflection towards this direction can be of great value for teachers' professional development.

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HEURISTIC STRATEGIES PROSPECTIVE TEACHERS USE IN ANALYSING STUDENTS' WORK

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In this paper I will describe with what specific heuristic strategies two prospective secondary school mathematics teachers, who are students towards the end of their academic education at the university, analyse a student's work and reconstruct possible thinking processes of this student. The research of these two prospective teachers reported in this paper is part of a qualitative study which investigates prospective teachers' behaviour of analysing students' work consisting of students' written answers to mathematical problem-solving tasks. The prospective teachers were questioned in clinical interviews and videotaped while analysing the students' work individually.

Keywords: heuristic strategies, prospective teachers, teacher education, teacher knowledge, analysis of students' work

INTRODUCTION

Examining students' work and their thinking in order to give feedback is an important part of effective teaching. For future teachers it is important to acquire diagnostic competence in order to understand and assess students' answers with the aim to make appropriate pedagogical and didactical decisions (Hußmann et al., 2007). This study draws on the process of analysing students' work. One of the main aspects of the study discussed in this paper is how prospective teachers analyse students' work and what they take into account. Particularly the following question is addressed: Which heuristic strategies do prospective teachers use in analysing students' work consisting of students' written answer to mathematical tasks?

THEORETICAL FRAMEWORK

Over the last couple of years many research projects have addressed teacher knowledge. Ball et al. (2008) discussed what makes Mathematical Knowledge for Teaching (MKT) special. They include among this, finding out what students have done, interpreting students' errors and assessing whether the thinking and approaches are mathematically correct and would work in general.

Examining students' work and thinking should be a part of everyday teaching practice (Borasi et al., 2002). Mathematics teachers "need to know how to [...] analyse students' solutions and explanations." (Hill et al., 2005, p. 372) Hill et al. (2005) include among the work of teaching mathematics, among others, "interpreting students' statements and solutions" (p. 372). The Teaching Principles of the National Council of Teachers of Mathematics (NCTM) Standards include "Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well." (NCTM, 2000, p. 2)

According to Hattie (2009) powerful teaching also includes that the teacher knows what the students understand, when they have misconceptions, and where they make errors.

In fundamental ways Schoenfeld (2011) views "teaching as a much more complex problem-solving activity" (p. 3). It is a challenging task for teachers to know how to deal with students' oral or written mathematical utterances, what to pay attention for (Crespo, 2000), and to understand what they are saying and doing (Wallach, 2005). Especially when students employ unusual strategies of solving a task, for example with nonstandard approaches which are unfamiliar to the teacher (Ball et al., 2008), students' thinking is not always directly obvious and easily comprehensible. Teachers have to figure out the students' thinking process and what they have done (ibid.). There is a need for teachers to use heuristic strategies to reconstruct possible thinking processes with the aim to understand the students' work. In the analysis of students' work possible thinking processes must be reconstructed in hypothetical, empathetic thinking by interpreting the work.

Crespo (2000) found in a study with elementary preservice teachers that they struggle with interpreting students' work. The elementary preservice teachers tend to evaluate the students' work immediately without analysing it carefully. In my study I had chosen prospective secondary school teachers towards the end of their academic education because I am interested in to what extent the academic education of secondary school teachers prepares for mathematical requirements needed in the analysis of students' solving processes.

Schoenfeld (2011) claims "that what people do is a function of their resources (their knowledge, in the context of available material and other resources), goals (the conscious or unconscious aims they are trying to achieve) and orientations (their beliefs, values, biases, dispositions, etc.)." (p. xiv) The attempt in this study is to regard analysing students' work as a function of knowledge and resources, of goals and of orientations. What people do in this study is analysing students' work. In the following I describe which kinds of knowledge could have influence on these analyses.

Analysing students' written work can be based on Pedagogical Content Knowledge (PCK), Content Knowledge (CK), and problem solving as mathematical competence (Figure 1). Shulman (1986) includes among Pedagogical Content Knowledge "the ways of representing and formulating the subject that make it comprehensible for others [...] [and] the conceptions and preconceptions that students of different ages and backgrounds bring with them" (Shulman, 1986, p. 9). Content Knowledge includes knowledge of the subject and its organising structures (Shulman, 1986). A teacher "need not only understand that something is so; the teacher must further understand why it is so" (Shulman, 1986, p. 9). Problem solving as mathematical competence is another important aspect for analysing students' written work as semiotic representation of the student's problem-solving activities. Reconstructing possible thinking processes of the student can be based on identifying specific problem-solving strategies in the student's work.

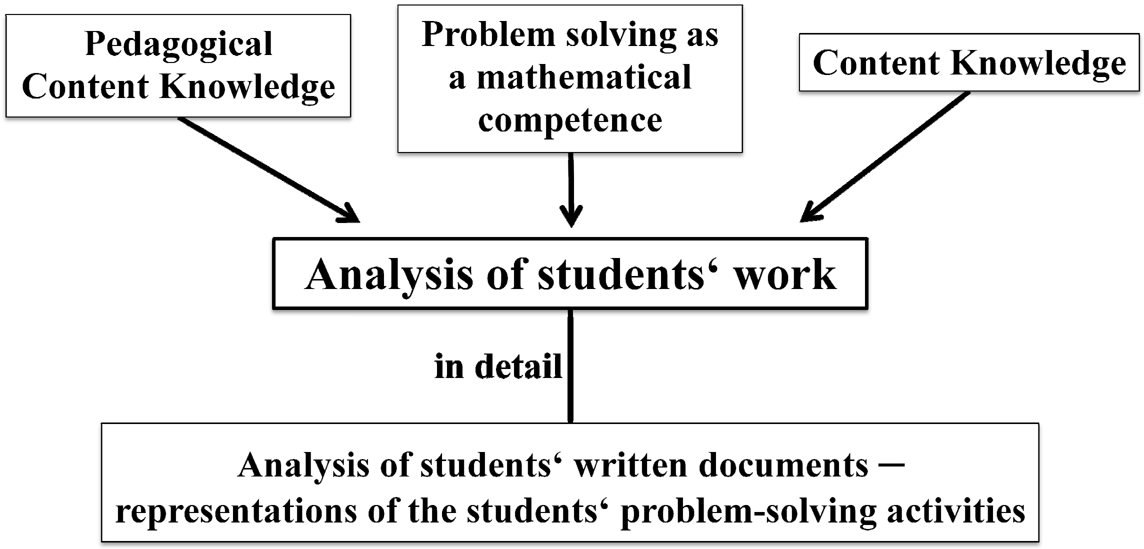


Figure 1. Analysis of students' work

In the process of analysing students' work it is in some cases necessary to change from one representation system to another, for example, if the student has chosen two different representation systems within his or her solution process. Duval (2006) distinguishes two different types of transformations of semiotic representations: treatments (transformation from one semiotic representation to another in the same register) and conversions (transformation from one semiotic representation to another in a different register).

METHOD OF INVESTIGATION AND ITS JUSTIFICATION

Research design and sample

The two cases described in this paper are situated within a qualitative study of 19 prospective secondary school mathematics teachers, towards the end of their academic education at the University of Oldenburg, Germany. In Germany the teacher education is composed of two parts. The first part is an academic education at the university and the second part is a practical training in teacher seminars and in school.

In a two step design prospective teachers were asked to solve three different mathematical tasks, and to analyse one student's written work to each of these tasks. More precisely, the prospective teachers should reconstruct possible thinking processes of the student and propose feedback to or support for the student. This should be done individually with unlimited time. The mathematical tasks have multiple ways of solving and differ in their mathematical content. Some of the students' written answers are fictive and others are real. These written answers of students to the different tasks include different ideas and ways of solving, e.g. graphic or algebraic.

The data includes on the one hand written work of the prospective teachers concerning their answers to the mathematical tasks and their analyses of students' work; on the other hand videotaped, transcribed material of oral comments while solving tasks and analysing students' work and a semi-structured interview afterwards. To understand their ideas in more detail, the prospective teachers were asked to explain their work and their hypotheses on the students' thinking processes in an interview. I had chosen this combination of written and oral form, because to one hand the participants could think more profoundly about different aspects to write down, and they were not in hurry to answer something quickly; on the other hand to get also data about what they perhaps have already thought concerning the student's work but did not write down. Additionally the prospective teachers are working with the students' work twice thereby they can get more ideas.

In this paper I describe the heuristic strategies two prospective teachers (Marc and Tim) show in analysing Lilly's written work to the task "Find the solution set", one of the three different mathematical tasks. I want to describe the analyses of these two participants because they use some identical strategies and some which are completely different from each other. In comparison with the other 17 participants of the whole study, the strategies of these two cases are also used by other participants.

The mathematical task and Lilly's answer, which were given to the participants, are shown in Figure 2.

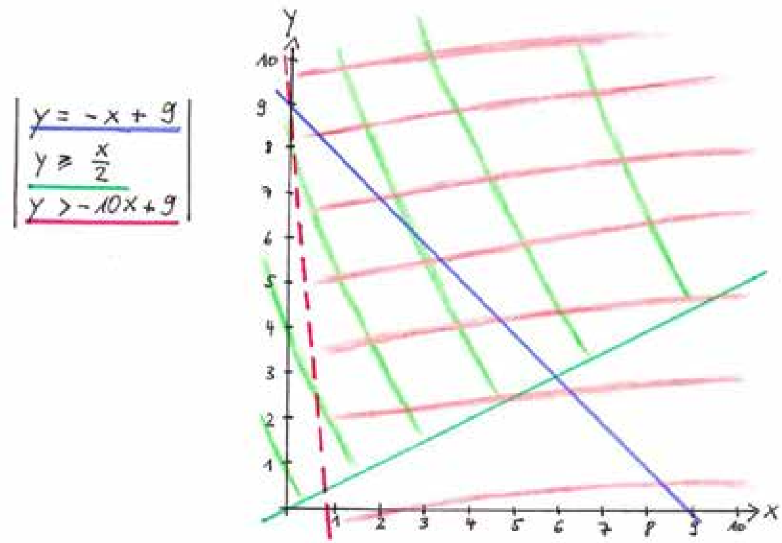
Mathematical problem "Find the solution set"

Find the solution set of $\left|\begin{array}{lcr} x+y & = & 9 \\ x-2y & \leq & 0 \\ 10x+y & > & 9 \end{array}\right|$ *where x and y are natural numbers.*

Describe your approach and document your way of solving as accurately as possible. What alternative approaches do you know? Carry them out.

Lilly's work to this task

Lilly has rearranged the system and created the following illustration in a Cartesian coordinate system.



What thinking process do you conjecture in Lilly's work?

Figure 2. Mathematical problem "Find the solution set" and Lilly's work

For this task content knowledge about the meaning of such a system, the solution set, and the concept of variables is needed. Depending on the chosen solution strategy more content knowledge is required, such as the substitution method or knowledge of geometric illustrations. German students are familiar with systems of linear equations, but not with dealing with linear inequalities. They have to transfer their problem-solving strategy for solving a system of linear equations to a system with inequalities or to think about another solution strategy where mathematical process skills will be needed.

Lilly has rearranged the inequalities and the equation for y by treatment within the algebraic register. She changes into the graphic register (conversion according to Duval, 2006) with her other representation. In the xy-plane the solution set of each inequality and equation is graphed as lines or areas for x and y $\in \mathbb{R}$. A possible reason why the line for y=-10x+9 is drawn as a dashed line could be that the set of the line is not included in the set of y>10x+9. The missing solution set of the complete system can be found graphically as the intersection of the three individual solution sets of the equation and the two inequalities. Since x, y $\in \mathbb{N}$, the solution set consists of the following ordered pairs: IL={(1/8),(2/7),(3/6),(4/5),(5/4),(6/3)}.

Data analysis

To find out heuristic strategies the prospective teachers use in analysing Lilly's work I go through the written and transcribed material in combination both line by line for analysing and interpreting significant statements and strategies with the aim to expose categories for heuristic strategies.

ANALYSES OF LILLY'S WORK

In the following I want to describe the strategies that two prospective teachers (the case of Marc and the case of Tim) use to analyse Lilly's work. The prospective teachers had unlimited time to analyse Lilly's work, both of them worked on it ten minutes. Before analysing the student's work the prospective teachers were asked to solve the mathematical task on their own.

Marc

Marc chooses in his own solution the substitution method (a target-aimed method for this task). He rearranges the equation for x and replaces it in the inequalities. His way of solving indicates the safe use of algebraic transformations and solving inequalities. In his solution process, he shows strength in formal-operational activities. Marc finds out two correct restrictions for x and for y (namely $0<x\leq 6$, $3\leq y<9$). However, he refers to the restrictions found for the variables separately, but not combined as ordered pairs and without regard to the condition $x+y=9$. Marc claims that he cannot write down a solution set.

Marc explains what thinking process he conjectures in Lilly's work after a few minutes looking at her written work and writing down some aspects. Marc mentions:

> Marc: She sees in y this function and tries to write down every equation, which is here [points at the system with equation and inequalities], as a function and tries to solve it with a drawing and yes, she has done it right. […]
>
> She is now thinking the solution set is everything which is on the line. At this it is all what is over it and for y3 it has to be definitely bigger, therefore she has drawn it as a dashed line, because it doesn't include this value thus this line.

Lilly has rearranged the inequalities and the equation for y. Hence he concludes that she will regard it as a function. A possible reason for this could be that her rearranged conditions remind him of the standard presentation of functions. Before commenting this orally he wrote down

- Lilly sees in y the function f(x), rearranges the equations for y
- equation=function

This indicates that he sets himself a framework namely the context function as a fundamental idea concerning Lilly's work. This fundamental idea of function occurs also at the end of his analysis.

In his written and in his oral analysis he numbers consecutively the three conditions for y in the algebraic representation of Lilly's work (y3 for the third condition). He hereby separates the conditions for y from each other (deliberately or not).

Marc tries to reconstruct the action by splitting up Lilly's work and analysing it separately. He splits up the solution in two ways: 1) At the beginning of his analysis he thinks in the algebraic register and changes into the graphic register (Duval, 2006). 2) Marc interprets the equation and the inequalities separately as straight lines and as designated areas in Lilly's graph.

He interprets elements of the solution locally like the graphed solution set of the equation and each inequality as lines or areas. In his analysis of Lilly's work he is influenced by his attitude towards the correctness of the work. Some of Marc's statements indicate that he thinks something is wrong but he is unsure. Nevertheless he takes it seriously and tries to understand Lilly's approach.

> Marc: What would happen now, if I take a value for x, which is smaller than one? […] She doesn't need to regard this, it is completely indifferent for her.

In this excerpt his analysis goes on with using a "what if"-strategy as occasion to think from Lilly's perspective. This approach does not advance him in reconstructing the thinking process. He rejects his strategy because he thinks that it is irrelevant. Later he thinks about how Lilly could go on working on this or how he would help her exactly.

> Marc: Perhaps one can go on working on this but I don't know it now. […] What would be, if I interpret it with my solution? So, I don't know, how do it with that, but if I take my solution to interpret it in connection with this solution.

Following this he tries to deduce the thinking process in Lilly's work with the help of his own solution. He shows his conditions ($0<x\leq 6$, $3\leq y<9$) in Lilly's work at the corresponding axes, but he does not respond to a combination of x-and y-values with respect to the solution set. Marc finds his conditions in the graphic representation, but he does not understand Lilly's representation. It can be said, that he does not succeed in understanding how the graphic representation of Lilly is related to the determination of the solution set of an algebraic system. His main problem is the mathematics. In the analysis of Lilly's work, he shows a weakness in the cross-linking by not associating his algebraic solving method with the graphic determination of the solution set of an algebraic system. So, he cannot complete Lilly's solution idea or reconstruct a possible thinking process completely even with the help of his own solution.

Marc can explain details, but he cannot relate the three conditions (equation and inequalities) in Lilly's work because he describes three solution sets. It shows that this is an analogical problem as in his own solution because he does not relate his two found conditions. Altogether he does not grasp the complete strategy of Lilly's approach, but rather the dealing with the separated conditions. He interprets some aspects of the solution locally.

Summary of Marc's strategies:

- sets himself a framework
- reconstructs the action by splitting up the work and analysing it separately
- deals with representations
- "what if"-strategy as occasion to think from Lilly's perspective
- uses his own solution to interpret her work

Tim

In his own solution, Tim makes use of the substitution method by rearranging the equation for x and replacing it in inequalities. Tim exploits two correct conditions for y ($y\geq 3$, $y<9$). He writes down a solution set for y ($y=\{3,4,5,6,7,8\}$) and for x

($x=\{1,2,3,4,5,6\}$) in a way that x and y are separated. He does not write down ordered pairs (x, y) regarding the condition $x+y=9$.

By analysing Lilly's work, Tim mentions:

Tim: First she tried to put the whole thing down to something familiar namely to a linear system of equation and then look at each as intersection of two lines.

When Tim describes Lilly's approach, he identifies Lilly's strategy which I call "drawing on something familiar".

Tim: This matches. This intersection point [points at P(6/3) in Lilly's graph] x equals 6 is ok and also y which is 3. Thus the points [points at his solution sets] are included.

Tim refers to intersection points. He checks if the intersection point (6/3) that he looked at in Lilly's graph is included in his own solution because he intends to verify Lilly's work. Then he states that his solution and Lilly's graphic representation is consistent. This excerpt is an indicator for the strategy "referring to his own solution".

He says that Lilly did not mark the solution set and that he does not quite see it. Following this he mentions:

Tim: She didn't understand the question of the task which is to find the solution set [points at the task] which fulfills this. This she didn't do exactly because [...].

By referring to Lilly's work Tim points out that Lilly's solution is not completed, when he explains that Lilly did not answer the task. He assumes that Lilly did not understand the question and takes the graphic representation into account. He points at the point (0/9) and says that this would be possible in this representation, but this does not fit to the fact $9>y$ (he points in the direction of the task), namely that it is only included up to the point (1/8) at which he is pointing. It indicates that he examines the correctness of the solution like he has done at the beginning of his analysis. Possibly Tim refers to his own solution again (however implicit) because he knows that the point (0/9) is not included in the solution set.

The same as in the case of Marc Tim applies the strategy of splitting up the solution. Thereby he focuses on single transformations and changes from algebraic register into graphic register. Later he mentions that something special is the dependence in the solution set. Take it together it has to be nine in each case.

Summary of Tim's strategies:

- identifies Lilly's strategy "drawing on something familiar"
- reconstructs the action by splitting up the work and analysing it separately
- deals with representations
- verifies Lilly's solution by referring to his own solution

Comparison of the two cases

Marc and Tim use some identical strategies to analyse Lilly's work (reconstruct the action by splitting up the work and analysing it separately; deal with representations). Both of them refer to their own solutions but in a different way. Marc uses his own solution to interpret Lilly's work whereas Tim refers to his solution to verify special aspects of Lilly's work.

Marc has difficulties in understanding Lilly's work and thinks something is wrong. However, he is unsure, so he takes it seriously and tries to understand Lilly's approach. He interprets details of her work locally, but cannot reconstruct a possible thinking process completely even with the help of his solution. Tim gives greater consideration to Lilly's strategy which he identifies in her work. Altogether he gives the impression that he rather verifies Lilly's work.

CONCLUSIONS

Concerning the answer of the research question the analyses of the two prospective teachers were analysed to find out their heuristic strategies by which they analyse a student's work and reconstruct possible thinking processes. The results of these cases show that analysing a student's work evokes specific heuristic strategies and that different heuristic strategies can be reconstructed. Marc and Tim use similar and different strategies in their analyses. Both of them use the strategy "refer to the own solution" which seems to be a typical strategy. "Reconstruct the action by splitting up the work and analysing it separately" as another heuristic strategy is one which is used frequently with the aim of understanding students' work, especially if the students employ unusual strategies of solving a task.

With the help of the whole study of all participants further strategies and their detailed descriptions can be investigated. All strategies of each prospective teacher will be regarded individually in order to find out different types and to identify different pattern. So far, the analyses of the data show that some prospective teachers have difficulties in analysing students' work and tend to evaluate the students' work immediately without analysing it in a deeper way. Other prospective teachers try to reconstruct the student's thinking processes carefully and do not only describe obvious activities of the student, for example describe not only written calculus, but also possible ideas of the students' solution strategy. These differences which may be caused in different orientations, strategies or differences in available knowledge need to be investigated. This can make contribution to the development of the academic teacher education by making proposals for example of analysing students' work what can be regarded in a didactic course at the university. Some prospective teachers can improve their analyses by knowing further strategies to be more flexible in their analyses of student's work. To pay attention to the mathematical thinking of students, try to make sense of what they are saying and doing is an important part of effective teaching and can help teachers to improve their practice and students' learning.

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RICHNESS AND COMPLEXITY OF TEACHING DIVISION: PROSPECTIVE ELEMENTARY TEACHERS' ROLEPLAYING ON A DIVISION WITH REMAINDER

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Division, one of the most difficult topics to teach in elementary mathematics, is rarely examined from the perspective of student teachers engaging with peers and making use of mathematical or pedagogical knowledge in teacher-pupil interaction. In this paper, we examine prospective elementary teachers preparing, realizing and discussing a role-play on division with remainder. Using the concept of "*structure of attention*"*, we offer a positive understanding of prospective teachers' performances in terms of navigating the richness and complexity of teaching division.*

INTRODUCTION

Division is known as one of the most difficult topics to teach in elementary mathematics (e.g. Salama, 1981; Silver *et al*., 1993). Thus, many studies examine preservice teachers' understandings of division (e.g Graeber & Tirosh, 1988; Ball, 1990; Simon, 1993; Campbell, 2002; Crespo & Nicol, 2006). Globally, researchers characterize those understandings in terms of actual or expected mathematical knowledge, stress on student teachers' difficulties, and occasionally offer alternatives to improve or extend their understandings of division and standard algorithms. However, those researches rarely examine student teachers as they engage with peers or make use of mathematical knowledge of division to teach (Simon, 1993).

Role-play involves staging a problematic situation with characters taking roles. It may be used to fulfill various objectives such as therapeutic objectives, personal and professional training objectives, or may be used as a pedagogical method (Mucchielli, 1983). The premise of role-play is to have persons, such as students, become active characters in a given situation. In one of our "mathematics method" courses, for example, students take the part of a teacher while others act as students, and they improvise around a mathematical task, a students' question or production, the use of teaching material, and so on (Lajoie & Pallascio, 2001; Lajoie, 2010). Since role-play asks student teachers to become active actors in different teaching situations, instead of simply imagining or analysing such situations, it provides an approach to research on mathematics teacher education that matches both of Simon (1993) demands.

In this paper, we report observations we made on a lesson in which student teachers prepared, performed and discussed a role-play on division with remainder. Bringing forth the richness and complexity of teaching division, we discuss how prospective teachers' *structure of attention* (Mason & Spence, 1993; Mason, 2003) played out in their actual use of resources such as mathematical knowledge, contextualization, or models such as the partitive and quotitive orientation to division.

BACKGROUND

Division with remainder presents a real mathematical challenge for prospective teachers. The division algorithm involves many arithmetical concepts, including positional, decimal numeral system, associativity, multiplication tables; while the remainder in itself opens to deeper questioning in terms of fractions, periodicity and infinite operations, and so on. In a somehow deficit perspective, previous research on the topic in the context of teacher education, giving attention to what might be missing rather than to the richness and potentiality of what is actually taking place, suggests that preservice teachers hardly make use of these concepts when confronted with division and its standard algorithm. More specifically, a gap is seen to exist between the conceptual and procedural levels (Silver, 1986) whereas:

> (…) the prospective teachers' conceptual knowledge was weak in a number of areas including the conceptual underpinnings of familiar algorithms, the relationship between partitive and quotitive division, the relationship between symbolic division and real-world problems, and identification of the units of quantities encountered in division computations (Simon, 1993, p. 233).

In this quotation, Simon also refers to two “primitive” models of division, the *partitive* (sharing) and the *quotitive* (grouping or measuring) models. As a way to student teachers’ understanding of division, Fischbein *et al*. (1985) use these models to observe that even with a “solid formal-algorithmic training”, students continue to be influenced by these intuitive models. Attachment to the models could explain a number of difficulties arising in unfamiliar problems, such as division by zero (Lajoie & Mura, 1998), because those models do not always constitute a solid conceptual basis (Simon, 1993). In addition, Kaput (1986) suggested an important mismatch between the dominant partitive experience of division and the quotitive approach generally used to teach division algorithms. Boulet (1998) confirmed this tendency in preservice teachers, noting that while the sharing model largely dominates when making sense of a division in real context, the measurement model clearly takes on when verbalising the algorithm. Finally, research pointed to student teachers’ struggle to interpret remainders or the fractional part of the quotient (Silver, 1986; Simon, 1993). With a closer attention to teaching situations, some highlighted multiple possible treatments of the remainder, and turned our attention to the importance of being able to deal with children’s ideas while making connections between the concrete, the symbolic and the algorithmic dimensions of the operation (Fang, Lee & Yang, 2012). Campbell (2002) observed that a quotitive disposition toward division also seems to impact future teachers' ability to make sense of the remainder, while confusions between the remainder and the quotient of a division may also appear when student teachers are confronted with unfamiliar tasks (e.g. reconstituting the remainder from a calculator). As we can see, literature on the topic is rich in conceptual analysis in relation with division. But less is said about the ability actually to call on, articulate and make us of these underlying concepts and rooting metaphors while teaching.

ELEMENTS OF A CONCEPTUAL FRAMEWORK

Teaching involves various kinds of knowing (e.g. Shulman, 1987), among which "pedagogical" and "content knowledge" attracted much attention in the community. But beside knowledge "about" teaching and mathematical concepts, a growing attention is given to what some call "know-how" or "knowing-to act in the moment" (Mason et Spence, 1999): the ability to draw on various knowledge in response to actual situations. Mason and Spence (1999) suggested that such ability "depends on the structure of attention in the moment, depends of what one is aware of" (p. 135). "Structure of attention" is concerned with what is attended to and how (Mason, 2003), and its development can be conceptualized at two levels: learners need to gain access to increasingly sophisticated structures of awareness and attain greater flexibility to allow their attention to be multiply structured. Experiencing situations in which a rich network of triggers and connections come about and can be rendered explicit is considered fundamental to this development.

> Educating this awareness is most effectively done by labelling experiences in which powers have been exhibited, and developing a rich network of connections and triggers so that actions 'come to mind'. No-one can act if they are unaware of a possibility to act; no-one can act unless they have an act to perform (Mason & Spence, 1999, p. 135).

Hence, it is important for the students to be in the presence of someone, like an instructor, who is aware of the awarenesses (Mason & Spence, 1999).

> Knowing is not a simple matter of accumulation. It is rather a state of awareness, of preparedness to see in the moment. That is why it is so vital for students to have the opportunity to be in the presence of someone who is aware of the awareness that constitute their mathematical 'seeing' (Mason & Spence, 1999, p. 151).

As an element of a conceptual framework underdevelopment in order to understand the competencies future teachers develop in "maths method courses" (e.g. Lajoie, 2010; Maheux & Lajoie, 2011), the concept of "structure of attention" enables us to look into their ability to make use of various knowledge. In this paper, we exemplify this and the potential outcome of such approach in a case on division with remainder.

METHOD (RESEARCH CONTEXT AND DESIGN)

As part of our research program on teacher education, we decided to investigate division with remainder in a study involving 40 preservice teachers enrolled in a 45 hours undergraduate course (3h weekly) on *Didactique de l'arithmétique au primaire* [primary arithmetic maths method]. No selection of the students was conducted, beside a general appreciation of the group (as whole) as typical of those we worked with for the last decade. Students were in their second of a four-year program, and had completed a first mathematics course. The course used for this study was designed around ten different *role-plays* (Lajoie, 2010) on various topics including numeration, operations and algorithms, fractions and decimal numbers.

Each *role-play* is organized in four moments. First, we introduce the 'theme' on which students will need to improvise (introduction time). Students then have about 30 minutes to prepare in small groups (preparation time). Third comes the play itself, where students chosen by the instructor (and coming from different preparation teams), come in front of the classroom and improvise, in an informed way (Maheux & Lajoie, 2011), a teacher-pupil interaction (play time). Finally, we have a whole classroom discussion (discussion time). Importantly, students thus have a preparation time to consider what might happen between a teacher and a pupil, but the role-play is essentially improvisational since there is no script and students learn only minutes before the play if they will be performing that day, and what role they have to take. We designed, taught (first author) and videotaped (second author) the three hours role-plays using two cameras (one capturing the whole classroom setting and interactions, the other following some teams more specifically). In this paper, we will focus on the role-play presented in Figure 1, which was also transcribed.

Role play theme

Fifth grade students (10-11 years old) who worked on a problem involving division with a remainder are sharing their answers. You observe mathematical difficulties in their approaches. You wish to engage with them in such a way that you can build up on their mishaps and help them move toward correct answers. Here is the problem they have been working on:

> A student from the other class made a mistake while dividing 18181 by 9. Here is his answer. Can you help find his error?
>
> $18181 \div 9 = 22{,}111$ remain 1.

Your students worked in team, and among them Team A and B arrived at the following explanations:

- Team A: The answer is 2020 remain 1/18181
- Team B: The answer is 2020 remain 0,111

Prepare yourselves to either play the role of a student from Team A or B, or that of the teacher who wants to help them starting from their reasoning.

Figure 1

That particular lesson, involving division with a remainder, took place midway into the term (when the students felt comfortable with role-play). As homework, the week before, they had to read Boulet's (1998) paper on the verbalisation of the division algorithm in terms of partitive or quotitive models, which were also discussed in class. The role-play (Figure 1) was designed to bring about multiple mathematical concepts potentially involved in (long hand) division with a remainder. This includes the fact that no context was provided for the division itself, so that preservice teachers would not be directed towards a partitive or quotitive model. To collect data, the moving camcorder mostly follows a team of four, part of which Justine will end up playing the role of the teacher. We thus focus on the interactions taking place in that team (during preparation time), on the play itself (play time), and on the whole group discussions (discussion time).

ANALYSIS

During preparation time, Justine's team quickly engaged in figuring out the correct mathematical answer to the division problem under investigation, and then turned to what should be done in regard with Team A's solution:

Cindy. We could ask them what they did, and they may answer that they have 1 remaining unit over the 18181. But that's not really it, because the unit, you have to divide it in 9, but you can't divide it in 9. (...)

Bella. When.. if you only look at this… it is part of the 18181… from the moment you decide to divide by 9… actually, when you see it written here it is still a unit part of your 18181, but when you write the result of the division, you can decide that it will stay as a unit, or you decide to write it as a remainder of 1 that you are going to divide by 9.

Justine. If you write 1, its because you decide not to divide it. You keep the whole unit.

Attending to the operation as presented, the students here stay in the symbolic domain, focusing on what appears as problematic in Team A's answer: their writing of the remainder. Inasmuch, students in this discussion did not *explicitly* make use of the sharing/partitive or grouping/quotitive models despite the classroom discussion that preceded. More so, they discuss the problem indistinctively using the "dividing in" and "dividing by" expressions which might refer to two different views on the division and its remainder. That is, as we can anticipate, exposure to the models through literature or classroom discussion does not spontaneously translate into resources for designing a teaching intervention.

When prompted by the second author (holding the camcorder), who asked them to explain what the remainder could represent, the students moved from the symbolic to the contextual level. They gave flesh to the problem in terms of "sharing candies between 9 children", clearly leaning toward a partitive orientation. If this contextualisation seemed to help them to confirm a relationship between the dividend (18181) and the remainder (1), Justine's team, however, did not anticipate the challenge of drawing on this model to verbalize the actual division. Nor did they consider where it would lead them, were they to continue the division.

As a result, during play time*,* when the role-play took place and Dominique – the student teacher chosen to play the pupil – tried using a similar context to explain her division, Justine did not react to her problematic verbalisation of the algorithm:

Dominique. So I want to divide 1 candy between 9 friends, but I can't [writes 0 in the quotient]. So I'll take the 8, and now I have 18 candies to split between 9. Each will have 2. Now I subtract and I take the 1. Now if I want to divide it again between 9… well I place a 0 here [in the quotient] and I'll put my 1 there. Now again I take the 8, 18 candies between 9 friends gives them 2 each. 18-18 is 0. I have 1 left to take. If I want to divide it between 9 friends it doesn't work [adds another 0 to the quotient, now 02020]. I have 1 candy left over the 18181, so 1/18181 [writes it next to the quotient and add "remain" between the two].

Justine. Ok, so what you are telling me… If we draw this like… Wait! Actually, what you are telling me is that I'm left with 2020… no, 2020 remaining 1 over the whole 18181. So your reasoning is good, but we'll illustrate that another way. […] Say you have a chocolate bar left. What did you do all the way through? You were dividing by?

Focusing on the remainder, she did not pick up on Dominique successively treating all "1" and "8" as units (she calls the first 1 in 18181 "one candy" instead of 1 group of 10000 candies, etc.). That is, she did not make use of the model, the context, nor the mathematical concepts at play to notice and act upon the problematic, highly procedural verbalisation of the algorithm. Justine responded to the student in the way she explored the problem in her group, the structure of her attention (to use Mason and Spence (1999)' expression for the condition of knowing-to act in the moment) being directed toward the problem of the remainder.

On the other hand, the context and partitive orientation to the problem are made use of when Justine guides Dominique to give meaning to 1 as a remainder. But when the time comes to consider 1/9 in relation with the quotient and what was previously the remainder, Justine's attention is all to the possibility of dividing this remainder which was at the center of her team's preparation. Hence, she does not pick up on Dominique (and her own!) confusion, calling 1/9 the remainder:

Dominique. So I have a chocolate bar and I divide it in 9 equal parts [drawing a 3x3 rectangle].
Justine. Yeah. I have… I have one left, one part … over nine. (…)
Dominique. Before I had 1/18181 but now I'm at 1/9. A remainder of 1/9.
Justine. So your remainder is 1 on 9, and could I… You agree that we divided the chocolate bar we had left in 9, so its like 1 divided by 9. (…)

This is particularly interesting because Justine actually faced the very same problem during preparation time, some time *before* they contextualized the problem. When the fraction 1/9 first came about, the second author, observing the team, asked Justine if 1/9 was the remainder. After a hesitation, she worked it out on paper and concluded that this was not the case: "there's nothing left, no remainder".

The question of the remainder came back minutes later, after Justine and her teammates move to Team B's solution ("2020 remain 0,111"). But first, we can see how confrontation with the decimal development took them to a procedural approach:

Justine: Here the students say that they have a remainder of 1, and because they cannot divide it anymore, they add a period because they are no more in the integer, they are in the decimal. They say "I add a zero, it give us one tenths. Like 10, I divide it by 9, it fits in only once."

In contrast, the first part of their preparation, with its concerns about the relations between the "1" and the "18181", now appears quite conceptual! We also see students moving back to the symbolic domain, trying to make sense of the division in algorithmic terms. And although they had just worked out a context (sharing candies) that could give meaning to their work on the decimals part (turning the remaining unit into ten tenth to be shared among 9 friends), they did not use it. And when asked to do so, Bella explained “You divide the candy in 9, and there is a little bit left”. This curious formulation (was the candy divided in 9 or in 10?), rephrased for her by the camera-holder as “divide it into *10*, what you have left is 0,1”, then led to an actual explanation of the residue: “You are always left with something, but maybe its one tenth, maybe one thousandth…”.

This brings to light not only the difficult task to linking the algorithm with a concrete situation, but also the particular challenge of articulating the remainder, the fractional part of the quotient, its decimal expression, and presence of a “remaining part” yet to be divided. This complexity was at the core of the role-play setup we presented to the students (during introduction time), involving many of the possible confusions, including dividing by 9 and dividing in 10 (so that sharing or grouping in 9 becomes possible). And indeed, when Justine plays with Dominique, she soon comes to a point where she asks her “pupil” to transform the fractional part of the quotient (1/9) in its decimal expression. As Dominique struggles, Justine offers to try and divide 100 by 90 instead: “you have bigger numbers but it’s the same”. Dominique then finds her way to mechanically proceed to the division (finding “1.11”), which Justine uses to emphasise the infinite development of the decimal part. Linking it with the answer, Dominique as a pupil was supposed to be answering to (“18181 ÷ 9 = 22,111 remain 1”, see Figure 1), Justine offers an interpretation of the “remain 1”. As she explains to Dominique, “there is still something to divide”, but “we don’t have a chocolate bar anymore”: the decimal part is thus something already divided, it belongs with the quotient (in contrast with Team B’s answer). After what she concludes to Dominique’s original division: “it's the same here.”

Along the conversation, Justine thus again, passes up the algorithm to keep focus on the decimal development of 100 ÷ 90, so much so that she also cuts through the fact that its quotient is actually ten times bigger than the one they were initially looking for. This could be analysed in terms of a lack of mathematical understanding, or as an effect of insufficient understanding of the division, its algorithm, of the pedagogical importance of making sense of computations, and so on. However, we must appreciate Justine’s consistency in attending to what emerges from the task at hand in the moment. The sensitivity to notice, the instinct to call upon, and the flexibility to make use while keeping track of one’s intention: those are complex, intricate dimension of attending that can only slowly develop. It seems possible, however, to correlate this development with the very design of the situations student teachers are presented with. Here, a task in which they were confronted with “extremes” in terms of possible interpretation of the remainder of a division, took them to an extended sense of the relations at play. During discussion time*,* when asked how they felt about the role-played interaction, students said “it was very good”, but agreed (Justine included) that there was something missing in the part where she began dealing with the 1/9 and the decimal part.

Summing up, we could say that Justine made use of mathematical understanding, the contextualisation and the intuitive models she actively experienced in preparation time, and also *as* they appeared in that moment. The problem as presented to the students (Figure 1) also seems to have largely oriented the students' preparation. So much so that it rendered invisible even what might appear as obvious aspects to consider (contextualization of the problem, verbalisation of the algorithm, interpretation of fractional part of the quotient), all elements the students discussed previously in whole class discussions and read about in the article we gave them.

CONCLUSION

The analysis we conducted in this study illustrates that the complexity of teaching a mathematical subject goes way beyond the complexity of the mathematical subject itself. The analysis allowed us to consider student teachers' ability to deal (before, while and after teaching) with the richness and complexity of a difficult subject in a positive way. That is, we came across the *multiple challenges* (prospective) teachers are presented with when preparing or actualizing teacher-pupils-like interactions on division with remainder: making use of models, contextualization and mathematical knowledge, making sense of the remainder and navigating through various forms of expressing the fractional part of the quotient, or attending to and verbalising the algorithm.

We think this is an important nuance when comparing with, for example, Simon's (1993) analysis stating that prospective teachers "do not make use of their concrete conceptual knowledge" to make sense of division computation and rather seem to "search for meaning within a narrow procedural space, reiterating rules for the procedure" (Simon, 1993, p. 248). As we see it, student teachers work with/in situations that strongly contribute to a structuring of their attention, which leads to the observed behaviour. An interesting question would then be: what may it take for the situation to allow students to have even more conceptual issues co-structuring their attention? (How) could mathematical understandings, contextualisation and models be made part of the situation so that student teachers actively use them in preparing or realizing an intervention? Giving the usual conditions of 3 hours x 15 weeks maths method courses, how readily available could those elements be so they play out as actual resources for students? How to ensure, in doing so, that those structuring resources do not become *too* constraining, leading student teachers to a technical view on preparing and performing mathematics instructional interactions?

It may feel easy to answer that student teachers should be *explicitly requested* to use various resources when they engage in preparation and the role-play. Although this might be the case, we find interesting to contrast what we observed in this study with similar claims. In a research design that used images and stories of real classroom teaching to help prospective teachers develop and practice the problem solving and decision-making skills they need, Crespo and Nicol (2006, p. 93) note:

> Although we acknowledge that this issue relates to the wording in this task, the preservice teachers' lack of mathematical exploration before designing a pedagogical response is concerning. In the context of teaching practice, mathematical challenges arise in much the same way as in this first version of the task; that is with no invitation or prompt to first explore and clarify one's own mathematical understanding.

In our case, although nothing was said about mathematically exploring the task, we could see how Justine's team is challenged right from the beginning of preparation time to develop an understanding of division with remainder. That is, before agreeing on a pedagogical response, the team took time to explore the situation mathematically. We suspect that the *goal* presented in the task (be ready to improvise on the theme, actually engaging with a "pupil") made the difference here. In other words, structure of attention may not only be only awareness to the possibilities a situation offers: it is also motivated, purposeful, and probably goal *orienting* as well (situat*ing* as much as situat*ed*). If this is the case, greater attention might be needed to the understanding of how prospective teachers are supported in the process. To put it bluntly, on this one aspect illustratively, even if mathematical exploration went on, this was clearly not enough to resolve all the mathematical confusions that came up. More so, these confusions arose even though our students already took a mathematics course focused on elementary mathematics the previous year. And actually, we could feel that, at the end of the lesson, many of our students where left with more *new* questions than new answers. Now, we wonder, might not this be the result of the expansion of those students' awareness of what is at stake? And may not this increasing awareness, fundamental to knowing to act in the moment, be the decisive outcome of teacher education? Another question being then: how can we provide, support, help students gaining means and confidence to navigate their queries?

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PATTERNS OF PARTICIPATION – A FRAMEWORK FOR UNDERSTANDING THE ROLE OF THE TEACHER FOR CLASSROOM PRACTICE

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Research on teachers' knowledge and beliefs has grown big in recent years. The larger parts of these fields are built on acquisitionist interpretations of human functioning. We explore the potentials of a participationist framework for understanding the role of the teacher for emerging classroom practices. The framework is built on social practice theory and symbolic interactionism and adopts a processual approach to understanding the role of the teacher. We use the framework in a qualitative study of two teachers with different prior experiences.

Key words: mathematics teachers, patterns of participation, classroom research.

Research on and with mathematics teachers has grown over the last decades. Among others, it has addressed the questions of how to reconceptualise the mathematical knowledge teachers need in instruction (Ball et al. 2008; Davis and Simmt 2006; Ma 1999; Rowland and Ruthven 2011; Rowland et al. 2009) and of the relationships between their conceptualisations of mathematics and its teaching and learning on the one hand and the classroom practices on the other (Leder et al. 2002; Maasz and Schlöglmann 2009; Rösken et al. 2011). This research on teachers' knowledge and beliefs generally interpret human functioning in acquisitionist terms. Knowledge and beliefs are considered entities that reside within the individual. They may be challenged socially, but such challenges are considered a result of an experiential encounter between the individual and an external reality.

This conceptualisation of the individual is somewhat at odds with other current attempts to interpret human functioning in more social terms and view learning and learning to teach as shifting modes of participation in socially established practices. For the purposes of the present paper we adopt a perspective more in line with this latter perspective as we use what we have called a patterns-of-participation framework to understand how the teacher contributes to the practices that emerge in mathematics classrooms (Skott forthcoming; Skott et al. 2011).

THE PATTERNS-OF-PARTICIPATION FRAMEWORK

Patterns-of-participation research (PoP) draws on social practice theory (Holland et al. 1998; Holland and Lave 2009; Lave 1996; Lave and Wenger 1991; Wenger 1998) and symbolic interactionism (Blumer 1969, 1980; Mead 1934) to develop processual and dynamic interpretations of the role of the teacher for classroom practice. The argument is that the locally social emerges as individuals view themselves from the outside and in the parlance of symbolic interactionism take on the attitudes of individual and generalised others. In interaction one interprets others' actions and actual and envisaged reactions to one's own conduct symbolically and adjusts one's actions accordingly. This does not necessarily imply becoming in line with one's immediate interlocutors' expectations. Rather it may involve reintroducing the perspective of for instance other individuals, a group of people who are significant for the context as seen by the person in question (e.g. a team of collaborating teachers), and what Holland and her colleagues (1998) call figured worlds (i.e. imagined as-if worlds such academia, games of Dungeons and Dragons and Alcoholics Anonymous; - or a reform discourse in mathematics education).

In Wenger's terms a community of practice is characterised by mutual engagement, a joint enterprise, and a shared repertoire; to participate in a practice is to engage in the negotiation of its meaning. When a teacher works with a team of colleagues to develop mathematics education or to find ways of addressing more generic educational problems she may be said to participate in a community of practice in Wenger's sense. However, the notions characterising a community of practice need to be stretched if they are to account for the relationship between a teacher and for instance the figured world of the reform in mathematics education. In spite of that we use 'participation' also to account for the latter situation. It is in this case a matter of negotiating meaning and positioning oneself in an internalised discourse about the teaching and learning of mathematics.

As teachers engage in immediate classroom interaction they draw on a range of other past and present practices and figured worlds, some of which relate to mathematics and its teaching and learning, while others do not. The task for PoP may then be rephrased as an attempt to disentangle the multiple practices and figured worlds in which the teacher participates during classroom interaction as well as their transformations and mutual relationships.

In this paper we use PoP to analyse the practices that evolve in two different classrooms at two different primary and lower secondary schools in Denmark. The two teachers are Susanne, working at Southern Heights, and Astrid, working at Eastgate. The question we address is what the patterns are in Susanne's and Astrid's participation in their mathematics classrooms?

THE STUDY OF SUSANNE AND ASTRID

The study of Susanne and Astrid is part of a larger study involving four other practising and prospective teachers. The study spans almost two years.

Susanne is 36 years old when she graduates as a teacher of mathematics from a city college in Denmark. She began teaching at Southern Heights four years prior to that without a degree in education, but she enjoyed teaching and after two years at the school she decided to enrol in a 2-year college programme for second-career, prospective teachers, specialising in mathematics. Formally, the course in mathematics deals not only with the subject itself, but is an integrated course in mathematics and mathematics education.

Astrid has taught for 18 years. During her pre-service education she studied most of the subjects in elementary school, but she specialized in music and physical education. After 9 years of teaching at different schools she gets a position at Eastgate, and begins to teach mathematics. At that time, all mathematics teachers at Eastgate, including Astrid, become involved in a teacher development programme involving four days of lectures and workshops and individual supervision by the teacher educator. Astrid enrols in a similar programme again 8 years later. By then she is also a mentor for prospective teachers in their practicum, she teaches at in-service programmes for other teachers, and she has repeatedly been invited to lecture at college on teaching methods in mathematics.

METHODS

The PoP framework invites analyses of the processual and dynamic character of teachers' participation in classroom interaction. Consequently we need a methodology that views instruction as continuous transformations of teachers' participation in classroom practice in view of broader social practices and figured worlds at the school in question and beyond. Also, we need an interpretive stance that views these practices as well as shifts in the teachers' engagement in them from the perspective of the teachers themselves.

Consequently we use a qualitative approach inspired by grounded theory (Charmaz 2006). We have previously used GT without the objectivist connotations associated with it, and we by no means consider ourselves free from theoretical prerequisites in the present study. However, we still use the coding schemes, constant comparisons, and memo writing of GT as flexible guidelines for theorising classroom processes. They have proved helpful not least as it is not apparent at the outset what and how practices and figured worlds are significant for the teacher in question. The openness of the analytical procedures in GT allows us to address these questions empirically.

The data on Susanne include observations of 12 lessons, six from before her graduation and six from five months after, as well as three semi-structured interviews (Kvale 1996) conducted before and after the observations.

The main data on Astrid are observations of four lessons and two 2-hour semi-structured interviews. Supplementary data include observation notes from her mentoring of three prospective teachers and group interviews with the prospective teachers on their experiences from the practicum. For our present purposes these are used as a supplementary perspective on Astrid's tales of herself as a professional. .

Classroom observations and interviews were video and audio-recorded, respectively, using Transana. The recordings were transcribed in full, and the data were coded moving back and forth between the transcripts and the recordings. The initial and focused codings resulted in 23 tentative categories (e.g. *curriculum*, *epistemological reflections*, *teaching college, teamwork,* and *helping students develop mathematical understanding*). The data were re-coded and memos were written while the categories were conceptualized in two steps, leading to a set of theoretical concepts, including *knowledge of mathematical teaching*, *the reform* and, *life story*, *participation in practices*, and wrote memos.

SUSANNE AT SOUTHERN HEIGHTS

In the interviews and observations Susanne engages in three significant practices and figured worlds beyond the one of the classroom. We describe these as *the tradition*, *the reform*, and *handling students*.

Susanne claims that her instructional approach is traditional. She tries to give explicit and precise directions for the students' subsequent individual work by presenting concepts and procedures for them to copy and follow. This is likely to create a calmer and quieter classroom than any alternative she can think of. She refers to her experiences as a student in secondary school as a source of inspiration for this, and also mentions her pre-service teacher education. At odds with the intentions of the teacher education programme, Susanne describes it as dominated by the teacher educator's exposition of proofs for the students to remember and copy. Susanne is not particularly fond of this approach to teacher education. Her criticism, however, is not directed against this way of working in mathematics. Rather, she suggests that prospective teachers should not spend their time studying the subject itself, but need to be "pumped full of great ideas for how to teach" (int. 1).

Susanne knows about the reform discourse from national curricular documents, from textbooks, and from the theoretical part of her college education, even though she does not think of the practices of the teacher education programme as in line with the reform. She refers to the dominant rhetoric of the programme as "college talk" and says that it focuses on student investigations and the use of manipulatives. Also, the students are to work independently, using informal methods before they are introduced to formal mathematics. However, Susanne is highly critical of the reform and associates "college talk" with what she describes as a pedagogy of "cut and paste", "fiddle and touch", and "cubes and gadgets". She finds it hard to see the mathematical potential in this, except for the emphasis on student understanding, "you know that doctrine that they need to understand and not just follow the rules" (int.1). Also, she thinks that in practice it takes too much time for the teacher to prepare lessons according to the reform and the resulting classroom atmosphere is bound to be too noisy. Susanne is aware the curricular documents are influenced by the reform, but she does not worry that her teaching is incompatible with the formal requirements, because she and her students follow a textbook scheme closely, in which "you can even smell the college talk" (int. 1).

Officially Susanne and her colleagues at Southern Heights work in teams, but the mathematics team meets rarely and irregularly, and when it does, they discuss practicalities and organizational issues. Susanne says that the school consists of “a lot of one-man armies, with each teacher running his own race” (int.1). In spite of that they share a concern for how to handle students, who are in some sort of trouble. Susanne is proud that the school “takes incredibly well care of” the students’ individual problems by using different organizational measures” (int. 1). For instance there is a special needs department for students with learning problems and an “observation class” for students, who are violating school norms. Susanne explains that she sometimes sends students off to the observation class, and that she refers some of the weaker ones to the special needs department. In line with this policy of separation, Susanne also separates the students who are not sent off to other departments into more manageable groups. For instance she asks students who are good in mathematics and who behave well to work alone outside the classroom. The remaining students are then a more homogeneous and manageable group.

Multiplication in grade 5

Susanne introduces the first lesson on multiplication in grade 5 by asking the students to suggest a one-digit and a two-digit number. 5 and 55 are suggested and she writes ‘5 × 55’ on the board. When she asks what this means, a girl, Mira, says that you have “Fifty-five five times” or “the reverse”. Susanne continues:

Susanne: Or the reverse, yes, or five fifty-five times. Exactly. Okay, but that means that I can say that now I take those five [points to ‘5’ on the board] five times first, and then afterwards I take them fifty times. That should be the same, right? Then I get fifty-five times altogether. It does not matter if I take fifty-five times at once, or whether I first take one pile and then the other pile and add them up, does it? So, let us do that. We begin by taking five five times [points to ‘5’ and the last ‘5’ in 55]. Five times five.

Dagmar: Twenty-five

Susanne: That is 25. And then this one, this is all the ones, so I write all the ones down here [writes ‘5’ underneath the ‘5 × 55’].

Dagmar: And the twos go down there? [Points to the left of ‘5’ in the result].

Susanne: Well, these are the tens, aren’t they? I add those to the next pile, because now I am to multiply the tens. Right. So in reality this is twenty, even though I have written ‘2’ up here, it is really …?

Dagmar: Twenty.

Susanne: It is really twenty, because it was twenty-five, wasn’t it [says twenty-five slowly, emphasizing both parts of the word]? But we just write ‘2’. Okay? Then I say, well really I say five times fifty, don’t I? I really say five times fifty, but we just do five times five.

Molly:	Well, it is 25, but//
Susanne:	Yes.
Molly:	But isn't it 125? [This may be Molly's suggestion for the result of the whole task].
Susanne:	No, because you need to add those two [points to the number carried]. Twenty-five and two?
Molly:	Twenty-seven.
Susanne:	Then it is twenty-seven. In reality it is two hundred and seventy, because it is five times fifty, this is what I says isn't it? But we did already put the ones down there, so we just write 27 [writes '27' in front of the '5' in the results line]. […]
Michael:	I don't understand this.
Susanne:	No, but then I try to explain it once more. [Repeats the explanation].

The introduction and the subsequent whole class examples to multiplication last more than half an hour.

Interpreting the above classroom episode, we consider Susanne's contributions to the classroom practices a result of the meaning she makes of the interactions that unfold. Doing so, Susanne draws on practices described previously.

In the lesson Susanne presents a multiplication algorithm. She emphasizes value of the digits several times, but in the process talks for instance about "the twos" instead of two tens. Several of the students suggest different results, and others complain that they do not understand. Susanne responds by going over the calculations again, but does so without explaining the value of each digit and without explaining the underlying mathematical reasoning.

Apparently Susanne attempts to introduce the procedure of the multiplication algorithm as well promote student understanding of how it works. In relation to *the tradition* and *the reform* as outlined previously, it seems as if the element of 'understanding' in *the reform* is inserted into a *traditional* instructional approach dominated by procedural competence. In the process it is transformed from being an outcome of the students' own mental activity (in *the* reform) to being transmitted by careful exposition.

Apart from sporadic attempts to supplement *the tradition* with an element of understanding there is a sharp discontinuity between Susanne's re-engagement in *the tradition* of school mathematics and *the reform* discourse. The two practices, then, do not merge to any great extent. Susanne at times inserts an element of understanding in isolated pockets of *the tradition*, possibly changing the meaning of understanding in the reform on the way. And she takes the organizational measures at the school further when asking students to go elsewhere to work, so as to have manageable group to teach according to *the tradition*, apparently changing the intention of supporting students with problems into handling problematic students in the process. But in general *the tradition* appears to be an almost monolithic structure and other practices function primarily by suggesting ways of handling issues at the outskirts of the main practice.

ASTRID AT EASTGATE

Astrid engages in two significant practices and figured worlds beyond the one of the classroom. We describe these as *the reform experiences* and *supporting students*.

Astrid's tales of her professional self at Eastgate has come to include her position as a mathematics teacher. This is primarily due to her participation in the two reform-oriented teacher development programmes. They were influential, not least "because we talked about teaching" and the need to understand the students thinking: "We got into mathematical thinking, and we have done this many times ever since" (int. 1). This includes an increased emphasis on student communication and the requirement that they explain not only their results, but also their solution methods.

Astrid's comment above refers to the spirit of collaboration between the mathematics teachers at Eastgate, "the Eastgate spirit" (int. 1). Astrid is enthusiastic about collegial collaboration, both when they jointly plan lessons and instructional sequences and when they discuss episodes from different teachers' classrooms.

One of the things Astrid has contributed to the collaboration with her colleagues is her collection of good teaching experiences. Following from the teacher development programme she has collected experiences with emphasising students' work with mathematical problem solving and other processes as well as some illustrating the role of task contexts for students' reactions to mathematics. These experiences have been discussed among the colleagues, and Astrid is still keen to use problem solving with the students.

Eastgate prioritises equity issues, not least as they relate to students with special needs. On the homepage it says that the school builds on the children's diverse abilities, in class as well as on the playground. Astrid agrees with the intention of including and supporting all children and says that she makes an effort to make instructional objectives for the children individually. In spite of her support, she suggests that the school's priorities come at the price of sometimes reducing the emphasis and level of the subject matter taught at Eastgate.

Multiplication in grade 5

Reintroducing multiplication in grade 5, Astrid asks the students to solve single-digit multiplications tasks. Subsequently the students are to draw rectangles of different sizes and find ways of determining the number of unit squares in each.

Astrid: If it was me, and I felt like a little (.), whew: I'm not good to keep track of too many numbers, so I might say, well, I just take such a little piece here [draw a small rectangle – 11 multiply 13]… we must remember two things; it is possible to make it into a multiplication task, and it is smart to multiply by 10.

Peter: Okay.

Astrid: Just think about it. Now, if you think, "ah, this is okay, I would like to do something more difficult", then you could for example say, "Well, I'd like to have" [draw a large square on the board – 24 multiply 32, Astrid whistles].

Olga: Frederik can do that one.

Astrid: Frederik can solve this one. You know, I think there are many of you who can solve this one.

Olga: Mostly Frederik

Jens: 100 times 100

Astrid: Okay. Now I want to go crazy, I want to try this. And so you can try to split it or count it – you can do whatever you want. The only thing is, NB, NB, NB [writes "!!" on the board], keep in mind what we have learned. You can make multiplication tasks and we have learned something like - it is very easy to multiply by 10.

Simon: I have an idea. If you find, what is here – you just count 1, 2, 3, and find it here, and multiplies them [referring to the length and width of the rectangle] – that is what you have to do.

Astrid: [holds out her arms in a gesture of approval of Simon's suggestion]. That is a way to do it. There are many ways to solve it. But before I hear any more pieces of good advice, I would like you all to try.

After the introduction, the students work individually for 20 minutes. They are to set tasks for themselves, and Astrid walks around among the students, asking questions like: “"How can you work this out?”, “Why do you do it like this?”, “If you do like this, how can you show it in the rectangle?” and “You have to write it down, so you can remember what you did”. Later the students show and discuss their different methods for finding the size of the rectangles.

The lesson aims to support the students' understanding of the idea of multi-digit multiplication and to work towards proficiency in carrying out a procedure. Some of the students' individual suggestions are shared at the end, but it is not apparent if this is to form the basis for a common approach. At the end of the lesson Astrid tells the students, that she has written a letter to their parents. The letter states a mathematical aim: The students must know the multiplication table up to ten; “the trick” of multiplying by 10; that a multiplication task can be written as an addition; and how to multiply one-digit numbers by two-digit numbers.

Astrid seems to draw on different prior practices in the episode above. She draws rectangles of different sizes on the board, and the students are to find the number of unit squares in each. Working individually the students are later to decide themselves how big a rectangle they want to work with as well as how they want to find the size. It is an open task that may be solved at many different levels. In that sense Astrid draws on the school's approach to inclusion. Discussing the task with the students in a whole class setting afterwards Astrid also focuses on getting all students to talk about their individual approaches. Astrid does not reject or openly approve of the various solutions, in effect avoiding prioritising any of them.

Astrid uses the rectangles, to support student understanding. She draws on her practical experiences with the reform in the teacher development programme and her collaboration with her colleagues. This seems evident from her emphasis on the students setting their own tasks and the lack of emphasis on a standard procedure, leading to an element of investigation. Also, as the students work, Astrid wants them to discuss mathematics and her focus is on the students reasoning.

There is one practice, however, that is conspicuously absent in this, the one of mathematics in a more traditional sense. Although there is a close connection between the drawing and an understanding of multi-digit multiplication and the related algorithms, it is not obvious if the students understand the mathematical point or if they simply are "counting squares". One can speculate if they understand that this is a possible road towards a general procedure for multi-digit multiplication.

CONCLUSION

The above two analyses are attempts to understand how teachers' participation in different present and prior practices relate differently to the ones that develop in the classroom. They are an alternative to interpreting teachers' contributions to classroom practice in terms of their knowledge and beliefs. In the case of Susanne at Southern Heights one practice is dominant while other and more reform-orientated practices are less obvious. In the case of Astrid at Eastgate several practices form a more equal pattern. Susanne's and Astrid's conditions are similar, but the participatory patterns differ. Doing away with the acquisitionist connotations of other lines of research, the patterns of participation framework has some promise for a better understanding of the dynamical nature of classroom interaction. PoP sheds light on the dynamic relationships between the teacher's engagement in the practices of the mathematics classroom and other, personally significant, past and present ones.

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INTEGRATING TECHNOLOGY INTO TEACHING: NEW CHALLENGES FOR THE CLASSROOM MATHEMATICAL MEANING CONSTRUCTION

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The paper examines the ways in which technological tools shape teachers' and students' activity and hence the meaning construction related to linear function in a Year 10 classroom. The Extended Mediational Triangle of Cole and Engeström *(1993) is used to analyze the aforementioned activity and to interpret conflicts and contradictions. Results show that students face difficulties to follow their teacher's effort for conceptual understanding via connections of different representations and mathematical context. The teacher's knowledge and flexibility helps him to exploit contradictions between his and students' objectives mostly productively. Technological tools are generally supportive to this direction but also giving rise to complications in teacher-students' communications.*

INTRODUCTION

The learning and teaching challenges of using technology in the mathematics classroom have been repeatedly addressed in Mathematics Education. In exploiting computational environments, teachers' main responsibility is to act as orchestra conductors, aiming at students' interaction with the provided artefacts in ways that allow mathematical meaning to collectively emerge. To this direction, they are expected to tailor scaffolding conditions, exploiting technology in ways that promote students' transpositions towards mathematical meanings. However, this is not a straightforward process, as complications may arise in its course, due to conflicting interests and understandings emerging in the context of the resultant classroom activity. This paper presents an attempt to explore such a situation, aiming to contribute to the wider discussion on students' conceptual understanding via the exploration with technological tools.

THEORETICAL BACKGROUND AND LITERATURE REVIEW

Mathematics teaching is characterized by complexity as it is framed by the classroom interactions, the tasks assigned to the students and the overall social context. Skott (2010) talks about teachers' patterns of participation in different practices that frame their teaching. In technology-based mathematics lessons the situation becomes even more complex, as the nature of tools and management issues complicate student – teacher interaction in moving from the technological to the mathematical objects (e.g., Marracci & Marriotti, in press).

Activity Theory embodies the individual and the society in a unity in a way that the individual acts on his/her society at the same time as he/she becomes socialized to it (Mellin Olsen, 1987). This interplay between the individual and the society could capture the systemic nature of mathematics teaching by addressing its complexity. Recently, Jaworski and Potari (2009) used the activity theory and in particular the Extended Mediational Triangle (EMT) (Figure 1) of Cole and Engeström (1993) to consider the role of the broader social frame in which classroom teaching is situated. In this study, the activity of a teacher and of his students is contrasted and certain contradictions are identified.

The topmost of the subtriangles represents the visible actions of the subject, in our case of the teacher and the students, who use a number of tools to reach a goal (the object). Engeström (1998) refers to this as the "tip of the iceberg" and argues that "the 'hidden curriculum' is largely located in the bottom parts of the diagram: in the nature of the rules, the community and the division of labour of the activity" (p. 79).

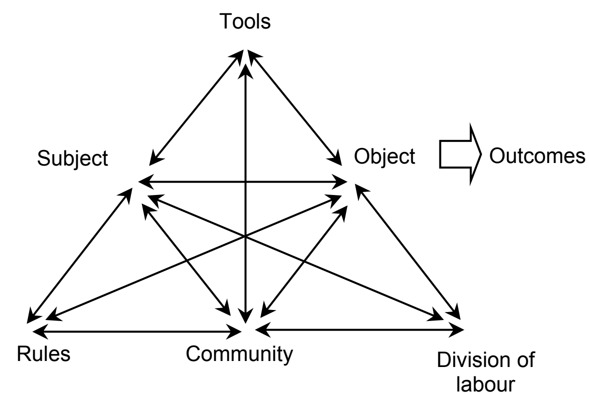

Figure 1: Extended Mediational Triangle

Students' participation in different communities (classroom, school, friends, parents etc.), each with its own rules and division of labour, has an impact on their classroom activity. Teacher's activity, on the other hand, based on the tasks and the tools s/he designs to achieve certain goals, is framed by the rules of the communities he belongs to.

In the present paper EMT is used to analyze the activity of the teacher and the activity of the students and to interpret conflicts and contradictions that the teacher faces in his attempts to support students using technological tools as a means to construct mathematical meaning.

Various attempts have been made to study technology integration into mathematics teaching, especially in the upper secondary school, as well as the particularities of this integration. For example, Biza (2011) investigated Year 12 students' understanding of the concept of tangent in computational contexts. She initially identified students' misconceptions with the concept itself and their difficulty in moving between representational systems. The teaching intervention employed was based on the usage of examples and of dynamic graphs. The classroom discussions analyzed showed limited taken-for-shared mathematics meanings between the teacher and the students as well as conflicts and fluctuations in students' arguments. Furthermore, the teacher's attempt to negotiate the construction of a shared mathematical meaning was not straightforward, fluctuating between orchestrating the classroom discussion, introducing new examples and changing the sequence of the examples.

Kendal and Stacey (2001) studied the teaching of derivatives by two secondary teachers to Year 11 students via a Computer Algebra System. One of the teachers relied predominately on lecturing and demonstrating while the other on exploiting children's ideas emerging during classroom discussions. The analysis of these discussions revealed that teachers' pedagogical choices were compatible with their conceptions about mathematics and its teaching as well as about technology usage for educational purposes. In particular, the first of the teachers exploited technology more and his students used effectively the technological artefacts to solve procedural problems. However, the students in the second class were more capable in dealing with conceptual issues related to derivatives.

Monaghan (2004) looked at how secondary mathematics teachers take advantage of digital technology in teaching. The participating teachers, who made moderate educational use of technology in their regular classes, encountered substantial difficulties in exploiting effectively technology and tended to encourage classroom activity that differed in structure from that employed otherwise. In particular, they showed preference for open-ended tasks, often requiring extensive investigative processes. They also expressed concerns for the nature of the mathematics involved and noticed that the students tended to concentrate on technological details to the expense of mathematics. The author concludes that students' interpretations of the tasks affected the emergent teachers' goals and the design of the teaching sessions.

In Trouche's (2004) study, Year 12 students were invited to deal with a demanding task in computational contexts, involving solving equations, studying function variation, finding limits and studying sequence variation. The students worked in groups and were expected to submit a research report by the end of the session. The researcher claims that the reported success of the activity should be attributed to the expertise of the teacher and to the highly motivated and intelligent students and that it is hard to carry out in "normal" classes due to the demanding instrumental orchestration required. Such an orchestration should allow students to first make sense of the problem, then to explore some special examples and to finally discuss relevant conjectures.

The studies presented above indicate that the usage of technology may facilitate students' classroom participation but it also gives rise to unexpected and unexplored challenges during the mediation process. These challenges might increase instead of decreasing the conflicts and contradictions present in the activity of the teacher and of the pupils, thus weakening and blurring the mathematical meaning construction process. The study presented below is an attempt to investigate how this takes place.

SETTINGS-METHODOLOGY

The study is an action research of a high school teacher with 20 years of teaching experience and his collaboration with four researchers. The teacher had just completed a Masters' program in Mathematics Education and participated initially as a teacher and later as a teacher educator in a number of professional development seminars related to the introduction of digital technologies in mathematics teaching. By returning to school, after a three-year school leave to complete his postgraduate studies, he wished to "implement" innovative ideas and approaches he came across during his studies. To this end, he introduced digital technologies into his teaching, promoted students' conceptual understanding through different representations and generally encouraged students to make connections across contexts and representations. These approaches were beyond students' experiences of mathematics teaching, which were textbook-based, over-emphasizing procedures and sophisticated techniques.

During the first months he faced a number of tensions related to students', colleagues' and parents' expectations and thus decided to inquire his teaching and investigate systematically its effectiveness. To this purpose, he established a cycle of planning-implementing-reflecting lessons, which was regularly discussed with the first author of this paper at the planning and reflection phases. Furthermore, central classroom incidents were placed under scrutiny and emerging issues were explored in weekly meetings of all four researchers (mostly through Skype). In the occasion reported here, the teacher wanted to investigate how he could help students to explore algebraic relations and in particular the linear function and its graphical use to solve algebraic equations and inequalities in the context of dynamic environments, such as Geometer's Sketchpad and Geogebra.

The first author of the paper observed and audiotaped his teaching in a Year 10 class (15-16 years old) for two months (27 students, 12 boys and 15 girls, for 17 teaching periods). In some of the sessions more than one audio recorder were used in order to capture a broader spectrum of the classroom activity. The main research question addressed here concerns the ways in which the available technological tools related to the teacher and the students' activity and thus the mathematical meaning construction. The data consisted of the transcribed classroom observations and the discussions both at school and at the meetings. Analysis was based on the identification of critical incidents, where contradictions and conflicts emerged and the teacher had to interpret and manage. The EMT was used to analyze the incidents, allowing the interpretation of these contradictions and the deepening of our understanding of what could be characterized as effective teaching management.

RESULTS

In both incidents reported below the students work in pairs or groups of three in the computers' room with a Geogebra file and a worksheet with technical instructions and mathematical tasks, both prepared by the teacher. The first comes from the first teaching session on linear functions while the second from the 5th and 6th sessions.

Incident 1: Teacher's management of students' unexpected responses

The teacher tries to use the possibilities offered by technology in order to allow students to identify many instances of the graph of a linear function. His goal is the students to explore general function properties by linking different representations.

For the needs of the introductory session on linear functions, the teacher prepared a Geogebra file consisting of a kinaesthetic representation of the function with two sliders, one for each parameter of the formula of the function. As the teacher explained just before the lesson: "They will manipulate the object [intuitively], gain a familiarity with this [make sense] and then make an interpretation of the object [typical meaning]". The use of technology influences his decision, as "it allows testing dynamic changes".

At the beginning of the session, he gives the students the worksheet and asks them to directly answer the first question: "Use the slider. What is the shape of the graph?" Students work in pairs:

1	S1:	Sir, when you say what is the shape, it is a triangle, that means if we use a specific [she means a specific value of the parameter *a*], it is true.
2	Teacher:	Which one is the graph?
3	S2:	A straight line.
4	Teacher:	The graph is this. Where is the triangle?
5	S1:	Here it is! Isn't this the triangle? [she points at the triangle defined by the line and the two axis, see a similar inscription at Figure 2]
6	S2:	I can see it too.
7	Teacher:	We are not interested in the axes. What if [they are] hidden? Hide the axes.
8	S2:	Yes, what if … but.
9	Teacher:	Just a minute
10	S1:	Sir, I understood it.
11	Teacher:	If you cover the axis, this is the shape; this is the function graph, you need to refer to this.
12	S3:	What do you mean 'what shape is the graph'?
13	S4:	Sir, you don't have good expression.

The environment in which students work combines paper-pencil and computer based activities and includes: the symbolic representation; the kinaesthetic representation; two sliders; and the open-ended question regarding the *shape* of the graph. Students face difficulties in the connection of the representations of the linear function, so a group of them concludes that the shape of the graph is a *triangle* (Figure 2). The teacher did not expect this response from the students. For him, a graph is a geometric expression of an algebraic relationship; therefore, the line on the screen is the *shape of the graph.* However, for the students, a *shape*, in both geometric and algebraic contexts, is the same: does not represent a relationship, but something that should have area. Students' prior experiences of "shapes" regard geometrical tasks, whereas for algebraic tasks the word "graph" is usually used. So, the students, failing to see the connection between the algebraic and the geometrical contexts of a linear function (transcript lines [12], [13]), they identify only a *triangle*. A linguistic conflict can be identified here between the teacher and the students: the graph of a linear function is a shape for the teacher, whereas students expect to see a geometric shape that has a surface. This interpretation gave rise to a new teaching goal for teacher –students' understanding of what *the shape of the graph* is. To this aim, he tries to make the mathematical objects transparent to the students through questions and features of the educational software (transcript lines [2], [4], [7]):

> In the first question, I say: "What shape is the graph?" They say: "It is a triangle." I understood that in this case they also saw the axes as part of the graph. So, I hid the axes and said: "Which is the graph?".

The teacher has chosen to negotiate different contexts (geometrical and algebraic) of the concept of linear function, with the intention to link the two, a process that was not easy for students. Finally, after the teacher used the Geogebra tool to hide the axes of the graph, students' responses ("Isn't that the graph?", "It is always a straight line.", "A line?") showed that they began to understand what he referred to.

Incident 2: Teaching goals versus students' explorations

In the second incident, the initial intention of the teacher was to engage students with the investigation of the properties of the linear function (f(x)=ax+β), especially regarding the role of a and β. To this aim he created a Geogebra file with a graphics window for the function graph and a spreadsheet window for the values of x and y (Figure 3).

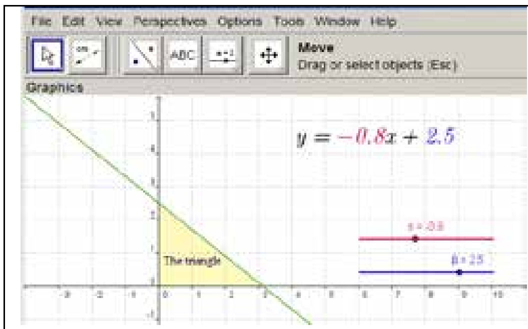

Figure 2: Geogebra file, Incident 1

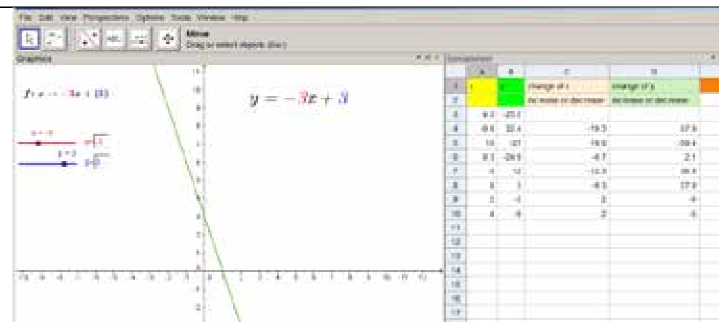

Figure 3: Geogebra file, Incident 2

The teacher has assigned to each group of students different values for a or β. In the spreadsheet, the first four values of x are random numbers, which change with F9 key, whereas the last four values of x are fixed and the same for every group. When the incident starts, students have already filled the columns of y, *change of x* and *change of y* by using the corresponding values of x and they are ready to deal with the task: "List as many observations you can regarding the results in the spreadsheet (try to be analytic in your description)". The teacher expects that some groups will observe that the change of y is proportional to the change of x and will try to connect the ratio of change of y over change of x with the parameter a or the slope of the graph. Any observation of this type might be helpful to him to introduce students to the monotonicity of functions at a later stage.

During the whole class discussion, a student working in a group with the function $f(x)=0.5x-3$ notices that the values of y have always the same sign with those of x.

The teacher reminds the students that, by pressing F9, the values of x change and that x can be any real number and asks them to think "whether for every value of x the value of y=0.5x-3 always have the same sign as x". Some students agree, others not. He takes up the challenge and attempts to make students investigate the problem, by encouraging them to make connections among different representations of the function and also to link the solutions of equations and inequalities with points of the graph. A student says that for "every negative value of x this will be true", another argues that "if x=0 then y=-3". Other students suggest that there are positive values of x, for which y is negative or positive, "all the values of x for which 0.5x<3 or 0.5x>3, respectively". However, there are students who fail to follow this suggestion. The teacher asks about the graph of the function and draws it roughly on the blackboard, following students' suggestions regarding the y-intercept and the positive slope of the line. Then, he asks students not participating in the previous conversation to propose positive values of x, for which y>0 or y=0 or y<0, fill a table with the corresponding values of y and plot some corresponding points on the graph.

The initial aim of the teacher was the students to relate the ratio of change of y over change of x with the parameter a or with the slope of the graph. To achieve this, he asks students to work with different values of a and β, in order to observe the pattern in the spreadsheet. However, the open-ended question in the worksheet drives students' observations to a different direction. By reflecting on his lesson after the session he notes: "The way that the question was posed in the worksheet might have been vague for what I had in mind". As in the first incident, he takes into account students' responses – although beyond his aims – and he creates a new teaching goal: to support students to give meaning to the graphical representation of the points (x, y) and relate them to the roots of the corresponding equation and inequality. Nevertheless, some students find it difficult to follow this shift:

> Instead of allowing all students to present their results, I grasped the first opportunity offered by the first student's response: "if for every value of x the value of y=0.5x-3 will always have the same sign as x" and I invited the class to work on this. I saw it as a good opportunity to discuss issues related to the solution of an equation and an inequality and its relation to the corresponding point on the graph. […] Students found it difficult to shift their attention from the problems they were working on to the specific case […]

Analyzing the two incidents by the EMT

The two incidents exemplify some of the observations across all the sessions offered by the teacher on linear functions. It seems that his *good* pedagogical intentions were challenged by discrepancies between his and his students' activities.

The EMT below (see Table 1) is used to interpret the context of the teaching/learning activity that took place across the observation.

Many of the teacher's decisions, choices and rules are deeply influenced by the Master's community. He strongly believes that conceptual understanding is the key for learning mathematics and he sets particular rules to provide for this:

> After the Master degree, for me, there are no students' simpleminded answers, which could exist because they would not study enough and, therefore, would not understand. My approaches on how I interpret what they do have certainly changed. I saw and realized how complicated things about maths and the various concepts are.

For this reason, he uses new practices and tools he came across, such as group communication and Geogebra tasks, interprets students' responses being informed by research literature and follows a different teaching structure from the one suggested by curriculum. Mathematical learning is being pursued by connections between different representations and contexts and the electronic environment and the questions in the worksheet aimed to facilitate students to make these connections.

Subject:	Teacher	Students
Object:	Should enable all students to: • connect symbolic, graphical and tabular representation and the graph of linear function • connect the geometric context and the algebraic context of linear function	• answer the questions of the worksheet.
Tools:	• Geogebra file • worksheet	• Geogebra file • worksheet
Community:	• classroom community • school community • master's community (researchers, schoolmates, professors) • wider educational community (other textbooks, the internet) • wider social community (relationships with students' parents)	• classroom community • school community • private mathematics lessons • friends • family • wider social community (team sports etc)
Rules:	• Different agenda from colleagues and curriculum: open-ended activities in agreement with findings of research literature - emphasis on students' conceptual understanding • group communication • teacher's requirements • exam requirements • pressures of the curriculum particularly with regard to time management • norms regarding working in mathematics and generally in class • norms regarding working in computer laboratory	• At private mathematics lessons and in other classes at the same school: Solving a great number of exercises for procedural understanding and practice. Teaching according to the book and curriculum structure. • examination requirements • peer pressures
Division of labour:	• planning the worksheet and Geogebra file • organization of students into groups • support for the students' teamwork • discussion with the whole class at the end of teamwork • consolidation of mathematical meanings	• working in groups • enhance personal meanings concerning the task • contribution to teamwork and the whole class discussion with the teacher

Table 1: EMT describing the context of the teaching/learning activity

However, it seems that the teacher has different objectives and rules from students, whose teaching/learning experience is within the traditional paradigm, which promotes practicing on exercises and developing procedural skills. As a result, although the connections of different contexts and representations are included in the teacher's objectives, these are not visible for the students in advance. In addition, these connections are not included in either students' objectives in practicing procedures of mathematics or school community's practices of teaching and learning.

This contradiction of objectives led, as a consequence, to a contradiction between the teacher's expectations and to what the students actually did. Later on, when the teacher noticed that his students did not make these connections (in discussing with the first author after lesson 10), he reflected:

> At the beginning, they had made these tasks with the parameters a and b… but possibly I did not manage to make these connections apparent… To make them apparent or to let them discover for themselves?... I do not know. This is why today they had this difficulty. They worked rather mechanically with the images, not realizing what the main point was.

DISCUSSION

The teacher's teaching goal is students' conceptual understanding via connections, which must be carried out through investigation. These connections are between different representations of functions (symbolical, graphical and tabular) as well as between different mathematical contexts (algebraic and geometrical). He incorporates technology into his everyday teaching and engages students in group-communication, practices that are new both to him and to his students. In class, the students start directly the investigation, without being informed about their teacher's objectives. The latter frequently experiences situations that he does not expect. He is flexible enough to hear students' voices and adapt his lesson plan accordingly. However, the students, being unfamiliar with this style of teaching and expecting a well-defined teaching agenda known in advance, aren't always able to follow him. Moreover, the exploration allowed by the technological tools makes them feel uncertain about the goals of their activity.

It could be argued that in a teaching approach where technological tools are exploited, students' activity becomes less controlled by the teacher. Furthermore, more layers of complication are added concerning the multiplicity of representations, the interaction with the environment and the dynamics of the emerging constructions. As a result, the meanings shaped by the students might frequently diverge from the meanings intended by the teacher. Many unexpected events may emerge from the students' reactions and the teacher needs to be flexible enough to adapt his/her planned goals and act in-the-moment. These actions require deep mathematical and pedagogical content knowledge for his attempts to effectively promote the mathematical meaning under construction. On the other hand, the openness of the situation, although might motivate students to get involved with the tasks, it is possible to be in conflict with other day-to-day classroom and institutional norms and practices that influence mathematical teaching and learning. Thus, the implementation of technology in the mathematics classroom requires careful consideration of teachers' scaffolding practices, which should aim to gradually introduce students into new ways of being involved with the mathematics classroom activity that promote genuine interactive construction of meaning, challenging them to sensitively explore resistances and resolve conflicts on the way.

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MTSK: FROM COMMON AND HORIZON KNOWLEDGE TO KNOWLEDGE OF TOPICS AND STRUCTURES

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This paper focuses on the mathematical knowledge a teacher needs to be able to teach. We give particular consideration to the principles underlying the subdomains making up the model, MTSK (presented in Carrillo, Climent, Contreras and Muñoz-Catalán, 2013), building on the category of Subject Matter Knowledge (Shulman, 1986). We define and analyse three subdomains: Knowledge of Topics (KoT), Knowledge of the Structure of Mathematics (KSM), and Knowledge of the Practice of Mathematics (KPM). We discuss the defining features of these categories, contrasting them with the model of MKT developed by Ball et al (2008), and using examples from our own experience as researchers in the area.

Keywords: MTSK, MKT, knowledge of topics, knowledge of the structure of mathematics, knowledge of the practice of mathematics.

FROM MKT TO MTSK

Ever since Shulman's (1986) seminal work, setting out the knowledge teachers bring into play in the exercise of their profession, a separation has been recognised between Subject Matter Knowledge and Pedagogical Content Knowledge, with seven different categories making up these two main domains. In their construct of Mathematical Knowledge for Teaching (MKT), Ball, Thames, and Phelps (2008), present a classification of mathematical knowledge, following Shulman (1986), and introducing six

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[1] This article represents part of the studies into teachers' knowledge by the SIDM group (from the Spanish 'Research Seminar into Mathematics Education') based at the University of Huelva, Spain. It comprises the following researchers: José Carrillo (coordinator), Nuria Climent, Luis C. Contreras, Miguel A. Montes, Álvaro Aguilar, Dinazar I. Escudero, Eric Flores and Enrique Carmona (University of Huelva), M. Cinta Muñoz-Catalán (University of Seville), Pablo Flores, Nielka Rojas and Elisabeth Ramos (University of Granada, Spain), C. Miguel Ribeiro, Rute Monteiro and C. Susana dos Santos (University of the Algarve, Portugal), Leticia Sosa and José L. Huitrado (University of Zacatecas, Mexico), Diana Vasco (State Technological University of Quevedo, Ecuador), and Emma Carreño (University of Piura, Peru).

different subdomains, *Common Content Knowledge* (CCK), *Specialised Content Knowledge* (SCK), *Horizon Content Knowledge* (HCK), making up Subject Matter Knowledge and *Knowledge of Content and Students* (KCS), *Knowledge of Content and Teaching* (KCT) and *Curricular Knowledge*, constituting Pedagogical Content Knowledge. We consider that SCK is particularly relevant as it is considered an area of knowledge exclusive to the profession of mathematics teaching. Despite representing a significant advance in our understanding of mathematics teachers' knowledge, the definition of this subdomain overlaps with others (Flores, E., Escudero, D.I. & Carrillo, J., 2013). Such problems of demarcation between subdomains led to the development of the MTSK framework (Carrillo et al. 2013).

In the following analysis we try to differentiate the specific knowledge that is held by teachers about a mathematical item from the perspective of a pupil (derivatives, for example) from the knowledge of that same item as an element of common knowledge.

Ball et al. (ibid) emphasise the mathematical demands entailed in teaching, which they exemplify with a subtraction computation. The example gives the correct answer to the subtraction 307-168 via the so-called 'borrowing' algorithm, then considers various typical wrong answers by pupils, to understand the cause of which requires special mathematical reasoning, and finally proposes other non-standard approaches which are often unfamiliar to the teacher. They state that anybody who knows how to solve the calculation can identify when a pupil's answer is incorrect, but that "skilful teaching requires being able to size up the source of a mathematical error" (ibid., p. 396). This kind of teacher knowledge, they add, is complemented by an ability to do such an analysis efficiently and fluently, and to see beyond the errors to the particular problems facing the pupils.

We would agree that identifying a wrong answer of this kind should be considered commonly held basic knowledge, and that it logically forms part of the teacher's knowledge by virtue of the demands of the work of teaching (Common Content Knowledge). However, in the process of analysing the error the teacher brings into play two different types of knowledge. First, in the case of the non-standard approaches, the teacher needs to ask him or herself, "What is going on mathematically in each case?" (ibid., p. 397). The answer to this question implies the mobilisation of intrinsically mathematical knowledge about the significance and implication of each step in the process of subtraction.

Second, in the case of a student arriving at the answer 261, the teacher has to consider "what line of thinking would produce this error" (ibid., p. 396), and reflect on the nature of the misunderstanding that gave rise to the mistake. In this case, the knowledge brought to bear is not only mathematical, as the item under consideration ceases to be mathematics itself and becomes the cognitive processes called upon when a pupil tackles a mathematics task.

In this paper our interest is in gaining a better understanding of the mathematical knowledge (in the sense of Subject Matter Knowledge (Shulman, 1986)) required by teachers in their day-to-day practice. We approach this interest from the model of Mathematics teacher's specialised knowledge (MTSK) which we present in Carrillo, Climent, Contreras and Muñoz-Catalán (2013, in this volume). MTSK offers a new perspective on the knowledge required in mathematics education, and whilst respecting Shulman's (1986) original division between *Subject Matter Knowledge* and *Pedagogical Content Knowledge*, it brings two fundamental aspects to the fore. First, it adopts the term "specialised" from the model of MKT by Ball et al (2008), but applies it to the whole of the new model. That is, instead of talking about specialised content knowledge, whereby the notion of 'specialised' is applied to content knowledge, the new model concerns the specialised nature of mathematics teacher's knowledge. Secondly, it shifts the focus of study onto the object of the teacher's reflection. Hence, in 'subject matter knowledge' we propose the subdomains *Knowledge of Topics* (KoT), *Knowledge of the Structure of Mathematics* (KSM), and *Knowledge of the Practice of Mathematics* (KPM). In the category *Pedagogical Content knowledge* we include the following subdomains: *Knowledge of Mathematics Teaching* (KMT), *Knowledge of Features of Learning Mathematics* (KFLM) and *Knowledge of Mathematics Learning Standards* (KMLS). For a full description of all these subdomains, see Carrillo et al. (2013), in this volume. Our interest in this paper is to describe the principle features of the subdomains within the first group, giving examples from each.

KNOWLEDGE OF TOPICS

In MTSK (Carrillo et al., 2013, in this volume), the subdomain *Knowledge of Topics* (KoT) represents a new way of viewing the subdomains *Common Content Knowledge* and *Specialized Content Knowledge* (SCK) from the model of MKT.

Ball et al. (2008) define *Common Content Knowledge (CCK)* as "the mathematical knowledge and skill used in settings other than teaching" (p. 399). We can equate this to the mathematics that can be found in mathematics (text) books (at any level). However, although this knowledge might be shared with other professions, we would argue that the teacher possesses a greater range and depth with regard to this knowledge by virtue of the simple fact that mathematics is the lifeblood of their work. This notion, as suggested above, inspires the idea that "specialised" spreads across the full range of domains in the model of MTSK (Carrillo et al., 2013, in this volume)

With this in mind, we can indicate other aspects of mathematical knowledge which should be included in this subdomain (KoT). To start with, there is the advanced knowledge needed to understand any particular topic. For example, if a teacher is explaining surface integrals at higher secondary level, they will need to know the concept of area, the density of rationals in the set of real numbers, and topological theory, all of which have in common the fact that without them, the topic cannot be understood and which are therefore required within the subdomain. By the same reasoning we would include non-curricular mathematical knowledge such as unconventional procedures for doing mental arithmetic.

Secondly, we also include knowledge about the different meanings a topic might have, as is the case with fractions (Llinares and Sánchez, 1997). A teacher who formulates different problems which can be solved via the same fraction but with different meanings demonstrates considerable reflection on the concept of fractions and their elements, which goes beyond mere problem solving and doing calculations with fractions. We also consider here phenomenological aspects associated with the knowledge of a mathematical item (Freudenthal, 1983; Rico 1997). In the model of MKT, all the above would be included within the subdomain *specialized content knowledge* (SCK). However, bearing in mind that the notion of specialised pertains to the full MTSK model, we consider that this kind of knowledge, being "pure subject matter knowledge" (that is, unalloyed with knowledge of students and pedagogy), belongs within KOT. Put simply, we consider KoT a subdomain containing advanced mathematical knowledge exclusive to the work of teaching.

To recapitulate, the defining features of KoT comprise advanced knowledge of school mathematics topics, along with knowledge of any different meanings involved, and corresponding phenomenological aspects.

KNOWLEDGE OF THE STRUCTURE OF MATHEMATICS

In this section, we focus on the second subdomain within 'subject matter knowledge' in the model of MTSK (Carrillo et al., 2013, in this volume) denominated *Knowledge of the Structure of Mathematics* (KSM).

This subdomain emerged as a result of reflecting on Horizon Content Knowledge (HCK) in the MKT model, which is defined as "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum” (Ball et al. 2008, p. 403). Initially, the authors were unsure whether this category belonged to subject matter knowledge or was spread across several categories. Later, Ball and Bass (2009) proposed subdividing HCK into three related dimensions: HCK - topics (HCK (T)) which concerns connections both within the field of mathematics and with other disciplines; HCK - practice (HCK (P)) dealing with how mathematics is constructed; and HCK - values (HCK (V)) specifying the main values when doing mathematics.

The consideration of how mathematics is interconnected internally is a significant component of KSM as it enables us to understand how teachers construct their mathematical knowledge. Likewise, knowledge of connections with other disciplines enables the teacher to devise mathematics problems drawing on other areas of knowledge. However, for the moment we will leave external connections to one side.

Martínez et al. (2011) consider that the *connections* teachers make between different areas of content (concepts or procedures), whether at the planning stage, during the execution of the lesson, or after the class, can be grouped into three types:

- Intraconceptual connections: connections between different ideas associated with a particular mathematical concept, constituting the essence of mathematics.
- Interconceptual connections: connections to different mathematical concepts.
- Temporal connections: connections between mathematical concepts at different stages of the curriculum, that is between what has been studied and what will be studied.

These connections derive from studies carried out by Fernández et al (2011), in which HCK is considered “not as another subdomain of MKT, but as a mathematical knowledge that actually shapes the MKT from a continuous mathematical education point of view” (p. 2646).

With regard to intraconceptual connections, we have become aware through successive analyses that the connections made are often so closely related within the same topic that we have been led to reconsider the appropriateness of including these within *Knowledge of the Structure of Mathematics* (KSM). The characteristics of such connections are closely linked to the knowledge of the meaning of a specific item, for which reason we consider them as pertaining to the subdomain *Knowledge of Topics* (KoT) presented above.

Interconceptual connections relate one mathematical topic to another or others. These connections are of a different order to intraconceptual connections as they underline that the mathematics teacher is establishing links between different mathematical items, in terms of both concepts and properties between them. The knowledge a teacher demonstrates in developing new mathematical items from existing knowledge is also included.

We include temporal connections in KSM, that is, connections to retrospective and prospective content relative to the current item of study. We can see a convergence here of the idea of *elementary mathematics from an advanced point of view, and advanced mathematics from an elementary point of view*. “These two notions enable two-way connections to be made – prospectively between elementary material at any particular level and its corresponding advanced treatment at later stages, and retrospectively between advanced material and its more basic treatment at lower levels,” (Carrillo et al., 2013, in this volume). In this context, and complementing the idea of connections, we include the concept of increasing complexity (and conversely, simplification). For example, the procedure for classifying two-dimensional shapes is the same at secondary level as it is at infant level, but if teachers at each level were to tackle this area from a perspective of continuity between the two educational levels, each would take a very different mathematical view. The secondary teacher, for example, could deal with the procedure from a complex perspective, suggesting classificatory systems based on two or more criteria at a time or which involved the use of inclusive groups. The infant teacher, for their part, might suggest classificatory strategies that use visual and manipulative cues one after the other to make different groups; they might even use a simplified strategy, such as *identifying* attributes and qualities, or *comparing* shapes (in order to help the pupils at this age to discard their tendency to view the world as a homogeneous whole), which we believe lie at the mathematical heart of the classificatory procedure. In other words, the cline from simple to complex is not cognitive but mathematical; the teacher’s task is to scrutinize the area of study so as to identify those contents which are close or connected and make up a mathematical framework.

Being able to superimpose a more advanced approach to a learning item onto a more basic approach to that item, or put another way, conceptualising an item from both a basic perspective and at the same time a more advanced perspective than that required by the curriculum at that point is one of the key specialised skills a teacher needs to develop.

As mentioned at the start of this section, KSM belongs to the category *Subject Matter Knowledge*, as does KoT. However, while the subdomain

KoT concerns in-depth knowledge of a specific topic (getting to its mathematical heart), KSM concerns knowledge of how topics are interrelated, and this kind of knowledge is based more on a global understanding of the mathematical structure connected to a concept. In the case of KoT, to make an analogy with building a house, we are talking about knowing how to put one brick on top of another, which brick to put and why. In the case of KSM, it is a question of taking a step back to see the girders and joists which make up the framework supporting the house.

KNOWLEDGE OF THE PRACTICE OF MATHEMATICS (KPM)

One of the dimensions of the HCK classification devised by Ball and Bass (2009) is knowledge of the ways of knowing and creating or producing in mathematics (syntactic knowledge). This includes aspects of mathematical communication, reasoning and checking, providing and applying definitions, making connections (between concepts, properties and so on), using correspondences and equivalences, deploying representations, arguing, generalising and exploring. Our perspective is that this categorization corresponds to *Knowledge of the Practice of Mathematics*, that is to say, conceptual knowledge about the rules of syntax themselves, and procedural knowledge about how to do mathematics, in addition to knowledge about the history of the discipline and its relation to other fields (Ball and McDiarmird, 1990). In this respect, we have included *Knowledge of the Practice of Mathematics* (KPM) as the third subdomain making up *Subject Matter Knowledge*.

Again, setting to one side connections with other areas of knowledge as mentioned above, this kind of knowledge is what in large part can be considered "mathematical logic". We understand that this knowledge, once again, is not exclusive to teaching, but is likely shared by anybody who has wondered about how mathematics is constructed. However, it is specialised in the sense that a teacher should know, for example, when a result with a double implication is fully or only partially demonstrated, and it is knowledge of this kind that enables the teacher to decide this. In short, then, *Knowledge of the Practice of Mathematics* (KPM) concerns reflecting upon ways of doing mathematics.

A group of primary teachers in a collaborative professional development project pondered the question of whether "the addition of a multiple of 2 and a multiple of 10 results in a multiple of 10" (Muñoz-Catalán, 2012). One of the teachers replied that it would need a general demonstration, as sometimes this was the case, but other times it was not. This is an example of knowledge about ways of demonstrating mathematics, which for this teacher seems to consist in checking whether an affirmation that she knows is not always true, is always true. This knowledge represents an idea of demonstration that is not valid in mathematics, but which we can say is *her* way of understanding what demonstrations are. However, this knowledge is not used in isolation in this case, but in conjunction with the knowledge the teacher has of natural numbers and their properties of divisibility, a fact which underlines how KPM is a subdomain which occurs across domains when the classroom treatment of any mathematical topic is analysed.

CONCLUSION

In this paper we give a detailed characterisation of the subdomains within the MTSK framework (Carrillo et al., 2013, in this volume), corresponding to the category of *Subject Matter Knowledge* proposed by Shulman (1986). We believe that the model of MTSK represents an advance in the process of categorising mathematics teacher's knowledge, which was initiated by Shulman and developed by Ball and the research group at the University of Michigan. It is the result of a considered analysis of the distinct dimensions of knowledge held by mathematics teachers, with the result that the subdomains meet classificatory criteria in terms of the object or aspect of the particular teaching item under consideration. The three subdomains considered here correspond to different objects within mathematics itself: *Knowledge of Topics* (KoT) refers to the knowledge of content itself; *Knowledge of the Structure of Mathematics* (KSM) concerns knowledge of the structure of the content; and *Knowledge of the Practice of Mathematics* (KPM) is knowledge about how mathematics is constructed.

Regarding the dimension of HCK relating to values, we wonder whether the central values inherent to mathematics are really epistemic in nature. Reconsidering the exemplification for this subdomain given by Ball and Bass (2009) which they characterise as 'core mathematical values and sensibilities' we find elements such as 'precision, care with mathematical language, consistency, parsimony, coherence and connections'. We believe that, in the absence of a better characterisation, the items included in this subdomain are conceptions and beliefs about mathematics – such as mathematical attitudes – and towards mathematics, and cannot thus be considered knowledge. Nevertheless, after reviewing numerous works studying knowledge for teaching, we think that extending the framework to include such considerations can only bring a finer-grained sensitivity to the analysis, providing a broader and deeper snapshot of the teacher under study, and consequently of their knowledge for teaching.

With respect to future lines of research relating to this model, we feel that the most important thing is to hone the defining features of the subdomains, working with concrete examples which help to clarify each category, and to develop suitable methodological questions. An additional area of great interest is the question of what it is to know something, and we would like to undertake a comparison of different approaches to understanding what knowledge is (Meel, 2003), with the hope of arriving at a definition of knowledge in the specific case of a mathematics teacher that would allow us to study the relation between knowledge and conceptions.

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PROSPECTIVE TEACHER'S SPECIALIZED CONTENT KNOWLEDGE ON DERIVATIVE

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Mathematics teacher knowledge has been widely studied, and recently a remarkable advancement has been reach with the proposal of the "Mathematical Knowledge for Teaching" (MKT) model for describing the complex of knowledge that a teacher should have to teach a specific mathematics topic. Nonetheless there are still questions to be addressed, such as, how or under what criteria can the MKT be assessed? How the teacher educators can help the prospective teacher to develop the different components of the MKT? How are related the different components of MKT? In this report, we have tackled, though partially, such questions, by advancing specific criteria to explore the prospective teachers' knowledge about the notion of derivative: common, specialized and extended knowledge. In this report we inform specifically on the specialized content knowledge.

BACKGROUND

The mathematical and didactical education of prospective teachers is a very pressing issue for teachers' educators. One of the problems that have raised a great deal of interest is to identify the didactic-mathematical knowledge required for the prospective teachers to teach mathematics. On this regard, a great deal of research has been conducted to identify the components of the web of knowledge that a teacher should know in order to develop his/her practice efficiently and to facilitate the students' learning.

Some researchers have proposed a number of alternatives where some features that make up the teachers' knowledge can be identified. The works of Shulman (1986), Fennema & Franke (1992) and Ball (2000), show a multifaceted vision on the construction of the knowledge required to teach. More recent researches such as those of Ball, Lubienski & Mewborn (2001), Llinares & Krainer (2006), Ponte & Chapman (2006), Philipp (2007), Sowder (2007), Ball, Thames & Phelps (2008), Hill, Ball & Schilling (2008) and Sullivan & Wood (2008), show the nonexistence of an universal agreement on a theoretical frame to describe the mathematics' teacher knowledge (Rowland & Ruthven, 2011). This fact is a cause of concern not only for the prospective teachers' education and for the professional development of inservice teachers, but also for the researchers' community, because it is important to establish a general understanding on what meanings entails the content knowledge and how it affects the practice of teaching. It is difficult to have a coherent approach for a program of teacher education if the role of the teacher knowledge, the features implied and how they interact in the mathematics teaching process (Petrou & Goulding, 2011) are not well understood. The question is: how to determine such didactic-mathematical knowledge based on models that include categories too "wide"? As Godino (2009) points out, the various models on the mathematical knowledge for teaching, informed by the researches in mathematics education, include categories too "wide" and disjoint, that calls for models that allow conducting a more precise analysis of each knowledge component that are put into effect in an effective teaching of mathematics. Besides, the latter will allow orienting the design of formative actions and the elaboration of tools to assess the mathematics teachers' knowledge. We consider that the teacher's primary responsibility is to facilitate learning; such a teacher is what we call effective.

In this report we offer a partial answer to questions such as: How or under which criteria can the MKT be assessed? How teachers' educators can help the prospective teachers to develop the different MKT components? How the different MKT components are related among them? To propose an answer to these questions, we use the theoretic tools provided by the Onto-Semiotic Approach (OSA) (Godino, Batanero & Font, 2007) to knowledge and instruction. We have designed and applied a questionnaire to explore some relevant features of the epistemic facet of the didactic-mathematical knowledge of prospective teachers, on the derivative, which includes, according to the Ball and colleagues' model, the common content knowledge, specialized content knowledge and extended content knowledge. Specifically, we focus on the specialized content knowledge, for which we propose two levels of analysis and different categories of analysis.

THE DIDACTIC-MATHEMATICAL MODEL

In this research we use the Didactic-Mathematical Knowledge Model (DMK) proposed by Godino (2009) within the Ontosemiotic approach to cognition and instruction (Godino, Batanero & Font, 2007). This model for the DMK includes six facets or dimensions for the didactic-mathematical knowledge, which are involved in the teaching and learning of mathematics specific topics: 1) *Epistemic*: components of the institutional implemented meaning (problems, languages, procedures, definitions, properties, justifications); 2) *Cognitive:* development of the personal meanings (learning); 3) *Affective*: the emotional states (attitudes, emotions, motivations) of each student regarding not only the mathematics objects but also the planed study process, and its distribution over time; 4) *Interactional*: sequence of interactions between the teachers and students, oriented at the fixation and negotiation of meanings; 5) *Mediational*: distribution over time of the technological resources used and distribution of time for the actions and processes involved; and 6) *Ecological*: system of relations with the social, political, economic context that underlies and affects the study process.

For each of the above facets, four levels of analysis are considered. These levels allow the analysis of the teacher's DMK according to the type of information required to take instructional decisions. The aforementioned levels are: 1) *Mathematical and didactical practices*; description of the actions performed to solve the mathematics tasks proposed to contextualize the content and to promote learning. The general lines of action of the teacher and students are also described; 2)

Configuration of objects and processes (mathematical and didactical), description of mathematics objects and processes that intervene during the mathematic practices, as well as those which emerge out of them. The purpose of this level is to describe the complexity of objects and meanings that intervene in the mathematics and didactics practices. Such complexity is an explanatory factor not only for the meaning conflicts but also for the learning progression; 3) *Norms and meta-norms*, identification of the web of rules, habits, norms that regulate and facilitate the study process, and that affect each facet and its interactions; and 4) *Suitability*, identification of possible improvements of the study process, that increment the didactic suitability.

Due to the fact that our research was carried out with prospective teachers we focus the analysis on the epistemic and cognitive dimensions. We, then, focus on the facets 1) Mathematical and didactical practices; and 2) Configuration of objects and processes taking into account the students' answers to some tasks. Investigating the levels three and four of the analysis described above is beyond the scope of this report.

Subjects and Context

The questionnaire was administered to a sample of 53 students enrolled in the final modules (sixth and eighth semester) of the degree in mathematics teaching offered by the Universidad Autónoma de Yucatán (UADY) in Mexico. This is a four-year degree (8 semesters). The School of Mathematics of the UADY is responsible for training teachers to work at higher secondary or university level in the state of Yucatan (Mexico). The 53 students who responded to the questionnaire had studied differential calculus in the first semester of their degree course, and they had subsequently completed other modules related to mathematical analysis (integral calculus, vector calculus, differential equations, etc.). They had also studied subjects related to the teaching of mathematics. These 53 students constituted the entire population of students with these characteristics in the University of Yucatán.

The EF-DMK-Derivative Questionnaire

The questionnaire, which we have call *EF-DMK-Derivative* (Pino-Fan, Godino, Font and Castro, 2012, pp. 298-299), is made up of seven tasks, and has been designed to assess certain relevant features of the epistemic facet of prospective secondary teachers' didactic-mathematical knowledge (DMK) on the derivative. According to Ball and colleagues model (Ball, Lubienski & Mewborn, 2001; Hill, Ball & Schilling, 2008) this epistemic facet comprises three types of knowledge: common content knowledge, specialized content knowledge and extended content knowledge.

Three criteria were considered for the selection of the seven tasks that make up the questionnaire. The first states that the tasks must include a wide range of meanings related to the derivative; the second states that, for the resolutions, some representation means should be use, and the third states that the type of knowledge must include: common content knowledge, specialized knowledge and extended knowledge. The description of the features and content assessed in every task included in the questionnaire, can be seen in Pino-Fan, et al., (2012, pp.299-301). The methodological choice to build and implement a written questionnaire has the advantage of being applied to a relatively large sample, compared with study based in interviews to a few students, although it is not possible to deepen in the evaluation of the others facets and nuances of the didactic-mathematical knowledge.

The Specialized Content Knowledge

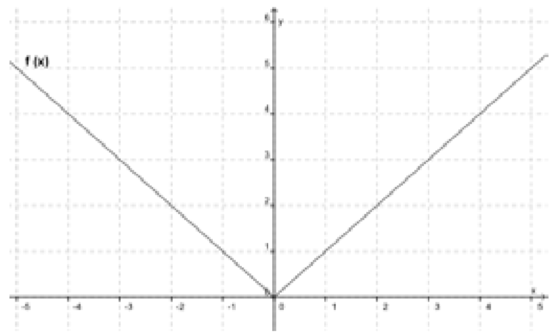
Ball, Thames and Phelps (2008) propose that the specialized content knowledge is the "mathematical knowledge and skill unique to teaching" (p. 400). This knowledge includes "how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Hill, Ball y Schilling, 2008, p. 377-378). We agree with this approach to the specialized content knowledge; nonetheless the question that arises is: what specific criteria allow us to analyze and to improve such knowledge required by the prospective teachers? One of the fundamental features of the specialized content knowledge tasks, included in the questionnaire, is the reflection carried out by the prospective teachers, on mathematical objects, its meanings and the complex relations among them. These web of complex relations are put into effect while teaching and learning mathematics. The relationship between objects and meanings are fixed and operationalized by means of the notion *configuration of objects and meanings* (Godino, et al., 2007). Such notion favors not only the systematic identification of a number of procedures to solve the mathematic tasks (including the identification of representations, concepts and properties), but also the identification of both, procedures justifications and properties used in solving them. Additionally, the aforementioned analysis, not only of the tasks but also of the didactic variables that intervene and orient the reflection, on both the possible generalizations, or particularizations, and the connections to other mathematical contents (Godino, 2009).

Thus, in our model for the didactic-mathematical knowledge, two levels are proposed for the specialized content knowledge. In the first level, the prospective teachers *should use not only, different* representations, concepts, propositions, procedures, and justifications, but the range of mathematical object' meanings of the mathematic concept under study – the notion of derivative –. The second level refers to the *teachers' competency to identify knowledge* (language elements, concepts/ definitions, properties/propositions, procedures and justifications) put into effect during the resolution of tasks on the derivative. It is clear that the specialized content knowledge implies the common content knowledge and some features of the extended knowledge.

The item a) of Task 2 (Fig. 1) assesses the common content knowledge, and the item b) assesses the extended content knowledge. The common content knowledge is used when the prospective teacher answers this item without providing any justifications nor using any representation. The extended content knowledge is assessed when the prospective teacher has to generalize the initial task about the derivability of the absolute value function at $x=0$, on the basis of valid justifications for the proposition "the graph of a derivable function cannot have peaks" by defining the derivative as the limit of the increment quotient. On one side, items b) and c) refer to the first level of the specialized content knowledge, because prospective teachers may solve them making use of both different representations (graphic, symbolic and verbal), providing valid justifications for their procedures. On the other side, item e) explores the second level of the specialized knowledge, because the prospective teacher must both, solve the aforementioned tasks and identify the web of knowledge that are put into effect in its resolution.

Task 2

Consider the function $f(x)=|x|$ and its graph.



a) For what values of x is $f(x)$ derivable?
b) If it is possible, calculate $f'(2)$ and draw a graph of your solution? If it is not possible, explain why.
c) If it is possible, calculate $f'(0)$ and draw a graph of your solution? If it is not possible, explain why.
d) Based on the definition of the derivative, justify why the graph of a derivable function cannot have 'peaks' (corners, angles).
e) What knowledge is put in to play when solving the above items of this task?

Figure 1: Task 2 from the *EF-DMK-Derivative* Questionnaire

Figure 2 shows Task 5, that assess the first level of the specialized knowledge, for prospective teachers must use different derivative meanings in its resolution: slope of a tangent line and, instant rate of change. At first glance it seems to be one of the "drill exercises" that are commonly found in the high school calculus text books, where it suffices to apply some theorems and propositions on the derivative to solve it. Due to the latter, both item a) and b), individually, assess features of the prospective teachers' common content knowledge. Nonetheless, the main task objectives are double: first, to explore, globally, the mathematical activity carried out by the prospective teachers, and, second, to test the connections or associations among the different derivative meanings established by them.

Task 5

Given the function $y=x^3-\frac{x^2}{2}-2x+3$

a) Find the points on the graph of the function for which the tangent is horizontal.
b) At what points is the instantaneous rate of change of y with respect to x equal to zero?

Figure 2: Task 5 from the *EF-DMK-Derivative* Questionnaire

RESULTS: ANALYSIS AND DISCUSSION

We will present the analysis of an answer provided by a prospective teacher (A), which exemplifies one of the solution typologies identified on the prospective teachers set of solutions to Task 2. On such analysis the primary objects (language, concepts, properties, procedures and justifications), and processes than intervene on both, the statement and on the tasks solutions, are identified. We base our analysis on the levels 1 and 2 of teacher's knowledge described above, which refer, both to *the didactical and mathematical practices*, and to the configuration of objects, respectively. In this section we do not present the "global" results of the entire questionnaire, not even the complete results for the two tasks that we presented. It is because we do pretend to show the usefulness of both the criteria and methodology of analysis proposed; on the other hand, the space constraints make it impossible to present the results in its entire extension.

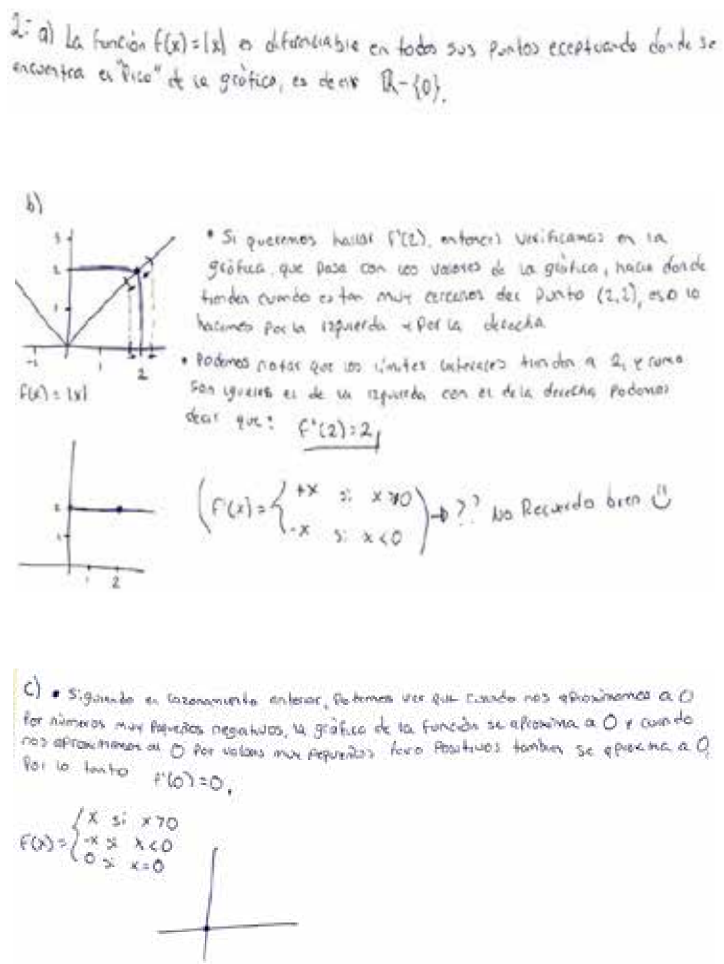
Analysis to Cognitive Configurations Subjacent of the Task 2

In regard to cognitive configuration used by the prospective teachers to provide a solution to Task 2, three types of resolutions were identified. Each type of configuration is associated to a specific configuration of objects and processes. We have named these three types of cognitive configuration as: *graphic-verbal, technic and formal.* A high percentage of prospective teachers, 88,6% and 54,7%, respectively, provided a configuration *graphic-verbal* to items a) and c) (e.g., "…it is not derivable at *x=0* because an infinite number of tangent lines can be traced, to the function on that point"). For section b), the majority of prospective teachers, 62, 3%, provided a *technic* configuration (using both the derivative rules and the definition of the absolute value function). One preservice teacher (1,9%) provided a *formal* solution, using the derivative meaning as an instant rate of change (limit of the increment quotient), for the first four items in the task.

Figure 3 shows the solution provided by the prospective teacher that has a *graphic-verbal configuration* associated. In regard to the mathematic practice performed by the prospective teacher (A), it can be observed in Fig.3 that he begins his solution process with a visual justification of the property "the derivability of an absolute value function". Generally speaking, his solution is made of verbal descriptions based on the graph of the function $f(x) = |x|$.

Cognitive Configuration:

In the solution provided by the prospective teacher (A), it is possible to identify the use of a great deal of linguistic features such as: the use of natural language (verbal descriptions), some symbolic entries such as "$\Re - \{0\}$" or the rule of derivation by parts $f'(x) = \begin{cases} x & si\, x > 0 \\ -x & si\, x < 0 \\ 0 & si\, x = 0 \end{cases}$ of an absolute value function [item c)]. In the same venue, student (A) uses graphic elements [items b) and c)] to "explain" his analysis. These linguistic elements refer to a group of concepts/definitions, propositions/properties that are illustrated in what follows.



a) The function $f(x)=|x|$ is differentiable at all points except where finds the "peak" of the graph, that is $\Re-\{0\}$.

b) If we want to find $f'(2)$ then we verify in the graph, what happens to the values of the graph, towards where it tends [the values] when they are very close of the point (2,2), we do it that for the left and for the right.

We can note [in the graph] that the lateral limits tend to 2, and as they are equal, both to the left and the right, we can say that: $f'(2)=2$

$\left(f'(x)=\begin{cases} +x & si\, x\geq 0 \\ -x & si\, x<0 \end{cases} \right) \rightarrow$?? I do not remember.

c) Following the above reasoning, we can see that when we approach to 0 by very smal negative numbers, the graphic of the function approaches to 0 and when we approach to 0 by very small positive numbers it also approaches to 0. Therefore $f'(0)=0$.

Figure 3: Graphic-verbal resolution of task 2. Student A

Among the concepts and definitions used by the prospective teacher, we can underline those of function (absolute value), domain (of the derivative function and represented by $\Re-\{0\}$), approximation (to a specific point of the absolute value function, in this case to *x = 2* and *x = 0,* taking values close to those points), and the derivative of the absolute value function (wrongly taken as $f'(x)=\begin{cases} x & si\, x>0 \\ -x & si\, x<0 \\ 0 & si\, x=0 \end{cases}$). The one side limits, and the bilateral limits, to the points *x = 2* and *x = 0,* though calculated correctly (if the question were to calculate the limits to the absolute function on such points), based on the graph, were incorrectly used by the prospective teacher (A), when calculating the derivative of the function in the points *x = 2* and *x = 0*. This misuse of one side limits, to calculate the derivative, seems to be based on the misinterpretation or misunderstanding of a proposition that refers to the relationship between continuity and derivability: a derivable function is always a continuous function, but a continuous function could be not derivable. This is put in evidence with the procedure and the ensuing justification to calculate *f' (2)*: "If we want to find *f'(2)*=2, then we verify in the graph what happens to the graph values, where do they tend when they are very close to (2, 2); we do it for both sides: left and right. We can see that the lateral limits tend to 2, and due they are the same, both on the left and on the right, we can say that f '(2) = 2".

The procedure and justification provided by the prospective teacher to calculate *f'(2)*=2 can also be seen in the graphic representation given to item b) (Fig.3). The misunderstanding is made more evident when the prospective teacher points out both the procedure and the justification to solve item c), which ask for a verification to be carried out on the derivability of the absolute value function at x=0. The student says: "Following the preceding reasoning, we can see that when we get close to 0, using very small negative numbers, the graph of the function gets close to 0; and when we get close to 0, using very small but positive numbers, the graph of the function also gets close to 0, thus $f'(0)=0$". Among other properties used in the student's solution we can highlight the derivability at zero of the absolute value function, which is visually justified, as follows, "The function $f(x)=|x|$ is derivable in every point except in those points where a "corner" is found on the graphic …".

The answers that we have included in this type of cognitive configuration, focus on procedures and justifications based on the visual analysis of the graphic features of the function, as in the example provided (Task 2). Another answer type, quite common, were those where the no derivability of the absolute value function, at x= 0, is justified by tracing an "infinity" number of tangents to the function at that point. Regarding item e), that assesses level two of the specialized content knowledge, it is clear that, as no previous instruction has been offered, to the prospective teachers, on the identification of previous and emerging knowledge involved in the task, only a limited number of concepts, such as function, absolute value and derivative at a point, were provided by the prospective teachers.

FINAL REFLECTIONS

The results obtained through the implementation of the questionnaire *EF-DMK-Derivative* show that the prospective teachers manifest difficulties to solve tasks related, not only to the specialized and extended content knowledge but also, with the common content knowledge. It is clear that the prospective teachers have a better performance when solving tasks that entail the use of the derivative as the slope of a tangent line. This was confirmed when the prospective teachers solve tasks such as the fifth (Fig.2) where their answers show a disconnection among the different derivative meanings. The manifested inadequacies of knowledge, justify the pertinence of designing specific formative actions in order to develop the epistemic facet of the didactic-mathematical knowledge on the derivative. The development could be accomplished first, by designing a teaching process for the derivative, which stresses the derivative global meaning (Pino-Fan, Godino y Font, 2011). Secondly, the two levels of the specialized content knowledge should be considered, both in its *application level* (use of linguistic elements, concepts, properties, procedures and justifications, as well as the use of different derivative partial meanings to solve the tasks) and in its *identification level*. The latter refers to the competency to identify mathematics objects, their meanings and the relation among them. This prospective teacher's competence would allow a suitable learning management of their future students.

These two levels for the specialized content knowledge are closely related, within the model DMK, to other facets for the teacher's knowledge. The level one, related to the application, is connected to the interactional and mediational facets (knowledge of content and teaching), because the mastery of this level of specialized content knowledge about specific topics, such as the derivative, gives the teacher the resources to perform efficiently his future professional tasks. The level two, identification, is related to the cognitive and affective facets (knowledge of content and students), because it entitles the teacher to detect (previously, during and after the teaching activity) features such as: the mathematical knowledge involved, mathematical objects and meanings, conflicts and mistakes that can arise to his/her future students. This identification competence may lead the prospective teacher to manage the students' learning in a more effective way. Finally, the Didactic-Mathematical Knowledge Model (DMK), offers the tool "configuration of primary mathematics objects" that allows analysing and categorizing some features of the epistemic facet of didactic-mathematical knowledge manifested by prospective teachers. In this report we have focused on the specialized content knowledge, how to analyse it and we have offered suggestions to improve it. The analysis presented herein, as a way of example, is what prospective teachers are expected to demonstrate at the (beginning of, during, and end of) second stage, and its development should begin since the early stages of teacher's professional training.

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DIDACTIC CODETERMINATION IN THE CREATION OF AN INTEGRATED MATH AND SCIENCE TEACHER EDUCATION: THE CASE OF MATHEMATICS AND GEOGRAPHY

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This paper presents an application of the Anthropological Theory of the Didactic to describe and analyse the genesis of an integrated mathematics and science pre-service teacher education. Reporting from the pre-experimentation phase, it is shown how the levels in the scale of didactic codetermination enable us to understand more clearly how integration is envisaged. We examine more closely the case of a bi-disciplinary teaching-module wherein math plays one part together with geography, and we demonstrate how the scale can be used to explore the precise nature of the intended interaction between the two disciplines.

INTRODUCTION TO THE AMBIGUITIES OF INTEGRATED EDUCATION

At the turn of the century, Czerniak, Weber, Sandmann, and Ahern (1999) made a literature review of science and mathematics integration. They concluded that a lot of "testimonials" existed for the positive benefits of integration, but few empirical studies actually supported this notion that an integrated curriculum is better than a well-designed traditional curriculum. They also emphasised that the term "integrated" was shrouded in ambiguity, with no clear distinction between the diverse labelling of the many-named phenomenon: interdisciplinary, multidisciplinary, cross-disciplinary, trans-disciplinary, thematic or blended, just to mention a few. Ten years later, Stinson, Harkness, Meyer, and Stallworth (2009) reported a similar lack of common characterizations when asking teachers to identify given scenarios as integrated or not. They further concluded that:

> "Potential gains from integration (i.e., time savings, improving on student achievement, improving student interest or motivation) are predicated on a common understanding for what integration means. At the very least, curricula or initiatives designed to foster integration must develop operational definitions for integration before laying claims to an integrated approach or product" (p.159)

The problems with specific labels like "multidisciplinary" or "interdisciplinary" are at least twofold: First, as indicated above, we have no commonly acknowledged definitions: In Andresen and Lindenskov (2009) p. 213-214, multidisciplinary was used to signify a cooperation with clear delimitation of the individual disciplines, proposedly in contrast to interdisciplinary where borders *between disciplines are claimed to be more or less cancelled*. In the same article, interdisciplinary is synonymous to cross-disciplinary, and trans-disciplinary is a radical form where no borders between disciplines are acknowledged. Completely opposite distinctions are found in Matthews, Adams, and Goos (2009), p.892, where interdisciplinary refers to curricula in which there is a mixture of science and mathematics although the *boundaries of the two disciplines remain visible*, and "integrated" is used to signify the lack thereof. A second, perhaps more profound problem is the principal inability of the labels to specify in any detail how the interaction between the disciplines is carried out, let alone what predicates the conditions of the interaction. For this reason too, we need a precise epistemological model of integrated mathematics and science education.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

In this paper we chose to use the term "integrated" to signify *any* educational setting where two or more institutionally established disciplines are intended to work together in order to bring about learning. It is thus used as an overarching name encompassing all the other labels which are usually employed to signify more specific ways of conducting integrated education. The Anthropological Theory of the Didactic (ATD) describes what happens in educational situations as situated in an institutional ecology. Such an ecology is described by a hierarchy of levels, cf. figure 1. This means that the conditions on one level depend on influences from other levels of the ecology. This interdependency is articulated using a scale of levels of determination (see Artigue & Winsløw, 2010; or Chevallard, 2004 for more details). In integrated education we are considering at least two such ecologies where an explicit decision has been made to cooperate. We can then identify at which level the decision has been made, at which level the cooperation is meant to take place etc. At most times it will be understood that the cooperation is initially defined at the disciplinary level when considering integrated mathematics and science education, but note that our definition of "integrated" also encompasses cooperation initiated on other levels e.g. the math teacher and the science teacher could agree to use "cooperative learning" (Kagan, 1989) during their respective lessons in the same class, thereby situating the integration on the pedagogy level. Focusing on integration at the discipline level, we have the following framework for investigating the disciplinary cooperation, indicated in figure 2. Above the discipline level, the two ecologies are in principle the same, because we are considering the disciplinary interaction *at the*

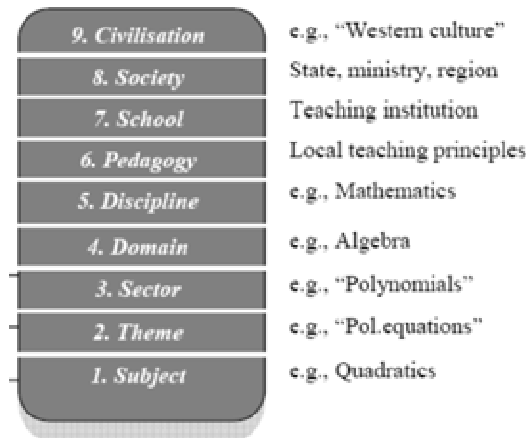

Figure 1: Mono-disciplinary ecology regarding quadratics. (Taken from Winsløw, 2011, p. 133)

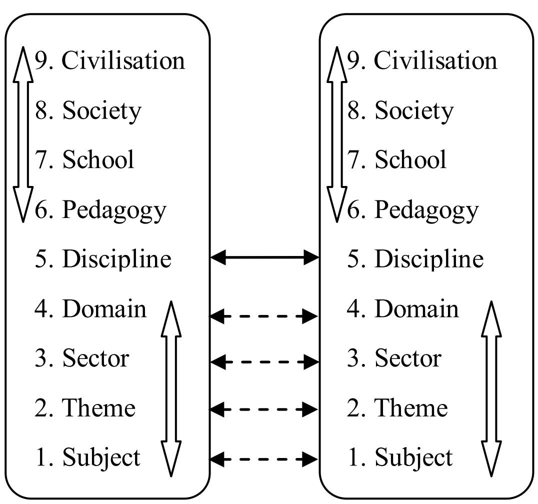

Figure 2: Levels of co-determination for bi-disciplinary ecology

same institution. It is important to note that "mathematics" and "geography" appears as disciplines in closely related, institutional contexts (such as lower secondary schools, universities, and "university colleges") and while these appearances are indeed different and bound to the institutions, they are also strongly linked. In this paper we will limit ourselves to look at disciplines inside one institutional context (university college) only. This means that influences from same-name disciplines in another institutional context, like universities, comes into our model at a higher level (mainly society). This we will call *second order influence*. It is now possible to model some of the central questions associated with the two integrated disciplines. The first type of questions concerns the "knowledge to be taught" (Bosch & Gascón, 2006):

> "What bodies of knowledge are chosen? How are they named? Why these ones and why with this kind of organisations? What are the reasons to these choices?" (p.56)

The answers to these are determined at the level of the discipline, and the levels above, as indicated by the leftmost vertical arrows.

The second type of questions concerns the levels below the disciplines, where the interaction is realised. These are usually more controlled by the teacher, but still constrained from above (Bosch & Gascón, 2006):

> "Why are mathematical contents divided in these or those particular blocks? Which are the criteria for this division and what kind of restrictions on the concrete activity of teachers and students does it cause?" (p.61)

While both types of questions are phrased the same way for mono-disciplinary education, they take on special meaning when more than one discipline is involved. Decisions taken from the perspective of one ecology have to be informed by the other. This is indicated on figure 2 by the horizontal arrows, where the solid one indicates the level at which the cooperation is formally defined, and the dotted ones signify the possibilities of interaction, whose existence and character may be further specified in a particular context, e.g. as part of the planning of an intended curriculum. This leads us to summarise the following research questions: What are the main features of the interplay between institutionalized ecologies in the planning of integrated math and science education? How can we study the "integratedness" in an inductive way, starting with actual and concrete plans for interaction, rather than with general rhetoric that tends to blur the detailed features? What conditions the planning and cooperation in integrated approaches?

OUR CONTEXT AND METHODS

Teacher education in Denmark is institutionally placed at so called "university colleges", which are higher education institutions independent from research universities. A consortium between the University of Copenhagen, University College Copenhagen and the Metropolitan University College was formed to construct an experimental teacher education program (called ASTE, Advanced

Science Teacher Education). The goal was to investigate, among other things, the synergistic effects of a multi-disciplinary science teacher education. The students are to become teachers of math and science in the lower secondary school, and the design of ASTE has been developed jointly by participating college and university professors. One of the main characteristics of the program is that large proportions of ordinary curricular items have been placed in bi-disciplinary teaching modules. In this paper we will apply the theoretical framework to study the development of the module named: "Geographical Information Systems, data analysis and modelling in geography", where parts of the math and geography contents are to be taught together. It should be noted that the geography discipline in Danish lower secondary school and at Danish teacher education colleges covers both physical and human geography, and the module we consider here also reflects that.

To shed light on the planning of this interplay we have conducted qualitative interviews with the five developers of this particular module; one math and geography college-professor from each university college (below referred to as CM_1, CM_2, CG_1 and CG_2) and one university-professor from the geography discipline (called UG) and with special interests in education. The institutional affiliations are rather complex in the ASTE-collaboration, but we will present them here because the institutional setting is of importance to our model: The two math college-professors are women, and although at the time of interviews they represented two separate colleges, one of them had only recently changed from one to the other. The two geography college professors, male and female, come from another university-college than the two math professors. It should be noted that it is only the college professors who are expected to do the actual teaching, and the programme is implemented in the *physical* institutional setting of a branch of one of the participating university colleges, to which only one of the college professors (math) belong. The ASTE program comes with its own *formal* institutional settings that are written down in general sections of the curriculum, parts of which are tailored specifically, while others are adopted directly from the ordinary institutional framework.

The interviews were conducted in August 2012, beginning with a pilot interview of one geography college-professor. Informed thereby a scheme of questions were designed, and it was decided to ask the interviewees in advance to think of 1) a concrete activity to undertake in the module and 2) if possible, try to think in broad terms of an "entire" plan of action for the module. This aims to follow the inductive approach, starting with the levels below the discipline level. The interviews were semi-structured and lasted approx. 40 minutes each. They centered on two distinct parts: The design of the specific module as situated in the framework of ASTE and further thoughts on the realisation thereof. There was a focus on the individual respondent's perceptions and experiences from the curriculum drafting work, seen in relation to their stance in the existing education system.

DATA HANDLING AND RESULTS

All the interviews were recorded electronically and subsequently inventoried minute for minute. The quotations in the following subsections are translated and transcribed from oral Danish by the authors, and are indexed according to their temporal placement in the interviews

Determination of curricular items for bi-disciplinary integration.

Most of the curricular items chosen from each of the disciplines (see figure 3), are recognizable to both mathematics and geography teachers, that is, they have suggestions about what an item from the other discipline could contribute to their own, but generally they do explicitly acknowledge their lack of expertise in the other discipline e.g.:

> My challenge is that I do not know much about the geography discipline in teacher education …so although I do understand the words, I have difficulties knowing what contents they represent, because I do not know much about geography as a discipline in teacher education. But on the other hand, it is also what makes such cooperation enormously exciting. It is exactly to become knowledgeable about the other disciplines. Are the problems they work with similar to those of math, and how is it about methods? This, I think, could be enormously exciting to get insightful about. (CM_1; 16:11-16:42)

From Geography:

- Knowledge, theories and problem from physical and human geography
- Skills at using geographical sources and methods
- IT as integrated part of the discipline in university college and lower secondary school
- Knowledge to further students geographic language and “bildung”
- Skills to utilize informal arenas of learning and employ investigative methods of inquiry

From Math:

- Geometry, specifically analytic, parametrizisations and trigonometry
- Using and evaluating appropriate representations
- Defining, structuring, mathematizing, interpreting and critique of mathematical models
- Skills at planning, organizing and evaluating teaching.

Figure 3: Curricular items chosen for the module: “GIS, data analysis and modelling in geography”

This supports the notion that the intended integration, instigated at the discipline level, is indeed formal, but there is the desire to make it real by getting to know the other discipline through cooperation, and this is even seen as a separate advantage. It is also evident that curricular items, formulated along the lines of “the use of IT” are recognizable because they are determined at the school and pedagogy level, which are common to the two disciplines. Then there are some items, the determination of which, are situated exclusively at the level of the discipline: “Using and evaluating appropriate representations” and “skills at using geographical sources and methods” are intrinsic to the disciplines, but they pertain to similar disciplinary categories, such as the use of abstract symbols or graphics to represent data.

One could wonder why, or why only, the geometry domain is mentioned from the math discipline, and not e.g. statistics or functions, which could go well together with “data analysis”. The interviewees agree that it is a practical and in a way arbitrary choice, because other domains could be made to work out just as well, but there is obviousness to the pairing of geometry and physical geography, which in our model can be seen as determined on the civilization level:

> "It is something about the measurement of the earth and maps, it all fits together very nicely, scales and similarity, it is nearly obvious! (CM_2 20:20-20:28)

The etymology of geo-metry and geo-graphy alludes to the kinship between the two and historically, one may contend that the choice of "maps" and "geometry" are the primary domains involved in the interaction, and is indeed natural and reflects culturally rooted views of the two disciplines and the links between them.

Co-determination of concrete activities in the module.

In fact, when it comes to respondents' ideas for teaching, they are all connected to the notion of maps: The reference to different kinds of maps, the making of maps, maps as a tool for investigation, the historical development of maps. This is determined both at the discipline level, but also conditioned by the requirements of the institution in which the pre-service teachers are going to teach (schools; this second order influence appears at the society level in our model):

> "Mathematics sometimes requires practical examples to illustrate what math is, and that is what you can do when working with mapmaking, ...because the math related to that, is also the math you use in lower secondary school" (CG_1 18:47-19:13)

In figure 4 we have shown an example of how to model the interdependencies surrounding the creation of a physical map, as the subject of a teaching activity. Using this model it is possible to identify possibilities and constraints that one discipline imposes on the other, to the enactment of such an activity: At the domain-level, the choice of geometry, applied to the measurement of the earth, precipitates the restriction to physical geography. This in turn determines what should be the types of maps to work with: e.g. physical or topographic instead of other more culturally oriented maps such as political maps. Then distances could be the information you would like to convey using the map, and the choice of plane geometry might be taken to avoid the time-consuming complication of earth curvature and the different heights of the landscape being drawn on the map, especially if triangles are to be the mathematical theme. Triangles are used when making the measurements of the land considered as a plane surface, and then, by similarity, transferred to the map:

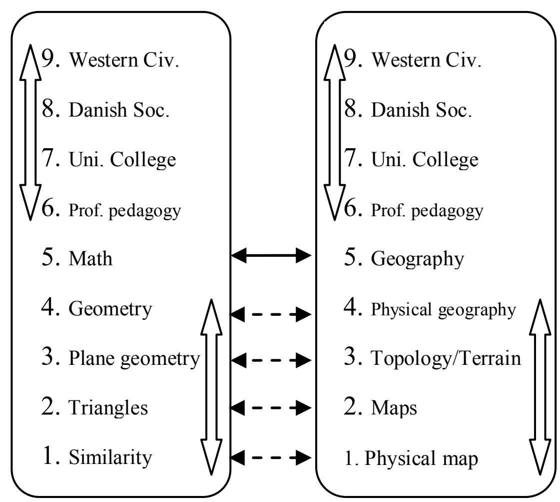

Figure 4 Example of bi-disciplinary ecology inferred from suggestions of interviewees

> "yes, that about surveying, at least triangulation and how maps has been made, and how maps look; I don't know how much surveying per se [they have in geography], when I

think about surveying, it is more from a mathematical point of view, when we go out and construct figures out there" (CM_2 14:31-15:50]

"It it clear that it would be tempting to measure on a sphere, meaning spherical geometry... but this [module] is not huge, so we have to be careful about how much we can achieve in the given time (CM_2 24:42-24:57)

It may appear curious that the math teachers so willingly lend their discipline to the making of maps and surveying, but it is a common feature of Danish teacher education that applied math is considered of quite high value, which is also directly reflected in the curriculum description of the mathematics discipline at university colleges:

"The history of the discipline, the discipline's function as a bearer of culture, and the application of the discipline, is an important part of its identity as a teaching discipline" Undervisningsministeriet (2011, appendix 2, section 3)

Looking at other suggestions to the possible contents of the module we find a lot of references to "problems of flooding and water flow" (CG_1 5:10, CM_1 19:20-19.37; 22:10, CM_2 15:20-16:00; CM_2 17:55-18:20). These are all strongly influenced by the society level:

"It would be really nice if it [the context for teaching] could be some kind of real problem" (CM_1 22:21-22:28)

"... but it [the problem of the activity] is strictly a concern for society, it is all concerns for society" (CM_2 18:30-18:35).

Let us now select two of the concrete examples: 1) Investigating the impact of a new national law, proscribing that no land may be farmed that is closer to a stream than 5 meters. 2) Investigating the flow of water through a lake or stream. Both open up a host of possible teaching avenues, both for math and geography. In figure 5 and 6 we have put this into our model to describe how decisions at higher levels of the scale will interact to produce the possible integrated practice at the subject level. In both examples we recognize the aim to use Geographical Information Systems as a tool to do investigations, but GIS does not appear as an object of knowledge in itself. Therefore it is not mentioned explicitly in the model. It is used at the praxis levels (thematic and subject), and it is conditioned at the discipline level:

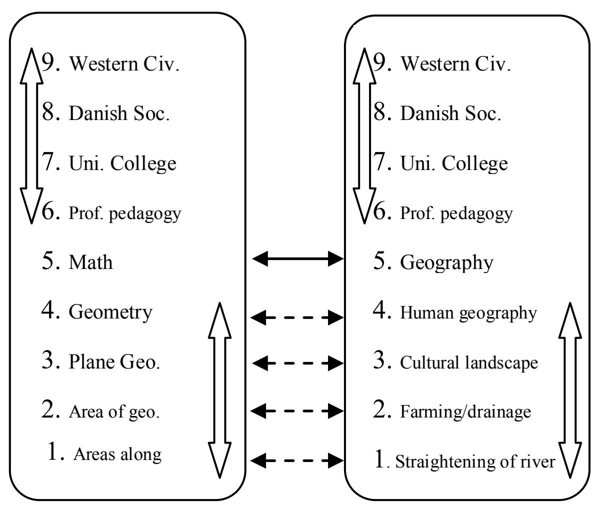

Figure 5 Example of bi-disciplinary ecology "Investigating impact of new law"

> "I think, it is because many who work with geographical information systems believe it is absolutely obvious that the whole world should know about it, because it is so incredibly smart and it appears in so many contexts" (CG_1 6:14-6:25)

It is worthwhile to notice that CG_1 expresses the desirability to have GIS included at the discipline level of math and geography, in the institutional ecology of the university college, whereas the reasons for the desire is said to come from parts of the society level, namely *those who work with GIS*. This is another example of a second order influence, which comes from an institutional context outside the university college (namely, from the same-name discipline in scientific institutions such as universities). Looking at the first avenue of teaching in figure 5 (example 1 above), we remark that it is directly determined on the society level as the idea originates from the consequences of a political decision. This clearly conditions the domain level to the geographical subfield of human geography, and the sector narrows it down to looking at culturally formed landscapes. Then what can be investigated mathematically is the area of farmland affected by the law, giving rise to the sector of plane geometry. One could hypothesize that the subsequent implications for the economy of the affected farmers could be mathematically considered, *but that would not be in strict accordance with the choice of geometry at the domain level*. As a consequence the theme and subject will, in regards to math, revolve around non-trivial calculations of area alongside curves, which could benefit from the aid of GIS. The geography part could draw on the area calculations and focus on issues of farming and the straightening of rivers (a classical subject of Danish human geography)

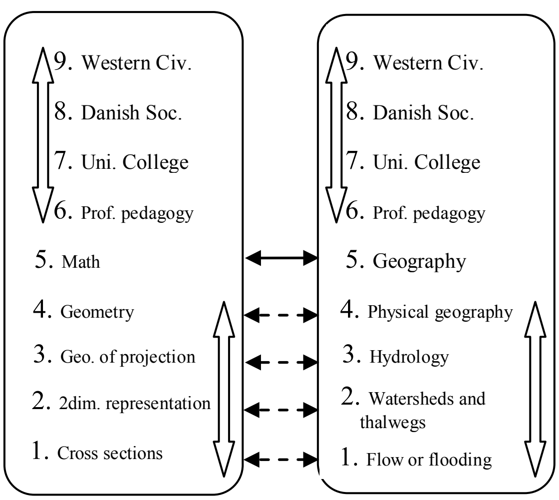

Figure 6 Example of bi-disciplinary ecology "Flow of water"

The second avenue of teaching (example 2, above) takes us into the "geometry of projections" sector of the chosen mathematical domain. (Figure 6) This is influenced by the choice of hydrology in the physical geographic domain, which has the study of moving water as an object. One theme could be thalwegs, in which two dimensional representations of river cross sections are extensively used. Also the cross section of the inlet and outlet of a lake, will determine the area of the lake surface. Horizontal cross sections of the lake landscape can be used to predict the extension of the lake for different flow rates:

> "We talked at some point about the flow through a lake, how the surface area, yeah, could be measured, but also how it changed in accordance with the flow in and out." (CM_1 19:13-19:31)

To construct the two dimensional representation, it is suggested that students physically go into the geotope and make the measurements using a mobile phone application, that can transfer data to a GIS system. Doing investigations in the field is an important part of geography, valued at the discipline level, and the processing of data to make the graphical representations, in this case, of river cross sections, is firmly rooted in the mathematical ecology at university colleges (school and discipline levels). The above quote may also allude to find some relationship or model of the changing lake surface area, expressed in terms of a function, as a pivot for the teaching activity. However, that would conflict with the choice of geometry at the domain level. This choice, if vigorously adhered to, seems indeed to impose rather strong restrictions on the lower levels:

> "You could easily get into functions here, and differential equations, if you look at the velocity with which the water runs from the lake. So that could be described in some dynamical systems, but we have nevertheless chosen that it [the module] should take another direction." [CM_1 29:15-29:39]

The two above analysed avenues of teaching both have common connection to the concept of "flow". This reminds us of what Wake (2011) calls a *bridging concept*, which "provide a driver to facilitate cross-disciplinary thinking" (p.1004). But we contend that the interaction among the different levels of didactic codetermination in the bi-disciplinary ecology provides a refined and more precise model of the idea reflected by the term "bridging concept".

CONCLUSION AND PERSPECTIVES

The way two disciplines, as situated in the institutional ecology of teacher education at university colleges, interact, when trying to establish integrated education, are determined by factors residing at levels above the one immediately considered. The interaction crosses the disciplinary boundaries, meaning e.g. that the domain level of one discipline will influence the theme level of the other. The route of influence, as expressed in interviews with the developers of the integrated math and geography teaching module, can be modelled by the levels of determination, and *it goes by way of the vertically and horizontally indicated directions*. That is, we have seen no determinations that appear to go, for example, directly from the pedagogy level of the math disciplinary ecology to the theme level of the geography ecology. But the possibility of such level crossing codetermination in integrated education needs to be further investigated. This question, and the more general one of seeing borders between disciplines as a criterion of demarcation, is by no means a trivial one when we look at the long ongoing debate about the nature of integrated education. In the illustrations used to represent our model we have what appear to be clear borders between the participating disciplines. This is to recognize that our model *does* operate with disciplines as distinct bodies of knowledge and this also reflect evident conditions in the institutional context studied. Indeed, the curriculum construction in ASTE begins with the existing disciplines, which define relevant positions in the institutional context of the university college. In that fashion the disciplines come before "big questions" even in the early planning phases. The model is not to be taken normatively, and it does not say to which degrees borders do, or should be discernible, in order to represent "true" integration. It serves simply to organise our analysis of how integration of disciplines takes place, or is planned to take place. Finally it allows us to situate second order influences from same-name "scientific disciplines" and "secondary school disciplines" which co-determine the planning and cooperation in integrated approaches at the university colleges participating in this project. An extension of the model to study these influences further appears of great interest to understand more globally the interplay between participating institutions.

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A REFERENCE FRAMEWORK FOR TEACHING THE STANDARD ALGORITHMS OF THE FOUR BASIC ARITHMETIC OPERATIONS: FROM THEORETICAL ANALYSIS TO TASK DESIGN

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The study presented in this paper offers a reference framework for characterizing primary school teachers' mathematical knowledge of the standard algorithms of the four basic arithmetic operations. The framework was first devised theoretically, from mathematical analysis of the algorithms and from past research on knowledge for teaching arithmetic operations with rational numbers. It was then applied to designing tasks for charactering and deepening the teachers' conceptual understanding of the algorithms. The paper contains examples of tasks related to the knowledge of mathematical principles underlying the algorithms, which were tested with a group of 46 primary school mathematics teachers.

Keywords: *standard algorithms of the four basic arithmetic operations, mathematical knowledge for teaching, task design.*

INTRODUCTION

The four standard algorithms of the four basic arithmetic operations is one of the main topics currently taught in primary school[1]. Much time is devoted to teaching this topic, but according to many studies, the pupils have difficulty in performing and understanding it (Fuson, 1992; Fuson et al., 1997; Kilpatrick et al., 2001). There is serious criticism of the way this subject is taught (e.g., Lee, 2007; Ma, 1999; McIntosh, 1998). Many researchers argue that it is taught in a rote manner that does not encourage conceptual understanding. In order to understand the roots of this problem in depth, several studies focusing on the teachers' knowledge of this subject were conducted (e.g., Hill & Ball, 2004; Hill et al., 2008; Kilpatrick, Swafford & Findell, 2001; Ma, 1999; Tchoshanov, 2011).

With few exceptions (e.g., Peled & Zaslavsky, 2008), the recent studies focus on particular algorithms. A framework for characterizing the knowledge for teaching the four standard algorithms as a holistic topic is still missing. The need for such a

[1] This article deals with the four standard algorithms being taught in Israel. These algorithms may have slightly different appearances in different countries.

framework manifests itself due to the broadly recognized importance of emphasizing the connectedness of mathematics (e.g., NCTM, 2000). In addition, and in line with research on other mathematical topics, such a framework is needed for systematic design of mathematical tasks with the potential to characterize and promote the teachers' knowledge of the subject. The present study aims at fulfilling this gap[2].

Specifically, the goal of the study is to identify, first theoretically, the core components of mathematical knowledge for teaching the standard algorithms of the four basic arithmetic operations, and then to empirically examine the feasibility of the framework by applying it to the design of tasks having the potential to capture variations in the teachers' knowledge.

In the next section we present and theoretically justify the framework. This is followed by examples of tasks related to one of the framework's components, the knowledge of mathematical principles underlying the algorithms. The article is concluded by remarks on applicability of the framework for research and practice.

REFERENCE FRAMEWORK

The framework presented in Table 1 is devised based on juxtaposition of two sources: (1) past studies on the knowledge for teaching the basic arithmetic operations with rational numbers, and (2) analysis of the algorithms in terms of the underlying mathematical principles.

	Component	Operative criteria for component examination
PK	**P**rocedural **K**nowledge	1. Demonstrated knowledge of the algorithms' steps 2. Performing correct computations
KP	**K**nowledge of mathematical **P**rinciples underlying each algorithm: place-value, number regrouping and distributive law	1. Recognizing mathematical principles, which are violated in students' incorrect performance 2. Explaining the stages of the algorithms using appropriate mathematical principles 3. Recognizing mathematical principles underlying different representation of the algorithms
KS	**K**nowledge of **S**imilarity between different algorithms based on the common underlying principles	1. Recognizing common principles underlying different algorithms 2. Classification mistakes in the use of different algorithms by pointing out the common mathematical principles violated
KR	**K**nowledge of different **R**epresentations of each algorithm and connections between them	"Translating" standard, vertical, algorithms to horizontal representations

Table 1: Reference framework for teaching the four elementary algorithms

[2] The paper is based on the PhD research of the first-named author conducted in the Technion under the supervision of the second-named author, Prof. Orit Zaslavsky and Dr. Irit Peled.

Noticeably, the first component corresponds to the notion of procedural understanding, and the next three – to conceptual understanding, as defined by Skemp (1987).

Procedural Knowledge (PK) refers to one's ability to accurately perform the algorithms. This component is not unique for teachers' knowledge. Ball, Hill and Bass (2005) claim that being able to perform the algorithms correctly is essential for teaching, but also insufficient.

Knowledge of mathematical Principles underlying each algorithm (KP) component is included in the framework because, mathematically speaking, the algorithms work based on certain mathematical laws, and it is reasonable to assume that knowledge of these laws is an important part of conceptual understanding the topic. Peled and Zaslavsky (2008) classified this type of knowledge as "local connections between procedures and conceptual knowledge" (p. 28). The principles for the framework were chosen as follows.

First, many researchers assert the role of the *place value principle* in the addition and subtraction algorithms (Ball, Hill & Bass, 2005; Kilpatrick, Swafford & Findell, 2001; Fuson, 1992; Ma, 1999; Thanheiser, 2010); some mention the principle as underlying also the multiplication (Ma, 1999; Kilpatrick, Swafford & Findell, 2001) and the division (Kilpatrick, Swafford & Findell, 2001) algorithms. Second, several researchers point out that the *number regrouping principle* is reflected in addition and subtraction algorithms (Kilpatrick, Swafford & Findell, 2001; Fuson, 1992; Ma, 1999), and it is possible to show that this principle is also involved in the multiplication and division algorithms. Third, Ma (1999) recognizes the *distributive law* as underlying the multiplication algorithm, and it is not difficult to see that it also underlies the division algorithm. In sum, the first two principles underlie all the four algorithms, and the third one – the multiplication and division algorithms.

Knowledge of Similarity (KS) between different algorithms based on the common underlying principles was considered by Peled and Zaslavsky (2008) as a kind of meta-knowledge "about a procedure which includes global aspects underlying a specific procedure or common to a number of procedures" (p. 31). They explored the iterative structure of the standard algorithms as an example of such meta-knowledge and argued that it may emerge in learners based on noticing the iterative structure of each algorithm.

In the proposed framework, we choose to focus on the similarity between the algorithms at the level of the above three mathematical principles. Consequently, there is a sort of hierarchy between the KP and KS components: the KP component is necessary (though not sufficient) for KS.

Our focus on the mathematical principles in the KS component is based on the two-fold argument, as follows. On one hand, a person who studied each algorithm separately and observed that a particular principle repeatedly appears, may use this observation as a semantic tool for expressing in-depth similarity between the algorithms (cf. Peled & Zaslavsky, 2008, for a compatible argument). On the other hand, a person who learned that the same principle underlies several algorithms may use this knowledge in order to unpack how the principle works in each algorithm.

Knowledge of different Representations (KR) *of each algorithm and connections between them* component is included in the framework because many researchers accent the importance of knowledge of different representations and models of the subject taught for its conceptual understanding (e.g., Ball, Hill & Bass, 2005; Davis & Simmt, 2006; Hiebert & Carpenter, 1992; Leikin & Levav-Waynberg, 2007; Ma, 1999). Among various representations and models of the four standard algorithms, we choose to focus on translating from the standard, vertical, representation of the algorithms to their horizontal representations. This is because for all four algorithms the translation process is rich with the opportunities to unpack the procedures' steps and reveal how the aforementioned mathematical principles work.

Each knowledge component may be manifested at different levels of depth. The operational criteria for identifying the teachers' knowledge on a *conceptual level* are presented in Table 1. A person may also possess only *rote-level* knowledge of the algorithms. In terms of our framework, the *rote level* of knowledge is manifested when a person, who was encouraged to explicitly use his or her knowledge of the underlying mathematical principles, exposed only the knowledge of the technical-computational aspects in discourse or performance. These two levels of knowledge are exemplified in the next section.

CHARACTERIZING THE COMPONENTS OF KNOWLEDGE BY MEANS OF TASKS

In this section we show how the framework can be applied to the design of tasks having the potential to reveal one's level of knowledge of the algorithms. Due to space constraints, we decided to focus only on the KP component. Three tasks that correspond to three operational criteria characterizing the KP component of knowledge are presented below. The tasks were tested in five 90-minute professional development workshops with two groups of elementary school teachers. Overall, 46 teachers participated in 10 workshops, which were conducted by the first-named author in the framework of a one-year professional development course. The teacher-participants possessed at least B.Ed. degree, had already taught mathematics (among other subjects) in elementary school and wished to be certified as elementary school mathematics teachers.

The principles underlying the four algorithms were discussed with the teachers prior to exposing them to the tasks. The teachers worked on Task 1 and Task 3 in groups of two to four participants; Task 2 was given for individual work. The performance of each group on each task was audio-taped and transcribed; all the written materials were collected. Additional insight on teachers' work on the tasks was received during individual interviews conducted after the end of the workshop's part dealing with the standard algorithms. The two answers accompanying Task 1 and Task 2 presented below are representative of the edges of the spectrum of the answers: the first points to the rote level and the second to the conceptual level of knowledge. Two answers related to Task 3 exemplify different degrees of conceptual-level knowledge. The examples were chosen in order to demonstrate the range of variations in the teachers' knowledge, which can be captured by the framework.

Task 1: Mistaken computations

The task corresponds to the first operational criteria for the KP component (see Table 1). The teachers were given a series of computations by imaginary pupils containing various violations of the place-value and regrouping principles in the four algorithms[3]. Four items presented in Figure 1 deal with violations of the place-value principle. For each computation, the teachers were required: (1) to explain the mistake, (2) suggest its possible reason(s), and (3) offer a suitable method to pedagogically treat it. They were asked first to discuss the above three requests, and then to put their agreed responses in writing.

Note that the teachers were not instructed to necessarily use the names of the principles in their written responses, though the principles were emphasized at the beginning of the workshop. Thus, a variation inherited in the task was related to the teachers' choice of whether or not to use the (conceptual) language of the mathematical principles or to use rote language.

```
                                                    12
   520         7444          213                 ______
 + 45        - 586         × 354                 1352|11
 ----        -----         -----               - 11
   970         1584          852                 --
                          + 1065                - 25
                             639                  22
                           -----                  --
                            2556                  32
```

Fig. 1: Examples of mistaken computations from Task 1

One group of teachers responded, in writing, to the multiplication item in Figure 1 as follows:

> Description of the mistake: The student didn't write the numbers at the right place.
>
> Possible reasons for the mistake: The student forgot to move the numbers (the partial products) to left side.
>
> Possible method to treat the mistake: To remind the technique of the multiplication algorithm.

Noticeably, the teachers choose not to mention the place-value principle in their response and considered only the technical aspect. They did the same with the rest of

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[3] Some computations were adopted from Zaslavsky (2003) and the others were offered by the first-named author based on mathematical analysis of the algorithms

the items. Moreover, the principles were not mentioned also in the audio-taped discussion. Thus, it can be suggested that the mistaken computations were treated by the teachers in a rote manner.

Another group of teachers responded to the task as follows:

Description of the mistake: The student doesn't understand the meaning of the number's place according the place-value principle.

Possible reason for the mistake: The student doesn't understand the place-value of the number 1065 (tens) and 639 (hundreds). The student doesn't understand the importance of adding the appropriate units at the stage of addition of the algorithm.

Possible method to treat the mistake: To ask the student to estimate the product quantity: $200 \cdot 300 = 60000$. Is it possible that $213 \cdot 354 = 2556$? To explain the student that 213 multiplied by 5 tens is 1056 tens, so we must to write the product at the place of tens, so at the next stage of the algorithm we can add appropriate units.

In this response we can see explicit reference to the place value principle. In "possible methods to treat the mistake" the teachers suggest a series of exercises aimed at raising the pupils' awareness of the place value principle indirectly, by means of estimating the results. Such an answer can be taken as a manifestation of the teachers' conceptual-level knowledge of KP.

Task 2: Unpacking the algorithms

The task corresponds to the second operational criteria for the KP component (see Table 1) and concerned the multiplication and division algorithms. As to multiplication, the teachers were offered the following scenario:

Imagine that you are sitting in the teachers' room. One of your colleagues asks you to explain as specifically as you can each step of the standard multiplication algorithm while solving an exercise "435 times 28." The colleague specifically asks you to point out the mathematical principles underlying the algorithm as she is going to teach the algorithm next week and wants to be sure that she knows not only the technical part.

The task was to individually write the explanation asked for by the colleague. The contrast between the responses of Teacher A and Teacher B, in terms of rote and conceptual levels of the KP component is apparent below.

$$\begin{array}{r} \times\ 435 \\ 28 \\ \hline 3480 \\ +\ 870\ \ \\ \hline 12180 \end{array}$$

Fig.2: A computation from Task 2

Teacher A (with the reference to Fig. 2):

8 multiplying by 435 $\longrightarrow$ 3480. We write the result right beneath.

2 multiplying by 435 $\longrightarrow$ 870. We move the result one place to the left (it's possible to write zero as a placeholder). Finally we adding the results and get 12180.

Teacher B (with reference to Fig. 2)

> First of all, you have to explain that we regroup 28 as 20 and 8 and multiply each part separately: $435 \cdot (20+8)$. We then have to remind the distributive law in order to explain the multiplication: $435 \cdot (20+8)$
>
> 8 units multiplied by 435 is 3480 units, and 2 tens multiplied by 435 are 870 tens. So we must write the results in the appropriate places. It's the key element of the explanation. Only after understanding this properly you can continue. Finally, you have to add the partial products and pay attention to the units.

In sum, the rote and conceptual level responses were observed at both Task 1 and Task 2. Note however, the following differences in task design: Task 1 addresses imaginary students and Task 2 – an imaginary colleague; Task 1 does not include explicit instruction to mention the mathematical principles underlying the algorithms and Task 2 does. Consequently, one can reasonably suggest the following. Those teachers who responded to at least one of the tasks in a conceptual manner may be characterized as those who possess conceptual-level knowledge of the KP component. Those teachers who responded to both tasks in rote manner, probably not only decided not to mention the principles in the given context, but indeed do not possess conceptual-level knowledge.

Task 3: "Translating" the algorithms into horizontal representation

The task was to "translate" all the stages of the computation presented in the standard, vertical, form, into a horizontal row of transformations and justify the validity of each equality sign in the row by the aforementioned mathematical principles. The task included items related to the subtraction, multiplication and division algorithms; each algorithm was dealt with in one 90-minute workshop. The "translating" step of the task corresponded to the KR component (see Table 1) and the "justifying" step corresponded to the third operational criteria for the KP component.

The division algorithm was reserved for the last workshop, so the teachers already knew how to deal with the task in the contexts of the subtraction and multiplication algorithms. Nevertheless, 7 out of 10 groups appeared to be unable to translate the division presented in Fig. 3 and returned empty sheets. Apparently, this occurred because the task was more demanding that the previous ones: only those who possess conceptual knowledge of the division algorithm could write something in response. Two examples below suggest that groups who did respond to the task possessed the required conceptual knowledge, although to different degrees.

```
  206
 ______
  2473 |12
 -24
 ____
   073
   -72
  ____
     1
```

Fig.3: A computation from Task 3

One group of the teachers responded to the task as follows:

Division 2473 by its parts	Performance of division

2473:12=2400:12+72:12+1:12=200+6+1:12

As can be seen, the teachers translated the computation using two row transformations and used their own expressions to explain them rather than the names of the aforementioned principles.

The response of another group follows.

Regrouping 2473	Distributive law of division	Performance of division	Performance of addition

2473:12= (2400+72+1):12=2400:12+72:12+1:12=200+6+1/12=206+1/12

This group translated the vertical computation using four row transformations and explained the first two in terms of the intended mathematical principles.

Based on the above written responses, follow-up interviews and a whole-group discussion, we know that the first group chose to translate the vertical computation into a horizontal form straightforwardly, by focusing on particular numbers in the vertical algorithm (i.e., 24, 72 and 1). Then when the teachers tried to explain each horizontal transformation as requested in the task, they observed that the names of the principles did not fit and used their own expressions. The second group began from recalling the relevant principles and decided how many and which transformation to write so they would fit the names of the principles. Consequently, we deem that the response of the second group reflects a higher degree of conceptual knowledge of the division algorithm, in terms of the offered framework, than that of the first one.

SUMMARY AND CONTRIBUTION

Ball, Hill and Bass (2005) point out the teachers' ability to unpack a mathematical subject into related sub-subjects as an important component of mathematics knowledge for teaching. In the context of the standard algorithms of the four basic arithmetic operations, such an unpacking still needs operational conceptualizing. A reference framework presented in this article offers one way of so doing. Based on the above theoretical and task design considerations, we argue that the proposed framework captures the essential components of teaching knowledge of the subject and can provide a solid basis for teacher preparation programs and for characterizing the teachers' knowledge. Specifically, the framework puts forward the mathematical principles underlying the algorithms as an overarching semantic/mathematical tool for identifying the components of knowledge needed for teaching the algorithms. Consequently, the study continues and advances the research venue started in the studies of Ball, Hill and Bass (2005); Ma (1999) and Peled and Zaslavsky (2008).

Furthermore, the framework includes operational criteria for characterizing the knowledge components at different levels, up to the conceptual level of teaching the algorithms, i.e., teaching with the potential to expose for learners deep mathematical similarities between the algorithms and the connections between their different representations. Thus, the framework addresses the call formulated in Davis and Simmt (2006) and Peled and Zaslavsky (2008). It has been demonstrated that the criteria, in turn, can serve for designing tasks for mathematics teachers' professional development. Three examples of tasks provided in this article have (hopefully) demonstrated the sensitivity of the framework to variations in the teachers' knowledge.

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DEVELOPING MATHEMATICS TEACHER EDUCATION PRACTICE AS A CONSEQUENCE OF RESEARCH

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The authors of this paper are three members of the Cambridge-based research team who developed the Knowledge Quartet (KQ), a theory of mathematics teacher knowledge, with a focus on classroom situations in which this knowledge is applied. At the same time as being researchers, the authors were elementary mathematics teacher education instructors. Despite many years' experience of preparing trainee teachers in elementary mathematics, they found that the KQ research had brought about new awareness of the importance of some components of mathematics didactics, as well as providing new tools for undertaking some aspects of their teacher educator role. The paper explores some of these awarenesses and tools in detail.

Keywords: teacher knowledge, mathematics teacher educator, Knowledge Quartet

INTRODUCTION

This paper is a contribution to a young field of research, which seeks to understand the ways in which teacher educators (specifically, mathematics teacher educators) can gain in wisdom, competence and effectiveness in their work. The state of the art has similarities with the emergence of mathematics teaching as a research field, fuelled by action research, in the 1980s: until then, the research gaze was on students rather than teachers. Likewise, researchers into mathematics teaching, themselves typically mathematics teacher educators, have only recently viewed themselves (or their work) as suitable objects of research, having previously attended to the knowledge and performance of their own 'students'. Even the goals of what we are calling mathematics teacher educator 'development' are, as yet, unclear. In a Special Issue of the *Journal of Mathematics Teacher Education*, Brown and Coles (2010) address the topic in a neutral way, as "change", and ask what it might mean to say that a mathematics teacher educator has "changed". One state with potential for productive change is awareness (Mason, 2008), specifically critical awareness of how mathematics teaching is being modelled, in pre-service teacher education in particular. Adler and Davis (2011) identify three such modelling practices - 'look at my practice', 'look at your practice' and 'look at (mathematics teaching) practice' - and argue that these offer different opportunities for learning about teaching. The first model presents the teacher educator him/herself as example, or role model, as a mathematics teacher might be inclined to do when cast in the role of trainee-mentor (Mason, 2008, p. 50). The second suggests explicit emphasis on critical, constructive examination of one's own teaching practice. It will become apparent that the changes in our own awareness and professional identity relate more to the third model, in which the teaching practice of others provokes awareness of possibilities for the development of one's own teaching.

In their survey of research in mathematics teacher education, Adler, Ball, Krainer, Lin and Novotna (2005) pointed to the importance of the research activity of mathematics teacher educators in helping them to understand and develop their own practice. This link between mathematics teacher educator research and practice is at the heart of this paper, and we turn to it now. The work of University Departments of Education is typically distributed across diverse programs and agendas, including a leading role in the education and professional preparation of prospective teachers. There can be, in the UK at least, and probably elsewhere, a fuzzy divide between faculty engaged in teacher preparation and those engaged in research. Thus, while teacher education is expected to be research-informed, this basis in scholarship most often rests on the research of academics other than those doing the 'training'. This state of affairs comes about for a number of reasons, and many faculty on both sides of the divide are very content with it. However, the purpose of this paper is to exemplify how mathematics teacher educators can benefit and learn from *their own* research activity, with direct relevance to their teacher education role. In the paper we reflect upon our own experience as education department faculty who have endeavoured to straddle the research-practice divide.

The paper begins with a brief account of a research project on mathematics teacher knowledge, to which each of us made a major contribution. The remainder of the paper is devoted to reflection on, and discussion of, some ways in which the research had a direct impact on our professional work with prospective teachers, thereby (we believe) making us 'better' teacher educators. Despite having, between us, over 70 years experience of preparing trainee teachers in elementary mathematics, we found that this particular research activity had given us new awarenesses of the importance of some components of mathematics didactics, as well as providing new tools for undertaking some aspects of our teacher educator role.

We conclude this introduction by articulating an (as yet) unspoken assumption: that knowledge for mathematics teacher education is not quite the same as knowledge for mathematics teaching. This must be the case because the students (student teachers, mathematics students) are intended to be learning different things. It has been pointed out that the 'community of practice' model of situated learning does not fit institutional learning very well, precisely because mathematics learners are not apprentice teachers. Mason captures the distinction between teacher and teacher educator in terms of the direction of attention:

> Teaching is about directing learner attention [...] about being aware of what learners are not yet aware of, and finding ways to prompt them to become aware. Educating teachers is about directing attention to practices and choices, constructs and theories which can inform choices when teaching (Mason, 2008, pp. 51-2).

The significance of such practices, choices, constructs and theories will become apparent in the narrative to come, in which we 'make public' the integration of our research into our teaching.

The Knowledge Quartet

In 2002-03, we undertook some empirical research into mathematics teachers' knowledge, in collaboration with two additional colleagues in Cambridge. Our approach to investigating the relationship between teacher knowledge and classroom practice was to observe and videotape novice teachers teaching. The participants were 12 graduate prospective ('trainee') elementary school teachers in our university faculty of education. We observed and videotaped two mathematics lessons taught by each participant. In the analysis of these videotaped lessons, we identified aspects of trainees' classroom actions that seemed to be informed by their *mathematics* subject matter knowledge or their *mathematical* pedagogical content knowledge (Shulman, 1986). We realised later that most of these related to choices made by the trainee, in their planning or more spontaneously. Each was provisionally assigned an 'invented' code, such as: 'choice of examples'; 'choice of representation'; 'adheres to textbook'; and 'decision about sequencing'. These were grounded in particular moments or episodes in the tapes. This provisional set of codes was rationalised and reduced (e.g. eliminating duplicate codes and marginal events) by negotiation and agreement in the research team. This inductive process generated 20[1] agreed codes, which were subsequently grouped into four broad, super-ordinate categories, or 'dimensions' – hence the 'Quartet'. The four dimensions and the corresponding contributory codes are shown in Table 1.

Dimension	Contributory codes
Foundation:	awareness of purpose; adheres to textbook; concentration on procedures; identifying errors; overt display of subject knowledge; theoretical underpinning of pedagogy; use of mathematical terminology.
Transformation:	choice and use of examples; choice and use of representation; use of instructional materials; teacher demonstration.
Connection:	anticipation of complexity; decisions about sequencing; making connections between procedures; making connections between concepts; recognition of conceptual appropriateness.
Contingency:	deviation from agenda; responding to students' ideas; use of opportunities; teacher insight during instruction.

Table 1: The Knowledge Quartet – dimensions and contributory codes

A brief conceptual outline of the KQ is as follows. The first dimension, *foundation*, consists of teachers' mathematics-related knowledge, beliefs and understanding, incorporating Shulman's (1986) classic taxonomy of *kinds* of knowledge without undue concern for the boundaries between them. The second dimension, *transformation*, concerns knowledge-in-action as demonstrated both in planning to teach and in the act of teaching itself. A central focus is on the representation of ideas to learners in the form of analogies, examples, explanations and demonstrations. The third dimension, *connection*, concerns ways that the teacher achieves coherence within and between lessons: it includes the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks. Our final dimension, *contingency*, is witnessed in classroom events that were not planned for. In commonplace language, it is the ability to 'think on one's feet'. More detailed conceptual accounts can be found in Rowland, Huckstep & Thwaites (2005), and in the book Rowland, Turner, Thwaites & Huckstep (2009). Related reports at previous CERME conferences include Huckstep, Rowland & Thwaites (2006). In this paper we also draw upon the longitudinal doctoral research project of the second author (Turner, 2010), in which the findings of the 2002-03 KQ project were applied for the first time; and on continuation projects (e.g. see Rowland, Jared & Thwaites, 2011) in which the scope and methodology of the KQ were extended.

WHAT WE LEARNED FROM THE KQ RESEARCH

We now proceed to describe some of the ways in which the research outlined above brought about new awarenesses, and enabled new approaches, in our professional work as elementary mathematics educators. This will be organised into sections corresponding to specific issues, topics and approaches about which we became more sensitive and knowledgeable as a consequence of the research.

The role of 'theory' within pre-service mathematics teacher education

When using the KQ to analyse the practice of beginning teachers, it was salutary to find that they did not draw on what we thought they had learned from our methods courses in the university to the extent that we might have hoped. The mathematical knowledge for teaching (MKT) of beginning teachers might be expected to be mainly *propositional* (Shulman, 1986), i.e. gained from their own mathematics education and from mathematics methods courses during teacher education programmes. There were a number of instances where situations categorised under the *foundation* dimension indicated that, once in the classroom, trainees did not draw on propositional knowledge addressed during their graduate teacher education course. Although there was evidence that this was held as propositional knowledge, these beginning teachers were frequently unable to draw on this knowledge and activate it in their early teaching. Two examples will illustrate this observation.

Amy. During her final school placement, in a lesson about counting with 4–5 year-old children, Amy asked her pupils to write nineteen on their white boards. Several children wrote '1P', at least one wrote '99' and many wrote '91'. The trainee teacher focused on the reversal of the nine but did not address the problem of digit order. During the post-lesson interview the trainee teacher was asked why she thought children had reversed the digits:

> Because you say nine first, then you say the teen that's why often they write the nine first they often want to write nine first then write it from right to left instead of left to right. (Amy)

Amy clearly knew about the problems children encounter in writing teen numbers (e.g., Anghileri, 2007), but did not apply this knowledge in her practice.

Kate used a number line to help children complete addition calculations such as '8 + 8' and '3 + 4' by beginning at one of the numbers and then counting on the second number. This pre-supposed that children had reached the 'count on' stage in addition. However, observation of the children's independent use of the number lines suggested that some were still at the 'count all' stage (Carpenter and Moser, 1984). Kate was asked if she remembered the stages children go through in learning addition:

> At first not knowing that you can just start at numbers, that you have to count the one, two, three … so you have to count three to get up to three before you can carry on. (Kate)

Although she knew that some children would not be able to understand the addition strategy of starting with one number and then counting on the second number, this propositional knowledge was not drawn on in Kate's teaching.

Analysis of further data using the KQ framework suggested that these beginning teachers became more able to draw on propositional MKT as they gained experience (Turner 2010, pp. 98ff). This suggested that teachers need experience, and focused reflection on their experience, in order to contextualise and make use of the propositional knowledge we present to them in the university. We should not be surprised or disappointed when we find beginning teachers not drawing on this knowledge. We learned that providing the KQ as a tool for reflecting on their teaching helps them to make links to this propositional knowledge and to apply it within the context of their practice.

The use of the KQ to structure review of, and reflection on, teaching

The KQ helped us to observe and to analyse the teaching of our elementary trainee teachers and to give detailed feedback which focused on the mathematical content of their lessons. This detailed analysis of teaching suggested the need to address more explicitly the importance of selecting appropriate examples and representations, as well as making connections and responding contingently to pupils, in our mathematics methods course. Guidelines based on the framework (Rowland et al, 2009, pp. 35-37) were also developed to support university and school-based colleagues working with elementary trainee teachers who were not mathematics education experts. These guidelines were presented and very well received during mentor training sessions at the university, and continue to be available to colleagues.

The usefulness of the framework for supporting observation of, and feedback on, mathematics teaching was explored in a study carried out between 2004 and 2008 (Turner, 2010). It was used as a tool to identify, analyse and chart developments in beginning teachers' MKT, and also as a tool to promote that development. As a tool for development, it was used to frame review discussions of mathematics teaching between teachers and the mathematics teacher educator (MTE) It was also used by the teachers to support individual reflection, helping them identify situations in which their MKT was revealed and to frame their written *reflective* accounts.

In the early phases of the study, the lesson review meetings were intensive and took the form of a stimulated recall interview. The researcher [the second author] used a KQ analysis of the lesson to determine questions to ask and comments to make when the teacher watched the videotape of their lesson. For example, a coding of *choice of examples* (CE) suggested stopping the videotape to ask whether the trainee teacher thought the examples they had used in their explanation of a mathematical procedure were the most appropriate, or whether they might have caused some confusion. The structure of these initial review meetings would be impossible to sustain across a large number of trainee teachers or with busy practicing teachers. The methods employed in the second stage of the study were therefore more appropriate as a model for scaling up the adoption of the KQ for structuring post-lesson review meetings. Lessons were again observed and videotaped however the review meeting was based on a 'broad sweep' KQ analysis of detailed field notes made while observing the lesson. The second author asked questions or commented on significant episodes which had been identified in the analysis and the teachers made observations in relation to the codes and dimensions of the framework with which they were now familiar.

The study also aimed to determine whether the KQ framework supported independent reflection on the mathematical content of teaching. Therefore, during the third phase teachers were not given feedback following their lessons, but were sent DVD copies of the lessons and asked to write reflective accounts independently, structured by the dimensions and codes of the KQ framework. A number of comments made by the teachers demonstrated that they found the framework useful when planning for, and reflecting on, their mathematics teaching. For example:

> I often find myself referring to it in my head when I am planning. …I think the most important effect is having the four headings, makes me more aware of what I am planning and teaching and why. You find yourself questioning yourself and justifying your decisions and choices, it makes you more purposeful in your choices, more precise. (Amy)

From this study we learned that the KQ can be used effectively to frame lesson reviews so that they focus on the MKT of teachers. We also learned that use of the KQ can help teachers to focus their independent reflection on the mathematical content of their teaching.

The role of representations and examples in mathematics teaching

Despite our experience as teacher educators, the KQ research gave us a new appreciation and understanding of the importance of *examples* in mathematics teaching. When teachers teach mathematics they choose and use examples all the time – the relevant code was present in our coding of every lesson. In fact our focus on examples came at an interesting time from a national and international research perspective. While we were building an emergent theory of teacher-chosen examples (e.g. Rowland, Thwaites & Huckstep, 2003), Watson and Mason (2005) were developing a theory of learner-generated examples, applying and extending the ideas of Ference Marton on variation theory. Both of these perspectives were represented in a PME Research Forum (Bills et al., 2006) and in a special issue of *Educational Studies in Mathematics* (Bills & Watson, 2008).

As a consequence of our own research, we realised and understood better the different purposes for which examples are used, and that the choice of examples is far from arbitrary – some examples 'work' better than others. These insights have had a significant effect on our practice in our role as mathematics teacher educators. So whilst formerly we might have spoken in a general way about the importance of choosing examples with care, we are now able to offer our trainee teachers a more analytical account of the choice and use of examples in mathematics teaching and learning. In particular, we identify and exemplify three broad categories of examples that were commonplace in our data, but which, we argue, teachers would do well to avoid. We labelled these categories: examples which confuse the role of variables; examples intended to illustrate a particular procedure, for which another procedure would be more sensible; and randomly generated examples. For details, see e.g. Rowland et al. (2009).

By way of illustration, we exemplify the first of these categories (confusing the role of variables) here, with two excerpts from the classroom data.

Kirsty was reviewing the topic of Cartesian co-ordinates with a class of 10 to 11-year-old pupils. Kirsty began by asking the children for a definition of co-ordinates. One child volunteered that “the horizontal line is first and then the vertical line”. Kirsty confirmed that this was essentially correct. She then moved on to assessing the pupils’ understanding of this key convention by asking them to identify the co-ordinates of a number of points as she marked them on a co-ordinate grid, projected onto a screen at the front of the classroom. Before marking the first point, she reminded them that “the x-axis goes first”. Kirsty’s first example was the point (1, 1). It is interesting to speculate reasons for Kirsty’s choice of this example, recognising that these ‘reasons’ might be of different types – pragmatic, pedagogical, affective and so on. In any case, the example would seem to be entirely ineffective in assessing what Kirsty wanted to determine: the children’s grasp of the significance of the order of the two elements of the ordered pair.

Michael’s lesson with a Year 4 class was about telling the time with analogue and digital clocks. One group was having difficulty with analogue quarter past, half past and quarter to. Michael intervened with this group, showing them first an analogue clock set at six o’clock. He then showed them a quarter past six and half past six. When asked to show half past seven on their clocks, one child put both hands on the seven. We can’t be sure, but the child’s inference from Michael’s demonstration example (half past six) seems reasonably clear. Of the twelve possible examples available to exemplify half-past, half past six is arguably the most unhelpful.

The role of *representations* in mathematics teaching has been extensively researched and theorised (e.g. Goldin, 2002). Nevertheless, our research yielded further insights that we were able to bring to our work with trainee teachers. These include the importance of the mathematical appropriateness of representations used for pedagogical purposes. We had observed the trainees’ propensity to choose representations on the basis of their superficial attractiveness at the expense of their mathematical relevance (Turner, 2008). In addition, we are now better placed to emphasise the interplay between choice of representations and choice of examples (e.g. Huckstep et al., 2006).

New uses of classroom video data within initial teacher education

The use of video in mathematics teacher education is well-established (e.g. Borko et al., 2008), and articulates well with case method teacher education pedagogy (Merseth, 1996). In England, the video resources that have been most in evidence in primary teacher education are of the kind developed by a government agency for 'National Numeracy Strategy' training (Askew et al, 2004). These tend to feature 'best practice' examples of 'model' lessons given by experienced teachers, presumably with the intention that other teachers will emulate their example. With the permission of the participants in our research, we use video clips from their lessons in a somewhat different way, and with a rather different purpose. These clips feature novice teachers, not 'experts', and as we observe them it is not hard even for trainee teachers to identify things that could be done differently, and maybe should be. We have written about some of these episodes elsewhere (e.g. Huckstep et al., 2006; Rowland, 2010), and there is insufficient space to describe them here. These video stimuli promote lively and thoughtful discussions about what seemed to be successful and what 'went wrong', and why, and what these trainees would do themselves to avoid the errors made (in their judgement) so as to improve the instruction. By contrast, we propose that when an expert teacher's lesson 'goes well', the ingredients of its success can often be invisible to the novice trainee. Using our research video data, and in other ways, we now use these authentic classroom scenarios to pose challenging mathematical and didactical problems, and to raise awareness and insight, in our university-based sessions with trainees.

CONCLUSION

Teachers and teacher educators often approach their professional development through action research. This entails investigating one's own practice, adapting it, and looking for evidence of the impact of this change. The development in our professional practice brought about by our research was a consequence of a very different process. We did not set out with the primary aim of developing our own practice. Rather, our focus was on the practice of trainee teachers as we tried to understand how their MKT was revealed and applied in the act of teaching. However, in investigating the practice of trainee teachers, we developed a way of understanding mathematics teaching which supported our own professional development as teacher educators in a number of different ways.

Developments in our understanding of beginning teachers' MKT, as revealed through KQ analysis of their practice, led to changes in the content of our methods courses, particularly in relation to the importance of examples and representations. We found that the MKT that was 'learned' by trainees in our methods courses was not always available to beginning teachers in their practice. However, we discovered that teachers can be supported in applying this knowledge by providing the KQ as a tool for focused reflection. We improved our teaching placement lesson reviews by using the KQ to focus discussion on the mathematical content of teaching, and began to induct school-based colleagues in the use of the KQ to support mentoring of trainees. We also presented the KQ framework to trainees themselves to support focused reflection on their mathematics teaching, so as to enable them to continue developing their MKT during school placements and after their mathematics methods courses were completed. We also developed new video resources for primary mathematics teacher education, and new ways of using them. Finally, a bonus in terms of professional development from participating in the KQ research related to the development of understanding and cohesion within the elementary mathematics teaching team.

These outcomes of our study illustrate the possibility of a symbiotic relationship between research into teaching and learning in classrooms and the professional development of teacher educators, demonstrating how the roles of researcher and of teacher educator can be complementary and mutually supportive.

Notes

1. In 2002 there were 18 codes in fact: two more were subsequently added in the light of new data.

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MATHEMATICS IN PRE-SERVICE TEACHER EDUCATION AND THE QUALITY OF LEARNING: THE PROCEPTUAL DIVIDE

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This paper presents two episodes of an exploratory study on a prototype of a mathematics curriculum for pre-service teacher education. The focus here is on the way in which students may operationalize mathematical concepts in rational arithmetic thus improving the mathematical quality of their learning. It is claimed in the major research that the curriculum should offer a mathematical matrix strong and flexible enough to enable them to manipulate and create the conditions to teach mathematics with quality as a generalist teacher. The main claim in this paper is the difference between those who relate and compress arithmetic procedures and those who remain in rote learning step-by-step arithmetic procedures, that is called the proceptual divide as showed.

Keywords: *Arithmetic's, curriculum, pre-service teacher education, proceptual divide, SOLO levels.*

INTRODUCTION

This paper presents an exploratory study of a prototype of a mathematics curriculum for pre-service teacher education. In this prototype it is argued that the mathematics curriculum in pre-service teacher education should offer a mathematical matrix strong and flexible enough to enable them to manipulate and create the conditions for students to learn mathematics.

Students who fail to transform rote-learn arithmetic situations into simpler structures by compressing them into thinkable concepts can't relate them to conceptual learning thus stopping them to generate new knowledge in the sense that Gray & Tall (1994) called the *proceptual divide*. The episodes exemplified in this paper are taken from arithmetic tutoring sessions.

To teach the intuitive arithmetic studied in basic education, future teachers need a foundation or construction following a rational method. This distinction between rational and practical arithmetic does not arise from the nature of the subject, but from the method by which it is structured, founded on a deductive theory based on the boundary between arithmetic and logic by adding scientific rigour to the ability of reasoning that future teachers should have.

This view is in line with the ideas of Tall (2007), in developing his theory of mathematical growth by changing focus from process to concept, called *procept, and* seen as:

> An elementary procept is the amalgam of three components: a process which produces a mathematical object, and a symbol which is used to represent either process or object. (Gray & Tall, 1994, p.6)

According to this focus on the complexity of mathematical thought (Tall, 2002) we are trying to understand and find how pre-service teachers operationalize mathematical concepts in arithmetic accessing evidences of the *proceptual divide*.

The implementation of a new mathematical curriculum allows to study, characterize and test that curriculum in various perspectives and at the same time, by analysing the student answers in the various tasks like homework's, quizzes and exams, using as categories the SOLO (*Structure of Observed Learning Outcomes*) model from Biggs and Collis (1982) and Biggs and Tang (2007).

Associating the evolution of their mathematical thinking using Tall theories of *Advanced Mathematical Thinking* (Tall, 2002, 2007) and the concepts of *procept*, *proceptual thinking* and *proceptual divide* (Gray & Tall, 1994) as frameworks to perceive the way students conceptualize mathematical concepts.

BACKGROUND

The introduction of a new mathematics curriculum for Basic Education (pre-service kindergarten, primary and elementary teachers) is reflected in teacher education either by the particular definition of a new kind of student (more proactive ones) or by the need for new methods of teaching and learning of mathematics. This opportunity to change the mathematics curriculum for teachers occurs under a combination of factors: amendments to study plans in higher education; the growing concern with the teaching of mathematics in particular due to the poor results of international studies like PISA (OECD) and a significant change in the mathematics curriculum in elementary education.

Considering all these factors we designed a mathematics curriculum for Basic Education at an institution of higher education in Portugal who wants to combine three levels of intervention: a solid mathematical foundation for all pre-service teachers; a comprehensive training for teaching mathematics connection among knowing and teaching mathematics; a didactics and pedagogical approach about what means to teach mathematics.

The curriculum has been developed taking into account the argument that the pre-service teachers need to have a better understanding of mathematical concepts because it helps them gain a greater understanding of the connections among different areas of mathematics and beyond, taking into account that this course is for generalist teachers.

TEACHING MATHEMATICS TO PRE-SERVICE TEACHERS

Issues related to mathematical preparation of future teachers have been investigated in view of training and teaching on education and not have as much importance as the subject of study for conceptual knowledge of these professionals. Studies on this topic have shown signs of concern, because this kind of mathematical knowledge is not present in many teachers (Veloso, 2004).

To Tall (1989) curriculum development should offer student contexts where they develop mathematical knowledge, leading to a significant growth of their mathematical reasoning. This arduous process of transitioning from a less formal mathematics to a more formalized understanding of mathematical processes and concepts, needs to be assessed by the teacher, both in terms of the complexity of thought highlighted and within the quality of learning.

The traditional concept of a mathematics curriculum structured gradually, starting with familiar elementary concepts and to a gradual complexity of structures has not worked for the simple reason that our brain does not function logically like a computer (Tall, 1989).

In addition, there are several conceptions that advocate that, on the one hand future mathematics teachers must have a broad base of didactics and pedagogical training and the knowledge of the mathematical content will be gained from the experience, on the other hand some argue that they should have a large mathematical training and that the pedagogical aspects are going to be acquired with the experience. In our view, both these dimensions should be balanced, and that should exist a compromise here.

The quality of learning

In this study we consider the quality of learning not only as the quantitative grade a student achieve when answering a question but also to the qualitative process of producing an answer using facts, concepts and skills to achieve the solution to that answer. But this is a complex issue because the quality of learning does not depend exclusively on the student, but there's other dimensions like the quality of the teaching itself, and other like is prior knowledge, motivation, self-regulated learning and so on.

In this paper we are experimenting the hierarchical SOLO model to identify only the process of producing an answer.

The SOLO model

Is the emphasis on the analysis of the quality of the responses from students that make the SOLO model interesting for the study. Throughout the development of the problems the focus is not on correct or incorrect answers, but in the structure (nature) of the responses, encoded in categories based on the SOLO levels enabling a more detailed description of the development of mathematical thought and the quality of their learning.

Table 1: Description of levels in the SOLO model relating them with the response indicators adapted from Biggs & Collis (1982) and from Ceia (2002)

	Mathematical thinking	Indicators
Extended abstract	Goes beyond the topic, make connections to other concepts and generalizations.	Theory, generalization, hypothesis, reflection. Maximum capacity, uses relevant data and interrelations. No felt need to give closed responses allowing possible alternatives.
Relational	Makes complex connections and synthesizes parts to the overall significance.	Compare, explain the causes, integrate, analyze, report, apply. High capacity, uses relevant data and interrelations. No inconsistencies within the subject, but closure is unique.
Multi-structural	Makes some connections but lack a unifying vision.	Enumerate, classify, describe, list, match, working with algorithms. Medium capacity, can isolate relevant data. Can achieve a different conclusion with the same data.
Uni-structural	Makes simple connections without identifying its importance.	Identify, memorize, perform simple procedures. Low capacity, only one relevant data. Jumps to conclusions on a single aspect.
Pre-structural	Provides information loose and disorganized, not related.	Can not relate. Minimal capacity, confused answer. Inconsistent responses.

This SOLO model becomes a tool which allows a framework which assists the implementation of an educational model based on the mathematical complexity of thought, in view of the quality of their learning and allows to avoid the emphasis on a single learning process.

The *proceptual divide*

The procedural approach of learning mathematics is rooted in positivistic frameworks where, through a set of predefined and outlined procedures, one gets an answer. This appears to be embedded on the idea that by doing sufficient numbers of similar (or even identical) exercises one gets the routines needed to learn mathematics. Although the short-term success of such approach is relatively obvious and educational policies adopt such measures, since they guarantee interesting statistical results (seen in the short term) - for example if a student preparing for an exam, repeating the exercise until exhaustion of previous examinations. In the long term it can be seen the flaws of such an approach, when that same student needs to relate content learned in previous years, so there is not a significant learning.

Gray & Tall (1994) use the concept of encapsulation of a process in a mental object, rooted in the work of Piaget to sustain cycles of building mental structures that in Piaget's theory are cycles of assimilation-accommodation, or reification in Sfard's theory.

The use of symbols, however, have a double meaning, introducing some ambiguity between the procedure and the concept. The way students deal with this ambiguity seems to be the root of a quality learning of mathematics.

> We characterize proceptual thinking as the ability to manipulate the symbolism flexibly as process or concept, freely interchanging different symbolisms for the same object. It is proceptual thinking that gives great power through the flexible, ambiguous use of symbolism that represents the duality of process and concept using the same notation. (Gray & Tall, 1994, p.6)

This combination of procedural and conceptual thoughts is called *proceptual thinking*. When there is an inability to relate these two types of thinking making it is impossible the development of conceptual thinking.

The dichotomy between those who can and those who can not overcome the barrier of procedural thinking is defined by *proceptual divide*. This is one of the biggest barriers and one of the factors that has most contributed towards the failure of teaching and learning mathematics (Gray & Tall, 1994).

Teaching arithmetic's

The foundation of number theory focuses on, in their essence, in two schools, an Formalist represented by Peano and Hilbert among others, and another Logic represented by mathematicians such as Cantor and Russell. In our curriculum there was no concern of using any of the current theory of the integers or an axiomatic model. What is required in the course is just an introduction to elementary mathematics. It must be therefore an ordination of the theory in such terms that every proposition is a logical consequence of propositions previously demonstrated. The conceptualization thereby requires the establishment of ideas or primitive concepts defined by axioms leading to choose an approach closer to a formalistic model like Peano axioms.

"The fundamental idea in the development of powerful thinking in mathematics is the compression of knowledge into thinkable concepts." (Tall, 2007, p.150) This compression of knowledge enable students to relate ideas, concepts and *procepts* allowing them to go beyond rote-learning, that a number of studies show that fails in an unfamiliar context, like is showed in the second episode in this paper.

METHODOLOGICAL APPROACH

The examples in this paper show the process of inquiring and reasoning in action. These episodes are taken from one larger ongoing study in which we analyse and evaluate the mathematics curriculum of pre-service teachers and their learning. The aim of this study was to try to understand the mental construction of mathematical reasoning and, more specifically, to see how a student thinks mathematically taking into account the *proceptual divide* notion, hoping to observe it in action.

This specific experiment was designed based on a tutorial interview in which two students (let's call them Ana and Maria) came to clarify doubts on some issues that had been asked in the exam.

These students attended, and failed, *Mathematics I*, and requested these tutoring sessions to clarify doubts for the upcoming exam, so they had already attended classes, either theoretical either practical matters about these episodes, having made various of the exercises about the issue. These two examples were taken from the exam questions, and have been solved in the tutoring sessions.

In these episodes one of us acted as a teacher and as a researcher and two individual sessions of one hour per student were observed, totalling four hours of work. Both episodes about one exercise of rational arithmetic.

One of the goals of this kind of exercises was to enhance not just the resolution of common arithmetic operations, but to establish a parallelism with aspects of a more formal mathematics, intending to develop mathematical reasoning.

By mathematical reasoning, or demonstration, we understand the combination or joining of two or more propositions to obtain new propositions by means of mathematical reasoning within a finite number of steps deduced from one or more propositions. The method worked is the complete induction (or method of recurrence) and to demonstrate that a given property is true for all integers is enough to demonstrate: (i) is true for 1, (ii) admitted as true for n, is also true for the successor of n (heredity).

Starting from the following proposition that is taken as an axiom:

> All property belonging to the integer 1 and the successor of an integer that enjoys this property belongs to all integers (principle of finite induction).

The operations studied were the addition, multiplication and exponentiation and are defined as:

1. Adding two integers - the operation that for each pair of integers a and b matches a given integer ($a + b$) according to the following conventions:

   [A1] $a+1=suc\ a$ [A2] $a+ suc\ b = suc\ (a+b)$

2. Multiplication of two integers - the operation that for each pair of integers a and b matches a given integer ($a.b$), according to the following conventions:

   [M1] $a\ .\ 1 = a$ [M2] $a\ .\ suc\ b = a\ .\ b + a$

3. Exponentiation - the operation that for each pair of integers a and n matches a given integer (a^n), according to the following conventions:

   [E1] $a^1 = a$ [E2] $a^{suc\ n} = a^n\ .\ a$

Thus, it is necessary that the student in the following problem, realize these recurrence concepts to solve: *Calculate, by recurrence, using the respective axioms* $(2+2)\ .\ 2^1$

The categories of analysis of the question and subsequent answers are based on SOLO levels and their attributes and this exercise has been classified as possibly *relational* indicating an orchestration between facts and theories involved, their actions and goals. These kinds of exercises were familiar to both students in the classroom environment although it was the first time they had covered this kind of procedure in mathematics.

To take a deeper analysis we used Tall theories covering an important aspect of the *proceptual divide* since the *procepts* here involved are ambiguous (operations can be viewed either in the field of elementary arithmetic or in the field of rational arithmetic).

EXPECTATIONS FROM THE PROCEPTUAL DIVIDE – TWO EPISODES

For Ana this issue had been solved naturally through elementary arithmetic and had not even noticed that, in the task she was asked to use the rational arithmetics, not understanding her low rating. The dialogue between Ana and Maria and myself was held in Portuguese.

Ana: So, but the account is not right? Gave 8 ...

Teacher: Yes, Ana, but read the statement again ...

Ana: Yes ... and ...

Teacher: We used the axioms?

Ana: What axioms? ... Ah! ... [reread the sentence, looking at the exam and then back at me with a satisfied air]

Teacher: ?

Ana: So I must do like in the classroom, with those axioms of operations, right?

Teacher: Right. [It is noted then a change in Ana, who quickly wrote on a sheet ... (2+2).2 (E1)... (2+ suc 1) . 2 (A1) ... (suc 2+1) . 2 (A2)... and so on until the right answer]

Teacher: Why you do not do that in the exam?

Ana: Well, I did not read the statement professor ... and it was so easy ...

After seeing her failure, Ana had no difficulty in solving the exercise. This is an example in which, looking at the question (not having read it entirely) and especially to the expression, used a familiar process of elementary arithmetic to solve (correctly) the question ... the issue is that she did not answer the question of the exam.

From the moment she really knows what to do, quickly solves the exercise using rational arithmetic's with axioms relating the properties studied by making a change in how she handles the mathematical objects.

In the exam she focused on the procedures looking for the expression in a disconnected way between number and operation, subsequently, by looking to the expression as a whole, she identified the respective axioms.

In this episode Ana easily surpassed the *proceptual divide* on this issue, rapidly changing their mathematical thinking from procedural to *proceptual* simply by reading again the question. When analysing the responses through the SOLO levels, Ana went from a response classified as a possible *uni-structural* level (in the reply to the exam) by jumping to fast for a conclusion, to a response in possible *relational* level (in the tutoring session) by generalizing within the given context using related aspects which corresponds to the level intended to the question. The next episode with Maria went up differently:

Maria: Professor I did not understand what it was is that I supposed to do in this exercise ...

Teacher: So, Maria?

Maria: asks to do math, but with what axioms?

Teacher: those who were worked in class, you don't remember?

...

Maria: but this was not just in lectures? I mean we had to decorate these axioms, it could come in a form ...

Teacher: why?

Maria: We've done a few of these exercises, and always with an operation only, never with several, so one cannot memorize everything ...

[after almost an hour to explain the resolution and how it should identify the propositions]

Maria: My head does not give much more, this is very complicated …

Teacher: Tell me why?

Maria: Only theory, when I teach kids, I will not teach so, teaching them only to do the math, none of these things successors and properties and axioms …

[and we continue with a discussion of what it was to be a teacher ...]

In this example, Maria was so attached to the procedure that after two sessions she still didn't realize that she could not rote this kind of exercise. Even the very procedural knowledge (she couldn't understand the procedure) was deficient because she could even identify the numeric expressions as more complex *procepts*. In this case the *proceptual divide* is in line with the ideas of Gray and Tall when they state that:

> This lack of a developing proceptual structure becomes a major tragedy for the less able which we call the proceptual divide. We believe it to be a major contributory factor to widespread failure in mathematics. (Gray & Tall, 1994, p.18)

At the end of the sessions, Maria could solve some issues, but with only two integers and one operation, having been scheduled a third session, with some exercises proposed by the teacher (which never came to happen until the next day of examination). The responses of Maria (on examination and tutoring session) were classified (according to the SOLO levels) as possibly *pre-structural* with great inconsistencies and later *uni-structural* by jumping to fast to conclusions, far from the outcome intended with the exercise.

FINAL REMARKS

In this paper we analysed two episodes in an attempt to expose the conceptualization of Gray and Tall on the *proceptual divide*. The work of Ana and Maria (like all his colleagues) is to overcome the barrier of elementary arithmetic and think mathematically about a set of properties that are common to the operations studied in any circumstance, through the generalization that is possible in the rational arithmetic and this separation was identified by their different characteristics.

The use of the SOLO levels to characterize the questions and answers has the limitation of identifying an image at the moment it is not possible, and in isolation, assess the students' mathematical thinking, hence the necessity to use other methods such as inquiry, to obtain a better understanding of the phenomena under study.

Ana's success depended more on concentration and ability to read the sentence, since after a few questions (directed to read of the sentence and not to the mathematical content), managed to solve that problem, and others - and subsequently had a positive rating the exam by solving a similar issue using other properties of operations so it's unclear that she crossed the *proceptual divide* by means of the intervention (which was not the aim of this paper) or that she already is using *propectual thinking* and the problem with the exercise was only due to misreading, Maria failed again in the exam, but in a similar issue she solved partially (only the addition, which was detached), revealing still be using only procedural thinking, having failed to overcome the barrier of *proceptual divide*.

Both examples are used to identify differences in the mathematical thinking related to the same exercise and require further and clearer evidences of the *proceptual divide* that could be an important mechanism for mathematics teachers and researchers in mathematics education.

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THE INFLUENCE OF EARLY CHILDHOOD MATHEMATICAL EXPERIENCES ON TEACHERS' BELIEFS AND PRACTICE

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This paper shows how particular forms of formative familial experiences provide prospective teachers with the intellectual tools necessary for undertaking critical analyses of both the received and intended curriculum. Data from a multiple case study shows that beliefs formed through early mathematical experiences stay with individuals to reveal themselves in subsequent beliefs and practice.

Key words: teachers, early childhood, beliefs, practice, norms.

INTRODUCTION

This study follows in the tradition of research into the relationship between teachers' espoused beliefs and their enacted practices (Beswick, 2007; Skott, 2009; Thompson, 1984). It draws on a multiple case study of the whole class interactive phases of the mathematics lessons of six English primary teachers and the rationales they offer for their actions. The data yielded two distinct groups of three teachers, essentially defined, as I explain below, by their early mathematical experiences. All six teachers were similarly qualified, were enthusiastic teachers of mathematics and considered, by their colleagues and others, as ambassadors for the subject. It came as a surprise, therefore, when my analyses highlighted not only substantial differences in practice-related beliefs and the enactment of those beliefs but also the ways in which early childhood influences had moulded such different teachers. Three elements of teachers' practice were identified in the larger study: mathematical intentions, pedagogical approaches and classroom norms. In this paper, due to reasons of space, I report on *classroom norms* (Yackel & Cobb, 1996), and on two of the six teachers, one representative of each group, to highlight the resonance between beliefs, practice, and early childhood experiences.

THEORETICAL BACKGROUND

A number of studies, both theoretical (Ernest, 1989) and empirical (Beswick, 2007; Furinghetti & Pehkonen, 2002; Skott, 2009), have highlighted the influential role of teachers' beliefs on classroom practice. In particular, Ernest argues that teachers' perspectives on the nature of mathematics influence the construction of their mental models of the subject and its teaching. Thus, although subject knowledge is important, it is not sufficient by itself to account for the differences between mathematics teachers. English teachers, the focus of this paper, are typically thought to hold beliefs more in accordance with traditional than reform practice, emphasising the mastery of symbols, skills and procedures (Andrews, 2007).

Warfield et al. (2005) argue that the relationship between "teachers' beliefs and their instruction is not as direct as sometimes thought" (p. 442), stressing that it is not unusual for individuals to hold contradictory beliefs, thereby making it difficult to determine how particular beliefs influence practice. This may be because teachers' mathematics-related beliefs draw on not only beliefs about mathematics and its teaching, but also beliefs about themselves as teachers and the classroom context in which it occurs. Moreover, beliefs about schools, teaching and mathematics will first be formed during childhood. Therefore, understanding teachers' beliefs, which here are construed to be "subjective, experienced based, often implicit knowledge" (Pehkonen & Pietilä, 2003, p. 2), and their genesis about mathematics is important if we are to understand the relationship between beliefs and observed practice.

We know that trainee teachers who experienced failure at school may develop beliefs and practices focused on protecting their students from the pain induced by such experiences. The opposite is also likely to be true; students who recall positive experiences as learners of mathematics will approach teaching positively. Thus, it seems reasonable to assume that mathematics teachers who learned procedural mathematics successfully may have difficulty accepting the validity of alternative practices; their experiences will foster beliefs that will underpin their approaches to mathematics teaching (Handal & Herrington, 2003). Moreover, Muir (2012) has shown how parents influence not only their children's beliefs and attitudes towards mathematics, but also their learning of the subject and the development of their self-efficacy. That is, there appears to be a clear link between parents' attitudes, perceptions and beliefs about mathematics and children's attitudes and performance in mathematics. Yet, little research has explored the nature of parental perceptions of and attitudes towards mathematics in general and its impact on their children's, children who subsequently become teachers, perceptions, values and understanding of the subject.

Consequently, this paper aims to address the following questions: How do primary teachers of mathematics conceptualise the whole class aspect of their work? With sub-questions: 1) What knowledge and beliefs underpin their actions? 2) In what ways do the espoused beliefs resonate with the enacted? 3) What justifications do they present for their actions?

DATA COLLECTION AND ANALYSIS

A number of studies (Thompson, 1984; Beswick, 2007) have shown that case study can facilitate our knowledge and understanding of the relationship between teachers' espoused beliefs and enacted practice. Thus, a multiple exploratory case study (Stake, 2002) was undertaken to examine individual teachers' perceptions of, and justifications for, what they believe they do in the whole class interactive phases of their mathematics lessons. This involved six primary teachers, each considered locally to be an effective teacher of mathematics or, importantly, an ambassador for the subject. Such an approach controlled for various teacher characteristics such as teacher confidence or indifference towards teaching the subject.

For each teacher, data collection involved an initial semi-structured interview, followed by between three and six, video-recorded, lesson observations. To examine the relationship between espoused and enacted practice recorded lessons were viewed jointly by me and the teacher concerned as components of repeated video stimulated recall interview (SRI). Data were analysed by means of constant comparison (Glaser & Strauss, 1967), a process whereby newly collected data from one lesson were compared with data collected from the previous lessons and interviews, and, in so doing, facilitates the development and refinement of theory. In this paper, due to limitations of space, I discuss two of these teachers; each one being representative of one of the two distinct groups that emerged from the larger study.

THE STUDY RESULTS AND DISCUSSION

In the following I present and discuss a summary of the data on each teacher's *background* and the *classroom norms* that emerged in the observations and explicitly emphasised by the teacher in the interviews that followed. The first section on their background is presented against the three broad headings that structured the interview analyses. Teacher utterances are italicised. These concern the following: 1) Mathematics as a subject; 2) Confidence in mathematics knowledge for teaching; 3) Being a teacher of mathematics.

Mathematics as a subject

Both teachers (Caz and Gary), stated they enjoyed mathematics as a child. However differences between these teachers were only highlighted when specific aspects of their enjoyment were discussed.

Caz believed she had a *natural talent* for mathematics and recalled how she assumed everyone else was enjoying mathematics just as she was. She could not understand why mathematics *wasn't so obvious to everybody at school*. She believed her enjoyment of mathematics stemmed from a family view in which mathematics was challenging but interesting. She spoke much about her engagement in exploring mathematics at home as a young child with her father and younger brother. Believing that the sort of *games found on the Nintendo DS today with puzzles and games and things, were similar to the things we used to do with pencil and paper together at home*. She had always enjoyed playing with, number and logic puzzles, and remained keen to engage her own class in interesting mathematics, like the exploration of the work of Fibonacci that she had experienced as a child with her family. She believed her father had a significant influence in how she viewed mathematics.

Gary in contrast described how he found mathematics unproblematic at school, and could describe very little about family influence regarding the subject. He remembered being good at the subject, and talked about *the rightness and wrongness* *of mathematics*. In particular he remembered *he enjoyed working through his text books, getting lots of ticks and feeling very motivated by the correctness of his neatly presented work*. He appeared to enjoy a procedural approach to learning mathematics, appreciating *small steps* and clearly defined levels of progress. Gary remembered learning and *memorising tricks* and talked about how *they worked for me,* and therefore used them in his approach with his class.

Confidence in Mathematical knowledge for Teaching

Both teachers were confident in their mathematical knowledge for teaching. Gary trained as a primary generalist with a mathematics specialism, whereas Caz gained a degree in early child psychology, before gaining her teacher status, where she studied children development and theories of learning which she often referred to in her interviews. Gary remembered how he found the specialism of his degree interesting, but did not remember anything in particular about his training, other than teaching approaches acquired during teaching practice.

Being a Teacher of Mathematics

When discussing being a teacher of mathematics, colleagues' utterances frequently referred to the latest official directives and exploited the vocabulary embedded in them. Admittedly, they were all mathematics specialists, so perhaps this should have been been expected. It could be argued that this acceptance and exploitation of vocabulary, only teachers would be expected to understand, reinforces primary teachers' professional identity, not least because "our identities are composed and improvised as we go about living our lives embodying knowledge and engaging in our contexts" (Connelly & Clandinin, 1999, p. 4). That is, continuing participation in this 'nationally led' vocabulary is not only a source of identity within the primary teacher community (Wenger, 1998) but the means by which they remain part of the primary teaching community.

In conclusion, the initial interviews revealed both interesting and pertinent characteristics about the project teachers. When analysing their backgrounds, views and beliefs, a dichotomy of experiences emerged: Caz was representative of one group, and Gary the other. The table below illustrates the very strong differences between these teachers' perception of mathematics.

Experientially-formed beliefs: Beliefs as learner, trainee and experienced teacher.	Teacher	
	Caz	Gary
Believes they have a natural talent for the subject and strongly influenced by family views about the subject	✓	
Found mathematics unproblematic in their own schooling of the subject		✓
Found mathematics challenging but enjoyable in their own learning of the subject	✓	

Enjoyed a mechanical approach to mathematics at school – the challenge of working through text books and levelled cards of questions		✓
Influenced by courses and training in how children learn or the learning of mathematics	✓	
Influenced by teachers they have worked with and Senior management team		✓
Concerns about children's engagement with the mathematical learning e.g. children groupings and how discussion develops learning	✓	
Concerns about children reaching targets and motivating children to work and achieve		✓
Believes the way mathematics is taught now is much better or more fun than when they were young	✓	✓

Table 1: Differences in perceived beliefs

The results resonate strongly with earlier research highlighting the connections between beliefs formed during the early learning of mathematics and practice (Thompson, 1984; Ernest, 1989). The characteristics of these two groups will be discussed below, but, crudely, the first group, represented by Caz, held a relational perspective on mathematics and its teaching, while the second, represented by Gary, illustrate an instrumental (Skemp, 1976). Moreover, the evidence indicates that the members of both groups still enjoyed the same things as when they were young, and that these formative beliefs are not only deep rooted but reflected in their perspectives on their own classrooms. This perspective will now be presented through the classroom norms emphasised by each teacher in observations and SRIs.

Classroom Norms

Classroom Norms (CN) emerged from the data in all cases of the study which identified a regular pattern to the way in which the teachers conducted their Whole Class Interaction (WCI) in mathematics lessons. Each individual was seen to behave and offer consistent perceptions for that behaviour thus establishing a classroom norm as described by Yackel & Cobb (1996) and Chazan et al. (2012). Utterances made by the teacher are presented in all cases in *italics*.

The teachers fall into the same two groups as presented earlier in their background influences. That is, 'the understanding that students are expected to explain their solutions and their ways of thinking is a social norm, whereas the understanding of what counts as an acceptable (mathematical) explanation is a sociomathematical norm' (Yackel & Cobb, 1996, p. 461). There are three main threads to the discussion of classroom norms which are presented in the table (2) below:

Structural Norms	highlight the lesson structures through which teachers present mathematics during WCI phases. For example, the emphasis made on explicit learning objectives and success criteria, discussion, and particular peculiarities of whole

	class phases, time and flexibility
Cognitive Norms	encompass the emphasis a teacher makes to developing children as enquirers and problem solvers of mathematics in general, the precision and fluency procedures, and whether an instrumental or a relational approach to teaching and learning is made in WC phases
Attitudinal Norms	illustrate the emphasis the teacher places on developing children's confidence, motivation to learning/mathematical learning, enjoyment of learning/mathematics and the teacher's actions and styles connected to *social* relationships emphasised by the teacher

Table 2: Classroom norm key threads

Structural norms appeared very similar between teachers, for example, both teachers presented learning objectives (LOs) on their boards, but their rationale for doing so was quite different. In so doing, they exploited routine behaviours familiar to them and their children. Caz, LOs were broad and, I argue, commensurate with her ambitions that her children should remember the mathematics and not the context. For example Caz had asked her class to increase the mass of cake sizes by ten, she *said I want them to remember we were multiplying by 10 and 100 and not learning about cakes!* She consistently encouraged her children to offer ideas and questions about the learning objectives (a social norm). However the manner in which children responded to questions indicated their awareness that they were expected to provide mathematical justifications or reasoning behind their contributions. In short, she presented mathematics as a way of thinking and behaving (sociomathematical norm).

In contrast Gary spent much time emphasising what he intended to be learnt, with between three and six LOs presented every lesson, meticulously going through each in detail. Such actions, while superficially mathematical, concerned the establishment of behavioural, rather than cognitive patterns of working and so, I argue, reflect a social norm, because they are no more than a 'telling' of what children are to learn. This approach was also seen in his use of success criteria (a list of how children will learn the objectives displayed). This is an important distinction and something likely to be hidden from Gary, who believed, as his institutional management team had reiterated, an effective teacher is one who ticks off each of a series of 'teaching skills'. For example: *go through learning objectives with the class – tick. Go through the vocabulary – tick*. It is not wrong it is simply reflective of instrumentally- rather than relationally-focused beliefs (Skemp, 1976).

Discussion was managed in different ways. For Caz discussion frequently included paired talk, questioning and argumentation. She expected children to think and make connections between the mathematics and real-life experiences, as described by Weber et al. (2008). Gary typically followed an Initiation Response and Feedback (IRF) format. The manner, in which this played out in enacted practice, was quick with rapid closed questions answered by selected students. The social norm was for children to sit quietly in front of their teacher and listen and wait to be asked.

To summarise these differences is to acknowledge that children develop learnt behaviours as either autonomous or dependent learners. Autonomous learners ask their teacher questions and offer ideas even if they might be wrong. They talk about their mistakes and misunderstandings publicly. Dependent learners are quiet, essentially passive, but highly attentive to their teacher. The key point however, is directly related to their teacher's approach (Lawson, 2004), and I would argue, their teacher's belief of what it is to be a learner.

Cognitive norms differed greatly between what the teachers perceived as games, and what they understood by whole class discussion. Games were played in both teachers' lessons, and in particular at the beginning. Caz used, for example, speed against the clock games for recalling facts, through perhaps dance routines to jog memory and paired games to develop calculation strategies and vocabulary. She said they *really enjoy playing those sorts of games* (competitive pairs). *They make it really hard for each other too* (with their questions). The emphasis for using games was to make their *children think and talk using new mathematical vocabulary* and *plan their strategies to win* thus providing an opportunity to behave mathematically (a sociomathematical norm).

Although Gary also used games his rationale was quite different. He frequently used structured tasks, such as writing out times-tables forwards and backwards, which he described as a game. He justified these as time fillers (social norm) and not a sociomathematical norm. At other times he exploited a game called 'popcorn', whereby he calls out a number and, in one variant of the game, children sit if the number is odd and stand if it is even. He varied it so to try to catch children out and, in observed lessons children seemed to enjoy the activity. Interestingly, Gary spoke about it as *breaking up the lesson, to get a bit of movement going, we even go into the millions of whole number, just associating a bit of quick thinking ...that's an even number, I need to stand up, odd numbers oh I sit down.* In such accounts we can see a social rather than mathematical norm where the emphasis was on having fun.

All teachers develop cultural routines and rituals that children come to know (Alexander, 2000) and two such rituals, concerned thinking time. Gary provided short opportunities, typically between three and seven seconds, for children to think about a question before answering. Caz provided several minutes for discussion through whole class or paired talk. Such distinctions typically permeated the lessons of each group.

There has been some discussion in primary education about what pace actually means. Official documentation in England (OfSTED, 2005) indicate that a fast pace is necessary during direct teaching. However, the confidence of the official version of pace is at odds with the literature, e.g. Alexander (2000) write that 'an observer may be deceived into concluding that pace of classroom talk equates with pace of pupil learning' (p. 430), perhaps a pointless exercise if it is not appropriate.

The belief of both project teachers, quite naturally, is that they do what they do because they believe their approaches are effective and educationally beneficial. Yet the research into WCI phases of mathematics lessons (Alexander, 2010) indicates otherwise. In conclusion, the pace and the relationship to the amount of thinking time given to children dichotomised the teachers. Although the time provided for thinking reflected a social norm in each classroom, the conceptions presented by each teacher highlighted differences in a mathematical emphases. Caz believed that children should co-construct their answer to develop mathematical thinking, whereas Gary believed *he* was structuring children's thinking.

Attitudinal norms were presented through their different emphasis on children's enjoyment, confidence and motivation of mathematics. When Caz, emphasised her desire for their children to enjoy mathematics, she did so in relation to their structuring their children's learning of mathematics. It was a cognitive tool rather than an end in itself. Consequently, her ambition reflected a sociomathematical rather than a social norm. Gary, however, discussed enjoyment in very different ways. While it could be argued that his desire for their lessons to be fun helped to maintain his children's focus and concentration, he believed that enjoyment of mathematics would lead to success and increased confidence, as found by Skott's (2009) research on teachers. Gary spoke of how 'target children' were asked lots of questions to build their confidence, for example. Frequently, saying *...oooh that was a very good answer* or *good girl* or *good boy*. Such actions reflect a social norm (Yackel & Cobb, 1996). A justification for his actions may lie in the fact that he paid great attention to the children he perceived as weak. Gary was focussed on progressing all his class two sub-levels in their curricular assessments (as instructed by his senior management) and so worked hard on those at the margin of that.

To summarise classroom norms, important similarities and differences between the teachers' beliefs and practices are highlighted. The classroom norms illustrated in these two classrooms seem in accordance with individual teacher's core beliefs about learning. What is of substantial interest, and an appropriate site for future research, is the clear distinction between the two groups of teachers was either constantly encouraging social norms or encouraging sociomathematical norms. The group, emphasising social norms focus on the achievement of particular behaviours which just happen to be in mathematics, not the mathematics itself. The opposite reflects a more relational learning that is created through sociomathematical norms that encourage a collective, co-constructed learning is rooted in mathematics.

CONCLUSION

Both teachers were, not only considered to be strong mathematically, but leaders of the subject, (according to local definitions) effective teachers of primary mathematics. Yet two distinct groups of teachers consistently emerged through their background perspectives and their classroom norms, despite the fact that all these teachers are well qualified, there remain substantial differences between them in respect to their early background experiences related to the subject, and how their subject knowledge plays out in the classroom.

This study indicates that what transpires during the whole class interactive phases of a lesson is far more complex than a simple analysis of subject knowledge can reveal (Skott, 2009). Teachers draw on core beliefs about mathematics and mathematics teaching that are frequently immune to change (Handal & Herrington, 2003). The findings of this study, suggest qualitatively different teacher characteristics. On the one hand are teachers who behave autonomously; teachers who mediate the constraints within which they work, and perceive learners to be autonomous. On the other hand teachers who appear dependent are mediated by the constraints within which they work, and emphasise dependent learners in the classroom norms. Of course, this is a simple summary that belies the layers of complexity of what an individual teacher chooses to do in any given set of circumstances. Yet it highlights a strong relationship between how they viewed, valued and played with mathematical ideas at an early age, and continue to manifest this approach in their own teaching.

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TEACHING PRACTICES TO ENHANCE STUDENTS' SELF-ASSESSMENT IN MATHEMATICS: PLANNING A FOCUSED INTERVENTION

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This paper focus on the planning process of a teaching intervention aimed at promoting students' self-assessment in mathematics. We present an interpretative case study of a collaborative group, planning assessment practices addressed to students' appropriation of assessment criteria in mathematics. The study allow us to: describe the main focus and factors considered in the planning processes; identify the key practices/strategies planned; and understand their foundations. It also highlights the importance of planning formative assessment practices in mathematics and gives some insight into that process.

Keywords: formative assessment practices, planning, self-assessment, assessment criteria, mathematics

INTRODUCTION

Self-assessment is a privileged form of formative assessment (Nunziati, 1990; Santos, 2008) that helps students to take greater responsibility for their own learning (Sadler, 1989) and leads to significant improvements in their achievement, particularly in mathematics (Brookhart *et al*., 2004). However, the development of self-assessment isn't easy and requires several conditions, which should be promoted by teachers, through formative assessment practices (Wiliam, 2011). To be effective, these practices should be carefully planned.

This study is part of a broader research [1], trying to understand assessment practices of teachers, aimed at promoting students' self-assessment in mathematics. These practices are addressed to: (i) an intentional oral communication during whole-class mathematical discussions; (ii) the appropriation of assessment criteria by students; (iii) the development of students' written self-assessments. They are integrated into a teaching intervention planned in a context of collaborative work involving the first author of this paper (researcher) and four mathematics teachers (grades 7 to 9). In this paper, we focus on the planning process of the teaching intervention, especially with regard to assessment practices addressed to appropriation of assessment criteria, during an initial period of 9 months. We consider the following research questions: (i) What were the main features of the planning process (level of planning, influencing factors and focus)? (ii) What planning decisions were made by the group regarding the definition and implementation of classroom practices? (iii) What reasons have substantiated such planning decisions?

Although the importance of teachers' practices is widely recognized in the context of formative assessment, very few research has been presented in the particular context of mathematics, namely at previous CERME conferences.

THEORETICAL FRAMEWORK

Teachers' practices can be viewed as the activities that they regularly conduct, with certain meanings and intentions, in their working context (Ponte & Chapman, 2006). When teachers develop assessment practices intended to enhance students' learning, we may speak of formative assessment. Formative assessment "accommodate[s] all the ways in which assessment can shape instruction" (Wiliam, 2011, p. 40), it involves elicitation, interpretation and use of evidence about students' learning to make founded decisions about teaching and learning (Wiliam, 2011).

There are some key strategies that are associated to the territory of formative assessment: (i) clarifying and sharing learning intentions and criteria for success; (ii) engineering effective classroom discussions, questions and learning tasks; (iii) providing feedback that moves learning forward; (iv) activating students as the owners of their own learning; (v) activating students as instructional resources for one another (Leahy, Lyon, Thompson & Wiliam, 2005). However, there isn't a *one-size-fits-all package* (Leahy *et al.,* 2005), so it is not enough for teachers to know these strategies. They need to study, and possibly discuss with others, how to implement these strategies in their own classrooms. Otherwise, "without this space for teachers' voices, it seems likely that formative assessment will be enacted more as a set of techniques rather than as a step towards a more dialogic form of teaching" (Hodgen, 2007, p. 1893).

Self-assessment is an internal process of regulation of own thinking and learning (Nunziati, 1990) that is vital for learning (Black & Wiliam, 1998). It includes monitoring and action: the student confronts what he/she did with what he/she was expected to do, acknowledging the differences between these two situations, and acts to reduce or eliminate them (Sadler, 1989; Santos, 2008). Therefore, assessment criteria are a reference and a needed condition to self-assessment, but they are just its starting point. They must be legitimate for students and allow them to understand what is expected of them (Hadji, 1994), they must be *appropriated* by students. However, this is quite rare and difficult since the meaning given by students to the criteria may be different from the one given by the teacher (Vial, 2012). Moreover, one must take into account the didactic tension: "the more clearly the teacher indicates the behaviour sought, the easier it is for students to display that behaviour without generating it from understanding" (Mason, 1998, p.2). So, it is necessary to create opportunities for students to really understand the criteria in the context of their work (Black & William, 1998) and to develop ways of promoting a state of *working-on*, instead of *working-through*, in mathematics classroom (Mason, 1998).

Classroom discussions are excellent opportunities for the development of formative assessment practices. In particular, teachers should engage students in discursive practices, encouraging them to develop, explain, justify and assess their ideas and those of colleagues (Henning *et al*., 2012); promote the establishment and respect for rules of interaction; and direct the focus of discussion, cautioning the development of important mathematical aspects (Chazan & Ball, 1995).

Given the complexity of effective formative assessment practices, teacher collaboration might be especially useful (Hargreaves, 1998). Besides that, careful planning is essential. In fact, planning processes are central in teachers' practices, but they are typically undervalued (Calderhead, 1996; Clark & Peterson, 1986; Shavelson & Stern, 1981).

Teacher planning is both a psychological process – in which teachers visualizes the future, inventories means and ends, and constructs a framework to guide their action – and a practical activity – the things that teachers do when they say that they are planning (Clark & Peterson, 1986). Planning include three phases: preactive (before teaching), interactive (during teaching) and postactive (after teaching) (Milner, 2001). Calderhead (1996) presents the main features of planning processes: (i) planning occurs at different and interconnected levels, from yearly and long-term plans to lesson plans, and may be seen as "a continuous process of re-examining, refining and adding to previous decisions" (p. 714); (ii) planning is mostly informal, teachers plan by mentality focusing on aspects that need their attention (iii) planning is creative, it does not follow a linear process from specified objectives to activities planned to accomplish them, it has a problem-finding and a problem-solving phase; (iv) planning is knowledge-based, teachers base their planning on different kinds of knowledge (for example, knowledge of subject matter and of students); (v) planning must allow flexibility, to adapt planned activities accordingly to the situations that might emerge; and 6) planning occurs within a practical and ideological context, since there are various factors that influence teachers' planning as policy expectations, textbooks or other materials being used, teachers' experiences and conceptions of mathematics teaching and learning.

METHODOLOGY

This is an interpretative case study (Yin, 2002), being the collaborative group considered as one case. Special attention is given to questions of "how" and "why" the collaborative group plan the teaching intervention aimed at promoting students' self-assessment in mathematics. The collaborative group was constituted to the broad research purpose. The four teachers were chosen to take part in the group, based on the following criteria: to evidence sensitivity concerning issues related to the research aim and openness to consider them in their professional practices; to have different professional experience.

In this study, collaboration is characterized by joint work, in order to provide mutual support and the achievement of goals (not necessarily the same) that benefit all (Boavida & Ponte, 2002). Participants must feel comfortable in their roles (not necessarily the same) and be attentive to the needs of others and open to negotiate understandings emerging from the collaborative effort (Hargreaves, 1998).

Collaborative group meetings lasted a total of 25 months and included the planning of the teaching intervention and assessment of the practices/strategies after implementation in classrooms. The planning process took as starting point the development of a shared understanding regarding the objectives and guidelines that frame the intervention, based on the discussion of various documents, especially associated with formative assessment, oral communication and collective discussions in mathematics classroom. Initially, the researcher played a key role in the collaborative group, being responsible for: (i) negotiate the general goals of the teaching intervention to be planned, (ii) propose documents and materials as basis for discussion and work, (iii) propose points for the agenda of the meetings. Over time, these roles were shared with teachers, as they were feeling more comfortable with the topics addressed and the work was putting focus more on planning and reflection on real classroom practices, rather than on theoretical perspectives.

Data collection includes participant observation of 15 meetings of the collaborative group (meetings were audio recorded; code M1 to M15 is used to identify each meeting), supported by document collection of teachers' planning materials. The process of data analysis included: the transcription of the meetings recorders; the selection of relevant parts of the texts, keeping the coherence of discourse; the interpretation of the texts reduced, taking into account who talks and his/her intention and the context. This interpretation involved the identification of perspectives, ways of being and acting presented in the participants' speech, which led to the identification of themes. In this paper, in particular, is not explicitly compared or contrasted the speech of each participant, but considered the speech of all participants in the construction of a collective discourse of the collaborative group. The transcripts presented in the following section were translated from Portuguese.

ANALYSIS AND RESULTS

Defining assessment criteria in mathematics

The assessment criteria are recognized, by the group, as essential to clarify what is expected of students in mathematics classroom and help them to self-assess accordingly:

Valter: It has to be very clear to them what they're supposed to do.

Researcher: Yeah.

Valter: Because that is what allows them the self-regulation. (...)

Filipa: We have to give him the assessment criteria (...) for him to know how he will be assessed (…) isn't it? (M5).

In particular with regard to the students' self-assessment, the group agreed that they tend to be not criteria-based (or use criteria different from the ones of the teacher):

> I think they have many difficulties ... when I ask them a self-assessment, they are not (...) being more criteria-based when a person asks them: (…) What do you think you can do to improve your learning? It is where ... I think they have difficulties (Filipa, M2).

The group have agreed that the assessment criteria should reveal the expectations concerning the role of students in collective mathematical discussions, in particular as regards to aspects diagnosed as source of resistance and difficulties for students:

[Example 1]

Researcher: I think that if this is a problem in your students [do not value the collective discussion], in the assessment criteria (...) it should be evident that one of the things that they are expected to do is (…) comment or ... whatever ... but ...

Filipa: Yeah, know how to assess...

Joana: Know how to justify his option. (...)

Filipa: ... criticize the other, criticize the speech of the other (M4)

[Example 2]

Filipa: I think it's very important [that they compare, analyse, relate ...]. Because I think that it is what they have more difficulties. (...)

Valter: The ability ... to be able to argue on the basis of the argument presented by another, counter-argue, I find it an asset. To seek to understand the other's perspective (M5).

Furthermore, the group stressed the need for the criteria to put a focus on mathematics: “It’s important that there is debate among students, but I have to emphasize that they have… there must be mathematics, if not they will not realize: Well, I participated a lot” (Valter, M4). Thus, the group defined assessment criteria in three domains of mathematics: Concepts and procedures; Strategies and processes of reasoning; Communication.

Working assessment criteria with students

For presenting the criteria, the collaborative group first use a table of descriptors for various levels of performance. But this table was considered too complex for students: “will they understand all this text? The 7th graders? I have some [students] of the 9th grade who will lose themselves (Filipa, M7)”. So it was created a simplified assessment grid, presenting a description of what is expected of students for each criterion. To prepare this grid, the group sought to use a language accessible to students:

Valter: They have a restricted code of language. (…)

Filipa: If you put “I make good oral statements”, they understand. (…)

Valter: The student must know to what he is answering. (M8)

Aware of the difficulties that some terms could still cause (particularly *systematic solving strategy*), the group considered the importance of negotiating its meaning with students, using concrete examples: “Or give an example (...) They [students] have to realize (...) I think that with an example is better” (Valter, M8).

Following the same assumptions, for a first approach to the assessment criteria in the classroom, the collaborative group decided to involve students in an assessment experience of work samples, using the assessment grid. Strengths and weaknesses should be identified in light of the criteria and a negotiation process should be developed, allowing students to propose changes, but without jeopardizing the key ideas considered by the group:

Researcher: ... the idea was also to leave (...) students, isn’t it? ... if necessary, change something. Let them even...

Joana: ... suggest…

Researcher: "So you think that here, perhaps, would be more clear to you this?".

Joana: Sure. (...)

Researcher: Without removing what we consider essential, isn’t it?

Joana: I think so. (M7)

In addition, the group realized the need to invest systematically in the appropriation of the criteria, as a whole, in the context for which they were planned – collective mathematical discussions with presentation of students’ work. To operationalize this idea, the group planned a first cyclical model of lessons, comprising: (i) performing the task in small groups; (ii) groups’ presentations and collective discussion; (iii) students’ self-assessment; and (iv) confrontation between students’ assessment and the one performed by the teacher, followed by whole-class discussion. In this model, self-assessment is developed by filling the assessment grid, aiming to reinforce the criteria by which students must guide and assess their performance. Self-assessment is asked to groups of students, rather than individually, to encourage discussion and simultaneously not expose individual cases of students who may eventually feel more constrained. The whole-class discussion about teacher’s and students’ assessments, including teacher feedback, was specially planned to open doors to the negotiation of meanings regarding the criteria and the clarification of what is expected of students, using concrete examples of effectiveness in the classroom, but without constraining students:

Researcher: I think the advantage (...) is that it only speaks...

Sofia: ... who wants, isn’t it?

Researcher: ... who feels comfortable to present his case (…) So, we defend cases of students who do not want to expose themselves...

Filipa: Sure.

Researcher: So, they don’t feel obligated to do it, but nevertheless they are being confronted with the assessment that the teacher did, and with the examples of others they can... (M9).

Assessing and rethinking practices

To assess the effectiveness of the model and rethink future steps, the group considered to stop/close the corresponding cycle, at proper moments, asking individual written reflections to students as self-assessment.

Filipa: … what do you think about (…) putting this [model] in all lessons of a [mathematical topic]? (...)

Valter: And, maybe, taking off occasionally to see what is already achieved....

Joana: Yeah. (...)

Researcher: (...) the idea is, later, as Valter said, that they no longer need it [the assessment grid] (...) Maybe it should be interesting (...) after the end of the topic, to ask for a written reflection, for example, no longer...

Valter: ... without the grid in front. (...)

Researcher: And then, through these reflections we can also see if it is necessary (...) to continue or not. (M9)

The group defined a new cyclical model of lessons, similar to the previous one but more flexible, for application when students show a reasonable understanding of the criteria. In this model, self-assessment of students is developed through a written reflection, which may be open or oriented depending on its main purpose:

Researcher: [We want that students] are able to reflect, self-assess their work, also to find strategies to improve (...) maybe in order to a reflection (…) be more useful to them (…) some guidelines may be provided (...)

Valter: This can go through (...) at some point ask for a reflection with some indications (...) And then, later, give again a reflection that is open (...) to see if things meanwhile were being internalized. (M14)

This phase of self-assessment and the next one (whole-class discussion of the assessments) might not happen every time, so they don’t become routine procedures that don’t raise reflection:

Filipa: … they [students] do a written reflection for each task and it ends the conversation! (…) is for their own good! (...)

Researcher: It depends (…) Because it may become a tedious process and they are always writing the same thing, you know? Instead of evoking reflection, be...

Filipa: Yes.

Sofia: It becomes routine. (M5)

DISCUSSION AND CONCLUSIONS

In this study, planning has gathered characteristics, mainly, of a long-term planning – a structure of what to do was defined – but also of other levels of planning – how to operationalize some practices/strategies was considered in some detail (Shavelson & Stern, 1981). Thus, different levels of planning were contemplated and have informed each other (Milner, 2001). Nevertheless, since the teaching intervention presupposed the introduction of innovative elements in teachers' practices, long-term planning emerged as the most significant in a first stage (Milner, 2001).

Planning was developed based on different types of knowledge and was influenced by several factors (Calderhead, 1996), in particular: (i) knowledge of mathematics curriculum, literature and teachers' conceptions about teaching and learning mathematics, in order to define the assessment criteria and the expected roles of teacher and students in the intervention lessons; (ii) teachers' previous experiences and knowledge about their students to inform about challenges and difficulties; (iii) research recommendations, namely about formative assessment, and teachers' conceptions and previous experiences to outline practices/strategies and ways of operationalizing them in mathematics classroom.

Regarding classroom practices planned by the collaborative group, they reveal some formative key-strategies (Leahy *et al.,* 2005) and are the result of the collaborative group work, trying to find a suitable way of implementing those broad strategies in the mathematics classroom. The planning process of these practices focus on three main areas: defining assessment criteria in mathematics; working assessment criteria with students; assessing and rethinking practices. Assessment criteria were defined as a powerful resource both to clarify what is expected of students (and indirectly of teacher) in intervention lessons, and to support students' self-assessment (Hadji, 1994). The definition of such criteria was, itself, guided by some criteria, namely: to take into account critical points (in relation to particular difficulties diagnosed) and to put a focus on mathematics. This led to criteria that meet mathematical skills and understandings recognized as essential in mathematics learning (NCTM, 2000): Concepts and procedures, Strategies and processes of reasoning; Communication.

Planning how to *work on* assessment criteria with students (instead of *work through*) has raised new challenges and concerns to the group (Mason, 1998). First, a simplified grid of assessment was prepared so that criteria would become accessible to students. Then, a process of negotiation was conceived, engaging students in the assessment of work samples, with discussion about the strong and weak aspects of each one. Recommendations of several authors were, thereby, considered and combined (Sadler, 1989; Santos, 2008; Wiliam, 2011). For students to understand the criteria in context (Black & William, 1998), a cycle model of practices/strategies was also planned, including students' self-assessment, using the grid, and whole-class discussion about teacher's and students' assessments. Later on, to assess the effectiveness of previous practices in promoting students' appropriation of assessment criteria, the group has planned to ask written reflections to students.

Ultimately, written reflections were integrated in a new model for intervention lessons, as a way to promote and regulate students' self-assessment, which takes into account the didactic tension and the importance of students *working-on* in mathematics classroom (Mason, 1998).

Practices planned by the collaborative group show potential to meet the conditions, identified by Sadler (1989), as supporters of the improvement of students' learning: sharing an idea of quality similar to the one of the teacher, continuous monitoring of work and access to a repertoire of alternative strategies that can be implemented to improve. Naturally, planning should continue, through a cycle of preactive, interactive, and postactive planning (Milner, 2001), readjusting the practices toward the promotion of students' self-assessment in mathematics.

This study presents an innovative character by placing the focus of teacher planning on formative assessment in mathematics and on ways of operationalize such assessment in the classroom, giving some insight into that process. It highlights the importance of, on the one hand, valuing formative assessment in mathematics teachers' planning and, on the other hand, developing a focused and intentional planning of teachers' assessment practices so that they effectively contribute to students' learning in mathematics.

NOTES

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CAPTURING PRE-SERVICE TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING

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This study investigated the change in the Mathematical Knowledge for the Teaching (MKT) of geometry for three preservice teachers over a period of seven months. Data was collected via pre- and post-surveys as well as interviews. Data collection items were specifically designed to elicit preservice teachers' MKT as they responded to items involving the analysis of student work and thinking. The results indicated that there was a significant decrease in preservice teachers' scores on items pertaining to analysis of student work and thinking. Preservice teachers relied on their past experiences as students and on their work experience while making pedagogical decisions. Recommendations for future research in preservice teacher education are discussed.

Keywords: Preservice Teacher Education, Mathematical Knowledge for Teaching, Geometry

INTRODUCTION

Until a few years ago, researchers in the U.S. defined teachers' subject matter knowledge using quantitative means such as number of subject specific courses taken or the teachers' scores on standardized tests (Even, 1993). However, the adequacy of these proxies for measuring mathematical knowledge for teaching has been the subject of controversy and debate (Hill, Rowan, & Ball, 2005). Ball et al. (2005) coined the term Mathematical Knowledge for Teaching (MKT) in order to distinguish a specialized body of knowledge of subject matter that is needed for teaching. MKT consists of not only knowledge of mathematics content, but also how that content is taught. It includes knowledge about which topics or concepts are easy for students to learn and which are difficult. While several studies at the elementary and middle school level have claimed that teachers' MKT is correlated to student achievement, such studies are rare at the secondary school level. One reason for this gap is that high school mathematics consists of a variety of topics and is very complex (McCrory, Ferrini-Mundy, Floden, Reckase, & Senk, 2010). Knowledge about how secondary students think about mathematics is also very limited.

This work is a part of my dissertation study in which I investigated the nature of MKT of secondary preservice teachers by eliciting their reactions to samples of student work included in written surveys, using a case study methodology (Somayajulu, 2012). The following research question guided the data collection and analysis:

> What factors do preservice teachers consider when making pedagogical decisions based on analysis of students' mathematical work and thinking?

THEORETICAL FRAMEWORK

In this article, I subscribe MKT to being described as “mathematical knowledge needed to carry out the work of teaching mathematics” (Ball et al., 2008, p. 395). For the purpose of this study I utilized the Knowledge for Algebra Teaching (KAT) (McCrory et al., 2010, 2012). Although the KAT framework has been designed for assessing algebra teaching, the teaching tasks and the knowledge categories that are highlighted by the framework are applicable to all the areas of mathematics, and in particular to geometry. This framework consists of a two dimensional matrix in which the rows represent the tasks that teachers perform while teaching mathematics and the columns represent the categories of knowledge required to perform those tasks. The specific categories can be seen in detail in Figure 1 below. For the purposes of the current study, I focused on the task of analysing student work and thinking and the knowledge bases required to successfully perform that task.

The framework also consists of three overarching categories: decompressing, bridging and trimming (McCrory et al., 2010). Decompressing, according to McCrory et al. (2010) is working from a more compressed understanding of mathematics to a more unsophisticated form. It includes “attaching meaning to symbols and algorithms that are typically employed by sophisticated mathematics users in automatic, unconscious ways” (McCrory et al., 2010, p. 38). Trimming is a process in which teachers present an advanced or sophisticated mathematical idea to students in a way that the fundamental nature of the topic is preserved but it is now less rigorous (McCrory et al., 2012). Bridging involves making connections between mathematical topics or between mathematics and other subject areas (McCrory et al., 2012) which is similar to Shulman’s (1986) notion of lateral and vertical curriculum knowledge. Decompressing and trimming are complementary activities; whereas the former draws on mathematical knowledge to unpack meaning of algorithms, equations, etc, the latter is how a teacher preserves fundamental meaning of advanced mathematical topics in the context of less-advanced classrooms (McCrory et al., 2012). The KAT framework guided the task selection for the surveys.

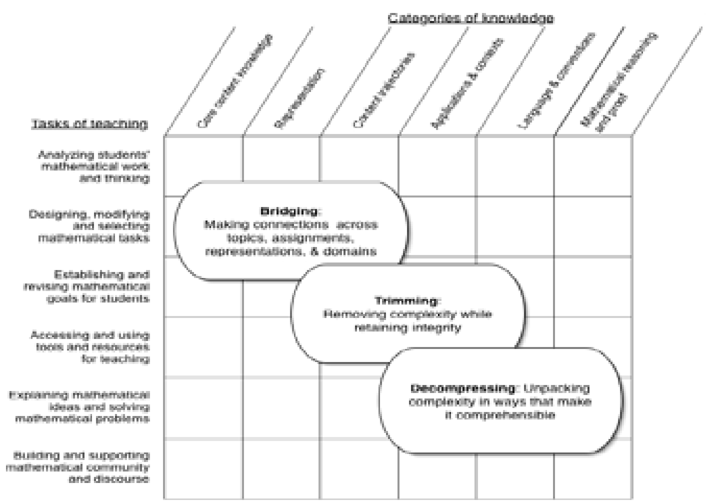

Figure 1: Knowledge for Algebra Teaching (McCrory et al., 2010, p. 58)

METHODOLOGY

Participants

The original dissertation study involved eight participants: four male and four female enrolled in the Master of Education program seeking licensure to teach 7th to 12th grade mathematics in the U.S. Six of the participants had a bachelor's degree in mathematics and two participants had a background in engineering. For the purpose of this study, I report on an in-depth case study involving three of the participants. The selection of these three participants was done based on Cooney, Shealy and Arvold's (1998) classification of preservice teachers. A brief description of the three case study participants is given below.

Cersei

Cersei had a background in industrial engineering prior to joining the M. Ed. Program. Cersei fell in the isolationist category (Cooney et al., 1998) in that she rejected the beliefs of others. The program had very little effect on Cersei's beliefs about mathematics teaching and learning.

Bran

Bran completed his bachelor's degree in mathematics prior to joining the program. Bran demonstrated the qualities of a naive connectionist (Cooney et al., 1998) in that he was receptive to other's beliefs but is not able to resolve the conflicts between his beliefs and other's beliefs.

Nedd

Nedd was a computer engineer before joining the program. Nedd was a naive idealist (Cooney et al., 1998). He readily accepted other's views and beliefs without questioning them. Unlike the other participants, Nedd opted for an extra year to complete the program.

Data Collection and Instruments

Data collection took place over a period of seven months via pre and post surveys and interviews. The surveys and interviews were designed to capture the preservice teachers' MKT as it pertained to the analysis of student work and thinking via the processes of decompressing, trimming, and bridging. The survey items were taken from previously recorded episodes of student work on geometric tasks consisting of instances of children's thinking and heuristic usage (Manouchehri, 2012). Additionally survey items were chosen from geometric topics common to the U. S. secondary school curriculum. The following is an example of one chosen task:

> A student was given the following problem: Consider a cube whose base area is 4 cm^2. If the area of the base increases to 16 cm^2, how much does the volume increase?
>
> The student replies by saying that the volume of the cube would increase by 4096 cm^3. How do you think the child arrived at this answer?

What techniques or tools may be used to help the child understand the solution?

What are some questions you can ask the student to further his understanding on the topic?

The content validity of the survey instrument was verified by obtaining feedback from practicing teachers, mathematics educators, and mathematics education graduate students. The surveys were piloted with graduate students in mathematics education and based on their responses and feedback, necessary modifications were made. Finally the surveys were administered to preservice teachers outside of the sample for this study. Based on their responses final changes were made to the language in order to remove any ambiguities.

Interviews were conducted after the administration of the surveys to gather further information of the participants' responses to the surveys. The interviews were approximately 90 minutes long. The interviews consisted of three parts: (1) obtaining background information from each participant, (2) a self efficacy scale (Tschannen-Moran, & Hoy, 2001), and (3) focused on exploration of survey responses. In this paper, I focus on (1) and (3).

Data Analysis

The data analysis was completed in two phases. During the first phase, the preservice teachers' responses to the surveys were analysed and coded. In order to do this, the survey responses were classified by dividing MKT along two categories of mathematical and pedagogical analysis. I utilized previously identified performance indicators from a study exploring knowledge for teaching (Manouchehri, 2011). The indicators are as follows:

Mathematical Analysis

Articulating basis for mathematical decisions that children make.

Identifying the strengths and weaknesses of ideas from a mathematical point of view.

Identifying content trajectories and using them to assess children's conceptions.

Identifying the sources of children's errors/misconceptions and reasoning how they could have resulted.

Developing mathematically sound instructional strategies.

Pedagogical Analysis

Identifying why certain pedagogical moves are appropriate to pursue with children based on their analysis of student work and thinking

Successfully addressing areas in which children would or would not be able to perform adequately.

Offer a rationale for why certain pedagogical choices should be implemented.

Identifying the advantages and disadvantages of use of instructional tools.

Based on these indicators, I categorized the responses on a continuum from Mathematically and Pedagogically Naïve to Mathematically and Pedagogically Mature. The scoring of the responses is explained in table 1. This resulted in a maximum possible score of 4 and a minimum possible score of 0.

Score	Mathematical Analysis	Pedagogical Analysis
0	Mathematically Naïve	Pedagogically Naïve
1	Mathematically Developing	Pedagogically Developing
2	Mathematically Mature	Pedagogically Mature

Table 1: Scoring Rubric

Since the number of prompts used differed from the pretest to the posttest, I calculated a percentage score for each participant (that is total points scored divided by total points attempted on a particular survey).

The second phase of data analysis involved analysing the interview data. The interviews were transcribed and coded based on the references they made while analysing student work. Once the interviews were coded, percentages of occurrences for each of the coded category were calculated. Those percentage scores were utilized to build an illustrative map of sources that the preservice teachers drew from while analysing student work and thinking. Finally interview data was compared with the responses to the surveys in order to triangulate conclusions. An example of an illustrative map is given below. The map describes the various factors of Cersei's MKT as exemplified through references while analyzing student work and thinking during the pre interview.

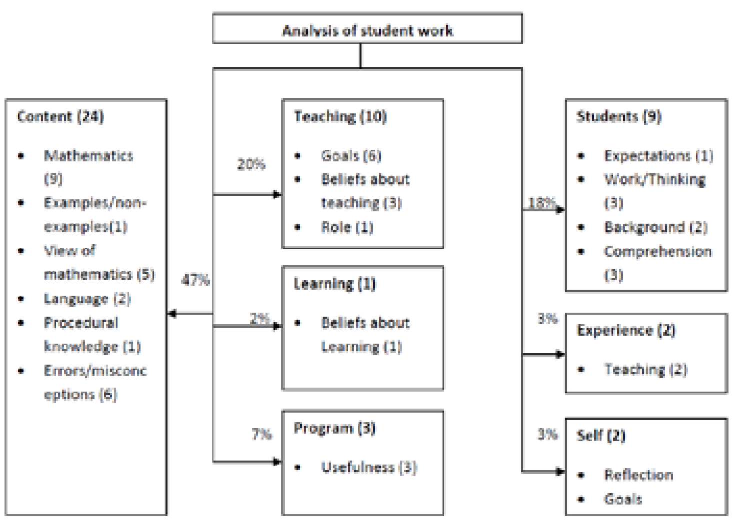

Figure 2: Example of an illustrative map highlighting factors influencing participants decision making

From the above example we see that Cersei made a total of 51 references to student work on geometry during her interview. While commenting on student work, Cersei referenced the geometric content 24 times and teaching 10 times. She also made 9 references to students. Other references were made to learning, experiences, M. Ed. Program and to self. These categories were further broken down into subcategories. So for example, while commenting on the content, Cersei referenced the mathematics 9 times, while she referred to the child's errors/misconceptions about 6 times.

Such maps were generated for both pre and post surveys for each of the three case study participants. These maps were then used to generate a list of important factors that the preservice teachers consider while making pedagogical decisions and hence what factors might be included in the teachers' MKT.

RESULTS

The findings for each of the three case study participants are discussed below.

Cersei

The major factors affecting Cersei's orientation toward teaching were her experiences both as a learner of the subject as well as her work experience, and her beliefs about the teaching and learning of mathematics. She viewed mathematics teaching as being similar to her job as an industrial engineer. When asked if being an engineer was different/similar to being a teacher she replied:

> Cersei: I was always the person that trained the new industrial engineers coming into the department. So I guess I've kind of been teaching all along without really recognizing it but really liked in that whole experience just to being kind of a mentor and helping so many get started into the department.

Analysis of Cersei's responses to the surveys revealed that there was a decrease in her scores for prompts pertaining to analysis of student work and thinking, decompressing, trimming and bridging. After augmenting her interview data with the survey data it was observed that even though her attention to student work had increased, there was a decrease in her attention to the mathematical content. Cersei was also not able to make specific connections to models of assessment and hence was unable to utilize them while analysing student work and thinking.

Bran

The main factors affecting Bran's orientation towards teaching were his experiences as a learner of mathematics as well as his knowledge of the subject matter. Bran acknowledged the connections between his college mathematics courses and high school mathematics and drew on this knowledge of content trajectories while attending to student work. For example, one of the questions on the survey asked the participants to choose two of three topics (similarity, transformations and right triangle trigonometry) such that it would foster student learning. Bran selected transformations and right triangle trigonometry. He did not choose similarity because he felt that it could be derived from transformations.

Researcher: You said similarity can be described through transformations, and how would you go about that?

Bran: I think I'd start with a shape. So a transformation of this would be to make each side longer. So I could keep this here, so I can make everything, like move these points twice as far away from each other. And so I could have two shapes that after the transformation, the shape is still similar. So I can tell that's a special case of transformation when the relationships between the sides and the angles are the same.

Comparison of Bran's pre and posttest survey scores revealed that there was an increase in his scores for prompts pertaining to the analysis of student work and thinking, trimming, and bridging. However, there was a decrease in his scores for prompts pertaining to decompressing. His interview data revealed that his attention to the mathematical content had increased. Bran also demonstrated an increased tendency to try and understand the reasoning behind the student work.

Nedd

Work experience was the biggest factor affecting Nedd's orientation to teaching. Nedd was of the opinion that his job as an engineer had already instilled in him the skills required to be a successful teacher. When I asked Nedd if being a teacher was similar to being an engineer he replied that they required the same set of skills.

Nedd: Yeah. So I think the biggest reason, the biggest thing is, umm, mostly continuous improvement. You know problem solving, adaptability, looking at all the inputs. You know, defining a problem, alternatives, what gets you to that next improvement. I mean, I think those are very drilled into me and I think that's a skill a teacher needs to have.

Nedd also showed a decrease in his scores pertaining to analysis of student work and thinking, decompressing and bridging. From the interview data, I observed that Nedd's attention to student work decreased. Instead the tasks pertaining to student work and thinking served as an avenue for Nedd to reflect on his own mathematical knowledge.

A cross examination of the three cases led to the following findings:

Preservice teachers had trouble decompressing their knowledge while analyzing student work and thinking.

Preservice teachers relied on their experiences while analysing student work and making pedagogical decisions.

Preservice teachers were unable to utilize learning based assessment models such as van Hiele or Pirie-Kieren (Pirie, & Kieren, 1994) to aid in evaluating student work as well as designing instructional tasks.

Knowledge of content and content trajectory had an impact on the participants' analysis of student work and their pedagogical decision-making.

DISCUSSION

Figure 3 illustrates the common factors affecting the preservice teachers' pedagogical decision making. Participants' past experiences along with their beliefs about the teaching and learning of mathematics, and their knowledge of the trajectory of the content were critical factors when making pedagogical decisions while analyzing student work and thinking.

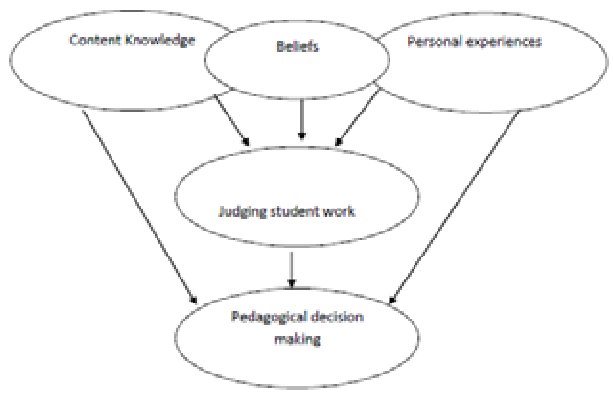

Figure 3: Forces influencing pedagogical decision making (Somayajulu, 2012, p. 282)

Elbaz (1983) demonstrated the interactions between teachers' personal theories and practice, often referred to as teachers' personal theorizing. This study exhibited similar results, where in the preservice teachers relied on their past experiences and their classroom observations of their mentor teachers while making pedagogical decisions. Of these, their past experiences as learners of mathematics and their work experiences had a prominent influence on their decision making. While referring to student work, the preservice teachers almost always made decisions independent of attention to student work and relying mostly on their experience as learners of mathematics. For the participants with engineering backgrounds, work experience was a constant influence on how they viewed the mathematical content.

Several studies have demonstrated the importance of the knowledge of content and its trajectory while making decisions (Aubrey, 1996; Kahan, Cooper, & Bethea, 2003). This study too demonstrated that such knowledge was critical in pedagogical decision making. Preservice teachers, who were not comfortable with the content, were not able to identify the concepts that were central to the topic and hence could not identify the trajectory of contents.

One major concern was that none of the participants were able to connect theory to practice. This concern has been documented by several researchers, in particular by Jaworski (2006). According to Jaworski (2006), this inability to connect theory to practice stems from the fact that even though theories are valuable tools for analysing student work, they do not offer any clear insights to teaching. I observed that none of the preservice teachers in the study were able to apply models of assessment to aid them in their analysis of student work and thinking. One reason for this is that it is unclear how we can measure teachers' development within the realm of a research based program. It is essential to develop an understanding of teachers' learning trajectories as they are exposed to new knowledge via teacher education programs.

Another concern is the lack of influence of the methods courses on the participants. One likely reason for this is that the methods courses were designed without much sensitivity towards helping the teachers' bridge theory and practice. In light of the findings of this study, I recommend that preservice teacher education programs should take into consideration the prior experiences of teachers as learners and build on them while designing activities that are more effective.

LIMITATIONS

A major limitation of this study was that the analysis was entirely based on the results of the surveys and the interviews. While I observed the participants in their methods courses, I did not analyse the discourse in those environments. Such analysis might help in providing deeper explanations of why certain changes took place.

The survey instrument that was developed covered a limited range of topics. Moreover the surveys were about 150 minutes long and as a result some of the participants left items blank, especially at the end of the post surveys.

RECOMMENDATIONS

There is a need for developing instruments that are capable of capturing the different aspects of MKT in geometry. Such instruments also need to cover a broader range of geometric topics. While doing so, steps need to be taken so that the surveys are not too long. Another recommendation for future work would be to consider the use of interviews as venues for learning and growth of preservice teachers.

Studies analysing classroom discourse and interactions amongst preservice teachers need further attention as they have potential to offer perspectives on how to sequence tasks to better aid in teacher preparation. There is a further need to analyse the interaction between content knowledge and teaching at the secondary school level.

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PRE-SERVICE TEACHERS' KNOWLEDGE AND BELIEFS: THEIR ASSOCIATION TO PRACTICE IN THE CONTEXT OF TEACHING FUNCTION WITH ANALOGIES

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The present study concentrates on pre-service teachers' knowledge and beliefs and their association to teaching practice in the context of teaching function with analogies. During the first phase of data collection, pre-service teachers completed a knowledge test concerning the definition of function, its types, and analogy. In the second phase, pre-service teachers were observed and videotaped during their teaching practice. In the last phase, interviews were conducted upon the completion of pre-service teachers' teaching practice to yield data about their beliefs about using analogies. Results revealed that pre-service teacher knowledge and beliefs about function and analogy strongly associated with the nature and the extent of analogy use in teaching the function concept.

Keywords: Function, analogy, pre-service teachers, teacher beliefs

INTRODUCTION

There is an agreement among mathematics educators on the importance of analogies or situations from everyday life within the verbal definitions of the function (Elia, Panaoura, Eracleous, & Gagatsis, 2007), the examples of functions from their applications in real life (Elia & Spyrou, 2006) or the tasks that aim to use the definitions of functions (Christou et al., 2005) because of their valuable opportunities for students to gain understanding in functional thinking (Cooney & Wilson, 1993). An analogy is drawn by transferring ideas from a *familiar* concept to an *unfamiliar* one (Thiele & Treagust, 1995). The familiar concept that provides basis features to interpret the unfamiliar concept is often called the source or analog; whereas the unfamiliar concept to which the information is transferred is called the target (Gentner, 1983). Researchers have basically emphasized a correspondence in some respects from source to target, thus underlined the "*relation*" and "*similarity*" in which the same system of relations holds across different objects (Eid, 2007; Heywood, 2002). Matching the familiar domain to the unfamiliar domain is achieved by accessing the similarities and differences of the domains, and then mapping similar characteristics between the two domains by indicating the breakdown points. Analogy generation is crucial for teaching, the origin of which grounds to the connections between the real world and the target concept that establishes as real world linkage, and which, in general terms, could be said to prompt visualization of abstract concepts that facilitate understanding (Thiele & Treagust, 1992). The sense of this perspective is clearly articulated in recent years, such as that of several mathematics educators (Fast, 1996; Richland, Holyoak, & Stigler, 2004). Analogical representations such as function machines attempt to move students from an action to a process conception (Selden and Selden 1992). To be able to manage this move, the quantity and quality of teachers' mathematical knowledge on functions and analogy are important aspects. The collection of articles written by Even (1990, 1993, 1998) and by other researchers (Hitt, 1998; Lloyd & Wilson, 1998; Sanchez & Llinares, 2003) pointed out subject matter knowledge and pedagogical content knowledge, as defined by Shulman (1987), essential for teaching the function concept. Moreover, characteristics of teachers' belief systems have also been linked to various components of these knowledge aspects.

Although a few studies investigated pre-service teachers' use of analogies in teaching function concepts (e.g., Ubuz et al., 2009), not much is known about teachers' knowledge, beliefs, and ability to use analogies in teaching functions that are fundamental components of their pedagogical content knowledge. Considering these facts, we aimed to investigate how pre-service teachers' knowledge and belief are associated with how and when they use analogies to teach the idea of functions. Response to this question can provide insight as to possible linkages between teachers' knowledge and beliefs about functions and analogy, and their teaching of functions with analogy.

METHODS

Context and Participants

The participants were all seven pre-service secondary mathematics teachers (PT1, PT2, PT3, PT4, PT5, PT6, and PT7) attending to Master of Science without Thesis Program at Middle East Technical University. All seven of them were the total number of the students in their last term attending to this program. Master of Science without Thesis Program is a three semester certificate program to teach mathematics at secondary school level (grades 9-12). The first three were male and the rest were female. All graduated already from the Department of Mathematics. They all had some previous teaching experience through their participation to the private tutoring programs. Data were collected during *Practice Teaching in Secondary Education* course provided at the last semester including 14 weeks. This course involves practice teaching in classroom environment for acquiring the required skills to become an effective mathematics teacher. Each week PTs spend their six class hours in a classroom environment at an arranged public secondary school, and two class hours at the university. In that two hours period at the university, PTs presented sample lessons one by one to their peers and the instructor. They were required to present the function concept and its types by generating analogies. At this stage, their knowledge on functions and analogy, and their images resulted from their previous experiences in school and university mathematics as well as the method courses offered at the Master of Science without Thesis Program. With regard to functions and analogy, method courses involved history of function, misconceptions about functions, definition of analogy, and importance of analogy for learning and teaching.

At the public school they taught two lessons with presence of the instructor (the first researcher) and the classroom teacher. At other times they presented lessons whenever the classroom teacher allowed them to do. Teaching at the university and the school constituted 30 percent of the course grade. Lesson plans constituted 15 percent of the course grade. While preparing the lesson plans, they mainly focused on objectives, materials, teaching techniques and the development process in the lesson.

Data Collection and Analysis

Knowledge Test: Knowledge test assessed the PTs in two major strands of knowledge prior to their teaching practice: subject matter knowledge and pedagogical knowledge. The item on subject matter knowledge was chosen to test their factual knowledge about functions, particularly the definitions of function and its types (one-to-one, onto, into, and one-to-one and into). The item on pedagogical knowledge was prepared to assess their knowledge on analogy, particularly the definition of it and its characteristics. They were required to complete the test in the presence of the research assistant of the course during the first week of their course and no time limit was imposed. At this week, all PTs, except PT7, were registered on the course, and ultimately the knowledge test data were driven from them. At the add-drop week PT7 registered to the course and were involved in the present study. Themes on PTs' definitions of functions were explored by considering the historical development of functions (Cooney & Wilson, 1993). Reviewing the development, it becomes obvious that development has moved in the direction of including various elements of functions, that is, of the concept of set, arbitrary correspondence, and the requirement that each value of the independent variable has a unique image. These elements on functions provided us to keep track of PTs' knowledge of the function concept and its types. In line with this, the correctness of their definitions was also explored. The underlying ideas of analogy contributed significantly to our analysis and interpretations of the responses given to "*What is analogy? Define and determine the main characteristics of it*". The responses were categorized under the following descriptors: (1) transferring the familiar domain to the unfamiliar domain, (2) accessing the similarities and differences of the domains, and (3) mapping similar characteristics between the two domains by indicating the breakdown points.

Teaching Practice: At the beginning of the Practice Teaching in Secondary Education course, function topics covered at the 9th grade were assigned to each participant to be presented using analogies, to provide an effective flow of lesson and to cover all topics relevant to functions. Each participant prepared two lesson plans about assigned topics to be presented at the classroom in the university. One of the lesson plans was on function concept and the other on its types. The first author (instructor of the course) and the third author (assistant of the course) observed and videotaped the PTs during their teaching practice experiences to be consulted in further analysis. The teaching practice data were collected from the observations of all the seven PTs. All PTs, except PT7, were observed on two different occasions in terms of teaching function concept and its types within an average duration of 30 minutes. PT7 was observed on a single occasion involving both teaching function concept and its types within 40 minutes period.

Content analysis (Philips & Hardy, 2002) was conducted to discern meaning in teachers' written and spoken expressions. Videotapes of 13 sessions were fully transcribed and considered line by line whilst annotated field notes were used as supplementary sources. The first phase of data analysis included detecting analogy-based teaching instances and identifying source analogies and the target concepts. A portion of the course was considered to be analogical if it was aligned with the working definition stated above and/or it was stated in the lesson as being analogical. Then the spotted cases was scrutinized concerning nature and extent of analogy use, considering analogical relationship, presentational format, level of enrichment, position, and limitations. The framework suggested by Thiele and Treagust (1994) served as a tool for analyzing the spotted cases. In addition, analogies were analyzed through the presence of any limitation and categorized as applicable or not applicable. The ones that are applicable were analyzed in terms of the presence of any stated warnings which highlights to the students where possible attribute mismatches may occur and categorized as specified or not specified. Furthermore, analogical instances were analyzed whether they are generated by listeners (as student) or presenters (as teacher). That is, while one PT as a teacher was presenting a lesson, the others were required to *participate* in that lesson as students. Thus, during the teaching practices all PTs had the opportunity to act as both students (i.e., listeners) and teachers (i.e., presenters).

Interview: A semi-structured interview was conducted by the third author upon the completion of PTs' teaching practice to yield data about each ones espoused (stated) beliefs about using analogies, lasting approximately 17 minutes for each case. The teaching practice of the PTs provide insights into the extent to which espoused beliefs (what they say) support their enacted beliefs (what they do) evidenced in the classroom.

While analyzing the responses given for "At which stage should analogies be used in the teaching process?", we delineated the position aspect of the theoretical framework that was used in analyzing the observations. The analysis brought forth three descriptors: (a) advance organizer, (b) embedded activator, and (c) post-synthesizer. The focus of the analysis on the responses given for "Who should construct the analogies?" was partially attributable to teachers' conceptions on the pedagogy of the analogy-generated lessons. Thus, our analysis of the interview responses focused on finding useful ways to think about the perspectives held by the PTs. This analysis traced perspective characteristic of the construction of analogies within two descriptors (Harrison & Coll, 2008): (a) student-generated analogies, and (b) teacher-generated analogies. The responses given to "*Is it appropriate to use analogies in teaching the function concept?"* were categorized in two descriptors: (a) appropriate and (b) inappropriate. The PTs' value of using analogies was pivotal in our analyses to categorize the responses because it signaled the insight into their conceptions about analogies and provided an opportunity to inquire into those conceptions in relation to the appropriateness of teaching functions via analogies. While analyzing the responses given to why questions subsequent to each question we did not rely solely on categorizing the aforementioned descriptors rather we sought for the essence of PTs' reasons of their beliefs.

Reliability of Coding

The initial classifications were undertaken by the last two researchers after repeated reading of the knowledge test responses, and teaching and interview transcriptions. Headings were created in relation to the theoretical framework. These analyzed data were then evaluated by the first author who was an expert in teaching functions and analogy. The analyzed data were then subject to discussion by the three members of the research team to further refine the headings. At this meeting, it was decided that all the data will be analyzed by the aid of the matrices comprising the PTs, headings, and descriptors. Subsequently, the independent analyses were carried out by the last two researchers using the matrices for the knowledge test, teaching practice, and interview data in conjunction with the first author's comments. Analyses of the analogies in teaching practice in relation to the two headings - position and level of enrichment - were germane to disagreement. The conflicts were driven from the different conceptualizations of embedded activator and post-synthesizer descriptors in the position heading and the extended descriptor of level of enrichment heading. After consulting with the first author and utilizing the review of literature, embedded activator was restricted to the analog domain be presented after the introduction of the target domain; post-synthesizer was restricted to the analog domain be presented following a complete treatment of the target; and level of enrichment was restricted to the detail of mapping (e.g., expressing the domain, range, process, and the univalence feature) rather than the degree of mapping (e.g., using one analog to express multiple targets and/or multiple analogs to express a single target).

RESULTS

Knowledge of Functions and Analogy

Definition of function and its types. The notion of function as a dependence relation between elements became dominant in the definitions provided for all six PTs. According to their definitions, relations pairing elements of the second set with one or several elements of the first set and each element in the first set has a unique image were considered functions. Analysis of the responses revealed that all of the six PTs approached the types of function from a modern perspective. Their definitions endeavored to reflect function's correspondence perspective referred to as relation between sets. Apparently, the concept of set becomes a fundamental element in the definitions. Further, it was noted in the definitions that the relation or correspondence need not involve numbers but could also involve relationships or correspondences between other elements that vary. Mostly definitions did not illustrate the requirement of a definite "law" correspondence. To summarize, Bourbaki's definition has remained dominant. The types of functions were also defined correctly referring to their specific characteristics.

Description of analogy. The underlying ideas of analogy under the following descriptors: (1) transferring the familiar domain to the unfamiliar domain, (2) accessing the similarities and differences of the domains, and (3) mapping similar characteristics between the two domains by indicating the breakdown points were emphasized by all PTs in different words. They underlined the fact that an analogy cannot hold all the shared attributes, rather the similarities can be built in terms of particular features of a concept.

Teaching Practice

Amongst the 45 analogies, 26 of them were teacher-generated and the remaining 19 were student-generated. Teacher-generated analogies refer to analogies that are constructed by the presenter either during or before the lesson. Student-generated analogies were developed by the listeners/participants either during or after the lesson mostly with the presenters' initiation during lesson.

Analogical Relationship. Results revealed that the vast majority of analogical relationships were functional (41 of 45, 91%) as they include the behavior of the source shared by the target concept. Only four (9%) analogies shared both functional and structural relationship. Structural-functional analogies were generated considering the spelling and meaning similarities of the terms while teaching types of functions.

Presentation Format. Results indicated that 31 of the 45 identified analogies (69%) were verbal, and only 14 (31%) had a pictorial representation together with the verbal representation. The 10 of the 14 pictorial-verbal analogies (71%) and 16 of the 31 verbal analogies (51%) were generated by the presenters of the lesson. This might be due to the fact that presenters tend to support their teaching to enhance the understanding of the function concept via pictorial-verbal analogies.

Position. Most of the analogies (23 of 45, 51%) were generated prior to the investigation of target concept that referred to as advanced organizer. Twelve of the 45 analogies (27%) acted as post-synthesizers while the remaining 10 (22%) were generated as embedded activators. The 20 of the 23 advance organizers (87%) and two of the 12 post-synthesizer (17%) were generated by the presenters of the lesson.

The Level of Enrichment. According to the "level of enrichment" criteria, it was observed that most of the analogies (20 of 45, 45%) generated were enriched analogies following extended analogies (15 of 45, 33%) and simple analogies (10 of 45, 22%). Further, three of the 10 simple analogies that primitively state that the target is like the source, 10 of the 15 (67%) extended analogies that indicate several shared attributes of a single source used to teach a variety of targets, or a variety of sources used to teach a single target, and 13 of the 20 (65%) enriched analogies in which some shared attributes between the source and target concepts for the analogical relations are stated, were generated by the presenters themselves. That is, simple analogies were generated mostly by the PTs who were student participants in the class, and enriched and extended analogies were generated mostly by the PTs who are the teachers of the class.

Limitation. Limitations play a central role in the teaching and learning with analogies by contributing to the conceptualization of the links between analog and target. Four of the 45 analogies were discarded from the analyses of this heading since these analogies were mathematically incorrect (see aforementioned section). Of the 41 analogies 24 (59%) were classified as not applicable due to having no limitation; 17 (41%) were reserved for having a limitation among which nine were not specified and eight were specified.

Epistemological Appropriateness. The epistemological appropriateness of the analogies was classified in terms of whether the domain, range, process, univalence, and other features of the target concept correctly mapped to the analog or not. Results documented that 37 of the 45 analogies (82%) were correct and four of the 45 (9%) analogies were incorrect. The rest four (9%) structural-functional analogies cannot be analyzed in terms of epistemological appropriateness as they corresponded to the target concept in terms of their spelling and verbal meaning. As expected, the most of the incorrect analogies (3 of 4, 75 %) were formed by the PTs as student participant with the presenters' initiation during lesson.

Beliefs

When to use analogy? PTs appeared to have an insight into the use of analogy from the introducing of a new concept to the developing an understanding for this new concept. All six PTs indicated that analogies could be used as an *embedded activator* since analogies can be presented as an example at a point after the definition of a concept or at a time when the mathematical content is becoming more abstract or difficult to the students.

Who should construct the analogies? PTs' preference for the teacher-generated analogies was evident during the interviews. The reasons can be illustrated with two underpinnings. In generating the analogy the PTs put emphasis on the importance of the subject matter knowledge and pedagogical knowledge. As teachers have knowledge both on subject and analogies, they can direct students more efficiently, emphasizing similarities and differences between the analog and the target. They deliberately viewed analogy generation as a difficult process since the analogy generation grounds explicitly on the connectedness between the source and target concepts.

The Use of Analogies in Teaching Functions. The use of analogies in teaching functions was greatly valued and instigated in the comments of all PTs except PT3. They clearly found them appropriate as functions are difficult for students and engenders anxiety among them as it was introduced for the first time at the 9th grade. They articulated that analogies help conceptualization because correspondence between analogy or familiar concept and function concept requires students to extend their understandings in a meaningful way.

DISCUSSION

This study's foregoing findings lend credence to the consistent associations between PTs' subject matter knowledge, pedagogical knowledge, beliefs, and their practices advanced by Fennema and Franke (1992). In terms of practical implications, the findings of this study clearly support the need for developing PT knowledge and beliefs as they have an association on PTs' teaching practice.

Knowledge test and teaching practice data revealed that all PTs tended to use similar definitions on functions. This implies that PTs were consistent in their knowledge and teaching practice. PTs' understandings clearly articulated that they were able to reconcile the analogical approach to functions with the prominent features of their own knowledge on functions. However, they were limited in their attempts to generate analogies in scientific contexts. This supplied evidence that generating an effective analogy requires not only knowledge of mathematics but also the knowledge of interdisciplinary subjects that can furthermore support the development of new mathematics concepts in real-life contexts. This thereby illustrates that PTs' knowledge of functions and other subjects acted as a filter for the interpretation and a springboard for epistemologically appropriate analogies.

PTs tend to think of analogy as a tool with the descriptors of transferring the familiar domain to the unfamiliar domain, accessing the similarities and differences of the domains, and mapping similar characteristics between the two domains by indicating the breakdown points. This indicates, for example, that the pre-service teachers knew the importance of stating the limitations of an analogy by putting particular emphasis on the breakdown points. This also was observed in the teaching practice where mostly all exhibited a fulfilled intention of discussing where their analogies break down. Only PT7 and PT4 exhibited an unfulfilled intention of discussing where their analogies break down. This might be attributable to two reasons: First, PT7 and PT4 might assume that their peers themselves were capable of building the analogical relationships between analog and target concepts and therefore they might not have teaching concerns. And secondly, although they, as a teacher, were indeed aware of indicating where the analogy breakdowns, they might fail to transform this knowledge and integrate it into the analogical context.

These foregoing findings lend credence to the consistent associations between PTs' espoused beliefs (what they say) and enacted beliefs (what they do). The more they valued and instigated the use of analogies in teaching functions in their comments, the higher number of analogies they generated in their teaching practices. Some of the PTs (i.e., PT1) underlined that there may be times where students should construct their own analogies and in their teaching practice they encouraged the classroom to think about the analogies and to propose new analogical relations to the classroom. Furthermore, PT3 stated that analogies definitely should not be used as a starting point for the introduction of a new concept and therefore as a teacher used analogies only as post-synthesizer.

"Analogies should not be used at the very beginning of the lesson. I mean, while introducing a new concept. Rather, they should be used somewhere during instruction while giving explanations. I mean, you should generate an analogy in order to score a goal, not to get into a position." PT3

Although association was evident between teachers' knowledge, beliefs, and teaching practice, what teachers teach is mediated by the external factors such as textbooks since teachers' thinking about functions is based on how functions are presented in school textbooks (Cooney & Wilson, 1993). Thus, future research can explore the textbooks and to what extent teachers can enhance the treatment presented by a textbook in order to accommodate particular objectives.

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UNDERSTANDING PROFESSIONAL DEVELOPMENT FROM THE PERSPECTIVE OF SOCIAL LEARNING THEORY

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This paper considers professional development (PD) from the perspective of Social Learning Theory (SLT). While a single component of SLT, self-efficacy, is frequently referred to in teacher education research, the full theory has not been used in this field previously. In this paper SLT is evaluated in terms of the extent to which it is able to describe and explain the phenomena of professional learning. An example of reform-oriented PD is used for this purpose. SLT provides an integrated view of learning, accounting for individual, cognitive aspects of learning, such as knowledge and beliefs as well as social and participatory aspects of learning. This study demonstrates that SLT provides a potentially useful theoretical approach in the context of teachers' professional learning.

Keywords: Professional development, professional learning, self-efficacy, Social Learning Theory

INTRODUCTION

It is argued that theories of professional development need to include both cognitive and social aspects of learning (Borko, 2004). However, theory has tended to focus on either cognitive or social perspectives. Cognitive perspectives have centred on notions of changes in teachers' beliefs or knowledge. Social perspectives have considered professional learning through participation (Lave & Wenger, 1991). Social Learning Theory (SLT) (Bandura, 1977) provides a theoretical approach that integrates cognitive aspects and social effects in learning. While an important sub-construct of SLT, self-efficacy, has been used extensively in theorising teacher education, there are no examples, to the author's knowledge, of research that draws on the full theory.

In this paper the potential use of SLT is explored by interpreting the effects of a professional development (PD) programme that was designed to support teachers in using student-centred problem-solving (SCPS) approaches in their teaching. SCPS is an approach to teaching that is characterised by student collaboration and discussion in the process of solving open-ended tasks and activities. This contrasts with traditional teacher-centred teaching, which is characterised by the teaching of mathematical methods and routines, this is the prevailing approach in England's secondary schools (Ofsted, 2008, 2012). The importance of SCPS is in fostering deeper understanding of mathematics as well as in improving motivation and engagement (Rocard, 2007).

Schoenfeld's (2002) criteria for assessing a model or theory were used to guide the evaluation of the viability of SLT in the context of PD. The criteria used are based on the qualitative assessment of the *descriptive* and *explanatory power* of the theory. In other words, to what extent can SLT effectively account for the observed phenomena? This is part of ongoing work in the development of SLT in the context of PD. Further assessments are being undertaken; this paper reports on the extent to which SLT can describe and explain PD. Schoenfeld (2002) explains *descriptive power* as the extent to which the theory captures '"what counts" in ways that seem faithful to the phenomena being described' (p. 488). *Explanatory power* refers to the extent to which theory provides a description of the way in which things work (p. 489). While Schoenfeld (2002) proffers further criteria, these two were chosen as the most important at this stage and are also consistent with Sriraman and English's (2010) functional criteria for theory.

SOCIAL LEARNING THEORY

From the perspective of SLT, the mechanism of learning and the formation of individual knowledge is through observation (Bandura, 1977). SLT posits a sophisticated mental modelling of observed behaviours and subsequent construction of novel behaviours, but this does not necessarily mean direct imitation of others' behaviours. Observational learning has been found to be an important mechanism in teacher development (Lortie, 2002), as has the importance of a mental model or a picture of the lesson as the teacher enters a classroom (Rowlands, Thwaites, & Jared, 2011). The way in which teachers choose and construct behaviours is influenced by the extent to which they believe they will be successful with a course of action in a particular context. This self-regulatory process within SLT is referred to as self-efficacy. It is the belief an individual has in the level of success they will experience when they act in certain ways in specific contexts. Self-efficacy reflects cognitive capacities and underlying skills, it also incorporates affective components such as confidence, motivation and willingness to innovate (Bandura, 1997). Previous research has found teachers' self-efficacy to be related to positive teaching behaviours and student achievement. Teachers with lower levels of efficacy are more pessimistic about student motivation and believe in strict classroom regulation and rely on extrinsic inducements and negative sanctions to get students to study (Woolfolk, Rosoff, & Hoy, 1990). Self-efficacy is an important dimension but it is the broader theoretical framework that is being considered in this study.

The core component of SLT, as referred to above, is observational learning and the mental modelling of observed behaviours. Teachers (re)construct behaviours to implement in classroom. Much behaviour, according to Bandura (1977, 1997), becomes routine and does not require prior modelling and planning. Teachers at the beginning of their careers observe and model the practice of other teachers, adapt them and (re)produce them in the classroom (Lortie, 2002), this is consistent with SLT. Feedback and response as well as self-assessment by the individual teacher influence the formation of their teaching behaviours (Lave & Wenger, 1991). In time, practices become largely routine (Bandura, 1997; Cuban, 2009; Wake, 2011). As teachers, we observe the largely traditional teaching of more experienced colleagues, we reconstruct this, knowing that it represents a safe and stable practice. Thus we enter into a well-established didactic contract (Brousseau, 1997) based on traditional and conservative teaching approaches. Wake (2011) argues that the introduction of SCPS approaches requires a change or renegotiation of this contract. According to SLT then, the facilitation of this renegotiation is reliant upon teachers having the appropriate pedagogical knowledge – in the form of mental models of possible and alternate practices, pedagogies and behaviours (Bandura, 1997) – combined with a level of self-efficacy in order to be able to implement such approaches (Guskey, 1988).

It is important to be reminded of the social and contextual effects that present a challenge to innovation. Within SLT these act in a regulatory way, mediated through self-efficacy. If our behaviours are challenged then we may begin to doubt their ultimate success, we may modify our behaviour. Similarly, as teachers, if we introduce an innovative pedagogy and it is challenged or it is responded to unfavourably by students, parents and colleagues, then it is likely that we will be less confident with the approach. Ultimately we may change our behaviour to an approach that we believe will be more acceptable. The kinds of teachers that persist with innovation have been shown to demonstrate high levels of self-efficacy in the context of teaching (Berman & McLaughlin, 1978). Beyond the effects of others in influencing behaviour, there are also the effects of the working conditions, the demands of the job, the nature of teaching and the institution in which is takes place. It has been recognised that these environmental effects also have a strong influence on the way teachers teach (Cuban, 2009; Leinhardt, 1988).

At a theoretical level, SLT appears to offer a useful framework for describing and explaining professional learning – from the above description it appears to have good *descriptive power*. In order to test this further and test SLT's *explanatory power,* a study involving a PD programme was undertaken. Three components were derived from SLT and through an iterative analysis of the data. These were, 1) *Teacher knowledge,* 2*) Teacher self-efficacy beliefs* and 3) *Social, contextual and environmental effects. Teacher knowledge* corresponds with aspects of data where the teacher refers to their knowledge and the effects on their knowledge in the context of the PD and their teaching. In terms of SLT, knowledge is conceived of as mental models of potential behaviours. If SLT provides a reasonable descriptive and explanatory capability then there may also be suggestion of observational learning and the modelling of behaviours. For *teacher self-efficacy beliefs,* examples might feature direct references to confidence, motivation or expressions of willingness to include innovative processes in their teaching. Alternatively, there might be negative effects on self-efficacy. *Social, contextual and environmental effects* refer to aspects of the PD and teachers' experience that are related to student, parent or colleague expectations that may have a bearing on what the teacher does in the classroom. In addition, the aspects that relate to the effects of the nature of the job of teaching are also accounted for in this component. The three components provide the means through which data taken from the PD example can be related to SLT.

THE PROFESSIONAL DEVELOPMENT

The PD programme was designed to support teachers in the teaching of SCPS. In England's secondary schools teaching is largely traditional – teacher-led and focussed on the teaching of methods (Ofsted, 2008, 2012). However, the English National Curriculum stresses the importance of problem-solving. This PD was designed to support the teaching of problem-solving. It was also designed to enable groups of teachers to work together with one teacher leading the PD. There are seven modules, each focusses on one aspect of pedagogy. Module themes include, promoting student collaboration and group work; developing questioning to promote student reasoning; and formative assessment – all in the context of using 'rich' and open-ended problem-solving tasks. All modules have the same structure: an *introductory session*, *into-the-classroom* and a *follow-up* session. In the *introductory session*, teachers consider the ideas in the module – for example, student collaboration – they attempt tasks and lesson activities provided with the PD materials, they watch edited videos of real lessons in which the focus of the module is exemplified. Teachers are encouraged to reflect on and discuss the ideas presented. They then collaboratively plan a lesson based on the tasks in the PD materials and teach the lesson in the *into-the-classroom* phase. In the *follow-up session*, teachers review and reflect upon their experience and look at further videos and materials. Each of the PD sessions lasts about one-hour. The professional learning theory underpinning the design is based on teachers' beliefs. The PD is intended to provide a supportive setting in which teachers can try out a different approach and have the opportunity to develop different beliefs – more oriented toward SCPS than traditional teacher-centred approaches (For more details about the design philosophy behind the PD materials see Swan, 2006). Observations made during a pilot study involving the evaluation of the PD indicated that this theoretical approach did not provide sufficient descriptive or explanatory power and a more sophisticated model was needed. This prompted the investigation of SLT as an alternative theoretical framework.

METHODOLOGY

The study took place in mathematics departments in four schools in England. The University of Nottingham has partnerships with schools for the purposes of pre-service training of teachers, 86 of the secondary schools were contacted and invited to take part in the study. Twelve schools expressed an interest in the project. After initial meetings with the schools, four schools agreed to take part. The PD programme took place over two terms. Schools completed a module (*introductory* and *follow-up sessions*) each term, all teachers in the department were encouraged to take part: most teachers (over 90%) attended the PD sessions.

This multi-site case study was guided by the need to address questions such as ‘how’ and ‘why’ in a naturalistic setting (Yin, 2009) and in the context of professional learning. The aim of the research was to develop an in-depth understanding of the effects of the PD on teachers’ beliefs and practices. Data were collected through questionnaires, observations of PD sessions, observations of lessons and interviews with individual teachers and heads of departments. Eleven teachers were involved in the video-study, three teachers in three of the schools and two from the smaller of the four schools. At the start of the project the teachers were asked to teach a problem-solving lesson, they were interviewed before and after the lesson, it was videoed and student work also collected. As soon as was convenient the mathematics departments ran the introductory session of the first PD module, this was video recorded and the head of department and PD leader were interviewed. Teachers then taught the lesson they had planned in the PD, this lesson was videoed and a post lesson interview was conducted. The same procedure was repeated in the follow-up session. A final problem-solving lesson was taught by the video-study teachers and they were interviewed again. They were asked about the aspects of the PD they had used in the observed lesson and the extent to which the PD influenced their views and the way they taught. The same process was repeated with a different PD module in the second term. Eleven teachers were observed and interviewed five times through the project. Data were collected using questionnaires from over 90% of the mathematics teachers (n=37) in all four schools. The research design involved the mathematics department as a whole but observing a smaller number of teachers in more detail in the video-study.

The overall strategy was built around drawing conclusions form the large amount of data, which included interview data, lesson video, video of PD sessions and questionnaires. The primary source of data for analysis was the interview data. Unstructured interviews were used in the initial rounds of post-lesson interviews, these were transcribed and analysed and semi-structured interviews were developed at each successive round of interviews. In this way a constant comparative approach was used in the data collection. A method was developed for systematically summarising lesson content and PD session content (Watson & Evans, 2012). Analysis was conducted with reference to what was observed in lessons and in PD sessions. Analyses of teacher interviews were cross-referenced with observations of lessons and the analysis of interviews with the head of department. This provided a network of triangulation and enhanced the trustworthiness of results. The three components of SLT, 1) *Teacher knowledge,* 2*) Teacher self-efficacy beliefs* and 3) *Social, contextual and environmental effects* were used to code the data in order to analyse the PD from the perspective of SLT. The aim was to assess the descriptive and explanatory power of SLT in this particular context.

RESULTS

In this section, a theoretical analysis of the PD is presented from the perspective of SLT. The results are presented in terms of the three themes described above.

Teacher knowledge

The most important aspect of teacher knowledge was the development of pedagogic knowledge. This was often very specific aspects of pedagogy and was often treated as models that could be adapted and developed for use in the teacher's own classroom. As an example of this, two schools used the PD module which focussed on student collaboration, teachers who took part in the PD were interested in approaches to groupwork. They were concerned with how to promote the full participation of students working collaboratively on an open-ended and unfamiliar problem. Of particular interest were techniques to assign different roles to the group members or setting up 'ground rules' for discussion.

> [Students] do not always know how to work in groups. To have the rules was good because they all knew they all had to take part. It was not one person who could take over and one person sit back and not say anything. They all knew that they had to listen to each other. They were all clear on expectations so they knew what was acceptable and what was not. (Sally, Hilltown School, year 8, high-ability).

The introduction of pedagogic moves and techniques, as in the above example, was often related to structuring a more student-centred lesson. The knowledge that teachers most often applied in observed lessons were associated with techniques and pedagogic passages they had observed in the PD sessions or that was described in the accompanying printed materials.

There were examples of observational learning. Video examples of lessons were shown in the PD sessions. It was quite common to see teachers use elements of what they had seen. For example, in the PD module on questioning and reasoning, there is a video of a lesson where the teacher, having given students a little time to work on an open-ended problem in pairs, asks students to explain their approaches while she summarises each idea in a sentence or two on the whiteboard. This was recreated in an observed lesson. The way in which teachers' acquired models of teaching was not limited to the videos. Detailed lesson plans also provided them with models and structures, which they would follow to help them in developing their own approaches to SCPS.

Teacher self-efficacy beliefs

Teachers were asked about their experiences of the PD and how their perspectives may have changed as a result. A common feature of their responses was concerned with their confidence in connection with incorporating SCPS into their teaching. In many cases, the PD had supported increased motivation and confidence to include SCPS in their teaching. In some examples, especially where the teacher was working with lower ability groups, it was evident that the teacher had become less confident.

The self-efficacy of each teacher was explored, through the study, with individual teachers, however it was also valuable to consider the heads of departments' perspectives of teachers and compare these with individual explanations.

They were asked to consider which teachers would find it more or less difficult to use the ideas of the PD and what sorts of characteristics were important. The issues they raised were to do with confidence and motivation.

> [It] is the comfort-zone that is quite hard for them to break out of. To have things like children working in groups, that is quite difficult. Children may be diverging in the ways they are doing things and they can't plan for every way they might do it. The teacher can't have a written answer to give them confidence in what is going to happen. They are going to have to think on their feet and follow the way the children are going. (Anne, Head of Mathematics, Norman Fletcher School).

> I would have said Barry, me, Kate, Lizzie do these activities with a certain degree of confidence, because I think we like to try things like that. And we don't mind if it goes wrong. (Deborah, head of mathematics, Barrington School)

The efficacious teachers were more willing to take risks in their classrooms. This was also supported by observations of lessons, the efficacious teachers seemed more at ease with a student-centred approach. "Loss of control" or "out of my comfort-zone" were phrases used by teachers in describing how they felt about the approach. The video-study teachers, particularly with higher ability group, felt that the PD programme had helped them become more confident in giving students greater control over their learning. This was supported by observations of their teaching over the course of the PD – teachers became more comfortable with the approach. This was also consistent with the observations of heads of departments.

There was no reference to changes in beliefs about mathematics or the teaching and learning of mathematics. It appeared that many teachers did not see beliefs about teaching as a barrier to taking a more open-ended approach. In the following the head of department explained that she thought that many of the teachers had appropriate beliefs to teach in the way suggested by the PD. The barriers appeared related to self-efficacy.

> They come with a lot of the points [to do with questioning to promote reasoning in the context of problem-solving] that are there anyway. It's just a case of consciously thinking about it. A lot of the things we know we don't necessarily put into practice (Anne, head of mathematics, Norman Fletcher School).

There were two aspects to self-efficacy, the first is that the more efficacious were more willing to experiment with their teaching. Second, the data suggests that the PD had an effect on teachers' self-efficacy. Often this was positive but if teachers had a bad experience with SCPS as part of the PD it could undermine confidence and diminish self-efficacy.

Social, contextual and environmental effects

There were references to the social, environmental and contextual factors that limited teachers' capacity to incorporate student-centred problem-solving into their teaching. Teachers often referred to the having to 'cover' or get through the scheme of work or curriculum in order to prepare students for examination. In addition, they suggested that because students were not used to working in a SCPS way – particularly in the lower ability classes – this presented barriers to teaching in that way. Some teachers and most of the heads of departments also expressed concerns about parental expectations in what mathematics teaching should be like. The approaches suggested in the PD were often perceived to be contrary to the expectations of students, parents and sometimes colleagues in the wider school. In one school, teachers were particularly concerned how the inspectorate would judge their teaching if they were observed teaching in the ways suggested in the PD programme. However, teachers, particularly those in the video-study, acknowledged that the PD and lessons they had tried out had given them space to experiment with and explore different ways of teaching. It had given them 'permission' to do something different.

All heads of departments explained how difficult it had been to fit in meeting time to hold the PD sessions. One school had to hold the sessions during lunch breaks another had to do the sessions in scheduled department meeting time but found issues that had been put to one side for the PD still needed to be communicated and discussed. At the same time they all valued the opportunity that the PD created: to meet, discuss and explore teaching and learning as a department. They felt that being part of the project had made them, as a department, think and try out lessons and different approaches. However, in one of the schools it was clear that only the teachers involved in the video-study were actually trying out the planned lesson with their own classes.

DISCUSSION

This preliminary analysis demonstrates that SLT has both *descriptive* and *explanatory power* in the context of this PD. This could be further developed through more analysis. Schoenfeld (2002) describes *descriptive power* in terms of the theory's capacity to capture "what counts" *or* the extent to which the theory takes the right factors into account? (p. 488). It appears that teacher knowledge, self-efficacy beliefs and the social and contextual effects, represent the 'right factors'. Schoenfeld (2002) contrasts description and explanation: 'It is one thing to say that people will or will not be able to do certain kinds of tasks, or even to describe what they do on a blow-by-blow basis; it is quite another thing to explain why' (p. 489). The analysis of SLT presented through this papers suggests SLT offers a viable explanation of professional learning from both the cognitive perspectives of teacher knowledge and beliefs as well as the social and situated aspects of professional learning. SLT also addresses learning processes in terms of observational learning which appears consistent with other studies (See, for example, Lortie, 2002).

CONCLUSION

SLT appears to offer a useful theoretical framework that integrates cognitive and social aspects of professional learning. SLT accounts for teachers' pedagogic knowledge in terms of mental models of behaviour that have been acquired through observational processes. The way in which pedagogic knowledge is applied in the classroom is influenced by the teachers' level of self-efficacy as well as the social and contextual setting. SLT could provide an improved theoretical framework that has the potential to enhance the design and evaluation of PD. However, this is a preliminary exercise in testing the potential of SLT. SLT has been considered against two criteria: *descriptive* and *explanatory power*. Schoenfeld (2002) suggests further criteria, further research is required to assess SLT against these additional criteria. Although, at this stage it is reasonable to conclude that SLT has potential in theorising teachers' professional learning.

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WHAT CAN WE LEARN FROM OTHER DISCIPLINES ABOUT THE SUSTAINABLE IMPACT OF PROFESSIONAL DEVELOPMENT PROGRAMMES?

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This paper deals with the sustainable impact of professional development programmes. While research on this issue is rather scarce in mathematics teacher education, some other domains like health care or development aid are well grounded in research results regarding this topic. This paper gives an insight into the other disciplines' knowledge concerning the impact of innovations and professional development programmes and the respective fostering factors. Moreover, possible implications for mathematics teacher education are discussed.

INTRODUCTION

The question of how to promote mathematics teachers' professional development is of great interest and discussed in various papers (e.g., Krainer & Zehetmeier, 2008; Loucks-Horsley, Stiles, & Hewson, 1996; Maldonado, 2002; Sowder, 2007; Zehetmeier, 2010; Zehetmeier & Krainer, 2011). In this context, the question of sustainability is of particular relevance. Despite its central importance for both, teachers and teacher educators, research on sustainable impact is generally lacking within teacher education disciplines (Datnow, 2006; Rogers, 2003).

However, a sound knowledge base concerning the issue of sustainability would be useful for understanding the long-term impact of teacher professional development programmes. At the same time, this knowledge would allow thorough discussions regarding implications for upcoming professional development programmes' planning, implementation, and evaluation. The aim of this paper is to provide other disciplines' knowledge concerning this issue. For this, an extensive literature was carried out; using qualitative content analysis (Mayring, 2003), relevant topics were identified and clustered. The following sections provide other disciplines' knowledge regarding factors that foster the sustainability of professional development programmes' impact.

This raises the question: Who are the others? Health care disciplines come with a relative long tradition of researching the topic of professional development's sustainable impact. This led to a widespread body of research findings concerning this issue. Besides the health care disciplines also research on development aid or public service evaluation has available interesting findings; they also can be used as focal points for discussing and reflecting sustainability in mathematics teacher education. Thus, this paper's literature review is based particularly on research findings from health care disciplines (e.g., Scheirer, 2005). Moreover, results from disciplines like development aid (e.g., van den Berg, 2006) or public service evaluation (e.g., Savaya, Elsworth, & Rogers, 2009) are provided.

THE OTHERS‘ FOSTERING FACTORS

Literature regarding conceptual or empirical knowledge of factors that may foster the sustainability of innovations is rather sparse (Johnson et al., 2004). However, “the question of what factors contribute to or detract from program sustainability is important because … it cannot be assumed that proven success in achieving its goals ensures a program’s continuation beyond its initial funding” (Savaya et al., 2009, p. 2). The question which factors help increase the likelihood of sustainability is particularly addressed in literature regarding the institutionalization of programmes within organizations: “This issue is of central importance when one is planning for program sustainability, when it is helpful to know what processes and other influences need to be considered to extend the delivery of program activities” (Scheirer, 2005, p. 324).

This paper uses a qualitative analysis of literature (Mayring, 2003): Eight central factors, which foster the sustainability of programmes, were categorized. The following factors are central, because they were found to be influential more often than other ones: perceived benefit, innovation champions, mutual fitting, institutional support, sufficient resources, networking, ownership, and integration of rules. The following paragraphs provide an overview concerning these central factors.

Perceived benefit

One of the central factors fostering the sustainability of programmes is “the perceived benefit from the programme” (Amazigo et al., 2007, p. 2080) for the people involved. This implies in particular that “attention to the needs, attitudes, and perceptions of adopters is critical to their sustained use of an innovation” (Johnson et al., 2004, p. 143). And further: “Users must perceive a benefit to the innovation beyond that of current practices. … Adopters are also more likely to sustain an innovation if they believe it is effective” (Johnson et al., 2004, p. 145). Baum et al. (2006) state that some “initiatives were often only felt to have happened because of the previous collaborations. ... In effect these had laid the seed bed on which future projects grew” (p. 262).

In particular, the “evidence that the model works … and the ability to document positive client outcomes” (Blasinsky et al., 2006, p. 721) represents a strong fostering factor. On the other hand, Scheirer (2005) highlights that these “benefits to staff members and/or clients … are readily perceived, but not necessarily documented via formal evaluation” (p. 339). This issue points to benefits and outcomes which may not have been intended or expected in a programme’s conception; and which may – thus – not be considered in project evaluations or research efforts.

Pluye, Potvin, Denis, Pelletier, and Mannoni (2005) found incentives to be a factor fostering the sustainability of innovations: "The promotion of personnel (into positions of greater responsibility and power) encouraged the routinization of innovations. … Adding concrete benefits to human resources also constitutes an incentive (for example, in the form of convenience or reduced effort)" (p. 125).

Innovation champions

Another central factor that supports the sustainability of programmes is "the presence of champions for an innovation" (Johnson et al., 2004, p. 138). Similarly, Scheirer (2005) highlights "the key role of a program champion" (p. 339). Also Savaya et al. (2009) state that "program champions who promote the program in the organization and the community can contribute to program sustainability" (p. 2).

These champions are "formal and informal leaders within adopting systems … who proactively promote an innovation from inside or outside of a system" (Johnson et al., 2004, p. 143). They "are critical to creating an environment that supports and facilitates sustaining innovations. … Such champions can serve as brokers on behalf of the innovation with other decisionmakers" (Johnson et al., 2004). Johnson et al. (2004) describe in detail: "Essential skills for innovation champions include communicating their commitment to the innovation, … engaging others, overcoming barriers, building infrastructure, thinking and learning reflectively, summarizing and communicating, coaching for sustainability, and building further organizational capacity to spread the innovation" (p. 144).

Blasinsky et al. (2006) point to the importance of staff members who are "already trained [in the programme]" and are "available not only to continue [the programme] but also to train others in the intervention" (p. 726).

Mutual fitting

Yet another central factor fostering sustainability is the fitting of innovations and adopting institutions. For example, "when program objectives fit with the values of the organization and staff" (Pluye et al., 2005, p. 125). Or "when cultural artifacts from program activities are shared with organizational artifacts" (Pluye et al., 2005, p. 125); here, artifacts are defined as myths, symbols, metaphors and rituals that express a set of organizational values, beliefs and feelings. Another kind of fitting is represented by "the adaptation of activities according to their context or environment" (Pluye et al., 2005); in this case, adaptation means the adjustment of activities regarding local contexts and environmental variations. In sum, this refers to introducing innovations into organisations without "disruption of the operating work flow" (p. 126).

Johnson et al. (2004) state that sustainability is fostered when innovative programmes are "compatible with the philosophical orientation … and internal agenda of users" (p. 145). Similarly, Scheirer (2005) claims for "a substantial fit with the underlying organization's mission and procedures" (p. 339). This challenges both the organisations' stability and flexibility: "The stability of an organization and its ability to change significantly contribute to the sustainability of new programs" (Savaya et al., 2009, p. 2).

Institutional support

Institutional support is another central factor that supports the sustainability of programmes. This can be mirrored by the "willingness of the organization to promote change" (Blasinsky et al., 2006, p. 726). Or when organisations take the risk of supporting innovative programme activities: Because then organisations "build confidence among actors involved in activities and encourage the routinization of programs" (Pluye et al., 2005, p. 124).

For this, the administration of organisations "must have the structures and capacity necessary to carry out administrative functions related to an innovation responsively, effectively, and efficiently" (Johnson et al., 2004, p. 144). In this regard, it is important to know that "systems that focus on strengthening administrative capacity to support an innovation during its initial implementation are more successful at sustaining the innovation once the initial trial ends" (Johnson et al., 2004, p. 144).

Sufficient resources

Yet another central factor fostering sustainability is the availability of resources. Johnson et al. (2004) state that "sustainability research clearly identifies resources as important to sustaining innovations" (p. 143). These resources include human, physical, technological, financial and informational resources (Pluye et al., 2005; Johnson et al., 2004). Sufficient resources can support the sustainability of programmes in the case of "equipment turnover (renewal of material resources when needed)" or of "turnover in key personnel (change of original personnel after an appropriate period of time)" (Pluye et al., 2005, p. 124). To ensure the availability of sufficient resources, programmes can "have multiple sources of funding", and/or "the project leaders can plan to raise resources for the future, when fund raising starts early on" (Savaya et al., 2009, p. 2).

Networking

Savaya et al. (2009) highlight the importance of networking: "Self-contained programs are less likely to be sustained than are programs that are well integrated with existing systems" (p. 2). In this regard, Pluye et al. (2005) state "that transparent communication between the actors is necessary to achieve congruence among objectives, to share cultural artifacts, and to take corrective actions, thus promoting routinization" (p. 125). For networking, some "positive relationships among key implementers" (Johnson et al., 2004, p. 138) are useful: "Collaboration between program developers and teachers who are implementing the program appeared to increase their commitment and desire to implement the new procedures. A supportive peer network among implementers of an innovation is also important for sustaining innovations" (Johnson et al., 2004, p. 138).

Ownership

Savaya et al. (2009) point to the factor ownership as being central for sustainability: They found “greater sustainability of programs that were developed and implemented with the involvement and support of community bodies” (p. 2). Also Johnson et al. (2004) indicate the importance of “ownership by … system stakeholders” (p. 138) as factor fostering the sustainability of innovative programmes. Similarly, Amazigo et al. (2007) point to the fostering influence of “community leaders [who] show appreciation” (p. 2080) for the programmes.

Integration of rules

Research findings of Johnson et al. (2004) suggest that the integration of rules is another fostering factor: “Policies and procedures … assure that the innovation remains part of the routine practice of the organization, even after the top management who advocated sustaining the innovation leaves the organization” (p. 143). For Yin (1981), sustainability is fostered when “program functions become part of job descriptions and prerequisites” or when “the use of innovation becomes part of statute, regulation, manual, etc.” (p. 63).

DISCUSSION AND IMPLICATIONS

This section links the others’ fostering factors to mathematics teacher education. Communalities can indicate possible affirmations and validations of our discipline’s knowledge. Differences may point to aspects worth being challenged and reconsidered.

Communalities

The other disciplines identified several factors that foster the sustainability of programmes (see the others’ factors above). In a meta-analysis concerning factors in the teacher education disciplines, Zehetmeier (2008) found yet similar, but not the same factors. For example, mutual fitting, ownership, and networking turned out to be central fostering factors in both the others’ and teacher education literature. Therefore, it seems reasonable to facilitate factors identified by both domains.

Zehetmeier and Krainer (2011) highlight in particular the outstanding relevance of contextual factors. Similarly, a study of Nickerson and Moriarty (2005) points to organizational conditions (e.g., teachers’ relationships with the school administration) being highly relevant for the further development of schools. Since contextual factors contribute particularly to sustainable impact, organisational development should be part of any professional development programme. This means, that not only mathematics teachers should be seen as a programme’s target group, but also the teachers’ contexts (e.g., colleagues, pupils, principals, parents, policies, etc…). Therefore, professional development and school development should be considered as concomitant processes. This relevance of contextual factors is also highlighted in the other disciplines (see section “institutional support”, above).

Rogers (2003) highlights that the diffusion of an innovation depends on different characteristics: Relative advantage, compatibility, complexity, trialability, and observability. Fullan (2001) describes similar characteristics (need, clarity, complexity, quality, and practicality) influencing the acceptance and impact of innovations. Relative Advantage includes the perceived advantage of the innovation (which is not necessarily the same as the objective one). Compatibility and need denote the degree to which the innovation is perceived by the adopters as consistent with their needs, values and experiences. Complexity and clarity include teachers’ perception of how difficult the innovation is to be understood or used. Trialability denotes the opportunity of participating teachers to experiment and test the innovation (at least on a limited basis). Quality and practicality make an impact on the change process. Observability points to the claim that innovations should be visible to other stakeholders. Therefore, when aiming at sustainable impact, the following implications should be considered: An innovation with greater relative advantage will be adopted more rapidly. This issue is also addressed by the other disciplines’ factor “perceived benefit” (see above). More complex innovations are adopted rather slowly, compared to less complicated ones. Innovations that can be tested in small steps represent less uncertainty and will be adopted as a whole more rapidly. High quality innovations that are easily applicable in practice are more rapidly accepted. Innovations which are visible to other persons and organisations are more likely to be rapidly accepted and adopted. These implications are closely linked to the others’ fostering factor “integrations of rules” (see above).

Shediac-Rizkallah and Bone (1998) categorized three groups of factors that foster or hamper programmes’ sustainability: (a) factors pertaining to the project; (b) factors within the organizational setting, and (c) factors in the broader community environment. Zehetmeier and Krainer (2011) try to reduce the multiple factors’ complexity by clustering them into three dimensions (the three Cs; see Krainer, 2006): Content (high level and balance of subject-related action and reflection), Community (high level and balance of individual and social activities, in particular fostering community-building within and outside the professional development programme), and Context (high level and balance of internal and external support). Thus, both domains acknowledge the rather complex system of factors and try to establish useful and suitable models. Therefore, if professional development programmes are aimed to be sustainable, it seems crucial to carefully consider and facilitate these fostering factors. If some of these factors are dependent from the programmes’ existence, then these factors may be substituted with alternative ones that are less or not at all connected to the programmes’ existence.

Differences

Leadership as fostering factor is not really a topic in the others' disciplines. Indeed, Johnson et al. (2004) point to "effective leadership" (p. 138) being a fostering factor. However, it remains unclear, what this notion may mean. By contrast, within the teacher education disciplines the issue of leadership is of great importance. The results of several studies suggest the central influence of school leadership to the (sustainable) impact of school innovation initiatives (e.g., Fullan, 2006; Owston, 2007): Fullan (2006) proposes a direct correlation between the sustainability of innovations a the new role of school leadership: "This new leadership, if enduring, large scale change is desired, needs to go beyond the successes of increasing student achievement and move toward leading organizations to sustainability"(p. 113). These new leaders focus on systemic relationships to foster sustainability not only on the individual level, but also on the levels of organisations or educational systems. "Such leaders widen their sphere of engagement by interacting with other schools in a process we call lateral capacity building. When several leaders act this way they actually change the context in which they work" (Fullan, 2006, p. 113). Fullan (2006) calls this new type of leadership "system thinkers in action" ("they have the capacity to be simultaneously on the dance floor and the balcony", p. 114). Similarly, Owston (2007) states: "Support from the school principal is another essential factor that contributes to sustainability" (p. 70). He distinguishes three types of administrative support: Neutral leaders (who meet innovations rather passive without promoting or prohibiting); Supportive principals (who create and support beneficial environments for innovations); And actively involved leaders (who are driving visionary ideas, identify personally with innovations and motivate other teachers for the innovation). Therefore, for programmes aiming at sustainable impact, it seems indicated to foster and support this kind of leadership.

The presence of an innovation champion as fostering factor is rather no big topic in mathematics teacher education disciplines. However, Cobb and Smith (2008) highlight the important role of "brokers" (as do Johnson et al., 2004, see above) for the development of a shared instructional vision of high quality mathematics instruction. They describe brokers as people "who can facilitate the development of a shared instructional vision by bridging between perspectives and agendas of different role groups. Brokers are people who participate at least peripherally in the activities of two or more groups, and thus have access to the perspectives and meanings of each group" (p. 238).

SUMMARY

This literature review revealed other disciplines' knowledge regarding the sustainability of professional development programmes' impact. With regard to mathematics teacher education the following implications can be deduced: Teachers, facilitators, and researchers of professional development programmes should

- plan for sustainability from the very start,
- take systematically into account the unintended and unexpected impacts,
- consider professional development and school development as concomitant processes,
- foster and support sustainable leadership,
- foster and support innovation champions,
- focus on factors that are less dependent from the programme's existence.

When discussing and researching professional development programmes' sustainable impact, the fostering factors are playing the central role. Knowing and being sensible for them is prerequisite for any conceptualization, implementation and evaluation of future professional development programmes which aim at sustainable impact. Thus, further, broader as well as deeper research of professional development programmes' sustainable impact and their respective fostering factors appears to be promising from both scientific and practical perspectives.

Another felicitous sentence concerning the complexity of fostering and hindering factors is provided by Slavin (2004): "With the many ways that innovations can be undone, it is perhaps more surprising when they do maintain over time than when they do not" (p. 61). Therefore, each programme has to carefully consider its respective fostering factors regarding the sustainability of impact, since each professional development programme has its own and particular objectives, contents, methods, and environments. Considering these factors in the programme's planning may help to establish sustainable impact.

NOTE

This paper is a modified and shortened version of Zehetmeier (in press). In this aforesaid paper, also the others' rationales, definitions, theories, research methods, and discussions are provided.

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SPECIALIZED CONTENT KNOWLEDGE OF MATHEMATICS TEACHERS IN UAE CONTEXT

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This study aimed to characterize the gap between the mathematics teachers' current knowledge of mathematics and the ideal mathematical understanding for teaching in the UAE context. Specifically, it aims to investigate this gap by focusing on the specialized content knowledge of mathematics teachers in UAE. A fairly representative sample was chosen from UAE public schools and a questionnaire including open-ended and multiple-choice questions was applied to 142 mathematics teachers from grade 1-12. The results suggest that there is quite a gap between the current and ideal levels in terms of understanding core mathematical ideas, interpreting student work and knowledge, understanding purpose of assessment and conceptualizing the development of mathematics curriculum.

INTRODUCTION

The purpose of this research project[1] was to characterize the mathematics teachers' knowledge of mathematics in UAE public schools in order to portray the gap between what mathematics teachers should know and what they already know. For this purpose, I sampled a number of teachers from different public schools of United Arab Emirates (UAE) and applied a comprehensive research-based mathematics questionnaire assessing their knowledge of and about mathematics.

RELEVANT LITERATURE AND RESEARCH QUESTIONS

Even though there is not much solid research-based evidence, it is the perception shared in the relevant research literature that teacher knowledge has an impact on students' understanding, especially in the field of mathematics education (Ball & Forzani, 2009). Such shared belief in this area puts extra responsibilities on the shoulders of researchers' to investigate the relationship between teacher knowledge and student success. However, before doing so, researchers need to learn more about the nature of teacher knowledge so that they can then use it to pursue the aforesaid endeavor. Therefore, in this study, I aimed to investigate the following research questions in a geographic region whose voice has not been heard well enough in the relevant literature, specifically UAE. The research questions for this study were:

1. How do in-service mathematics teachers understand and think about fundamental mathematical ideas in major strands (geometry, algebra, measurement, probability and statistics, and numbers)"?
2. What is the nature of the gap between 'what teachers currently know' and 'what they should know' to teach mathematics effectively in UAE schools?

In light of these research questions the current study aimed to contribute to the field's understanding of mathematics teacher knowledge and its nature. To pursue these research questions I used the following framework and methodology. The data analysis is still in progress but some of the results will be shared in this article.

THEORETICAL FRAMEWORK

Searching through the relevant research literature gave me the chance to identify the types of knowledge (in mathematics) and the ideal qualities of mathematics teachers with respect to these knowledge types. Following on Shulman's (1986) previously proposed knowledge types, Ball and her colleagues in a number of studies (e.g., Ball, Lubienski, & Mewborn, 2001; Hill, Schilling, & Ball, 2004; Hill, Ball, & Schilling, 2008; Ball, Thames, & Phelps, 2008) identified the types of knowledge teachers should have that is required in teaching mathematics, called mathematical knowledge for teaching (MKT). In this framework, MKT consists of two major components, namely subject matter knowledge (SMK) and pedagogical content knowledge (PCK). PCK consists of knowledge of content and students, knowledge of content and teaching and knowledge of curriculum. SMK consists of common content knowledge, mathematical knowledge at the horizon and specialized content knowledge (SCK). The subcomponent SCK is the focus of attention for this study since it is directly related to teachers' mathematics subject matter knowledge and it is the most essential one. SCK refers to the "mathematical knowledge that allows teachers to engage in particular *teaching* tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems" (Hill et al., 2008, p.378). This account of SCK was initially described in Ball et al. (2005). In explaining SCK, Ball and her colleagues working on teacher knowledge suggested that teachers should have some SCK that is required to manage everyday tasks of teaching mathematics. Even though these tasks of teaching mathematics vary a lot and these scholars did not provide a complete list in a single publication, it was possible to compile a list of these everyday tasks of teaching from their writings that have previously appeared in different publications (e.g., Ball, Lubienski, & Mewborn, 2001; Hill, Schilling, & Ball, 2004; Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2008; Ball, Thames, & Phelps, 2008). Below is a list of tasks mathematics teachers need to deal with regularly, which require special kind of knowledge (especially SCK) on the part of the teachers. According to Ball and her colleagues mathematics teachers who have SCK have the knowledge of or ability to do the following:

> **SCK#1:** Unpack" mathematical knowledge in order to provide meaning for learners; **SCK#2:** Have knowledge of interpretations and contexts; knowledge of common errors; diagnosing errors in student work; **SCK#3:** Design mathematically accurate explanations that are comprehensible and useful for students; **SCK#4:** Know mathematical explanations for common rules, procedures or algorithms; represent mathematical ideas (and operations) carefully; **SCK#5:** Provide explanations and justifications for mathematical ideas and procedures; evaluate mathematical explanations; present mathematical ideas; **SCK#6:**

Generate examples; find an example to make a specific mathematical point; **SCK#7:** Analyze mathematical treatments in textbooks; Appraise and adapt the mathematical content of textbooks; deploy mathematical definitions or proofs in accurate yet also grade-level-appropriate ways; **SCK#8:** Build connections among mathematical ideas; know underlying mathematical structures; **SCK#9:** Interpret and make mathematical and pedagogical judgments about students' questions, solutions, problems and insights (both predictable and unusual); respond to students' "why" questions; **SCK#10:** Assess aspects of understandings students show; listen to and interpret students' responses; analyze student work; pose questions; (+recognize what is involved in using a particular representation; select representations for particular purposes); **SCK#11:** Analyze a superficial understanding of an idea; be able to respond productively to students' mathematical questions and curiosities; evaluate the plausibility of students' claims (often quickly); attend to ambiguity of specific words; make mathematical practices explicit; **SCK#12:** Choose and develop useable definitions; use mathematically appropriate and comprehensible definitions; **SCK#13:** Connect a topic being taught to topics from prior or future years; sequence ideas; **SCK#14:** Think about multiple representations; map between a physical or graphical model, the symbolic notation and the operation or process, and make connections among the representations; link representations to underlying ideas and to other representations; construct and/or link non-symbolic representations of mathematical subject matter; use mathematical notation and language and critique its use; **SCK#15:** Inspect equivalencies; **SCK#16:** Know alternative solution methods, and claims; evaluate mathematical methods, claims and (alternative) solutions.

In the current study, this list of competencies is considered to be "what mathematics teachers should know." To learn about teachers' current competencies in each of the above issues (what they currently know), I designed a questionnaire consisting of questions with several subquestions per competency, the details of which are given in the following section. Such design helped me to identify the gap in between what mathematics teachers already know and what they should know.

METHOD

A questionnaire testing the aforesaid competencies was generated based on the relevant research literature and my own experiences about teacher knowledge, and applied to a sample of teachers. The sample was chosen from among all public schools throughout UAE. I used proportional stratified sampling to sample 100 schools out of 499 by considering cycles, gender, and region. Even though I sampled 100 schools out of all UAE public schools I could only work with volunteered teachers from 55 of those 100 schools because of bureaucratic limitations. The distribution of participant teachers is illustrated in Table 1.

City name / Cycle	Cycle 1 (Gr.1-5)	Cycle 2 (Gr.6-8)	Cycle 3 (Gr.9-12)	Common Cycle (Gr.1-12)	Total
Al Ain	12	25	11	17	65
Western Region	0	1	4	22	27
Abu Dhabi	12	23	15	0	50
Total	24	49	30	39	142

Table 1: Participant mathematics teacher distribution based on cycle and region

This sample is about the 10% of the whole school population and mathematics teacher population. The sample of teachers is fairly representative of the public school mathematics teacher population. The participants were 85 male (60%) and 57 (40%) female mathematics teachers.

SOME OF THE QUESTIONS ASKED IN THE QUESTIONNAIRE

Once those competencies (as laid out in the Theoretical Framework section) were identified, I developed a questionnaire targeting each competency (group). Then these questions were piloted on mathematics teacher groups online the results of which helped in revising and finalizing the questionnaire. Some of the questions from this questionnaire and their corresponding competencies are given within the results.

DATA ANALYSIS PROCEDURE

The gathered data included two main sections; one is about demographics and the other is about mathematical knowledge. Demographic information is analyzed to highlight the background characteristics of the participant teachers, which is partly shared in the Method section. The data about mathematical knowledge was mainly qualitative but to save time in overall data analysis process, I transformed this data into quantitative form and make the necessary analyses using SPSS. For example, in analyzing the participant answers about the division of fractions problems (targeting SCK#4, as explained in page 8 of this paper), the responses were coded as follows: “1 = "Completely wrong answer and/or rationale", 2 = "Says "it is right" but no rationale", 3 = "Turn it into invert-multiply and find solution", 4 = "Size or division matches rationale", 5 = "Size and division match rationale", 9 = "No answer". Once such coding is completed for this question, I then checked the frequencies of different responses and then move into the qualitative analysis of these responses to learn more about the nature of participants’ SCK. A detailed qualitative and quantitative analyses are still ongoing. Because of space limitations, the nature of this data analysis is briefly included in the paper. Some of the results are highlighted and briefly discussed in the following section.

BRIEF DISCUSSION OF SOME OF THE MAJOR RESULTS

The current study revealed that in-service mathematics teachers in UAE have significant problems regarding mathematics content, analyzing student work, curricular issues and assessment. This also suggests that there is quite a gap between where they are and the ideal SCK that they should have. Therefore, this paper only highlights the weaknesses of the participant teachers to reveal the nature of the aforesaid gap. The results about these issues are briefly explained below.

ISSUES RELATED TO CONTENT

Participant in-service mathematics teachers can mostly carry out the basic mathematical procedures (e.g., finding solution of a basic division problem), which is a strength on their part, but most of them have serious problems with interpreting the conceptual meanings embedded in those procedures. For example, in analyzing whether a given 3-dimensional graphical representation for $g(x, y) = xy$ as in Figure 1 represents a function (targeting SCK#13), almost 90% of the teachers mentioned that a 3-dimensonal graph cannot represent a function, which is also one of the major misconceptions seen among students. In earlier grades whether a given graph represents function is tested through vertical line test (VLT). However, when the given representation is a three-dimensional (3D) graph, as given in higher grade mathematics classes, the participants had a hard time applying this VLT to the given graph. They even left VLT aside and think that a 3D graph cannot represent a function. This seems to be because it is a challenging task for them to connect a topic being taught in early grades to topics from future years (a requirement for SCK#13).

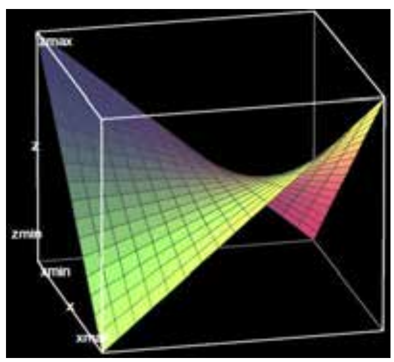

Figure 1: A given 3-dimensional graphical representation for $g(x, y) = xy$. [Note that this question was originally developed in a project, conducted within Mid-Atlantic Center, supervised by Prof. Kathleen Heid from The Pennsylvania State University in USA]

The participant teachers, for example, analyze equations through a single lens, mostly through algebra, as opposed to referring to several lenses including geometry. In analyzing a given first degree equation, $2x+4 = 3x+8$ (targeting SCK#15), about 95% of the teachers found its solution as $x=-4$, which is a strength on the teachers' part, but could not provide another way to analyze it (e.g., as a point of intersection of two lines). This suggests that their analysis of equivalencies is limited (opposite to the requirement of SC#15) and they did not seem to be able to interpret equivalencies, such as $2x+4 = 3x+8$, by referring to different domains like analytic geometry.

One final example that illustrates a challenge for participants regarding conceptual meanings of mathematical ideas is the way they approach to the concept of parallelism. In one of the questions, the participants were given a drawing-a-parallelogram scenario in a dynamic geometry software environment (DGS) (with pictures as illustrated in Figure 2 in order). In this scenario the participants were told that a student, Ahmed, was going through the following steps. In a dynamic geometry environment, called Wingeom, a line is drawn (passing through two arbitrary points, A and B), then a point (out of line AB) is put on screen (point C), and the program is asked to draw a parallel line (passing through that outside point) to the initially drawn line. Then the DGS draws a parallel line passing through the initial point (point C) but it automatically puts another point on this newly drawn line (point D).

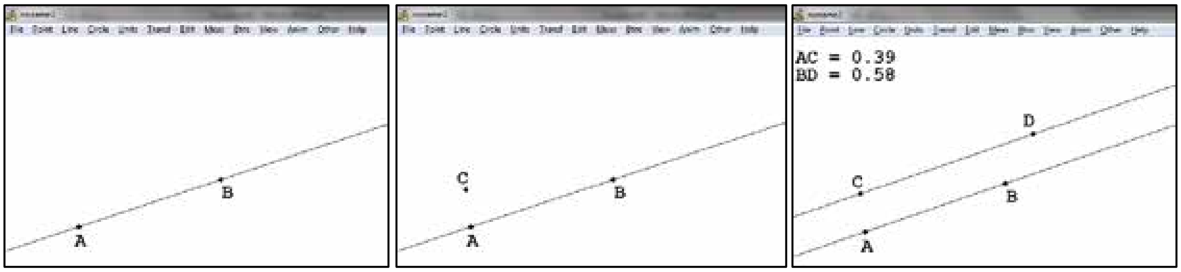

Figure 2: Pictures given in question #9 targeting SCK#9.

They are then asked: "*After going through Step#3 and finding the measures for AC and BD, he got puzzled. He thought to himself, "If Wingeom did not draw the parallel line based on the distances between A and C, and B and D, how did it decide on how to draw the parallel line CD? On what basis did the program created this extra point (point D) automatically?" If Ahmed was in your class, how would you explain the answers to these two questions mathematically to Ahmed?*"

It is obvious from the participants' responses that they could only think about parallelism by focusing on the distance between the two lines or the ratios of distances *AB*, *AC*, *CD* and *DB*. A sample response from one of the participants is as follows: "I confess utter confusion as to the reason Wingeom would create such a pt. D at exactly the place it did. However, it could have picked a random point on $\overleftrightarrow{AB}$ and gone 0.3 inches away to put a point D and then draw the parallel line. It may have created point *D* so that the ratio of $\overline{AB}$ to $\overline{CD}$ was equal to *AC* to *DB* but without knowing the values of $\overline{AB}$ and $\overline{CD}$. I cannot verify such a claim."

The above response suggests that this teacher focused his attention to the distance between the two lines as well as the ratios of certain distances in the given scenario, whereas he ignored the fact that parallelism also requires same sloped lines. The participant teachers had difficulty in interpreting and making mathematical and pedagogical judgments about hypothetical unusual student questions and in responding to students' "why" questions - a requirement for SCK#9.

As seen in these given examples, interpreting the mathematical meanings of core mathematical ideas is quite a challenge for the participant mathematics teachers.

ISSUES RELATED TO ANALYSIS OF STUDENT WORK

The participants' analysis of student work is not at a desired level when it comes to an alternative solution method. For example, when given a scenario about a fraction division solution, like $\frac{3}{8} \div \frac{1}{3} = \frac{9}{24} \div \frac{8}{24} = \frac{9}{8}$ (called common denominator algorithm), they only turned it into a well-known invert-and-multiply algorithm to check whether it is accurate or not instead of analyzing the accuracy of the given method. This suggests that their ability to provide mathematical explanations for common algorithms, which is a requirement for SCK#4, falls short in reaching the ideal level.

The study also revealed that the participant mathematics teachers are also quite weak in identifying possible student errors for a given problem situation. They were asked to talk about possible student errors for the following problem targeting SCK#2:

Question #2: *A computer game store is having a sale. They have advertised 10% off everything in the store. They also have just purchased a new shipment of computer games. These games cost the store 32.11AED each. They want to price the game so that they will make at least a 40% profit, even at the sale price. What is the lowest regular selling price for the game that will allow this profit? (Bair & Rich, 2011)*

a) *If a student brings this question to your mathematics class, to what extend would you feel confident (or comfortable) in analyzing this problem situation before you actually solve it?*

   ☐*5-highly confident* ☐ *4-somewhat confident* ☐ *3-confident*
   ☐*2-little confident* ☐ *1-not confident*

b) *What mathematical knowledge or understandings are required to solve this problem? [Please be specific in your answer. For example, saying that it requires algebra or geometry is not informative! If you need, you can solve the problem here and then talk about the mathematical components of it!]*

c) *What common mathematical errors (or mistakes) would you expect from students in solving such a problem?*

In approaching such a problem, instead of focusing on the student errors, participants mostly solved the problem first, and then talk very superficially about possible student errors such as “students will most like make calculation errors.” This suggests that the participant mathematics teachers need significant support in how to analyze student work in mathematics classes. Such support can be in a form where they are given opportunities to “listen” to the student ideas, interpret those ideas, and think about what it means to think like a student in certain problem situations.

ISSUES RELATED TO ASSESSMENT

The study revealed an interesting result about mathematics teachers’ understanding of assessment. When participants were asked to choose from among three assessment items so as to test student understanding of infinity (given below), about 95% of the participant teachers chose the easiest to handle representation to ask, Example 3. A typical response given by the participants who chose Example 2 or 3 as the best one to ask students are: “Since sets are given side by side [in Example 1], students can’t really analyze them” and “Because it is well cleared [in Example 3] to the student that both groups are infinite.”

Question #10 (targeting SCK#10): *Which of the following sample questions would be nice to assess student understanding of infinity? In answering the question please also consider the way the questions are represented. (Tsamir & Dreyfus, 2002)*

☐ *1-Example#1* ☐ *2-Example#2* ☐ *3-Example#3*

Example #1: Consider the following sets. Which one do you think has more elements than the other?

$A = \{1, 2, 3, 4, 5, 6, \ldots\}$; $B = \{1, 4, 9, 16, 25, 36, \ldots\}$

Example #2: Consider the following sets. Which one do you think has more elements than the other?

$$A = \{1, 2, 3, 4, 5, 6, \ldots\}$$
$$B = \{1, 4, 9, 16, 25, 36, \ldots\}$$

Example #3: Consider the following sets. Which one do you think has more elements than the other?

$$A = \{1, 2, 3, 4, 5, 6, \ldots\}$$
$$B = \{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, \ldots\}$$

Such focus on the teachers' part suggests that the participants think about assessment in mathematics as a way to help students as opposed to as a way to test knowledge of students. One of the purposes of assessment is to figure out how and to what extent students understand a targeted concept (Airasian & Russell, 2008) and a teacher can assess students' knowledge through probing questions that do not include any hints about the solution of the problem. The fact that almost every participant choose the easiest-to-handle question to ask students to figure out how students think about infinity suggests that they seem to misinterpret the purpose of assessment in mathematics classes.

ISSUES RELATED TO CURRICULUM

From curricular standpoint, almost all of the participant teachers follow the traditional ways to explain curricular decisions made in the books. For example, they think about trapezoid by referring to exclusive definitions (definition #1 as given below) rather than inclusive definitions (definition #2 as given below).

Question** #7 **(targeting SCK#7): *A well-known mathematics educator in USA, Zal Usiskin, checked through the geometry textbooks used in USA since 1800s and realized that there are mainly two definitions given for "trapezoid" as shown below.*

<u>Definition #1:</u> Trapezoid is a quadrilateral with exactly one pair of parallel sides.

<u>Definition #2:</u> Trapezoid is a quadrilateral with at least one pair of parallel sides.

A) *If we accept Definition #1 which one of the following figure(s) would be considered as trapezoids? Mark all that apply.*

☐ *1-Parallelogram* ☐ *2-Rectangle* ☐ *3-Rhombus* ☐ *4-Square* ☐ *5-Isosceles Trapezoid*

Why? .....

B) *If we accept Definition #2 which one of the following figures would be considered as trapezoids? Mark all that apply.*

☐ *1-Parallelogram* ☐ *2-Rectangle* ☐ *3-Rhombus* ☐ *4-Square* ☐ *5-Isosceles Trapezoid*

Why? .....

***C)** If you were to use one of these definitions to teach students in your math classes, which definition would you use?*

☐ *1) Definition#1* ☐ *2) Definition#2*

Why? .....

In addition to this, when they need to make curricular decisions about the sequence of mathematical topics (e.g., which topic should come first, triangles or circles?), their decisions do not reflect a solid understanding of those mathematical concepts. For example, most of them do not know that having the knowledge of circle is necessary to make sense of triangles, and therefore, it may be appropriate to teach circles first and then triangles. Therefore, their responses suggest that their analysis of the mathematical treatments in textbooks and deploying mathematical definitions in accurate and grade-level-appropriate ways, as suggested by SCK#7, need to be improved in ways that allow them to critique mathematical definitions and their treatments in textbooks.

IMPLICATIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

Considering the above general results about mathematics teacher knowledge, there seems to be a considerably negative large gap between where teachers are and where they should be – teachers currently teaching mathematics in UAE are quite weak in understanding the core mathematical ideas they teach, in interpreting and analyzing student work, in the assessment of understanding mathematical ideas, and finally in making and criticizing curricular decisions. Their SCK needs to be improved in order to close this gap. Therefore, considering their background information it seems quite reasonable to think that teachers who currently teach mathematics in UAE public schools need considerable professional development support in the aforesaid areas. The nature of this support will depend on identifying the weaknesses of the teachers, which was partially done through the current research study, and then preparing professional development programs that specifically address those needs and weaknesses throughout long term programs.

These results also suggest that teachers are not prepared well throughout the Education and Science faculties of the universities in the Gulf region. This is not to say that they do not learn anything from those programs. It is rather saying that the experiences teachers gain from their undergraduate education does not seem to support them well for the aforesaid areas. Obviously teaching the subject for many years (some teachers have the experience of 20 years of teaching) or having extra credentials (e.g., master, PhD) did not help them improve their understanding of the core mathematical ideas either. Neither the academic background nor the teaching experience that they had helped them in answering even very simple questions about core mathematical ideas such as drawing a triangle (e.g., one question was about investigation of whether given three side measures helps one draw a triangle) or interpreting a first degree equation.

Finally, the mathematics teacher competencies should be carefully reconsidered and revised. The SCK construct seems to be useful in targeting certain competencies for mathematics teachers regarding their content knowledge and in generating questions to test those competencies. However, to what extend these competencies and testing of them would tell us about student success is not clear. The connection between student success and these competencies is beyond the scope of this paper.

NOTES

1. The current paper describes part of a major research study funded by United Arab Emirates University with the fund number 31D000 for the years 2010-2012 and includes some preliminary results about mathematics teacher knowledge.

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PROFESSIONAL DEVELOPMENT PROGRAM IN FORMATIVE ASSESSMENT

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Key words; Professional development, formative assessment, mathematics education

BACKGROUND

Several studies have demonstrated that substantial learning gains are possible when teacher introduce effective formative assessment into their classroom practice (e.g. Black & Wiliam, 1998; Hattie, 2009; National Mathematics Advisory Panel, 2008), but a strong research base supporting how effectively help teachers to implement a formative assessment practice is lacking (Schneider & Randel, 2010;Wiliam 2010).

Effective formative assessment (FA) can be conceptualized as practice based on an adherence to the "big idea" of using evidence about student learning to adjust instruction to better meet student needs, and a competent use of the following five key strategies (Wiliam 2010);

1. Clarifying, sharing and understanding learning intentions and criteria for success;
2. Engineering effective classroom discussions, questions, and tasks that elicit evidence of learning;
3. Providing feedback that moves learners forward;
4. Activating students as instructional resources for one and another;
5. Activating students as the owners of their own learning.

PROJECT DESCRIPTION

We are part of a larger project that uses an experimental design, with 40 randomly selected teachers in grade 4 and 7, to investigate the impact of a comprehensive (one day a week in 20 weeks) professional development program (PDP) in formative assessment on teachers' implementation of FA and on student achievement and motivation. In our part of the project we will investigate in which ways the teachers change their classroom practice, with respect to FA. An aim is to evaluate to what extent the PDP can be used to help teachers develop their formative assessment practice. Another aim is to contribute to the understanding of factors that are significant in the support of teachers' implementation of effective FA. The teachers' classroom practices have been observed and teachers have been interviewed before and after the PDP. The teachers' have also answered a questionnaire as an evaluation of the PDP. The framework of FA suggested by Wiliam and colleagues (see above) form the basis for analysis of the teachers' practices before and after the PDP.

EXPECTED RESULTS

We expect to identify interesting changes with respect to FA in the participating teachers' classroom practices, over time and between teachers. The identification of similarities, differences and patterns will help us understand the reasons for these possible changes, and offer indications for improved FA-programs in the future. The collection of data is completed and in February we expect to be able to present some preliminary results.

DISCUSSION – THE PROJECT CONTINUES

Based on the results and conclusions from this project, professional development program in formative assessment will be developed and researched.

On the poster, the content will be presented with the same structure as above; this will be done by text, photos and pictures. Preliminary results might be presented with tables.

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STATISTICAL KNOWLEDGE AND TEACHING PRACTICES OF ELEMENTARY SCHOOL TEACHERS IN THE CONTEXT OF COLLABORATIVE WORK

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Keywords: *Teachers' knowledge; Teachers' practices, Statistics education*

SUMMARY OF THE POSTER

Statistics is an important topic in today's world. An active participation in society requires the ability to analyze and interpret statistical data represented in many different ways. In addition, statistics has an important role in several school subjects and supports the development of students' critical reasoning (Batanero, Godino & Roa, 2004). This leads statistics to receive an increased emphasis in the school curriculum. The main aim of its teaching is the development of students' statistical literacy, from an early age (Ponte & Sousa, 2010).

Teachers' mathematical knowledge is an important research topic (Ball, Hill & Bass, 2005). Groth (2007) indicates that, among all the mathematical topics, usually it is in statistics that teachers have weakest knowledge. Since teachers' knowledge is one of the most influential bases for effective teaching, it is necessary to investigate the knowledge needed to teach this topic and how it may develop (Fennema & Franke, 1992; Groth, 2007). As students' learning derives from the way teachers plan lessons, conduct such lessons, and reflect on the whole process of teaching and learning, the study of teachers' practices is also crucial. As teachers' practices are deeply intertwined with teachers' knowledge for teaching, thus this study aims at understanding the development of specialized knowledge for teaching and of knowledge regarding practical instruction of elementary school teachers in terms of statistics education, in their mutual relationship, in a collaborative work context.

The study is based on a collaborative context since this is an important support for teachers to deal with professional problems (Ponte & Serrazina, 2004) as well as an useful strategy for carrying out investigations into professional practice (Boavida & Ponte, 2002). In a collaborative environment the process of joint reflection is likely to support teachers' professional development, as it allows an analysis and discussion of their practices, leading to the clarification of some aspects and contributing for new actions to emerge.

The study follows a qualitative and interpretative methodology, with three case studies of grade 3 and 4 teachers of two different schools. The working sessions group will focus on collaborative discussion and reflection on current curriculum guidelines, as well as on articles and other documents relating to the teaching and learning of Statistics in elementary grades. During these sessions teachers plan lessons for their students which will be screened (via videotaped excerpts) and discussed in order to promote a constant joint reflection. It is expected that collaborative work will promote the development of teacher´s knowledge for teaching. Data collection includes participant observation of collaborative work sessions and of teachers' classes (with audio and video recording); semi-structured interviews with participating teachers, one at the beginning of investigation and another at the end (with audio recording) and the collection of written documents (teachers' diaries, documents produced by teachers and students during classes and documents produced in the collaborative group). Data analysis will be made according to categories of analysis related to teachers' knowledge and practices and statistical education issues, taking into account the theoretical framework and the research questions.

The poster, supported by graphical elements, will present the goals and main ideas of the theoretical framework regarding both teachers' knowledge and practices and statistical education. It will also describe the method used in the study and present some preliminary results.

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THE TRIGONOMETRIC FUNCTIONS - CONCEPT IMAGES OF PRE-SERVICE MATHEMATICS TEACHERS

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Abstract. According to the present Croatian curriculum for secondary education, substantial time is allocated to the trigonometric functions of real argument. However, results of the Croatian State Matura show that students' performance on tasks involving this mathematical content is unsatisfactory. In order to gain a deeper insight into causes of this discrepancy in the context of the Croatian educational tradition, we have conducted a survey of basic concepts related to trigonometric functions of university mathematics education students in Croatia.

Key words: pre-service teachers, concept image, trigonometric functions

THE STUDY AND RESULTS

In our study, we were focused on concepts such as the radian measure, as well as sine, cosine and tangent function and the way they are used in solving simple trigonometric equations and inequalities (e.g. $\sin x < \cos x$, $\tan x > -1$). In order to describe dominant cognitive structure associated with the concept, the results obtained are analyzed against the theoretical framework of concept image vs. concept definition (Tall & Vinner, 1981) and the notion of procept (Gray & Tall, 1994) which is as well concerned with predominance of procedural knowledge over conceptual. Our research questions were: What kinds of concept images of radian and of trigonometric functions do pre-service teachers have? What kind of understanding of radian and trigonometric functions do pre-service teachers have: dominantly procedural or dominantly conceptual? In our study, we have collected data over entire populations of 79 students of the 3rd year and 26 students of the 5th year of university mathematics education programmes at the largest Croatian university. They completed a one-hour questionnaire with 13 open-ended questions addressing their understanding of the concepts mentioned above. Students' responses to each question were classified and the code plan has been developed accordingly. Based on the most typical concept images detected and strategies used to solve given problems, 8 students (4 from each group) were selected for semi-structure interviews on the same and some additional questions. Interviews were audio recorded and then analyzed.

In our poster, we present findings with major contribution to the research questions posed. Results suggest that pre-service teachers relate radian measure dominantly to rectangular coordinate system, circle trigonometry, as well as to trigonometric functions, whereas right triangle trigonometry and degree measure make significant part of their concept images of sine and cosine functions. On the contrary, tangent function is seen dominantly only as a ratio of sine and cosine, without referring to its geometric interpretation in circle trigonometry. Pre-service teachers refer to π as the unit for radian measurements, while real numbers not being of the form $q\pi, q \in \mathbb{Q}$, are not recognized as the radian measures of an angle. Moreover, degree angle measure dominates students' conceptions of angle measure. For example, to find a length of an arc subtending central angle given in radians, they preferably convert radians into degrees, neglecting the definition of a radian measure. This reveals also a students' need to use conversion formulas, emphasizing their procedural knowledge. Dominance of procedural knowledge over conceptual is also seen regarding basic trigonometric inequalities. Some of these findings on Croatian dataset confirm earlier findings in Fi, 2006, Topçu, Akkoç, Yılmaz & Önder, 2006, and Chin & Tall, 2012. Our results evidently show that a change in Croatian initial mathematics teacher education should be made to promote pre-service teachers' conceptual knowledge on radian measure and trigonometry functions of real argument. Radian measure should be more explicitly related to angles and arc length, while values of basic trigonometric functions (especially of tangent function) should be recognized as coordinates of points in rectangular coordinate system. Our study is the first research on students' understanding of trigonometry concepts in Croatia.

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MATHEMATICS TEACHERS' UNDERSTANDING AND INTERPRETATION OF THEIR OWN LEARNING AND CLASSROOM PRACTICE

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SUMMARY

Educational systems around the world are at the moment in the center of numerous reforms in order to improve schools and students achievement (Roesken, 2011). In the Swedish context the reforms requires interpretation and implementation of changed curricula grounded in learning and teaching paradigms that may challenge and question mathematics teachers´ present beliefs and teaching practices. In order to meet these requirements teachers play a central role in gaining adequate professional development; to participate in and obtain qualified and continuing learning opportunities (Borko, 2004). Several contextual conditions are identified by international research as promoting principles for teacher learning; providing ongoing opportunities for teachers to collaborate and learn together, content focused school-based professional developmental activities based on experiences and incorporating inquiries into teachers own practice and learning (Loucks-Horsley, Love, Stiles, Mundry & Hewson, 2010, Borko, 2004). The aim of this study is to examine Swedish mathematics teachers understanding and interpretation of their own learning and classroom practice during learning activities designed according to the consensus criteria mentioned above.

Two teachers in a local mathematics team participated in a 3 year-long PD-project, initiated by the teachers, the school-leader and a local resource teacher, who also joint facilitated the learning activities. The PD-initiative was raised after mutual identification of an escalating need for improving students result in mathematics. The intervention was designed by the researcher, the teachers, a resource teacher and the school leader and consisted of further improving the mathematics teachers' content knowledge and pedagogical knowledge through a number of PD activities (i.e. workshops, network meetings, collegial reflections) on a regular basis.

In 2010 semi-structured interviews with two participating teachers were conducted and video-recorded in order to capture the teachers' understanding and interpretation of their classroom practice and their own learning needs. During 2010 and 2011 artifacts were collected from practice, for example classroom videos of teaching sequences concerning number sense and arithmetic, students work, assessment data, teachers planning. In 2012 Video Stimulated Reflection (VSR) was used on the same participating teachers. Through VSR, supplemented with artifacts from practice, the participating teachers collaboratively reflected upon the film-clips from 2010 concerning (a) the classroom practice and (b) their own pedagogical reasoning as mentioned above.

Preliminary findings indicates (and to some extent confirms previous research) that in this meta-cognitive process of sharing reflections on authentic video-recordings, supplemented with other artifacts from practice, the teachers interpret their own learning and classroom practice and become aware of their own progress and learning needs. They also re-interpret the students' conceptual understanding, learning and learning needs. The collaborative conversations and the contextual setting of the intervention seem to be motivators and key-factors for establishing a systematic approach to teachers' reflection on their own practice, teaching concerns and needs in relation to their students' learning needs. *During the autumn of 2012 the interconnected model (Clarke and Hollingsworth, 2002) is adapted to the data in order to analyze the complexity of teachers' professional growth.*

This study has implications for research into the professional learning of mathematics teachers as well as for the design of school-based PD. Even though an international need for mathematics teachers' professional development is identified and expressed, we know little about the precise challenges, needs and possibilities related to mathematics teachers' continuing professional learning in and for practice in local contexts. Jaworski (2011) stresses that "we know much less than we should about teachers' learning from experience; whether teachers learn, what they learn and what supports learning from experience." (p. 11).

Keywords: Professional development, mathematics teacher learning, artifacts from practice

The poster-content will be presented using the IMRAD-model. Short, but clear and concisely introduction (I) and statement of aims. The method section (M) will be brief. The result section (R) will be the major part, visualized and illustrated with cites and representations in the middle of the poster. The discussion section (D) will be brief and sited on the right side of the poster.

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THE MATHEMATICAL KNOWLEDGE FOR TEACHING

A view from the Onto-Semiotic Approach to Mathematical Knowledge and Instruction

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Keywords: teachers' training, teacher's knowledge, mathematical knowledge for teaching, onto-semiotic approach, didactic-mathematical model.

One of the problematics that has drawn a lot of attention from both researches community and policy makers alike, is the identification and characterization of the knowledge web that a mathematics teacher should have in order to teach effectively and to facilitate their students' learning on specific mathematics topics. One proposal on the teachers' knowledge that is widely accepted, is the model called "Mathematical Knowledge for Teaching (MKT)", developed by Ball and colleagues (Ball, Lubienski & Mewborn, 2001; Hill, Ball y Schilling, 2008; Ball, Thames & Phelps, 2008). This proposal is a remarkable advancement for describing the complex of knowledge that a teacher should have to teach mathematics. Nonetheless, despite the advances that the MKT model represents, there are still questions to be addressed, such as: how to identify the teachers' didactic-mathematical knowledge when the teachers' knowledge models include categories too wide? Specifically, under what criteria can the MKT be evaluated? How can the teachers be supported to acquire or to develop the MKT components? In general, as Godino (2009) points out, both the MKT model and the others various models on the mathematical knowledge for teaching, informed by the researches in mathematics education, include categories too "wide" and disjoint, that call for models that allow conducting a more precise analysis of each knowledge component that are put into effect in an effective teaching of mathematics. The latter will allow orienting to the design of formative actions and the elaboration of tools to assess the mathematics teachers' knowledge.

Thus, in this work, based on both the Onto-Semiotic Approach to Mathematical Knowledge and Instruction (OSA) (Godino, Batanero & Font, 2007) and the categories of didactic-mathematical teacher's knowledge (Godino, 2009), we propose a model called "Didactic-Mathematical Knowledge (DMK)". This model proposes six facets or dimensions to analyse the teacher's didactic-mathematical knowledge about a specific mathematical topic (Godino et al, 2011, p. 278-279): 1) *Epistemic Facet:* The intended and implemented institutional meaning for a given mathematical content, that is, the set of problems, procedures, concepts, properties, language, and arguments included in the teaching and its distribution over the time; 2) *Cognitive facet:* Students' levels of development and understanding of the topic, and students' strategies, difficulties, and errors as regards the intended content (personal meaning); 3) *Affective facet:* Students' attitudes, emotions, and motivations regarding the content and the study process; 4) *Mediational facet:* Didactic and technological resources available for teaching and the possible ways to use and distribute these resources over time; 5) *Interactional facet:* Possible organisations of the classroom discourse and the interactions between the teacher and the students that help solve the students' difficulties and conflicts; 6) *Ecological facet:* Relationships of the topic with the official curriculum, other mathematical themes and with the social, political, and economical settings that support and condition the teaching and learning. These facets reinterpret and organize the different components of the MKT. Furthermore, to each of the said facets the OSA provides theoretical and methodological tools that allow more detailed analysis. For example, for the epistemic and cognitive facets the tool "objects and processes configuration" is proposed, which refers to the detailed and systematic description of the linguistic elements, concepts, propositions, procedures and arguments, involved in the mathematical activity. An example of application of this tool can be seen in Pino, et al. (2012). The relationships between the components of MKT and the facets and levels of analysis of the DMK will be graphically illustrated in the poster.

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MATHEMATICAL KNOWLEDGE FOR TEACHING AS A MEASURE OF COHERENCE IN INSTRUCTION MATERIALS PRODUCED BY TEACHERS ON THE INTERNET

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KEYWORDS

Didactical Engineering, Mathematical Knowledge for Teaching, Teacher-shared instructions on the Internet

SUMMARY

In the poster, a plan for upcoming research is presented as a result of two previous studies. A first study revealed a discrepancy between instruction materials in mathematics shared by teachers through the Internet (Liljekvist, 2012, in preparation). In that study the didactical message was analysed, such as, topic-specific features related to syllabus, mathematical ideas, competencies, cognitive demands, and degree of devolution. Further on, an analysis was conducted on in what way these task-features could be understood as representations of didactical engineering. In a second study, lesson plans produced by student teachers were analysed and coherence was used as a measure of student teachers' Mathematical Knowledge for Teaching (MKT) (van Bommel, 2012a, 2012b). The study indicated that MKT influenced the degree of coherence within the produced lesson plans.

In the suggested upcoming research the coherence and lack of coherence within the analysed instruction materials made by teachers will be analysed using the MKT framework. In the poster, two examples will be presented to show in what way MKT could explain the degree of coherence within each example. Furthermore, the concept Didactical engineering and the framework MKT will be outlined.

CONCEPTUAL FRAMEWORK

Didactical engineering "deals with the production of the possible and available meanings of a students' activity" (Herbst & Kilpatrick, 1999, p. 7), hence, a teacher has the didactical responsibility to outline a task, or adopt it (Brousseau, 1997, Kieran, 1998) and thereby orchestrate a didactical situation to address, for instance, goals in syllabus.

MKT can be described as the mathematical knowledge needed to carry out the work of teaching mathematics (Ball, Phelps, Thames, 2008). The MKT model consists of 6 domains and complemented the PCK notion suggested by Shulman (1986). As a result, items were developed to be able to study the relationship between teacher knowledge and pupils' achievement. In this study the MKT model will be used to study the relationship between teachers' apparent knowledge and coherence in their produced materials. The relevance of the connections between features within the produced instruction materials, determine the degree of coherence.

RESEARCH QUESTION

How can the degree of coherence in teacher-shared instruction material on the Internet, be understood in terms of MKT?

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PRE-SERVICE ELEMENTARY TEACHERS' PROCEDURAL KNOWLEDGE OF THE GREATEST COMMON FACTOR AND LEAST COMMON MULTIPLE

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Pre-service elementary teachers in the United States demonstrated on a survey and follow-up interviews what I categorized as either superficial procedural knowledge or deep procedural knowledge of the greatest common factor (GCF) and least common multiple (LCM). Those exhibiting deep procedural knowledge varied the processes used to find the GCF and LCM based on the numbers' representation. Those exhibiting superficial procedural knowledge followed the same procedure regardless of the numbers' representation. Statistically significant evidence suggested that pre-service elementary teachers who define the GCF (and LCM) through the relationships between two or more numbers and their GCF (and LCM) also demonstrate deep procedural knowledge of the GCF (and LCM).

REVIEW OF LITERATURE

Zazkis and Campbell (1996) investigated pre-service elementary teachers' understanding of elementary number theory concepts and determined that their respondents showed a disposition towards procedural thinking, even when they displayed a conceptual understanding of the topic. Hiebert and Lefevre (1986) characterize procedural knowledge as memorization of facts and algorithms used to solve mathematical tasks (1986). Star (2005) later argued for a reconceptualization of procedural knowledge and asserted that a distinction must be made between those learners with superficial and deep procedural knowledge: “There are subtle interactions among the problem's characteristics, one's knowledge of procedures, and one's problem-solving goals that might lead a solver to implement a particular series of procedural actions” (p. 409). If a solver possesses a superficial knowledge of the procedures s/he may fall back on the known standard procedure to solve the problem, regardless if it is the most efficient process. If the solver instead possesses deep procedural knowledge, s/he may use various techniques to produce a solution that best matches the form of the problem.

FINDINGS

Through use of a survey instrument and interviews, I assessed 48 pre-service elementary teachers’ knowledge of the GCF and the LCM. The respondents demonstrated what I categorized as either superficial procedural knowledge or deep procedural knowledge. Items on the survey asked the participants to determine the GCF and LCM of two numbers given various representations of the numbers, such as the numbers' prime factorizations, lists of each number's factors, or lists containing the first 10 multiples of the numbers. The participants classified as demonstrating superficial procedural knowledge of the GCF and LCM applied the same procedure to each survey item regardless of the numbers' representation. Those exhibiting deep procedural knowledge varied the processes used to determine the GCF and LCM and applied more efficient methods based on the numbers' representation. For instance, a survey item asked respondents to find the GCF of two numbers that were represented by lists containing all of their factors. Those that I classified as demonstrating superficial procedural knowledge did not utilize this list, found the prime factorization for each number, and then used these prime factorizations to determine the GCF. Those that I classified as demonstrating deep procedural knowledge exhibited an understanding that the representation of two numbers as lists of their factors is transparent with respect to their GCF, and determined the GCF by finding the largest factor shared on each list of factors.

The survey also asked the participants to define the GCF and LCM. The responses fell into the following two categories: (1) those describing the relationship between two or more numbers and their GCF and LCM, and (2) those detailing a process that can determine the GCF and LCM for two or more whole numbers involving the numbers' prime factorizations. Using Fisher's exact test, I compared the relationship between the respondents' definitions of the GCF and LCM with the form of procedural knowledge that they demonstrated. The data revealed statistically significant evidence ($p=0.03$) suggesting that pre-service elementary teachers who define the GCF correctly through relationships also exhibit deep procedural knowledge of the GCF. Similarly, the data revealed statistically significant evidence ($p=0.04$) suggesting that pre-service elementary teachers who define the LCM correctly through relationships also exhibit deep procedural knowledge of the LCM.

POSTER FORMAT

In the poster I will explain my methodology, display the survey instrument with typical participant responses, and discuss my rational for classifying these responses as demonstrating superficial or deep procedural knowledge.

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THE RELATIONSHIPS BETWEEN THE TRADITIONAL BELIEFS AND PRACTICE OF MATHEMATICS TEACHERS AND TEIR STUDENTS' ACHIEVEMENTS

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The importance of teachers' beliefs for students' learning is highlighted from different sources showing that teachers' beliefs affect their teaching approach that, in turn, affects students' achievement. The studies of mathematics teachers' beliefs in Latvia brought out a contradiction between teachers' constructivist beliefs on teaching and learning and their traditional routine work while revealing match between some traditional beliefs on teaching and learning and traditionally-oriented instructional practice. The aim of the present study is to explore the possible relationships between the traditional beliefs and practice of mathematics teachers and their students' achievement in mathematics. For this purpose the Latvian data from two international research projects were analyzed. The sample included 190 mathematics teachers and their 2828 students from grade 9 representing different regions of Latvia, schools with different programs of education, rural areas and cities. The results suggest that the traditional beliefs of teachers are connected with lower students' achievement in mathematics test, while teachers' traditionally oriented self-reported practice is positively related to the achievement of their students.

Key words**: teachers' espoused beliefs, reported practice, students' achievement.**

THEORETICAL FRAMWORK

Belief entails individual, seldom – stable subjective knowledge that includes person's feelings or care (Pehkonen, 1994). The given study is focused on teachers' beliefs. We will single out teachers' espoused beliefs from all the diverse other beliefs of teachers. Teachers' espoused beliefs (what is said) about teaching are what teachers think about the impact of teaching in general, as well as their understanding of how children learn (McMullen, Elicker et al., 2006). It is important to pay attention, along with teachers' espoused beliefs, to their reported practice. In the given research, exactly due to these reasons survey on reported practice followed in a common package with the survey of teachers' beliefs. Teachers' reported practice (what is done) is teacher's recognition of methods used in class and the frequency of using them (almost every lesson, at about half of lessons, in some lessons, never).

The world educational research has recently manifested a dichotomous division of teachers' beliefs and approaches to teaching and learning: *the traditional beliefs* and *the constructivist beliefs* (OECD, 2009).

The aim of the present research is to make out whether there is correlation between the *traditional* orientation in *teachers' espoused beliefs* and reported practice and the achievements in mathematics of these teachers' students. Two research questions were set: 1) what relationships exist between the traditional espoused beliefs of teachers on teaching mathematics and their students' achievement in mathematics? 2) What relationships exist between the traditional orientation of teachers in self-reported practice and their students' achievements in mathematics?

METHOD

The given research has used data of two international studies: Singapore National Education Institute project "Non-Cognitive Skills and Singapore Learners – an international comparison" and NorBa project "Nordic-Baltic comparative research in mathematics education". The following methods of statistical analysis were used for data processing: Kolmogorov-Smirnov test to assess the distribution of data, descriptive statistics, frequencies, two step cluster analysis, factor analysis, hi-quadrangle criterion, Wilcoxon criterion, Kruskal-Wallis criterion, Mann-Whitney criterion as well as Cronbach Alpha to assess the reliability.

RESULTS

The research data base shows that there exist correlation between Latvian teachers' espoused traditional beliefs on teaching mathematics and their students' achievements in doing mathematics tasks: the more distinct teachers' espoused traditional beliefs, the lower their students' achievements. On the other hand teachers' traditional inclinations in their reported practice have a positive impact on their students' achievements. Completely different tendencies of teachers' espoused beliefs and reported practice may be accounted for by the fact that the major factor of influence on learner's achievement is not the teacher's beliefs but readiness to change, i.e. to change his/her beliefs as well as an ability to adapt his/her practice to the learners' intellectual level, needs, motivation. At the same time these teachers continue to use traditional methods gradually introducing new ones in their work. Despite the fact that the education philosophy reflected in the State Education standard is oriented toward the process of learning, in reality the mathematics education in Latvia is measured by students' achievement that, in turn, is characteristic of the traditional paradigm of education. Indeed, after every three years every learner in Latvia must take a compulsory centralized test or state examination in mathematics, regular mathematics Olympiads have been organized on local and state level. Learners' achievements in Olympiads and examinations are the basis of mathematics teacher ratings. This may account for the great influence of traditional reported practice on learners' achievements.

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TEACHER LEARNING WITHIN THE CONTEXT OF LESSON STUDY

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Lesson study is a professional development strategy that is grounded in daily practice. A team of teachers collaboratively studies teaching and learning through examining single lessons. By means of live classroom observations and post-lesson discussions, student learning is related to the lesson design and the course of the lesson. Lesson study cases from Dutch secondary schools will be presented during the CERME. In those cases, the professional learning of (mathematics) teachers is captured and will be related to the core of lesson study - observation and discussion - completed with teaching and collaborative planning.

Keywords: lesson study; teacher learning; secondary education

Lesson study is a professional development strategy in which teachers collaboratively study teaching and learning by means of live classroom observations and post-lesson discussions (e.g., Fernandez & Yoshida, 2004; Saito, 2012). Lesson study originated over a century ago in Japan where it is widely viewed as the foremost professional development strategy for teachers (Fernandez & Yoshida, 2004; Stigler & Hiebert, 1999). By the end of the '90s, lesson study gained worldwide attention and spread over different Western countries.

The goal of my research is to gain understanding of what and how teachers learn when participating in lesson study. On the basis of previous research, Lewis (2009) presents a schematic that can be used to address the impact of lesson study on teachers and instruction. The schematic explicitly links lesson study to teacher learning but it does not elucidate the influence of different activities in the lesson study process on teacher learning. This has become an important focus of my research.

By means of a single case study, we investigated a secondary school mathematics teacher's learning outcomes - particularly changes in pedagogical content knowledge (PCK) - and related those to teachers' learning activities within the context of lesson study (Van Smaalen, Verhoef, Yoshida, & Pieters, 2012). This exploratory study shows (a) the importance of live classroom observation for developing knowledge of student learning, and (b) the significance of imagination when creating a (research) lesson.

In a current follow-up study, which started in 2011, we broadened our view from PCK development to professional learning: the development of knowledge, skills, and habits of mind necessary to professional thinking and practice, and (intentions for) changes in practice itself. Fifteen secondary school teachers - including nine mathematics teachers - participate in this study. To gain a deeper understanding of teachers' learning - especially the relation between lesson study activities and teacher learning outcomes - we asked the participants to fill in a learner report (e.g., Van Kesteren, 1993) after each lesson study activity (planning, teaching or observing, and discussing). Besides this, we recorded the teachers' joint reflection at the end of each lesson study cycle. We are currently analyzing the data that has been collected so far (data belonging to six lesson study cycles), describing diversity of learning experiences (i.e. learning outcomes related to lesson study activities) using qualitative survey analysis (Jansen, 2010).

During the CERME, I will present the results of both the single case study and the follow-up study. The poster starts with the conceptual framework (i.e., lesson study and professional learning) and the research questions. Next, the method will be presented, in particular the learner report and qualitative survey analysis. Then the results will be described by picturing the diversity of learning outcomes (e.g., knowledge of student learning) and their relation to lesson study activities (i.e., collaborative planning, teaching, live classroom observation, and post-lesson discussion). The poster ends with the findings of the study (e.g., the importance of live classroom observation) and points for discussion.

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USAGE OF TASKS DURING PROFESSIONAL DEVELOPMENT OF IN-SERVICE MATHEMATICS TEACHERS

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In the poster we describe the searching process of the mathematical tasks and design materials optimal usage for in-service teacher during professional development course. Based on the analysis of needs and interconnected model (Clark & Hollingsworth, 2002) we interpret teachers' usage of tasks and identify their way of learning. The main focus is on the transformation between learning in content knowledge and pedagogical content knowledge (Ball et al., 2008). According to the findings we suggest the new design of the continuing professional development course.

Key words: teachers' learning, CPD, inquiry based tasks

FOCUS OF THE POSTER

Our poster presents some study findings in which we were piloting different approaches of in-service mathematics teachers' usage of student oriented tasks during continuing professional development (CPD) course. The content of CPD was focused on introduction of constructivists' approach to teaching, development of competences and inquiry based learning supported by ICT. According to the usage of tasks we identified several approaches to in-service teachers' learning. Our research question was: "How was teachers' usage of tasks evolving during CPD course and what supported this change?"

THEORETICAL FRAMEWORK

Since 2009, there have been established new CPD courses for in-service teachers. In Slovak context most of the CPD courses were designed in implicit model. In the observed CPD courses we firstly focused on identification of necessary content knowledge (CK) and then we connected it with appropriate pedagogical content knowledge (PCK) (Shulman, 1986, Ball *et al*., 2008). We analysed the trajectory of teachers' learning according to the interconnected model (Clark & Hollingsworth, 2002) and based on the findings we redesigned the model of the new planned CPD course.

METHODS AND DATA ANALYSIS

In our study three groups of lower secondary mathematics teachers 28 teachers in total were observed during the five whole day CPD sessions. One of the researchers was a lecturer of the CPD course. Some of the parts of the courses were videotaped.

Teachers were supposed to create and submit written assignments that contain some part of lesson planning after each session and the final assignment. All the written assignments were analysed. All sessions were transcribed and coded. Based on the data we identified events from CPDs that were in any way “critical” in connections with usage of tasks by the teachers. These events were interpreted within our theoretical framework.

INDICATION AND JUSTIFICATION OF THE CONTENT

The poster consists of introduction, theoretical framework, methodology, results that include examples of tasks and discussion with new design of the CPD. We observed that the settings of CPD and tasks usage in many cases were not sufficient for changes within domain of practice as well as domain of consequence. However, some teachers demonstrate positive results of inquiry tasks development. Based on these findings and theoretical framework form project PRIMAS we present new design of CPD model that more accurately addresses the difficulties that teachers had while implementing the innovative curriculum material into practice.

FORMAT OF THE POSTER

The poster will have a prescribed rectangular format 70 cm x 90 cm and will be printed on one piece of paper. Data will be optically organised and clearly described.

1. This work was supported in part by the EU, within the 7FP project, under grant agreement 244380 "PRIMAS – Promoting Inquiry In Mathematics And Science Education.”

2. The actual poster presented at CERME8 may be obtained from the author by emailing them to <jsunderlik@ukf.sk; sceretkova@ukf.sk>

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PROFESSIONAL MATHEMATICS TEACHER IDENTITY: AN EXAMPLE WITH TELESECUNDARIA SYSTEM TEACHERS IN MEXICO

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The aim of this poster is to discuss about the constitution of professional identity, taking into account that the profesional identity is shaped through the participation in a collaborative learning network. The participants are Mexican teachers that belong to the telesecundary system, who work in rural or marginalized urban zones. The online work promotes a reflection about mathematics tasks that are in textbooks and the way that they have to apply in the contexts from her/his students.

INTRODUCTION

This study focuses on the analysis of the relationship between teachers' identity and teachers' practices. If the social environment is considered a contextual variable that has an impact on how a person acts, how does it come to affect identity formation?

The aim of our work is to identify the "professional mathematics teacher identity" of Mexican teacher of "Telesecundaria system" and create a collaborative learning network, which helps and also increases his professional identity.

The poster will include three sections: first the research problem, second the theoretical framework and finally the methodological advances.

CONCEPTUAL FRAMEWORK

Based on reviewing research into professional identity in relation to teachers and teacher education, Beijaard, Meijer & Verloop (2004) concluded that often lacks a clear definition of professional identity.

The notion of mathematics teacher identity is considered a social construct that describes not only a human group but also determines their actions. It means that the teachers' identity is not only the possession of a defined set of assets required for the profession. For instance, Gee (2001) says: "Being recognized as a certain kind of person, in a given contexts, is what I mean…by identity" (p.99). Identities are not stable but dynamic and situated, emerging in talk in different situations of everyday life. Ponte and Chapman (2008) suggest that this construct, professional mathematics teacher identity, may be seen as the teacher's ''professional self'' or an instance of a social identity. We consider that professional mathematics teacher identity emerges and changes in significant social experiences that define identity indicators. The analysis of the development of this construct allows also to recognize and explain the actions of teachers from their perspective.

METHODOLOGY

The participants in this investigation are teachers that work in an educational program in secondary level called "Telesecundaria". This program offers coverage in urban or rural areas in the secondary level (25% of the population) and students receive instruction in all subjects from a single teacher. Instruction is delivered through three mechanisms: television broadcasts, teachers and texts. Sometimes the teachers do not identify themselves as mathematics teachers. Rather, they must master the disciplinary knowledge of all subjects that are included in the school curriculum.

We will use reflective narratives (Bjuland, Cestari & Borgersen, 2012) to identify the identity of the teachers from four areas: personal history, their initial training in their daily work and professional development spaces in which they have participated as well as from their participation in the collaborative learning network.

Telesecundaria's teachers around Mexico as well as researchers and teacher educators will be those that constitute the collaborative network of teachers. The working methodology is to conduct a joint reflection on the design of learning tasks that are based on their textbooks for using in the classroom. This research encourages teachers to question the use of mathematics concepts, strategies and tools designed to affect the learning of their students.

FINAL DISCUSSION

The analyzed data suggest that there is not unique professional identity of telesecundaria's teachers. The diversity of the initial training of teachers influences how them guide the students' learning. Teachers should make adjustments to the curriculum according to the telesecundaria's context and student difficulties. In our opinion, the study of professional mathematics teacher identity provides a different approach to the professional development of teachers. It implies an approximation of the teacher's reality from their perspective, to explain from "there" their practices.

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INTERACTION SUITABILITY ANALYSIS WITH PROSPECTIVE MATHEMATICS TEACHERS

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We study how the development of four professional tasks focused on the analysis of mathematics class interactions, may influence the reflection and evaluation of own practice done by future secondary mathematics teachers. We describe the tasks and we show its specific impact on these reflections.

KEY WORDS: Interaction suability analysis, Teacher development

INTRODUCTION

The training program for prospective teachers of mathematics at the University of Barcelona is looking to develop various competencies, including competence didactic analysis. Our study is assumed as a teaching experiment in which design, plan, implement and evaluate a set of professional tasks. Four professional tasks focused on the analysis of the interactions in the mathematics classroom will be presented, by using case studies, with theoretical underpinnings. It is intended that future teachers develop tools for critical thinking and research attitudes. In task 1, future teachers must review the process of solving a geometrical problem, developed by a couple of students aged 16, with reference to the analysis of interactions proposed by Cobo & Fortuny (2010). In the second, was asked to analyze an episode of one class of measurement developed with students of 12-13 years, with reference to another episode analyzed using the proposal from Vanegas, Font y Giménez (2009). The third task is the analysis of a debate in a class of beginning algebra, having as a reference the proposal given by Vanegas, Giménez & Font (2012), by characterizing democratic mathematical practices. In the fourth task, we did a joint reflection among future teachers and trainers, on the notion of quality of interaction analysis, by using an ontosemiotic approach (OSA).

To analyze how the proposed tasks have influenced the thinking process of the practice of future teachers, we identify written productions, class discussions, and particularly in the Master Thesis (MT): a)if it appears a collaborative work and reasoning, b) if teaching purposes are clearly described, c) the type of didactical configuration is identified, d) the importance of recognizing the teacher's role in generating teaching situations that allow the development of mathematical democratic practices is revised and it appear some criteria used to assess the quality of their practice.

DISCUSSION AND CONCLUSIONS

All students interested in the analysis of the interaction in the reflection of their own practice, as an important aspect of class management analysis. The first reflections immediately after the school practice, are characterized by superficial and descriptive comments focused on the action, with little justification argument.

> "… in the activity with more implication, in which the students participate more actively was a "role-play", in which students are points. The teacher says an inequality with two unknown, then the students raise…" (Student A)

When analyzing MT productions, value judgments prevail. 88% of prospective teachers recognize different performances which has enabled or not promoted the interaction. Only 24% identify concrete situations of their own practice as justification. 16% of the students allude to the quality of the interaction to reflect on how to improve aspects of their own practice. The following example refers to both the proposed mathematical democratic practices (Task 3) as a means of intervention raised in Task 2.

> "This is to allow more time for students to think about how to get to the solution of problems... On the other hand, it has also changed the role of the teacher giving students proper tracks looking to reflect rather than closer to the procedure or solution … The role of the teacher can encourage students' reasoning or else cut their reflections. So the teacher has to act as a guide but let the student take responsibility at all times" (Student B)

Only one student proposes a creative graph to represent the evaluation of different aspects of their practice, using some of the criteria proposed by the EOS and noting what was their level of achievement in different practice sessions.

Finally, we recognize that the use of different theoretical framework for analyzing the interaction allowed some improvements in the reflection of the actual practice of future teachers. Meanwhile, they were considered more variables to characterize the interaction, were incorporated in certain cases the categories proposed by the authors studied, and it is reflected on the importance of analyzing the development of these processes in math class. We suggest that more time is needed to have more influence, and the need to have a common discussion about own reflection.

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