The multi-dimensional Morse-Smale index

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Outline

1. The multidimensional Morse–Smale index problem
   - Joint program with B. Ørsted, A. Portaluri, N. Waterstraat, C. Zhu
   - Achievements by S. Smale, J. Simons, J. Deng and C. Jones
2. Our view to classical Morse theory
   - The Yiming Long program
   - From J.C.F. Sturm to M. Morse, W. Ambrose, H.M. Edwards, J.J. Duistermaat, R.C. Swanson, C. Zhu
3. What new can we contribute?
   - Standard model in strong symplectic Hilbert space
   - Analysis in weak symplectic Banach spaces
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The multidimensional Morse–Smale index problem

Data: M compact orientable smooth Riemannian manifold,
$\dim M = m \in \mathbb{N}$, $\partial M = \Sigma$ (classical $m = 1$)
$\pi\colon \mathcal{E} \to M$ Hermitian vector bundle of fiber dimension d
$\mathcal{I}(x,y) := \mathcal{I}_i(x,y) + \mathcal{I}_b(x|_\Sigma, y|_\Sigma)$, symmetric bilinear form on $\Gamma(\mathcal{E})$

Definition

(**a**) $\mathcal{I}$ m-dimensional *Morse-Smale index form* of order $k \geq 1 :\Longleftrightarrow$
(i) $\mathcal{I}_i(x,y) =_{\text{locally}} \int_M \sum_{|J|,|J'|\leq k} \langle a_{J,J'} \partial^{|J|} x/\partial\xi^J \,,\, \partial^{|J'|} y/\partial\xi^{J'} \rangle$ for
$\operatorname{supp}(x),\ \operatorname{supp}(y) \subset U \subset M$ coordinate patch,
$a_{J,J'} \in C(M, \mathrm{gl}(d,\mathbb{C}))$, J, J' multi-indices of length m with $|J|, |J'| \leq k$,
$\bigoplus$ suitable assumptions (differentiability, symmetry).
(ii) $\mathcal{I}_b$ symmetric bilinear form on

$$V := \ker P \subset H^{k-1/2}(\Sigma; \mathcal{E}|_\Sigma),\ P \text{ pseudodiff. projection.}$$

(**b**) *Morse index* $\mu^-(\mathcal{I}) := \max\{\dim Z \mid \mathcal{I} \text{ neg. definite on } Z \subset \Gamma(\mathcal{E})\}$.
Finite only if $\forall_{|J|,|J'|=k}\ a_{J,J'}$ positive definite.

Problem: $\{\mathcal{I}_s\}_{0\leq s\leq 1}$ curve of index forms $\Longrightarrow \mu^-(\mathcal{I}_1) - \mu^-(\mathcal{I}_0) = ?$

1. Precise description of the multidimensional index forms.
2. How do they arise from variational problems $\{c\colon M \to W\}$ with energy type functional and W Riemannian or semi-Riemannian?
3. Investigate $\mathcal{I} \mapsto A \in \mathrm{Diff}^{2k,\mathrm{sa}}$ s.t. $\forall_{x,y\in\Gamma(\mathcal{E})}\, \mathcal{I}(x,y) = \int_M \langle Ax, y\rangle$, subjected suitable b.c.
4. Deformation properties of spectral invariants of s.a. extensions of ell. op's of order $2k$.
5. Prove for general M (not necessarily star-shaped domain $\subset \mathbb{R}^m$)

$$\mu^-(\mathcal{I}_1) - \mu^-(\mathcal{I}_0) \overset{!?}{=} -\mathrm{SF}\{A_s, V_s\} \overset{BBB,CZ\,2014}{=} \mathrm{MAS}\{C_s, V_s\},$$

   where $\{A_s\}, \{V_s\}, \{C_s\}$ associated curves of symmetric elliptic differential operators, self-adjoint boundary conditions, and Cauchy data spaces.
6. New case studies for $k = 1$ and $m = 1, 2$, and for symmetric spaces.

The achievements by Smale, Simons, Deng and Jones

S. Smale, 1965: k, d, m arbitrary; $\mathcal{I}(x,y) := \int_M \langle Ax, y\rangle$;
A sym. strongly ell. op., order $2k$, weakUCP; b.c.: generalized Dirichlet
$\Longrightarrow \quad \exists_{\varepsilon>0} \forall_{\{g_s\}_{0\le s\le 1}\ \varepsilon-\text{contraction}} \quad \mu^-(\mathcal{I}) = \sum_{0<s\le 1} \alpha(s) < \infty$.

Definition (Smale's shrinking)

(**a**) $\{g_s\}_{0\le s\le 1}$ smooth ε-type *contraction* of $M :\iff$
(i) $\{g_s\}$ 1-parameter family of diffeomorphisms of M into itself with $g_0 = \text{id}$ and $(p,s) \mapsto g_s(p) \in C^\infty$;
(ii) $M_s := \operatorname{im} g_s \supsetneqq \operatorname{im} g_{s'} =: M_{s'}$ for $s < s'$, and $\operatorname{vol}(M_1) < \varepsilon$.
(**b**) $\alpha(s) :=$ dimension of generalized Dirichlet solutions over M_s.
(**c**) s (or rather $\Sigma_s := \partial M_s$) *conjugate* $:\iff \alpha(s) > 0$.

J. Simons, 1968: M minimal variety immersed in $\widetilde{M}$ with $\dim \widetilde{M} = n$; i.e., $k = 1, \mathcal{E} := \mathcal{N}(M), d = n - m$, A Jacobi op., b.c. Dirichlet. So $\mu^-(M) := \mu^-(\mathcal{I})$ calculable for all minimal surfaces, and closed minimal varieties in spheres S^n.
J. Deng, C. Jones, 2011: $M \subset \mathbb{R}^m$ star shaped; $k = 1, d = 1$, wider class of b.c. Gaps in the interesting proof can be repaired.

Numerical simulation: triumph and failure

Critical points analysis: $\ell : H^1(\Lambda(M)) \to \mathbb{R}$ length or energy

- 1905 H. Poincaré: On any convex surface there exists a simple closed geodesic (after Jacobi and Hadamard).
- 1951 L.A. Lyusternik, A.I. Fet: On any closed Riemannian manifold there exist a closed geodesic.
- 1951 J.P. Serre: On any closed Riemannian manifold M with $p, q \in M$ and $p \neq q$ there exist infinitely many disjoint geodesics from p to q.
- 1969 D. Gromoll, W. Meyer: $H^*(M; \mathbb{R})$ sufficiently complicated $\Rightarrow$ #(prime closed geodesics) = ∞.
- 1992 J. Franks: On any surface of genus 0 there exist infinitely many geometrically distinct closed geodesics.
- 2009/10 H. Duan, Y. Long: On some compact simply connected Riemannian manifolds (like S^3, S^4) there exist at least two geometrically distinct closed geodesics.

The 1D set-up by C. Zhu

After W. Ambrose (1961), H. M. Edwards (1964), J. J. Duistermaat (1976), and R. C. Swanson (1978): Morse index $=$ #(eigenvalues) $=$ SF $=$ MAS $=$ #(conjugate points)

Data: $m = k = 1$, $c\colon [0,1] \to W$ geodesic, W Riemannian (or semi-R), $\dim W = d$, $E(c) := \int_0^1 f(t, c(t), \dot{c}(t))dt$. Study $D^2E(c)$ via

$$\mathcal{I}_i(x,y) := \int_0^1 \left(\langle p\frac{d}{dt}x\,,\,\frac{d}{dt}y\rangle + \langle qx\,,\,\frac{d}{dt}y\rangle + \langle q^*\frac{d}{dt}x\,,\,y\rangle + \langle rx, y\rangle\right) dt,$$

$$\mathcal{I}_b((x(0),x(1)),(y(0),y(1)) := Q((x(0),x(1)),(y(0),y(1)),$$

$\mathcal{E} := c^*(TW) \otimes \mathbb{C}$, $p,q,r \in C^0([0,1], \mathrm{gl}(d,\mathbb{C}))$, $p \in C^1$, $\begin{pmatrix} p & q \\ q^* & r \end{pmatrix}$ Hermitian;

b.c.: $(x(0),x(1)),(y(0),y(1)) \in V \subset \mathbb{C}^{2d}$, $Q : V \times V \to \mathbb{C}$.

$\mathcal{I}_i \mapsto A := -\frac{d}{dt}(p\frac{d}{dt} + q) + q^*\frac{d}{dt} + r$, domain $\mathcal{D}$ defined by (V, Q)

Results: $A_{\mathcal{D}}$ closed unbounded self-adjoint Fredholm operator.

Canonical lower order variation $\Rightarrow \mathrm{SF}\{A_{s,\mathcal{D}_s}\} = \mu^-(\mathcal{I}_i + \mathcal{I}_b)$.

Symplectic analysis $\Rightarrow \mathrm{SF}\{A_{s,\mathcal{D}_s}\} = \mathrm{MAS}\{\mathrm{Graph}\big(\Phi(A_s)(1)\big), \mathrm{Lag}(sQ)\}$ in $\mathbb{C}^{4d}$ canon. symplectic.

Standard model, I

(X,ω) strong symplectic complex Hilbert space, $\omega(x,y) = \langle Jx, y\rangle$ with $J^* = -J$ and $J^2 = -I$. $X^\pm := \ker(J \mp iI)$ *symplectic splitting*

Theorem

(i) $\forall\lambda \in \mathcal{L}(X,\omega)\ \exists U\colon X^+ \to X^-$ *unitary with* $\lambda = \mathrm{Graph}(U)$.
(ii) $(\lambda,\mu) \in \mathcal{F}^2\mathcal{L}(X,\omega) \iff UV^{-1} - I_{X^-} \in \mathcal{F}$.
(iii) $\mathrm{MAS}(\lambda(s),\mu(s))_{s\in[0,1]} := \mathrm{SF}_{(0,\infty)}\left(U_s V_s^{-1}\right)_{s\in[0,1]}$ *well defined.*
(iv) $\forall_{k\in\mathbb{N}}\exists\,\varphi_k\colon \pi_{2k-1}(\mathcal{F}^2\mathcal{L}(X,\omega)) \xrightarrow{\cong} K^1(S^{2k-1}) \cong \mathbb{Z}$ *with* $\varphi_1 = \mathrm{MAS}$.

EXAMPLE:
$\mathcal{H}$ complex separable Hilbert space, A closed symmetric operator

Definition

(i) $\beta_A := \mathrm{dom}(A^*)/\,\mathrm{dom}(A)$ strong symplectic Hilbert space with $\gamma\colon\ \mathrm{dom}(A^*) \to \beta_A$, $\langle\gamma(x),\gamma(y)\rangle := \dots$, $\omega(\gamma(x),\gamma(y)) := \langle A^*x, y\rangle - \langle x, A^*y\rangle$.
(ii) $\mathrm{CD}(A) := \{\gamma(x) \mid x \in \ker A^*\}$ Cauchy data space

Standard model II: strength and limitations

Theorem (von Neumann, 1930, via Krein-Vishik-Birman, to I. Gelfand)

A_D self-adjoint extension $\iff$ $[D] \subset \beta_A$ Lagrangian.

Theorem (BBB& Wojciechowski, 1986)

M compact smooth mf. with $\partial M = \Sigma$, A linear symmetric elliptic differential operator $\Longrightarrow \mathrm{CD}(A)$ Lagrangian (in $L^2(\Sigma), \beta_A, \dots$).

Theorem (BBB& Furutani, 1998)

Let A_D self-adjoint Fredholm extension, $\{C_s\}_{s\in[0,1]}$ C^0 curve in $\mathcal{B}(\mathcal{H})$, and $\ker(A^* + C_s + \varepsilon) \cap \mathrm{dom}(A) = \{0\}$ (weak inner UCP), then
(i) $\{\mathrm{CD}(A + C_s), [D]\}$ C^0 curve of Fredholm pairs of Lagrangians and
(ii) $\mathrm{SF}\{(A + C_s)_D\}_{s\in[0,1]} = \mathrm{MAS}\{\mathrm{CD}(A + C_s), [D]\}_{s\in[0,1]}$.

WARNING: strength and limitations

How weak symplectic Banach spaces arise

Data: M cpt. smooth mf. with $\partial M =: \Sigma$

$A(s)\colon C_0^\infty(M;E) \to C_0^\infty(M;E)$, $s \in [0,1]$ curve of symm. ell. (first) order diff. ops., $D(s) = \ker(P(s)\circ\gamma)$ curve of well-posed b.c.

$\Rightarrow \{A(s)_{D(s)}\}$ curve of closed unbounded sa. Fredholm ops.

What fixed? $\begin{cases} \beta_{A(s)}\,, & \text{not controllable} \\ H^1(M;E) \text{ and } H^{1/2}(\Sigma;E|_\Sigma) \cong H^1(M;E)/H_0^1(M;E), \\ \text{yes, but weak symplectic!} \end{cases}$

On $L^2(\Sigma;E|_\Sigma)$ strong $\omega(s)_{\text{Green}}(x,y) := -\langle J(s)x,y\rangle_{L^2}$

On $H^{1/2}(\Sigma;E|_\Sigma)$ induced weak $\omega(s)(x,y) := \omega(s)_{\text{Green}}(x,y)$ $= -\langle J'(s)x,y\rangle_{H^{1/2}}$ with compact $J'(s) = (I+|B|)^{-1/2}J(s)$, B formally self-adjoint elliptic of first order on Σ

Challenges:

- $J'(s)^2 \neq -I$, so $H^{1/2} \neq \ker(J'(s)-iI) \oplus \ker(J'(s)+iI)$;
- Continuous variation of $\mathrm{CD}(A(s))$?
- How to define $\mathrm{MAS}\{\mathrm{CD}(A(s)),\gamma(D(s))\}$?
- What spectral flow formula?

The counterexamples to keep in mind

Examples (blocking direct generalizations *strong* $\to$ *weak*)

1. No symplectic splitting: Let $(X,\omega) := \lambda \oplus \lambda^*$ and $\lambda := \ell^p$ ($p \in (1,+\infty) \setminus \{2\}$). Then X is a strong symplectic Banach space, but there is no splitting $X = X^+ \oplus X^-$ such that $\mp i\omega|_{X^\pm} > 0$, and $\omega(x,y) = 0$ for all $x \in X^+$ and $y \in X^-$.
2. Annihilator not always involutive on closed subspaces: Let (X,ω) be a weak symplectic Hilbert space and $\omega(x,y) = \langle Jx, y\rangle$. Let V be a proper closed linear subspace of X such that $V^\perp \cap JX = \{0\}$. Then $V^\omega = J^{-1}V^\perp = \{0\}$ and $V^{\omega\omega} = X \neq V$.
3. Fredholm pair of Lagrangians with negative index: Let X be a complex Hilbert space and $X = X_1 \oplus X_2 \oplus X_3$ an orthogonal decomposition with $\dim X_1 = n \in \mathbb{N}$ and $X_2 \simeq X_3$. Then we can find a skew-self-adjoint injective, but not surjective J such that $\omega(x,y) = \langle Jx, y\rangle$ becomes a weak symplectic form on X and $\lambda_\pm = \{(\alpha, \pm\alpha); \alpha \in X_2\}$ becomes a pair of complementary Lagrangian subspaces of $X_2 \oplus X_3$ by identifying X_2 and X_3, and, in fact, a pair of Lagrangians of X with $\mathrm{ind}(\lambda_+, \lambda_-) = -n$.

A naturally looking General Spectral Flow Formula

Proposition (BBB, M. Lesch, C. Zhu, 2011)

Data: $A(s)_{s\in[0,1]}$ *curve of ell. diff. ops., order k*
$A_m(s;\sigma)\colon D_m(s;\sigma) := H_0^{\sigma+k}(M;E) \longrightarrow H^\sigma(M;E)$
$A_M(s;\sigma)\colon D_M(s;\sigma) := H^{\sigma+k}(M;E) \longrightarrow H^\sigma(M;E)$
$Q(s;\sigma+\frac{1}{2})\colon D_M(s;\sigma)/D_m(s;\sigma) \longrightarrow \gamma(\ker A_M(s;\sigma))$ *Calderón proj.*
$\dim\ker A_m(s;0)=\ell, \quad \dim\ker A_m^t(s;0)=\ell'$ *constant,*
$\Longrightarrow \{\operatorname{im} Q(s;\sigma)\}$ *continuous for* $\sigma \geq 1/2-k$.

Theorem (BBB, C. Zhu, February 2015, 130 pages)

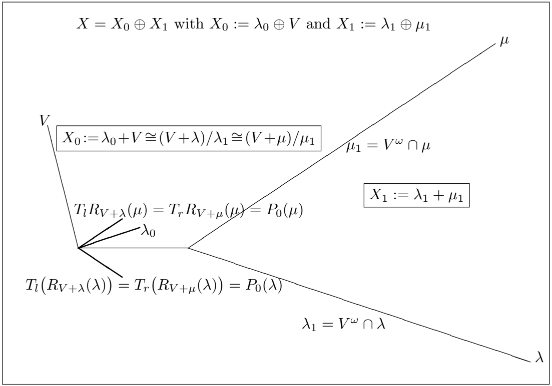
$\mathrm{SF}\{A(s)_{D(s)}\}_{0\le s\le 1} \;=\; \mathrm{MAS}\{\mathrm{CD}(A(s)),\gamma(D(s))\}_{0\le s\le 1}$ *admitting smooth variation of symmetric operator* $A(s)$*, continuous variation of Fredholm domain* $D(s)=\ker(P_s\circ\gamma)$*,* $P(s)\in \mathrm{Grass}_{\mathrm{sa}}(A(s))$*, constant ghosts' dimensions.*

NOTE 1: L^2 formula requires continuity of domain both in L^2 and $H^{1/2}$.
H^σ formula requires continuity of domain only in H^σ for $\sigma>0$.
NOTE 2: Hörmander index (correction for replacement of path in $\mathcal{F}^2\mathcal{L}$)

Weak symplectic geometry - Key lemmata I

(X,ω) symplectic vector space, $(\lambda,\mu)\in\mathcal{F}_0^2\mathcal{L}(X,\omega) \implies$

1) Natural decomposition $X = X_0\oplus X_1$, $X_0 := V+\lambda_0$, $\dim X_0<\infty$, $W := V+\lambda$ closed co-isotropic; admits sympl. reduction $R_W(\lambda) := \big((\lambda+W^\omega)\cap W\big)/W^\omega \subset W/W^\omega$

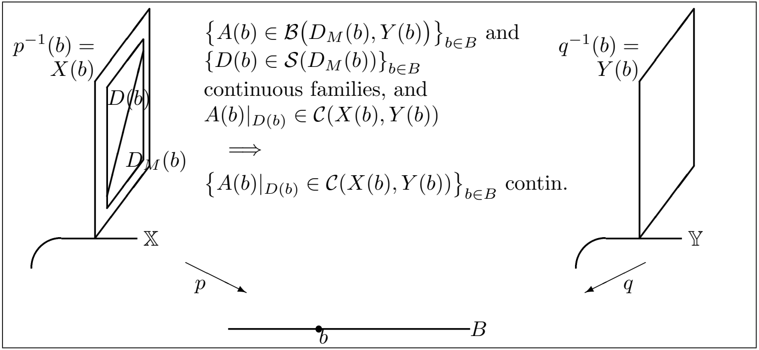


2) Challenge: $\exists W$? and parametrization $W(s)/W(s)^{\omega(s)}$ bundle?

Weak symplectic geometry - Key lemmata II

3) X Banach space, B top. space, $M, N\colon B \to \mathcal{S}(X)$ continuous. Then $M + N$ continuous $\Leftrightarrow$ $M \cap N$ continuous (following NEUBAUER 1968).

4) Following NICOLAESCU, LESCH: B topological space, $p\colon \mathbb{X} \to B$, $p_1 : \mathbb{X}_1 \subset \mathbb{X} \to B$, $q\colon \mathbb{Y} \to B$ Banach bundles



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