# Algebra in Cuneiform

Introduction to an Old Babylonian Geometrical Technique

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**Algebra in Cuneiform**

Introduction to an Old Babylonian Geometrical Technique

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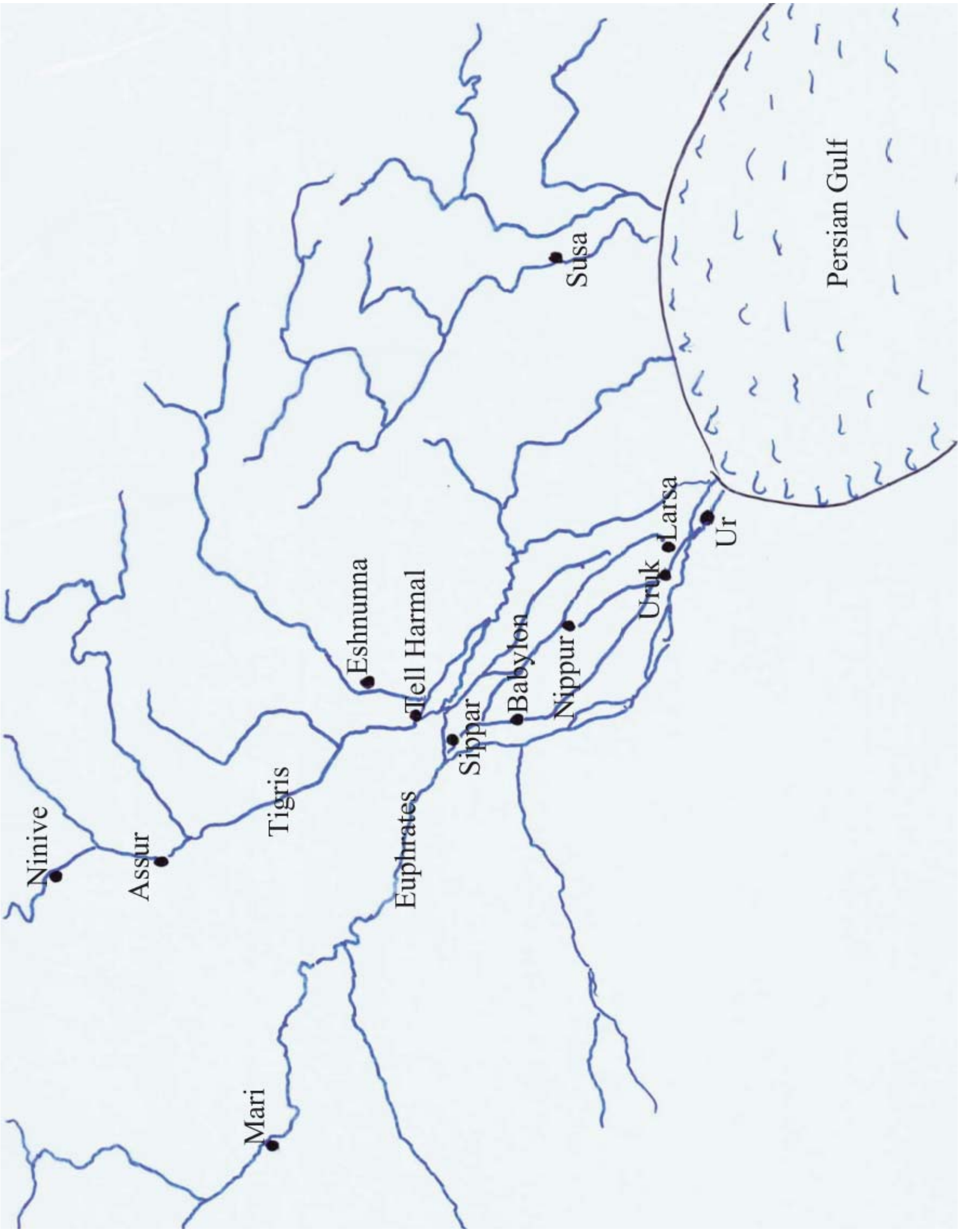
In memory of Peter Damerow



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**Boxes**

# Preface

This book presents an important aspect of Babylonian mathematics, namely the technique or discipline usually known as “Babylonian algebra”. This “algebra” is the earliest example of advanced mathematics that has come down to us, for which reason it is spoken of in most general expositions of the history of mathematics. However, most of these expositions rely on translations and interpretations going back to the 1930s. The present book, in contrast, builds on recent research.

The traditional interpretation made it possible to establish a list of the results obtained by the Babylonians; of the calculations they were able to perform; and, so to speak, of the formulas they knew. But since its starting point was contemporary mathematical thought it was not able to reconstruct the *different* thinking which hides behind the Babylonian results. The aim of the present book is to highlight that difference, and thus to show that *mathematics can be thought in several ways*.

A first version of the book was written for students of the Danish high school system in 1998; another version – revised and augmented – appeared in French in 2010. This, as well as the present further updated version, addresses itself to those who are interested in the history of mathematics without possessing necessarily mathematical competence beyond what is acquired in high school. Teachers may use it together with their students at various levels.

A first approach (in teaching as well as private study) may concentrate on the first-degree equation TMS XVI #1, and the basic second-degree equations, that is, BM 13901 #1 and #2, YBC 6967 and TMS IX #1 and #2. The Introduction and Chapters 5 to 7 provide a general overview.

In order to get deeper into the matter one may read the other texts from Chapters 1 and 2, and the texts TMS IX #3, AO 8862 #2, BM 13901 #23 and YBC 6504 #4 from Chapter 3.

Those who become passionate may read all the texts from Chapters 1 to 4, and then try their teeth on the texts from Appendix A.

In Appendix B, those who know the rudiments (or more) of Babylonian language and grammar will find transliterations of most of the texts from Chapters 1 to 4 and Appendix A.

# Introduction

## The issue – and some necessary tools

### *“Useless mathematics”*

At some moment in the late 1970s, the Danish Union of Mathematics Teachers for the pre-high-school level asked its members a delicate question: to find an application of second-degree equations that fell inside the horizon of their students.

One member did find such an application: the relation between duration and counter numbers on a compact cassette reader (thus an application that at best the parents of today’s students will remember!). That was the only answer.

Many students will certainly be astonished to discover that even their teachers do not know why second-degree equations are solved. Students as well as teachers will be no less surprised that such equations are taught since 1800 BCE without any possible external reference point for the students – actually for the first 2500 years without reference to possible applications *at all* (only around 700 CE did Persian and Arabic astronomers *possibly* start using them in trigonometric computation).

We shall return to the question why one taught, and still teaches, second-degree equations. But first we shall have a look at how the earliest second-degree equations, a few first-degree equations and a single cubic equation looked, and examine the way they were solved. We shall need to keep in mind that even though some of the problems from which they are derived look practical (they may refer to mercantile questions, to fortification ramps and to the division of fields), then the mathematical substance is always “pure”, that is, deprived of any immediate application outside mathematics itself.

### Rudiments of general history

Mesopotamia ("Land between the rivers") designates since Antiquity the region around the two great rivers Euphrates and Tigris – grossly, contemporary Iraq. Around 3500 BCE, the water level in the Persian Gulf had fallen enough to allow large-scale irrigation agriculture in the southern part of the region, and soon the earliest "civilization" arose, that is, a society centred on towns and organized as a state. The core around which this state took shape was constituted by the great temples and their clergy, and for use in their accounting this clergy invented the earliest script (see the box "Cuneiform writing", page 4).

The earliest script was purely ideographic (a bit like modern mathematical symbolism, where an expression like $E = mc^2$ can be explained and even pronounced in any language but does not allow us to decide in which language Einstein thought). During the first half of the third millennium, however, phonetic and grammatical complements were introduced, and around 2700 BCE the language is unmistakeably Sumerian. From then on, and until c. 2350, the area was divided into a dozen city-states, often at war with each other for water resources. For this reason, the structure of the state was transformed, and the war leader ("king") displaced the temples as the centre of power. From around 2600 a professional specialization emerges, due to wider application of writing. Accounting was no longer the chore of the high officials of temple and king: a new *profession*, the scribes, taught in school, took care of that.

Around 2340, an Akkadian conqueror subdued the whole of Mesopotamia (Akkadian is a Semitic language, from the same language family as Arabic and Hebrew, and it had been amply present in the region at least since 2600). The Akkadian regional state lasted until c. 2200, after which followed a century of competing city states. Around 2100, the city-state of Ur made itself the centre of a new centralized regional state, whose official language was still Sumerian (even though most of the population, including the kings, probably spoke Akkadian). This "neo-Sumerian" state (known as Ur III) was highly bureaucratized (perhaps more than any other state in history before the arrival of electronic computers), and it seems that the place-value number notation was created in response to the demand of the bureaucracy for convenient calculational instruments (see the box "The sexagesimal place-value system", page 7).

In the long run, the bureaucracy was too costly, and around 2000 a new phase of smaller states begins. After another two centuries a new phase of centralization centred around the city of Babylon sets in – from which moment it is meaningful to speak of southern and central Mesopotamia as "Babylonia". By now (but possibly since centuries),

**Rudiments of general history**

Sumerian was definitively dead, and Akkadian had become the principal language – in the south and centre the Babylonian and in the north the Assyrian dialect. None the less, Sumerian survived in the environment of learned scribes – a bit like Latin in Europe – as long as cuneiform writing itself, that is, until the first century CE.

The phase from 2000 until the definitive collapse of the Babylonian central state around 1600 is known as the "Old Babylonian" epoch. All texts analyzed in the following are from its second half, 1800 to 1600 BCE.

## *The first algebra, and the first interpretation*

Before speaking about algebra, on should in principle known what is meant by that word. For the moment, however, we shall leave aside this question; we shall return to it in the end of the book; all we need to know for the moment is that algebra has to do with equations.

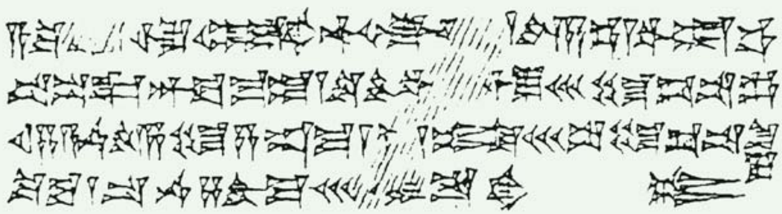


*Figure 1*. The cuneiform version of the problem BM 13901 #1

Indeed, when historians of mathematics discovered in the late 1920s that certain cuneiform texts (see the box "Cuneiform writing", page 4) contain "algebraic" problems, they believed everybody knew the meaning of the word.

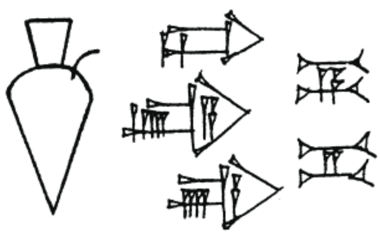
Let us accept it in order to enter their thinking, and let us look at a very simple example extracted from a text written during the 18th century BCE in the transliteration normally used by Assyriologists – as to the function of *italics* and SMALL CAPS, see page 20 and the box "Cuneiform writing", page 4 (Figure 1 shows the cuneiform version of the text):

**Cuneiform writing**

From its first beginning, Mesopotamian writing was made on a flattened piece of clay, which was then dried in the air after the inscription (a "tablet"). In the fourth millennium, the signs were drawings made by means of a pointed stylus, mostly drawings of recognizable objects representing simple concepts. Complex concepts could be expressed through combination of the signs; a head and a bowl containing the daily ration of a worker meant "allocation of grain" (and later "to eat"). The signs for numbers and measures, however, were made by vertical or oblique impression of a cylindrical stylus.

With time, the character of the script changed in two ways. Firstly, instead of tracing signs consisting of curved lines one impressed them with a stylus with sharp edges, dissolving the curved lines into a sequence of straight segments. In this way, the signs seem to be composed of small wedges (whence the name "cuneiform").

In the second half of the third millennium, numerical and metrological signs came to be written in the same way. The signs became increasingly stylized, loosing their pictographic quality; it is then not possible to guess the underlying drawing unless one knows the historical development behind the sign. Until around 2000 BCE, however, the variations of characters from one scribe to another show that the scribes knew the original drawings. Let us for instance look at the character which initially depicted a vase with a spout (left). In the middle we see three third-millennium variants of the same character (because the script was rotated 90 degrees to the left in the second millennium, it is habitual to show the third-millennium script in the same way). If you know the origin, it is still easy to recognize the underlying picture. To the right we



**Cuneiform writing**

see two Old Babylonian variants; here the picture is no longer suggested.

The other change concerns the use of the way the signs were used (which implies that we should better speak of them as "characters"). The Sumerian word for the vase is DUG. As various literary genres developed alongside accounting (for instance, royal inscriptions, contracts and proverb collections), the scribes needed ways to write syllables that serve to indicate grammatical declinations or proper nouns. This syllabic system served also in the writing of Akkadian. For this purpose, signs were used according to their approximate phonetic value; the "vase" may thus stand for the syllables *dug*, *duk*, *tug* and *tuk*. In Babylonian writing, the Sumerian sign might also serve as a "logogram" or "word sign" for a word meaning the same as DUG – namely *karpatum*.

Words to be read as logograms or in Sumerian are transliterated in SMALL CAPS; specialists (cf. Appendix B) often distinguish Sumerian words whose phonetic value is supposed to be known, which are then written in s p a c e d  w r i t i n g, from those rendered by their "sign name" (corresponding to a *possible* reading), which are written as SMALL CAPS. Phonetic Akkadian writing is transcribed as *italics*.

Assyriologists distinguish "transcriptions" from "transliterations". A "transcription" is an intended translation into Akkadian written in Latin alphabet. In a "transliteration" each cuneiform character is rendered separately according to its presumed phonetic or logographic value.

**1.** A.ŠÀ$^{l[am]}$ *ù mi-it-ḫar-ti ak-m*[*ur-m*]*a* 45-E 1 *wa-ṣi-tam*
**2.** *ta-ša-ka-an ba-ma-at* 1 *te-ḫe-pe* [3]0 *ù* 30 *tu-uš-ta-kal*
**3.** 15 *a-na* 45 *tu-ṣa-ab-ma* 1-⌊E] 1 ÍB.SI$_8$ 30 *ša tu-uš-ta-ki-lu*
**4.** *lìb-ba* 1 *ta-na-sà-aḫ-ma* 30 *mi-it-ḫar-tum*

The unprepared reader, finding this complicated, should know that for the pioneers it was almost as complicated. Eighty years later we understand the technical terminology of Old Babylonian mathematical texts; but in 1928 it had not yet been deciphered, and the numbers contained in the texts had to provide the starting point.[1]

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[1] However, around 1930 one had to begin with texts that were much more complex than the one we consider here, which was only discovered in 1936. But the principles were the same.

The most important contributions in the early years were due to Otto

It was already known that these numbers were written in a place-value system with base 60 but without indication of absolute order of magnitude (see the box "The sexagesimal system", page 7). We must suppose that the numbers appearing in the text are connected, and that they are of at least approximately the same order of magnitude (we remember that "1" may mean one as well as 60 or $\frac{1}{60}$). Let us therefore try to interpret these numbers in the following order:

$45'$ ($=\frac{3}{4}$) — $1°$ — $1°$ — $30'$ ($=\frac{1}{2}$) — $30'$ — $15'$ ($=\frac{1}{4}$) —
$45'$ — $1°$ — $1°$ — $30'$ — $1°$ — $30'$.

In order to make the next step one needs some fantasy. Noticing that $30'$ is $\frac{1}{2}\cdot 1$ and $15' = (30')^2$ we may think of the equation

$$x^2+1\cdot x = \frac{3}{4} .$$

Today we solve it in these steps (neglecting negative numbers, a modern invention):

$$\begin{aligned} x^2+1\cdot x = \tfrac{3}{4} &\Leftrightarrow x^2+1\cdot x+(\tfrac{1}{2})^2 = \tfrac{3}{4}+(\tfrac{1}{2})^2 \\ &\Leftrightarrow x^2+1\cdot x+(\tfrac{1}{2})^2 = \tfrac{3}{4}+\tfrac{1}{4} = 1 \\ &\Leftrightarrow (x+\tfrac{1}{2})^2 = 1 \\ &\Leftrightarrow x+\tfrac{1}{2} = \sqrt{1} = 1 \\ &\Leftrightarrow x = 1-\tfrac{1}{2} = \tfrac{1}{2} . \end{aligned}$$

As we see, the method is based on addition, to both sides of the equation, of the square on half the coefficient of the first-degree term ($x$) – here $(\frac{1}{2})^2$. That allows us to rewrite the left-hand side as the square on a binomial:

$$x^2+1\cdot x+(\tfrac{1}{2})^2 = x^2+2\cdot\tfrac{1}{2}\cdot x+(\tfrac{1}{2})^2 = (x+\tfrac{1}{2})^2 .$$

This small trick is called a "quadratic completion".

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Neugebauer, historian of ancient mathematics and astronomy, and the Assyriologist François Thureau-Dangin.

### The sexagesimal place-value system

The Old Babylonian mathematical texts make use of a place-value number system with base 60 with no indication of a "sexagesimal point". In our notation, which also employs place value, the digit "1" may certainly represent the number 1, but also the numbers 10, 100, ..., as well as 0.1, 0.01, ... . Its value is determined by its distance from the decimal point.

Similarly, "45" written by a Babylonian scribe may mean 45; but it may also stand for $\frac{45}{60}$ (thus $\frac{3}{4}$); for $45\cdot60$; etc. No decimal point determines its "true" value. The system corresponds to the slide rule of which engineers made use before the arrival of the electronic pocket calculator. This device also had no decimal point, and thus did not indicate the absolute order of magnitude. In order to know whether a specific construction would ask for 3,5 $m^3$, 35 $m^3$ or 350 $m^3$ of concrete, the engineer had recourse to mental calculation.

For writing numbers between 1 and 59, the Babylonians made use of a vertical wedge (𒁹) repeated until 9 times in fixed patterns for the numbers 1 to 9, and of a *Winkelhaken* (a German loanword originally meaning "angular hook") (𒌋) repeated until 5 times for the numbers 10, 20, ..., 50.

A modern reader is not accustomed to reading numbers with undetermined order of magnitude. In translations of Babylonian mathematical texts it is therefore customary to indicate the order of magnitude that has to be attributed to numbers. Several methods to do that are in use. In the present work we shall employ a generalization of the degree-minute-second notation. If 𒌋𒐊 means $\frac{15}{60}$, we shall transcribe it 15´, if it corresponds to $\frac{15}{60\cdot60}$, we shall write 15˝. If it represents $15\cdot60$, we write 15\`, etc. If it stands for 15, we write 15 or, if that is needed in order to avoid misunderstandings, 15°. 𒌋𒐊 understood as $10+5\cdot60^{-1}$ will thus be transcribed 10°5´

| | |
|---|---|
| 𒌍 | understood as 30´ thus means $\frac{1}{2}$. |
| 𒐏𒐊 | understood as 45´ means $\frac{3}{4}$. |
| 𒌋𒐖 | understood as 12´ means $\frac{1}{5}$; understood as 12\` it means 720. |
| 𒌋 | understood as 10´ means $\frac{1}{6}$. |
| 𒌋𒐋𒐏 | may mean 16\`40 = 1000 or 16°40´ = $16\frac{2}{3}$, etc. |
| 𒁹𒐏 | may mean 1\`40 = 100, 1°40´ = $1\frac{2}{3}$, 1´40˝ = $\frac{1}{36}$, etc. |

Outside school, the Babylonians employed the place-value system exclusively for intermediate calculations (exactly as an engineer used the slide rule fifty years ago). When a result was to be inserted into a contract or an account, they could obviously not allow themselves to be ambiguous; other notations allowed them to express the precise number they intended.

Comparing the ancient texts and the modern solution we notice that the same numbers occur in almost the same order. The same holds for many other texts. In the early 1930s historians of mathematics thus became convinced that between 1800 and 1600 BCE the Babylonian scribes knew something very similar to our equation algebra. This period constitutes the second half of what is known as the "Old Babylonian" epoch (see the box "Rudiments of general history", page 2)

The next step was to interpret the texts precisely. To some extent, the general, non-technical meaning of their vocabulary could assist. In line 1 of the problem on page 5, *ak-mur* may be translated "I have heaped". An understanding of the "heaping" of two numbers as an addition seems natural and agrees with the observation that the "heaping" of 45´ and 15´ (that is, of $\frac{3}{4}$ and $\frac{1}{4}$) produces 1. When other texts "raise" (*našûm*) one magnitude to another one, it becomes more difficult. However, one may observe that the "raising" of 3 to 4 produces 12, while 5 "raised" to 6 yields 30, and thereby guess that "raising" is a multiplication.

In this way, the scholars of the 1930s came to choose a purely arithmetical interpretation of the operations – that is, as additions, subtractions, multiplications and divisions *of numbers*. This translation offers an example:[2]

1. I have added the surface and (the side of) my square: 45´.
2. You posit 1°, the unit. You break into two 1°: 30´. You multiply (with each other) [30´] and 30´:
3. 15´. You join 15´ to 45´: 1°. 1° is the square of 1°. 30´, which you have multiplied (by itself),
4. from 1° you subtract: 30´ is the (side of the) square.

Such translations are still found today in general histories of mathematics. They explain the numbers that occur in the texts, and

[2] A literal retranslation of François Thureau-Dangin's French translation. Otto Neugebauer's German translation is equivalent except on one point: where Thureau-Dangin translated "1°, the unit", Neugebauer proposed "1, the coefficient". He also transcribed place-value numbers differently.

they give an almost modern impression of the Old Babylonian methods. There is no fundamental difference between the present translation and the solution by means of equations. If the side of the square is $x$, then its area is $x^2$. Therefore, the first line of the text – the problem to be solved – corresponds to the equation $x^2+1\cdot x = \frac{3}{4}$. Continuing the reading of the translation we see that it follows the symbolic transformations on page 6 step by step.

However, even though the present translation as well as others made according to the same principles explain the numbers of the texts, they agree less well with their words, and sometimes not with the order of operations. Firstly, these translations do not take the geometrical character of the terminology into account, supposing that words and expressions like "(the side of) my square", "length", "width" and "area" of a rectangle denote nothing but unknown numbers and their products. It must be recognized that in the 1930s that did not seem impossible *a priori* – we too speak of $3^2$ as the "square of 3" without thinking of a quadrangle.

But there are other problems. The most severe is probably that the number of operations is too large. For example, there are two operations that in the traditional interpretation are understood as addition: "to join to" (*waṣābum*/DAḪ, the infinitive corresponding to the *tu-ṣa-ab* of our text) and "to heap" (*kamārum*/GAR.GAR, from which the *ak-mur* of the text). Both operations are thus found in our brief text, "heaping" in line 1 (where it appears as "add") and "joining" in line 3.

Certainly, we too know about synonyms even within mathematics – for instance, "and", "added to" and "plus"; the choice of one word or the other depends on style, on personal habits, on our expectations to the interlocutor, and so forth. Thureau-Dangin, as we see, makes use of them, following the distinctions of the text by speaking first of "addition" and second of "joining"; but he argues that there is no conceptual difference, and that nothing but synonyms are involved – "there *is* only one multiplication", as he explains without noticing that the argument is circular.

Synonyms, it is true, can also be found in Old Babylonian mathematics. Thus, the verbs "to tear out" (*nasāḫum*/ZI) and "to cut off" (*ḫarāṣum*/KUD) are names for the same subtractive operation:

they can be used in strictly analogous situations. The difference between “joining” and “heaping”, however, is of a different kind. No text exists which refers to a quadratic completion (above, page 6) as a “heaping”. “Heaping”, on the other hand, is the operation to be used when an area and a linear extension are added. These are thus distinct operations, not two different names for the same operation. In the same way, there are two distinct “subtractions”, four “multiplications”, and even two different “halves”. We shall come back to this.

A translation which mixes up operations which the Babylonians treated as distinct may explain why the Babylonian calculations lead to correct results; but they cannot penetrate their mathematical thought.

Further, the traditional translations had to skip certain words which seemed to make no sense. For instance, a more literal translation of the last line of our small problem would begin “from the inside of 1°” (or even “from the heart” or “from the bowels”). Not seeing how a number 1 could possess an “inside” or “bowels”, the translators tacitly left out the word.

Other words were translated in a way that differs so strongly from their normal meaning that it must arouse suspicion. Normally, the word translated “unity” by Thureau-Dangin and “coefficient” by Neugebauer (*waṣīum*, from *waṣûm*, “to go out”) refers to something that sticks out, as that part of a building which architects speak about as a “projection”. That must have appeared absurd – how can a number 1 “stick out”? Therefore the translators preferred to make the word correspond to something known in the mathematics of their own days.

Finally, the order in which operations are performed is sometimes different from what seems natural in the arithmetical reading.

In spite of these objections, the interpretation that resulted in the 1930s was an impressive accomplishment, and it remains an excellent “first approximation”. The scholars who produced it pretended nothing more. Others however, not least historians of mathematics and historically interested mathematicians, took it to be the unique and final decipherment of “Babylonian algebra” – so impressive were the results that were obtained, and so scary the perspective of being forced to read the texts in their original language. Until the 1980s, nobody noticed that certain apparent synonyms represent distinct operations[3].

### *A new reading*

As we have just seen, the arithmetical interpretation is unable to account for the words which the Babylonians used to describe their procedures. Firstly, it conflates operations that the Babylonians treated as distinct; secondly, it is based on operations whose order does not always correspond to that of the Babylonian calculations. Strictly speaking, rather than an interpretation it thus represents a control of the correctness of the Babylonian methods based on modern techniques.

A genuine interpretation – a reading of what the Old Babylonian calculators thought and did – must take two things into account: on one hand, the results obtained by the scholars of the 1930s in their "first approximation"; on the other, the levels of the texts which these scholars had to neglect in order to create this first approximation.

In the following chapters we are going to analyse a number of problems in a translation that corresponds to such an interpretation. First some general information will be adequate.

#### Representation and "variables"

In our algebra we use $x$ and $y$ as substitutes or names for unknown *numbers*. We use this algebra as a tool for solving problems that concern other kinds of magnitudes, such as prices, distances, energy densities, etc.; but in all such cases we consider these other quantities as represented by numbers. For us, numbers constitute the *fundamental representation*.

For the Babylonians, the fundamental representation was geometric. Most of their "algebraic" problems concern rectangles with length,

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[3] Nobody, except perhaps Neugebauer, who on one occasion observes (correctly) that a text makes use of a wrong multiplication. In any case it must be noticed that neither he nor Thureau-Dangin ever chooses a wrong operation when restituting the missing part of a broken text.

width and area[4], or squares with side and area. We shall certainly encounter a problem below (YBC 6967, page 46) that asks about two unknown *numbers*, but since their product is spoken of as a "surface" it is evident that these numbers are *represented* by the sides of a rectangle.

An important characteristic of Babylonian geometry allows it to serve as an "algebraic" representation: it always deals with *measured* quantities. The measure of its segments and areas may be treated as *unknown* – but even then it exists as a numerical measure, and the problem consists in finding its value.

**Units**

Every measuring operation presupposes a metrology, a system of measuring units.; the numbers that result from it are concrete numbers. That cannot be seen directly in the problem that was quoted above on page 8; mostly, the mathematical texts do not show it since they make use of the place-value system (except, occasionally, when given magnitudes or results are stated). In this system, all quantities of the same kind were measured in a "standard unit" which, with very few exceptions, was not stated but tacitly understood.

The standard unit for *horizontal distance* was the NINDAN, a "rod" of c. 6 m[5]. In our problem, the side of the square is thus

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[4] More precisely, the word translated "length" signifies "distance"/ "extension"/"length", while that which is translated "width" means "front"/"forehead"/"head". They refer to the idea of a long and narrow irrigated field. The word for the area (*eqlum*/A.ŠÀ) originally means "field", but in order to reserve it for technical use the texts use other (less adequate) words when speaking of genuine fields to be divided. In what follows, the term will be translated "surface", which has undergone a similar shift of meaning, and which stands both for the spatial entity and its area.

A similar distinction is created by other means for lengths and widths. If these stand for "algebraic" variables they are invariably written with the logograms UŠ and SAG̃; if used for general purposes (the length of a wall, a walking distance) they may be provided with phonetic complements or written syllabically as *šiddum* and *pūtum*.

[5] In the absence of a sexagesimal point it is in principle impossible to know

$\frac{1}{2}$ NINDAN, that is, c. 3 m. For *vertical distances* (heights and depths), the basic unit was the KÙŠ, a "cubit" of $\frac{1}{12}$ NINDAN (that is, c. 50 cm).

The standard unit for *areas* was the SAR, equal to 1 NINDAN$^2$. The standard unit for volumes had the same name: the underlying idea was that a base of 1 NINDAN$^2$ was provided with a standard thickness of 1 KÙŠ. In agricultural administration, a better suited area unit was used, the BÙR, equal to 30` SAR, c. 6½ ha.

The standard unit for *hollow measures* (used for products conserved in vases and jars, such as grain and oil) was the SÌLA, slightly less than one litre. In practical life, larger units were often used: 1 BÁN = 10 SÌLA, 1 PI = 1` SÌLA, and 1 GUR, a "tun" of 5` SÌLA.

Finally, the standard unit for *weights* was the shekel, c. 8 gram. Larger units were the *mina*, equal to 1` shekel (thus close to a pound)[6] and the GÚ, "a load" equal to 1`` shekel, c. 30 kilogram. This last unit is equal to the talent of the Bible (where a talent *of silver* is to be understood).

### Additive operations

There are two additive operations. One (*kamārum*/UL.GAR/GAR.GAR), as we have already seen, can be translated "to heap *a* and *b*", the other (*waṣābum*/DAḪ) "to join *j* to *S*". "Joining" is a concrete operation which conserves the identity of *S*. In order to understand what that means we may think of "my" bank deposit *S*; adding the interest *j* (in Babylonian called precisely *ṣibtum*, "the joined", a noun derived from the verb *waṣābum*) does not change its identity as *my* deposit. If a geometric operation "joins" *j* to *S*, *S* invariably remains in place, whereas, if necessary, *j* is moved around.

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whether the basic unit was 1 NINDAN, 60 NINDAN or $\frac{1}{60}$ NINDAN. The choice of 1 NINDAN represents what (for us, at least) seems most natural for an Old Babylonian calculator, since it already exists as a unit (which is also true for 60 NINDAN but not for $\frac{1}{60}$ NINDAN) and because distances measured in NINDAN had been written without explicit reference to the unit for centuries before the introduction of the place-value system.

[6] It is not to be excluded that the Babylonians thought of the *mina* as standard unit, or that they kept both possibilities open.

"Heaping", to the contrary, may designate the addition of abstract numbers. Nothing therefore prevents from "heaping" (the number measuring) an area and (the number measuring) a length. However, even "heaping" often concerns entities allowing a concrete operation.

The sum resulting from a "joining" operation has no particular name; indeed, the operation creates nothing new. In a heaping process, on the other hand, where the two addends are absorbed into the sum, this sum has a name (*nakmartum*, derived from *kamārum*, "to heap") which we may translate "the heap"; in a text where the two constituents remain distinct, a plural is used (*kimrātum*, equally derived from *kamārum*); we may translate it "the things heaped" (AO 8862 #2, translated in Chapter 3, page 63).

### Subtractive operations

There are also two subtractive operations. One (*nasāḫum*/ZI), "from $B$ to tear out $a$", is the inverse of "joining"; it is a concrete operation which presupposes $a$ to be a constituent part of $B$. The other is a comparison, which can be expressed "$A$ over $B$, $d$ goes beyond" (a clumsy phrase, but which maps the structure of the Babylonian locution precisely). Even this is a concrete operation, used to compare magnitudes of which the smaller is not part of the larger. At times, stylistic and similar reasons call for the comparison being made the other way around, as an observation of $B$ falling short of $A$ (note 25 discusses an example).

The difference in the first subtraction is called "the remainder" (*šapiltum*, more literally "the diminished"). In the second, the excess is referred to as the "going-beyond" (*watartum*/DIRIG).

There are several synonyms or near-synonyms for "tearing out". We shall encounter "cutting off" (*ḫarāṣum*) (AO 8862 #2, page 63) and "make leave" (*šutbûm*) (VAT 7532, page 68).

### "Multiplications"

Four distinct operations have traditionally been interpreted as multiplication.

First, there is the one which appears in the Old Babylonian version of the multiplication table. The Sumerian term (A.RÁ, derived from the

Sumerian verb RÁ, "to go") can be translated "steps of". For example, the table of the multiples of 6 runs:

1 step of 6 is 6
2 steps of 6 are 12
3 steps of 6 are 18
...

Three of the texts we are to encounter below (TMS VII #2, page 33, TMS IX #3, page 59, and TMS VIII #1, page 82) also use the Akkadian verb for "going" (*alākum*) to designate the repetition of an operation: the former two repeat a magnitude $s$ $n$ times, with outcome $n{\cdot}s$ (TMS VII #2 line 18; TMS IX #3, line 21); TMS VIII #1 line 1 joins a magnitude $s$ $n$ times to another magnitude $A$, with outcome $A+n{\cdot}s$.

The second "multiplication" is defined by the verb "to raise" (*našûm*/ÍL/NIM). The term appears to have been used first for the calculation of volumes: in order to determine the volume of a prism with a base of $G$ SAR and a height of $h$ KÙŠ, one "raises" the base with its standard thickness of 1 KÙŠ to the real height $h$. Later, the term was adopted by analogy for all determinations of a concrete magnitude by multiplication. "Steps of" instead designates the multiplication of an abstract number by another abstract number.

The third "multiplication" (*šutakūlum*/GU$_7$.GU.$_7$), "to make $p$ and $q$ hold each other" – or simply because that is almost certainly what the Babylonians thought of, "make $p$ and $q$ hold (namely, hold a rectangle)"[7] – is no real multiplication. It always concerns two line segments $p$ and $q$, and "to make $p$ and $q$ hold" means to construct a rectangle contained by the sides $p$ and $q$. Since $p$ and $q$ as well as the area $A$ of the rectangle are all measurable, almost all texts give the numerical value of $A$ immediately after prescribing the operation – "make 5 and 5 hold: 25" – without mentioning the numerical multiplication of 5 by 5 explicitly. But there are texts that speak separately about the numerical multiplication, as "$p$ steps of $q$", after

[7] The verbal form used would normally be causative-reciprocative. However, at times the phrase used is "make $p$ together with $q$ hold", which seems to exclude the reciprocative interpretation.

prescribing the construction, or which indicate that the process of "making hold" creates "a surface"; both possibilities are exemplified in AO 8862 #2 (page 63). If a rectangle exists already, its area is determined by "raising", just as the area of a triangle or a trapezium. Henceforth we shall designate the rectangle which is "held" by the segments $p$ and $q$ by the symbol ⊏⊐($p$,$q$), while □($a$) will stand for the square which a segment $a$ "holds together with itself" (in both cases, the symbol designate the configuration as well the area it contains, in agreement with the ambiguity inherent in the concept of "surface"). The corresponding numerical multiplications will be written symbolically as $p\times q$ and $a\times a$.

The last "multiplication" (*eṣēpum*) is also no proper numerical multiplication. "To repeat" or "to repeat until $n$" (where $n$ is an integer small enough to be easily imagined, at most 9) stands for a "physical" doubling or $n$-doubling – for example that doubling of a right triangle with sides (containing the right angle) $a$ and $b$ which produces a rectangle ⊏⊐($a$,$b$).

**Division**

The problem "what should I raise to $d$ in order to get $P$?" is a *division problem*, with answer $P\div d$. Obviously, the Old Babylonian calculators knew such problems perfectly well. They encountered them in their "algebra" (we shall see many examples below) but also in practical planning: a worker can dig $N$ NINDAN irrigation canal in a day; how many workers will be needed for the digging of 30 NINDAN in 4 days? In this example the problem even occurs twice, the answer being $(30\div4)\div N$. But division was no separate *operation* for them, only a problem type.

In order to divide 30 by 4, they first used a table (see Figure 2), in which they could read (but they had probably learned it by heart in school[8]) that IGI 4 is 15´; afterwards they "raised" 15´ to 30 (even

[8] When speaking of a "school" in the Old Babylonian context we should be aware that we only know it from textual evidence. No schoolroom has been identified by archaeologists (what was once believed to be school rooms has turned out to be for instance store rooms). We therefore do not know whether the scribes were taught in palace of temple schools or in the private homes of

| | | | |
|---|---|---|---|
| Of 1, its 2/3 | 40 | 27, its IGI | 2 13 20 |
| Its half | 30 | 30, its IGI | 2 |
| 3, its IGI | 20 | 32, its IGI | 1 52 30 |
| 4, its IGI | 15 | 36, its IGI | 1 40 |
| 5, its IGI | 12 | 40, its IGI | 1 30 |
| 6, its IGI | 10 | 45, its IGI | 1 20 |
| 8, its IGI | 7 30 | 48, its IGI | 1 15 |
| 9, its IGI | 6 40 | 50, its IGI | 1 12 |
| 10, its IGI | 6 | 54, its IGI | 1 6 40 |
| 12, its IGI | 5 | 1, its IGI | 1 |
| 15, its IGI | 4 | 1 4, its IGI | 56 15 |
| 16, its IGI | 3 45 | 1 12, its IGI | 50 |
| 18, its IGI | 3 20 | 1 15, its IGI | 48 |
| 20, its IGI | 3 | 1 20, its IGI | 45 |
| 24, its IGI | 2 30 | 1 21, its IGI | 44 26 40 |
| 25, its IGI | 2 24 | | |

*Figure 2*. Translation of the Old Babylonian table of reciprocals (IGI)

for that tables existed, learned by heart at school), finding 7°30′[9].

Primarily, IGI *n* stands for *the reciprocal of n as listed in the table* or at least as easily found from it, not the number $\frac{1}{n}$ abstractly. In this way, the Babylonians solved the problem $P \div d$ via a multiplication $P \cdot \frac{1}{d}$ *to the extent that this was possible*.

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a master scribe instructing a handful of students; most likely, many were taught by private masters. The great number of quasi-identical copies of the table of reciprocals that were prepared in order to be learned by heart show, however, that future scribes were not (or not solely) taught as apprentices of a working scribe but according to a precisely defined curriculum; this is also shown by other sources.

[9] It may seem strange that the multiplication of IGI 4 by 30 is done by “raising”. Is this not a multiplication of a number by a number? Not necessarily, according the expression used in the texts when IGI 4 has to be found: they “detach” it, The idea is thus a splitting into 4 equal parts, one of which is detached. It seems that what was originally split (when the place-value system was constructed) was a length – namely 1` [NINDAN], not 1 [NINDAN]. This Ur-III understanding had certainly been left behind; but the terminological habit had survived.

However, this was only possible if $n$ appeared in the IGI table. Firstly, that required that $n$ was a "regular number", that is, that $\frac{1}{n}$ could be written as a finite "sexagesimal fraction".[10] However, of the infinitely many such numbers only a small selection found place in the table – around 30 in total (often, 1 12, 1 15 and 1 20 are omitted "to the left" since they are already present "to the right").

In practical computation, that was generally enough. It was indeed presupposed that all technical constants – for example, the quantity of dirt a worker could dig out in a day – were simple regular numbers. The solution of "algebraic" problems, on the other hand, often leads to divisions by a non-regular divisor $d$. In such cases, the texts write "what shall I posit to $d$ which gives me $A$?, giving immediately the answer "posit $Q$, $A$ will it give you".[11] That has a very natural explanation: these problems were constructed backwards, from known results. Divisors would therefore always divide, and the teacher who constructed a problem already knew the answer as well as the outcome of divisions leading to it.

**Halves**

$\frac{1}{2}$ may be a fraction like any other: $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. This kind of half, if it is *the half of something*, is found by raising that thing to 30′. Similarly, its $\frac{1}{3}$ is found by raising to 20′, etc. This kind of half we shall meet in AO 8862 #2 (page 63).

But $\frac{1}{2}$ (in this case necessarily the half of something) may also be a "natural" or "necessary" half, that is, a half that could be nothing else. The radius of a circle is thus the "natural" half of the diameter: no other part could have the same role. Similarly, it is by necessity the

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[10] And, tacitly understood, that $n$ itself can be written in this way. It is not difficult to show that all "regular numbers" can be written $2^p \cdot 3^q \cdot 5^r$, where $p$, $q$ and $r$ positive or negative integers or zero. 2, 3 and 5 are indeed the only prime numbers that divide 60. Similarly, the "regular numbers" in our decimal system are those that can be written $2^p \cdot 5^q$, 2 and 5 being the only prime divisors of 10.

[11] The expression "posit to" refers to the way simple multiplication exercises were written in school: the two factors were written one above the other (the second being "posited to" the first), and the result below both.

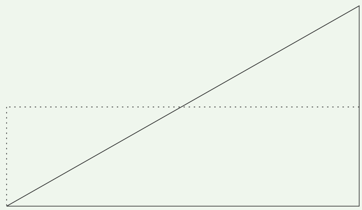

Figure 3

exact half of the base that must be raised to the height of a triangle in order to give the area – as can be seen on the figure used to prove the formula (see Figure 3).

This "natural" half had a particular name (*bāmtum*), which we may translate "moiety". The operation that produced it was expressed by the verb "to break" (*ḫepûm*/GAZ) – that is, to bisect, to break in two equal parts. This meaning of the word belongs specifically to the mathematical vocabulary; in general usage the word means to crush or break in any way (etc.).

**Square and "square root"**

The product $a \cdot a$ played no particular role, neither when resulting from a "raising" or from an operation of "steps of". A square, in order to be something special, had to be a geometric square.

But the geometric square did have a particular status. One might certainly "make *a* and *a* hold" or "make *a* together with itself hold"; but one might also "make *a* confront itself" (*šutamḫurum*, from *maḫārum* "to accept/receive/approach/welcome"). The square seen as a geometric configuration was a "confrontation" (*mitḫartum*, from the same verb)[12]. Numerically, its value was identified with the length of the side. A Babylonian "confrontation" thus *is* its side while it *has* an area; inversely, our square (identified with what is contained and not with the frame) *is* an area and *has* a side. When the value of a "confrontation" (understood thus as its side) is found, another side which it meets in a corner may be spoken of as its "counterpart" –

[12] More precisely, the Babylonian word stands for "a situation characterized by the confrontation of equals".

*meḫrum* (similarly from *maḫārum*), used also for instance about the exact copy of a tablet.

In order to say that *s* is the side of a square area *Q*, a Sumerian phrase (used already in tables of inverse squares probably going back to Ur III, see imminently) was used: "by *Q*, *s* is equal" – the Sumerian verb being ÍB.SI$_8$. Sometimes, the word ÍB.SI$_8$ is used as a noun, in which case it will be translated "the equal" in the following. In the arithmetical interpretation, "the equal" becomes the square root.

Just as there were tables of multiplication and of reciprocals, there were also tables of squares and of "equals". They used the phrases "$n$ steps of $n$, $n^2$" and "by $n^2$, $n$ is equal" ($1 \leq n \leq 60$). The resolution of "algebraic" problems often involves finding the "equals" of numbers which are not listed in the tables. The Babylonians did possess a technique for finding approximate square roots of non-square numbers – but these *were* approximate. The texts instead give the exact value, and once again they can do so because the authors had constructed the problem backward and therefore knew the solution. Several texts, indeed, commit calculational errors, but in the end they give the square root of the number that should have been calculated, not of the number actually resulting! An example of this is mentioned in note 43, page 77.

## *Concerning the texts and the translations*

The texts that are presented and explained in the following are written in Babylonian, the language that was spoken in Babylonia during the Old Babylonian epoch. Basically they are formulated in syllabic (thus phonetic) writing – that which appears as *italics* on page 5. All also make use of logograms that represent a whole word but does not indicate neither the grammatical form not the pronunciation (although grammatical complements are sometimes added to them); these logograms are transcribed in SMALL CAPS (see the box "Cuneiform writing", page 4). With rare exceptions, these logograms are borrowed from Sumerian, once the main language of the region and conserved as a scholars' language until the first century CE (as Latin in Europe until recently). Some of these logograms correspond to technical expressions already used as such by the Sumerian scribes;

IGI is an example. Others serve as abbreviations for Babylonian words, more or less as *viz* in English, which represents the shorthand for *videlicet* in medieval Latin manuscripts but is pronounced *namely*.

As already indicated, our texts come from the second half of the Old Babylonian epoch, as can be seen from the handwriting and the language. Unfortunately it is often impossible to say more, since almost all of them come from illegal diggings and have been bought by museums on the antiquity market in Baghdad or Europe.

We have no direct information about the authors of the texts. They never present themselves, and no other source speaks of them. Since they knew to write (and more than the rudimentary syllabic of certain laymen) they must have belonged to the broad category of scribes; since they knew to calculate, we may speak about them as "calculators"; and since the format of the texts refers to a didactical situation, we may reasonably assume that they were school teachers. [13].

All this, however, results from indirect arguments. Plausibly, the majority of scribes never produced mathematics on their own beyond simple computation; few were probably ever trained at the high mathematical level presented by our texts. It is even likely that only a minority of school teachers *taught* such matters. In consequence, and because several voices speak through the texts (see page 32), it is often preferable to pretend that it is the text itself which "gives", "finds", "calculates", etc.

The English translations that follow – all due to the author of the book – do not distinguish between syllabically and logographically written words (readers who want to know must consult the transliterations in Appendix B). Apart from that, they are "conformal" – that is, they are faithful to the original, in the structure of phrases[14]

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[13] On the problem of the "school", see note 8, page 16, and page 107.

[14] In Akkadian, the verb comes in the end of the phrase. This structure allows a number to be written a single time, first as the outcome of one calculation and next as the object of another one. In order to conserve this architecture of the text ("number(s)/operation: resulting number/new operation"), this final position of the verb is respected in the translations, ungrammatical though it is. The reader will need to be accustomed (but non-English readers should not learn it so well as to use the construction independently!).

as well as by using always distinct translations for words that are different in the original and the same translation for the same word every time it occurs unless it is used in clearly distinct functions (see the list of "standard translations" on page 139). In as far as possible the translations respect the non-technical meanings of the Babylonian words (for instance "breaking" instead of "bisecting") and the relation between terms (thus "confront itself" and "confrontation" – while "counterpart" had to be chosen unrelated of the verbal root in order to respect the use of the same word for the copy of a tablet).

This is not to say that the Babylonians did not have a technical terminology but only their everyday language; but it is important that the technical meaning of a word be learned from its uses within the Old Babylonian texts and not borrowed (with the risk of being badly borrowed, as has often happened) from our modern terminology.

The Babylonian language structure is rather different from that of English, for which reason the conformal translations are far from elegant. But the principle of conformality has the added advantage that readers who want to can follow the original line for line in Appendix B (the bibliographic note on page 159 indicates where the few texts not rendered in the appendix were published).

In order to avoid completely illegible translations, the principle is not followed to extremes. In English one has to choose whether a noun is preceded by a definite or an indefinite article; in Babylonian, as in Latin and Russian, that is not the case. Similarly, there is no punctuation in the Old Babylonian texts (except line breaks and a particle that will be rendered ":"), and the absolute order of magnitude of place-value numbers is not indicated; minimal punctuation as well as indications of order of magnitude (′, ` et °) have been added. Numbers that are written in the original by means of numerals have been translated as Arabic numerals, while numbers written by words (including logograms) have been translated as words; mixed writings appear mixed (for instance, "the 17th", and even "the 3rd" for the third).

Inscribed clay survives better than paper – particularly well when the city burns together with its libraries and archives, but also when discarded as garbage. None the less, almost all the tablets used for what follows are damaged. On the other hand, the language of the mathematical texts is extremely uniform and repetitive, and therefore it is often possible to reconstruct damaged passages from parallel passages on the same tablet. In order to facilitate reading the reconstructions are only indicated in the translations (as ¿...?) if their exact words are not completely certain. Sometimes a scribe has left out a sign, a word or a passage when writing a tablet which however can be restored from parallel passages on the same or closely kindred tablets. In such cases the restitution appears as ⟨...⟩ (the original editions of the texts give the complete information about destroyed and illegible passages and scribal omissions). Explanatory words inserted into the texts appear within rounded brackets (...).

Clay tablets have names, most often museum numbers. The small problem quoted above is the first one on the tablet BM 13901 – that is, tablet #13901 in the British Museum tablet collection. Other names begin AO (Ancient Orient, Louvre, Paris), VAT (Vorderasiatische Texte, Berlin) or YBC (Yale Babylonian texts). TMS refers to the edition *Textes mathématiques de Suse* of a Louvre collection of tablets from Susa, an Iranian site in the eastern neighbourhood of Babylon.

The tablets are mostly inscribed on both surfaces (“obverse” and “reverse”), sometimes in several columns, sometimes also on the edge; the texts are divided in lines read from left to right. Following the original editions, the translations indicate line numbers and, if actual, obverse/reverse and column.

# Chapter 1

## Techniques for the first degree

Our main topic will be the Old Babylonian treatment of second-degree equations.[15] However, the solution of second-degree equations or equation systems often asks for first-degree manipulations, for which reason it will be useful to start with a text which explains how first-degree equations are transformed and solved.

### *TMS XVI #1*

**1.** The 4th of the width, from the length and the width I have torn out, 45´. You, 45´
**2.** to 4 raise, 3 you see. 3, what is that? 4 and 1 posit,
**3.** 50´ and 5´, to tear out, posit. 5´ to 4 raise, 1 width. 20´ to 4 raise,
**4.** 1°20´ you ⟨see⟩,[16] 4 widths. 30´ to 4 raise, 2 you ⟨see⟩, 4 lengths. 20´, 1 width, to tear out,
**5.** from 1°20´, 4 widths, tear out, 1 you see. 2, the lengths, and 1, 3 widths, heap, 3 you see.
**6.** IGI 4 detach, 15´ you see. 15´ to 2, lengths, raise, 30´ you ⟨see⟩, 30´ the length.
**7.** 15´ to 1 raise, 15´ the contribution of the width. 30´ and 15´ hold.
**8.** Since "The 4th of the width, to tear out", it is said to you, from 4, 1 tear out, 3 you see.

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[15] As in the case of "algebra" we shall pretend for the moment to know what an "equation" is. Analysis of the present text will soon allow us to understand in which sense the Old Babylonian problems can be understood as equations.

[16] "you ⟨see⟩" translates *ta-*⟨*mar*⟩. The scribe thus does not omit a word, he uses the first syllable (which happens to carry the information about the grammatical person) as a logogram for the whole word. This is very common in the texts from Susa, and illustrates that the use of logograms is linked to the textual genre: only in mathematical texts can we be reasonably sure that no other verbs beginning with the syllable *ta* will be present in this position.

**9.** IGI 4 de⟨tach⟩, 15′ you see, 15′ to 3 raise, 45′ you ⟨see⟩, 45′ as much as (there is) of widths.

**10.** 1 as much as (there is) of lengths posit. 20, the true width take, 20 to 1′ raise, 20′ you see.

**11.** 20′ to 45′ raise, 15′ you see. 15′ from $^{30}{}_{15'}$ tear out,

**12.** 30′ you see, 30′ the length.

This text differs in character from the immense majority of Old Babylonian mathematical texts: it does not state a problem, and it solves none. Instead, it gives a didactic explanation of the concepts and procedures that serve to understand and reduce a certain often occurring equation type.

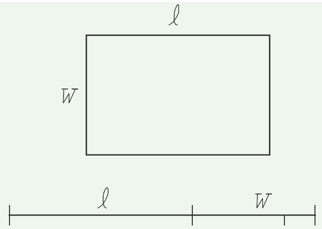


*Figure 4*. The geometry of TMS XVI #1

Even though many of the terms that appear in the translation were already explained in the section "A new interpretation", it may be useful to go through the text word for word.

Line 1 formulates an equation: *The 4th of the width, from the length and the width I have torn out, 45′.*

The equation thus concerns a length and a width. That tells us that the object is a rectangle – from the Old Babylonian point of view, the rectangle is the simplest figure determined by a length and a width alone[17]. Concerning the number notation, see the box "The

[17] A right triangle is certainly also determined by a length and a width (the legs of the right angle), and these two magnitudes suffice to determine it (the third side, if it appears, may be "the long length"). But a triangle is always introduced as such. If it is not practically right, the text will give a sketch.

The word "practically" should be taken note of. The Babylonians had no concept of the angle as a measurable quantity – thus, nothing corresponding sexagesimal system”, page 7. If $\ell$ is the length and $w$ the width, we may express the equation in symbols in this way:

$$(\ell + w) - \frac{1}{4}w = 45' \ .$$

Something, however, is lost in this translation. Indeed, *the length and the width* is a condensed expression for a “heaping”, the symmetric addition of two magnitudes (or their measuring numbers; see page 14). The length is thus not prolonged by the width, the two magnitudes are combined on an equal footing, independently of the rectangle. The sole role of the rectangle is to put its dimensions at disposal as unknown magnitudes (see Figure 4).

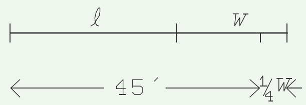


*Figure 5*. “The equation” of TMS XVI #1

Once the length and the width have been “heaped”, it is possible to “tear out” $\frac{1}{4}w$, since this entity is a part of the width and hence also of the total. To “tear out”, as we remember, is the inverse operation of “joining”, and thus the removal of a magnitude from another one of which it is a part (see Figure 5).

Line 1 shows the nature of a Babylonian equation: a combination of measurable magnitudes (often, as here, geometric magnitudes), for which the total is given. Alternatively the text states that the measure of one combination is equal to that of another on, or by how much one exceeds the other. That is not exactly the type of equation which is taught in present-day school mathematics, which normally deals with pure numbers – but it is quite similar to the equations manipulated by engineers, physicists or economists. To speak of “equations” in the Babylonian context is thus not at all anachronistic.

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to our “angle of 78°”. But they distinguished clearly “good” from “bad” angles – we may use the pun that the opposite of a *right angle* was a *wrong angle*. A *right angle* is one whose legs determine an area – be it the legs of the right angle in a right triangle, the sides of a rectangle, or the height and the average base of a right trapezium.

Next, lines 1 and 2 ask the student to multiply the 45′ (on the right-hand side of the version in symbols) by 4: *You, 45′ to 4 raise, 3 you see*. To "raise", we remember from page 15, stands for multiplying a concrete magnitude – here the number which represents a composite line segment. The outcome of this multiplication is 3, and the texts asks a rhetorical questions: *3, what is that?*

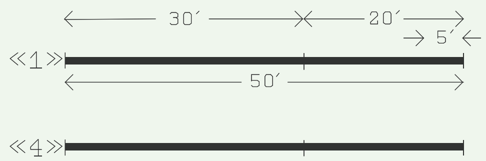


*Figure 6*. Interpretation of TMS XVI, lines 1 – 3

The answer to this question is found in lines 2 – 5. *4 and 1 posit*: First, the student should "posit" 4 and 1. To "posit" means to give a material representation; here, the numbers should probably be written in the appropriate place in a diagram (Figure 6 is a possible interpretation). The number «1» corresponds to the fact that the number 45′ to the right in the initial equation as well as the magnitudes to the left are all used a single time. The number «4» is "posited" because we are to explain what happens when 45′ and the corresponding magnitudes are taken 4 times.

*50′ and 5′, to tear out, posit*: the numbers 50′ and 5′ are placed on level «1» of the diagram. This should surprise us: it shows that the student is supposed to know already that the width is 20′ and the length is 30′. If he did not, he would not understand that $\ell + w = 50'$ and that $\frac{1}{4}w$ (that which is to be torn out) is 5′. For the sake of clarity not only the numbers 50′ and 5′ but also 30′ and 20′ are indicated at level «1» in our diagram even though the text does not speak about them.

Lines 3 – 5 prove even more convincingly that the student is supposed to know already the solution to the problem (which is thus only a quasi-problem). The aim of the text is thus not to find a solution. As already stated, it is to explain the concepts and procedures that serve to understand and reduce the equation.

These lines explain how and why the initial equation

$$(\ell+w)-\tfrac{1}{4}w = 45'$$

is transformed into

$$4\ell+(4-1)\,w = 3$$

through multiplication by 4.

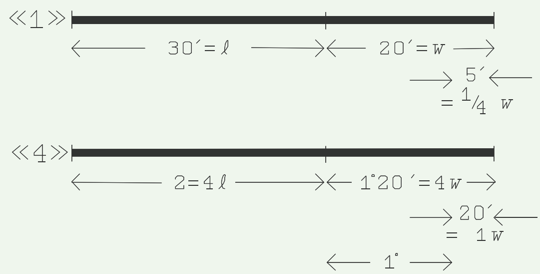


*Figure 7.* Interpretation of TMS XVI, lines 3 – 5

This calculation can be followed in Figure 7, where the numbers on level «1» are multiplied by 4, giving thereby rise to those of level «4»:

*5′ to 4 raise, 1 width*: 5′, that is, the $\frac{1}{4}$ of the width, is multiplied by 4, from which results 20′, that is, one width.

*20′ to 4 raise, 1°20′ you ⟨see⟩, 4 widths*: 20′, that is, 1 width, is multiplied by 4, from which comes 1°20′, thus 4 widths.

*30′ to 4 raise, 2 you ⟨see⟩, 4 lengths*: 30′, that is 1 length, is multiplied by 4. This gives 2, 4 lengths.

After having multiplied all the numbers of level «1» by 4, and finding thus their counterparts on level «4», the text indicates (lines 4 and 5) what remains when 1 width is eliminated from 4 widths: 20′, *1 width, to tear out, from 1°20′, 4 widths, tear out, 1 you see*.

Finally, the individual constituents of the sum $4\ell+(4-1)\,w$ are identified, as shown in Figure 8. *2, the lengths, and 1, 3 widths, heap, 3 you see*: 2, that is, 4 lengths, and 1, that is, $(4-1) = 3$ widths, are added. This gives the number *3*. We have now found the answer to the question of line 2, *3 you see. 3, what is that?*

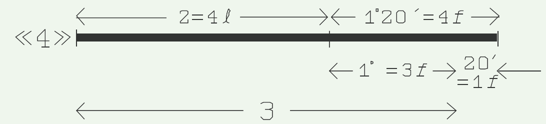


*Figure 8*. Interpretation of TMS XVI, line 5

But the lesson does not stop here. While lines 1–5 explain how the equation $(\ell+w)-\frac{1}{4}w = 45'$ can be transformed into $4\cdot\ell+(4-1)\cdot w = 3$, what follows in lines 6–10 leads, through division by 4, to a transformation of this equation into

$$1\cdot\ell+\frac{3}{4}\cdot w = 45' \;.$$

For the Babylonians, division by 4 is indeed effectuated as a multiplication by $\frac{1}{4}$. Therefore, line 6 states that $\frac{1}{4} = 15'$: *IGI 4 detach, 15′ you see*. IGI 4 can be found in the table of IGI, that is, of reciprocals (see page 17).

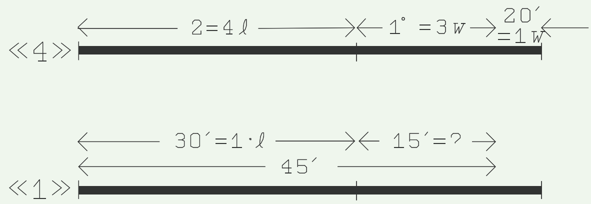


*Figure 9*. Interpretation of TMS XVI, lines 6–12

Figure 9 shows that this corresponds to a return to level «1»:

*15′ to 2, lengths, raise, 30′ you ⟨see⟩, 30′ the length*: 2, that is, 4 lengths, when multiplied by $\frac{1}{4}$ gives 30′, that is, 1 length.

*15′ to 1 raise, 15′ the contribution of the width*. (line 7): 1, that is, 3 widths, is multiplied by $\frac{1}{4}$, which gives 15′, the contributions of the width to the sum 45′. The quantity of widths to which this contribution corresponds is determined in line 8 and 9. In the meantime, the contribution of the length and the width are memorized: *30′ and 15′ hold* – a shorter expression for *may you head hold*, the formulation used in other texts. We notice the contrast to the material taking note of the numbers 1, 4, 50′ and 5′ by "positing" in the beginning.

The contribution of the width is thus 15′. The end of line 9 indicates that the number of widths to which that corresponds – the *coefficient* of the width, in our language – is $\frac{3}{4}$ (= 45′): *45′ as much as (there is) of widths*. The argument leading to this is of a type known as "simple false position"[18].

Line 8 quotes the statement of the quasi-problem as a justification of what is done (such justifications by quotation are standard): *Since "The 4th of the width, to tear out", it is said to you*. We must therefore find out how much remains of the width when $\frac{1}{4}$ has been removed.

For the sake of convenience, it is "posited" that the quantity of widths is 4 (this is the "false position"). $\frac{1}{4}$ of 4 equals 1 (the text gives this number without calculation). When it is eliminated, 3 remains: *from 4, 1 tear out, 3 you see*.

In order to see to which part of the falsely posited 4 this 3 corresponds, we multiply by $\frac{1}{4}$. Even though this was already said in line 6, it is repeated in line 9 that $\frac{1}{4}$ corresponds to 15′: *IGI 4 de⟨tach⟩, 15′ you see*.

Still in line 9, multiplication by 3 gives the coefficient of the width as 45′ (= $\frac{3}{4}$): *15′ to 3 raise, 45′ you ⟨see⟩, 45′ as much as (there is) of widths*.

Without calculating it line 10 announces that the coefficient of the length is 1. We know indeed from line 1 that a sole length enters into the 45′, without addition nor subtraction. We have thus explained how the equation $4\cdot\ell+(4-1)\cdot w = 3$ is transformed into

$$1\cdot\ell+\frac{3}{4}\cdot w = 45' \ .$$

The end of line 10 presents us with a small riddle: what is the relation between the "true width" and the width which figures in the equations?

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[18] "Simple" because there is also a "double false position" that may serve to solve more complex first-degree problems. It consists in making two hypotheses for the solution, which are then "mixed" (as in alloying problems) in such a way that the two errors cancel each other (in modern terms, this is a particular way to make a linear interpolation). Since the Babylonians never made use of this technique, a "false position" always refers to the "simple false position" in what follows.

The explanation could be the following: a true field might measure 30 [NINDAN] by 20 [NINDAN] (c. 180 m by 120 m, that is, $\frac{1}{3}$ BÙR), but certainly not 30′ by 20′ (3 m by 2 m). On the other hand it would be impossible to draw a field with the dimensions 30×20 in the courtyard of the schoolmaster's house (or any other school; actually, a sand-strewn courtyard is the most plausible support for the diagrams used in teaching). But 30′ by 20′ would fit perfectly (we know from excavated houses), and this order of magnitude is the one that normally appears in mathematical problems. Since there is no difference in writing between 20 and 20′, this is nothing but a possible explanation – but a plausible one, since no alternative seems to be available.

In any case, in line 11 it is found again that the width contributes with 15′, namely by multiplying 20′ (1 width) by the coefficient 45′: *20′ to 45′ raise, 15′ you see*.

In the end, the contribution of the width is eliminated from 45′ (already written ${}_{30}{}^{15}$, that is, as the sum of 30′ and 15′, in agreement with the partition memorized in the end of line 7). 30′ remains, that is, the length: *15′ from ${}^{30}{}_{15}$′ tear out, 30′ you see, 30′ the length*.

All in all, a nice pedagogical explanation, which guides the student by the hand crisscross through the subject "how to transform a first-degree equation, and how to understand what goes on".

Before leaving the text, we may linger on the actors that appear, and which recur in most of those texts that state a problem together with the procedure leading to its solution.[19] Firstly, a "voice" speaking in the first person singular describes the situation which he has established, and formulates the question. Next a different voice addresses the student, giving orders in the imperative or in the second person singular, present tense; this voice cannot be identical with the one that stated the problem, since it often quotes it in the third person, "since he has said".

In a school context, one may imagine that the voice that states the

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[19] The present document employs many logograms without phonetic or grammatical complements. Enough is written in syllabic Akkadian, however, to allow us to discern the usual scheme which, in consequence, is imposed upon the translation.

problem is that of the school master, and that the one which addresses the student is an assistant or instructor – "edubba texts"[20], literary texts about the school and about school life, often refer to an "older brother" whose task it is to give instructions. However, the origin of the scheme appears to be different. Certain texts from the early 18th century begin "If somebody asks you thus, 'I have ...' ". In these texts the one who asks is a hypothetical person not belonging to the didactical situation – a pretext for a mathematical riddle. The anonymous guide is then the master, originally probably to be identified with a master-surveyor explaining the methods of the trade to his apprentice.

### *TMS VII #2* [21]

**17.** The fourth of the width to the length I have joined, its seventh
**18.** until 11 I have gone, over the heap
**19.** of length and width 5´ it went beyond. You, 4 posit;
**20.** 7 posit; 11 posit; and 5´ posit.
**21.** 5´ to 7 raise, 35´ you see.
**22.** 30´ and 5´ posit. 5´ to 11 raise, 55´ you see.
**23.** 30´, 20´, and 5´, to tear out, posit. 5´ to 4
**24.** raise, 20´ you see, 20 the width. 30´ to 4 raise:
**25.** 2 you see, 2, lengths. 20´ from 20´ tear out.
**26.** 30´ from 2 tear out, 1°30´ posit, and 5´ to ¿50´, the heap of length and width, join?
**27.** 7 to 4, of the fourth, raise, 28 you see.
**28.** 11, the heaps, from 28 tear out, 17 you see.
**29.** From 4, of the fourth, 1 tear out, 3 you see.
**30.** IGI 3 detach, 20´ you see. 20´ to 17 raise,
**31.** 5°40´ you see, 5°40´, (for) the length. 20´ to 5´, the going-beyond, raise,

[20] The Sumerian word É.DUB.BA means "tablet house", that is, "school".

[21] This text is rather intricate. Who finds it too opaque may skip it and eventually return to it once familiarized with the Babylonian mode of thought.

**32.** 1′40″ you see, 1′40″, the to-be-joined of the length. 5°40′, (for) the length,
**33.** from 11, heaps, tear out, 5°20′ you see.
**34.** 1′40″ to 5′, the going-beyond, join, 6′40″ you see.
**35.** 6′40″, the to-be-torn-out of the width. 5′, the step,
**36.** to 5°40′, lengths, raise, 28′20″ you see.
**37.** 1′40″, the to-be-joined of the length, to 28′20″ join,
**38.** 30′ you see, 30′ the length. 5′ to 5°20′
**39.** raise: 26′40″ you see. 6′40″,
**40.** the to-be-torn-out of the width, from 26′40″ tear out,
**41.** 20′ you see, 20′ the width.

This is the second, difficult problem from a tablet. The first, easy one (found on page 126 in English translation) can be expressed in symbols in this way:

$$10\cdot(\tfrac{1}{7}[\ell+\tfrac{1}{4}w]) = \ell+w\ .$$

After reduction, this gives the equation

$$\ell\cdot 10 = 6\cdot(\ell+w)\ .$$

This is an "indeterminate" equation, and has an infinity of solutions. If we have found one of them $(\ell_o, w_o)$, all the others can be written $(k\cdot\ell_o, k\cdot w_o)$. The text finds one by taking the first factor to the left to be equal to the first factor to the right (thus $\ell = 6$), and the second factor to the right to be equal to the second factor to the right (thus $\ell+w = 10$, whence $w = 4$). Afterwards the solution that has been tacitly aimed at from the beginning is obtained through "raising" to 5′ (the "step" $\tfrac{1}{7}[\ell+\tfrac{1}{4}w]$ that has been "gone" 10 times). Indeed, if $\ell = 6$, $w = 4$, then the "step" is 1; if we want it to be 5′ (which corresponds to the normal dimensions of a "school rectangle", $\ell = 30'$, $w = 20'$), then the solution must be multiplied by this value. All of this – which is not obvious – is useful for understanding the second problem.

The first problem is "homogeneous" – all its terms are in the first degree in $\ell$ and $w$. The second, the one translated above, is inhomogeneous, and can be expressed in symbols in this way:

$$11\cdot(\tfrac{1}{7}[\ell+\tfrac{1}{4}w) = [\ell+w]+5'\ .$$

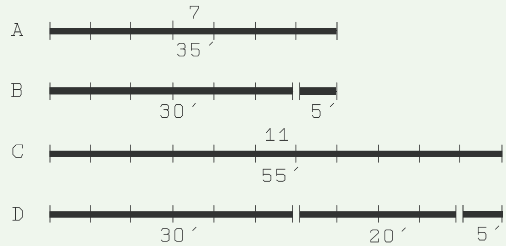


*Figure 10*. Interpretation of TMS VII, lines 21 – 23

We take note that $\frac{1}{4}w$ is "joined" to the length; that we take $\frac{1}{7}$ of the outcome; and that afterwards we "go" this segment 11 times. What results "goes beyond" the "heap" of length and width by 5′. The "heap" is thus no part of what results from the repetition of the step – if it were it could have been "torn out".

The solution begins with a pedagogical explanation in the style of TMS XVI #1, the preceding quasi-problem. Reading well we see that the 5′ which is "raised" to 7 in line 21 must be the "step" $\frac{1}{7}[\ell+\frac{1}{4}w]$ – the raising is a verification that it is really the 7th – and not the "going-beyond" referred to in line 20. Once again the student is supposed to understand that the text is based on the rectangle ⊏⊐(30′,20′). Having this configuration in mind we will be able to follow the explanation of lines 21 to 23 on Figure 10: when the "step" 5′ is "raised" to 7, we get 35′ (A), which can be decomposed as $\ell$ and $\frac{1}{4}w$ (B). When it is "raised" to 11 we find 55′ (C), which can be decomposed as $\ell$, $w$, and 5′ (D).

Next follows the prescription for solving the equation; is it still formulated in such a way that the solution is supposed to be known. "Raising" to 4 (lines 23 to 25) gives the equivalent of the symbolic equation

$$11\cdot(\tfrac{1}{7}[4\ell+4\cdot\tfrac{1}{4}w]) = 4\cdot([\ell+w]+5') \ .$$

Not having access to our symbols, the text speaks of $\frac{1}{4}w$ as 5′, finds that $4\cdot\frac{1}{4}w$ is equal to 20′, and identifies that with the width (line 24); then $4\ell$ appears as 2, said to represent lengths (line 25).

Now, by means of a ruse which is elegant but not easy to follow, the equation is made homogeneous. The text decomposes $4\ell+w$ as

$$(4-1)\ell-5'+(w-w)+(\ell+w+5')$$

and "raises" the whole equation to 7. We may follow the calculation in modern symbolic translation:

$$11\cdot([4-1]\ell-5'+0+[\ell+w+5'])=(7\cdot4)\cdot([\ell+w]+5')$$
$$\Leftrightarrow\ 11\cdot([4-1]\ell-5')=(28-11)\cdot([\ell+w]+5')$$
$$=17\cdot([\ell+w]+5')$$
$$\Leftrightarrow\ 11\cdot(\ell-\tfrac{1}{3}\cdot5')=\tfrac{1}{3}\cdot17\cdot(\ell+w+5')$$
$$\Leftrightarrow\ (\ell-1'40'')\cdot11=5°40'\cdot(\ell+w+5')$$

However, the Babylonians did not operate with such equations; they are likely to have inscribed the numbers along the lines of a diagram (see Figure 11); that is the reason that the "coefficient" $(4-1)$ only appears in line 29.

As in the first problem of the text, a solution to the homogeneous equation is found by identification of the factors "to the left" with those "to the right" (which is the reason that the factors have been inverted on the left-hand side of the last equation): $\ell-1'40''$ (now called "the length" and therefore designated $\lambda$ in Figure 11) thus corresponds to 5°40′, while $\ell+w+5'$ (referred to as "the heap" of the new length $\lambda$ and a new width $\phi$, that is, $\lambda+\phi$) equals 11; $\phi$ must therefore be $11-5°40'=5°20'$. Next the text determines the "to-be-joined" (*wāṣbum*) of the length, that is, that which must be joined to the length $\lambda$ in order to produce the original length $\ell$: it equals 1′40″, since $\lambda=\ell-1'40''$. Further it finds "the to-be-torn-out" (*nāsḫum*) of the width, that is, that which must be "torn out" from $\phi$ in order to produce $w$. Since $\ell+w+5'=11$, $w$ must equal $11-\ell-5'=11-(\lambda+1'40'')-5'=(11-\lambda)-(1'40''+5')=\phi-6'40''$; the "to-be-torn-out" is thus 6′40″.

But "joining" to $\lambda$ and "tearing out" from $\phi$ only gives a *possible* solution, not the one which is intended. In order to have the values for $\ell$ and $w$ that are aimed at, the step 5′ is "raised" (as in the first problem) to 5°40′ and 5°20. This gives, respectively, 28′20″ and 26′40′; by "joining" to the former its "to-be-joined" and by "tearing out" from the latter its "to-be-torn-out" we finally get $\ell=30'$, $w=20'$.

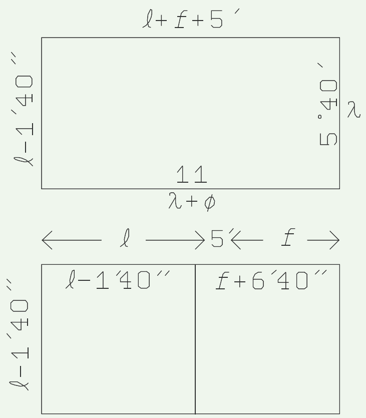


*Figure 11*. The resolution of TMS VII #2

We must take note of the mastery with which the author avoids to make use in the procedure of his knowledge of the solution (except in the end, where he needs to know the "step" in order to pick the solution that is aimed at among all the possible solutions). The numerical values that are *known* without being *given* serve in the pedagogical explanations; afterwards, their function is to provide *names* – having no symbols like $\ell$ and $\lambda$, the Babylonian needs to use identifications like "the length 30′" and "the length 5′40″" (both are lengths, so the name "length" without any qualifier will not suffice).

Numerical values serve as identifiers in many texts; none the less, misunderstandings resulting from mix-up of *given* and *merely known* numbers are extremely rare.

# Chapter 2

# The fundamental techniques for the second degree

After these examples of first-degree methods we shall now go on with the principal part of Old Babylonian algebra – postponing once more the precise determination of what “algebra” will mean in a Babylonian context. In the present chapter we shall examine some simple problems, which will allow us to discover the fundamental techniques used by the Old Babylonian scholars. Chapter 3 will take up more complex and subtle matters.

## *BM 13901 #1*

**Obv. I**

**1.** The surface and my confrontation I have heaped: 45′ is it. 1, the projection,

**2.** you posit. The moiety of 1 you break, 30′ and 30′ you make hold.

**3.** 15′ to 45′ you join: by 1, 1 is equal. 30′ which you have made hold

**4.** from the inside of 1 you tear out: 30′ the confrontation.

This is the problem that was quoted on page 5 in the Assyriologists’ “transliteration” and on page 8 in a traditional translation. A translation into modern mathematical symbolism is found on page 6.

Even though we know it well from this point of view, we shall once again examine the text and terminology in detail so as to be able to deal with it in the perspective of its author.

Line 1 states the problem: it deals with a *surface*, here a square, and with its corresponding *confrontation*, that is, the square configuration parametrized by its side, see page 19. It is the appearance of the “confrontation” that tells us that the “surface” is that of a square.

“Surface” and “confrontation” are *heaped*. This addition is the one that must be used when dissimilar magnitudes are involved, here an

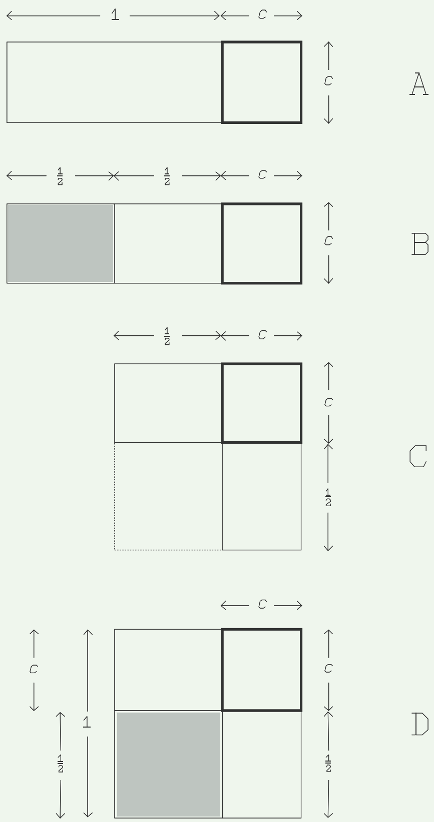


*Figure 12*. The procedure of BM 13901 #1, in slightly distorted proportions

area (two dimensions) and a side (one dimension). The text tells the sum of the two magnitudes – that is, of their measuring numbers: 45′. If $c$ stands for the side of the square and $\square(c)$ for its area, the problem can thus be expressed in symbols in this way:

$$\square(c)+c = 45' \ (= \tfrac{3}{4}) \ .$$

Figure 12 shows the steps of the procedure leading to the solution as they are explained in the text:

A: *1, the projection, you posit*. That means that a rectangle $\sqsubset\!\sqsupset(c,1)$ is drawn alongside the square $\square(c)$. Thereby the sum of a length and an area, absurd in itself, is made geometrically meaningful, namely as a rectangular area $\sqsubset\!\sqsupset(c,c+1) = \frac{3}{4} = 45'$. This geometric interpretation explains the appearance of the "projection", since the rectangle $\sqsubset\!\sqsupset(c,1)$

"projects" from the square as a projection protruding from a building. We remember (see page 10) that the word was originally translated as "unity" or "coefficient" simply because the translators did not understand how a number 1 could "project"

B: *The moiety of 1 you break*. The "projection" with adjacent rectangle $\sqsubset\!\sqsupset(c,1)$ is "broken" into two "natural" halves.

C: *30′ and 30′ you make hold*. The outer half of the projection (shaded in grey) is moved around in such a way that its two parts (each of length 30′) "hold" the square with dotted border below to the left. This cut-and-paste procedure has thus allowed us to transform the rectangle $\sqsubset\!\sqsupset(c,c+1)$ into a "gnomon", a square from which a smaller square is lacking in a corner.

D: *15′ to 45′ you join: 1*. 15′ is the area of the square held by the two halves (30′ et 30′), and 45′ that of the gnomon. As we remember from page 13, to "join" one magnitude to another one is an enlargement of the latter and only possible if both are concrete and of the same kind, for instance areas. We thus "join" the missing square, completing in this was the gnomon in order to get a new square. The area of the completed square will be $45'+15' = 1$.

*by 1, 1 is equal*. In general, the phrase "by $Q$, $s$ is equal" means (see page 20) that the area $Q$ laid out as a square has $s$ as one of its equal sides (in arithmetical language, $s = \sqrt{Q}$). In the present case, the text thus tells that the side of the completed square is 1, as indicated in D immediately to the left of the square.

*30′ which you have made hold from the inside of 1 you tear out*. In order to find the side $c$ of the original square we must now remove that piece of length $\frac{1}{2} = 30'$ which was added to it below. To "tear out" $a$ from $H$, as we have seen on page 14, is the inverse operation of a "joining", a concrete elimination which presupposes that $a$ is actually a part of $H$. As observed above (page 10), the phrase "from the inside" was omitted from the early translations, being meaningless as long as everything was supposed to deal with abstract numbers. If instead the number 1 represents a segment, the phrase does make sense.

*30′ the confrontation*. Removing from 1 the segment $\frac{1}{2} = 30'$ which was added, we get the initial side $c$, the "confrontation", which is hence equal to $1-30' = 30' = \frac{1}{2}$ (extreme left in *D*).

That solves the problem. In this geometric interpretation, not only the numbers are explained but also the words and explanations used in the text.

The new translation calls for some observation. We take note that no explicit argument is given that the cut-and-paste procedure leads to a correct result. On the other hand it is intuitively clear that it must be so. We may speak of a "naive" approach – while keeping in mind that *our* normal way to operate on equations, for instance in the example solving the same problem on page 6, is no less naive. Just as the Old Babylonian calculator we proceed from step to step without giving any explicit proof that the operations we make are justified, "seeing" merely that they are appropriate.

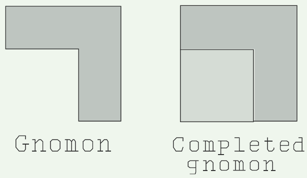


*Figure 13.*

The essential stratagem of the Old Babylonian method is the completion of the gnomon as shown in Figure 13. This stratagem is called a "quadratic completion"; the same term is used about the corresponding step in our solution by means of symbols:

$$\begin{aligned} x^2+1\cdot x = \tfrac{3}{4} \quad &\Leftrightarrow\ x^2+1\cdot x+(\tfrac{1}{2})^2 = \tfrac{3}{4}+(\tfrac{1}{2})^2 \\ &\Leftrightarrow\ x^2+1\cdot x+(\tfrac{1}{2})^2 = \tfrac{3}{4}+\tfrac{1}{4} = 1 \\ &\Leftrightarrow\ (x+\tfrac{1}{2})^2 = 1 \end{aligned}$$

However, the name seems to apply even better to the geometric procedure.

It is obvious that a negative solution would make no sense in this concrete interpretation. Old Babylonian algebra was based on tangible quantities even in cases where its problems were not really practical. No length (nor surface, volume or weight) could be negative. The only idea found in the Old Babylonian texts that approaches negativity is that a magnitude can be *subtractive*, that is, pre-determined to be torn out. We have encountered such magnitudes in the text TMS XVI #1 (lines 3 and 4 – see page 25) as well as TMS VII #2 (line 35, the “to-be-torn-out of the width” – see page 34). In line 25 of the latter text we also observe that the Babylonians did not consider the outcome of a subtraction of 20′ from 20′ as a *number* but, literally, as *something not worth speaking of*.

Certain general expositions of the history of mathematics claim that the Babylonians did know of negative numbers. This is a legend based on sloppy reading. As mentioned, some texts state for reasons of style not that a magnitude $A$ exceeds another one by the amount $d$ but that $B$ falls short of $A$ by $d$; we shall encounter an example in BM 13901 #10, page 48, see note 25. In his *mathematical commentaries* Neugebauer expressed these as respectively $A-B = d$ and $B-A = -d$ ($A = B+d$ and $B = A-d$ would have been closer to the ancient texts, but even Neugebauer had his reasons of style). In this way, mathematicians who only read the translations into formulas and not the explanations of the meaning of these (and certainly not the translated texts) found their “Babylonian” negative numbers.

As the French Orientalist Léon Rodet wrote in 1881 in when criticizing modernizing interpretations of an ancien Egyptian mathematical papyrus: “For studying the history of a science, just as when one wants to obtain something, ‘it is better to have business with God than with his saints’”.[22]

## *BM 13901 #2*

**Obv. I**

**5.** My confrontation inside the surface I have torn out: 14`30 is it. 1, the projection,

**6.** you posit. The moiety of 1 you break, 30′ and 30′ you make hold,

**7.** 15′ to 14`30 you join: by 14`30°15′, 29°30′ is equal.

**8.** 30′ which you have made hold to 29°30′ you join: 30 the confrontation.

[22] Léon Rodet, *Journal asiatique*, septième série **18**, p. 205.

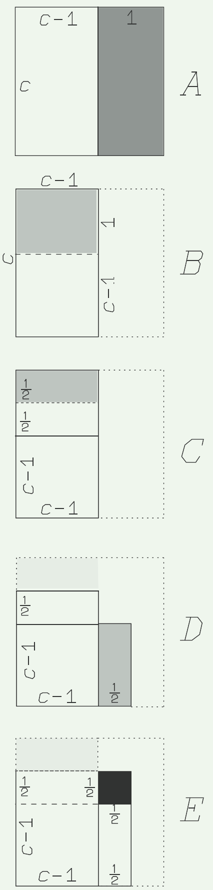


*Figure 14*. The procedure of BM 13901 #2

This problem, on a tablet which contains in total 24 problems of increasing sophistication dealing with one or more squares, follows immediately after the one we have just examined.

From the Old Babylonian point of view as well as ours, it is its "natural" counterpart. Where the preceding one "joins", this one "tears out". The basic part of the procedure is identical: the transformation of a rectangle into a gnomon, followed by a quadratic complement.

Initially the problem is stated (line 5): *My confrontation inside the surface I have torn out: 14`30 is it*. Once again the problem thus concerns a square area and side, but this time the "confrontation" $c$ is "torn out".

To "tear out" is a concrete subtraction by removal, the inverse of the "joining" operation, used only when that which is "torn out" is part of that magnitude from which it is "torn out"[23]. The "confrontation" $c$ is thus seen as part of (the inside of) the area. Figure 14A shows how this is possible: the "confrontation" $c$ is provided with a width (a "projection") 1 and thereby changed into a rectangle ⊏⊐($c$,1), located inside the square. This rectangle (shaded in dark grey) must thus be "torn out"; what remains after we have eliminated ⊏⊐($c$,1) from □($c$) should be 14`30. In modern symbols, the problem corresponds to

[23] The inverse of the "heaping" operation, on the other hand, is no subtraction at all but a *separation into constitutive elements*. See note 54, page 106.

$$\square(c)-c = 14`30\ .$$

Once more, we are left with a rectangle for which we know the area (14`30) and the difference between the length ($c$) and the width ($c-1$) – and once more, this difference amounts to 1, namely the “projection”.

*1, the projection, you posit*. In Figure 14B, the rectangle $⊏⊐(c,c-1)$ is composed of a (white) square and a (shaded) “excess” rectangle whose width is the projection 1.

*The moiety of 1 you break*. The excess rectangle, presented by its width 1, is divided into two “moieties”; the one which is detached is shaded in Figure 14C.

Cutting and pasting this rectangle as seen in Figure 14D we once again get a gnomon with the same area as the rectangle $⊏⊐(c,c-1)$, that is, equal to 14`30.

*30′ and 30′ you make hold, 15′*. The gnomon is completed with the small square (black in Figure 14E) which is “held” by the two moieties. The area of this completing square equals 30′×30′ = 15′.

Next, the area of the completed square and its side are found: *15′ to 14`30 you join: by 14`30°15′, 29°30′ is equal*.

Putting back the “moiety” which was moved around, we find the side of the initial square, which turns out to be 29°30′+30′ = 30: *30′ which you have made hold to 29°30′ you join: 30 the confrontation*.

We notice that this time the “confrontation”of the square is 30, not 30′. The reason is simple and compelling: unless $c$ is larger than 1, the area will be smaller than the side, and we would have to “tear out” more than is available, which evidently cannot be done. As already explained, the Babylonians were familiar with “subtractive magnitudes”, that is, magnitudes that are predetermined to be “torn out”; but nothing in their mathematical thought corresponded to our negative numbers.

We also notice that the pair (14`30°15′, 29°30′) does not appear in the table of squares and square roots (see page 20); the problem is thus constructed backwards from a known solution.

## *YBC 6967*

**Obv.**

**1.** The *igibûm* over the *igûm*, 7 it goes beyond
**2.** *igûm* and *igibûm* what?
**3.** You, 7 which the *igibûm*
**4.** over the *igûm* goes beyond
**5.** to two break: 3°30´;
**6.** 3°30´ together with 3°30´
**7.** make hold: 12°15´.
**8.** To 12°15´ which comes up for you
**9.** 1` the surface join: 1`12°15´.
**10.** The equal of 1`12°15´ what? 8°30´.
**11.** 8°30´ and 8°30´, its counterpart, lay down.

**Rev.**

**1.** 3°30´, the made-hold,
**2.** from one tear out,
**3.** to one join.
**4.** The first is 12, the second is 5.
**5.** 12 is the *igibûm*, 5 is the *igûm*.

Second-degree problems dealing with rectangles are more copious than those about squares. Two problem types belong to this category; others, more complex, can be reduced to these basic types. In one of these, the area and the sum of the sides is known; in the other, the area and their difference are given.

The above exercise belongs to the latter type – if we neglect the fact that it does not deal with a rectangle at all but with a pair of numbers belonging together in the table of reciprocals (see page 17 and Figure 2). *Igûm* is the Babylonian pronunciation of Sumerian IGI, and *igibûm* that of IGI.BI, "its IGI" (the relation between the two is indeed symmetric: if 10´ is IGI 6, then 6 is IGI 10´).

One might expect the product of *igûm* and *igibûm* to be 1; in the present problem, however, this is not the case, here the product is supposed to be 1`, that is, 60. The two numbers are represented by the sides of a rectangle of area 1` (see line F.9); the situation is depicted

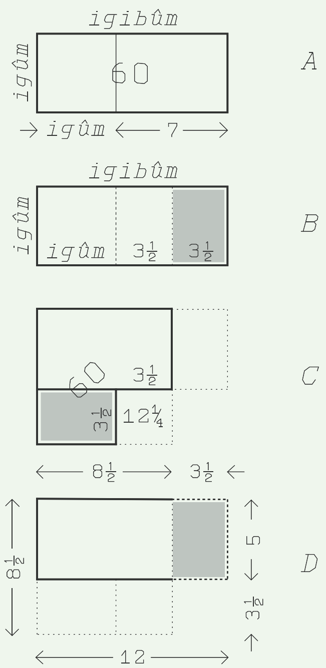


*Figure 15*. The procedure of YBC 6967

in Figure 15A. Once more we thus have to do with a rectangle with known area and known difference between the length and the width, respectively 1` and 7.

It is important to notice that here the "fundamental representation" (the measurable geometric quantities) serves to represent magnitudes of a different kind: the two numbers *igûm* and *igibûm*. In our algebra, the situation is the inverse: *our* fundamental representation is provided by the realm of abstract numbers, which serves to represent magnitudes of other kinds: prices, weights, speeds, distances, etc. (see page 11).

As in the two analogous cases that precede, the rectangle is transformed into a gnomon, and as usually the gnomon is completed as a square "held" by the two "moieties" of the excess (lines F.3 – 10). The procedure can be followed on the Figures 15B and 15C.

The next steps are remarkable. The "moiety" that was detached and moved around (the "made-hold", that is, that which was "made hold" the complementary square) in the formation of the gnomon is put back into place. Since it is *the same* piece which is concerned it must in principle be available before it can be "joined". That has two consequences. Firstly, the "equal" 8°30′ must be "laid down[24]" twice, as we see in Figure 15D: in this way, the piece can be "torn out" from one (leaving the width *igûm*) and "joined" to the other (giving the length *igibûm*). Secondly, "tearing-out" must precede "joining" (lines R.1–3), even though the Babylonians (as we) would normally prefer to add before subtracting – cf. BM 13901 #1–2: the first problem adds the side, the second subtracts: *3°30′, the made-hold, from one tear out, to one join*.

In BM 13901 #1 and #2, the complement was "joined" to the gnomon, here it is the gnomon that is "joined". Since both remain in place, either is possible. When 3°30′ is joined to 8°30′ in the construction of the *igibûm*, this is not the case: if one magnitude stays in place and the other is displaced it is always the latter that is "joined". Differently from our addition and the "heaping" of the Babylonians, "joining" is no symmetric operation.

### *BM 13901 #10*

**Obv. II**

**11.** The surfaces of my two confrontations I have heaped: 21°15′.
**12.** Confrontation (compared) to confrontation, the seventh it has become smaller.
**13.** 7 and 6 you inscribe. 7 and 7 you make hold, 49.
**14.** 6 and 6 you make hold, 36 and 49 you heap:
**15.** 1\`25. IGI 1\`25 is not detached. What to 1\`25
**16.** may I posit which 21°15′ gives me? By 15′, 30′ is equal.

[24] The verb in question (*nadûm*) has a broad spectrum of meanings. Among these are "to draw" or "to write" (on a tablet) (by the way, the word *lapātum*, translated "to inscribe", has the same two meanings). Since what is "laid down" is a numerical value, the latter interpretation could seem to be preferable – but since geometrical entities were regularly identified by means of their numerical measure, this conclusion is not compulsory.

**17.** 30′ to 7 you raise: 3°30′ the first confrontation.
**18.** 30′ to 6 you raise: 3 the second confrontation.

We now return to the tablet containing a collection of problems about squares, looking at one of the simplest problems about two squares. Lines 11 and 12 contain the statement: the sum of the two areas is told to be 21°15′, and we are told that the second "confrontation" falls short of the first by one seventh.[25] In symbols, if the two sides are designated respectively $c_1$ and $c_2$:

$$\square(c_1)+\square(c_2) = 21°15' \ , \ \ c_2 = c_1 - \tfrac{1}{7}c_1 \ .$$

Formulated differently, the ratio between the two sides is as 7 to 6. This is the basis for a solution based on a "false position" (see page 31). Lines 13 and 14 prescribe the construction of two "model squares" with sides 7 and 6 (making these sides "hold", see Figure 16), and finds that their total area will be 49+36 = 1`25. According to the statement, however, the total *should* be 21°15′; therefore, the area must be reduced by a factor 21°15′/1`25. Now 1`25 is no "regular" number (see page 18) – that it, it has no IGI: *IGI 1`25 is not detached*. We must thus draw the quotient "from the sleeves" – as done in lines 15–16, where it is said to be 15′ (that is, $\frac{1}{4}$). However, if the area is reduced by a factor 15′, then the corresponding sides must be reduced by a factor 30′: *By 15′, 30′ is equal*. Remains finally (lines 17 and 18) to "raise" 7 and 6 to 30′.

The first "confrontation" thus turns out to be 7·30′ = 3°30′, and

[25] Here we see one of the stylistic reasons that would lead to a formulation in terms of falling-short instead of excess. It might as well have been said that one side exceeds the other by one sixth, but in the "multiplicative-partitive" domain the Babylonians gave special status to the numbers 4, 7, 11, 13, 14 and 17. In the next problem on the tablet, one "confrontation" is stated to exceed the other by one seventh, while it would be just as possible to say that the second falls short of the first by one eighth.

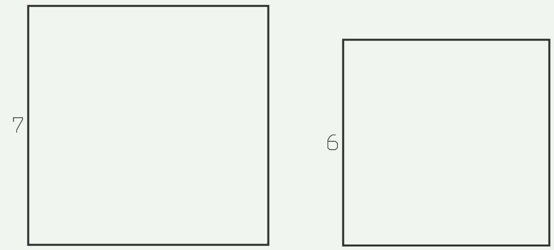


*Figure 16*. The two squares of BM 13901 #10

the second 6·30′ = 3[26].

### *BM 13901 #14*

**Obv. II**

**44.** The surfaces of my two confrontations I have heaped: 25′25″.
**45.** The confrontation, two-thirds of the confrontation and 5′, NINDAN.
**46.** 1 and 40′ and 5′ over-going 40′ you inscribe.
**47.** 5′ and 5′ you make hold, 25″ inside 25′25″ you tear out:

**Rev. I**

**1.** 25′ you inscribe. 1 and 1 you make hold, 1. 40′ and 40′ you make hold,
**2.** 26′40″ to 1 you join: 1°26′40″ to 25′ you raise:
**3.** 36′6″40‴ you inscribe. 5′ to 40′ you raise: 3′20″
**4.** and 3′20″ you make hold, 11″6‴40⁗ to 36′6″40‴ you join:
**5.** by 36′17″46‴40⁗, 46′40″ is equal. 3′20″ which you have made hold
**6.** inside 46′40″ you tear out: 43′20″ you inscribe.

---

26 One might believe the underlying idea to be slightly different, and suppose that the original squares are subdivided into 7×7 respectively 6×6 smaller squares, of which the total number would be 1`25, each thus having an area equal to $\frac{21°15'}{1`25}$ = 15′ and a side of 30′. However, this interpretation is ruled out by the use of the operation “to make hold”: Indeed, the initial squares are already there, and there is thus no need to construct them (in TMS VIII no 1 we shall encounter a subdivision into smaller squares, and there their number is indeed found by “raising” – see page 83).

**7.** IGI 1°26′40″ is not detached. What to 1°26′40″
**8.** may I posit which 43′20″ gives me? 30′ its *bandûm*.
**9.** 30′ to 1 you raise: 30′ the first confrontation.
**10.** 30′ to 40′ you raise: 20′, and 5′ you join:
**11.** 25′ the second confrontation.

Even this problem deals with two squares (lines F.II.44–45).[27] The somewhat obscure formulation in line 45 means that the second "confrontation" equals two-thirds of the first, with additional 5′ NINDAN. If $c_1$ and $c_2$ stands for the two "confrontations", line 44 informs us that the sum of the areas is $\square(c_1)+\square(c_2) = 25'25''$, while line 45 states that $c_2 = 40'\cdot c_1+5'$.

This problem cannot be solved by means of a simple false position in which a hypothetical number is provisionally assumed as the value of the unknown – that only works for homogeneous problems[28]. The numbers 1 and 40′ in line 46 show us the way that is actually chosen: $c_1$ and $c_2$ are expressed in terms of a *new magnitude*, which we may call $c$:

$$c_1 = 1\cdot c\ ,\ \ c_2 = 40'\cdot c+5'\ .$$

That corresponds to Figure 17. It shows how the problem is reduced to a simpler one dealing with a single square $\square(c)$. It is clear that the area of the first of the two original squares ($\square(c_1)$) equals 1×1 $\square(c)$, but that calculations has to wait until line R.I.1. The text begin by considering $\square(c_2)$, which is more complicated and gives rise to several contributions. First, the square $\square(5')$ in the lower right corner: *5′ and 5′ you make hold, 25″*. This contribution is eliminated from the sum 25′25″ of the two areas: *25″ inside 25′25″ you tear out: 25′ you*

[27] This part of the tablet is heavily damaged. However, #24 of the same tablet, dealing with three squares but otherwise strictly parallel, allows an unquestionable reconstruction.

[28] In a simple false position, indeed, the provisionally assumed number has to be reduced by a factor corresponding to the error that is found; but if we reduce values assumed for $c_1$ and $c_2$ with a certain factor – say, $\frac{1}{5}$ – then the additional 5′ would be reduced by the same factor, that is, to 1′. After reduction we would therefore have $c_2 = \frac{2}{3}c_1+1'$.

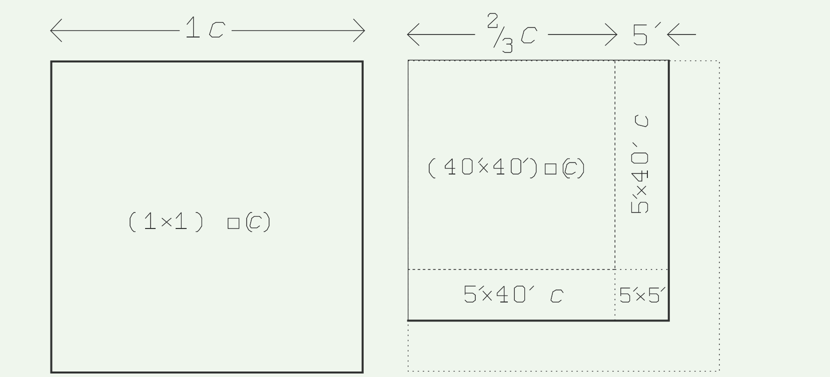

*Figure 17*. The two squares of BM 13901 #14

*inscribe*. The 25´ that remains must now be explained in terms of the area and the side of the new square □($c$).

□($c_1$), as already said, is 1×1 = 1 times the area □($c$): *1 and 1 you make hold, 1*[29]. After elimination of the corner 5´×5´ remains of □($c_2$), on one hand, a square □(40´$c$), on the other, two "wings" to which we shall return imminently. The area of the square □(40´$c$) is (40´×40´) □($c$) = 26´40″ □($c$): *40´ and 40´ you make hold, 26´40″*. In total we thus have 1+26´40″ = 1°26´40″ times the square area □($c$): *26´40″ to 1 you join: 1°26´40″*.

Each "wing" is a rectangle ⊏⊐(5´,40´$c$), whose area can be written 5´·40´$c$ = 3´20″$c$: *5´ to 40´ you raise: 3´20″*. All in all we thus have the equation

$$1°26´40″□(c)+2·3´20″c = 25´ .$$

This equation confronts us with a problem which the Old Babylonian author has already foreseen in line R.I.2, and which has caused him to postpone until later the calculation of the wings. In modern terms, the equations is not "normalized, that is, the coefficient of the second-degree term differs from 1. The Old Babylonian calculator might correspondingly have explained it by stating in the terminology of TMS XVI that "as much as (there is) of surfaces" is not one – see the left part of Figure 18, where we have a sum of α

[29] This meticulous calculation shows that the author thinks of a *new* square, and does not express □($c_2$) in terms of □($c_1$) and $c_1$.

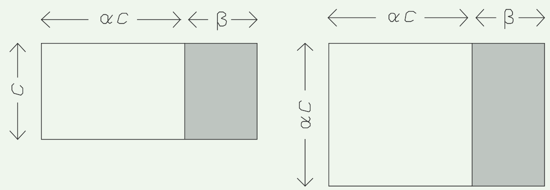


*Figure 18*. Transformation of the problem $\alpha\square(c)+\beta c=\Sigma$

square areas (the white rectangle $\sqsubset\!\sqsupset(c,\alpha c)$) and β sides, that is, the shaded rectangle $\sqsubset\!\sqsupset(c,\beta)$), corresponding to the equation

$$\alpha\square(c)+\beta c=\Sigma$$

(in the actual case, $\alpha = 1°26'40''$, $\beta = 2\cdot3'20''$, $\Sigma = 25'$). This prevents us from using directly our familiar cut-and-paste procedure. “Breaking” β and making the two “moieties” “hold” would not give us a gnomon.

The Babylonians got around the difficulty by means of a device shown in the right-hand side of Figure 18: the scale of the configuration is changed in the vertical direction, in such a way that the vertical side becomes $\alpha c$ instead of $c$; in consequence the sum of the two areas is no longer $\Sigma$ (= 25′) but $\alpha\Sigma$ (= $1°26'40''\cdot25'$ = $36'6''40'''$): *1°26′40″ to 25′ you raise: 36′6″40‴ you inscribe*. As we see, the number β of sides is not changed in the operation, only the value of the side, namely from $c$ into $\alpha c$[30].

In modern symbolic language, this transformation corresponds to a multiplication of the two sides of the equation

$$\alpha c^2+\beta c=\Sigma$$

by α, which gives us a normalized equation with unknown the $\alpha c$:

[30] This device was used constantly in the solution of non-normalized problems, and there is no reason to suppose that the Babylonians needed a specific representation similar to Figure 18. They might imagine that the *measuring scale* was changed in one direction – we know from other texts that their diagrams could be very rough, mere structure diagrams – nothing more than was required in order to guide thought. All they needed was thus to multiply the sum Σ by α, and that they could (and like here, would) do before calculating β.

$$(\alpha c)^2+\beta\cdot(\alpha c) = \alpha\Sigma ,$$

an equation of the type we have encountered in BM 13901 #1. We have hence arrived to a point where we can apply the habitual method: "breaking" the shaded rectangle and make the two resulting "moieties" "hold" a quadratic complement (see Figure 19; the outer "moiety" is lightly shaded in its original position and more heavily in the position to which it is brought). Now, and only now, does the calculator need to know the number of sides in the shaded rectangle of Figure 18 (that is, to determine β). As already said, each "wing" contributes 5′·40′ = 3′20″ sides. If the calculator had worked mechanically, according to fixed algorithms, he would now have multiplied by 2 in order to find β. But he does not! He knows indeed that the two wings constitute the excess that has to be "broken" into two "moieties". He therefore directly makes 3′20″ and 3′20″ "hold", which produces the quadratic complement, and "joins" the resulting area 11″6‴40⁗ to that of the gnomon 36′6″40‴: *3′20″ and 3′20″ you make hold, 11″6‴40⁗ to 36′6″40‴ you join:* [...] *36′17″46‴40⁗*.

36′17″46‴40⁗ is thus the area of the completed square, and its side √36′17″46‴40⁗ = 46′40″: *by 36′17″46‴40⁗, 46′40″ is equal*. This number represents 1°26′40″·$c$+3′20″; therefore, 1°26′40″$c$ is 46′40″−3′20″ = 43′20″: *3′20″ which you have made hold inside 46′40″ you tear out: 43′20″ you inscribe*. Next, we must find the value of $c$. 1°26′40″ is an irregular number, and the quotient 46′40″/1°26′40′ is given directly as 30′:[31] *IGI 1°26′40″ is not detached. What to 1°26′40″ may I posit which 43′20″ gives me? 30′ its bandûm*.

In the end, $c_1$ and $c_2$ are determined, $c_1$ = 1·$c$ = 30′[32], $c_2$ = 40′·$c$+5′ = 25′: *30′ to 1 you raise: 30′ the first confrontation. 30′ to 40′ you raise: 20′, and 5′ you join: 25′ the second confrontation*.

The problem is solved.

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[31] The quotient is called BA.AN.DA. This Sumerian term could mean "that which is put at the side", which would correspond to way multiplications were performed on a tablet for rought work, cf. note 11, page 18.

[32] That the value of $c_1$ is calculated as 1·$c$ and not directly identified with $c$ confirms that we have been working with a *new* side $c$.

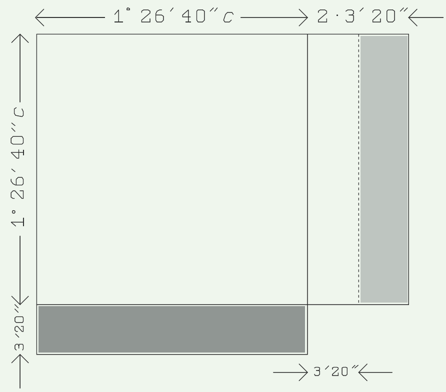


*Figure 19*. BM 13901 #14, the normalized problem

## *TMS IX #1 and #2*

**#1**

**1.** The surface and 1 length I have heaped, 40′. ¿30, the length,? 20′ the width.

**2.** As 1 length to 10′ the surface, has been joined,

**3.** or 1 (as) base to 20′, the width, has been joined,

**4.** or 1°20′ ¿is posited? to the width which 40′ together with the length ¿holds?

**5.** or 1°20′ toge⟨ther⟩ with 30′ the length holds, 40′ (is) its name.

**6.** Since so, to 20′ the width, which is said to you,

**7.** 1 is joined: 1°20′ you see. Out from here

**8.** you ask. 40′ the surface, 1°20′ the width, the length what?

**9.** 30′ the length. Thus the procedure.

**#2**

**10.** Surface, length, and width I have heaped, 1. By the Akkadian (method).

**11.** 1 to the length join. 1 to the width join. Since 1 to the length is joined,

**12.** 1 to the width is joined, 1 and 1 make hold, 1 you see.

**13.** 1 to the heap of length, width and surface join, 2 you see.

**14.** To 20′ the width, 1 join, 1°20′. To 30′ the length, 1 join, 1°30′.

**15.** ¿Since? a surface, that of 1°20′ the width, that of 1°30′ the length,

**16.** ¿the length together with? the width, are made hold, what is its

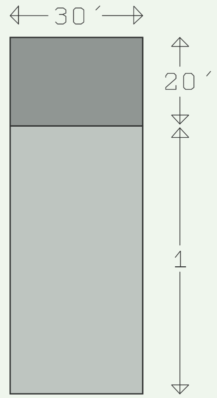


*Figure 20*. TMS IX, #1

name?
**17.** 2 the surface.
**18.** Thus the Akkadian (method).

As TMS XVI #1, sections #1 and #2 of the present text solve no problem[33]. Instead they offer a pedagogical explanation of the meaning to ascribe to the addition of areas and lines, and of the operations used to treat second-degree problems. Sections #1 and #2 set out two different situations. In #1, we are told the sum of the area and the length of a rectangle; in #2, the sum of area, length and width is given. #3 (which will be dealt with in the next chapter) is then a genuine problem that is stated and solved in agreement with the methods taught in #1 and #2 and in TMS XVI #1.

Figure 20 is drawn in agreement with the text of #1, in which the sum of a rectangular area and the corresponding length is known. In parallel with our symbolic transformation

$$\ell{\cdot}w+\ell = \ell{\cdot}w+\ell\cdot 1 = \ell\cdot(w+1)\ ,$$

[33] The tablet is rather damaged; as we remember, passages in ¿...? are reconstructions that render the meaning (which can be derived from the context) but not necessarily the exact words of the original.

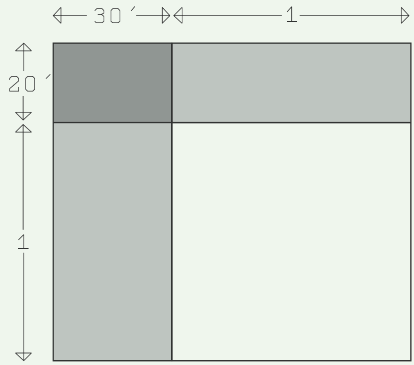


*Figure 21*. TMS IX, #2

the width is extended by a “base[34]”. That leads to a whole sequence of explanations, mutually dependent and linked by “or ... or ... or”, curiously similar to how we speak about the transformations of an equation, for example

$$\text{“}2a^2-4 = 4,\text{ or } 2a^2 = 4+4,\text{ or } a^2 = 4,\text{ or } a = \pm\sqrt{4} = \pm 2\text{”} .$$

Line 2 speaks of the “surface” as 10´. This shows that the student is once more supposed to know that the discussion deals with the rectangle ⊏⊐(30´,20´). The tablet is broken, for which reason we cannot know whether the length was stated explicitly, but the quotation in line 6 shows that the width was.

In the end, lines 7–9 shows how to find the length once the width is known together with the sum of area and length (by means of a division that remains implicit).

#2 teaches how to confront a more complex situation; now the sum of the area and both sides is given (see Figure 21). Both length and width are prolonged by 1; that produces two rectangles ⊏⊐($\ell$,1) and ⊏⊐($w$,1), whose areas, respectively, are the length and the width. But it also produces an empty square corner ⊏⊐(1,1). When it is filled we have a larger rectangle of length $\ell+1$ (= 1°30´), width $w+1$

[34] The word KI.GUB.GUB is a composite Sumerian term that is not known from elsewhere and which could be an *ad hoc* construction. It appears to designate something stably placed on the ground.

(= 1°20′) and area $1+1=2$; a check confirms that the rectangle "held" by these two sides is effectively of area 2.

This method has a name, which is very rare in Old Babylonian mathematics (or at least in its written traces). It is called "the Akkadian (method)". "Akkadian" is the common designation of the language whose main dialects are Babylonian and Assyrian (see the box "Rudiments of general history"), and also of the major non-Sumerian component of the population during the third millennium; there is evidence (part of which is constituted by the present text) that the Old Babylonian scribe school took inspiration for its "algebra" of the practice of an Akkadian profession of surveyors (we shall discuss this topic on page 114 ). The "Akkadian" method is indeed nothing but a *quadratic completion* albeit a slightly untypical variant, that is, the basic tool for the solution of all mixed second-degree problems (be they geometric or, as with us, expressed in number algebra); and it is precisely this basic tool that is characterized as the "Akkadian (method)"

# Chapter 3

# Complex second-degree problems

The preceding chapter set out the methods used by the Babylonians for the solution of the fundamental second-degree problems – cut-and-paste, quadratic completion, change of scale. However, as inherent in the term “fundamental”, the Babylonians also worked on problems of a more complex nature. Such problems are in focus in the present chapter, which first takes up the third section of the text of which we have just examined the two introductory pedagogical sections.

## *TMS IX #3*

**#3**

**19.** Surface, length, and width I have heaped, 1 the surface. 3 lengths, 4 widths heaped,
**20.** its 17th to the width joined, 30´.
**21.** You, 30´ to 17 go: 8°30´ you see.
**22.** To 17 widths 4 widths join, 21 you see.
**23.** 21 as much as of widths posit. 3, of three lengths,
**24.** 3, as much as lengths posit. 8°30´, what is its name?
**25.** 3 lengths and 21 widths heaped.
**26.** 8°30´ you see
**27.** 3 lengths and 21 widths heaped.
**28.** Since 1 to the length is joined and 1 to the width is joined, make hold:
**29.** 1 to the heap of surface, length, and width join, 2 you see,
**30.** 2 the surface. Since the length and the width of 2 the surface,
**31.** 1°30´, the length, together with 1°20´, the width, are made hold,
**32.** 1 the joined of the length and 1 the joined of the width,
**33.** make hold, ¿1 you see.? 1 and 1, the various (things), heap, 2 you see.
**34.** 3 ..., 21 ..., and 8°30´ heap, 32°30´ you see;
**35.** so you ask.

**36.** ... of widths, to 21, that heap:
**37.** ... to 3, lengths, raise,
**38.** 1`3 you see. 1`3 to 2, the surface, raise:
**39.** 2`6 you see, ¿2`6 the surface?. 32°30′ the heap break, 16°15′ you ⟨see⟩.
**40.** {...}. 16°15′ the counterpart posit, make hold,
**41.** 4`24°3′45″ you see. 2`6 ¿erasure?
**42.** from 4`24°3′45″ tear out, 2`18°3′45″ you see.
**43.** What is equal? 11°45′ is equal, 11°45′ to 16°15′ join,
**44.** 28 you see. From the 2nd tear out, 4°30′ you see.
**45.** IGI 3, of the lengths, detach, 20′ you see. 20′ to 4°30′
**46.** {...} raise: 1°30′ you see,
**47.** 1°30′ the length of 2 the surface. What to 21, the widths, may I posit
**48.** which 28 gives me? 1°20′ posit, 1°20′ the width
**49.** of 2 the surface. Turn back. 1 from 1°30′ tear out,
**50.** 30′ you see. 1 from 1°20′ tear out,
**51.** 20′ you see.

Lines 19 and 20 present a system of two equations about a rectangle, one of the first and one of the second degree. The former is of the same type as the one explained in TMS XVI #1 (see page 25). the second coincides with the one that was examined in section #2 of the present text (see page 55). In symbolic translation, the equation system can be written

$$\tfrac{1}{17}(3\ell+4w)+w = 30' \ , \quad \sqsubset\!\sqsupset(\ell,w)+\ell+w = 1 \ .$$

In agrement with what we have seen elsewhere, the text multiplies the first-degree equation by 17 (using the Akkadian verb "to go", see page 15), thus obtaining integer coefficients (*as much as*):

$$3\ell+(4+17)\,w = 3\ell+21w = 17\cdot 30' = 8°30' \ .$$

This is done in the lines 21–25, while the lines 26 and 27 summarize the result.

Lines 28–30 repeat the trick used in section #2 of the text (see Figure 21): the length and the width are prolonged by 1, and the square that is produced when the two "joined[35]" "hold" is "joined"

[35] As the "to-be-joined" of page 33, this noun (*wuṣubbûm*) is derived from the

to the “heap” $\sqsubset\!\sqsupset(\ell, w)+\ell+b$; out of this comes a “surface 2”, the meaning of which is again explained in lines 30–33.

The lines 34–37 are very damaged, too damaged to be safely reconstructed as far as their words are concerned. However, the numbers suffice to see how the calculations proceed. Let us introduce the magnitudes $\lambda = \ell+1$ and $\phi = w+1$. The text refers to them as the length and width “of the surface 2” – in other words, $\sqsubset\!\sqsupset(\lambda,\phi) = 2$. Further,

$$\begin{aligned} 3\lambda+21\phi &= 3\cdot(\ell+1)+21\cdot(w+1) \\ &= 3+21+3\ell+21w \\ &= 3+21+8°30' \\ &= 32°30' \, . \end{aligned}$$

In order to facilitate the understanding of what now follows we may further introduce the variables

$$L = 3\lambda \, , \quad W = 21\phi$$

(but we must remember that the text has no particular names for these – in contrast to λ and ϕ which do have names; we now speak *about*, not *with* the Babylonian author). Lines 36–39 find that

$$\sqsubset\!\sqsupset(L, W) = (21\cdot 3)\cdot 2 = 1`3°\cdot 2 = 2`6° \, ;$$

summing up we thus have

$$L+W = 32°30' \, , \quad \sqsubset\!\sqsupset(L, W) = 2`6° \, .$$

We have now come to line 39, and arrived at a problem type which we had not seen so far: A rectangle for which we know the area and the *sum* of the two sides.

Once again, a cut-and-paste method is appealed to (see Figure 22). As before, the known segment is “broken” together with the rectangle which goes with it. In the present situation, this segment is the sum of $L$ and $W$. This rectangle is composed from $\sqsubset\!\sqsupset(L, W)$, traced in full, and a square $\square(L)$ to its right, drawn with a dotted line. Next, we let the two “moieties” of this segment “hold” a square (lines 39–40). As we see, that part of the original rectangle $\sqsubset\!\sqsupset(L, W)$ which falls outside

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verb “to join”.

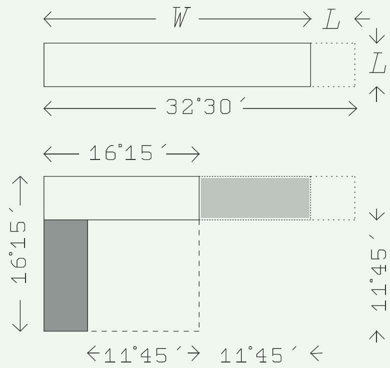


*Figure 22*. The cut-and-paste method of TMS IX #3

the new square can just be fitted into it so as to form a gnomon together with that part which stays in place. In its original position, this piece appears in light shading, whereas it is darkly shaded in its new position.

One part of the new square □(16°15′) is constituted by the gnomon, whose area results from recombination of the original rectangle ⊏⊐$(L, W)$; this area is hence 2\`6. We also know the area of the outer square, 16°15′×16°15 = 4\`24°3′45″ (lines 40 and 41). When the gnomon is "torn out" (lines 41 and 42), 2\`18°3′45″ remains for the square contained by the gnomon. Its side (that which "is equal") is 11°45′, which must now be "joined" to one of the pieces 16°15′ (which gives us $W$) and "torn out" from the other, its "counterpart" (which gives us $L$). This time, however, it is not *the same* piece that is "joined" and "torn out"; there is hence no reason to "tear out" before "joining", as in YBC 6967 (page 48), and the normal priority of addition can prevail. Lines 43–44 find $W = 28$ and $L = 4°30′$. Finally, the text determines first $\lambda$ and $\phi$ and then $\ell$ and $w$ – we remember that $L = 3\lambda$, $\lambda = \ell+1$, $W = 21\phi$, $\phi = w+1$. Since 28 has no IGI, line 48 explains that $21 \cdot 1°20′ = 28$.

## *AO 8862 #2*

**I**

**30.** Length, width. Length and width
**31.** I have made hold: A surface I have built.
**32.** I turned around (it). The half of the length
**33.** and the third of the width
**34.** to the inside of my surface
**35.** I have joined: 15.
**36.** I turned back. Length and width
**37.** I have heaped: 7.

**II**

**1.** Length and width what?
**2.** You, by your proceeding,
**3.** 2 (as) inscription of the half
**4.** and 3 (as) inscription
**5.** of the third you inscribe:
**6.** IGI 2, 30´, you detach:
**7.** 30´ steps of 7, 3°30´; to 7,
**8.** the things heaped, length and width,
**9.** I bring:
**10.** 3°30´ from 15, my things heaped,
**11.** cut off:
**12.** 11°30´ the remainder.
**13.** Do not go beyond. 2 and 3 make hold:
**14.** 3 steps of 2, 6.
**15.** IGI 6, 10´ it gives you.
**16.** 10´ from 7, your things heaped,
**17.** length and width, I tear out:
**18.** 6°50´ the remainder.
**19.** Its moiety, that of 6°50´, I break:
**20.** 3°25´ it gives you.
**21.** 3°25´ until twice
**22.** you inscribe; 3°25´ steps of 3°25´,
**23.** 11°40´25″; from the inside
**24.** 11°30´ I tear out:
**25.** 10´25″ the remainder. ⟨By 10´25″, 25´ is equal⟩.

**26.** To the first 3°25′
**27.** 25′ you join: 3°50′,
**28.** and (that) which from the things heaped of
**29.** length and width I have torn out
**30.** to 3°50′ you join:
**31.** 4 the length. From the second 3°25′
**32.** 25′ I tear out: 3 the width.
**32a.** 7 the things heaped.
**32b.** 4, the length
3, the width
12, the surface

The two first words of the first line (I.30) tell us that we are dealing with a figure that is fully characterized by its length and its width, that is, with a rectangle (cf. page 26) – or rather with a rectangular field: references to surveyors' practice can be found in the text (for instance, *I turned around it* in line I.32 probably means that the surveyor, after having laid out a field, has walked around it; in I.36 he *turned back*).

Before studying the procedure, we may concentrate on certain aspects of the formulation of the text. In line I.31 we see that the operation "to make hold" does not immediately produce a numerical result – since the measures of the sides are still unknown, that would indeed be difficult. The text only says that a "surface" has been "built"; we are probably meant to understand that it has been laid out in the terrain. Later, when two known segments are to "hold" (lines II.13–14, and perhaps II.21–22), the numerical determination of the area appears as a distinct operation, described with the words of the table of multiplication. Finally, we observe that the text defines the outcome of a "heaping" multiplication as a plural, translated "the things heaped", and that the normal alternating pattern of grammatical person is not respected.

The text, almost certainly from Larsa, seems to be from c. 1750 BCE and thus to belong to the early phase of the adoption of algebra by the southern scribe school (see page 117). These particularities may therefore give us information about the ideas on which it was based – such ideas were to become less visible once the language and format became standardized.

The topic of the problem is thus a rectangle. Lines I.36–37 tell us that the "heap" of its length and width is 7, while the lines I.32–35 state that "joining" half of the length and one third of the width to the "surface" produces 15[36]:

$$⊏⊐(\ell, w)+\frac{1}{2}\ell+\frac{1}{3}w = 15\ ,\quad \ell+w = 7\ .$$

The upper part of Figure 23 illustrates this situation, with 2 and 3 "inscribed as inscription" of $\frac{1}{2}$ respectively $\frac{1}{3}$ of the "projections" 1[37] of the length and the width (lines II.2–5); the heavily drawn configuration thus has an area equal to 15.

The solution *could* have followed the pattern of IX #3 (page 59). By introducing an "extended length" $\lambda = \ell+\frac{1}{3}$ and an "extended width" $\phi = w+\frac{1}{2}$, and adding (according to the "Akkadian method") the rectangle $⊏⊐(\frac{1}{2},\frac{1}{3})$ which is lacking in the corner where 2 and 3 are "inscribed", we would have reduced the problem to

$$⊏⊐(\lambda,\phi) = 15+⊏⊐(\frac{1}{2},\frac{1}{3}) = 15°10'\ ,$$
$$\lambda+\phi = 7+\frac{1}{2}+\frac{1}{3} = 7°50'\ .$$

However, the present text does not proceed like that – Old Babylonian algebra was a flexible instrument, not a collection of recipes or

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36 We should observe that the half that appears here is treated as any other fraction, on an equal footing with the subsequent third. It is not a "moiety", and the text finds it through multiplication by 30′, not by "breaking".

Let us also take note that the half of the length and the third of the width are "joined" to the "surface", not "heaped" together with it. A few other early texts share this characteristic. It seems that the surveyors thought in terms of "broad lines", strips possessing a tacitly understood breadth of 1 length unit; this practice is known from many pre-Modern surveying traditions, and agrees well with the Babylonian understanding of areas as "thick", provided with an implicit height of 1 KÙŠ (as inherent in the metrology of volumes, which coincides with that for areas – see page 13). The "projection" and "base" of BM 13901 and TMS IX #1 are likely to be secondary innovations due to the school – different schools, indeed, and therefore different words. They allowed to think of segments as truly one-dimensional while still permitting their transformation into rectangles with width 1.

37 The absence of this notion from the text should not prevent it from using it as a technical term of general validity.

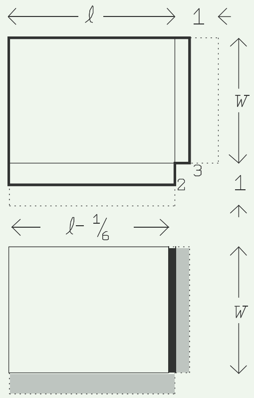


*Figure 23.* The reduction of AO 8862 #2

algorithms to be followed to the letter. The text finds the half of 7 (that is, of the sum of the length and the width) and "brings" the outcome 3°30′ to "the things heaped, length and width". "To bring" is no new arithmetical operation – the calculation comes afterwards. The text must be understood literally, the rectangle $\sqsubset\!\sqsupset(\ell+w,\frac{1}{2})$ (represented by the number 3°30′) is brought *physically* to the place where length and width (provided with widths $\frac{1}{2}$ and $\frac{1}{3}$) are to be found. In this way it becomes possible to "cut off" the rectangle $\sqsubset\!\sqsupset(\ell+w,\frac{1}{2})$ – as long as it was elsewhere that would make no sense. In bottom of Figure 23, the area that is eliminated is drawn shaded and black: the rest, in white, will be equal to 11°30′.

In this operation, it is obvious that the (shaded) half of the length that had been "joined" according to the statement has been eliminated. However, more than the (equally shaded) third of the width has disappeared. How much more precisely?

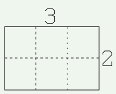


*Figure 24*

It would be easy to subtract 20′ (= $\frac{1}{3}$) from 30′ (= $\frac{1}{2}$), but that may not have been deemed sufficiently informative.[38] In any case, the text introduces a detour by the phrase "Do not go beyond!" (the same verb as in the "subtraction by comparison"). A rectangle $\sqsubset\!\sqsupset(3,2)$ is constructed (perhaps one should imagine it in the corner where 2 and 3 are "inscribed" in Figure 23; in any case Figure 24 shows the situation). Without further argument it is seen that the half (three small squares) exceeds the third (two small squares) by one of six small squares, that is, by a sixth – another case of reasoning by "false position". Exceptionally, IGI 6 is not "detached" but "given" (namely by the table of reciprocals).

We thus know that, in addition to the third of the width, we have eliminated a piece $\sqsubset\!\sqsupset(w,10')$ (drawn in black); if $\lambda = \ell - 10'$, we therefore have

$$\lambda + w = 7 - 10' = 6°50' \ , \quad \sqsubset\!\sqsupset(\lambda, w) = 11°30' \ .$$

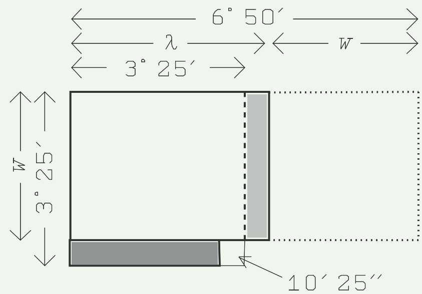


*Figure 25*

Once more we therefor have a rectangle of which we know the area and the sum of length and width. The procedure is the same as in the final part of TMS IX #3 – see Figure 25; the area that is to be displaced is shown again in light shading in the position from where it is to be taken and in heavy shading where it has to be placed. The only difference is terminological: in TMS IX #3, the two "moieties"

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[38] Alternatively, the trick used by the text could be a reminiscence of the ways of surveyors not too familiar with the place-value system; or (a third possibility) the floating-point character of this system might make it preferable to avoid it in contexts where normal procedures for keeping track of orders of magnitude (whatever these normal procedures were) were not at hand.

are "made hold", here they are "inscribed" – but since a multiplication of a number by a number follows immediately, the usual construction of a rectangle (here a square) must be intended (lines II.13 – 14)[39].

In the end, the final addition of the side of the square precedes the subtraction, as in TMS IX #3. Once more, indeed, it is not *the same* piece that is involved in the two operations; there is therefore no need to make it available before it is added.

## *VAT 7532*

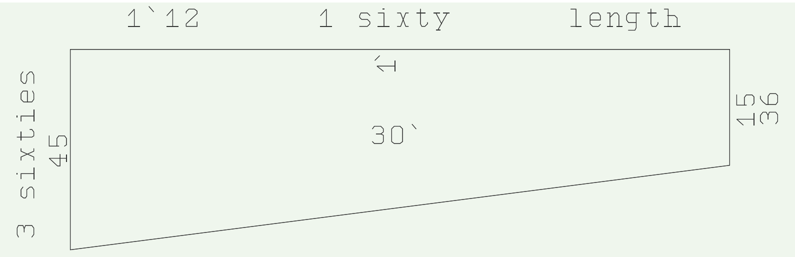


*Figure 26*. The diagram of VAT 7532
The "upper width" is to the left

**Obv.**

**1.** A trapezium. I have cut off a reed. I have taken the reed, by its integrity

**2.** 1 sixty (along) the length I have gone. The 6th part

**3.** broke off for me: 1`12 to the length I have made follow.

**4.** I turned back. The 3rd part and $\frac{1}{3}$ KÙŠ broke off for me:

**5.** 3 sixty (along) the upper width I have gone.

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[39] It is not quite to be excluded that the text does not directly describe the construction but refers to the inscription twice of 3°25´ on a tablet for rough work, followed by the numerical product – cf, above, note 11 (page 18); in that case, the construction itself will have been left implicit, as is the numerical calculation in other texts. Even the "inscription" of 2, followed by its IGI (II.3 and 6) might refer to this type of tablet. Then, however, one would expect that the "detachment" of the IGI should follow the inscription immediately; moreover, the inscription of 3 in line II.4 is not followed at all by "detachment" of its IGI, which after all speaks against this reading of the lines II.3 – 6 and II.21 – 22.

**6.** With that which broke off for me I enlarged it:
**7.** 36 (along) the width I went. 1 BÙR the surface. The head (initial magnitude) of the reed what?
**8.** You, by your proceeding, (for) the reed which you do not know,
**9.** 1 may you posit. Its 6th part make break off, 50′ you leave.
**10.** IGI 50′ detach, 1°12′ to 1 sixty raise:
**11.** 1`12 to ⟨1`12⟩ join: 2`24 the false length it gives you.
**12.** (For) the reed which you do not know, 1 may you posit. Its 3rd part make break off,
**13.** 40′ to 3 sixty of the upper width raise:
**14.** 2` it gives you. 2` and 36 the lower width heap,
**15.** 2`36 to 2`24 the false length raise, 6``14`24 the false surface.
**16.** The surface to 2 repeat, 1`` to 6``14`24 raise
**17.** 6````14```24`` it gives you. And $\frac{1}{3}$ KÙŠ which broke off
**18.** to 3 sixty raise: 5 to 2`24, the false length,
**19.** raise: 12`. $\frac{1}{2}$ of 12` break, 6` make encounter,

**Rev.**

**1.** 36`` to 6````14```24`` join, 6````15``` it gives you.
**2.** By 6````15```, 2``30′ is equal. 6` which you have left
**3.** to 2``30`` join, 2``36` it gives you. IGI 6``14`24,
**4.** the false surface, I do not know. What to 6``14`24
**5.** may I posit which 2``36 gives me? 25′ posit.
**6.** Since the 6th part broke off before,
**7.** 6 inscribe: 1 make go away, 5 you leave.
**8.** ⟨IGI 5 detach, 12′ to 25 raise, 5′ it gives you⟩. 5′ to 25′ join: $\frac{1}{2}$ NINDAN, the head of the reed it gives you.

This problem also deals with a field – yet with a field which the surveyor would only encounter in dream (or rather, in a nightmare). “Real life” enters through the reference to the unit BÙR, a unit belonging to practical agricultural administration, and through the reference to measuring by means of a reed cut for this purpose; its length ($\frac{1}{2}$ NINDAN) corresponds indeed to a measuring unit often used in practical life and called precisely a “reed” (GI in Sumerian). One may also imagine that such reeds would easily break. Finally, the use of the numeral “sixty” shows us one of the ways to express numbers unambiguously.

Everything else, however – that is, that the area of the field is known before it is measured, and also the ways to indicate the measures of the pieces that break off from the reed – shows which ruses the Old Babylonian school masters had to make use of in order to produce second-degree problems having some taste of practical life.

For once, Figure 26 reproduces a diagram that is traced on the tablet itself. In general, as also here, diagrams are only drawn on the tablets when they serve to clarify the statement; they are never used to explain the procedure. On the other hand, Figure 26 shows once more that the solution is known in advance: the numbers 1`, 45 and 15 are indeed the measures of the sides expressed in NINDAN.

We thus undertake to measure the trapezium by means of a reed of unknown length $R$. We manage to measure 1` reed lengths along the length of the trapezium before the reed loses a sixth of its length and is reduced to $r = \frac{5}{6}R$. What remain of the length turns out to be 1`12$r$ (F.2–3).

Then the reed breaks for the second time. According to lines F.4 and 5, the measure of the "upper width" (to the left)[40] is 3`$z$, where $z = \frac{2}{3}r - \frac{1}{3}$ KÙŠ is the length of the reed after this second reduction.

The piece that broke off last is put back into place, and the "(lower) width" (evidently to the right) is paced out (line F.7) as 36 $r$. Finally we learn that the area of the field is 1 BÙR = 30` SAR (1 SAR = 1 NINDAN$^2$, see page 13). We are asked to find the original length of the reed – its "head" in the sense of "beginning".

Lines F.9–11 determine the length in units $r$ by means of a false position: if $R$ had been equal to 1, then $r$ would have been 50′; conversely, $R$ must correspond to $r$ multiplied by IGI 50′ = 1°12′. 1` steps of $R$ thus correspond to 1`12·$r$, and the complete length will be

[40] The position of the "upper" width to the left is a consequence of the new orientation of the cuneiform script (a counterclockwise rotation of 90°) mentioned in the box "Cuneiform writing". On tablets, this rotation took place well before the Old Babylonian epoch, as a consequence of which one then wrote from left to right. But Old Babylonian scribes knew perfectly well that the *true* direction was vertically downwards – solemn inscriptions on stone (for example Hammurabi's law) were still written in that way. For reading, scribes may well have turned their tablets 90° clockwise.

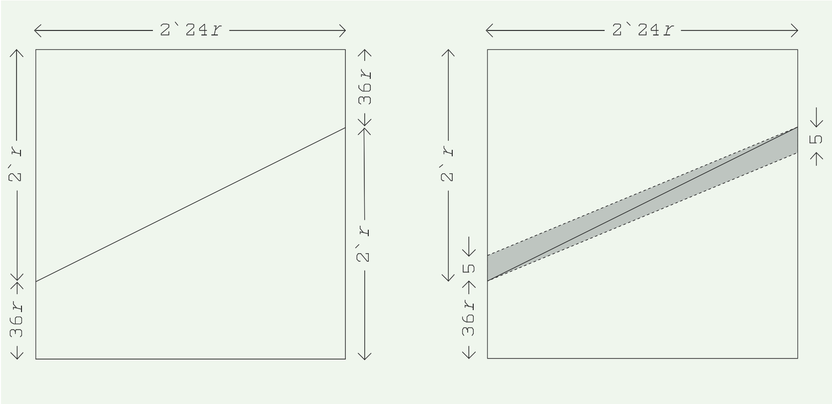


*Figure 27*. The doubled trapezium of VAT 7532

$$1`12\cdot r+1`12\cdot r = 2`24\cdot r.$$

The text speaks of 2`24 as the “false length”, that is, the length expressed in units *r*.

Another false position is applied in line F.12. The text posits 1 for the length *r* of the reed once shortened, and deducts that what remains after the loss of $\frac{1}{3}$ must be equal to 40´. Leaving aside the extra loss of $\frac{1}{3}$ KÙŠ, the false upper width (the upper width measured in units *r*) is thus 40´ times 3 sixties, that is, 40´·3` = 2`. In other words, the upper width measures 2`*r* – still leaving aside the missing piece of $\frac{1}{3}$ KÙŠ.

Since line F.7 indicates that the false (lower) width is 36, we thus know – with the same reserve concerning the missing $\frac{1}{3}$ KÙŠ – the three sides that will allow us to determine the area of the trapezium in units □(*r*).

Yet the text does not calculate this area: *The surface to 2 repeat*. Instead it doubles the trapezium so as to form a rectangle (see the left part of Figure 27), and the lines F.14–16 calculate the area of this rectangle (the “false surface”), finding 6``14`24 (in the implicit unit □(*r*)).

It the reed had not lost an ulterior piece of $\frac{1}{3}$ KÙŠ, we might now have found the solution by means of a final false position similar to that of BM 13901 #10 (see page 48): according to line F.7, the area of the field is 1 BÙR, the doubled area hence 2 BÙR = 1`` NINDAN$^2$ (F.16: *The surface to 2 repeat, 1``*). However, things are more complicated here. For each of the 3` steps made by the twice shortened reed a piece of $\frac{1}{3}$ KÙŠ is missing from our calculation, in total thus 3`·$\frac{1}{3}$ KÙŠ = 1` KÙŠ = 5 NINDAN (1 KÙŠ = $\frac{1}{12}$ NINDAN): *And* $\frac{1}{3}$ *KÙŠ which broke off to 3 sixty raise: 5* (F.17–18). Therefore the area of the real field does not correspond to what we see to the left in Figure 27 but to that which remains after elimination of the shaded strip to the right. The area of this strip is 5·2`24 *r* = 12`*r*: *5 to 2`24, the false length, raise: 12`*. The relation between the "false surface" and that of the doubled real trapezium can now be expressed by the equation

6``14`24□(*r*)–12`*r* = 1`` .

This non-normalized equation is solved in the usual way. First it is multiplied by 6``14`24: *1`` to 6``14`24 raise 6````14```24`` it gives you* (F.16–17). That leads to the normalized equation

□(6``14`24 *r*)–12`·(6``14`241 *r*) = 6````14```24``

or, with *s* = 6``14`241 *r* as unknown,

□(*s*)–12*s* = 6````14```24``.

From here onward, the procedure coincides with that of BM 13901 #2 (page 43), with a small variation in the end. The calculations can be followed in Figure 28.

The area 6````14```24`` corresponds to the rectangle of (height) *s* and breadth *s*–12`. Half of the excess of the height over the breadth is "broken" and repositioned as seen in the diagram: lightly shaded in the original positions, heavily shaded where it is moved to. The construction of the completing square is described with one of the synonyms of "making hold", namely "to make encounter" (F.19). After the usual operations we find that *s* = 6``14`24 *r* = 2``36`, and in line R.5 that *r* = 25´. We observe, however, that the "moiety" that was moved around is not put back into its original position, which would have reconstituted *s* in the vertical direction. Instead, the other "moiety", originally left in place, is also moved, which allows a

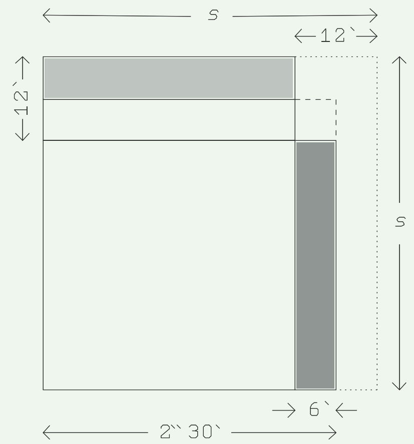


*Figure 28*

horizontal reconstitution $s$ = 6``14`24 $r$ = 2``36: *6` which you have left to 2``30`` join, 2``36` it gives you.*[41]

In the lines R.6–8, the calculator introduces a third false position: if $R$ had been equal to 6, then $r$ would be 5. The difference of 1 between $R$ and $r$ is $\frac{1}{5}$ of $r$ or $12'$ times $r$. Now the true value of $r$ is $25'$; in order to obtain $R$ we must hence "join" $12'\cdot25' = 5'$ to it. Therefore $R = 25'+5' = 30' = \frac{1}{2}$ NINDAN.

One might believe this problem type to be one of the absolute favourites of the Old Babylonian teachers of sophisticated mathematics. We know four variants of it differing in the choice of numerical parameters. However, they all belong on only two tablets sharing a number of terminological particularities – for instance, the use of the logogram $\frac{1}{2}$ for the "moiety", and the habit that results are "given", not (for example) "seen" or "coming up". Both tablets are certainly products of the same locality and local tradition (according to the orthography based in Uruk), and probably come from the same school or even the same hand. A simpler variant with a rectangular field, however, is found in an earlier text of northern origin, and also in a

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[41] This distinction between two halves of which one is "left" is worth noticing as another proof of the geometric interpretation – it makes absolutely no sense unless understood spatially.

text belonging together with the trapezium variants; if not *the* favourite, the broken reed was probably *a* favourite.

## *TMS XIII* [42]

**1.** 2 GUR 2 PI 5 BÁN of oil I have bought. From the buying of 1 shekel of silver,
**2.** 4 SILÀ, each (shekel), of oil I have cut away.
**3.** $\frac{2}{3}$ mina of silver as profit I have seen. Corresponding to what
**4.** have I bought and corresponding to what have I sold?
**5.** You, 4 SILÀ of oil posit and 40, (of the order of the) mina, the profit posit.
**6.** IGI 40 detach, 1´30″ you see, 1´30″ to 4 raise, 6´ you see.
**7.** 6´ to 12`50, the oil, raise, 1`17 you see.
**8.** $\frac{1}{2}$ of 4 break, 2 you see, 2 make hold, 4 you see.
**9.** 4 to 1`17 join, 1`21 you see. What is equal? 9 is equal.
**10.** 9 the counterpart posit. $\frac{1}{2}$ of 4 which you have cut away break, 2 you see.
**11.** 2 to the 1st 9 join, 11 you see; from the 2nd tear out,
**12.** 7 you see. 11 SILÀ each (shekel) you have bought, 7 SILÀ you have sold.
**13.** Silver corresponding to what? What to 11 ¿SILÀ? may I posit
**14.** which 12`50 of oil gives me? 1`10 posit, 1 mina 10 shekel of silver.
**15.** By 7 SILÀ each (shekel) which you sell of oil,
**16.** that of 40 of silver corresponding to what? 40 to 7 raise,
**17.** 4`40 you see, 4`40 of oil.

This is another problem which, at superficial reading, seems to reflect a situation of real practical (here, commercial) life. At closer inspection, however, it turns out to be just as artificial as the preceding broken-reed question: a merchant has bought $M$ = 2 GUR 2 PI 5 BÁN (= 12`50 SÌLA) of fine oil (probably sesame oil). We are not

[42] As TMS VII #2, this problem is rather difficult. It offers an astonishing example of application of the geometrical technique to a non-geometrical question.

told how much he paid, but the text informs us that from the quantity of oil which he has bought for one shekel he has cut away 4 SÌLA, selling what was left ($v = a-4$) for 1 shekel; $a$ and $v$ are thus the reciprocals of the two prices – we may speak of them as "rates" of purchase and sale. Moreover, the total profit $w$ amounts to $\frac{2}{3}$ *mina* = 40 shekel of silver. For us, familiar with algebraic letter symbolism, it is easy to see that the total purchase price (the investment) must be $M \div a$, the total sales price $M \div v$, and the profit in consequence $w = M \div v - M \div a$. Multiplying by $a \cdot v$ we thus get the equation

$$M \cdot (a-v) = F \cdot av \ ,$$

and since $v = a-4$, the system

$$a-v = 4 \ , \ \ a \cdot v = (4M) \div w \ .$$

This system – of the same type as the one proposed in YBC 6967, the *igûm-igibûm* problem (page 46) – is indeed the one that is solved from line 8 onward. Yet it has certainly not been reached in the way just described: on one hand because the Babylonians did not have our letter symbolism, on the other because they would then have found the magnitude $(4M) \div w$ and not, as they actually do, $(4 \div w) \cdot M$.

The cue to their method turns up towards the end of the text. Here the text first finds the total investment and next the profit *in oil* (4`40 SÌLA). These calculations do not constitute a proof, since these magnitudes are not among the data of the problem. Nor are they asked for, however. They must be of interest because they have played a role in the finding of the solution.

Figure 29 shows a possible and in its principles plausible interpretation. The total quantity of oil is represented by a rectangle, whose height corresponds to the total sales price in shekel, and whose breadth is the "sales rate" $v$ (SÌLA per shekel). The total sales price can be divided into profit (40 shekel) and investment (purchase price), and the quantity of oil similarly into the oil profit and the quantity whose sale returns the investment.

The ratio between the latter two quantities must coincide with that into which the quantity bought for one shekel was divided – that is, the ratio between 4 SÌLA and that which is sold for 1 shekel (thus $v$).

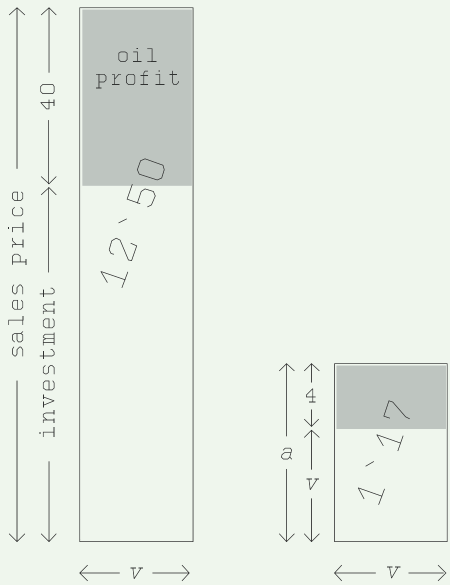


*Figure 29*. Geometric representation of TMS XIII

Modifying the vertical scale by a factor which reduces 40 to 4, that is, by a factor $4 \div w = 4 \div 40 = 6'$, the investment will be reduced to $v$, and the area to $(4 \div w) \cdot M = 1`17$. In this way we obtain the rectangle to the right, for which we know the area ($a \cdot v = 1`17$) and the difference between the sides ($a - v = 4$), exactly as we should. Moreover, we follow the text in the order of operations, and the oil profit as well as the investment play a role.

On the whole, the final part of the procedure follows the model of YBC 6967 (and of other problems of the same type). The only difference occurs in line 10: instead of using the "moiety" of $a - v$ which we have "made hold" in line 8, $a - v$ is "broken" a second time. That allows us to "join" first (that which is joined is already at disposal) and to "tear out" afterwards.

In YBC 6967, the *igûm-igibûm* problem (page 46), the geometric quantities served to represent magnitudes of a different nature, namely abstract numbers. Here, the representation is more subtle: one segment represents a quantity of silver, the other the quantity of oil corresponding to a shekel of silver.

## *BM 13901 #12*

**Obv. II**

**27.** The surfaces of my two confrontations I have heaped: 21´40″.
**28.** My confrontations I have made hold: 10´.
**29.** The moiety of 21´40″ you break: 10´50″ and 10´50″ you make hold,
**30.** 1´57″21{+25}‴40⁗[43] is it. 10´ and 10´ you make hold, 1´40″
**31.** inside 1´57″21{+25}‴40⁗ you tear out: by 17″21 {+25}‴40⁗, 4´10″ is equal.
**32.** 4´10″ to one 10´50″ you join: by 15´, 30´ is equal.
**33.** 30´ the first confrontation.
**34.** 4´10″ inside the second 10´50″ you tear out: by 6´40″, 20´ is equal.
**35.** 20´ the second confrontation.

With this problem we leave the domain of fake practical life and return to the geometry of measured geometrical magnitudes. However, the problem we are going to approach may confront us with a possibly even more striking case of representation.

This problem comes from the collection of problems about squares which we have already drawn upon a number of times. The actual problem deals with two squares; the sum of their areas is given, and so is that of the rectangle "held" by the two "confrontations" $c_1$ and $c_2$ (see Figure 30):

$$\square(c_1)+\square(c_2) = 21'40'' \ ,$$
$$⊏⊐(c_1,c_2) = 10' \ .$$

[43] By error, line 30 of the text has 1´57″46‴40⁗ instead of 1´57″21‴40⁗; a partial product 25 has been inserted an extra time, which shows that the computation was made on a separate device where partial products would disappear from view once they had been inserted. This excludes writing on a clay surface and suggests instead some kind of reckoning board.

The error is carried over in the following steps, but when the square root is taken it disappears. The root was thus known in advance.

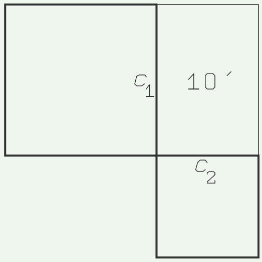


*Figure 30*. The two squares and the rectangle of BM 13901 #12

The problem could have been solved by means of the diagram shown in Figure 31, apparently already used to solve problem #8 of the same tablet, which can be expressed symbolically as follows:

$$\square(c_1)+\square(c_2) = 21'40'' \;,\; c_1+c_2 = 50'.$$

However, the author chooses a different method, showing thus the flexibility of the algebraic technique. He takes the two *areas* $\square(c_1)$ and $\square(c_2)$ as *sides* of a rectangle, whose area can be found by making 10′ and 10′ "hold" (see Figure 31):

$$\square(c_1)+\square(c_2) = 21'40'' \;,\; \sqsubset\!\sqsupset(\square(c_1),\square(c_2)) = 10'{\times}10' = 1'40'' \;.$$

In spite of the geometric character of the operations the Babylonians were thus quite aware that the area of a rectangle whose sides are the squares $\square(c_1)$ and $\square(c_2)$ coincides with that of a square whose side is the rectangle $\sqsubset\!\sqsupset(c_1,c_2)$ – which corresponds to our arithmetical rule $p^2{\cdot}q^2 = (pq)^2$.

We now have a rectangle for which we know the area and the sum of the two sides, as in the problems TMS IX #3 (page 59) and AO 8862 #2 (page 63). The solution follows the same pattern, but with one inevitable difference: this procedure can only give us $\square(c_1)$ and $\square(c_2)$; in order to know $c_1$ and $c_2$ we must find out what "is equal by" them. The calculations can be followed on Figure 32.

What is to be taken note of in this problem is hence that it represents *areas* by line segments and the *square of an area* by an area. Together with the other instances of representation we have encountered, the present example will allow us to characterize the Old Babylonian technique as a *genuine algebra* on page 105.

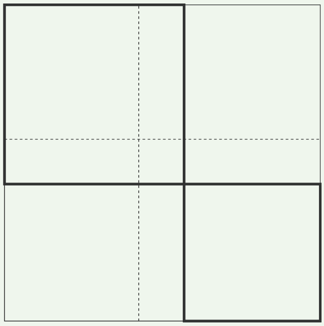

*Figure 31*. The diagram that corresponds to BM 13901 #8

## *BM 13901 #23*

**Rev. II**

**11.** About a surface, the four widths and the surface I have heaped, 41´40˝.
**12.** 4, the four widths, you inscribe. IGI 4 is 15´.
**13.** 15´ to 41´40˝ you raise: 10´25˝ you inscribe.
**14.** 1, the projection, you join: by 1°10´25˝, 1°5´ is equal.
**15.** 1, the projection, which you have joined, you tear out: 5´ to two
**16.** you repeat: 10´, NINDAN, confronts itself.

Whereas the previous problem illustrates the "modern" aspect of Old Babylonian mathematics, the present one seems to illustrate its archaic side – even though they come from the same tablet

This is no real contradiction. The present problem #23 is *intentionally* archaic. In other words, it is *archaizing* and not truly archaic, which explains its appearance together with the "modern" problems of the same collection. The author is not modern and archaic at the same time, he shows his virtuosity by playing with archaisms. In several ways, the formulations that are used here seem to imitate the parlance of Akkadian surveyors. The text speaks of the *width* of a square, not of a "confrontation"; further, this word appears in syllabic writing, which is quite exceptional (cf. note 4, page 12). The introductory

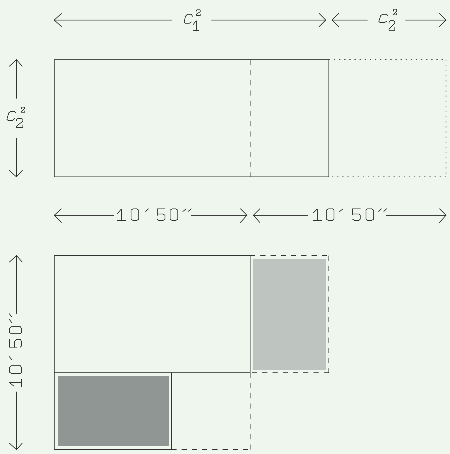


*Figure 32*. The procedure used to solve the rectangle problem

phrase “About a surface”[44] seems to be an abbreviated version of the characteristic formula introducing a mathematical riddle: “if somebody asks you thus about a surface ...” (cf. pages 33, 117, 119 and 137). The expression “the four widths”[45]reflects an interest in what is *really there* and for what is *striking*, an interest that characterizes riddles in general but also the mathematical riddles that circulated among the mathematical practitioners of the pre-Modern world (see page 112). Even the method that is used is typical of riddles: the use of an astonishing artifice that does not invite generalization.

The problem can thus be expressed in the following way:

$$4c+\square(c) = 41'40'' \ .$$

Figure 33 makes clear the procedure: $4c$ is represented by 4 rectangles $\sqsubset\!\sqsupset(1,c)$; the total $41'40''$ thus corresponds to the cross-shaped configuration where a “projection” protrudes in each of the four

[44] In the original, the word is “surface” marked by a phonetic complement indicating the accusative. An accusative in this position is without parallel, and seems to allow no interpretation but the one given here.

[45] For once, the determinate article corresponds to the Akkadian, namely to an expression which is only used to speak about an inseparable plurality (such as “the four quarters of the world” or “the seven mortal sins”).

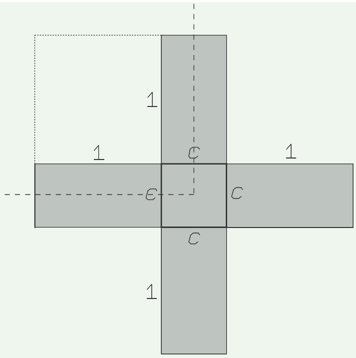


*Figure 33*. The procedure of BM 13901 #23

principal directions. Lines 12–13 prescribe to cut out $\frac{1}{4}$ of the cross (demarcated by a dotted line) and the "joining" of a quadratic complement □(1) to the gnomon that results. There is no need to "make hold", the sides of the complement are already there in the right position. But it is worthwhile to notice that it is the "projection" itself that is "joined": it is hence no mere number but a quadratic configuration identified by its side.

The completion of the gnomon gives a square with area 1°10′25″ and thus side 1°5′. "Tearing out" the "projection" – now as a one-dimensional entity – we find 5′. Doubling this result, we get the side, which turns out to be 10′. Here again, the text avoids the usual term and does not speak of a "confrontation" as do the "modern" problems of the collection; instead it says that 10′ NINDAN "confronts itself".

This method is so different from anything else in the total corpus that Neugebauer believed it to be the outcome of a copyist's mixing up of two problems that happens to make sense mathematically. As we shall see below (page 116), the explanation is quite different.

The archaizing aspect, it should be added, does not dominate completely. Line 12, asking first for the "inscription" of 4 and stating afterwards its IGI, seems to describe the operations on a tablet for rough work that were taught in school (see note 39, page 68, and page 128).

## *TMS VIII #1*

**1.** The surface 10′. The 4th of the width to the width I have joined, to 3 I have gone ... over
**2.** the length 5′ went beyond. You, 4, of the fourth, as much as width posit. The fourth of 4 take, 1 you see.
**3.** 1 to 3 go, 3 you see. 4 fourths of the width to 3 join, 7 you see.
**4.** 7 as much as length posit. 5′ the going-beyond to the to-be-torn-out of the length posit. 7, of the length, to 4, ¿of the width?, raise,
**5.** 28 you see. 28, of the surfaces, to 10′ the surface raise, 4°40′ you see.
**6.** 5′, the to-be-torn-out of the length, to four, of the width, raise, 20′ you see. $\frac{1}{2}$ break, 10′ you see. 10′ make hold,
**7.** 1′40″ you see. 1′40″ to 4°40′ join, 4°41′40″ you see. What is equal? 2°10′ you see.
**8.** 10′ ¿...? to 2°10′ join, 2°20′ you see. What to 28, of the surfaces, may I posit which 2°20′ gives me?
**9.** 5′ posit. 5′ to 7 raise, 35′ you see. 5′, the to-be-torn-out of the length, from 35′ tear out,
**10.** 30′ you see, 30′ the length. 5′ the length to 4 of the width raise, 20′ you see, 20 the length (mistake for width).

In BM 13901 #12 we saw how a problem about squares could be reduced to a rectangle problem. Here, on the contrary, a problem about a rectangle is reduced to a problem about squares.

Translated into symbols, the problem is the following;

$$\tfrac{7}{4}\,w-\ell = 5' \;,\;\; ⊏⊐(\ell,w) = 10'$$

("to 3 I have gone" in line 1 means that the "joining" of $\frac{1}{4}\,w$ in line 1 is repeated thrice). The problem could have been solved in agreement with the methods used in TMS IX #3 (page 59), that is, in the following way:

$$7w-4\ell = 4\cdot 5' \;,\;\; ⊏⊐(\ell,w) = 10'$$

$$7w-4\ell = 20' \;,\;\; ⊏⊐(7w,4\ell) = (7\cdot 4)\cdot 10' = 28\cdot 10' = 4°40'$$

$$7w = \sqrt{4°40'+(\frac{20'}{2})^2} + \frac{20'}{2} = 2°20 \;,$$

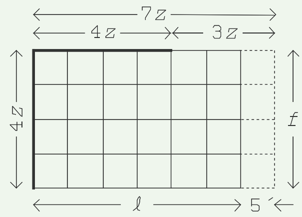


*Figure 34*. The method of TMS VIII #1

$$4\ell = \sqrt{4°40' + \left(\frac{20'}{2}\right)^2} - \frac{20'}{2} = 2$$

$$w = 20' \ , \quad \ell = 30' \ .$$

However, once again the calculator shows that he has several strings on his bow, and that he can choose between them as he finds convenient. Here he builds his approach on a square whose side ($z$) is $\frac{1}{4}$ of the width (see Figure 34). In that way, the width will equal 4, understood as $4z$ (*You, 4, of the fourth, as much as width posit*), and the length prolonged by 5´ will be equal to 7, understood as $7z$ (*7 as much as length posit*). Line 4 finds that the rectangle with sides $7z$ and $4z$ – in other words, the initial rectangle prolonged by 5´ – consists of $7 \cdot 4 = 28$ small squares $\square(z)$[46]. These 28 squares exceed the area 10´ by a certain number of sides ($n \cdot z$), the determination of which is postponed until later. As usual, indeed, the non-normalized problem

$$28\square(z) - n \cdot z = 10'$$

is transformed into

$$\square(28z) - n \cdot (28z) = 28 \cdot 10' = 4°40' \ .$$

Line 6 finds $n = 4 \cdot 5' = 20'$, and from here onward everything follows the routine, as can be seen on Figure 35: $28z$ will be equal to 2°20´,

46 The use of a "raising" multiplication shows that the calculator does not construct a new rectangle but bases his procedure on a subdivision of what is already at hand – see the discussion and dismissal of a possible alternative interpretation of the procedure of BM 13901 #10 in note 26, page 50.

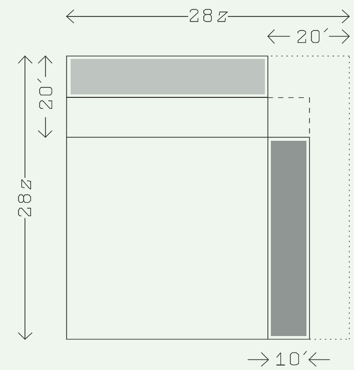


*Figure 35*. Resolution of the normalized equation of TMS VIII #1

and $z$ hence to 5′.[47] Therefore, the length $\ell$ will be $7{\cdot}5'-5' = 30'$, and the width $w$ $4{\cdot}5' = 20'$.

## *YBC 6504 #4*

**Rev.**

**11.** So much as length over width goes beyond, made encounter, from inside the surface I have torn out:

**12.** 8′20″. 20′ the width, its length what?

**13.** 20′ made encounter: 6′40″ you posit.

**14.** 6′40″ to 8′20″ you join: 15′ you posit.

**15.** By 15′, 30′ is equal. 30′, the length, you posit.

So far, everything we have looked at was mathematically correct, apart from a few calculational and copying errors. But everybody who practises mathematics sometimes also commits errors in the argument; no wonder then that the Babylonians sometimes did so.

The present text offers an example. Translated into symbols, the problem is the following:

47 Line 10 speaks of this as *5′ the length* – namely the side of the small square. Some other texts from Susa also speak of the side of a square as its "length".

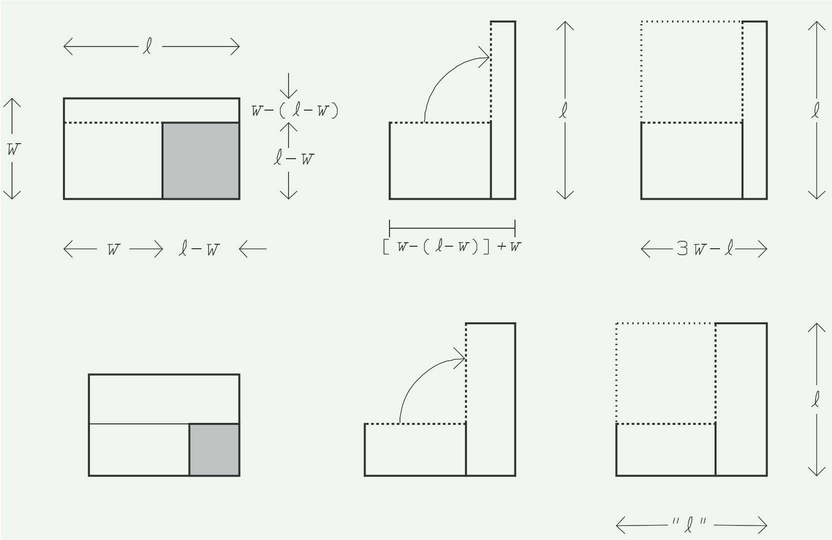


*Figure 36*. The cut-and-paste operations of YBC 6504 #4

$$\sqsubset\!\sqsupset(\ell, w)-\square(\ell-w) = 8'20'' \ , \quad w = 20' \ .$$

Astonishingly, the length is found as that which "is equal by" $\sqsubset\!\sqsupset(\ell, w)-\square(\ell-w)+\square(w)$ – that is, after a transformation and expressed in symbols, as $\sqrt{(3w-\ell)\cdot\ell}$ .

The mistake seems difficult to explain, but inspection of the geometry of the argument reveals its origin (see Figure 36). On top the procedure is presented in distorted proportions; we see that the "joining" of $\square(w)$ presupposes that the mutilated rectangle be cut along the dotted line and opened up as a pseudo-gnomon. It is clear that what results from the completion of this configuration is *not* $\square(\ell)$ but instead – if one counts well – $\sqsubset\!\sqsupset(3w-\ell,\ell)$. Below we see the same thing, but now in the proportions of the actual problem, and now the mistake is no longer glaring. Here, $\ell = 30'$ and $w = 20'$, and therefore $\ell-w = w-(\ell-w)$. In consequence the mutilated rectangle opens up as a true gnomon, and the completed figure corresponds to $\square(\ell)$ – but only because $\ell = \frac{3}{2}\,w$.

This mistake illustrates an important aspect of the "naive" geometry: as geometric demonstrations in general it demands scrupulous attention if one will not risk to be induced into error by what is "immediately" seen. The rarity of such errors is evidence of the high competence of the Old Babylonian calculators and shows that they were almost always able to distinguish the *given magnitudes* of a problem from what more they knew about it.

# Chapter 4

## Application of quasi-algebraic techniques to geometry

We still have not decided what to mean by "algebra". Any distinction between Old Babylonian "algebra" and "quasi-algebra" must therefore remain preliminary – a hypothesis that will allow us to collect the observations that in the end will serve in a more systematic discussion.

Be that as it may, all problems dealt with in Chapters 1 – 3 can be translated into modern algebraic symbols (albeit with a certain loss of information). On the whole the same can be said about the methods used to resolve them.

Such a translation will not be possible in the problems that are analyzed in the present chapter. There is, however, a fairly close connection between the methods that are applied here and those which we know from the preceding chapters. In this sense at least it seems legitimate to speak of them as "quasi-algebraic.

### *VAT 8512*

**Obv.**

**1.** A triangle. 30 the width. In the inside two plots,
**2.** the upper surface over the lower surface, 7` went beyond.
**3.** The lower descendant over the upper descendant, 20 went beyond.
**4.** The descendants and the bar what?
**5.** And the surfaces of the two plots what?
**6.** You, 30 the width posit, 7` which the upper surface over the lower surface went beyond posit,
**7.** and 20 which the lower descendant over the upper descendant went beyond posit.
**8.** IGI 20 which the lower descendant over the upper descendant went beyond
**9.** detach: 3´ to 7` which the upper surface over the lower surface went beyond

**10.** raise, 21 may your head hold!
**11.** 21 to 30 the width join: 51
**12.** together with 51 make hold: 43`21
**13.** 21 which your head holds together with 21
**14.** make hold: 7`21 to 43`21 join: 50`42.
**15.** 50`42 to two break: 25`21.
**16.** The equal of 25`21 what? 39.
**17.** From 39, 21 the made-hold tear out, 18.
**18.** 18 which you have left is the bar.
**19.** Well, if 18 is the bar,
**20.** the descendants and the surfaces of the two plots what?
**21.** You, 21 which together with itself you have made hold, from 51
**22.** tear out: 30 you leave. 30 which you have left
**23.** to two break, 15 to 30 which you have left raise,
**24.** 7`30 may your head hold!

**Edge**

**1.** 18 the bar together with 18 make hold:
**2.** 5`24 from 7`30 which your head holds
**3.** tear out: 2`6 you leave.

**Rev.**

**1.** What to 2`6 may I posit
**2.** which 7` which the upper surface over the lower surface went beyond gives me?
**3.** 3°20′ posit. 3°20′ to 2`6 raise, 7` it gives you.
**4.** 30 the width over 18 the bar what goes beyond? 12 it goes beyond.
**5.** 12 to 3°20′ which you have posited raise, 40.
**6.** 40 the upper descendant.
**7.** Well, if 40 is the upper descendant,
**8.** the upper surface is what? You, 30 the width,
**9.** 18 the bar heap: 48 to two break: 24.
**10.** 24 to 40 the upper descendant raise, 16`.
**11.** 16` the upper surface. Well, if 16` the upper surface,
**12.** the lower descendant and the lower surface what?
**13.** You, 40 the upper descendant to 20 which the lower descendant over the upper descendant goes beyond
**14.** join, 1` the lower descendant.
**15.** 18 the bar to two break: 9

**16.** to 1` the lower descendant raise, 9`.
**17.** 9` the lower surface.

Many Old Babylonian mathematical problems deal with the partition of fields. The mathematical substance may vary – sometimes the shape of the field is irrelevant and only the area is given together with the specific conditions for its division; sometimes, as here, what is asked for is a division of a particular geometric shape

Already before 2200 BCE, Mesopotamian surveyors knew how to divide *a trapezium* into two equal parts by means of a parallel transversal; we shall return in a short while to how they did it. A similar division of a triangle cannot be made exactly without the use of irrational numbers – which means that it could not be done by the Old Babylonian calculators (except with approximation, which was not among the normal teaching aims).

The present problem deals with a variant of the triangle division which *can* be performed exactly. As we see in lines F.1–3 and as shown in Figure 37, a triangular field is divided into two parcels (an "upper surface" and a "lower surface") by a "bar", that is, a parallel transversal. For simplicity we may assume the triangle to be rectangular. It is almost certain that the author of the text did as much, and that the "descendants" are thus part of the side; but if we interpret the "descendants" as heights, the calculations are valid for an oblique triangle too.

The two parcels are thus unequal in area. However, we know the difference between their areas, as well as the difference between the appurtenant "descendants". The solution makes use of an unsuspected and elegant ruse and may therefore be difficult to follow.

Lines F.8–10 "raise" the IGI of the difference between the two "descendants" to the difference between the two "surfaces". This means that the text finds the width of a rectangle whose length corresponds to the difference between the partial heights and whose area equals the difference between the partial areas. This width (which is 21) is first memorized and then "joined" to the triangle.

The outcome is a triangle with an attached rectangle – all in all the trapezium shown in Figure 37. When prolonging the bar, producing a parallel transversal of the trapezium, we discover that is divides the

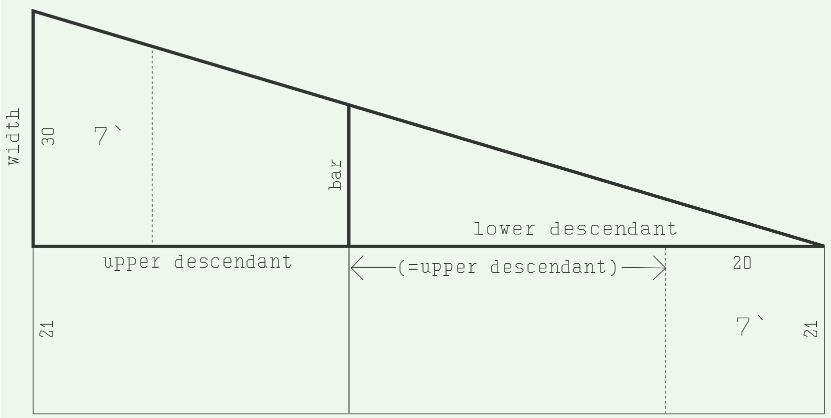


*Figure 37*. The triangular divided field of VAT 8512, with the auxiliary rectangle

trapezium into two equal parts – and that is the problem the surveyors had known to solve for half a millennium or more.

Lines F.11–16 show how they had done: the square on the bisecting transversal is determined as the average between the squares of the parallel sides. The operations that are used ("making hold" and "breaking") show that the process is really thought in terms of geometric squares and average. Figure 38 shows why the procedure leads to the correct result. By definition, the average is equidistant from the two extremes. Therefore the gnomon between 21 and 39 must equal that between 39 and 51 ($39^2-21^2 = 51^2-39^2$); half of these gnomons – the two parts of the shaded trapezium – must therefore also be equal. In the first instance this only concerns a trapezium cut out along the diagonal of a square, but we may imagine the square drawn long (into a rectangle) and perhaps twisted into a parallelogram; none of these operations changes the ratio between areas or parallel linear extensions, and they allow the creation of an arbitrary trapezium. This trapezium will still be bisected, and the sum of the squares on the parallel sides will still be twice that of the parallel transversal.

We may take note that the operation of "drawing long" is the same as that change of scale in one direction which we have encountered in the solution of non-normalized problems, and which was also

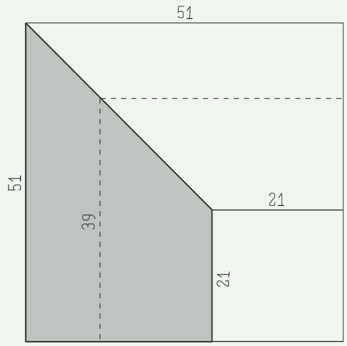


*Figure 38*. The bisection of the trapezium of BM 8512

used in TMS XIII, the oil trade (see page 75); we shall meet it again in a moment in the present problem.

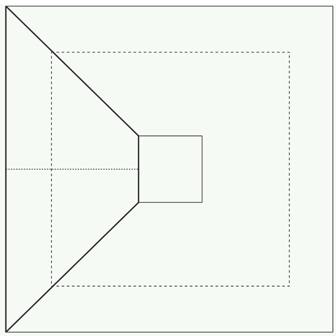

*Figure 39*. The bisection of the trapezium explained by concentric squares

Possibly the rule was first found on the basis of concentric squares (see Figure 39) – the geometric configuration represented by two or several concentrically nested squares was much appreciated in Babylonian mathematics and may have been so already in the third millennium (as it remained popular among master builders until the Renaissance); the principle of the argument evidently remains the same.

Line F.17 thus finds the bisecting transversal; it turns out to be 39, and the "bar" between the two original parcels must therefore be $39-21 = 18$.

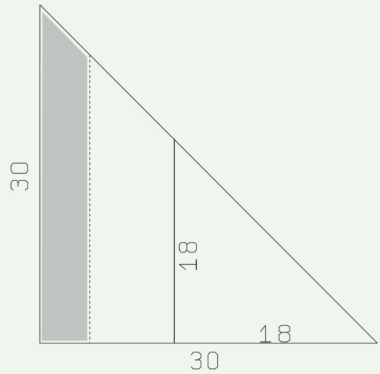


*Figure 40*

The next steps may seem strange. Lines F.21–22 appear to calculate the width of the triangle, but this was one of the given magnitudes of the problem. This means no doubt that we have effectively left behind Figure 37, and that the argument is now based on something like Figure 38. When we eliminate the additional width 21 we are left with a triangle that corresponds to the initial triangle but which is isosceles – see Figure 40.

In order to find the "upper descendant" the text makes the false position that the shortened and isosceles triangle is the one we are looking for. Its length (the sum of the "descendants") is then equal to the width, that is, to 30. In order to find the true triangle we will have to change the scale in the direction of the length.

Lines F.23–24 calculate that the area of the false triangle is 7`30. The two areas in white are equal, and their sum must be $2\cdot(\frac{1}{2}\cdot(18\cdot 18))$ = 5`24. The shaded area – which corresponds to the difference between the two parcels – must therefore be 7`30–5`24 = 2`6 (edge 1–3).

But we know that the difference is 7` and not 2`6. Lines R.1–3 therefore establish that the difference 2`6 that results from the false position must be multiplied by 3°20′ if we are to find the true difference 7`. Since the width is already what it should be, it is the length and the "descendants" that must be multiplied by this factor. The "upper descendant" will thus be 3°20′·(30–18) = 40 (line R.6). Afterwards everything is quite simple; it could have been even simpler, but the road that is chosen agrees better with the pedagogical style which we know for example from TMS XVI #1, and it is probably more fruitful from a didactical point of view.

The way this problem is solved certainly differs from what we have encountered so far. But there are also common features that become more conspicuous in a bird's eye view.

This change of scale in one direction we already know as an algebraic technique. A no less conspicuous *difference* – the absence of a quadratic completion, that is, of the "Akkadian method" – points to another family characteristic: the introduction of an auxiliary figure that is first "joined" and then "torn out".

Less evident but fundamental is the "analytic" character of the methods. Since Greek Antiquity, the solution of a mathematical problem is called "analytic" if it starts from the presupposition that the problem is *already* solved; that allows us to examine – "to analyze" – the characteristics of the solution in order to understand how to construct it.[48]

A solution by equation is always analytical. In order to understand that we may look again at our modern solution of TMS XIII, the oil trade (page 74). According to the starting hypothesis, the quantity of SÌLA that is bought for 1 shekel of silver is a known number, and we call it $a$. We do the same with the sales rate (which we call $v$). The total investment is hence $M \div a$, the total sales price $M \div v$, and the profit therefore $w = \frac{M}{v} - \frac{M}{a}$. Then we multiply by $v \cdot a$, and so forth.

[48] The antithesis of the "analytical" method is the "synthesis, in which the solution is constructed directly, after which this solution is shown to be indeed valid. This is the proof style of Euclid's *Elements*, and since Antiquity it is the consistent complaint that this makes it more difficult than necessary to understand the work: the student sees well that each step of a proof is correct, and therefore has to accept the end result as irrefutable; but one does not understand the reasons which make the author take the single step, and in this way the author appears shrewd rather than really pedagogical. Since Antiquity Euclid (or his predecessors) have also been suspected to have first found their constructions and proofs by means of analysis, constructing the solution in the second instance but hiding their traces.

That is, we treat *a* and *v* as if they were known numbers; we pretend to have a solution and we describe its characteristics. Afterwards we derive the consequences – and find in the end that $a = 11$, $v = 7$.

Even the Old Babylonian cut-and-paste solutions are analytic. Presupposing that we know a solution to the oil problem we express it as a rectangle of area 12`50, of which a part of length 40 corresponds to the oil profit. Then we examine the characteristics of this solution, and find the normalization factor by which we should multiply in order to get a difference 4 between the sides, and so on.

The solution to the present problem is also analytical. We *presuppose* that the triangle has been completed by a rectangle in such a way that the prolonged "bar" divides resulting trapezium in equal parts, after which we calculate how much the width of the rectangle must be if that shall be the case; and so on. Even though it has its justification, the distinction "algebra" (problems that are easily translated into modern equations) and "quasi-algebra" seems less important in the perspective of the Old Babylonian texts than in ours.

### *BM 85200+VAT 6599 #6*

Face I

**9.** An excavation. So much as the length, that is the depth. 1 the dirt I have torn out. My ground and the dirt I have heaped, 1°10′. Length and width, 50′. Length, width, what?

**10.** You, 50′ to 1, the conversion, raise, 50′ you see. 50′ to 12 raise, 10 you see.

**11.** Make 50′ confront itself, 41′40″ you see; to 10 raise, 6°56′40″ you see. Its IGI detach, 8′38″24‴ you see;

**12.** to 1°10′ raise, 10′4″48‴ you see, 36′, 24′, 42′ are equals.

**13.** 36′ to 50′ raise, 30′, the length. 24′ to 50′ raise, 20, the width; 36′ to 10 raise, 6, the depth.

**14.** The procedure.

This is a problem of the third degree, coming from a tablet that has been broken into two parts, one of which is in London and one in Berlin (whence the composite name). It deals with a parallelepipedal

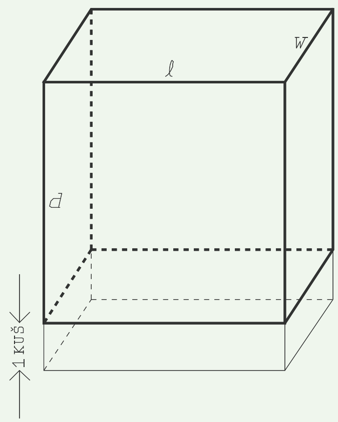


*Figure 41*. The excavation extended 1 KÙŠ downwards

"excavation", of length $\ell$ [NINDAN], width $w$ [NINDAN] and depth $d$ [KÙŠ]. The length is equal to the depth, but because of the use of different metrologies in the two directions that means that $d = 12\ell$.

Further the sum of the length and the width is [$\ell + w =$] 50′, and the sum of the volume of dirt that has been "torn out", that is, dug out[49] and the "ground" (the base) is [$\ell{\cdot}w{\cdot}d + \ell{\cdot}w =$] 1°10′. This latter equation can be transformed into $\ell{\cdot}w{\cdot}(d+1) =$ 1°10′ – that is, if the excavation had been dug 1 KÙŠ deeper, its volume would have equalled 1°10′ [$\text{NINDAN}^2{\cdot}$KÙŠ] (see Figure 41)[50].

The solution is based on a subtle variant of the false position (in its proper form this method would not serve, since the problem is not homogeneous – see note 28, page 51). "The position" consists in the construction of a "reference cube" with the side $\ell + w$. In horizontal measure, its side is $1{\cdot}50' = 50'$ [NINDAN], since "the conversion" of NINDAN into NINDAN asks for a multiplication by 1. In vertical measure, it is $12{\cdot}50' = 10$ KÙŠ, since "the conversion" of NINDAN into

49 The text uses the same verb "to tear out" as for the subtractive operation.

50 The statement also refers to "1 the dirt that I have torn out", but this information is not used. It is another example of a magnitude that is *known but not given*; knowing its numerical value allows the teacher to make a distinction between the real excavation ("1 the dirt") and the volume of the excavation extended downwards by 1 KÙŠ ("1°10′, the dirt").

KÙŠ implies a multiplication by 12 (both conversions take place in line 10).

Lines 11–12 find the volume of the reference cube to be 6°56′40″. This volume is contained 10′4″48‴ times in the extended excavation.

We should now imagine that the sides of the extended excavation are measured by the corresponding sides of the reference cube. If $p$ is the number of times the length $\ell$ is measured by 50′ NINDAN, $q$ the number of times the width $w$ is measured by 50′ NINDAN, and $r$ the number of times the depth $d$+1 KÙŠ is measured by 10 KÙŠ (= 50′ NINDAN), then

$$p \cdot 50' + q \cdot 50' = \ell + w = 50',$$

and therefore

$$p + q = 1;$$

further

$$r \cdot 10 = d+1 = 12\ell + 1 = 12 \cdot p \cdot 50' + 1 = 10p + 1\ ,$$

whence

$$r = p + \frac{1}{10} = p + 6';$$

and finally

$$p \cdot q \cdot r = 10'4''48'''\ .$$

We therefore have to express 10′4″48‴ as the product of three factors $p$, $q$ and $r$ that fulfil these conditions. That is what the text does in line 12, where the factors appear as the "equals" 36′, 24′ and 42′. Afterwards, line 13 finds $\ell$, $w$ and $d$.

The factorisation seems to be drawn from the teacher-magician's sleeves, and that is probably how it has actually been produced, just like the various square roots and quotients. Since the solution was known beforehand, that would be easy. But it is also possible to find it by systematic reasoning, beginning with simple numbers – one must simply express 10``4`48 (= $2^6 \cdot 3^4 \cdot 7$) as the product of three numbers $P$, $Q$ and $R$ where $P+Q = 60$, $R = P+6$[51]. Knowing the general

[51] In order to have integers we here introduce $P = 60p = 1`p$, $Q = 1`q$,

character of Old Babylonian mathematics we may even claim that the text can only allow itself to draw the answer from the sleeves because it would be *possible* (albeit somewhat laborious) to find it without magic. Let us first assume that $P = 1$; then, since $P+Q = 60$, $Q$ will be 59, which is impossible; the hypotheses $P = 2$ and $P = 3$ can be rejected for analogous reasons; $P = 4$ gives $R = 10$, which is also excluded – 10``4`48 contains no factor 5; $P = 5$ is impossible in itself; $P = 6$ gives $Q = 54$ and $R = 12$, which must be rejected, both because the factor 7 is missing and because control shows the product not to be what is required. The next value $P$ which does not lead to impossible values for $Q$ or $R$ is 12, but it must be rejected for the same reasons; $P = 18$ is impossible because the product is only around half of what is needed. $P = 24$ and $P = 30$ must be rejected for the same reasons as $P = 6$. Finally we arrive at $P = 36$, a value that fits. If we had counted prime factors it would have been even easier, but nothing suggests that the Babylonians knew that technique.

It must be emphasized, however, that this method only works because a simple solution *exists*. Thereby the problem differs fundamentally from those of the second degree, where a good approximation to that which “is equal” would give an almost correct solution (and the Babylonians knew well to find approximate square roots even though they did not do it in their algebra problems). The Babylonians were thus not able to solve cubic problems in general as they could solve second-degree problems – for that, one had to wait for the Italian algebraist of the 16th century CE.

Our text speaks of *three* “equals” which are not even equal. This usage evidently represents a generalization of an idea coming from the sides of the square and the cube. There is nothing strange in such a generalization – our own notion of the “roots” of an equation comes in the same way from early Arabic algebra, where the fundamental equations were formulated in terms of an amount of money and its square root. As this origin was forgotten the word came to be understood as a designation for the value of the unknown that satisfies the equation.

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$R$ = 1`$r$. Then $PQR$ = 1```$pqr$ = 10``4`48.

Other problems from the same tablet speak of a single “equal”; that is the case when the volume of the excavation measured by the reference parallelepiped (not always a cube) must be factorized as $p^3$ or as $p^2 \cdot (p+1)$. Tables indeed exist for these two functions, and in these $p$ appears precisely as “the equal”; the latter table had the name “equal, 1 joined” – see page 135.

As in the second-degree algebra, the treatment of the third-degree problems is analytic – what we have just looked at is a typical representative of the category: one presupposes that a solution exists and draws the consequences from what can then be stated. In the same way, every solution by means of a false position is analytic – it begins by the hypothesis of a solution.

Apart from that, only rather peripheral characteristics connect the second and the third degree: the terminology for operations, the use of tables, the fundamental arithmetical operations.

Other problems on the same tablet (all dealing with parallelepipedal “excavations”) are reduced to problems of the second or even the first degree. These are solved by the techniques we already know, and never by factorization. The Babylonians were thus aware of possessing another (and in their opinion, as we see) better technique, and they knew perfectly the difference between problems that can be solved by their algebraic techniques and those which do not yield to such attacks. But they seem not to have seen this difference as fundamental – the mathematical genre that is defined by the contents of the tablet is rather “excavation problems”, just as the genre defined by BM 13901 must be understood as “square problems” even though one of the problems is reduced to a rectangle problem. Once more, the distinction between “algebra” and “quasi-algebra” seems to be secondary, less important than the classification of problems according to the object they consider.

### *BM 15285 #24*

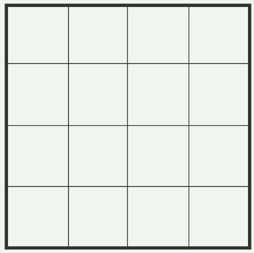


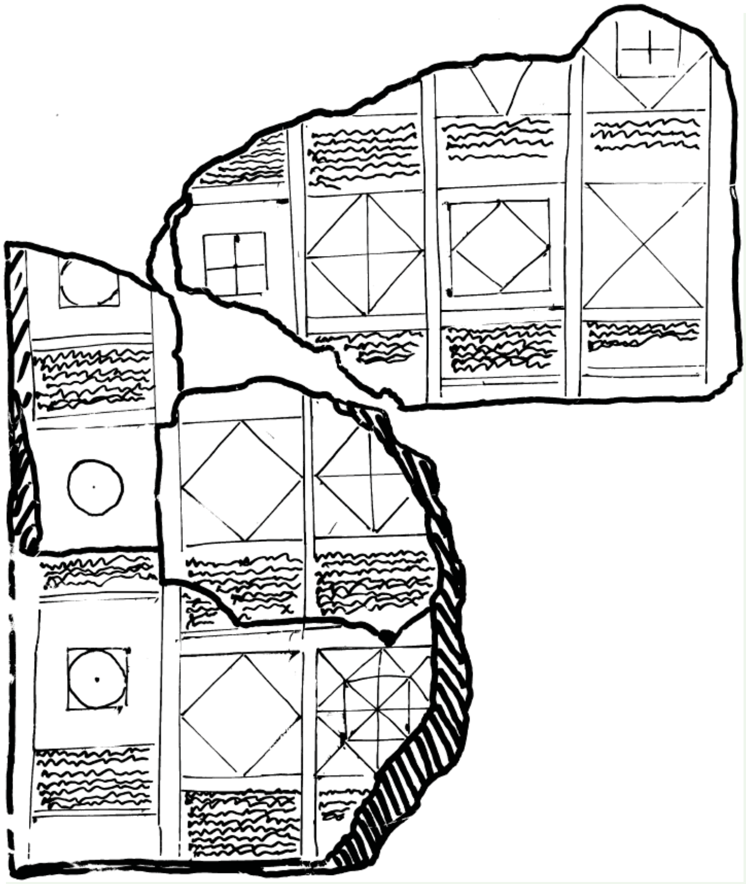
*Figure 42*

1. 1 UŠ the confrontation.
2. Inside, 16 confrontations
3. I have laid down. Their surface what?

The small problem that precedes is extracted from a tablet containing some 40 problems on subdivisions of a square with side 1 UŠ = 1` NINDAN – the surviving fragments of the tablet contain 31 problems. All are accompanied by diagrams showing the actual subdivision (often necessary for understanding the sometimes very concise enunciations). Figure 42 shows the diagram accompanying the present problem, Figure 43 shows the obverse of the principal fragment (problem #24 is found on its reverse).

The above text does not explain the procedure – none of the problems on the tablet does so. It is obvious, however, that there is no need for algebraic thinking here. It is no less evident that the technique used to calculate the coefficients in the problem BM 13901 #10 (page 48) will also serve here.

In line 3 it is seen that the verb translated “to lay down” may mean “draw”, cf. note 24, page 48.



Saggs

*Figure 43* The principal fragments of the tablet BM 15285. After H. W. F., 1960. “A Babylonian Geometrical Text”. *Revue d’Assyriologie* **54** (1960), 131–146

# Chapter 5

# General characteristics

### *Drawings?*

All the texts that were discussed above were illustrated by geometric drawings. However, only two of the tablets carried geometric diagrams, and in both cases these illustrated the problem statement, not the procedure.

Many aspects of the procedures are inexplicable in the traditional arithmetical interpretation but naturally explained in a geometrical reading. In consequence, *some* kind of geometry must have participated in the reasoning of the Babylonians. It is not very plausible, however, that the Babylonians made use of drawings quite like ours. On the contrary, many texts give us reasons to believe that they were satisfied with rudimentary structure diagrams; see for example page 53 on the change of scale in one direction. The absence of particular names for $L = 3\lambda$ and $w = 21\phi$ in TMS IX #3 (see page 61) also suggests that no new diagram was created in which they could be identified, while $\lambda$ and $\phi$ *could* be identified as sides of the "surface 2".

After all, that is no wonder. Who is familiarized with the Old Babylonian techniques will need nothing but a rough sketch in order to follow the reasoning; there is not even any need to perform the divisions and displacements, the drawing of the rectangle alone allows one to grasp the procedure to be used. In the same way as we may perform a mental computation, making at most notes for one or two intermediate results, we may also become familiar with "mental geometry", at most assisted by a rough diagram.

A fair number of field plans made by Mesopotamian scribes have survived; the left part of Figure 44 shows one of them. They have precisely the character of structure diagrams; they do not aim at being faithful in the rendering of linear proportions, as will be seen if we compare with the version in correct proportions to the right. In that respect they are similar to Figure 26, whose true proportions can be

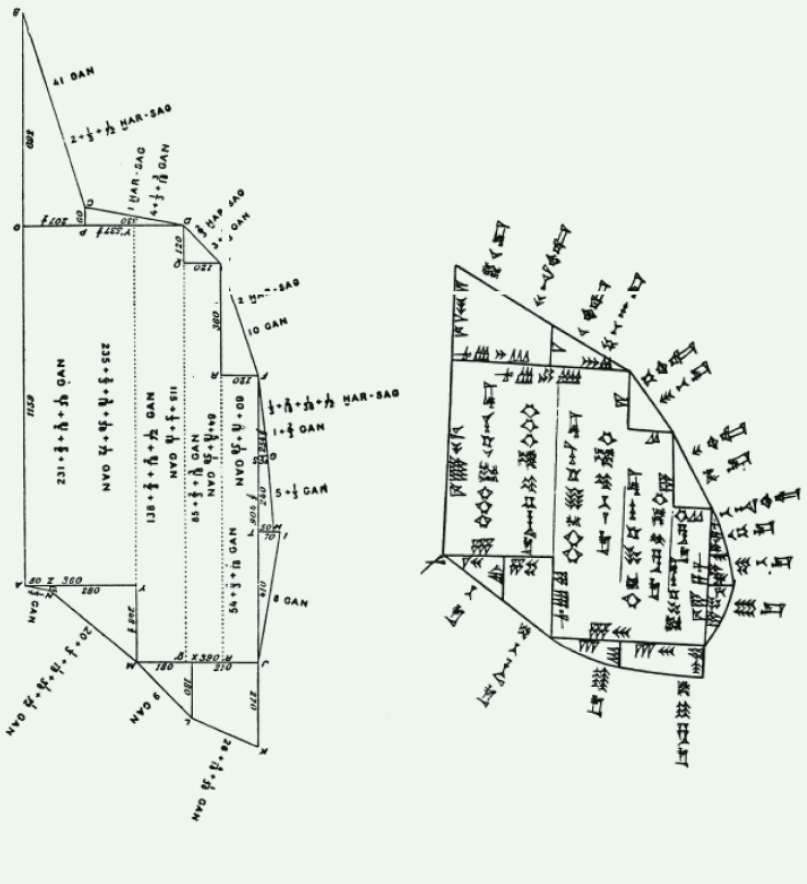


*Figure 44*. A neo-Sumerian field plan (21st century BCE), left as drawn on the tablet, right redrawn in correct proportions. After F. Thureau-Dangin, “Un cadastre chaldéen”. *Revue d’Assyriologie* **4** (1897–98), 13–27.

seen in Figure 27 – pages 68 and 71, respectively. Nor are they interested in showing angles correctly, apart from the “practically right” angles that serve area calculations and therefore have a *structural* role.

Practising “mental geometry” presupposes that one has first trained concrete geometry; real drawings of some kind must thus have existed. However, cut-and-paste operations are not easily made on a clay tablet. The dust abacus, used by Phoenician calculators in the first millennium BCE and then taken over by Greek geometers,[52] is much more convenient for this purpose. Here it is easy to cancel a part of a figure and to redraw it in a new position. A school-yard strewn with sand (cf. page 32) would also be convenient.

In the same way, dust or sand appears to have served in the first steps of learning the script. From this initial phase we know the tablets on which are inscribed the models the students are supposed to reproduce in order to learn the cuneiform characters. From *the next* phase we also have the clay tablets written by the students – but from the first phase the work of students has left no archaeological traces, which means that these will probably have been drawn in sand or dust. There is therefore no reason to be astonished that the geometrical drawings from the teaching of algebra and quasi-algebra have not been found.

## *Algebra ?*

Until now, for reasons of convenience and in agreement with the majority of historians of mathematics, we have spoken of an Old Babylonian "algebra" without settling the meaning one should ascribe to this modern word in a Babylonian context, and without trying to explain why (or whether) a geometrical technique can really be considered an "algebra".

On our way, however, we have accumulated a number of observations that may help us to form a reasoned opinion (at times hinting at the role these observations are going to play in the argument).

At first it must be said that the modern algebra to which the Old Babylonian technique might perhaps be assimilated is precisely *a technique*, namely the practice of equations. Nothing in the Old Babylonian texts allows us to assume that the Babylonians possessed the slightest hint of something like the algebraic *theory* which has developed since the 16th century (concerning the link between

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52 The Greek word for the abacus, ἄβαξ, is borrowed from a Phoenician root from which comes words for "dust" and "flying away".

coefficients and roots, etc.) – nor *a fortiori* to equate what they did with what professional mathematicians today call algebra (group theory and everything building on or extending that domain). The algebra of today which we should think of is what is learned in school and expressed in equations.

We have seen above (page 27) in which sense the Old Babylonian problem statements can be understood as *equations*: they may indicate the total measure of a combination of magnitudes (often but not always geometric magnitudes); they may declare that the measure of one combination equals that of another one; or that the former exceeds or falls short of the latter by a specified amount. The principle does not differ from than of any applied algebra, and thus not from the equations on which an engineer or an economist operate today. In this sense, the Old Babylonian problem statements are true equations.

But there is a difference. Today's engineer *operates on* his equations: the magnitudes he moves from right to left, the coefficients he multiplies, the functions he integrates, etc. – all of these exist only as elements of the equation and have no other representation. The operations of the Babylonians, on the contrary, were realized within a *different* representation, that of measured geometric quantities.[53]

With few exceptions (of which we have encountered none above) the Old Babylonian solutions are *analytic*. That also approaches them to our modern equation algebra. Beyond that, most of their procedures are "homomorphic" though not "isomorphic" analogues of ours, or at least easily explained in term of modern algebra.

These shared characteristics – statements shaped as equations, analysis, homomorphic procedures – have induced many historians of mathematics to speak of a "Babylonian algebra" (*seduced*, certain critics have said during the last 40 years). But there is a further reason

[53] Only first-degree transformations like those of TMS XVI #1 and TMS IX #3 may be seen as constituting a partial exception; TMS XVI #1 is indeed an explanation of how operations directly on the words of the equation are to be understood in terms of the geometric representation. Once that had been understood, TMS IX #3 could probably operate directly on the level of words. But TMS XVI #1 is no problem solution, and in TMS IX #3 the first-degree transformation is subordinate to geometric operations.

for this characterization, a reason that may be more decisive although it has mostly gone unnoticed.

Today's equation algebra possesses a neutral "fundamental representation" (see page 11): abstract numbers. This neutral representation is an empty container that can receive all kinds of measurable quantities: : distances, areas, electric charges and currents, population fertilities, etc. Greek geometric analysis, on the other hand, concerns nothing but the geometric magnitudes it deals with, these represent nothing but what they are.

In this respect, the Babylonian technique is hence closer to modern equation algebra than is Greek analysis. As we have seen, its line segments may represent areas, prices (better, inverse prices) – and in other texts numbers of workers and the number of days they work, and the like. We might believe (because we are habituated to confound the abstract geometric plan and the paper on which we draw) that geometry is less neutral than abstract numbers – we know perfectly well to distinguish the abstract number 3 from 3 pebbles but tend to take a nicely drawn triangle for the triangle itself. But even if we stay in our confusion we must admit that from the *functional point of view*, the Old Babylonian geometry of measured magnitudes is also an empty container.

Today's equation algebra is thus a *technique to find* by means of the *fiction that we have already found* (analysis) followed by the manipulation of unknown magnitudes as if they were known – everything within a representation that is functionally empty (namely, the realm of abstract numbers). Replacing *numbers* with *measurable geometric quantities* we may say the same about the Old Babylonian technique – with a small reserve to which we shall return presently. If the modern technique is understood as an "algebra" in spite of its immense conceptual distance from group theory and its descendants, it seems reasonable to classify the Old Babylonian technique as we have encountered it in Chapters 1–3 under the same heading.

That does not mean that there are no differences; there are, and even important differences; but these are not of a kind that would normally be used to separate "algebra" from what is not algebra.

Apart from the representation by a geometry of measurable magnitudes, the most important difference is probably that Old

Babylonian second- (and higher-)degree algebra had no practical application – not because it could not have for reasons of principle (it could quite well) but because no practical problem within the horizon of an Old Babylonian working scribe asked for the application of higher algebra. All problems beyond the first degree are therefore artificial, and all are constructed backwards from a known solution (many first-degree problems are so, too). For example, the author begins with a square of side 10′ and then finds that the sum of the four sides and the area is 41′40″. *The problem* which he constructs then states this value and requires (with a formula that was in favour among the calculators of the Middle Ages but which is also present in TMS XVI and TMS VII) that the sides and the area be "separated" or "scattered"[54].

This kind of algebra is very familiar today. It allows teachers and textbook authors to construct problems for school students for which they may be sure of the existence of a reasonable solution. The difference is that *our* artificial problems are supposed to train students in techniques that will later serve in "real-life" contexts.

What we do not know is the candour with which certain Old Babylonian texts speak of the value of magnitudes that in principle are supposed not to be known. However, since the text distinguish clearly between *given* and *merely known* magnitudes, using the latter only for identification and pedagogical explanation, this seemingly deviating habit first of all illustrates the need for a language in which to describe the procedure – an alternative to the $\ell$, $\lambda$ and $L$ of our algebra and the "segment *AB*" of our geometry. Since the texts represent the "teacher's manual" (notwithstanding the "you" that pretends to address the student), we cannot exclude that the true oral exposition to students would instead make use of a finger pointing to the diagram ("this width here", "that surface there"). Nor can we claim that things will really have occurred like that – we have no better window to the didactical practice of Old Babylonian mathematics than what is offered by TMS XVI #1 (page 25).

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[54] See TMS XVI #2 line 16 and TMS VII #1 line 4 (below, pages 125 and 126); the two terms seem to be synonymous. This "separation" or "dispersion", which is no subtraction, is the inverse operation of "heaping".

# Chapter 6

# The background

What we now know about Old Babylonian algebra – its flexibility, its operational power in the solution of sophisticated though practically irrelevant problems, the competence of those who practised it – leaves unanswered the enigma of its existence. Since this enigma is now almost 4000 years old, we may hope to learn something about our own epoch through a reflection on the situation in king Hammurabi's century.

## *The scribe school*

Old Babylonian mathematics was not the high-status diversion of wealthy and highly intelligent amateurs, as Greek mathematicians were or aspired to be. According to the format of its texts it was taught in the scribe school – hardly to all students, not even among those who went through the full standard curriculum, but at least to a fraction of future scribes (or future scribe school masters only?).

The word "scribe" might mislead. The scribe certainly knew to write. But the ability to calculate was just as important – originally, writing had been invented as subservient to accounting, and this subordinated function with respect to calculation remained very important. The modern colleagues of the scribe are engineers, accountants and notaries.

Therefore, it is preferable not to speak naively of "Babylonian mathematicians". Strictly speaking, what was taught number- and quantity-wise in scribe school should not be understood primarily as "mathematics" but rather as *calculation*. The scribe should be able to *find the correct number*, be it in his engineering function, be it as an accountant. Even problems that do not consider true practice always concern measurable magnitudes, and they always ask for a numerical answer (as we have seen). It might be more appropriate to speak of the algebra as "pure calculation" than as (unapplied and hence) "pure" mathematics. The preliminary observations of page 1 should thus be thought through once again!

That is one of the reasons that many of the problems that have no genuine root in practice none the less speak of the measurement and division of fields, of the production of bricks, of the construction of siege ramps, of purchase and sale, and of loans carrying interest. One may learn much about daily life in Babylonia (as it presented itself to the eyes of a professional scribe) through the topics spoken of in these problems, even when their mathematical substance is wholly artificial.

If we really want to find Old Babylonian "mathematicians" in an approximately modern sense, we must look to those who *created* the techniques and discovered how to *construct* problems that were difficult but could still be solved. For example we may think of the problem TMS XIX #2 (not included in the present book): to find the sides $\ell$ and $w$ of a rectangle from its area and from the area of another rectangle ⊏⊐($d$, ❐($\ell$)) (that is, a rectangle whose length is the diagonal of the first rectangle and whose width is the cube constructed on its length). This is a problem of the eighth degree. Without systematic work of theoretical character, perhaps with a starting point similar to BM 13901 #12, it would have been impossible to guess that it was bi-biquadratic (our term of course), and that it can be solved by means of a cascade of three successive quadratic equations. But this kind of theoretical work has left no written traces.

### *The first purpose: training numerical calculation*

When following the progression of one of the algebraic texts – in particular one of the more complicated specimens – one is tempted to trust the calculations – "it is no doubt true that IGI 6°56′40″ is 8′38″24‴, and if that was not the case, the modern edition of the text would certain have inserted a footnote" (certain writing errors have indeed been corrected above, so all calculations *should* be correct). The reader who has been more suspicious will, on the other hand, have received a good training in sexagesimal arithmetic.

That illustrates one of the functions of algebra in the curriculum: it provided a pretext for training the manipulation of difficult numbers. The aim of the school being the training of professional routines, intensive cultivation of sexagesimal arithmetic was obviously welcome.

This observation can be transferred to our own epoch and its teaching of second-degree equations. Its aim was never to assist the copying of gramophone records or CDs to a cassette tape. But the reduction of complicated equations and the ensuing solution of second-degree equations is not the worst pretext for familiarizing students with the manipulation of symbolic algebraic expressions and the insertion of numerical values in a formula; it seems to have been difficult to find alternatives of more convincing direct practical relevance – and the general understanding and flexible manipulation of algebraic formulas and the insertion of numerical values in formulas *are* routines which are necessary in many jobs.

### *The second purpose: professional pride*

The acquisition of professional dexterity is certainly a valid aim, even if it is reached by indirect means. Yet that was not the only purpose of the teaching of apparently useless mathematics. Cultural or ideological functions also played a role, as shown by the "edubba texts" (above, page 33), texts that served to shape the professional pride of future scribes.

Quite a few such texts are known. They speak little of everyday routines – the ability to handle these was too elementary, in order to be justified the pride of a scribe had to be based on something more weighty. To read and write the Akkadian mother tongue in syllabic writing did not count much. But to write Sumerian (which only other scribes would understand), that was something! To know and practice all the logograms, not least their occult and rare meanings, that would also count!

To find the area of a rectangular field from its length and width was also not suited to induce much self-respect – any bungler in the trade could do that. Even the determination of the area of a trapezium was too easy. But to find a length and a width from their sum and the area they would "hold" was already more substantial; to find them from data such as those of AO 8862 #2, or the nightmarish informations of VAT 7532 – that would allow one to feel as a *real* scribe, as somebody who could command the respect of the non-initiates.

We have no information about Sumerian and mathematics being used for social screening of apprentice-scribes – one of the functions of such matters in the school of today: Since the scribe school was no public school with supposedly equal access for everybody, there was hardly any need to keep the "wrong" people out by indirect means. However, even in recent times dead languages have also fulfilled a cultural role beyond that of upholding a social barrier. Since the Renaissance and for centuries, Latin (and "Latinity" as an emblem of elite culture) was part of the self-confidence of European administrative and juridical institutions; from that point of view, the mathematical formation of engineers was seen (by those who were in possession of Latin culture and had adopted its norms) rather as a proof of cultural and moral inferiority. Since the 18th century, however, mathematical competence and dexterity (at best, competence and dexterity *beyond* what was necessary) were essential components of the professional identity of engineers, architects and officers.[55]

Even analysis of the cultural function of "advanced" Old Babylonian mathematics may thus teach us something about our own epoch.

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[55] In the 19th century, precisely these three groups provided the bulk of subscribers to the *Journal des mathématiques élémentaires* and similar periodicals. The *Ladies' Diary*, published from 1704 until 1841 and rich in mathematical contents, could also aim at a social group that was largely excluded from Oxford-Cambridge and public-school Latinity and Grecity, to which even genteel women had no access.

# Chapter 7

# Origin and heritage

One way to explain socio-cultural structures and circumstances argues from their function: if the scribe school expended much effort to teach advanced mathematics and even more on teaching Sumerian, and if it continued to do so for centuries, then these activities must have had important functions – if not as direct visible consequences then indirectly. We have just seen an explanation of that kind.

Another way to explain them – no alternative but rather the other side of the coin – is based on historical origin. Who had the idea, and when? Or, if no instantaneous invention is in focus, how did the phenomenon develop, starting from which earlier structures and conditions? In our particular case: if the invention was not made in the scribe school, where did the inspiration come from, and how did the activity perhaps change character because of the transplantation into a new environment where it came to fulfil new functions?

Over the last 40 years, our knowledge about Mesopotamian third-millennium mathematics has advanced much, in particular concerning the determination of rectangular or quasi-rectangular areas. We may now confidently assert that the reason that we have found no third-millennium texts containing algebra problems is that *there were non*.

This contradicts the traditional belief that everything in Mesopotamia must date from times immemorial. Certainly, we are in the "Orient" where everything, as one knows, is without age and without development (and in particular without progress) – in "the West" at least a conviction "without age and without development".

## *The origin: surveyors' riddles*

On the contrary, the algebra of the Old Babylonian scribe school is no continuation of century- (or millennium-)old school traditions – nothing similar had existed during the third millennium. It is one expression among others of the new scribal culture of the epoch. In principle, the algebra might have been invented within the school environment – the work on bilingual texts and the study of Sumerian grammar from an Akkadian point of view certainly were. Such an origin would fit the fact that the central vocabulary for surveying and part of that used in practical calculation is in Sumerian or at least written with Sumerian logograms ("length", "width", IGI, "be equal by"), while the terms that characterize the algebraic genres as well as that which serves to express *problems* is in Akkadian.

However, an invention within the scribe school agrees very badly with other sources. In particular it is in conflict with the way problems and techniques belonging to the same family turn up in Greek and medieval sources. A precise analysis of all parallel material reveals a very different story – the material is much too vast to allow a complete presentation of the argument here, but part of it is woven into the following discussion.

The surveyors of central Iraq (perhaps a wider region, but that remains a hypothesis in as far as this early epoch is concerned) had a tradition of geometrical riddles. Such professional riddles are familiar from other pre-modern environments of mathematical practitioners (specialists of commercial computation, accounting, master builders, and of course surveying) whose formation was based on apprenticeship and not taken care of by a more or less learned school. As an example we may cite the problem of the "hundred fowls" which one finds in numerous Chinese, Indian, Arabic and European problem collections from the Middle Ages:

> Somebody goes to the market and buys 100 fowls for 100 dinars. A goose costs him 3 dinars, a hen 2 dinars, and of sparrows he gets 3 for each dinar. Tell me, if you are an expert calculator, what he bought![56]

There are many solutions. 5 geese, 32 hens, and 63 sparrows; 10 geese, 24 hens, and 66 sparrows; etc. However, when answering a riddle, even a mathematical riddle, one needs not give an exhaustive solution, nor give a proof (except the numerical proof that the answer

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[56] This is an "average" variant. The prices may vary, and also the species (mostly but not always birds are traded). As a rule, however, the problem speaks about 100 animals and 100 monetary units. There are mostly three species, two of which cost more than one unit while the third costs less.

fulfils the conditions)[57] Who is able to give *one* good answer shows himself to be a competent calculator "to the stupefaction of the ignorant" (as says a manual of practical arithmetic from 1525).

Often the solution of a similar riddle asks for the application of a particular trick. Here, for instance, one may notice that one must buy 3 sparrows each time one buys a goose – that gives 4 fowls for 4 dinars – and 3 sparrows for each two hens – 5 fowls for 5 dinar,

Such "recreational problems" (as they came to be called after having been adopted into a mathematical culture rooted in school, where their role was to procure mathematical fun) had a double function in the milieu where they originated. On one hand, they served training – even in today's school, a lion that eats three math teachers an hour may be a welcome variation on kids receiving 3 sweets a day. On the other, and in particular (since the central tricks rarely served in practical computation), they allowed the members of the profession to feel like "truly expert calculators" – a parallel to what was said above on the role of Sumerian and "too advanced" mathematics for the Old Babylonian scribes.

At some moment between 2200 and 1800 BCE, the Akkadian surveyors invented the trick that was later called "the Akkadian method", that is, the quadratic completion; around 1800, a small number of geometrical riddles about squares, rectangles and circles circulated whose solution was based on this trick. A shared characteristic of these riddles was to consider solely elements that are directly present in the figures – for instance *the* side or *all four* sides of a square, never "3 times the area" or "$\frac{1}{3}$ of the area". We may say that the problems are defined without coefficients, of, alternatively, with "natural" coefficients.

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[57] Who wants to, can try to find the full solution with or without negative numbers (which would stand for selling instead of buying), and demonstrate that it does represent an exhaustive solution under the given circumstances. That was done by the Arabic mathematician Abū Kāmil around 900 CE. In the introduction to his treatise about the topic he took the opportunity to mock those practitioners deprived of theoretical insight who gave an arbitrary answer only – and who thus understood the question as a riddle and not as a mathematical *problem*.

If ${}_4c$ stands for “the 4 sides” and $\square(c)$ for the area of a square, $d$ for the diagonal and $\sqsubset\!\sqsupset(\ell, w)$ for the area of a rectangle, the list of riddles seems to have encompassed the following problems:

$$c+\square(c) = 110$$
$${}_4c+\square(c) = 140$$
$$\square(c)-c = 90$$
$$\square(c)-{}_4c = 60 \ (?)$$

$$\ell+w = \alpha\ ,\ \ \sqsubset\!\sqsupset(\ell, w) = \beta$$
$$\ell-w = \alpha\ ,\ \ \sqsubset\!\sqsupset(\ell, w) = \beta$$

$$\ell+w = \alpha\ ,\ \ (\ell-w)+\sqsubset\!\sqsupset(\ell, w) = \beta$$
$$\ell-w = \alpha\ ,\ \ (\ell+w)+\sqsubset\!\sqsupset(\ell, w) = \beta;$$

$$d = \alpha\ ,\ \ \sqsubset\!\sqsupset(\ell, w) = \beta\ .$$

Beyond that, there were problems about two squares (sum of or difference between the sides given together with the sum of or difference between the areas); a problem in which the sum of the perimeter, the diameter and the area of a circle is given, and *possibly* the problem $d-c = 4$ concerning a square, with the pseudo-solution $c = 10$, $d = 14$; two problems about a rectangle, already known before 2200 BCE, have as their data, one the area and the width, the other the area and the length. That seems to be all [58].

These riddles appear to have been adopted into the Old Babylonian scribe school, where they became the starting point for the development of the algebra as a genuine discipline. Yet the school did not take over the riddle tradition as it was. A riddle, in order to provoke interest, must speak of conspicuous entities (*the* side, *all four* sides, etc.); a school institution, on the other hand, tends to engage in systematic variation of coefficients – in particular a school which, like that of the Mesopotamian scribes since the invention of writing in the fourth millennium, had always relied on very systematic

[58] In the Old Babylonian texts, a closed group consisted of the four rectangle problems where the area is given together with the length; the width; the sum of these; or their difference. One may presume that the completion trick was first invented as a way to make this group grow from two to four members.

variation.[59] In a riddle it is also normal to begin with what is *most* naturally there (for instance the four sides of a square) and to come afterwards to derived entities (here the area). In school, on the contrary, it seems natural to privilege the procedure, and therefore to speak first of that *surface* which eventually is to be provided with a "projection" or a "base".

Such considerations explain why a problem collection about squares like BM 13901 moves from a single to two and then three squares, and why all problems except the archaizing #23, "the four sides and the area" invariably speak of areas before mentioning the sides. But the transformation does not stop there. Firstly, the introduction of coefficients asked for the introduction of a new technique, the change of scale in one direction (and then different changes in the two directions, as in TMS IX #3); the bold variation consisting in the addition of a volume and an area gave rise to a more radical innovation: the use of factorization. The invention of these new techniques made possible the solution of even more complicated problems.

On the other hand, as a consequence of the drill of systematic variation, the solution of the fundamental problems became a banality on which professional self-esteem could not be built: thereby work on complicated problems became not only a possibility but also a cultural necessity.

One may assume that the orientation of the scribal profession toward a wide range of practices invited the invention of problems outside abstract surveying geometry where the algebraic methods could be deployed – and therefore, even though "research" was no aim of the scribal school, to exploit the possibilities of *representation*. It is thus, according to this reconstruction, the transfer to the school that

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[59] Who only practises equation algebra for the sake of finding solutions may not think much of coefficients – after all, they are mostly a nuisance to be eliminated. However, Viète and his generation made possible the unfolding of *algebraic theory* in the seventeenth century by introducing the use of general symbols for the coefficients. Correspondingly, the Old Babylonian teachers, when introducing coefficients, made possible the development of *algebraic practice* – without the availability and standardized manipulation of coefficients, no free representation is possible.

gave to the cut-and-paste technique the possibility to become the *heart of a true algebra.*

Other changes were less momentous though still conspicuous. In the riddles, 10 was the preferred value for the side of the square, remaining so until the sixteenth century CE; the favourite value in school was 30′, and when an archaizing problem retained 10 it was interpreted as 10′.[60] Finally, as explained above (page 33), the hypothetical "somebody" asking a question was replaced by a professorial "I".

BM 13901 #23 (page 79), retaining "the four widths and the surface" (in that order) and the side 10 while changing its order of magnitude is thus a characteristic fossil pointing to the riddle tradition. Even its language is archaizing, suggesting the ways of surveyors not educated in the scribe school. Taking into account its position toward the end of the text (#23 of 24 problems, #24 being the most intricate of all), we may see it as something like "last problem before Christmas".

It appears that the first development of the algebraic discipline took place in the Eshnunna region, north of Babylon, during the early decades of the 18th century;[61] from this area and period we have a number of mathematical texts that for once have been regularly excavated and which can therefore be dated. By then, Eshnunna was a cultural centre of the whole north-central part of Iraq; Eshnunna also produced the first law-code outside the Sumerian south. The text $Db_2$-146 (below, page 135) comes from a site belonging to the Eshnunna kingdom.

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[60] In order to see that 10 (and 30) had precisely this role one has to show that 10 was not the normal choice in other situations where a parameter was chosen freely. Collation of many sources shows that 10 (respectively 30 in descendants of the school tradition) was the preferred side not only of squares but also of other regular polygons – just as 4, 7, 11, etc. can be seen to have been favourite numbers in the multiplicative-partitive domain but only there, cf. note 25 (page 49).

[61] Eshnunna had been subdued by Ur III in 2075 but broke loose already in 2025.

In c. 1761 Eshnunna was conquered by Hammurabi and destroyed. We know that Hammurabi borrowed the idea of a law-code, and can assume that he brought enslaved scholars back. Whether he also brought scholars engaged in the production or teaching of mathematics is nothing but a guess (the second-millennium strata of Babylon are deeply buried below the remains of the first-millennium world city), but in any case the former Sumerian south took up the new mathematical discipline around 1750 – AO 8862 (above, page 63), with its still unsettled terminology and format, seems to represent an early specimen from this phase.

Problems from various sites in the Eshnunna region (for once regularly excavated and therefore dated) deal with many of the topics also known from later – the early rectangle variant of the "broken-reed" problem mentioned on page 73 is from one of them. Strikingly, however, there is not a single example of *representation*. AO 8862, on the other hand, already contains an example, in which a number of workers, their working days and the bricks they have produced are "heaped". It does not indicate the procedure, but clearly the three magnitudes have to be represented by the sides of a rectangle and its area multiplied by a coefficient. A large part of the Eshnunna texts start "If somebody asks you thus ...", found neither in AO 8862 nor in any later text (except as a rudiment in the archaizing BM 13901 #23).

Not much later, we have a number of texts which (to judge from their orthography) were written in the south. Several text groups obey very well-defined canons for format and terminology (not the same in all groups), demonstrating a conscious striving for regularity (the VAT- and Str-texts all belong here). However, around 1720 the whole south seceded, after which scribal culture there was reduced to a minimum; mathematics seems not to have survived. From the late 17th century we have a fair number of texts from Sippar, somewhat to the north of Babylon (BM 85200+VAT 6599 is one of them), and another batch from Susa in western Iran (the TMS-texts), which according to their terminology descend from the northern type first developed in Eshnunna. And then, nothing more ... .

### *The heritage*

Indeed, in 1595 a Hittite raid put an end to the already weak Old Babylonian statal and social system. After the raid, power was grasped by the Kassites, a tribal group that had been present in Babylonia as migrant workers and marauders since Hammurabi's times. This caused an abrupt end to the Old Babylonian epoch and its particular culture.

The scribe school disappeared. For centuries, the use of writing was strongly reduced, and even afterwards scholar-scribes were taught as apprentices within "scribal families" (apparently bloodline families, not apprenticeship formalized as adoption).

Even sophisticated mathematics disappeared. The social need for practical calculation, though reduced, did not vanish; but the professional pride of scholar-scribes now built on the appurtenance to a venerated tradition. The scribe now understood himself as somebody who *knew to write, even literature*, and not as a calculator; much of the socially necessary calculation may already now have rested upon specialists whose scanty literary training did not qualify them as "scribes" (in the first millennium, such a split is fairly certain).

The 1200 years that follow the collapse of the Old Babylonian cultural complex have not left a single algebra text. In itself that does not say much, since only a very small number of mathematical texts even in the vaguest sense have survived (a few accounting texts, traces of surveying, some tables of reciprocals and squares). But *when* a minimum of mathematical texts proper written by scholar-scribes emerges again after 400 BCE, the terminology allows us to distinguish that which had been transmitted within their own environment from that which was borrowed once again from a "lay" environment. To the latter category belongs a small handful of problems about squares and rectangles. They contain no representation, no variation of coefficients, nothing sophisticated like the "broken reed" or the oil trade, only problems close to the original riddles; it would hardly be justified to speak of them as representatives of an "algebra".

These late texts obviously do not inform us, neither directly nor indirectly, about the environment where the riddles had been transmitted, even though a continuation of the surveyors' tradition is the most verisimilar hypothesis. Sources from classical Antiquity as well as the Islamic Middle Ages at least make it clear that the tradition that had once inspired Old Babylonian algebra had survived despite the disappearance of its high-level offspring.

The best evidence is offered by an Arabic manual of practical geometry, written perhaps around 800 CE (perhaps later but with a terminology and in a tradition that points to this date), and known from a Latin twelfth-century translation.[62] It contains all the problems ascribed above to the riddle tradition except those about two squares and the circle problem – in particular the problem about "the four sides and the area", in the same order as BM 13901 #23, and still with solution 10 (not 10′). It also conserves the complex alternation between grammatical persons, the hypothetical "somebody" who asks the question in many of the earliest school texts, the exhortation to keep something in memory, and even the occasional justification of a step in the procedure by means of the quotation of words from the statement as something which "he" has said. Problems of the same kind turn up time and again in the following centuries – "the four sides and the area" (apparently for the last time) in Luca Pacioli's *Summa de Arithmetica* from 1494, "the side and the area" of a square in Pedro Nuñez's *Libro de algebra en arithmetica y geometria* from 1567 (in both cases in traditional riddle order and with solution 10).

In Greek mathematics, "algebraic" second-degree problems are rare but not totally absent. One is of particular interest: in one of the components of the text collection known collectively as *Geometrica* (attributed traditionally but mistakenly to Heron), "the four sides and the area" turns up again, though with the variation that "the four sides" have become "the perimeter". Here, the geometric description is so precise that we can even decide the orientation of the diagram – the rectangle representing the four sides is joined *below*, see Figure 45. The text speaks explicitly of the rectangle that represents 4$c$ as "four feet".

Since the discovery of Babylonian algebra, it has often been claimed that one component of Greek theoretical geometry (namely, Euclid's *Elements* II.1-10) should be a translation of the results of

[62] The *Liber mensurationum* ascribed to an unidentified Abū Bakr "who is called Heus" and translated by Gerard of Cremona.

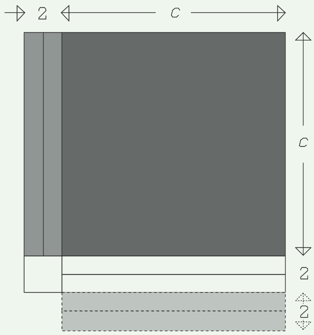


*Figure 45*. “The area joined to the perimeter” of the *Geometrica*

Babylonian algebra into geometric language. This idea is not unproblematic; Euclid, for example, does not solve problems but proves constructions and theorems. The geometric interpretation of the Old Babylonian technique, on the other hand, would seem to speak in favour of the hypothesis.

However, if we align the ten theorems *Elements* II.1 – 10 with the list of original riddles we make an unexpected discovery: all ten theorems can be connected directly to the list – they are indeed demonstrations that *the naive methods of the riddle tradition can be justified according to the best theoretical standards of Euclid’s days*. In contrast, there is *nothing* in Euclid that can be connected to the innovations of the Old Babylonian school. Its algebra turns out to have been a blind alley – not *in spite of* its high level but rather because of this level, which allowed it to survive only in the very particular Old Babylonian school environment.

The extraordinary importance of the *Elements* in the history of mathematics is beyond doubt. None the less, the most important influence of the surveyors’ tradition in modern mathematics is due to its interaction with medieval Arabic algebra.

Even Arabic algebra seems to have originally drawn on a riddle tradition. As mentioned above (page 97), its fundamental equations deal with an amount of money (a “possession”) and its square root. They were solved according to rules without proof, like this one for the case “a possession and ten of its roots are made equal to 39 dinars”:

you halve the roots, which in this question are 5. You then multiply them

> with themselves, from which arises 25; add them to 39, and they will be 64. You should take the root of this, which is 8. Next remove from it the half of the roots, which is 5. Then 3 remains, which is the root of the possession. And the possession is 9.

Already the first author of a treatise on algebra which we know (which is probably the first *treatise* about the topic[63]) – al-Khwārizmī, from the earlier 9th century CE – was not satisfied with rules that are not based on reasoning or proof. He therefore adopted the geometric proofs of the surveyors' tradition corresponding to Figures 12, 14, 22 and, first of all, the characteristic configuration of Figure 33. Later, mathematicians like Fibonacci, Luca Pacioli and Cardano saw these proofs as the very essence of algebra, not knowing about the polynomial algebra created by al-Karajī, as-Samawʾal and their successors (another magnificent blind alley). In this way the old surveyors' tradition conquered the discipline from within; the word *census*, the Latin translation of "possession", came to be understood as another word for "square". All of this happened in interaction with *Elements* II – equally in debt to the surveyors' tradition, as we have just seen.

Even though the algebra of the cuneiform tablets was a blind alley – glorious but blind all the same – the principles that it had borrowed from practitioners without erudition was thus not. Without this inspiration it is difficult to see how modern mathematics could have arisen. As it has been said about God: "If he did not exist, one would have had to invent him".

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[63] The quotation is borrowed from this treatise, rendered in "conformal translation" of the Latin twelfth-century version (the best witness of the original wording of the text).

# Chapter 8

# A moral

A moral? How? What does morality have to do with mathematics and its history?

Firstly, “a moral” – that of a fable – is not the same thing as *morality*. The moral of a fable represents the meditation that offers itself after the reading, “what can we learn from this?” In this sense, not only fables but also texts that tell history have often had the aim to suggest a moral – at least since the time of Herodotus and the Hebrew scribes who told the events of the times of Saul and David (or the fables about these presumed events).

In this sense too, the history of mathematics, and histories of mathematics, have their morals. The first interpretation of Old Babylonian algebra carried the implicit message that *they* had the same kind of mathematics as *we*. They only did not have that wonderful algebraic symbolism that has allowed us to go even further; and they also had not “discovered” the negative numbers (which in second-hand recycling was transformed into a conviction that they *had* discovered them). *They* had not yet progressed as far *we* have, but they were *on the same track* – the only track, the track toward us. With an easily deducted corollary: the fact that our track is the only track is a guarantee that what we do coincides with progress, and that all the others – other civilizations, and school students who have not yet understood – must learn to follow it. Another corollary, perhaps not quite as close at hand, nor however too far-fetched: what holds for mathematics might hold for other aspects of civilization: *we* are progress incarnate and verified.

This message disappears with the new interpretation. Old Babylonian mathematics certainly has many similarities with contemporary “world mathematics” – probably more than any other foreign mathematical culture (we build so directly on ancient Greek and medieval Arabic mathematics that we cannot consider them “foreign”). But the differences are conspicuous, be it concerning its methods, be it regarding aims and mode of thought. What we can learn from the new interpretation is thus that *mathematics can be thought in different ways*, and that one should always listen to the other (the other epoch studied by the historian, or the partner of the teacher, that is, the student) before deciding what this other must have thought and should think. If mathematics can be thought in different ways, then there is no guarantee that ours is in all respects *the* best possible – not even for ourselves, and even less in impersonal and supra-historical generality. However, by listening we may come to understand better our own practice and mode of thought, and to better ponder whether ours is one of the fruitful ways – perhaps even *which* fruits it promises.

The progress found in the history of mathematics is not a one-way motor road (in any case a thing never seen outside the world of metaphors!). In an image formulated by the historian of mathematics Moritz Cantor in 1875, it is to be compared to a river landscape with so many streams – streams which, with bendings and turnings, bifurcations and reunifications, have a tendency to run in the same direction, towards the same ocean. *If* progress exists in the history of civilizations, it will be of the same kind.

# Appendix A

# Problems for the reader

The problems presented in Chapters 1 to 4 were so different one from the others that it was necessary to accompany each of them by a copious commentary. In order to allow the reader who likes so to penetrate some Old Babylonian texts without being held firmly by the hand, this appendix contains problems in translation only or at most accompanied by the most necessary clarifications. Some are counterparts of problems that were presented above and come from the same tablets.

## *TMS XVI #2*

**13.** The 4th of the width to that by which the length goes beyond the width, to join,
**14.** 15´. You, 15´ to 4 raise, 1 you see, what is it?
**15.** 4 and 1 posit.
**16.** 15´ scatter. 10´, the going-beyond, and 5´, the joined, posit. 20´, the width,
**17.** to 10´, the going-beyond, join, 30´ the length, and 20´, to tear out, posit. 5´ to 4 raise,
**18.** 20´ you see. 20´, the width, to 4 raise, 1°20´ you see.
**19.** 30´, the length, to 4 raise, 2 you see. 20´, the width,
**20.** from 1°20´ tear out, 1 you see. 1
**21.** from 2, the lengths, tear out, 1 you see, what is it?
**22.** From 4, of the fourth, 1 tear out, 3 you see. IGI 4 detach, 15´ you see.
**23.** 15´ to 3 raise, 45´ you see, as as much as (there is) of widths posit. Posit to tear out.
**24.** 1 as as much as (there is) of lengths posit. [...] 1 take, to 1 length
**25.** raise, 30´ you see. 20´ the width, 20´ to 45´, (as much as (there is) of) widths, raise,
**26.** 15´ you see, 15´ to 15´ join, 30´ you see, 30´ the length.

Commentary: see #1 of the same tablet, page 25.

### *TMS VII #1*

**1.** The 4th of the width to the length I have joined, its 7⟨th⟩ to 10 I have gone,
**2.** as much as the heap of length and ⟨width⟩. You, 4 posit; 7 posit;
**3.** 10 posit; 5′ to 7 raise, 35′ you see.
**4.** 30′ and 5′ single out. 5′, the step, to 10 raise,
**5.** 50′ you see. 30′ and 20′, posit. 5′, the step, to 4, of the fourth of the width,
**6.** raise: 20′ you see, 20′, the width. 30′ to 4, of the fourth,
**7.** raise, 2 you see. 2 posit, lengths. 20′ from 20′ tear out,
**8.** and from 2, 30′ tear out, 1°30′ you see.
**9.** From 4, of the fourth, 1 tear out, 3 {...} you see.
**10.** IGI 3 detach, 20′ you see. 20′ to 1°30′ raise:
**11.** 30′ you see, 30′ the length. 30′ from 50′ tear out, 20′ you see, 20′ the width.
**12.** Turn back. 7 to 4, of the fourth, raise, 28 you see.
**13.** 10 from 28 tear out, 18 you see. IGI 3 detach,
**14.** 20′ you see. 20′ to 18 raise, 6 you see, 6 (for) the length.
**15.** 6 from 10 tear out, 4 (for) the width. 5′ to 6 raise,
**16.** 30′ the length. 5′ to 4 raise, 20′ you see, 20′ the ⟨width⟩.

Commentary: see #2 of the same tablet, page 33.

### *VAT 8389 #1*

**Obv I**

**1.** From 1 BÙR 4 GUR of grain I have collected,
**2.** from 1 second BÙR 3 GUR of grain I have collected.
**3.** grain over grain, 8`20 it went beyond
**4.** My plots I have accumulated: 30`.
**5.** My plots what?
**6.** 30`, the BÙR, posit. 20`, the grain which he has collected, posit.
**7.** 30`, the second BÙR, posit.

**8.** 15`, the grain which he has collected,
**9.** 8`20 which the grain over the grain went beyond,
**10.** and 30` the accumulation of the surfaces of the plots posit:
**11.** 30` the accumulation of the surfaces of the plots
**12.** to two break: 15`.
**13.** 15` and 15` until twice posit:
**14.** IGI 30`, of the BÙR, detach: 2″.
**15.** 2″ to 20`, the grain which he has collected,
**16.** raise, 40′ the false grain; to 15` which until twice
**16a.** you have posited,
**17.** raise, 10` may your head hold!
**18.** IGI 30, of the second BÙR, detach, 2″.
**19.** 2″ to 15`, the grain which he has collected,
**20.** raise, 30′ the false grain; to 15 which until twice
**20a.** you have posited, raise, 7`30.
**21.** 10` which your head holds
**22.** over 7`30 what goes beyond? 2`30 it goes beyond.
**23.** 2`30 which it goes beyond, from 8`20
**24.** which the grain over the grain goes beyond,

**Obv. II**

**1.** tear out: 5`50 you leave.
**2.** 5`50 which you have left
**3.** may your head hold!
**4.** 40′, the change, and 30′, the change,
**5.** accumulate: 1°10′. The IGI I do not know.
**6.** What to 1°10′ may I posit
**7.** which 5`50 which your head holds gives me?
**8.** 5` posit. 5` to 1°10 raise.
**9.** 5`50 it gives to you.
**10.** 5` which you have posited, from 15` which until twice
**11.** you have posited, from one tear out,
**12.** to one join:
**13.** The first is 20`, the second is 10`.
**14.** 20` (is) the surface of the first plot, 10` (is) the surface of the second plot.
**15.** If 20` (is) the surface of the first plot,
**16.** 10` the surface of the second plot, their grains what?
**17.** IGI 30`, of the BÙR, detach: 2″.

**18.** 2″ to 20`, the grain which he has collected,
**19.** raise, 40′. To 20`, the surface of the first plot,
**20.** raise, 13`20 the grain of 20`, the surface of the meadow.
**21.** IGI 30`, of the second BÙR, detach: 2″.
**22.** 2″ to 15`, the grain which he has collected, raise, 30′.
**23.** 30′ to 10`, the surface of the second plot
**24.** raise, 5 the grain of the surface of the second plot.
**25.** 13`30 the grain of ¿the surface? of the first plot
**26.** over 5 the grain of ¿the surface? of the second plot
**27.** what goes beyond? 8`20 it goes beyond.

This problem belongs on one of two twin tablets, containing a total of ten problems about the rent paid for two parcels of a field. On one parcel the rent is 4 GUR of grain per BÙR, on the other it is 3 GUR per BÙR. The present problem informs us also that the total area is 30` (SAR = 1 BÙR), and that the difference between the total rents of the two parcels is 8`20 (SÌLA). The other problems give, for instance, the two areas, or the difference between the areas together with the total rent.

As explained on page 13, the BÙR and the GUR are units belonging to practical life. In order to work in the place-value system we need to convert them into the standard units SAR and SÌLA (1 BÙR = 30` SAR, 1 GUR = 5` SÌLA); as we see, the difference between the two rents is already given SÌLA, and the total area in SAR.

A modern reader may find it strange that the two rents per BÙR, which in lines I.1-2 are given in GUR (per BÙR), are translated into SÌLA in lines I.6–7 without multiplication; in general, as we see, the text skips no intermediate step. The explanation is that the conversion is made by means of a "metrological table" (probably a table learned by heart). Precisely because such conversions had to be made so often, scribes had tables which not only stated the converted values of the practical units but also of their multiples. However, they had no tables for combined conversions, and therefore the final conversion into SÌLA per SAR asks for calculation.

The modern reader may also wonder that the text does not indicate once for all the value of the BÙR in SÌLA and its IGI. Once more the reason is that the text describes the Old Babylonian calculational technique: the calculator writes on a small tablet for rough work the three numbers 20 (20` SÌLA per BÙR), 30 (30˝ SAR per BÙR) and 2 (2″, IGI 30`) – and afterwards, by means of the multiplication table, the product 40 (20`·2″ = 40′ SÌLA per SAR).

A small explanation may be necessary in order to facilitate understanding of the procedure: first the text determines what the difference between the two rents would be if the two parcels had been equal in area, that is, 15` SAR each. This difference is not large enough – it is 2`30 SÌLA too small – and therefore the first parcel must be enlarged. Each time a SAR is transferred from the second to the first parcel, the difference grows by 40′+30′ SÌLA (the two "modifications" of II.4[64]); the number of SAR that must be transferred is then found by division.

In the end comes a numerical verification. Such verifications are not rare in the Old Babylonian texts, even though their presence is no general norm.

## *VAT 8390 #1*

**Obv 1**

**1.** Length and width I have made hold: 10` the surface.
**2.** The length to itself I have made hold:
**3.** a surface I have built.
**4.** So much as the length over the width went beyond
**5.** I have made hold, to 9 I have repeated:
**6.** as much as that surface which the length by itself
**7.** was made hold.
**8.** The length and the width what?
**9.** 10` the surface posit,
**10.** and 9 (to) which he has repeated posit:
**11.** The equalside of 9 (to) which he has repeated what? 3.

---

[64] The tablet is damaged on this point, but the traces of signs that remain could well come from the word *takkirtum*, which means "change" or "modification" but does not occur in other mathematical texts. In any case, this philological doubt does not touch the interpretation of the mathematical procedure.

**12.** 3 to the length posit
**13.** 3 to the width posit.
**14.** Since “so much as the length over the width went beyond
**15.** I have made hold”, he has said
**16.** 1 from 3 which to the width you have posited
**17.** tear out: 2 you leave.
**18.** 2 which you have left to the width posit.
**19.** 3 which to the length you have posited
**20.** to 2 which ⟨to⟩ the width you have posited raise, 6.
**21.** IGI 6 detach: 10´.
**22.** 10´ to 10` the surface raise, 1`40.
**23.** The equalside of 1`40 what? 10.

Obv. II

**1.** 10 to 3 which to the length you have posited
**2.** raise, 30 the length.
**3.** 10 to 2 which to the width you have posited
**4.** raise, 20 the width.
**5.** If 30 the length, 20 the width,
**6.** the surface what?
**7.** 30 the length to 20 the width raise, 10` the surface.
**8.** 30 the length together with 30 make hold: 15`.
**9.** 30 the length over 20 the width what goes beyond? 10 it goes beyond.
**10.** 10 together with 10 make hold: 1`40.
**11.** 1`40 to 9 repeat: 15` the surface.
**12.** 15` the surface, as much as 15` the surface which the length
**13.** by itself was made hold.

As support for the interpretation, a diagram may serve (Figure 46). Then the text almost explains itself, in particular if one keeps in mind BM 13901 #10 (page 48) and BM 15285 #24 (page 98).

One should take note of the use of the multiplicative operations “make hold”, “raise” and “repeat”. That “making hold” really implies a *construction* is underlined in I.3, as we have also seen in AO 8862 #2 (page 63). The “raising” in I.20 and II.7 is of special interest: it finds the area of rectangles, but as these are already in place, there is no need to construct them. Therefore the area is merely *calculated*.

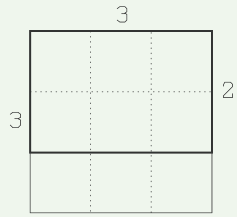



*Figure 46*. The geometry of VAT 8390 #1

## *VAT 8520 #1*

**Face**

**1.** The 13th from the heap of the *igûm* and the *igibûm*
**2.** to 6 I have repeated, from the inside of the *igûm*
**3.** I have torn out: 30′ I have left. 1 the surface. The *igûm* and the *igibûm* what?
**4.** Since "the thirteenth of the heap of the *igûm* and the *igibûm*
**5.** to 6 I have repeated, from the inside of the *igûm*
**6.** I have torn out: 30′ I have left", he has said,
**7.** 13, of the thirteenth, posit; 6 to which he has repeated posit;
**8.** 1, the surface, posit; and 30′ which he has left posit.
**9.** From 13, of the thirteenth, 6 to which he has repeated
**10.** tear out. 7 you leave.
**11.** 7 which you leave and 6 to which you have repeated,
**12.** may your head hold!
**13.** 7 to 6 raise, 42 to 1, the surface, raise, 42.
**14.** 42, may your head hold!
**15.** 13, of the thirteenth, to 30′ which he has left
**16.** raise, 6°30′ to two break: 3°15′.
**17.** 3°15′ together with 3°15′ make hold: 10°33′45″.
**18.** To 10°33′45″, 42 which your head holds
**19.** join, 52°33′45″.
**20.** The equal of 52°33′45″ what? 7°15′.
**21.** 7°15′ and 7°15′, its counterpart, lay down:
**22.** 3°15′, the made-hold, from one tear out, to the other join:
**23.** The first is 10°30, the other is 4.
**24.** What to 7, which your head holds, should I posit
**25.** which 10°30′ gives me? 1°30′ posit. 1°30′ to 7 raise,
**26.** 10°30′ it gives you. 1°30′ which you have posited is the *igûm*.

**27.** IGI 6, which your head holds, detach, 10´.
**28.** 10´ to 4 raise, 40´ is the *igibûm*.
**29.** Since 1°30´ is the *igûm*, 40´ is the *igibûm*, the surface is what?
**30.** 1°30´, the *igûm*, to 40´, the *igibûm*, raise, 1 is the surface.
**31.** 1°30, the *igûm*, and 40´, the *igibûm*, heap: 2°10´.

**Revers**

**1.** The thirteenth of 2°10´ what? 10´.
**2.** 10´ to 6 repeat: 1, from 1°30,
**3.** the *igûm*, tear out: 30´ you leave.

Like YBC 6967 (page 46), this problem deals with a number pair from the table of reciprocals. Both texts speak of their product as "the surface", in agreement with the geometric representation. But there is a difference: this time the product is 1, not 1` as in YBC 6967.

As regards the mathematical structure and the procedure, one may compare with TMS IX #3 (page 59).

## *Str 368*

**Face**

**1.** I have taken a reed, its measure I do not know.
**2.** 1 KÙŠ I have cut off. 1 sixty (steps along) the length I have gone.
**3.** (With) what I have cut off I have enlarged it
**4.** with 30 (steps) of that (along) the width I have gone.
**5.** 6`15 is the surface. The head (initial length) of the reed what?
**6.** You, by your proceeding,
**7.** 1` and 30 posit. (For) the reed which you do not know
**8.** 1 posit, to 1 sixty which you have gone
**9.** you raise: 1` is the false length.
**10.** 30 to this 1 raise, 30 is the false width.
**11.** 30, the false width to 1`, the false length,
**12.** raise, 30` the false surface.
**13.** 30` to 6`15, the true surface,

**Revers**

**1.** raise: 3```7``30` it gives you.
**2.** 5´ which you have cut off to the false length raise,
**3.** 5 it gives you. 5 to the false width raise,

4. 2`30 it gives you. $\frac{1}{2}$ of 2`30 break, 1`15
5. 1`15 make encounter, 1``33`45
6. to 3```7``30` join, 3```9``3`45.
7. What is equal? 13`45 is equal.
8. 1`15 which you have made encounter to the inside join,
9. 15` it gives you. IGI 30`, the false surface, detach, 2″.
10. 2″ to 15` raise, 30′ is the head of the reed.

This is the rectangle version of the “broken reed” (see page 117), similar to VAT 7532. In this variant, the field is rectangular, and the reed breaks a single time only.

## *YBC 6504 #1*

**Face**

1. So much as length over width goes beyond, I have made confront itself, from the inside of the surface
2. I have torn it out: 8′20″. Length over width 10′ goes beyond.
3. By your proceeding, 10′ you make hold:
4. 1′40″ to 8′20″ you join: 10′ you posit.
5. Half of 10′ you break: 5′ you posit.
6. 5′ you make hold: 25″ you posit.
7. 25″, the surface, to 10′ you join: 10′25″ you posit.
8. By 10′25″, 25′ is equal. 5′ to 25′ you join:
9. 30′, the length, you posit. 5′ from 25′ your tear out:
10. 20′, the width, you posit.

This problem deals with the same mutilated rectangle as #4 of the same tablet (see page 84): Together, indeed, the four problems of tablet represent an interesting variant of the closed group where the “surface” of a rectangle is given together with the length; with the width; with the sum of the sides; or with their difference (see note 58, page 114). In the present tablet, the “surface” is replaced everywhere by the same mutilated rectangle.

In this first problem, we know the side of the square that has been “torn out”. It is therefore easily reduced to the type we know from

YBC 6967 (page 46). In following the operations one should keep in mind that the number 10′ occurs in two different roles.

Exceptionally in this type, the "joining" of 5′ precedes the "tearing out". The tablet seems to belong to the same early phase and text group as AO 8862, and it shares this particularity with three texts from Eshnunna (thus belonging to an even earlier phase). It seems indeed that the school is responsible for the request that operations should always be concretely meaningful, just as it was responsible for outlawing broad lines – this request is not evidence of "a primitive intellect not yet ready for abstraction", as has been supposed, but of a critical mind reflecting upon how to justify what is done.

### *YBC 6504 #3*

**Revers**

**1.** So much as length over ⟨width⟩ goes beyond, made encounter, from inside the surface I have torn out,
**2.** 8′20″. 30′ the length, its width what?
**3.** 30′ made encounter: 15′ you posit.
**4.** 8′20″ from inside 15′ you tear out, 6′40″ you posit.
**5.** Half of 30′ you break:
**6.** 15′ made encounter: 3′45″ you posit.
**7.** 3′45″ to 6′40″ you join: 10′25″ you posit.
**8.** By 10′25″, 25′ is equal. 15′ from 25′ you tear out:
**9.** 10′ you posit. 10′ from 30′ you tear out:
**10.** 20′, the width, you posit.

This is the third problem from the same tablet. It makes use of a ruse which is both elegant and far from every routine (see Figure 47): elimination of the mutilated rectangle from the square $\square(\ell)$ on the length leaves a remainder that can be decomposed as a square $\square(\ell-w)$ and a rectangle $\sqsubset\!\sqsupset(\ell-w,30')$. These can be reconfigured as a gnomon, as shown in the diagram. We may look at the process as a "change of variable" – the problem now concerns a square $\square(\ell-w)$ and 30 of its sides, and its solution follows the book for such problems.

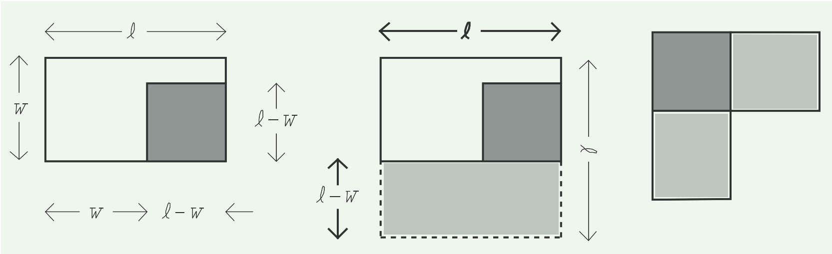



*Figure 47*. The geometry behind YBC 6504 #3, in slightly distorted proportions

## *BM 85200+VAT 6599 #23*

**Revers I**

**19.** An excavation. So much as I have made confront itself, and 1 KÙŠ, going beyond, that is the depth. 1°45′ of dirt I have torn out.

**20.** You, 5′, going beyond, to 1, the conversion, raise, 5′ you see; to 12 raise, 1 you see.

**21.** 5′ make confront itself, 25″ you see. 25″ to 1 raise, 25″ you see. IGI 25 detach,

**22.** 2\`24 you see. 2\`24 to 1°45′ raise, 4\`12 you see.

**23.** from “equal, 1 joined”, 6 ¿1? is/are equal(s). 6 to 5′ raise, 30′ you see, confronts itself. 6 (error for 7) the depth.

**24.** The procedure.

This problem comes from the same tablet as the “excavation problem” BM 85200+VAT 6599 #6 that was dealt with above (page 94), and its solution follows the same principles. Now the “ground” is square, and the depth exceeds the side by 1 KÙŠ. As “reference body” a cube of side 1 KÙŠ is chosen, which allows the use of a table of $n^2\cdot(n+1)$, called “equal, 1 joined”. Such tables have been found.

## *Db$_2$-146*

**Obv.**

**1.** If, about a (rectangle with) diagonal, (somebody) asks you

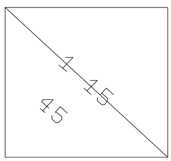
**2.** thus, 1°15 the diagonal, 45′ the surface;

**3.** length and width corresponding to what? You, by your proceeding,

**4.** 1°15´, your diagonal, its counterpart lay down:
**5.** make them hold: 1°33´45″ comes up,
**6.** 1°33´45 ¿may your? hand ¿hold?
**7.** 45´ your surface to two bring: 1°30 comes up.
**8.** From 1°33´45″ cut off: {...} 33´45″ the remainder.
**9.** The equal of 3´45″ take: 15´ comes up. Its half-part,
**10.** 7´30″ comes up, to 7´30″ raise: 56″15‴ comes up
**11.** 56″15‴ your hand. 45´ your surface over your hand,
**12.** 45´56″15‴ comes up. The equal of 45´56″15‴ take:
**13.** 52´30″ comes up, 52´30″ its counterpart lay down,
**14.** 7´30″ which you have made hold to one
**15.** join: from one
**16.** cut off. 1 your length, 45 the width. If 1 the length,
**17.** 45 the width, the surface and the diagonal corresponding to what?
**18.** You, by your making, the length make hold:
**19.** 1 comes up ... may your head hold.

**Rev**

**20.** ...: 45´, the width, make hold:
**21.** 33´45″ comes up. To your length join:
**22.** 1°33´45″ comes up. The equal of 1°33´45″ take:
**23.** 1°15´ comes up. 1°15´ your diagonal. Your length
**24.** to the width raise, 45´ your surface.
**25.** Thus the procedure.




This is one of the texts from the Eshnunna region, and thus belongs to the earliest phase (and as we see, it uses the phrase "to one join, from one cut off", not respecting the "norm of concreteness"). With fair precision it can be dated to c. 1775 BCE. The problem is one of the riddles which the Old Babylonian school borrowed from the Akkadian surveyors (see pages 112 and 114); it turns up, solved in precisely the same way, in a Hebrew manual from 1116 CE, that is, 1900 years later. In the text we see several reminiscences of this origin – for instance the introductory passage "If, about a (rectangle with)

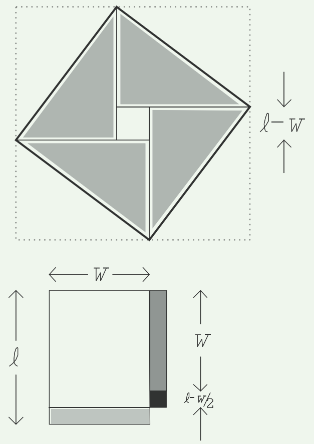



*Figure 48*. The geometry of $Db_2$-146

diagonal, (somebody) asks you thus" and the reference to the square on the length in line 21 simply as "your length"; both features reverberate in BM 13901 #23.

Lines 1 – 9 find the difference between the length and the width of the rectangle; the method is shown in the upper part of Figure 48. Afterwards, the sides are found from this difference and the area by the procedure which we already know perfectly well, for instance from YBC 6967 (see page 46), and which corresponds to the lower diagram in the figure.

The "hand" of lines 6 and 11 is a reference to the reckoning board on which the calculator performed his additions and subtractions. The "half-part" of line 9 (*muttatum*) is a synonym for "moiety".

In the end we have a proof with an unmistakeable trace of the "Pythagorean rule" in abstract formulation (*the length make hold*, without the usual idnetification of its numerical value).

# Appendix B

## Transliterated texts

For readers who already know at least the rudiments of the Babylonian language, this appendix gives transliterated versions of most of the texts translated in Chapters 1–4 and in Appendix A, together with a list of the words that appear together with the standard translation used in the English versions of the texts (see the explanation on page 21). All transliterations are taken from Jens Høyrup, *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin*, New York: Springer, 2002. The philological notes have been left out. The present standard translations are, with a few exceptions, the same as those used in this volume.

### *Key to vocabulary and standard translations*

A.RÁ: steps of
A.ŠÀ (~*eqlum*): surface
*alākum* (~RÁ): to go
*amārum*: to see
AN(.TA/NA) (~*elûm*): upper
*ana* (~.RA): to
*annikīam*: here
*aššum*: since
*atta* (*ina epēšika*): you (by your proceeding)
BAL: conversion
*bāmtum*: moiety
BÁN: BÁN
*bandûm*: bandûm
*banûm*: to build
*bêrum*: to separate
BÙR: BÙR
DAḪ (~*waṣābum*): to join
DAL (~*tallum*): bar
DIRIG (~*watartum*): going-beyond
DIRIG, UGU ... (~*eli* ... *watārum*): go beyond, over ...
DU$_7$.DU$_7$: to make encounter
DU$_8$ (~*paṭārum*): detach
*eṣēpum* (~TAB): to repeat
*elēnu*: over-going
*elûm*: come up (as a result)
EN.NAM (~*minûm*): what
*epēšum* (~KÌD): to proceed/procedure
*ezebum* (~TAG$_4$): leave
GABA(.RI) (~*meḫrum*): counterpart
GAM (~*šuplum*): depth
G̃AR (~*šakānum*): to posit
G̃AR.G̃AR (~*kamārum*): to heap/heap
GARIM (~*tawirtum*): parcel
GAZ (~*ḫepûm*): to break

GI (~*qanûm*): reed
GI.NA (~*kīnum*): true
GÍN (~*šiqlum*): shekel
GU$_7$(.GU$_7$) (~*šutakūlum*): to make hold
GUR: GUR
*ḫarāṣum*: to cut off
*ḫaṣābum* (~KUD): to break off
*ḫepûm* (~GAZ): to break
ḪI.A: various (things)
ÍB.SI$_8$ (substantif): the equal
.E *Q* *C* ÍB.SI$_8$: by *Q*, *c* is equal
ÍB.TAG$_4$ (~*šapiltum*): remainder
IGI (~*igûm*): *igûm*
IGI *n*: IGI *n*
IGI.BI (~*igibûm*): *igibûm*
*igibûm* (~IGI.BI): *igibûm*
*igûm* (~IGI): *igûm*
ÍL (~*našûm*): to raise
*imtaḫḫar* (<*maḫārum*): confronts itself
*ina* (~.TA): from
*inūma*: as
*ištēn* ... *ištēn*: one ... one
*ištēn* ... *šanûm*: the first ... the second
*ištu*: out from
(*n*-)KAM: the *n*th (of a sequence)
*itti* (~KI): together with
*kamārum* (~G̃AR.G̃AR, UL.GAR): to heap
KI (~*qaqqarum*): ground
*kī maṣi*: corresponding to what
KI(.TA) (~*šaplûm*): lower
KI.GUB.GUB: base
*kīma*: as much as (there is) of
*kīam*: thus
*kimrātum* (<*kamārum*): the things heaped
*kīnum* (~GI.NA): true
KUD: to break off
*kullum*: to hold
*kumurrûm* (<*kamārum*; ~G̃AR.G̃AR): heap
KÙŠ (~*ammatum*): KÙŠ
*la* (~NU): not
*lapātum*: to inscribe
*leqûm*: to take
*libbum*: inside
LUL (~*sarrum*): false
*-ma*: “:”
*ma.na* (~*manûm*): mina
*maḫārum*: to confront
*makāsum*: to collect (rent etc.)
*mala*: so much as
*manātum*: contribution
*maṭûm*: to be(come) small(er)
*meḫrum* (<*maḫārum*; ~GABA(.RI)): counterpart
*mindatum*: measure
*mīnûm* (~EN.NAM): what
*mišlum* (~ŠU.RI.A): half
*mitḫartum* (<*maḫārum*; ~LAGAB; ~ÍB.SI$_8$): confrontation
*muttarittum*: descendant
*muttatum*: half-part
*nadānum* (~SUM): to give
*nadûm*: to lay down
*nakmartum* (<*kamārum*): heap
*nasāḫum* (~ZI): to tear out
*nāsḫum* (<*nasāḫum*): the to-be-torn-out
*našûm* (~ÍL): to raise
*nēmelum*: profit

*nēpešum* (<*epēšum*): procedure
NIGIN (~*šutakūlum*): to make hold
NIM (~*našûm*): to raise
NINDAN: NINDAN
NU (~*la*, *ul*(*a*)): not
*paṭārum* (~DU$_8$): to detach
PI: PI
*qabûm* (~DUG$_4$): to say
*qanûm* (~GI): reed
*qaqqarum* (~KI): ground
*qātum*: hand
*ramānišu*: itself
*rēška likīl*: may your head retain!
*rēšum* (~SAG̃): head
SAḪAR (~*eperum*): dirt
*saḫārum*: to turn around
SAG̃ (~*rēšum*): head
SAG̃.DÙ (~*santakkum*): triangle
SAG̃.KI.GUD: trapezium
SAG̃(.KI): width
*sapāḫum*: to scatter
SAR (~*mūšarum*): SAR
*sarrum* (~LUL): false
SÌLA (~*qa*): SÌLA
SUM (~*nadānum*): to give
*ṣiliptum*: diagonal
*ša*: which / that of (etc.)
*šakānum* (~G̃AR): to posit
*šâlum*: to ask
ŠÁM: buying
*šanûm*: second
*šapiltum* (~ÍB.TAG$_4$): remainder
ŠE (~*šeʾum*): grain
*šeʾum* (~ŠE): grain
*šiqlum* (~GÍN): shekel
ŠU.RI.A (~*mišlum*): half
*šulmum*: integrity
*šumma*: if
*šumum*: name
*šūšum*: sixty
*šutakūlum* (<*kullum*; ~GU$_7$): to make hold
*šutamḫurum* (<*maḫārum*): to make confront itself
*šūtbum*: to make go away (<*tebûm*)
TA.ÀM: each
TAB (~*eṣēpum*): to repeat
TAG$_4$ (~*ezēbum*): to leave
*takīltum* (<*kullum*): made-hold
*takkirtum* (<*nakārum*): modification
*tammar* (<*amārum*; ~IGI.DU$_8$/PA(D)): you see
*târum* (~NIG̃ÍN): to turn back
*tawirtum* (~GARIM): parcel
TÚL.SAG̃: excavation
*u*: and
UL.GAR (~*kamārum*): to heap / heap
*ul*(*a*) (~NU): not
UŠ: length
*waṣābum* (~daḫ): to join
*wabālum*: to bring
*wāṣbum* (<*waṣābum*): the to-be-joined
*wāṣītum*: projection
*watārum* (~DIRIG): to go beyond
*wuṣubbûm* (<*waṣābum*): the joined
ZA.E (KÌD.DA/TA.ZU.DÈ) (~*atta* ...): you (by your proceeding)
ZI (~*nasāḫum*): to tear out

## *AO 8862 #2*

**I**

**30.** uš saĝ uš ù saĝ
**31.** *uš-ta-ki-il$_5$-ma* a.šà$^{lam}$ *ab-ni*
**32.** *a-sà-ḫi-ir mi-ši-il$_5$* uš
**33.** *ù ša-lu-uš-ti* saĝ
**34.** *a-na li-bi* a.šà-*ia*
**35.** [*ú*-]-*ṣi-ib-ma* 15
**36.** [*a-t*]*u-úr* uš *ù* saĝ
**37.** [*ak*-]*mu-ur-ma* 7

**II**

**1.** uš *ù* saĝ *mi-nu-um*
**2.** *at-ta i-na e-pe-ši-i-ka*
**3.** [2 *n*]*a-al-p*[*a*]*-at-ti mi-iš-li-im*
**4.** [*ù*] 3 *na-al-pa-ti*
**5.** [*ša*-]*lu-uš-ti ta-l*[*a*]*-pa-at-ma*
**6.** igi 2-bi 30 *ta-pa-tar-ma*
**7.** 30 a.rá 7 3,30 *a-na* 7
**8.** *ki-im-ra-tim* uš *ù* saĝ
**9.** *ub-ba-al-ma*
**10.** 3,30 *i-na* 15 *ki-i*[*m*]*-ra-ti-i-a*
**11.** *ḫu-ru-uṣ$_4$-ma*
**12.** 11,30 *ša-pi-il$_5$-tum*
**13.** *l*[*a*] *wa-t*[*ar*] 2 *ù* 3 *uš-ta-kal-ma*
**14.** 3 a.rá 2 6
**15.** igi 6 ĝál 10 *i-na-di-kum*
**16.** 10 *i-na* 7 *ki-im-ra-ti-i-ka*
**17.** uš *ù* saĝ *a-na-sà-aḫ-ma*
**18.** 6,50 *ša-pi-il$_5$-tum*
**19.** *ba-a-š*[*u*] *ša 6,50 e-ḫe-pe-e-ma*
**20.** 3,25 *i-na-di-ku*
**21.** 3,25 *a-di ši-ni-šu*
**22.** *ta-la-pa-at-ma* 3,25 a.rá 3,25
**23.** 11,40,[25] *i-na li-bi*
**24.** 11,30 *a-na-sà-aḫ-ma*

**25.** 10,25 *ša-pi-il$_5$-tum* ⟨10,25.e 25 íb.si$_8$⟩
**26.** *a-na* 3,25 *iš-te-en*
**27.** 25 *tu-ṣa-am-ma* 3,50
**28.** *ù ša i-na ki-im-ra-at*
**29.** uš *ù* sag̃ *a*[*s*]-*sà-aḫ-ma*
**30.** *a-na* 3,50 *tu-ṣa-am-ma*
**31.** 4 uš *i-na* 3,25 *ša-ni-im*
**32.** 25 *a-na-sà-aḫ-ma* 3 sag̃
**32a.** 7 *ki-im-ra-tu-ú*
**32b.** 4 uš
12 a.šà
3 sag̃

## *BM 13901 #1, #2, #10, #12, #14 and #23*

**Obv. I**

**#1**

**1.** a.šà$^{l[am]}$ *ù mi-it-ḫar-ti ak-m*[*ur-m*]*a* 45.e 1 *wa-ṣi-tam*
**2.** *ta-ša-ka-an ba-ma-at* 1 *te-ḫe-pe* [3]0 *ù* 30 *tu-uš-ta-kal*
**3.** 15 *a-na* 45 *tu-ṣa-ab-ma* 1-⌊e] 1 íb.si$_8$ 30 *ša tu-uš-ta-ki-lu*
**4.** *lìb-ba* 1 *ta-na-sà-aḫ-ma* 30 *mi-it-ḫar-tum*

**#2**

**5.** *mi-it-ḫar-ti lìb-bi* a.šà [*a*]*s-sú-uḫ-ma* 14,30.e 1 *wa-ṣi-tam*
**6.** *ta-ša-ka-an ba-ma-at* 1 *te-ḫe-pe* 30 *ù* 30 *tu-uš-ta-kal*
**7.** 15 *a-*⌈*na* 14,30 *tu-ṣa-*]*ab-ma* 14,30,15.e 29,30 íb.si$_8$
**8.** 30 *ša tu-uš-ta-ki-lu a-na* 29,30 *tu-ṣa-ab-ma* 30 *mi-it-ḫar-tum*

**Obv. II**

**#10**

**11.** a.šà *ši-ta mi-it-ḫa-ra-ti-ia ak-mur-ma* 21,15
**12.** *mi-it-ḫar-tum a-na mi-it-ḫar-tim si-bi-a-tim im-ṭi*
**13.** 7 *ù* 6 *ta-la-pa-at* 7 *ù* 7 *tu-uš-ta-kal* 49
**14.** 6 *ù* 6 *tu-uš-ta-kal* 36 *ù* 49 *ta-ka-mar-ma*
**15.** 1,25 igi 1,25 *ú-la ip-pa-ṭa-ar mi-nam a-na* 1,25
**16.** *lu-uš-ku-un ša* 21,15 *i-na-di-nam* 15.e 30 íb.si$_8$
**17.** 30 *a-na* 7 *ta-na-ši-ma* 3,30 *mi-it-ḫar-tum iš-ti-a-at*
**18.** 30 *a-na* 6 *ta-na-ši-ma* 3 *mi-it-ḫar-tum ša-ni-tum*

**#12**

**27.** a.šà *ši-ta mi-it-ḫa⟨-ra⟩-ti-ia ak-mur-ma* 21,40
**28.** *mi-it-ḫa-ra-ti-ia uš-ta-ki-il*$_5$*-ma* 10
**29.** *ba-ma-at* 21,40 *te-ḫe-pe-ma* 10,50 *ù* 10,50 *tu-uš-ta-kal*
**30.** 1,57,21{+25},40.e 10 *ù* 10 *tu-uš-ta-kal* 1,40
**31.** *lìb-bi* 1,57,21{+25},40 *ta-na-sà-aḫ-ma* 17,21{+25},40.e 4,10 íb.si$_8$
**32.** 4,10 *a-na* 10,50 *iš-te-en tu-ṣa-ab-ma* 15.e 30 íb.si$_8$
**33.** 30 *mi-it-ḫar-tum iš-ti-a-at*
**34.** 4,10 *lìb-bi* 10,50 *ša-ni-im ta-na-sà-aḫ-ma* 6,40.e 20 íb.si$_8$
**35.** 20 *mi-it-ḫar-tum ša-ni-tum*

**#14**

**44.** a-šà *ši-ta mi-it-ḫa-ra-ti-ia ak-mur-ma* ⸢25,]25
**45.** *mi-it-ḫar-tum ši-ni-pa-at mi-it-ḫar-tim* [*ù* 5 nind]an
**46.** 1 *ù* 40 *ù* 5 [*e-le-nu* 4]0 *ta-la-pa-at*
**47.** 5 *ù* 5 [*tu-uš-ta-kal 25 lìb-bi 25,25 ta-na-sà-aḫ-ma*]

**Rev. I**

**1.** [25 *ta-la-pa-at* 1 *ù* 1 *tu-uš-ta-kal* 1 40 *ù* 40 *tu-uš-ta-kal*]
**2.** [26,40 *a-na* 1 *tu-ṣa-ab-ma* 1,26,40 *a-na* 25 *ta-na-ši-ma*]
**3.** [36,6,40 *ta-la-pa-at* 5 *a-na* 4]0 *t*[*a-na-ši-ma* 3,20]
**4.** [*ù* 3,20 *tu-uš-ta-kal* 11,6,40] *a-na* 3[6,]6,40 [*tu-ṣa-ab-ma*]
**5.** [36,17,46,40.e 46,40 íb.si$_8$ 3,]20 *ša tu-uš-ta-ki*[*-lu*]
**6.** [*lìb-bi* 46,40 *ta-na-sà-aḫ-*]*ma* 43,20 *ta-la-pa-a*[*t*]
**7.** [igi 1,26,40 *ú-la ip-pa-t*]*a-ar mi-nam a-na* 1,2[6,4]0
**8.** [*lu-uš-ku-un ša* 43,20 *i-n*]*a-di-nam* 30 *ba-an-da-šu*
**9.** [30 *a-na* 1 *ta-na-ši-ma* 30] *mi-it-ḫar-tum iš-ti-a-at*
**10.** [30 *a-na* 40 *ta-na-ši-ma* 20] *ù* 5 *tu-ṣa-ab-ma*
**11.** [25 *mi-it-ḫar-t*]*um ša-ni-tum*

**Rev. II**

**#23**

**11.** a.šà$^{lam}$ *p*[*a*]*-a*[*-at er-bé-et-tam ù* a.š]à$^{lam}$ *ak-mur-ma* 41,40
**12.** 4 *pa-a-at er*[*-bé-e*]*t-tam t*[*a-la-p*]*a-at* igi 4 g̃ál.bi 15
**13.** 15 *a-na* 41,40 [*ta-n*]*a-ši-ma* 10,25 *ta-la-pa-at*
**14.** 1 *wa-ṣi-tam tu-ṣa-ab-ma* 1,10,25.e 1,5 íb.si$_8$
**15.** 1 *wa-ṣi-tam ša tu-iṣ-bu ta-na-sà-aḫ-ma* 5 *a-na ši-na*
**16.** *te-ṣi-ip-ma* 10 nindan *im-ta-ḫa-ar*

## *BM 15285 #24*

**1.** [1 UŠ *mi-i*]*t-ḫa-ar-tum*
**2.** *lìb-ba* 16 *mi-it-ḫa-ra-tim*
**3.** *ad-di* a.šà.bi en.nam

## *BM 85200+VAT 6599 #6 and #23*

**Obv. I**

**#6**

**9.** túl.sağ *ma-la* uš GAM-*ma* 1 saḫar.ḫi.a ba.zi KI$^{ri}$ *ù* saḫar.ḫi.a UL.GAR 1,10 uš *ù* sağ 50 uš sağ en⟨.nam⟩
**10.** za.e 50 *a-na* 1 bal *i-ši* 50 *ta-mar* 50 *a-na* 12 *i-ši* 10 *ta-mar*
**11.** 50 *šu-tam*⟨*-ḫir*⟩ 41,40 *ta-mar a-na* 10 *i-ši* 6,56,40 *ta-mar* igi-*šu du*$_8$*.a* 8,38,24 *ta*⟨*-mar*⟩
**12.** *a-na* 1,10 *i-ši* 10,4,48 *ta-mar* 36 24 42 íb.si$_8$
**13.** 36 *a-na* 50 *i-ši* 30 uš 24 *a-na* 50 *i-ši* 20 sağ 36 *a-na* 10 6 GAM
**14.** [*n*]*e-pé-šum*

**Rev. I**

**#23**

**19.** túl.sağ *ma-la uš-tam-ḫir ù* 1 kùš dirig GAM-*ma* 1,45 saḫar.ḫi.a [ba].zi
**20.** za.e 5 dirig *a-na* 1 bal *i-ši* 5 *ta-mar a-na* 12 *i-š*[*i* 1⌟ *ta-mar*
**21.** 5 *šu-tam*⟨*-ḫir*⟩ 25 *ta-mar* 25 *a-na* 1 *i-ši* 25 *ta-mar* igi ⌞25 du$_8$.a]
**22.** 2,24 *ta-mar* 2,24 *a-na* 1,45 *i-ši* 4,12 [*ta-mar*]
**23.** *i-na* íb.si$_8$ 1 daḫ.ḫa 6 ¿1? íb.s[i$_8$] ⌞6 *a-na* 5⌟ *i-*[*ši* 30] *ta*⟨*-mar*⟩ *im*⟨*-ta-ḫar*⟩ 6$^{sic}$ GAM
**24.** *ne-pé-š[um]*

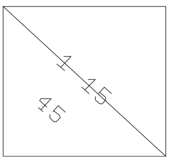
## ***$Db_2$-146***

**Obv.**

**1.** *šum-ma ṣí-li-ip-ta-a-am i-ša-lu-ka*
**2.** *um-ma šu-ú-ma* 1,15 *ṣí-li-ip-tum* 45 a.šà
**3.** *ši-di ù* sag̃.ki *ki ma-a-sí at-ta i-na e-pé-ši-ka*
**4.** 1,15 *ṣí-li-ip-ta-ka me-ḫe-er-šu i-di-i-ma*
**5.** *šu-ta-ki-il-šu-nu-ti-i-ma* 1,33,45 *i-li*
**6.** 1,33,45 šu KU.U$^{¿}$.ZU/BA$^{?}$
**7.** 45 a.šà-*ka a-na ši-na e-bi-il-ma* 1,30 *i-li*
**8.** *i-na* 1,33,45 *ḫu-ru-ús-ma* {1,}33,45(sic) *ša-pí-il-tum*
**9.** ib.sí 3,45 *le-qe-e-ma* 15 *i-li mu-ta-su*
**10.** 7,30 *i-li a-na* 7,30 *i-ši-i-ma* 56,15 *i-li.*
**11.** 56,15 šu-*ka* 45 a.šà-*ka e-li* šu-*ka*
**12.** 45,56,15 *i-li* ib.si 45,56,15 *le-qe-ma*
**13.** 52,30 *i-li* 52,30 *me-ḫe-er-šu i-di-i-ma*
**14.** 7,30 *ša tu-uš-ta-ki-lu a-na iš-te-en*
**15.** *ṣí-ib-ma i-na iš-te-en*
**16.** *ḫu-ru-úṣ* 1 uš-*ka* 45 sag̃.ki *šum-ma* 1 *uš*
**17.** 45 sag̃.ki a.šà *ù ṣí-li-ip-ti ki ma-ṣí*
**18.** [*at-ta i-na e-p*]*é-ši-ka ši-da šu-ta-ki-il-ma*
**19.** [1 *i-li* ...] *re-eš-ka li-ki-il*

**Rev.**

**20.** [...]-*ma* 45 sag̃.ki *šu-ta-ki-il-ma*
**21.** 33,45 *i-li a-na ši-di-ka ṣí-ib-ma*
**22.** 1,33,45 *i-li* ib.si 1,33,45 *le-*[*qe*]*-ma*
**23.** 1,15 *i-li* 1,15 *ṣí-li-ip-*[*ta*]*-ka* uš-*ka*
**24.** *a-na* sag̃.ki *i-ši* 45 a.šà-*ka*
**25.** *ki-a-am ne-pé-šum*



## *TMS VII #1 and #2*

**#1**

**1.** $4^{\text{-}at}$ sağ *a-na* uš daḫ $7^{\text{-}\langle ti\rangle}$*-šu* a-na 10 [*al-li-ik*]
**2.** *ki-ma* UL.GAR uš *ù* ⟨sağ⟩ za.e 4 ğar 7 [ğar]
**3.** 10 ğar 5 *a-rà*[65] 7 *i-ší* 35 *ta-mar*
**4.** 30 *ù* 5 *be-e-er* 5 a.rá *a-na* 10 *i-ší*
**5.** 50 *ta-mar* 30 *ù* 20 ğar 5 a.rá *a-na* 4 *re-*⟨*ba-ti*⟩ sağ
**6.** *i-ší-ma* 20 *ta-mar* 20 sağ 30 *a-na* 4 *re-ba-*⟨*ti*⟩
**7.** *i-ší* 2 *ta-mar* 2 ğar uš 20 *i-na* 20 zi
**8.** *ù i-na* 2 30 zi 1,30 *ta-mar*
**9.** *i-na* 4 *re-ba-ti* 1 zi 3{,20} *ta-mar*
**10.** igi 3 *pu-ṭú-*⟨*úr*⟩ 20 *ta-mar* 20 *a-na* 1,30 *i-ší-ma*
**11.** 30 *ta-mar* 30 uš 30 *i-na* 50 zi 20 *ta-mar* 20 sağ
**12.** *tu-úr* 7 *a-na* 4 *re-ba-*⟨*ti*⟩ *i-ší* 28 *ta-mar*
**13.** 10 *i-na* 28 zi 18 *ta-mar* igi 3 *pu-*⟨*ṭú-úr*⟩
**14.** 20 *ta-*⟨*mar*⟩ 20 *a-na* 18 *i-ší* 6 *ta-mar* 6 uš
**15.** 6 *i-na* 10 zi 4 sağ 5 *a-na* 6 [*i-š*]*í*
**16.** 30 uš 5 *a-na* 4 *i-ší* 20 *ta-*⟨*mar*⟩ 20 ⟨sağ⟩

**#2**

**17.** $4^{\text{-}at}$ sağ *a-na* uš daḫ $7^{\text{-}ti}$[*-šu*]
**18.** *a-di* 11 *al-li-ik* ugu [UL.GAR]
**19.** uš *ù* sağ 5 dirig za.e [4 ğar]
**20.** 7 ğar 11 ğar *ù* 5 dirig [ğar]
**21.** 5 *a-na* 7 *i-ší* 3[5 *ta-mar*]
**22.** 30 *ù* 5 ğar 5 *a-na* 1[1 *i-ší* 55 *ta-mar*]
**23.** 30 20 *ù* 5 zi ğar 5 [*a-n*]*a* 4
**24.** *i-ši* 20 *ta-*⟨*mar*⟩ 20 sağ 30 *a-na* 4 *i-ší-ma*
**25.** 2 *ta-mar* 2 uš 20 *i-na* 20 zi
**26.** 30 *i-na* 2 zi 1,30 ğar *ù* 5 *a-*⸢*na* ...]
**27.** 7 *a-na* 4 *re-*⟨*ba-ti*⟩ *i-ší-ma* 28 *ta-mar*
**28.** 11 UL.GAR *i-na* 28 zi 17 *ta-mar*
**29.** *i-na* 4 *re-*⟨*ba-ti*⟩ 1 zi 3 ⸤*ta*]*-mar*
**30.** igi 3 *pu-ṭú-*⟨*úr*⟩ 20 *ta-*⟨*mar*⟩ 20 ⸤*a-na*] 17 *i-*⟨*ší*⟩

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[65] I owe this correction of the published transliteration to Christine Proust, who has examined the tablet.

**31.** 5,40 *ta-*⟨*mar*⟩ 5,40 [u]š 20 *a-na* 5 dirig *i-ší*

**32.** 1,40 *ta-*⟨*mar*⟩ 1,40 *wa-ṣí-ib* uš 5,40 uš

**33.** *i-na* 11 UL.GAR zi 5,20 *ta-mar*

**34.** 1,40 *a-na* 5 dirig daḫ 6,40 *ta-mar*

**35.** 6,40 *n*[*a*]*-sí-iḫ* sağ 5 a.rá

**36.** *a-na* 5,40 uš *i-ší* 28,20 *ta-mar*

**37.** 1,40 *wa-ṣí-ib* uš *a-na* 28,20 [daḫ⌉

**38.** 30 *ta-mar* 30 uš 5 *a-*[*na* 5,20]

**39.** *i-ši-ma* 26,40 *t*[*a-mar* 6,40]

**40.** *na-sí-iḫ* sağ *i-na* [26,40 zi]

**41.** 20 *ta-mar* 20 sa[ğ]

## *TMS VIII #1*

**1.** [a.šà 10 4-*at* sağ *a-na* sağ daḫ] *a-na* 3 *a-li-*⌈*ik* ¿... ... ...? ugu]

**2.** [uš 5 dir]iğ za.e [4 *r*]*e-ba-ti ki-ma* sağ ğar *re-b*[*a-at* 4 *le-qé* 1 *ta-mar*]

**3.** [1 *a-na*] 3 *a-li-ik* 3 *ta-mar* 4 *re-ba-at* sağ *a-na* 3 d[aḫ 7 *ta-mar*]

**4.** ⌈7⌉ *ki-ma* uš ğar 5 dirig *a-na na-sí-iḫ* uš ğar 7 uš *a-na* 4 [¿sağ? *i-ši*]

**5.** 28 *ta-mar* 28 a.šà 28 *a-na* 10 a.šà *i-ší* 4,40 *ta-mar*

**6.** [5⌉ *na-sí-iḫ* uš *a-na* 4 sağ *i-ší* 20 *ta-mar* $\frac{1}{2}$ *ḫe-pe* 10 *ta-mar* 10 NIGIN

**7.** [1,40] *ta-mar* 1,40 *a-na* 4,40 daḫ 4,41,40 *ta-mar mi-na* íb.si 2,10 *ta-ma*[*r*]

**8.** [10 ¿s]i$_8$.si$_8$? *a-na* 2,10 daḫ 2,20 *ta-mar mi-na a-na* 28 a.šà ğar *šà* 2,20 *i-na-*[*di-n*]*a*

**9.** [5 ğar] 5 *a-na* 7 *i-ší* 35 *ta-mar* 5 *na-sí-iḫ* uš *i-na* 35 zi

**10.** [30 *ta-*]*mar* 30 uš 5 uš *a-na* 4 sağ *i-ší* 20 *ta-mar* 20 {uš} ⟨sağ⟩

## *TMS IX #1, #2 and #3*

**#1**

**1.** a.šà *ù* 1 uš UL.GAR 4[0?30 uš? 20 sag̃]
**2.** *i-nu-ma* 1 uš *a-na* 10 ⌈a.šà daḫ]
**3.** *ú-ul* 1 KI.GUB.GUB *a-na* 20 [sag̃ daḫ]
**4.** *ú-ul* 1,20 *a-na* sag̃ *šà* 40 *it-*⌈*ti* uš ⌈NIGIN g̃ar?]
**5.** *ú-ul* 1,20 *it-*⟨*ti*⟩ 30 uš NIG[IN] 40 *šum-*[*šu*]
**6.** *aš-šum ki-a-am a-na* 20 sag̃ *šà qa-bu-ku*
**7.** 1 daḫ-*ma* 1,20 *ta-mar iš-tu an-ni-ki-a-am*
**8.** *ta-šà-al* 40 a.šà 1,20 sag̃ uš *mi-nu*
**9.** [30 uš *k*]*i-a-am ne-pé-šum*

**#2**

**10.** [a.šà uš *ù* sag̃ U]L.GAR 1 *i-na ak-ka-di-i*
**11.** [1 *a-na* uš daḫ] 1 *a-na* sag̃ daḫ *aš-šum* 1 *a-na* uš daḫ
**12.** [1 *a-na* sag̃ d]aḫ 1 *ù* 1 NIGIN 1 *ta-mar*
**13.** [1 *a-na* UL.GAR uš] sag̃ *ù* a.šà daḫ 2 *ta-mar*
**14.** [*a-na* 20 sag̃ 1 da]ḫ 1,20 *a-na* 30 uš 1 daḫ 1,30
**15.** [⌈*aš-šum*? a.š]à *šà* 1,20 sag̃ *šà* 1,30 uš
**16.** [⌈uš *it-ti*? sa]g̃ *šu-ta-ku-lu mi-nu šum-šu*
**17.** 2 a.šà
**18.** *ki-a-am ak-ka-du-ú*

**#3**

**19.** a.šà uš *ù* sag̃ UL.GAR 1 a.šà 3 uš 4 sag̃ UL.GAR
**20.** [17]-*ti-šu a-na* sag̃ daḫ 30
**21.** [za.]e 30 *a-na* 17 *a-li-ik-ma* 8,30 [*t*]*a-mar*
**22.** [*a-na* 17 sag̃] 4 sag̃ daḫ-*ma* 21 *ta-mar*
**23.** [21 *ki-*]*ma* sag̃ g̃ar 3 *šà-la-aš-ti* uš
**24.** [3 *ki*]-*ma* uš g̃ar 8,30 *mi-nu šum-šu*
**25.** [3] uš *ù* 2[1 sa]g̃ UL.GAR
**26.** 8,30 *ta-mar*
**27.** [3] uš *ù* 21 sag̃ UL.[GAR]
**28.** [*aš-šum* 1 *a-na*] uš daḫ [*ù* 1 *a*]-*na* sag̃ daḫ NIGIN-*ma*
**29.** 1 *a-na* UL.GAR a.šà uš *ù* sag̃ daḫ 2 *ta-mar*
**30.** [2 a.]šà *aš-šum* uš *ù* sag̃ *šà* 2 a.šà
**31.** [1,30 uš *it*]-*ti* 1,20 sag̃ *šu-ta-ku-lu*
**32.** [1 *wu-ṣú-*]*bi* uš *ù* 1 *wu-ṣú-bi* sag̃

**33.** [NIGIN ¿1 *ta-mar*? 1 *ù* 1 ¿...?] ḫi.a UL.GAR 2 *ta-mar*
**34.** [3 ... 21 ... *ù* 8,30 UL.GAR] 32,30 *ta-mar*
**35.** [*ki-a*]*-am ta-šà-al*
**36.** [...].TI sağ *a-na* 21 UL.GAR-*ma*
**37.** [...] *a-na* 3 uš *i-ší*
**38.** [1,3 *ta-mar* 1,3 *a*]*-na* 2 a.šà *i-ši-ma*
**39.** [2,6 *ta-mar* ¿2,6 a.šà?] 32,30 UL.GAR *ḫe-pé* 16,15 *ta-*⟨*mar*⟩
**40.** {1[6,15 *ta-*]*mar*} 16,15 gaba ğar NIGIN
**41.** 4,[24,]3,45 *ta-mar* 2,6 [¿erasure?]
**42.** *i-na* 4,[2]4,3,45 zi 2,18,3,45 *ta-mar*
**43.** *mi-na* íb.si 11,45 íb.si 11,45 *a-na* 16,15 daḫ
**44.** 28 *ta-mar i-na* 2-kam zi 4,30 *ta-mar*
**45.** igi 3-*ti* uš *pu-ṭúr* 20 *ta-mar* 20 *a-na* 4,[30]
**46.** {20 *a-na* 4,30} *i-ši-ma* 1,30 *ta-mar*
**47.** 1,30 uš *šà* 2 a.š[à *mi-na*] *a-na* 21 sağ [*lu-uš-ku-un*]
**48.** *šà* 28 *i-na-di*[*-na* 1,20 ğ]ar 1,20 sağ
**49.** *šà* 2 a.šà *tu-úr* 1 *i-na* 1,[30 zi]
**50.** 30 *ta-mar* 1 *i-na* 1,20 z[i]
**51.** 20 *ta-mar*

## *TMS XIII*

**1.** 2(gur) 2(pi) 5 bán ì.ğiš šám *i-na* šám 1 gín kù.babbar
**2.** 4 silà ta.àm ì.ğiš *ak-ší-iṭ-ma*
**3.** $\frac{2}{3}$ *ma-na* {20 še} kù.babbar *ne-me-la a-mu-úr ki ma-ṣí*
**4.** *a-šà-am ù ki ma-ṣí ap-šu-úr*
**5.** za.e 4 silà ì.ğiš ğar *ù* 40 *ma-na ne-me-la* ğar
**6.** igi 40 *pu-ṭúr* 1,30 *ta-mar* 1,30 *a-na* 4 *i-ší* 6 *ta-mar*
**7.** 6 *a-na* 12,50 ì.ğiš *i-ší-ma* 1,17 *ta-mar*
**8.** $\frac{1}{2}$ 4 *ḫi-pi* 2 *ta-mar* 2 NIGIN 4 *ta-mar*
**9.** 4 *a-na* 1,17 daḫ 1,21 *ta-mar mi-na* íb.si 9 íb.si
**10.** 9 gaba ğar $\frac{1}{2}$ 4 *šà ta-ak-ší-ṭú ḫi-pi* 2 *ta-mar*
**11.** 2 *a-na* 9 1-kam daḫ 11 *ta-mar i-na* 9 2-kam zi
**12.** 7 *ta-mar* 11 silà ta.àm *ta-šà-am* 7 silà *ta-ap-šu-úr*
**13.** kù.babbar *ki ma-ṣí mi-na a-na* 11 [¿silà? *lu-uš-ku*]*-un*

**14.** *šà* 12,50 ì.ĝiš *i-na-ad-di-na* 1,[10 ĝar 1 *m*]*a-na* 10 gín k[ù.babbar]
**15.** *i-na* 7 silà ta.àm *šà ta-pa-aš*-[*šà-ru* ì.ĝiš]
**16.** *šà* 40 kù.babbar *ki ma-ṣí* 40 *a-na* 7 [*i-ší*]
**17.** 4,40 *ta-mar* 4,40 í.ĝiš

## *TMS XVI #1*

**1.** [4-at saĝ *i-na*] uš *ù* saĝ zi 45 za.e 45
**2.** [*a-na* 4 *i-ší* 3 *ta*]-*mar* 3 *mi-nu šu-ma* 4 *ù* 1 ĝar
**3.** [50 *ù*] 5 zi ⸢ĝar⸣ 5 *a-na* 4 *i-ší* 1 saĝ 20 *a-na* 4 *i-ší*
**4.** 1,20 *ta-*⟨*mar*⟩ 4 saĝ 30 *a-na* 4 *i-ší* 2 *ta-*⟨*mar*⟩ 4 uš 20 1 saĝ zi
**5.** *i-na* 1,20 4 saĝ zi 1 *ta-mar* 2 uš *ù* 1 3 saĝ UL.GAR 3 *ta-mar*
**6.** igi 4 *pu-*[*ṭú-ú*]*r* 15 *ta-mar* 15 *a-na* 2 uš *i-ší* [3]0 *ta-*⟨*mar*⟩ 30 uš
**7.** 15 *a-na* 1 *i-ší* [1]5 *ma-na-at* saĝ 30 *ù* 15 *ki-il*
**8.** *aš-šum* 4-*at* saĝ *na-sà-ḫu qa-bu-ku i-na* 4 1 zi 3 *ta-mar*
**9.** igi 4 *pu-*⟨*ṭú-úr*⟩ 15 *ta-mar* 15 *a-na* 3 *i-ší* 45 *ta-*⟨*mar*⟩ 45 *ki-ma* [saĝ]
**10.** 1 *ki-ma* uš ĝar 20 gi.na saĝ *le-qé* 20 *a-na* 1 *i-ší* 20 *ta-mar*
**11.** 20 *a-na* 45 *i-ší* 15 *ta-mar* 15 *i-na* $^{30}{}_{15}$ [zi]
**12.** 30 *ta-mar* 30 uš

## *VAT 7532*

**Obv.**

**1.** saĝ.ki.gud gi kid gi *e*[*l-qé-ma i-na š*]*u-u*[*l*]*-m*[*i*]*-šu*
**2.** 1 *šu-ši* uš *al-li-i*[*k* igi 6 ĝá]l
**3.** *iḫ-ḫa-aṣ-ba-an-ni-ma* 1,12 *a-na* u[š] *ú-r*[*i*]*-id-di*
**4.** *a-tu-úr* igi 3 ĝál *ù* $\frac{1}{3}$ kùš *iḫ*[*-ḫa-aṣ-ba-a*]*n-ni-ma*
**5.** 3 *šu-ši* saĝ an.na *al-li-*[*ik*]
**6.** *ša iḫ-ḫa-aṣ-ba-an-ni ú-te-er-šum-*[*m*]*a*
**7.** 36 saĝ *al-li-ik* 1(bùr)$^{\text{iku}}$ a.šà saĝ gi en.nam

**8.** za.e kìd.da.zu.dè gi *ša la ti-du-ú*
**9.** 1 ẖé.g̃ar igi 6 g̃ál-*šu ẖu-ṣú-ub-ma* 50 *te-zi-ib*
**10.** igi 50 du$_8$-*ma* 1,12 *a-na* 1 *šu-ši* nim-*ma*
**11.** 1,12 *a-na* ⟨1,12⟩ daẖ-*ma* 2,24 uš lul in.sum.
**12.** gi *ša la ti-du-ú* 1 ẖé.g̃ar igi 3 g̃ál-*šu ẖu-ṣú-ub*
**13.** 40 *a-na* 3 *šu-ši ša* sag̃ an.na nim-*ma*
**14.** 2 in.sum 2 *ù* 36 sag̃ ki.ta g̃ar.g̃ar
**15.** 2,36 *a-na* 2,24 uš lul nim 6,14,24 a.šà lul
**16.** a.šà *a-na* 2 $^{e}$tab 1 *a-na* 6,14,24 [n]im
**17.** 6,14,24 in.sum *ù* $\frac{1}{3}$ kùš *ša iẖ-h*[*a-as*]-*bu*
**18.** *a-na* 3 *šu-ši* nim-*ma* 5 *a-na* 2,24 uš lul
**19.** [n]im-*ma* 12 $\frac{1}{2}$ 12 gaz 6 du$_7$.du$_7$

**Rev.**

**1.** 36 *a-na* 6,14,24 daẖ-*ma* 6,15 in.sum
**2.** 6,15.e 2,30 íb.si$_8$ 6 *ša te-zi-bu*
**3.** *a-na* 2,30 daẖ 2,36 in.sum igi 6,14,24
**4.** a.šà lul nu.du$_8$ *mi-nam a-na* 6,14,24
**5.** ẖé.g̃ar *ša* 2,36 in.sum 25 ẖe.g̃ar
**6.** *aš-šum* igi 6 g̃ál *re-ša-am iẖ-ẖa-aṣ-bu*
**7.** 6 *lu-pu-ut-ma* 1 *šu-ut-bi* 5 *te-zi-ib*
**8.** ⟨igi 5 du$_8$-*ma* 12 *a-na* 25 nim 5 in.sum⟩ 5 *a-na* 25 daẖ-*ma* $\frac{1}{2}$ nindan sag̃ gi in.sum

## *VAT 8389 #1*

**Obv. I**

**1.** *i-na* bùr$^{iku}$ 4 še.gur *am-ku-us*
**2.** *i-na* bùr$^{iku}$ *ša-ni*[*-im*] 3 še.gur *am-*[*ku-us*]
**3.** *še-um* ugu *še-im* 8,20 *i-ter*
**4.** garim$^{ia}$ g̃ar.g̃ar-*ma* 30
**5.** garim$^{ú\text{-}a}$ en.nam
**6.** 30 *bu-ra-am* g̃ar.ra 20 *še-am ša im-ku-sú* g̃ar.ra
**7.** 30 *bu-r*[*a-a*]*m ša-ni-am* g̃ar.ra
**8.** [1]5 *š*[*e-am š*]*a im-ku-sú*
**9.** [8],20 [*š*]*a še-um* ugu *še-im i-te-ru* g̃ar.ra
**10.** *ù* 30 *ku-mur-ri* a.šà garim.meš g̃ar.ra-*ma*

**11.** 30 *ku-mur-ri* a.šà garim.meš
**12.** *a-na ši-na ḫe-pé-ma* 15
**13.** 15 *ù* 15 *a-di si-ni-šu* g̃ar.ra-*ma*
**14.** igi 30 *bu-ri-i*[*m p*]*u-ṭur-ma* 2
**15.** 2 *a-na* 20 š[e *š*]*a im-ku-su*
**16.** íl 40 *še-um* l[ul] *a-na* 15 [*š*]*a a-d*[*i*] *ši-ni-šu*
**16a.** *ta-aš-ku-nu*
**17.** íl 10 *re-eš-ka* [*l* ]*i-ki-il*
**18.** igi 30 *bu-ri-im ša-ni-i*[*m*] *pu-ṭur-ma* 2
**19.** 2 *a-na* 15 *še-im ša im-ku-sú*
**20.** íl 30 *še-um* lul *a-na* 15 *ša a-di ši-ni-šu*
**20a.** *ta-aš-ku-nu* íl 7,30
**21.** 10 *ša re-eš-ka ú-ka-lu*
**22.** ugu 7,30 *mi-nam i-ter* 2,30 *i-ter*
**23.** 2,30 *ša i-te-ru i-na* 8,20
**24.** *ša še-um* ugu *še-im i-te-ru*

**Obv. II**

**1.** *ú-sú-uḫ-ma* 5,50 *te-zi-ib*
**2.** 5,50 *ša te-zi-bu*
**3.** *re-eš-ka li-ki-il*
**4.** 40 *ta-ki-i*[*r-tam*] *ù* 30 [*ta-ki-ir*⌉*-tam*
**5.** g̃ar.g̃ar-*ma* 1,10 *i-gi-a-a*[*m ú-ul i-de*]
**6.** *mi-nam a-na* 1,10 *lu-uš-ku-*[*un*]
**7.** *ša* 5,50 *ša re-eš-ka ú-ka-lu i-na-di-nam*
**8.** 5 g̃ar.ra 5 *a-na* 1,10 íl
**9.** 5,50 [*i*]*t-ta-di*[*-k*]*um*
**10.** 5 *ša* [*ta-aš*]*-ku-nu i-na* 15 *ša* ⌊*a-di*⌋ *ši-ni-šu*
**11.** *ta-aš-ku-nu i-na i*[*š*]*-te-en ú-sú-uḫ*
**12.** *a-na iš-te-en ṣí-im-ma*
**13.** *iš-te-en* 20 *ša-nu-um* 10
**14.** 20 a.šà garim *iš-te-at* 10 a.šà garim *ša-ni-tim*
**15.** *šum-ma* 20 a.šà garim *iš-te-at*
**16.** 10 a.šà garim *ša-ni-tim še-ú-ši-n*[*a*] en.nam
**17.** igi 30 *bu-ri-im pu-ṭur-ma* 2
**18.** 2 *a-na* 20 *še-im ša im-ku-s*[*ú*]
**19.** íl 40 *a-na* 20 a.šà garim *i*[*š-te-at*]
**20.** íl 13,20 *še-um ša* 20 [a.šà garim]
**21.** igi 30 *bu-ri-im ša-ni*[*-im pu-ṭur-m*]*a* 2
**22.** 2 *a-na* 15 *še*[*-im ša im-ku-sú* í]l 30

**23.** 30 *a-na* 10 a[.šà garim *ša-ni-tim*]
**24.** íl ⌊5⌋ *še-*[*u*]*m* [*ša* 10 a.šà garim *ša-ni-tim*]
**25.** 13,20 [*še-um* ¿*ša*/a.šà$^?$ garim *iš-te-at*]
**26.** ugu [5] *še*[*-im* ¿*ša*/a.šà$^?$ garim *ša-ni-tim*]
**27.** *mi-nam i-ter* [8,20 *i-ter*]

## *VAT 8390 #1*

**Obv. I**

**1.** [uš *ù* sag̃] *uš-ta-ki-il-ma* 10 a.šà
**2.** [uš *a*]*-na ra-ma-ni-šu uš-ta-ki-il-ma*
**3.** [a.šà] *ab-ni*
**4.** [*ma*]*-la* uš ugu sag̃ *i-te-ru*
**5.** *uš-ta-ki-il a-na* 9 *e-ṣi-im-ma*
**6.** *ki-ma* a.šà*-ma ša* uš *i-na ra-ma-ni-šu*
**7.** *uš-t*[*a*]*-ki-lu*
**8.** uš *ù* sag̃ en.nam
**9.** 10 a.šà g̃ar.ra
**10.** *ù* 9 *ša i-ṣi-pu* g̃ar.ra*-ma*
**11.** íb.si$_8$ 9 *ša i-ṣi-pu* en.nam 3
**12.** 3 *a-na* uš g̃ar.ra
**13.** 3 *a-n*[*a* s]ag̃ g̃ar.ra
**14.** *aš-šum ma-*[*la* uš] ugu sag̃ *i-te-ru*
**15.** *uš-ta-k*[*i-il*] *iq-bu-ú*
**16.** 1 *i-na* ⌊3 *ša a-n*]*a* sag̃ *ta-aš-ku-nu*
**17.** *ú-*[*sú-uḫ-m*]*a* 2 *te-zi-ib*
**18.** 2 *ša t*[*e-z*]*i-bu a-na* sag̃ g̃ar.ra
**19.** 3 *ša a-na* uš *ta-aš-ku-nu*
**20.** *a-na* 2 *ša* ⟨*a-na*⟩ sag̃ *ta-aš-ku-nu* íl 6
**21.** igi 6 *pu-ṭur-ma* 10
**22.** 10 *a-na* 10 a.šà íl 1,40
**23.** íb.si$_8$ 1,40 en.nam 10

**Obv. II**

**1.** 10 *a-na* 3 *š*[*a a-na* uš *ta-aš-ku-nu*]
**2.** íl 30 uš
**3.** 10 *a-na* 2 *ša a-na* sag̃ *ta-aš*[*ku-nu*]

**4.** íl 20 sağ
**5.** *šum-ma* 30 uš 20 sağ
**6.** a.šà en.nam
**7.** 30 uš *a-na* 20 sağ íl 10 a.sà
**8.** 30 uš *it-ti* 30 *šu-ta-ki-il-ma* 15
**9.** 30 uš ugu 20 sağ *mi-nam i-ter* 10 *i-ter*
**10.** 10 *it-ti* [10 *šu*]*-ta-ki-il-ma* 1,40
**11.** 1,40 *a-na* 9 *e-ṣi-im-ma* 15 a.šà
**12.** 15 a.šà *ki-ma* 15 a.šà *ša* uš
**13.** *i-na ra-ma-ni-šu uš-ta-ki-la*

## *VAT 8512*

**Obv.**

**1.** [¿sağ.dú 30 sağ *i-na li-ib-bi ši-it-ta*? *t*]*a-wi-ra-tum*
**2.** [¿...? a.šà an.ta ugu a.šà] ki.ta 7 *i-tir*
**3.** *m*[*u-tar-ri-tum* ki.ta ugu *mu-tar-ri-tim*] an.ta 20 *i-tir*
**4.** *mu-tar-ri-d*[*a*]-⌈*tum ù pi-i-i*]*r-kum mi-nu-*[*u*]*m*
**5.** *ù* a.š[a] *ši-it*[*-ta ta-wi*]*-ra-tum mi-nu-u*[*m*]
**6.** *at-ta* 30 sağ ğar.ra 7 *ša* a.šà an.ta ugu a.šà ki.ta *i-te-ru* ğar.ra
**7.** *ù* 20 *ša mu-tar-ri-t*[*um* k]i.ta ugu *mu-tar-ri-tim* an.ta *i-te-ru* ğ[ar.r]a
**8.** igi 20 *ša mu-tar-ri-tum* ki.ta ugu *mu-tar-ri-tim* an.ta *i-te-ru*
**9.** *pu-ṭur-ma* 3 *a-na* 7 *ša* a.šà an.ta ugu a.šà ki.ta *i-te-ru*
**10.** íl 21 *re-eš-ka li-ki-il*
**11.** 21 *a-na* 30 sağ *ṣi-ib-ma* 51
**12.** *it-ti* 51 *šu-ta-ki-il-ma* 43,21
**13.** 21 *ša re-eš-ka ú-ka-lu it-ti* 21
**14.** *šu-ta-ki-il-ma* 7,21 *a-na* 43,21 *ṣi-ib-ma* 50,42
**15.** 50,42 *a-na ši-na ḫe-pé-ma* 25,21
**16.** íb.si$_8$ 25,21 *mi-nu-um* 39
**17.** *i-na* 39 21 *ta-ki-il-tam ú-sú-uḫ-ma* 18
**18.** 18 *ša te-zi-bu pi-ir-kum*
**19.** *ma šum-ma* 18 *pi-ir-kum*
**20.** *mu-tar-ri-da-tum ù* a.šà *ši-i*[*t-ta ta-wi-ra-tim mi-nu-um*]

**21.** *at-ta* 21 *ša a-na r*[*a-ma-ni-šu tu-uš-ta-ki-lu i-na* 51]
**22.** *ú-sú-uḫ-ma* 30 *te-z*[*i-ib* 30 *ša te-zi-bu*]
**23.** *a-na ši-na ḫe-pé-ma* 1[5 *a-na* 30 *ša te-zi-bu* íl]
**24.** 7,30 *re-eš*[*-ka li-ki-il*]

**Edge**

**1.** 18 *pi-i*[*r-kam it-ti* 18 *šu-ta-ki-il-ma*]
**2.** 5,24 [*i-na* 7,30 *ša re-eš-ka ú-ka-lu*]
**3.** *ú-sú-*[*u*]*ḫ-ma 2,6 te-*[*zi-ib*]

**Rev.**

**1.** *mi-nam a-na* 2,6 *lu-uš*[*-ku-un*]
**2.** *ša* 7 *ša* a.šà [an.ta ugu] a.šà ki.ta *i-*[*te-ru*] *i-na-di-nam*
**3.** 3,20 g̃ar.ra 3,20 *a-na* 2,6 íl 7 *it-ta-di-kum*
**4.** 30 sag̃ ugu 18 *pi-ir-ki mi-nam i-tir* 12 *i-tir*
**5.** 12 *a-na* 3,20 *ša ta-aš-ku-nu i-ši* 40
**6.** 40 *mu-tar-ri-tum* an.ta
**7.** *ma šum-ma* 40 *mu-tar-ri-tum* an.ta
**8.** a.šà an.ta *mi-nu-um at-ta* 30 sag̃
**9.** 18 *pi-ir-kam ku-mur-ma* 48 *a-na ši-na ḫe-pé-ma* 24
**10.** 24 *a-na* 40 *mu-tar-ri-tim* an.ta íl 16
**11.** 16 a.šà an.ta *ma šum-ma* 16 a.šà an.ta
**12.** *mu-tar-ri-tum* ki.ta *mi-nu-um ù* a.šà ki.ta *mi-nu-um*
**13.** *at-ta* 40 *mu-tar-ri-tam* an.ta *a-na* 20 *ša mu-tar-ri-tum* ki.ta ugu *mu-tar-ri-tim* an.ta *i-te-ru*
**14.** *ṣi-ib-ma* 1 *mu-tar-ri-tum* ki.ta
**15.** 1[8] *pi-ir-kam a-na ši-na ḫe-pé-ma* 9
**16.** *a-na* 1 *mu-tar-ri-tim* ki.ta íl 9
**17.** 9 a.šà ki.ta

## *YBC 6504*

**Obv.**

**#1**

**1.** [*ma-l*]*a* uš ugu sag̃ SI íb.s[i$_{8}$ *i-na lìb-ba* a.šà]
**2.** [ba.z]i-*ma* 8,20 uš ugu sag̃ [10 SI]
**3.** ⸢*i-na*⸣ *e-pe-ši-k*[*a*] 10 *tu-uš-t*[*a-kal-ma*]
**4.** 1,⸢40⸣ *a-na* 8,20 bí.d[aḫ*-ma* 10⸣ i[n.g̃a]r

**5.** šu.ri.a 10 *te-ḫe-ep-p*[*e-m*]*a* 5 in.ĝar
**6.** 5 *tu-uš-ta-kal-ma* 25 in.ĝar
**7.** 25 a.šà *a-na* 10 bí.daḫ-*ma* 10,25 in.ĝar
**8.** 10,25.e 25 íb.si$_8$ 5 *a-na* 25 b[í.d]aḫ-*ma*
**9.** 30 uš in.ĝar 5 *i-na* 25 ba.zi-*ma*
**10.** 20 saĝ in.ĝar

**#2**

**11.** *ma-la* uš ugu saĝ SI íb.si$_8$ *i-na lìb-ba* a.šà ba.zi-*ma*
**12.** 8,20 uš *ù* saĝ ĝar.ĝar-*ma* 50 *i-na e-pe-ši*[-*ka*]
**13.** 50 *tu-uš-ta-kal-ma* 41,40 in.ĝar
**14.** ⸤41.40 *a-na*] 8,20 bí.daḫ-*ma* 50 in.ĝ[ar]
**15.** igi ⸢5 ĝá]l *ta-pa-ṭar-m*[*a* 1]2 in[.ĝar]
**16.** 12 *a-na* 50 *ta-na-aš-ši*[-*ma* 1]0 in.[ĝar]
**17.** [šu.ri⸣.a 50 *te-ḫe-ep-pe-ma* [2]5 in.[ĝar]
**18.** 25 *tu-uš-ta-kal*[-*ma* 10,2⸣5 in.ĝar
**19.** 10 *i-na* 10,2[5 ba.zi-*m*]*a* 25 in.[ĝar]
**20.** 25.e 5 í[b.si$_8$] 5 *a-na* 25 bí[.daḫ-*ma*]
**21.** 30 uš in.ĝar
**22.** 5 *i-na* 25 ba.zi-*ma*
**23.** 20 saĝ in.ĝar

**Rev.**

**#3**

**1.** ⸢*ma-*]*la* uš ugu ⟨saĝ⟩ SI du$_7$.du$_7$ *i-na lìb-ba* a.šà ba.zi
**2.** 8,20 30 uš saĝ.bi en.nam
**3.** 30 du$_7$.du$_7$-*ma* 15 in.ĝar
**4.** 8,20 *i-na lìb-ba* 15 ba.zi-*ma* 6,40 in.ĝar
**5.** šu.ri.a 30 *te-ḫe-ep-pe-ma* 15 in.ĝar
**6.** 15 du$_7$.du$_7$-*ma* 3,45 in.ĝar
**7.** 3,45 *a-na* 6,40 bí-daḫ-*ma* 10,25 in[.ĝar]
**8.** 10,25.e 25 íb.si$_8$ 15 *i-na* 25 ba.zi-[*ma*]
**9.** 10 in.ĝar 10 *i-na* 30 ba.zi-*ma*
**10.** 2[0 sa]ĝ in.ĝar

**#4**

**11.** *ma-la* uš u[g]ù saĝ SI du$_7$.du$_7$ *i-na* a.šà ba.z[i$^{¿}$-*ma*$^{?}$]
**12.** 8,20 20 saĝ uš.bi en.nam
**13.** 20 du$_7$.du$_7$-*ma* 6,40 in.ĝar
**14.** 6,⸤40 *a*⸥-*na* 8,20 bí.daḫ-*ma* 15 in.ĝar
**15.** 15.e 30 íb.si$_8$ 30 uš in.ĝar

## *YBC 6967*

**Obv.**

**1.** [igi.b]i *e-li* igi 7 *i-ter*
**2.** [igi] *ù* igi.bi *mi-nu-um*
**3.** *a*[*t-t*]*a* 7 *ša* igi.bi
**4.** ugu igi *i-te-ru*
**5.** *a-na ši-na ḫe-pé-ma* 3,30
**6.** 3,30 *it-ti* 3,30
**7.** *šu-ta-ki-il-ma* 12,15
**8.** *a-na* 12,15 *ša i-li-kum*
**9.** [1 a.ša$^{!}$]$^{a\text{-}am}$ *ṣí-ib-ma* 1,12,15
**10.** [íb.si$_{8}$ 1],12,15 *mi-nu-um* 8,30
**11.** [8,30 *ù*] 8,30 *me-ḫe-er-šu i-di-ma*

**Rev.**

**1.** 3,30 *ta-ki-il-tam*
**2.** *i-na iš-te-en ú-su-uḫ*
**3.** *a-na iš-te-en ṣí-ib*
**4.** *iš-te-en 12 ša-nu-um* 5
**5.** 12 igi.bi 5 *i-gu-um*

# Bibliographic note

The largest batch of Old Babylonian mathematical texts has been published (with German translation) in

> Otto Neugebauer, *Mathematische Keilschrift-texte*. I-III. Berlin: Julius Springer, 1935, 1935, 1937. Reprint Berlin etc.: Springer, 1973,

and most of them also (with French translation) in

> François Thureau-Dangin, *Textes mathématiques babyloniens*. Leiden: Brill, 1938.

The above texts BM 13901, AO 8862, VAT 7532, YBC 6504, VAT 8512, VAT 8520, BM 85200+VAT 6599, BM 15285, VAT 8389, VAT 8390 and Str 368 are all contained in one as well as the other[66]. Neugebauer's edition contains a very substantial commentary, that of Thureau-Dangin (meant to be economically accessible) only a general introduction.

Other texts are found in

> Otto Neugebauer & Abraham Sachs, *Mathematical Cuneiform Texts*. New Haven, Connecticut: American Oriental Society, 1945.

The text YBC 6967 comes from this work.

All texts from Susa (TMS) come from

> Evert M. Bruins & Marguerite Rutten, *Textes mathématiques de Suse*. Paris: Paul Geuthner, 1961.

The text $Db_2$-146 comes from a journal publication,

> Taha Baqir, "Tell Dhiba'i: New Mathematical Texts". *Sumer* **18** (1962), 11 – 14, pl. 1 – 3.

Neugebauer's and Thureau-Dangin's editions are solid and dependable, as are their commentaries. However, when using Neugebauer's

[66] However, neither of the two volumes contains more than the principal fragment of BM 15285. A new edition based on the three fragments that are known today can be found in Eleanor Robson, *Mesopotamian Mathematics 2100 – 1600 BC. Technical Constants in Bureaucracy and Education*. Oxford: Clarendon Press, 1999.

*Mathematische Keilschrift-Texte* one should remember to consult the corrections that are given in volumes II and III – a pioneer work cannot avoid to formulate hypotheses and to propose interpretations that afterwards have to be corrected. Evidently the commentaries are based on the arithmetical interpretation of the algebraic texts, the originators of this interpretation being precisely Neugebauer and Thureau-Dangin.

The edition of the Susa texts is much less reliable. Too often, and in the worst sense of that word, the French translation and the mathematical commentary are fruits of the imagination. Even the translations of logograms into syllabic Akkadian are sometimes misleading – for instance, the logogram for "joining" is rendered by the Akkadian word for "heaping". Everything needs to be controlled directly on the "hand copy" of the cuneiform text. (In other words: the edition is almost useless for non-specialists, even for historians of mathematics who do not understand the Old Babylonian tradition too well; several general histories of mathematics or algebra contain horrendous mistakes going back to Evert Bruins's commentary.)

The basis for most of what is new in the present book compared to the original editions – the geometric interpretation, the relation between the school and the practitioners' tradition, the historical development – is set out in

Jens Høyrup, *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and Its Kin*. New York: Springer, 2002.

This volume also contains editions of almost all the texts presented above with an interlinear English translation and with philological commentary and precise indication of all restitutions of damaged signs (the exceptions are TMS XVI #2, Str 368 and VAT 8520 #1). At least until further notice, large extract s can be found on Google Books.

# Index

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