What is a number – what is a concept – who has a number concept?

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Lieber Peter,

Ich wurde hier eingeladen, um ein Thema zu präsentieren, das wir entweder schon diskutiert haben, oder das ich mit dir hätte diskutieren können.

Es fällt mir etwas schwer. Vieles hätte ich ja mit dir sehr gern diskutiert, aber wärst du dann auch daran interessiert gewesen? Gewiß, ich erinnere mich an etwas, daß du vor vielleicht zwanzig Jahren gesagt hast: "Wenn ich sage, daß etwas mich nicht interessiert, glaubt mir keiner. Aber ich kann sagen, daß ich keine Zeit dafür habe". Und das war ja der Punkt, nicht nur in den letzten sieben Jahren, wo du dich vermutlich noch mehr als früher auf die dringlichen Arbeiten konzentriert hast, sondern mindestens schon seit der Gründung des Instituts.

Also muß ich auf etwas zurückgreifen, daß wir tatsächlich diskutiert haben, und wo wir also nicht einig waren – sonst hätte es ja keine Diskussion gegeben sondern nur (wie sehr oft!) ein freundliches Gespräch. Auch dies Zurückgreifen ist tatsächlich nicht leicht. Meine eigenen Argumente habe ich hoffentlich mehr oder wenig verstanden, aber du würdest vermutlich behaupten, daß ich die deinen, und besonders deine Gegenargumente, nicht kapiert habe – und zum Teil ist das zweifellos wahr, während ich damals auch behauptete, du hättest die Meinigen schlecht verstanden. Leibniz' Traum, daß die Philosophen sich hinsetzen und alles algebraisch-genau durchrechnen sollten – der ist bekanntlich ein Traum geblieben. Aber trotzdem, schwierig oder nicht ... Wie Beethoven in einer Quartett-Partitur schrieb, "Muß es sein? Muß es sein? Es muß sein!".

Wir haben immer auf Deutsch diskutiert, glaube ich. Um zu betonen, daß es heute zwischen uns eine wahre Diskussion nicht mehr geben kann sondern nur eine asymmetrische Fiktion, werde ich jetzt die Sprache wechseln.

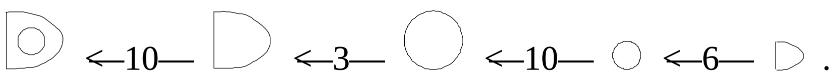
Peter and I both came to Mesopotamian mathematics from debates about the didactics of mathematics; from Piaget – and of course from Marxism. There were differences, however. Peter's Marxism was very Hegelian, while mine was closer to Engels. The evening some four decades ago I was in the S-Bahn and read Engels's explanation in the middle of volume 2 of *Das Kapital* [Marx 1885: 268*f*] that he had been forced to edit arguments from numerical examples strongly,

> So sattelfest Marx als Algebraiker war, so ungeläufig blieb ihm das Rechnen mit Zahlen, namentlich das kaufmännische, trotzdem ein dickes Konvolut Hefte existirt, worin er sämmtliche kaufmännische Rechnungsarten selbst in vielen Exempeln durchgerechnet hat

– then I laughed. I have a suspicion that Peter would not have shared my appreciation and would have looked for something deeper in the numerical examples.

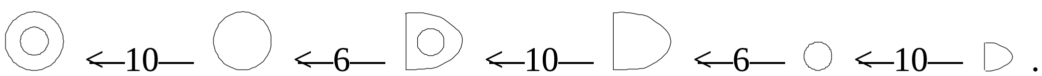
The same difference revealed itself in our approach to Piaget. None of us fell for his infatuation with group "theory" (at least I never heard Peter referring to it, and I certainly did not). But our thinking about *concepts*, though inspired for both of us by Piaget, diverged. The concept of "concepts" is near at hand in the reading of Piaget. His title *La causalité physique chez l'enfant* became *The Child's Conception of Physical Causality* in translation (other titles were changed correspondingly), and one volume in the "Jean Piaget Symposium Series" carries the title *Conceptual Development: Piaget's Legacy* [Scholnick et al 1999]. Here, Peter maintained in one of our discussions (as I remember it) that the fourth-millennium inventors and users of protoliterate writing in Uruk had no number concept because, firstly, there is no evidence that they mastered an arithmetical structure encompassing addition as well as multiplication (what amounts to practical multiplication may well have been seen as repeated addition);[1] secondly, because of the way their metro-numerical notations were structured, of which I shall give the necessary partial information here (everything of course builds on the results obtained by Peter and Robert Englund [1985], with Jöran Friberg in the background).

First there is the "Še-system", used for measuring quantities of grain (I leave out the "sub-unit part"):



A couple of variant systems in which small markings are added to the signs were probably used for particular kinds of grain (or use of grain in particular processes?).

Then there is "System S" ("S" for "sexagesimal"), the main number system:

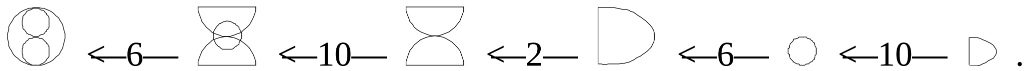


While the Še-system can be used to indicate quantity as well as quality (even though the sign ŠE may be added as a determinative in order to avoid confusion

[1] Here of course group theory creeps in, but not in Piaget's metaphorical ways.

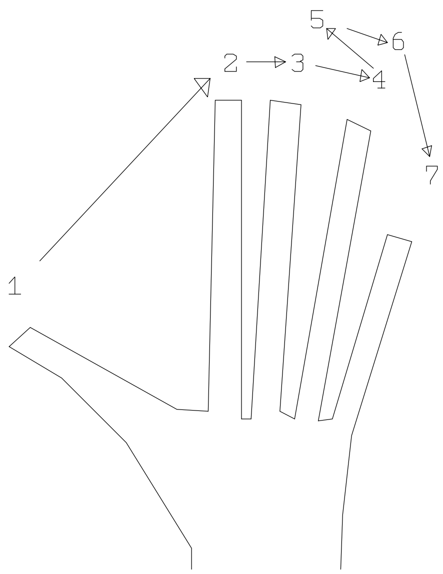
with the same signs used in System S), system S basically designates quantity only, quality being determined separately ("2 sheep"). In this sense, System S is a system for abstract numbers.

A number sequence with more restricted use is "System B", the bisexagesimal system:



Until the level 60, we see, it coincides with the sexagesimal system. It was apparently used for particular purposes – the counting of grain rations, perhaps also of milk products – and possibly, according to a single text, fresh fish. A system B* derived by markings is often used without indication of what is counted – "vermutlich weil das System B* einen so spezifischen Anwendungsbereich besaß, daß eine nähere qualitative Kennzeichnung des erfaßten Gegenstandes entfallen konnte" [Damerow & Englund 1985: 18].

In spite of this explanation, Peter tended to see the existence of systems like Še and of a particular counting system like B as evidence that the protoliterate administrators possessed nothing he would accept as a "number concept".



Being also inspired by Piaget, and having made many experiments and observations of my own dunring the 1970s on the topic, I agreed (and agree) with Peter that speaking of a "number concept" presupposes a certain amount of structure. The intuitive ability to distinguish three items from four without counting may perhaps be seen as an "arithmetical ability" – even though I would hesitate before using even this characterization until we have evidence that this ability contributes to the genesis of a genuine number concept. Nor would I speak of a number concept as long as children have learned the number jingle but do not discover a problem when towards the end they "count" in circle, or as long as they have no objections to the "proof" that they have 7 fingers on one hand made by means of a backward step; both change at the time when cardinality and ordinality are merged into a single structure, and when the child knows immediately that there must be more flowers than roses in the garden, without wishing to count them.

But my demands to a "number concept" do not go much further. From my experiences with teaching and explaining mathematics I have reached the conviction that concepts are dynamic structures; they grow in fullness as more and more connections are operationally integrated. That is probably also fairly Hegelian (or Hegel on his feet), and probably Peter would not have disagreed if that was what we had discussed. Possibly, our disagreement was only about where to put the lower limit for the number of integrated operations. In any case, this is the reason that I would not take the presence or possible absence of a multiplicative component (distinct from repeated addition) as a criterion by which the presence of a number concept can be decided – remembering also that even Euclid's definition of multiplication (*Elements* VII, def. 15) refers to repeated addition.

Peter tended to regard the existence of metrological sequences where quantity and quality are merged a proof that no concept at least of abstract number could be present. On that account I tend to follow Engels, according to whom (*Dialektik der Natur* [1962: 496]) "100.000 Dampfmaschinen [prove the principle] nicht mehr als Eine". I also remember my first physics teacher explaining to us (we were 11 years old by then) that "density is measured in pure number"; I have no doubt that this teacher possessed a well-developed number concept himself, but he may have found it too difficult for us to understand a ratio g/cm^3. So, for me "2 sheep" proves that the concept of abstract number was there,[2] even though its use was no more compulsory than the explicitation of the unit for my physics teacher once it was decided that densities were spoken of.

Similarly, I would see the existence of the bisexagesimal system not as proof that the Uruk-IV administrators had no *unified* number concept but as an early parallel to the particular brick metrologies of the late third millennium, and thus as evidence that they were skilfully adapting their mathematics to bureaucratic standard procedures.

A final disagreement of ours about number concepts concerned the implications to draw from something told by Igor M. Diakonoff [1983: 88]:

> The most curious numeral system which I have ever encountered is that of Gilyak, or Nivkhi, a language spoken on the river Amur. Here the forms of the numerals are subdivided into no less than twenty-four classes, thus the numeral '2' is *mex* (for spears, oars), *mik* (for arrows, bullets, berries, teeth, fists), *meqr* (for islands, mountains, houses, pillows), *merax* (for eyes, hands, buckets, footprints), *min* (for boots), *met'*

[2] It had probably been present in spoken language since long: the difference in structure between the Še- and the S-sequence suggests that the latter was formed when writing was introduced so as to agree with a pre-existing sequence of oral numerals.

(for boards, planks), *mir* (for sledges) etc., etc.

Peter tended to see even that as evidence that no unified number concept was present; I, instead, would observe, as Diakonoff does in the next sentence, that "the root is *m(i)-* in all cases" and find nothing more than a highly elaborate parallel to German *ein* *Mann* / *eine* *Frau*. Perhaps we could sum up the whole thing in this way: According to Peter, we should be aware that proto-literate administrators (etc.) did not think according to our modern patterns; in my view, even *we* deviate from these ideologically prescribed patterns much more often than we usually admit. I am not in general a follower of Bruno Latour, but I tend to agree that *we have never been modern*, or at least never as modern as we believe (perhaps interpreting Latour's phrase in a way he himself would not accept).

Peter might well have argued that I have misunderstood everything he said (and I, vice versa). This is quite plausible; this matter of disagreement was never a serious concern of ours, we usually marked out our points of view briefly and then went on to more productive dialogue where we could learn from each other, by sharing information and from mutual critical questioning. That was much more important for both of us – but it is difficult tell it in a way that makes a story. In spite of all efforts since Voltaire, war is much more conspicous in historiography than peace; it even was in Voltaire's own historical writings.

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