[Review of] Menso Folkerts, The Development of Mathematics in Medieval Europe: The Arabs, Euclid,Regiomontanus. (Variorum Collected Studies Series CS811). Aldershot: Ashgate, 2006

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Published in:
Aestimatio: Critical Reviews in the History of Science

Publication date:
2007

Document Version
Early version, also known as pre-print

Citation for published version (APA):
Høyrup, J. (2007). [Review of] Menso Folkerts, *The Development of Mathematics in Medieval Europe: The Arabs, Euclid,Regiomontanus*. (Variorum Collected Studies Series CS811). Aldershot: Ashgate, 2006. *Aestimatio: Critical Reviews in the History of Science*, *4*, 122-141.

***The Development of Mathematics in Medieval Europe: The Arabs, Euclid, Regiomontanus* by Menso Folkerts. (Variorum Collected Studies Series CS811)**

Aldershot: Ashgate, 2006. xii+340 pp. Indexes of names and manuscripts. ISBN 0–86078–957–8. Cloth.

This is Menso Folkerts' second Variorum-volume. The first volume was published in 2003[1]; it contained papers dealing with the properly Latin tradition in European mathematics, that is, the kind of mathematics which developed (mainly on the basis of agrimensor mathematics and the surviving fragments of Boethius's translation of the *Elements*) before the twelfth-century Arabo-Latin and Greco-Latin translations. This second volume deals with aspects of the development which took place after this decisive divide, from c. 1100 to c. 1500.

Few scholars, if any, know more than Folkerts about medieval Latin mathematical manuscripts. It is therefore natural that the perspective on mathematics applied in the papers of the volume is *mathematics as a body of knowledge*, in particular as transmitted in and between manuscripts. To the extent *mathematics as an activity* is an independent perspective, it mostly remains peripheral, being dealt with through references to the existing literature – exceptions are the investigations of what Regiomontanus and Pacioli *do* with their Euclid, in articles (VII) and (XI); or it is undocumented, as when Jordanus de Nemore's *De numeris datis* is stated to have been "probably used as a university textbook for algebra" (VIII, p. 413). There should be no need to argue, however, that familiarity with the ascertained body of knowledge is fundamental for the application of *any* perspective, and whoever is interested in medieval Latin mathematics can learn from the book.

It is somewhat more unfortunate that Folkerts tends to describe the mathematics he refers to through the modern interpretation. To say, for

[1] See the review [Høyrup, forthcoming/a].

instance, that the *Liber augmentis et diminutionis* shows "how linear equations with one unknown or systems of linear equations with two unknowns may be solved with the help of the rule of double false position" (I, p. 5) does not help the reader who is not already familiar with the kind of problems to which this rule was applied to understand that the treatise contains no equations but problems which modern footnotes express in terms of linear equations.[2]

Since many of the articles are surveys, they touch by necessity on topics outside Folkerts' own research interest. In such cases, Folkerts tends to mention existing disagreements or hypotheses instead of arguing for a decision (even in cases where one may suspect him to have an opinion of his own);[3] this is certainly a wise strategy, given the restricted space for each topic; but the reader should be aware that this caution does not imply that existing sources do not allow elaboration or decision.

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[2] For instance, "Somebody traded with a quantity of money, and this quantity was doubled for him. From this he gave away two dragmas, and traded with the rest, and it was doubled for him. From this he gave away four dragmas, after which he traded with the rest, and it was doubled. But from this he gave away six dragmas, and nothing remained for him" – translated from [Libri 1838: I, 326]. Seeing this simply as "an equation" also misses the point that it may just as legitimately be seen (for example) as a system of three equations with three unknowns (the successive amounts traded with).

Actually, the treatise solves this problem (and many others) not only through application of a double false position but also by reverse calculation and by means of its *regula* (that which Fibonacci calls the *regula recta*, first-degree *res*-algebra).

[3] One example: In (I, n. 13) it is said that the author of a reworking of al-Khwārizmī's algebra could be Guglielmo de Lunis. This hypothesis is quite widespread. It is not mentioned that the only two independent sources which inform us about a translation of the work (whether Latin or Italian) made by Guglielmo (Benedetto da Firenze and Raffaello Canacci, Lionardo Ghaligai depending on Benedetto), both quote it in a way which appears to exclude the identification. I guess Folkerts knows both sources.

With one exception, all articles in the volume turn around the tradition and impact of the *Elements*, and/or the figure of Regiomontanus. Unlike many Variorum volumes, several articles are not published in their original form but rewritten so as to encompass recent results. In total, 12 articles are included:

(I) “Arabic Mathematics in the West”, a revised translation of a paper first published in German in 1993 (16 pp.), deals with the arithmetic of Hindu numerals, algebra, Euclidean geometry (*Elements, Data, Division of Figures*), spherics and other geometrical topics (Archimedean works on the circle and the sphere, conics, practical mensuration). Given the short space, this can obviously be little more than a (very useful) bibliographic survey.

(II) “Early Texts on Hindu-Arabic Calculation” (26 pp.), first published in 2001, falls in two parts. The first part (6 pp.) is a general survey, covering the Indian introduction of the decimal place value system and its diffusion into the Arabic world; some of the major Arabic texts describing the system; the early Latin redactions of *Dixit algorizmi*; and the most important Latin algorism texts from the thirteenth and fourteenth centuries. The second part (17 pp.) is a detailed description of *Dixit algorizmi*, the earliest Latin reworking of the translation of al-Khwārizmī’s treatise of the topic. Of this reworking, two manuscripts exist, the second one discovered by Folkerts, who also published a critical edition [Folkerts 1997].

(III) “Euclid in Medieval Europe” (64 pp.) is a completely revised version of a paper first published in 1989. The first half of the article describes all known European-medieval translations and redactions, from Boethius until the mid-sixteenth century; it also includes a brief discussion of the Arabic versions. The second half is a “list of all known Latin and vernacular manuscripts up to the beginning of the sixteenth century that contain the text of Euclid’s *Elements* or reworkings, commentaries and related material”.

(IV) “Probleme der Euklidinterpretation und ihre Bedeutung für die Entwicklung der Mathematik” (32 pp.) was originally published in 1980. An initial chapter covers the same ground as the first part of (III), but with more emphasis on the character of the various versions of the *Elements*. Chapters 2 and 3 look at how late Ancient as well as Arabic and Latin commentators and mathematicians concentrated on specific aspects of the work: proportion theory, the parallel postulate, the theory of irrationals.

(V) “Die mathematischen Studien Regiomontans in seiner Wiener Zeit” (36 pp.) was originally published in 1980. It deals with a phase in Regiomontanus’s mathematical development of which little had been known. In Folkerts’ words, it shows that “laborious work on details may still allow one to find many mosaic cubes which, admittedly, do not change the picture of Regiomontanus the mathematician completely, but still allows to make it much more distinct”. At first, Regiomontanus’s “Wiener Rechenbuch” is analyzed, a manuscript from Regiomontanus’s hand written between 1454 and c. 1462 (Codex Wien 5203), containing original work as well as borrowed texts (at times, however, apparently rewritten in Regiomontanus’s own words); next Folkerts traces which treatises on *Visierkunst* (the practical mensuration of wine casks) Regiomontanus must have possessed or known, using the posthumous catalogues of Regiomontanus’s library and those parts of the codex Plimpton 188 which once belonged to Regiomontanus. Finally, Folkerts digs out from the same Plimpton codex evidence that the algebraic knowledge which Regiomontanus shows to possess in the correspondence with Bianchini (etc.) was already his in 1456 (including matters which are now known to have been current in Italian fourteenth-century *abbaco* algebra but not found in the *Liber abbaci* nor in al-Khwārizmī / JH). Even the symbolism of which Regiomontanus makes use after 1462 turns up in the Plimpton codex, both in passages that stem from Regiomontanus’s hand and in others for which he is probably not responsible.

(VI) “Regiomontanus’ Role in the Transmission and Transformation of Greek Mathematics” (26pp.) was originally published in 1996. After some biographical information it presents Regiomontanus’s “programme”, the leaflet that lists the works Regiomontanus intended to print on his own press (plans which were never realized because of his sudden death). Beyond some of Regiomontanus’s own writings, it includes in particular the *Elements*, Archimedes’s works, Menelaos’s and Theodosios’s spherics, Apollonios’s *Conics*, Jordanus de Nemore’s *Elements of Arithmetic* and *On Given Numbers*, Jean de Murs’ *Quadripartitum numerorum*, and the *Algorismus* *demonstratus*. The *Programme* is supplemented by Regiomontanus's Padua lecture from 1464, which refers to many of the same works, but also to Diophantos. Next Folkerts uses manuscripts which have been in Regiomontanus's possession, his annotations etc., to detect how much Regiomontanus actually knew about the authors and works he mentions – which was indeed much. Only in the case of the *Conics* it is not certain that he was familiar with more than the beginning of the work as translated by Gherardo da Cremona.

The final pages of this article present various number-theoretical, geometric and determinate and indeterminate algebraic problems not coming from Greek sources but present in the *Wiener Rechenbuch*; in a problem collection in the Plimpton manuscript, in Regiomontanus's hand and apparently from 1456; in the manuscript *De triangulis*; and in the letters exchanged with Giovanni Bianchini, Jacob von Speyer and Christian Roder. Some of the geometric problem solutions make use of algebraic techniques.

The discussion of approximations to the square root of a number $n = a^2+r$ on p. 109 invites a commentary. The *Rechenbuch* as well as the Plimpton collectiom offer the usual first approximation $\sqrt{n} \approx n_1 = a+\frac{r}{2a}$. The Plimpton collection then gives a second, supposedly better approximation $n_2 = a+\frac{4a^2+2r-1}{(4a^2+2r)\cdot 2a}$, about which Folkerts says that it is not clear where it comes from. Actually, the formula is wrong – it reduces to $a+\frac{4a^2-1}{8a^3}$ when $r = 0$, not to a. However, iteration of the procedure which yields n_1 gives $\tilde{n}_2 = a+\frac{(4a^2+2r)\cdot r-r^2}{(4a^2+2r)\cdot 2a}$, which coincides with the Plimpton second approximation for $r = 1$. In the present context one might believe Regiomontanus to deal only with an example where $r = 1$, and the general formula as such to be a reconstruction due to Folkerts. However, in (VIII, p. 422), Regiomontanus is quoted for the observation that the second approximation cannot be applied to all numbers, which is obviously not true for the approximation $\tilde{n}_2$. Regiomontanus must therefore be presumed to be at least co-responsible for the mistake.

Folkerts quotes the *Rechenbuch* for a different second approximation, *viz* $n_2 = \frac{n}{n_1} : 2$. This is obviously a misprint for $n_2 = (n_1+\frac{n}{n_1}) : 2$. By the way, a bit of calculation shows this n_2 and what was called $\tilde{n}_2$ above to be algebraically equivalent.

(VII) “Regiomontanus’ Approach to Euclid” (16 pp.) is a completely revised translation of an article first published in German in 1974. Its first half elaborates in greater depth the Euclidean aspect of the previous article and the presentation of the posthumous catalogues of Regiomontanus’s *Nachlaß* from article (V). The second half analyses Regiomontanus’s endeavour “to establish a correct text of Euclid” mainly based on mathematical critique of the Campanus version but drawing also on “Version II” (formerly known as “Adelard II”). As summed up by Folkerts (p. 10), Regiomontanus’s aim was “to establish a mathematically correct text (not to be understood in modern text-critical sense of a reconstruction of the original text)”, as was indeed “typical for Regiomontanus”.

(VIII) “Regiomontanus’ Role in the Transmission of Mathematical Problems” (18 pp.) was first published in 2002. It broadens the range of problem types with respect to those discussed in the end of article (VII), and says more about the way the problems are solved; the sources are the Plimpton problem collection, the correspondences, and the *Wiener Rechenbuch*. In particular, a number of problems going back to the Italian *abbaco* tradition are presented.

Several of these problems turn up again in the following decades in mathematical writings from southern Germany, first in a manuscript copied by Fridericus Amann in 1461 – at times with the same numerical parameters. Folkerts concludes that “Fridericus Amann must have learned something of the contents of MS Plimpton 188 soon after it was finished” (p. 414), and that “Regiomontanus played a crucial role in transmitting mathematical knowledge from Italy to Central Europe in the 15th century”. Given that even the problems in the Plimpton manuscripts are copied from an earlier source, this seems to the present reviewer to be at least a daring conclusion – see also note 17 and preceding text.

Some observations should be made. Firstly, on p. 418 it is stated that nos. 16–32 of the Plimpton collection ask for a number and serve as examples for al-Khwārizmī’s six problem types. This seems to be a typographical mistake (for 16–21?).[4] Secondly, the erroneous second-order approximation to a square root from the Plimpton collection is repeated on p. 422, whereas the one from the *Rechenbuch* is correct this time. Finally, on p. 419, something is wrong in the presentation of a "special arithmetical problem" – probably already in the original.[5]

(IX) "Leonardo Fibonacci's Knowledge of Euclid's *Elements* and of Other Mathematical Texts" (25 pp.) was still to appear when the volume was prepared (it was eventually published in Fall 2005). Going through the *Liber abbaci*, the *Pratica geometrie*, the *Flos*, the letter to Master Theodorus and the *Liber quadratorum* Folkerts traces the mathematical works that are used with "due reference" as well as those which are used without recognition of the borrowing. Euclid is quoted very often, Archimedes, Ptolemy, Menelaos, Theodosios and the *agrimensores* occasionally, but Arabic authors not (with the sole exception of *Ametus filius*, i.e., Aḥmad Ibn Yūsuf).[6]

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[4] According to Folkerts, no. 22 deals with compound interest (but illustrates al-Khwārizmī's fourth type), and nos. 27 and 30 are, respectively, of the types "purchase of a horse" and "give and take".

[5] The problem, from the Plimpton collection, states that "somebody wants to go as many miles as he has dinars. After every mile the dinars he possesses are doubled, but he loses 4 dinars. At the end he has 10 dinars". Folkerts solves this without making use of the magnitude of the remainder (the algebra involved cannot correspond to anything Regiomontanus would do), finding that the man starts with 4 dinars – but in that case he will be left with 4 dinars after each doubling and subtraction, never with 10. Regiomontanus has a marginal note that the problem has to be solved "in a reversed order", which Folkerts suggests might mean by "trial and error". This is not likely, stepwise backward calculation was a standard method for such "nested-box" problems. Going backwards from 10 dinars, we get the successive remainders 7, $5^1/_2$, $4^3/_4$, $4^3/_8$, ... The data of the problem are thus inconsistent (if rendered correctly), which Regiomontanus does not seem to have noticed.

[6] Reviewer's observation: Since Fibonacci tells regularly that methods are of Arabic origin, this *could* mean that he made his apparent borrowings from Abū Kāmil, al-Karajī etc. indirectly. However, his obvious verbatim copying

The last part of the article raises the question “Which version of Euclid did Leonardo use?”. Often Fibonacci seems to quote from memory – the same proposition may be formulated in different words in the *Liber abbaci* and the *Pratica*, none of formulations agreeing with any known Latin or Arabic version. Elsewhere, it is clear that Fibonacci uses the Latin translation from the Greek.

(X) “Piero della Francesca and Euclid” (22 pp.) was first published in 1996. The article starts by sketching the story of the Arabo-Latin *Elements* with emphasis on Campanus and by a brief general description of Piero’s mathematical works based on [Davis 1977]. Turning then to the use of Euclid, Folkerts shows that even Piero is fond of citing Euclid (mostly the *Elements*, but in *De prospective pingendi* also the *Optics*). There is no doubt that Piero used the Campanus version – he cites Campanus twice, and uses some of his additional propositions. However, Piero’s words and terminology often differ from those of Campanus, in a way which reflects Piero’s background in the *abbaco* tradition – both in the *Libellus de quinque corporibus regularibus*, which was originally written in Italian but is only extant in Latin translation, and in the *Trattato d’abaco*. Folkerts supposes this to reflect lack of familiarity “with the style used in scientific mathematical works” (p. 302) and not the use of a non-Campanus version. He points out that Piero’s numbering of certain propositions from book XV show that the manuscript he used is not among those known today.

The end of the article examines other citations in Piero’s mathematical writings (Vitruvius, Ptolemy, Archimedes, Theodosios) and the possible sources for his treatment of semi-regular solids – for which Jean de Murs’ *De arte mensurandi* might be one, but not the only source.

(XI) “Luca Pacioli and Euclid” (13 pp.) was originally published in 1998. Within the framework of a short biography concentrating on Pacioli’s interaction with Euclid, it discusses the traces of his translation of Euclid

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from Gherardo da Cremona’s translations of al-Khwārizmī [Miura 1981] and Abū Bakr [Høyrup 1996: 55] weakens the argument – at times Fibonacci clearly did not want to reveal his sources.

into the vernacular, the excerpts from the *Elements* in the *Summa de Arithmetica Geometria Proportioni et Proportionalita* from 1494 (drawn from the Campanus tradition), and his Latin edition of a purportedly restored Campanus text in 1509.

The vernacular translation turns out to have probably been made before first part of the *Divina proportione*, i.e., before 1497. The arithmetical part of the *Summa* contains excerpts from *Elements* V,[7] the geometrical part excerpts from books I–III, VI and XI. The material is transformed in a way which should be suited for a public with practical but only modest theoretical interests: the Euclidean material is brought in the beginning of sections – thus serving as "theoretical" underpinning for what follows – but there is no clear separation between definitions and enunciations, and proofs are mostly replaced by explanations with reference to diagrams.

The definitions from book I, as well as all excerpts from book XI, are rendered rather freely. The rest of the excerpts from book I, and those from books II–III and VI, are very close to the Campanus text. They cannot have been taken over from Pacioli's vernacular translation, since they agree rather precisely with passages in the manuscript BN Florence, Palatino 577, probably from c. 1460.[8]

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[7] These excerpts, dealt with previously by Margherita Bartolozzi and Raffaella Franci [1990], are not discussed further by Folkerts.

[8] This appears from the presentation to have been established/controlled by Folkerts himself. For the statement that the "geometrical section of Pacioli's *Summa* agrees in the other parts, too, with that Florence manuscript" (p. 226)), Folkerts refers to [Picutti 1989].

Because of the widespread unconditional acceptance of the thesis of this paper, meant to convince readers that Pacioli, *in claimed contrast to other abbacus writers*, was a vile plagiarist, the reviewer would like to make some observations. It is written in a strong and explicit anti-clerical key, which may be quite understandable in an Italian context, but is in itself no argument for its reliability – nor of course for the opposite (cf. Libri's wonderfully and similarly engaged *Histoire des sciences mathématiques en Italie*, still valuable after more than 150 years). However, without further control one

Folkerts' comparison of Pacioli's edition of the Campanus text with the *editio princeps* from 1482 shows that the proper corrections are minor, and that the main difference consist in the addition of comments (introduced, it is true, by the word *castigator*, which suggests that they are meant to be understood as corrections). In total, Folkerts counts 136 additions, 42 of which are more than 10 lines long; mostly, "Pacioli confines himself to explaining terms or individual steps within a proof or construction" (p. 228), at times he "makes remarks that are not immediately necessary for the

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should probably not trust the accuracy of an author who claims [Picutti 1989: 76] that Pacioli divides his text into chapters instead "distinctions": actually, the chapters are subdivisions of the distinctions, the distinctions are indicated in the titles, and the actual distinction as well as the chapter are indicated in the running head of all pages, in the 1494 edition of the *Summa* as well as the second edition from 1523; Picutti can have examined none of them seriously. Without endorsing the peer-review hysteria the reviewer also asks himself why Picutti only published in the Italian edition of *Scientific American* and never substantiated his assertions in a professional journal.

On the other hand it is obvious from a reproduced passage that Pacioli sometimes used either Palatino 577 or a precursor manuscript – and since Pacioli has diagrams which are omitted in the Palatino manuscript (as admitted by Picutti), Pacioli either used this manuscript creatively, or he borrowed from a precursor where the diagrams were present (the one shown in the reproduction is not in Fibonacci's *Pratica*, at least not in the Boncompagni edition [1862]). Elsewhere in the *Summa*, however, misprints in the lettering of the diagrams can be corrected by means of the Boncompagni edition of the *Pratica*. Pacioli evidently felt free to copy without acknowledging his sources explicitly, while stating in the initial unfoliated *Sommario* that most of his volume has been taken from Euclid, Boethius, Fibonacci, Jordanus, Blasius of Parma, Sacrobosco and Prosdocimo de' Beldomandi. Fibonacci, Piero, and many other writers in the *abbaco* traditions borrowed as freely and gave neither specific nor general reference when the name of the source carried no prestige. Only renewed scrutiny of the Palatino manuscript will reveal whether Pacioli also copied directly from Fibonacci's *Pratica* or *only* indirectly.

understanding of the theorem, but are suggested by it" (p. 229); it can be assumed "that the edition of Euclid contained elements from Pacioli's mathematical lectures" (p. 230).

(XII) "Algebra in Germany in the Fifteenth Century" (18 pp.) was never published before. Its theme was already touched at in articles (V), (VI) and (VIII), but here the perspective is broadened. Some of the essential sources for the arguments have been published, but much material remains in unpublished manuscripts, and a survey like the present one is certainly needed – also in order to create a background for further research.

The article starts by presenting the background in Italian *abbaco* algebra. This presentation, as explained, is built on [Franci & Toti Rigatelli 1985], which must now be considered partially outdated.[9] The claim (p. 3) that Piero della Francesca "contributed not only to perspective but also to algebra", and that therefore and for other reasons Luca Pacioli "has enjoyed unmerited fame, for his algebra contains nothing new of any value" is unwarranted. After all, Piero – truly impressing as he is as a geometer – repeated without distinction traditional nonsense along with valuable material in his algebra, thus obviously copying without making always control or calculations; Pacioli reflected on the algebraic material he borrowed, exactly as he reflected on his Euclidean borrowings.[10]

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[9] Its aim – thus [Franci 2002: 82 n.2] – was "to shed light on the algebraic achievements of the Italian algebraists of the Middle Ages, rather than to investigate their sources and internal links"; it even precedes a paper [Franci & Toti Rigatelli 1988] which Franci (*ibid.*) characterizes as a "first summary".

[10] Firstly, Piero repeated those false rules for higher-degree equations which had circulated at least since Paolo Gherardi (1328) – for instance [ed. Arrighi 1970: 139] solving the problem "cubes equal to things and number" (in modern symbols, "$\alpha x^3 = \beta x+n$") as if it had been "*censi* equal to things and number" ("$\alpha x^2 = \beta x+n$"). Rules which hold in specific cases only (as pointed out by Dardi da Pisa in 1344) are stated by Piero as universally valid – thus for instance *ibid*, p. 146. Secondly (cf. [Høyrup, forthcoming/b] and [Giusti 1993: 205]), Piero copies a long sequence of rules for quotients between algebraic powers, in which "roots" take the place of negative powers, the

The treatment of Germany begins with a presentation of Regiomontanus's contributions, with particular emphasis on his symbolism – the thesis being that Regiomontanus "was central for the transmission of Italian ideas about algebra to Central Europe" (p. 3).

According to Folkerts, Regiomontanus uses the following symbols or abbreviations:[11] a superscript *r* or *R* provided with a curl indication abbreviation for *res* or *radix*, following after the coefficient; a superscript *c*, also provided with a curl and following the coefficient, for *census*; he connects the two sides of the equation with a long horizontal stroke (which may thus be read as an equality sign in the function of equation sign); a sign for minus has been interpreted as *ī* (that is, *in*) followed by the curl meaning

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first negative power being identified with "number" (the rules appear to go back to a treatise written by Giovanni di Davizzo in 1339). Cf. also Enrico Giusti's characterization of the algebraic Piero [1991: 64, translation JH] as "a copyist who does not even notice, witness the very high number of repetitions of cases that were already treated (13 out of a total of 61) that what he was writing had already been copied one or two pages before" and as "an author [...] who did little more than to collect whatever cases he might find in the various authors at his disposition, without submitting them to accurate examination".

Pacioli points out explicitly [1494: I, 150^{r}] that no generally valid rule had so far been found for cases where the three algebraic powers are not separated by "equal intervals" (he was not the first to point it out, a similar observation is made in the *Latin algebra* [ed. Wappler 1887: 11] – see below, note 16 and appurtenant text). Pacioli also stays aloof of the confusion between negative powers and roots. He does include [1494: I, 67^{v}, $143^{\text{r–v}}$] a terminology where "*nth* root" stands for the *n*th (positive) power of the *cosa*, but since this system identifies the "first root" with the *cosa*, it is likely to be an outgrowth of the al-Khwārizmīan use of *root* (namely the square root of the *māl*/*census*) for the first power – an outgrowth of which Pacioli is *not* the inventor, since he describes the system for completeness' sake.

[11] These are only described in words by Folkerts, but see the depictions in [Curtze 1902: 232*f*, 278–80], [Cajori 1928: 95*f*] and [Tropfke/Vogel et al 1980: 281].

us, ιq̄.[12] However, the shapes shown in a photo in [Cajori 1928: 96] from the calculations made for a letter to Bianchini – *viz* m͡ŋ, at times becoming r͡ŋ – look more like pen variants of the traditional Italian shape m̂,[13] while a page from the Plimpton manuscript[14] uses the shape  twice but the shape  (meaning *mī(us)*) four times. The same page shows the abbreviation for *res* once superscript but more often on the line (and even more often the full word *cosa*). All in all, Regiomontanus symbols (mostly used as mere abbreviations) are much less fixed than Folkerts' description would let us believe.

In his Vienna period, as pointed out, Regiomontanus copied al-Khwārizmī's algebra (in Gherardo's translation) and Jean de Murs's *Quadripartitum numerorum* and annotated both carefully. As concerns the algebraic problems contained in the Plimpton collection, *De triangulis* and the correspondences, Folkerts grosso modo restricts himself to a cross-reference to articles (V), (VI) and (VIII).

Afterwards, a number of other fifteenth-century German writings are presented or mentioned briefly: The (mostly non-algebraic) problems added to the *Algorismus ratisbonensis* by Fridericus Amann and the algebra written by Amann in 1461, both Bayerische Staatsbibliothek, Clm 14908;[15] from Dresden, C 80, a "Latin algebra" as well as a "German algebra" from 1481 which "seems to depend on the 'Latin algebra'" (p. 9);[16] marginal notes

[12] Thus not only Folkerts but also the redrawings in [Tropfke/Vogel et al 1980: 206] and [Vogel 1954: Tafel VI].

[13] This shape is found, e.g., in Vatican Library, Chigiana, M.VIII.170, written in Venice in c. 1395. A reduction of the equally classical shape ⓜ is definitely less likely.

[14] Reproduced in high resolution on the web address http://columbia.edu/cgi-bin/dlo?obj=ds.Columbia-NY.NNC-RBML.6662&size=large.

[15] The problems were published in [Vogel 1954], the algebra in [Curtze 1895: 49–73].

[16] The former was published in [Wappler 1887], the latter in [Vogel 1981]. The codex was in the possession of Widman, and the *Latin algebra* was used by him. Since the *German algebra* makes abundant use both of a fraction-like

in the same manuscript made by Johannes Widmann, and the same author's *Behende und hubsche Rechenung auff allen kauffmanschafft* from 1489; the writings of Andreas Alexander (b. c. 1470), pupil of a certain Aquinas, a Dominican friar from whom also Regiomontanus says to have learned; the *Initius algebras* which *may* have been written by Alexander or by Adam Ries; Ries's (non-algebraic) *Rechenbuch* as well as the two editions of his *Coss* (1524, and after 1543); Rudolff's *Coss* (1525) and Stifel's *Arithmetica integra* (1544); and the Cistercian Conrad Landvogt (c. 1450 to after c. 1500), whom Folkerts himself has brought to light.

Folkerts bases his claim regarding Regiomontanus's central role in the transmission on various pieces of evidence. Firstly, the algebraic problems in the Plimpton collection have the heading *Regule de cosa et censo sex sunt capitula, per que omnis computatio solet calculari*; Amann gives the title *Regule dela cose secundum 6 capitola*. The similarity is not striking; moreover, if Amann had copied Regiomontanus, he would have had no reason

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notation for monomials known from Italian writings (see below, text around note 20) and of the phrases "mach mir die rechnung"/"Und moch des gleichen rechnung alzo" corresponding to the Italian "fammi questa ragione"/"così fa le simiglianti", none of which have are found in the *Latin algebra*, the *German algebra* must either draw on several sources of inspiration, or it must share a precursor with the *Latin algebra* rather than depend on it (or both).

That it must depend on several sources was indeed already observed by Vogel [1981: 10]. To Vogel's observations can now be added not only that the fraction-like notation for monomials is of Italian origin but also that the strange term and abbreviation for the fourth power (*wurczell von der worczell*/"root of the root") looks like a cross-breed between Piero's negative powers and Pacioli's alternative notation (see note 10). The idea to provide the fifth case (the one with a double solution) with three examples also corresponds to what can be found in Italy (Jacopo da Firenze as well as Dardi) – the original point being that one case requires the additive solution, one the subtractive solution, and one is satisfied by both.

The use of "root of root" in passages of the *German algebra* that are parallel to passages where the *Latin algebra* has the regular repeated *zensus*-abbreviation ʒʒ suggest that these parallels are due to the sharing of a common source rather than to direct translation.

whatever to restore Italian grammar (*dela cose* instead of *de cosa*). A close common source, on the other hand, is very likely.[17]

Secondly, Regiomontanus is supposed to have invented his own symbolism, and Amann to have borrowed it; since Amann appears to have visited Vienna in 1456, on which occasion Folkerts thinks that "there are good reasons to assume that he met Regiomontanus there and at this meeting [...] learnt of his symbols" (p. 8; Regiomontanus was 20 years old by then, while Amann must have been close to 50). Amann's symbols for *res*/*cosa* and *zensus* are indeed fairly similar to those of Regiomontanus. However, in (V, p. 201*f*) Folkerts indicates that parts of the Plimpton manuscript which appear *not* to be written by Regiomontanus also use symbols, one section exactly the same; there Folkerts points out that this might represent a precursor for Regiomontanus's symbolism. In that place, it is true, the symbols are not superscript, but even this is hardly an innovation due to Regiomontanus (nor is it, as we have observed, a constant habit of his): superscript symbols following the coefficient (the square meaning *censo* sometimes above, but *co* for *cosa* always following) were also used by Pacioli in a manuscript finished in 1478 (Vatican, Vat. Lat. 3129), which also (for example on fol. 67^{v}) uses the horizontal stroke as an equation sign (but ⓜ for minus).[18] Since superscript □ and *co* (and sometimes *cen* for *censo*) written above the coefficient are also used in the Italian manuscript Vat. Lat. 10488 written in 1424, for instance (original foliation) fol. 36^{v}, 38^{v}, 92^{r-v},

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[17] Indeed, the two examples from Regiomontanus's text which are reproduced on the web (see note 14) coincide substantially with those of Amann – much more so, indeed, than they would have done if Amann had reproduced from memory what he had discussed with Regiomontanus (see imminently), but much less than if he had translated from the Plimpton manuscript. One difference is informative: In Regiomontanus's text, there is a reference to the principle that when equals are added to equals, equals result. This Euclidean argument for the traditional "restoration" operation is absent from Amann's text, and thus likely to be Regiomontanus's own contribution – and an early manifestation of his characteristic approach.

[18] For a discussion of the stroke as equation sign in Pacioli's *Summa*, see [Cajori 1928: 110*f*].

Pacioli can not be suspected of having taken his inspiration from Regiomontanus.[19] Actually, this notation is likely to be borrowed from Maghreb algebra – cf. for instance [Tropfke/Vogel et al 1980: 376].

A different, fraction-like notation was used by Dardi of Pisa,[20] and also in the draft manuscript *Trattato di tutta l'arte dell'abacho* from c. 1334: $\frac{12}{c}$ stands for 12 *cose*, $\frac{4}{\text{ç}}$ for 4 *censi*. The same notation is used in the *German algebra* in C 80.[21] All in all, it is possible but not at all certain that some later cossists learned their symbolism (or part of it) from Regiomontanus. It is certain, however, that not all of them did, and equally certain that Regiomontanus did not invent it.

Thirdly, it is said on p. 9 that the "order of the [equation] types, which is elsewhere varied, is the same in the 'German algebra' in MS C 80 and in the Regiomontanus text in MS Plimpton 188. This cannot be a coincidence". Evalution of this statement is difficult because Folkerts gives no exact information about the presentation of the cases in MS Plimpton 188. However, in (VIII, p. 418) it is stated that "the Latin text in the Plimpton manuscript, which describes the six forms of equations, agrees word-for-word with the German translation that Fridericus Amann wrote five years later"; this simply means (see [Curtze 1895: 50]) that the order for these six fundamental cases is the standard order of Italian *abbaco* algebra – which certainly differs from the order of al-Khwārizmī, Abū Kāmil and Fibonacci;

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[19] Vat. Lat. 10488 sometimes uses $\widehat{m}$, sometimes $\widehat{me}$ for minus.

[20] [Høyrup 2007: 170] argues that this symbolism, found in the two earliest manuscripts, was already used in Dardi's original from 1344.

[21] With a set of symbols for the algebraic powers which is neither identical with what can be found in Italian treatises nor with those of Regiomontanus, Amann or the *Latin algebra*; see the facsimiles in [Vogel 1981: Tafel 1-3], and the comparison in [Vogel 1981: 11] (where it should be observed that the symbolic notation ascribed to Robert of Chester and the year 1150 refers to marginal notes in C 80 and to an appendix to Robert's translation found in 15th-century manuscripts from the South-German area).

In the very last problem of the *German algebra* [ed. Vogel 1981: 43], a different (but equally Italian) notation is used: a superscript ^{c} (for *cosa*), above or following the coefficient.

the same order is found in the *Latin algebra* as well as the *German algebra* from C 80. The agreement concerning the fundamental cases thus only indicates common roots in the *abbaco* tradition, and nothing more.

Then there are 18 more cases, either homogeneous or reducible to the second degree. They are found in the *Latin algebra* [Wappler 1887: 12*f*] as well as the *German algebra* [Vogel 1981: 22] from C 80 (the *Latin algebra* has one more, corrupt and not provided with illustrating example, which its compiler *alibi inveni*/"found elsewhere" [Wappler 1887: 12]. These share not only their order (which may perhaps be of Italian origin but which is certainly not common) but also the numerical parameters. *This* can certainly not be a coincidence, even though the cases themselves were all familiar in *abbaco* algebra since the early 14th century. Regiomontanus also has 18 more cases, and most of them coincide with those of the two algebras from C 80, and for these they follow the same order. But if Folkerts' transcription in modern symbols in (V, note 150) is to be relied upon, two cases are different: no. 12 is $ax^4+bx^2 = cx^3+dx^2$, while agreement with the algebras in C 80 would require $ax^4 = cx^3+dx^2$; no. 14 is $ax^2 = \sqrt{b}$, whereas agreement would ask for $ax^2 = \sqrt{(bx^2)}$. The latter deviation could at a pinch be a miswriting due to Folkerts or his typographer, the former not. Once more, the evidence seems to suggest shared inspiration rather than copying from Regiomontanus.

Summing up, Folkerts description of fifteenth-century German algebra is certainly indispensable for any further discussion of the topic, by listing all known important and several though not all minor manuscript sources and by pointing to many of the parameters that have to be taken into account; thus, only the use of Folkerts' text allowed the reviewer to grasp and sift the material well enough to formulate his objections. But Folkerts' conclusion is premature, and sometimes contradicted by precise inspection of the sources. It is therefore likely to be mistaken: Italian *abbaco* algebra appears to have inspired and spurred the German development not through a single but through multiple channels.[22] However, *no* definite conclusions can be drawn

[22] Further evidence for this, beyond the parameters discussed by Folkerts and the observations made in note 16, comes from the German standard spellings *coss*, *zensus* and *unze*. They point to inspiration from northern Italy (cf.

before manuscripts are gauged against the essential parameters both on the Italian and the German side. Unfortunately, few of the printed editions of Italian *abbaco* manuscripts that have been published during the last fifty years have bothered much about symbolism-like abbreviations and non-geometric marginal diagrams. It is to be hoped that Folkerts overview may contribute to changing this state of affairs!

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[Rohlfs 1966: I, 201*f*, 284, 388]), where *cossa*/*chossa*, *zenso* and *onzia* are common, Genoa rather than Venice. Regiomontanus, in MS Plimpton 188, writes *cosa*, the *German algebra* has *cossa*. The *Latin algebra*, as mentioned in note 16, uses an abbreviation for the second power (ʒ) which is derived from the spelling *zensus*.

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