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ROSKILDE UNIVERSITETSCENTER
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Jacopo da Firenze and the Beginning of Italian Vernacular Algebra

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FILOSOFI OG VIDENSKABSTEORI PÅ ROSKILDE UNIVERSITETSCENTER

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To Mohamed Souissi
and
in memoriam Khalil Jaouiche

Revised contribution to the International Workshop
The Origins of Algebra: From al-Khwarizmi to Descartes
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Referee: Aksel Haaning

First discovery – and neglect

In [1929], Louis Karpinski published a short description of “The Italian Arithmetic and Algebra of Master Jacob of Florence, 1307”. Among other things he pointed out that the algebra chapter of the treatise in question presents the algebraic “cases” in a different order than al-Khwārizmī, Abū Kāmil, and Leonardo Fibonacci, and that the examples that follow the rules are also different than those of the same predecessors. Karpinski did not mention explicitly the absence of geometric proofs of the rules, nor that the examples differ from those of the other authors already in general style, not only in detailed contents; but close reading of Karpinski’s text and excerpts from the manuscript leave little doubt on either account.

I have not been able to discover any echo whatsoever of this publication. This may have at least three reasons.

Firstly, 1929 fell in a period where the interest in European medieval mathematics was at a low ebb – probably the lowest since the Middle Ages, at least since 1840. From 1920 to c. 1948 (from the death of Moritz Cantor to the beginning of Marshall Clagett’s work in the field), the total number of scholarly publications dealing with Latin and European vernacular mathematics does not go much beyond the dozen.

Secondly, the existence of the distinct *abbaco* mathematical tradition was not recognized, although Karpinski had already described another *abbaco* treatise in [1910]. As early as [1900: 166], it is true, Cantor had spoken of the existence throughout the fourteenth century of two coexisting “schools” of mathematics, one “geistlich” (“clerical”, i.e., universitarian), the other “weltlich oder kaufmännisch” (“secular or commercial” and derived from Leonardo Fibonacci’s work); part of Cantor’s basis for this (but only a modest part) was Libri’s edition [1838: III, 302–349] of a major section of what has now been recognized as Piero della Francesca’s *Trattato d’abaco*[1] (which Cantor, accepting Libri’s wrong dating, had located in the fourteenth century). Eneström [1906] had done what he could to make a fool of Cantor by twisting his words[2]. Attentive reading would easily

[1] On the identification of Libri’s manuscript with the very manuscript from which Arrighi [1970] made his edition, see [Davis 1977: 22*f*].

[2] Arguing from his own blunt ignorance of the institution within which university mathematicians moved, he rejected the epithet “clerical” as absurd (“Sacrobosco und Dominicus Clavasio waren meines Wissens nicht Geistliche”; actually, all university scholars were at least in lower holy orders, as evident from the familiar fact that they

have exposed Eneström's arrogant fraud; but the kind of knowledge that would have been required for that had come to be deemed irrelevant for historians of mathematics and hence forgotten, and Sarton [1931: 612*f*] not only cites Eneström's article but embraces the whole thesis uncritically.

Thirdly, like Cantor, Karpinski took the continuity from Fibonacci onward for granted, and concluded that the

> treatise by Jacob of Florence, like the similar arithmetic of Calandri, marks little advance on the arithmetic and algebra of Leonard of Pisa. The work indicates the type of problems which continued current in Italy during the thirteenth to the fifteenth and even sixteenth centuries, stimulating abler students than this Jacob to researches which bore fruit in the sixteenth century in the achievements of Scipione del Ferro, Ferrari, Tartaglia, Cardan and Bombelli.

Only those interested in manifestations of mathematical stagnation – thus Karpinski invited to conclude – would gain anything from looking deeper into Jacopo's treatise.

The manuscripts

Whatever the reason, nobody seems to have taken interest in the treatise before Van Egmond inspected it in the mid-seventies during the preparation of his global survey of Italian Renaissance manuscripts concerned with practical mathematics [1980]. By then, the autonomous existence of the *abbaco* tradition in the fourteenth and fifteenth centuries was well-established; but Van Egmond noticed that the manuscript which Karpinski had examined (Vatican ms. Vat. Lat. 4826, henceforth **V**) could be dated by watermarks to the mid-fifteenth century, and that the algebra chapter (and certain other matters) were missing from two other manuscripts containing Jacopo's *Tractatus algorismi* (Florence, Riccardiana Ms. 2236, henceforth **F**; and Milan, Trivulziana Ms. 90, henceforth **M**).[3] Because **M** can be dated by watermarks to c. 1410, some 40 years before **V**, and since **V** contains rules for the fourth degree not present in the algebra

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were submitted to canonical jurisdiction). Because Fibonacci is supposed to be spoken of as a merchant only in late and unreliable sources (it was no part of Cantor's argument that he was one, although Cantor does refer to him elsewhere in pseudo-poetical allusions as the "learned merchant" – pp. 85*f*, 154; yet the very preface to the *Liber abbaci* speaks of Fibonacci's commercial travelling), and because merchants' mathematics teaching was supposed never to treat useless problems like the "100 fowls", no "commercial" school could have been inspired by Fibonacci and teach such useless problems.

[3] An edition of **F** was prepared by Annalisa Simi [1995]. A critical edition of **F** and **M** by the late Jean Cassinet and Annalisa Simi has not yet appeared.

of Paolo Gherardi's *Libro di ragioni* from 1328, Van Egmond decided [personal communication] "that the algebra section of Vat.Lat. 4826 [was] a late 14th-century algebra text that [had] been inserted into a copy of Jacopo's early 14th-century algorism by a mid-15th-century copyist".

Close textual analysis of **V** shows that this manuscript is very coherent in style as well as regarding the presence of various characteristic features both in the chapters that are shared with **F** and in those that are not; **F**, on the other hand, is less coherent.[4] Van Egmond's explanation of the differences between the two versions must therefore turned around: **V** is a quite faithful descendant of Jacopo's original (or at least of the common archetype for **F** and **V**), whereas **F** (and its cousin **M**) is the outcome of a process of rewriting and abridgement, an adapted version apparently meant to correspond to the curriculum of the abbacus school as described in a fifteenth-century document [ed. Arrighi 1967].

Internal evidence shows that **V** is a meticulously made (but not a blameless) library copy made from another meticulous copy;[5] seeming setoffs from Provençal orthography suggest that preceding steps in the copying process (if

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4 See [Høyrup 2001]. Repetition of the details of the extensive argument would lead too far; but let me list a few points that on the whole speak for themselves:

- in one place **F** refers to a diagram that is only present in **V**;
- in another problem, the illustrating diagram in **F** is so different from what is needed that Simi inserts a "(sic!)"; the diagram in **V** corresponds to the description of the situation in the text;
- one problem in **F** starts "egli è un terreno lo qual è ampio 12 braccia, cioè uno muro, et è alto braccia 7 ed è grosso braccia 1 et $^1/_4$"; the counterpart in **V** starts "egli è uno muro, el quale è lungho 12 braccia e alto sette. Et grosso uno et $^1/_4$". The solution in both speaks of a the wall presented in **V**.
- **V** tells regularly that the first-order approximation to an irrational square root is approximate, and regularly also gives a (mistaken but easily explainable) second-order approximation. Occasionally, **F** also mentions the approximate character of the first-order formula; but in one place it believes it may be true, while in another it mixes up the wrong second-order formula found in **V** with a correct formula, which makes the whole thing quite nonsensical.
- In **V**, the commercial partnership serves (both in sections that have a counterpart in **F** and in those that have none) as a general model for proportional partition; in **F**, this trick is mostly avoided – but in one place it is not.

As we shall see, descendant treatises show that the algebra section in **V** must antedate 1328 by so much that 1307 seems a quite reasonable date.

5 On fol. 46^v we find what according to its contents is a marginal note indicating that the list of silver coins has been forgotten by mistake and comes later. But the note is *not* in the margin but within the normal text frame, which shows it to be copied.

any there are) can have been no less meticulous.[6] All in all it is thus legitimate to treat **V** as identical with Jacopo's treatise from 1307 apart from a few errors and omissions.

The algebra section

The algebra section proper of **V** runs from fol. 36[v] to fol. 43[r]. It is followed by an alligation problem dealing with grain and solved without algebra, and four problems which we would consider algebraic but whose solutions do not make use of *cosa, censo,* etc. One of them is an irreducible problem of the fourth degree, which is solved correctly. We shall return to this group below.

The rules

The algebra section proper gives rules for the following cases – *C* stands for *censo*, *t* for *thing* (*cosa*), *n* for *number* (*numero*):[7]

(1) $\alpha t = n$	(3) $\alpha C = \beta t$	(5) $\beta t = \alpha C+n$
(2) $\alpha C = n$	(4) $\alpha C+\beta t = n$	(6) $\alpha C = \beta t+n$

(7) $\alpha K = n$	(12) $\alpha K = \beta C+\gamma t$	(17) $\alpha CC+\beta K = \gamma C$
(8) $\alpha K = \beta t$	(13) $\alpha CC = n$	(18) $\beta K = \alpha CC+\gamma C$
(9) $\alpha K = \beta C$	(14) $\alpha CC = \beta t$	(19) $\alpha CC = \beta K+\gamma C$
(10) $\alpha K+\beta C = \gamma t$	(15) $\alpha CC = \beta C$	(20) $\alpha CC+\beta C = n$
(11) $\beta C = \alpha K+\gamma t$	(16) $\alpha CC = \beta K$	

The first six cases are the traditional first- and second-degree cases, familiar since al-Khwārizmī's *Kitāb al-jabr wa'l-muqābalah*. The remaining are all either homogeneous or reducible to second-degree problems, and thus nothing new compared to what was done in the Arabic world since centuries. As already mentioned, the order of the six fundamental cases differs, both from that of al-Khwārizmī (extant Arabic text as well as Latin translations) and Abū Kāmil (both have 3-2-1-4-5-6) and from that of Fibonacci (who has 3-2-1-4-6-5).

Another noteworthy characteristic is that all cases are defined as non-

[6] In one place, moreover, the text of **V** should transform $4\sqrt{54}$ into a pure square root; instead we find a blank, and in the margin the words "così stava nel'originale spatii". Obviously, the author did not want to compute 16×54 mentally but postponed – and forgot; and all intermediate copyists have conserved the blank.

[7] The Latin treatises (the translations of al-Khwārizmī as well as Fibonacci) would refer to the numbers as *dragmas*, but this idiom is absent from Jacopo's formulation of the rules. Similarly, Jacopo refers to the first power of the unknown as *thing* (*cosa*), never as *root* (*radix*) as do the Latin treatises.

normalized problems, and that the first step of each rule is thus a normalization. In the Latin treatises, all cases except "roots equal number" (where the normalized equation *is* the solution)[8] are defined as normalized problems, and the rules are formulated correspondingly. (All also teach how to proceed when a non-normalized problem is encountered, but this is done outside the regime of rules.)

The examples

To each of the first six cases, Jacopo gives at least one, sometimes two or three examples. For the remaining cases, only the rules and no examples are given. In translation[9] the statements of these problems run as follows:

1a. Make two parts of 10 for me, so that when the larger is divided in the smaller, 100 results from it.

1b. There are three partners, who have gained 30 libre. The first partner put in 10 *libre*. The second put in 20 *libre*. The third put in so much that 15 *libre* of this gain was due to him. I want to know how much the third partner put in, and how much gain is due to (each) one of those two other two partners.

2. Find me two numbers that are in the same proportion as is 2 of 3: and when each (of them) is multiplied by itself, and one multiplication is detracted from the other, 20 remains. I want to know which are these numbers.

3. Find me 2 numbers that are in the same proportion as is 4 of 9. And when one is multiplied against the other, it makes as much as when they are joined together. I want to know which are these numbers.

4a. One lent to another 100 *libre* at the term of 2 years, to make (up at) the end of year. And when it came to the end of the two years, then that one gave back to him *libre* 150. I want to know at which rate the *libra* was lent a month.

4b. There are two men that have *denari*. The first says to the second, if you gave me 14 of your *denari*, and I threw them together with mine, I should have 4 times as much as you. The second says to the first: if you gave me the root of your *denari*, I should have 30 *denari*. I want to know how much each man had.

5a. Make two parts of 10 for me, so that when the larger is multiplied against the smaller, it shall make 20. I ask how much each part will be.

5b. Somebody makes two voyages, and in the first voyage he gains 12. And in the second voyage he gains at that same rate as he did in the first. And when his voyages were completed, he found himself with 54, gains and capital together.

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[8] Fibonacci actually defines even this case in normalized form – but gives no example and thus escapes the absurdity.

[9] Here as everywhere in the following, translations into English are mine if nothing else is indicated. In the present case I use the translation from [Høyrup 2000] with minor emendations.

I want to know with how much he set out.[10]

5c. Make two parts of 10 for me, so that when one is multiplied against the other and above the said multiplication is joined the difference which there is from one part to the other, it makes 22.[11]

6. Somebody has 40 *fiorini* of gold and changed them to *venetiani*. And then from those *venetiani* he grasped 60 and changed them back into *fiorini* at one *venetiano* more per *fiorino* than he changed them at first for me. And when he has changed thus, that one found that the *venetiani* which remained with him when he detracted 60, and the *fiorini* he got for the 60 *venetiani*, joined together made 100. I want to know how much was worth the *fiorino* in *venetiani*.

The first observation to make is that none of the problems are stated in terms of numbers, *things* and *censi* (afterwards, of course, a "position" is made identifying some magnitude with the *thing*; without this position, no reduction to the corresponding case could result). In the Latin treatises, in contrast, the first examples are always stated in the same terms as the rules.

Second, we notice that some of the pure-number examples follow the pattern of the "divided ten", familiar since al-Khwārizmī's treatise and abundantly represented in the *Liber abbaci*. Others, however, are of a type with no such precedent: those where the ratio between two unknown numbers is given;[12] for any given polynomial equation with a single unknown it is of course easy to create an example of this kind, thereby adding cheap apparent complexity.

Further, we should be struck by the abundant presence of problems apparently related to commercial activities – *muʿāmalāt*-problems ("problems dealing with social life"), in the classification of Arabic mathematics. The only problem of this type found in the Latin algebra translations is the one where an given sum of money is distributed evenly first among a an unknown number x of people, next among x+1 [ed. Hughes 1986: 255], with given difference between the shares in the two situations. Among the problems treated in the algebra section of the *Liber abbaci* at most some 8 percent belong to the *muʿāmalāt* category: 4 variants of the problem type just mentioned, one problem treating of the purchase of unspecified goods, and one referring to interest and

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10 Both solutions are shown to be valid.

11 This example serves to demonstrates that one of the two solutions may be false (unless, as we would say, the difference between the two numbers can be counted as negative).

12 There is an analogue of Jacopo's superficially similar problem (1a) in al-Khwārizmī's treatise [ed. Hughes 1986: 248], repeated by Abū Kāmil [ed. Sesiano 1993: 360]; but like Jacopo's (1a) these problems speak of division, not of "proportion", and like Jacopo's they are primarily divided-ten problems.

commercial profit.

Finally, we should take notice of the square root of an amount of real money in (4b); this is without parallel even in the non-algebraic chapters of the *Liber abbaci*, where *muʿāmalāt* problems abound.

Peculiar methods

In the main, the methods used by Jacopo to solve the problems of course coincide with what we know from the Latin works. But some differences may be observed here and there. We may look at the solution to (1b) – a paradigmatic example of how to break a butterfly on the algebraic wheel – in which several idiosyncrasies are represented (fols 36^v–37^r):

> Do thus, if we want to know how much the third partner put in, posit that the third put in a thing. Next one shall aggregate that which the first and the second put in, that is, *libre* 10 and *libre* 20, which are 30. And you will get that there are three partners, and that the first puts in the partnership 10 *libre*. The second puts in 20 *libre*. The third puts in a thing. So that the principal of the partnership is 30 *libre* and a thing. And they have gained 30 *libre*. Now if we want to know how much of this gain is due to the third partner, when we have posited that he put in a thing, then you ought to multiply a thing times that which they have gained, and divide in the total principal of the partnership. And therefore we have to multiply 30 times a thing. It makes 30 things, which you ought to divide in the principal of the partnership, that is, by 30 and a thing, and that which results from it, as much is due to the third partner. And this we do not need to divide, because we know that 15 *libre* of it is due to him. And therefore multiply 15 times 30 and a thing. It makes 450 and 15 things. Hence 450 numbers and 15 things equal 30 things. Restore each part, that is, you shall remove from each part 15 things. And you will get that 15 things equal 450 numbers. And therefore you shall divide the numbers in the things, that is, 450 in 15, from which results 30. And as much is the thing. And we posited that the third partner put in a thing, so that he comes to have put in 30 *libre*. The second 20 *libre*. The first 10 *libre*. And if you should want to know how much of it is due to the first and to the second, then remove from 30 *libre* 15 of them which are due to the third. 15 *libre* are left. And you will say that there are 2 partners who have gained 15 *libre*. And the first put in 10 *libre*. And the second put in 20 *libre*. How much of it is due to (each) one. Do thus, and say, 20 *libre* and 10 *libre* are 30 *libre*, and this is the principal of the partnership. Now multiply for the first, who put in 10 *libre*, 10 times 15 which they have gained. It makes 150. Divide in 30, from which results 5 *libre*. And as much is due to the first. And then for the second, multiply 20 times 15, which makes 300 *libre*. Divide in 30, from which results 10 *libre*, and as much is due to the second partner. And it is done, and it goes well. And thus the similar computations are done.

Let us first concentrate on the start of the procedure, the one that leads to the determination of what the third partner put in. It makes use of the "partnership rule", a special case of the rule of three: the share of each partner in the profit is found as the product of his share of the capital first multiplied by the total profit, next divided by the total capital of the partnership,

$$p_i = \frac{c_i \cdot P}{C} \ .$$

The second part of the procedure, the one determining the shares of the first two partners by means of a ficticious new partnership, illustrates a feature of Jacopo's text that was already mentioned above: his recurrent use of the commercial partnership as a general model or functionally abstract representation within which all kinds of proportional distributions can be made.

Other idiosyncrasies

The use of the term "restore" (*ristorare*, corresponding to Arabic *jabara*) is another departure from the Latin algebra writings, in which it designates the cancellation of a subtracted term by addition. In Abū Bakr's *Liber mensurationum* translated by Gherardo da Cremona [ed. Busard 1968] it is also used a couple of times (#7, #55) in the function of multiplicative completion, changing $^2/_5$ and $^1/_4$ into 1 through multiplication by $2^1/_2$ and 4, respectively. Cancellation of an additive term, on the other hand, is nowhere spoken of in this way but instead as *opponere*, the Latin equivalent of *qabila* (whence *muqābalah*), whereas Jacopo uses "restore" in this way repeatedly.

Opporre is absent from Jacopo's text, but that probably does not mean that it contains no equivalent of *qabila* / *muqābalah*. Indeed, in Raffaello Canacci's *Ragionamenti d'algebra* [ed. Procissi 1954: 302] we read, in a passage ascribed to Guglielmo de Lunis, that *elmelchel* (the neighbour of *geber* (i.e., *jabr*) in the text and thus certainly a transcription of *al-muqābalah* in seemingly Mozarabic pronunciation) means "exempio hovvero aghuaglamento", "exemple or equation". This term (in the form *raoguaglamento*) is used in the end of example (5b).

A final characteristic by which Jacopo's treatise differs from all the Latin algebra writings is the complete absence of geometric proofs for the correctness of the algorithms by means of which the cases 4–6 are solved.

The *fondaco* problems

As mentioned above, Jacopo's treatise contains four problems which we would consider algebraic but which do not make use of the technique of *thing* and *censo* (fols 43^v–45^v). All deal with the yearly wages of the manager of a *fondaco* or warehouse. Their statements run as follows:

a. Somebody stays in a warehouse 3 years, and in the first and third year together he gets in salary 20 *fiorini*. The second year he gets 8 *fiorini*. I want to know accurately what he received the first year and the third year, each one by itself.

b. Somebody stays in a warehouse 4 years, and in the first year he got 15 *fiorini* of gold. The fourth he got 60 *fiorini*. I want to know how much he got the second year and the third at that same rate.

c. Somebody stays in a warehouse 4 years. And in the first year and the fourth together he got 90 *fiorini* of gold. And in the second year and the third together he got 60 *fiorini* of gold. I want to know what resulted for him, each one by itself.

d. Somebody stays in a warehouse 4 years. And in the first year and the third together he got *fiorini* 20 of gold. And in the second and the fourth year he got *fiorini* 30 {...} of gold. I want to know what was due to him the first year and the second and the third and the fourth. And that the first be such part of the second as the third is of the fourth.

Obviously, we are missing some information which Jacopo takes for granted. The solutions to (a) shows what:

> Do thus, and let this always be in your mind, that the second year multiplied by itself will make as much as the first in the third. And do thus, multiply the second by itself, in which you say that he got 8 *fiorini*. Multiply 8 times 8, it makes 64 *fiorini*. Now you ought to make of 20 *fiorini*, which you say he got in the first and third year together, {...} two parts which when multipli⟨ed⟩ one against the other makes 64 *fiorini*. And you will do thus, that is that you always halve that which he got in the two years. That is, halve 20, 10 result. Multiply the one against the other, it makes 100. Remove from it the multiplication made from the second year which is 64, 36 is left. And of this find its root, and you will say that one part, that is, the first year, will be 10 less root of 36. And the other part, that is, the second year, will be 8 *fiorini*. And the third will be from 10 less root of 36 until 20 *fiorini*, which are *fiorini* 10 and added root of 36. And if you want to verify it, do thus and say: the first year he gets 10 *fiorini* less root of 36, which is 6. Detract 6 from 10, 4 *fiorini* is left. And 4 *fiorini* he got the first year. And the second year he got 8 *fiorini*. And the third he got *fiorini* 10 and added root of 36, which is 6. Now put 6 *fiorini* above 10 *fiorini*, you will get 16 *fiorini*. And so much did he get the third year. And it goes well. And the first multiplied against the third makes as much as the second by itself. And such a part is the second of the third as the first of the second. And it is done.

The beginning of this solution provides the clue: the yearly wages are tacitly supposed to increase in geometric progression. When this is taken into account, all four problems possess unique solutions, which are found correctly in the text. In (a), it is used that the wages of the three years fulfil the condition

$$S_1 \cdot S_3 = S_2 \cdot S_2 = 64 .$$

At the same time, $S_1+S_3 = 20$. This problem *could* be solved by means of (*al-jabr*) algebra – it is indeed of the same type as (5a) above. But the text offers an alternative, a purely numerical algorithm – which coincides with the solution to the corresponding rectangle problem given by Abū Bakr (and in the tradition of geometric rectangle riddles since this tradition is first attested in the Old Babylonian clay tablets).

(b) first finds the quotient p of yearly increase (without giving it any name) as $\sqrt[3]{S_4/S_1}$, and then finds S_2 and S_3 as $p \cdot S_1$ and $p^2 \cdot S_1$, respectively. (d) finds p as $(S_2+S_4)/(S_1+S_3)$ (again without telling *what* is found) and next S_1 as $(S_1+S_3)/(1+p^2)$. Both solutions are straightforward for anybody who possesses a fair understanding of the nature of the ascending algebraic powers as a geometric series, but less straightforward for the one who knows his algebra through al-Khwārizmī or Fibonacci alone.

(c) is more complex. The solution makes use of the identity

$$S_1 \cdot S_4 = S_2 \cdot S_3 = \frac{(S_2+S_3)^3}{3(S_2+S_3)+(S_1+S_4)} ,$$

which can explained by the transformations ($S_1 = a$)

$$\frac{(S_2+S_3)^3}{3(S_2+S_3)+(S_1+S_4)} = \frac{a^3 p^3 (1+p)^3}{a(3p+3p^2+1+p^3)} = a^2p^3 = a \cdot ap^3 = ap \cdot ap^2$$

– something which certainly requires more than a merely “fair” understanding of the nature of the ascending algebraic powers as a geometric series. Who understood this (no explanation in the text suggests that Jacopo himself understood, fond though he elsewhere is of giving pedagogical explanations) will have had no difficulty in seeing how the cases (7) through (20) in the algebra proper could be solved either directly or by reduction to appropriate second-degree cases.

Abbreviations and notation

It is a general and noteworthy characteristic of Jacopo’s algebra (or at least of manuscript **V**, but there are good reasons to believe the manuscript to be true to the original in this regard) that it avoids all abbreviations in the technical algebraic terminology, as if the author was conscient of introducing a new field of knowledge where readers would be unfamiliar with the terminology and therefore unable to expand abbreviations correctly.[13] *A fortiori*, nothing in his

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[13] In a table listing the fineness of coins, *meno* is abbreviated (as was the standard); in the rest of the text, this abbreviation will be looked for in vain. Abbreviations for *radice*, *cosa*, and *censo* are equally absent. In contrast, terms that are not part of the algebraic technical vocabulary (*moltiplicare*, *libra*, *compagnia*, etc.) are regularly abbreviated.

It may then seem strange that nothing is said in the beginning of the algebra chapter algebra reminds even vaguely of algebraic symbolism or syncopation. Early in the treatise, however, we find an unusual variant of the Roman numerals – for instance in the explanation of 400,000 as $\overset{\text{m}}{\text{cccc}}$. This way to put the "denominator" above the thing being denominated coincides exactly with the algebraic notation found in Maghreb from the twelfth century onward [Abdeljaouad 2002: 11*f*; Souissi 1969: 92 n. 2] – but since the same system was also used by Diophantos and other ancient Greeks to write multiples of aliquot parts, and by Middle Kingdom Egyptian scribes for the writing of large numbers, the similarity remains suggestive and nothing more for the time being.

Whence?

Jacopo's algebra is not derived, neither from Fibonacci nor from the Latin translations of al-Khwārizmī (or Abū Kāmil) – that much should already be clear. Its ultimately roots in Arabic *al-jabr* are no less certain. In consequence, Jacopo's algebra confronts us with a hitherto unknown channel to the Arabic world and its mathematics.

This conclusion raises two difficult questions. Firstly, Jacopo's algebra, if fundamentally different from the Latin translations of al-Khwārizmī and Abū

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explaining what *cosa* and *censo* mean. The reason could be that an introduction to the chapter has disappeared during transmission – other chapters start by announcing what comes next – for instance,

– Abiamo dicto dele multiplicationi et dele divisioni et de tucto quello che intorno a ciò è di necessità. Ora lasciamo questo, et dirremo per propria et legitima forma et regola sopre tucti manere de numeri rocti [...].
– Abiamo dicto de rotti abastanza, però che dele simili ragioni de rotti tucte se fanno a uno modo e per una regola. E però non ne diremo più al punte. Et incominciaremo ad fare et ad mostrare alcune ragioni secondo che appresso diremo. Se ci fosse data alcuna ragione nela quale se proponesse tre cose [...].
– In nomine Domini amen. Qui appresso incominciaremo, et dirremo de tucte maniere de mesure. Et primamente dirremo del tundo ad conpasso [...].
– In Christi nomine amen. Qui sonno sotto scripte tucte maniere de leghe de monete. Et similmente tucti allegamenti de oro, argento et ramo [...].

The algebra chapter, in contrast, simply begins by stating the first rule (fol. 36^{v}),

> Quando le cose sonno eguali al numero, si vole partire el numero nelle cose, et quello che ne vene si è numero. Et cotanto vale la cosa.

However, *after* the example to the sixth rule we read (fol 42^{r})

> Qui finischo le sey regole conposte con alquanti assempri. Et incomincia l'altre regole che sequitano le sopradicte sey como vederete.

Seemingly, the text presupposes that these six rules have already been spoken of as such.

Kāmil, must also be fundamentally different from the Arabic originals, and his Arabic inspiration must therefore be of a different kind; secondly, his treatise contains no single Arabism, and direct use of Arabic sources on his part can thus be safely excluded. We must therefore ask, firstly, which kind of Arabic material provided his ultimate inspiration? And secondly, where in the Romance-speaking world did he find an environment using this material actively?

The two questions must be addressed one by one. Let us first look at a larger range of Arabic algebraic writings in relation to the parameters where Jacopo's algebra differs from the Latin treatises.

The order of cases

As stated above, al-Khwārizmī as well as Abū Kāmil present the six fundamental cases in the order 3-2-1-4-5-6 (Jacopo's order being 1-2-3-4-5-6). This "classical order" recurs in ibn al-Bannā'’s presentation of the cases in the *Talkhīṣ* [ed., trans. Souissi 1969], in al-Qalaṣādī's *Kašf* [ed., trans. Souissi 1988], in ibn Badr's *Ikhtiṣār al-jabr wal'l-muqābalah* [ed., trans. Sánchez Pérez 1916], and in ibn al-Yāsamīn's ʿ*Urjuza fi'l-jabr wa'l-muqābalah* (see [Souissi 1983: 220–223]).

Al-Karajī arranges things differently. In the *Kāfī* [ed., trans. Hochheim 1878] as well as the *Fakhrī* [Woepcke 1853], his order is 1-3-2-4-5-6. The same pattern is found in al-Samawʿal and al-Kašī [Djebbar 1981: 60*f*] and in Bahā'-al-Dīn al-ʿĀmilī's *Khulāṣah al-ḥisāb* [ed., trans. Nesselmann 1842] from c. 1600. In his solution of the equations, ibn al-Bannā' follows the pattern 3-2-1-4-6-5 (that of the *Liber abbaci*).

Jacopo's order is referred to by al-Māridīnī in his commentary to ibn al-Yāsamīn's ʿ*Urjūza* from c. 1500 as the one that is used "in the Orient", and it is indeed that of al-Miṣṣīṣī, al-Bīrūnī, al-Khayyāmī and Šaraf al-Dīn al-Ṭūsī [Djebbar 1981: 60]. But al-Qurašī, born in al-Andalus in the thirteenth century and active in Bejaia, has the same order [Djebbar 1988: 107].

Normalization

Al-Khwārizmī's original text, as the Latin translations, defines all cases except "things made equal to number" in normalized form and gives corresponding rules.[14] This also applies to ibn Turk's [ed., trans. Sayılı 1962] and Thābit's

[14] The Arabic manuscript published first by Rosen [1831] and later by Mušarrafa & Aḥmad [1939] defines the cases in non-normalized form, even though its rules presuppose normalized equations. However, Gherardo's extreme grammatical faithfulness in other respects attests to his reliability even on this account. The different pattern of the Arabic text is thus an innovation – an adaptation of the original to changing customs within

[ed., trans. Luckey 1941] demonstrations of the correctness of the rules, and to al-Khayyāmī's algebra [ed., trans. Rashed & Djebbar 1981]. Al-Karajī's *Kāfī* confronts us with a mixed situation: the three simple cases (1–3) are non-normalized (definitions as well as rules); case (4) is defined as non-normalized, but its rule presupposes normalization; the two remaining composite cases are presented only through normalized paradigmatic examples, and the formulations of the rules presuppose this normalization. The *Talkhīs* and the *Kašf* treat the simple cases as the *Kāfī*; they give no explicit definitions for the composite cases, but state rules that presuppose normalization. Ibn Badr defines gives non-normalized definitions for all cases, and corresponding rules for the simple cases; his rules for the composite cases presuppose normalization. Only Bahā' al-Dīn makes everything in non-normalized form, as does Jacopo.

Examples

Basic examples formulated in the same terms as the rules, i.e., of a *māl* ("possession", the equivalent of Jacopo's *censo*) and its *jidhr* ("[square] root"), are found in almost all the Arabic works I have looked at – al-Khwārizmī's, Abū Kāmil's and al-Khayyāmī's treatises, in al-Karajī's *Kāfī* and *Fakhrī*, in al-Qalaṣādī's *Kašf* and in ibn Badr's *Ikhtiṣār*. Only ibn al-Bannā''s *Talkhīṣ* and Bahā' al-Dīn's *Khulāṣah* contain no examples of this kin – but the *Talkhīṣ* contains no examples at all.[15]

The divided ten turns up everywhere, from al-Khwārizmī and Abū Kāmil to Bahā' al-Dīn. Problems where two unknown numbers are given in proportion are as absent from the Arabic treatises I have inspected as from the Latin ones.

Abū Kāmil, like al-Khwārizmī, deals with the division of a given amount of money between first x, then $x+p$ men, but apart from that none of them treat of *mu'āmalāt*-problems in the properly algebraic parts of their treatises. Most other treatises keep *mu'āmalāt* matters wholly apart from their algebra. The only exceptions among the works I have inspected are the *Fakhrī* and ibn Badr's and Bahā' al-Dīn's treatises. Ibn Badr, after a large number of divided-ten and *māl-jidhr* problems, has others dealing with the remuneration of a principal,

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the field. Indeed, comparison of the published Arabic version with Gherardo's and Robert of Chester's Latin translations shows that it has must have been submitted to at least three successive revisions – see [Høyrup 1998].

[15] The *Khulāṣah* does contain a first-degree problem about a *māl*, but apparently meant to stand for real money.

dowries,[16] the mixing of grain, the distribution of booty among soldiers, travels of couriers, and reciprocal gifts (three or four of each type). Of Bahā' al-Dīn's illustrations of the six fundamental cases, two deal with pure numbers and four with feigned *muʿāmalāt* (that is, with "recreational") problems. In a later chapter listing 9 problems that can be resolved by more than one method, the share of recreational problems is the same.

Square roots of real money

One of Jacopo's problems – (4b), the only one of his *muʿāmalāt* problems that belongs to a familiar recreational type – refers to the square root of an amount of real money. From a purely formal point of view this is highly traditional, the basic *al-jabr* cases being defined as problems dealing with a *māl* or "possession" and its square root, and treating the known number as a number of dirham. But already in al-Khwārizmī's time this had become a formality. It is true that he states not only the root when it has been found but also the *māl*, remembering thus that once this had been the real unknown quantity of the problem. But stating the case "*māl* made equal to number" in normalized form (and defining first the *root* as one of the numbers and next the *māl* as the product of this number by itself [ed. Hughes 1986: 233*f*]) he clearly shows to consider the *root* as the fundamental unknown – in perfect agreement indeed with his later identification of the root with the *šay'* or *thing*. From al-Khwārizmī onward we may thus claim that the *root* was a square root of formal, not real money.

Roots of real money are absent from almost all of the Arabic algebra writings I have examined – the only exceptions being al-Karajī, who in the *Fakhrī* once takes the root of an unknown price and twice of unknown wages, and ibn Badr, who twice takes the root of a dowry. However, the *Liber mahamaleth*, a Latin composition made in Spain during the twelfth century, contains at least two algebraic problems of the kind: in one, the square roots of a capital and a profit are taken, in another the square root of a wage [Sesiano 1988: 80, 83].

In order to find copious square roots of real entities (not only money but also, for instance, a swarm of bees, the arrows fired by Arjuna, or a horde of elephants) we have to go to India.

Commercial calculation within algebra

Jacopo employs the rule of three as a tool for algebraic computation; further, he uses the commercial partnership as a functionally abstract representation for

[16] Principal as well as dowry are designated *māl*, but the problem texts show that real invested money and real dowries are meant.

proportional distributions. I have never noticed anything similar in an Arabic treatise – al-Khwārizmī *presents* the rule of three in a separate chapter after the algebra proper and before the geometry, but this is a different matter.

Jabr and *muqabalah*

Jacopo's use of the equivalent of *jabr* and of the likely equivalent of *muqābalah* differs from al-Khwārizmī's use of the terms (which is also the main usage of Abū Kāmil, and that of ibn al-Bannā', al-Qalaṣādī and Bahā' al-Dīn). However, the Arabic usage is far from uniform, as can already be seen from the various Latin translations.

Firstly, Abū Bakr's *Liber mensurationum*, whose multiplicative use of *restaurare* was mentioned above, uses the phrase *restaura et oppone* repeatedly in situations where no subtraction is to be made. The meaning of "opposition" is clearly in concordance with Canacci's explanation, namely *to form a (simplified) equation* – and thus with Jacopo's usage. Even in Abū Kāmil's *Algebra* the same phrase turns up time and again with the same sense (see the index in [Sesiano 1993]). Similarly ambiguities are found in ibn Badr [Sánchez Pérez 1916: 24 n. 1].

In the *Fakhrī* [Woepcke 1853: 64], *jabr* may be additive as well as subtractive, just as in Jacopo's treatise. *Muqābalah*, on its part, is explained to be the formation of a simple equation where one or two terms is equal to one or two terms (three at most in total). In the *Fakhrī* [ed., trans. Hochheim 1878: III, 10], *jabr* is also said to include multiplicative completion (as in the *Liber mensurationum*). For the rest, this text seems to be ambiguous (as far as can be judged from the translation); perhaps it means to leave subtractive balancing unnamed and uses *muqābalah* as the *Fakhrī*, perhaps this latter term is meant to designate the removal which leads to the formation of the simplified equation.[17]

Geometric proofs

Geometric proofs for the correctness of the rules for the three composite cases are found in al-Khwārizmī and ibn Turk, and (with new ones add) in Abū Kāmil and in the *Fakhrī*. They are absent from the *Kāfī*, from the treatises belonging

[17] As Saliba [1972] has argued, the meaning of the *Fakhrī* appears to be the original one; the ambiguity in the *Kāfī* illustrates the way in which the new interpretation as the subtractive counterpart of *jabr* can have come about.

Raffaello Cannacci, in the passage where he explains *elmelchel* to stand for "exemple or equation", states that *elchel* (*al-qābila*, according to the parallel) stands for "opposition", explained to be the simplified equation.

to the Maghreb school (ibn al-Bannā’, al-Qalaṣādī) and from those of ibn Badr and Bahā’ al-Dīn.

Polynomial algebra and geometric progressions

I have seen nothing similar to Jacopo’s four *fondaco* problems in Arabic works, and never received a positive answer when asking others who might know better. But the basic underlying theory – that which also allows one to see that Jacopo’s cases (7) through (20) can be solved – was known at least since al-Karajī and al-Samaw’al,[18] and part of it was inherent in all writings that presented the sequence of algebraic powers as a geometric progression and also stated the rules for multiplying binomials.[19]

Summing up

Almost every seeming idiosyncrasy we find in Jacopo can be found in Arabic writings (the exceptions being the use of the rule of three and the partnership structure as tools for algebra, the examples asking for numbers in given proportion, and the idea that wages increase by default in geometric progression). But they never occur together in treatises I have inspected. Those who are furthest removed from Jacopo are al-Khwārizmī and Abū Kāmil. The exponents of the Maghreb school are somewhat closer (in their omission of geometric proofs and, hypothetically, in the similarity between their algebraic notation and Jacopo’s multiplicative writing of Roman numerals). But Jacopo’s order of cases, his use of the *jabr*- and *muqābalah*-equivalents, his square roots of real money and his ample use of *mu*‘*āmalāt*-problems within the algebra links him to (some middle ground between) al-Karajī’s writings, ibn Badr’s possibly Iberian *Compendium of Algebra*, the certainly Iberian *Liber mahamaleth*, and Bahā’ al-Dīn’s *Essence of the Art of Calculation*; his consistent presentation of non-normalized cases is only shared with the latter much younger work. In other, more explicit words: We do not know the kind of Arabic algebra that provided him with his ultimate inspiration, but it was certainly different from those (scholarly or “high”) currents

18 In the *Fakhrī*, al-Karajī makes use of the formula for the third power of a binomial [Woepcke 1853: 58]. At first he exemplifies it on $(2+3)^3$, next he uses it to show that $\sqrt[3]{2}+\sqrt[3]{54} = \sqrt[3]{128}$.

19 With hindsight, not only “part” but all that is required to resolve all of Jacopo’s *fondaco* problems was implied. But hindsight may amount to historiographical blindness: Cardano’s solution to the third-degree equation is “implied” in Old Babylonian “algebra”, in the sense that he combines tricks that were in use in that discipline; but it took more than three millennia to discover it.

that have so far been investigated by historians of mathematics.

The next generation

We ought now to concentrate on the second part of the "whence" question: where in the Romance-speaking world did Jacopo find an environment actively engaged in algebra?

However, an answer to this question (indirect or negative as it will be) can only be given if we look closely at the still extant Italian expositions of algebra from the immediately following generation.

One of these (**G**) is contained in Paolo Gherardi's *Libro di ragioni,* written in Montpellier in 1328.[20] Two others are contained in an *abbaco* manuscript from Lucca from c. 1330 [ed. Arrighi 1973], a conglomerate written by several hands. Its fols 80^v–81^v (pp. 194–197) contains a section on "le reghole dell'aligibra amichabile" (henceforth **L**); another section on "le reghole della chosa con asenpri" is found on fols 50^r–52^r (pp. 108–114; henceforth **C**).

Somewhat later but so closely related to one or more members of the first generation that they can inform us about it are two other items: **A**, a *Trattato dell'Alcibra amuchabile* from c. 1365 [ed. Simi 1994]; and **P**, an anonymous *Libro di conti e mercatanzie* [ed. Gregori & Grugnetti 1998] kept today in the Biblioteca Palatina of Parma and probably compiled in the Tuscan-Emilian area – according to problems dealing with interest in the years immediately after [13]89–95.

All of these depend to some extent on what we know from **V**, that is, on Jacopo. The first vernacular algebra that does not depend on him – and the earliest vernacular work dedicated exclusively to algebra – is the *Algibraa argibra*, according to one manuscript written by an otherwise unidentified Master Dardi from Pisa in 1344 (henceforth **D**).

The scheme on p. 18 summarizes some important features of these presentations of algebra. If a work has a rule for a particular case, it is marked R if the rule is true; X if it is false and constructed merely as an illegitimate imitation of the solution to a similar-looking second-degree problem; and S if the rule is valid only in a special case modelled after Jacopo's example (4a) from which the rule has been guessed (S_n if stated for the normalized case). The presence of examples is indicated E, marked by subscript digits (E_{12} thus indicates that two examples are given; E_1 and E_2 in the same row but different columns indicate that examples differ, E_1 and E_{1*} that they are identical apart from the

[20] Published by Gino Arrighi in [1987] – the chapter on algebra separately with translation and mathematical commentary by Van Egmond in [1978]; mentioned above, p. 3.

Case	**V**	**G**	**L**	**C**	**A**	**P**	**D**
$\alpha t = n$	1.R,E_{12},n	1.R,E_{1*},n	1.R,E_1,n	1.R,E_{1*},**p**	1.R,E_{12},n	1.R,E_{1*},**p**	1.R,E_{1**},**p**
$\alpha C = n$	2.R,E_1,**p**	2.R,E_2,n	2.R,E_2,n	2.R,E_{2*},n	2.R,E_1,**p**	2.R,E_2[a],**p**	2.R,E_3,**p**
$\alpha C = \beta t$	3.R,E_1,**p**	3.R,E_{1*},n	3.R,E_{1*},**p**	3.R,E_2,**p**	3.R,E_1,**p**	3.R,E_1,**p**	3.R,E_{2*},**p**
$\alpha C+\beta t = n$	4.R,E_{12},n	4.R,E_{1*},n	4.R,E_{1*},n	4.R,E_{1**},n	4.R,E_{12},n	4.R,E_{1*},**p**	4.R,E_3**p**
$\beta t = \alpha C+n$	5.R,E_{123},n	5.R,E_{2*},n	5.R,E_{2**},**p**	5.R,E_{2***}[b],n	5.R,E_{123},n	5.R,E_{2*},**p**	5.R,E_{1*45},**p**
$\alpha C = \beta t+n$	6.R,E_1,n	6.R,E_2,n	6.Omitted[c]	6.R,E_3,n	6.R,E_1,n	6.R,E_2,**p**	6.R,E_4[d],**p**
$\alpha K = n$	7.R,**p**	7.R,E_1,**p**	7.R,n	7.R,**p**	7.R,E_1,**p**	7.R,E_2,**p**	7.R,E_3,**p**
$\alpha K = \beta t$	8.R,**p**	9.R,E_1,**p**	8.R,n	8.R,**p**	8.R,E_1,**p**	9.R,E_1[e],**p**	8.R,E_2,**p**
$\alpha K = \beta C$	9.R,**p**	10.R,E_1,**p**	9.R,**p**	9.R,**p**	9.R,E_1,**p**	10.R,E_1,**p**	9.R,E_2,**p**
$\alpha K+\beta C = \gamma t$	10.R,n	15.R,E_1,n	10.R[f],**p**	14.R,n	15.R,n	15.R,E_1,**p**	14.R,E_{1*},**p**
$\beta C = \alpha K+\gamma t$	11.R,n		11.R,n	15.R,n	16.R,n		
$\alpha K+\gamma t = \beta C$					14.R,E_1,n	16.R,E_1,**p**	15.R,E_{234},**p**
$\alpha K = \beta C+\gamma t$	12.R,n	11.R,E_1,n	12.R[g],n	16.R,**p**	10.R,E_1,n	11.R,E_1,**p**	16.[h]R,E_2,**p**
$\alpha K = \sqrt{n}$		8.R,E_1,**p**			11.R,E_1,**p**	8.R,E_1,n	21.R,E_2,**p**
$\alpha K = \beta t+n$		12.X,E_1,n			12.X,E_1[i],n	12.X,E_1,**p**	
$\alpha K = \beta C+n$		13.X,E_1,n			13.X,E_1,n	13.X,E_1,**p**	
$\alpha K = \gamma t+\beta C+n$		14.X,E_1,n				14.X,E_1,n	
$\alpha CC = n$	13.R,n		13.R,**p**	11.R,**p**	17.R,n	17.R,E_1,**p**	11.R,E_2,**p**
$\alpha CC = \beta t$	14.R,**p**			12.R,**p**	18.R,**p**	18.R,E_1,p	12.R,E_2,**p**
$\alpha CC = \beta C$	15.R,**p**			13.R,**p**	19.R,**p**	19.R,E_1,**p**	13.R,E_2,**p**
$\alpha CC = \beta K$	16.R,**p**			10.R,**p**	20.R,**p**	22.E_1,**p**	10.[j]R,E_1,**p**
$\alpha CC+\beta K = \gamma C$	17.R,n				21.R,n		
$\beta K = \alpha CC+\gamma C$	18.R,n				22.R,n		
$\alpha CC = \beta K+\gamma C$	19.R,n				23.R,n		
$\alpha CC+\beta C = n$	20.R,n				24.R,n		
$\alpha CC+n = \gamma$C						20.R[k],E_1,n	
$\alpha C = \sqrt{n}$						21.R,E_1,n	
$\alpha C = n+\sqrt{\nu}$						23.X,E_1,**p**	
$\alpha K+\beta C+\gamma t = n$						24.S_n,E_1	A1.S,E_1,**p**
$\alpha CC+\beta K+\gamma C+\delta t = n$						25.S,E_1,n	A2.E_1,**p**

a. With the difference that $^1/_3+^1/_4$ has been replaced by $^7/_{12}$.

b. In the end of the solution, the compiler of **C** tinkers with the double solution which was present in his original. In the short collection of further illustrative examples, **C** also has the problem E_1 of **V**.

c. Absent; but since the ensuing text refers to “6 reghole”, this is clearly by involuntary omission.

d. E_4 in this line is closely related to E_3.

e. With a copying error in the statement which might look like being inspired by E_2.

f. The rule should read “Quando li chubi ⟨e li censi⟩ sono egualj alle cose [...]”.

g. The rule should read “Quando li chubi sono egualj ⟨a’ censi⟩ e alle chose [...]”.

h. Formulated $\beta C+\gamma t = \alpha K$.

i. Correcting a lacuna in the statement, which should read “Trouami 2 numeri che tale parte sia l’uno dell’altro come 2 di 3 e, multiprichato il primo per se medesimo et poi ⟨per⟩ quello numero faccia tanto quanto e più 12”.

j. Formulated $\beta K = \alpha CC$.

k. With a copying error, “traendone” instead of “più”.

choice of numerical parameters). The letters “p” and “n” indicate whether the division by which the equation is normalized is expressed as “partire per” or “partire in”; we shall see that this “neutral mutation” is an interesting parameter.[21] *K* stand for *cubo*, *C* for *censo*, *CC* for *censo di censo*, *t* for *cosa*, *n* for *numero* (in whatever spellings the manuscripts may use), and Greek letters for coefficients (implied by the plurals *cubi*, *censi*, and *cose*). We notice immediately that all works have the six fundamental cases in the same characteristic “non-Latin” order as Jacopo.

Paolo Gherardi

Let us first take up the column for **G**, Gherardi’s algebra from 1328, composed in that very town where Jacopo had written 21 years before him. Gherardi, as we see, follows Jacopo fairly closely in the six fundamental cases. The differences are the following:

– Gherardi never gives more than one example;
– he replaces Jacopo’s pure-number example for case (2) with a different pure-number example;
– in example (4), he divides the amount borrowed by 5;
– in Jacopo’s example (5b), he changes the given numbers in such a way that the result becomes irrational, and omits the second solution even though the rule mentions it;
– he replaces Jacopo’s example (6) by a pure-number version of the problem of dividing a given quantity (here 100), first among x, then among $x+p$ (here $x+5$) persons and adding the two results: $\frac{100}{t}+\frac{100}{t+5}=20$. The description of the procedure refers to a number diagram[22]

```
100      1 cosa
100      1 cosa piu 5
```

[21] Etymologically, “partire *a* in *b*” refers to the division of the quantity *a* into *b* equal parts, and “partire *a* per *b*” to the numerical computation; but I have never remarked any reference to the “parts” in question in any Italian writing from the epoch dividing “in” – the etymology must already have been forgotten. Any systematic choice of one or the other formulation (for instance, Jacopo dividing always the product of circular diameter and perimeter *in* 4 in order to find the area, and the perimeter *per* $3^{1}/_{7}$ in order to find the diameter) therefore points to a source in time or space where the distinction was still semantically alive.

[22] The diagram is actually missing from the manuscript, but it can be reconstructed from the verbal description and coincides with what is known from later manuscripts – see [Van Egmond 1978: 169 n. 11].

in a way (with “cross-multiplication” and all the other operations needed to add fractions) that implies underlying operations with the “formal” fractions $\frac{100}{1\ cosa}$ and $\frac{100}{1\ cosa\ piu\ 5}$.

Further on, major differences turn up:

– Gherardi leaves out all fourth-degree cases;
– he introduces $\alpha K = \sqrt{n}$ as a case on its own;
– he introduces three irreducible third-degree cases, giving false rules fashioned after those for the second degree – solving for instance the case $\alpha K = \beta t+n$ as if it had been $\alpha C = \beta t+n$;
– all higher-degree rules are illustrated by examples, all of which are pure-number problems of the kind that could easily be constructed *ad hoc* (finding two or three numbers in given proportion so that ...).

The illustrations to the false rules all lead to solutions containing irrational roots. This allowed the fraud to go undetected, since no approximate value of these solutions was computed – this was not the custom, even Jacopo when finding correctly a monthly interest of $\sqrt{600}$–20 *denari* in his example (4a) left it there.

The Lucca manuscript

L and **C** are closer to **V**, and largely to be described as somewhat free abridgments of Jacopo’s algebra. The changes they introduce in the numerical parameters of some examples do not change the character of these. Two of the examples where Gherardi differs from Jacopo are shared with **L**, but both are too simple to suggest particular affinity.

Trattato dell’Alcibra amuchabile

In those cases and problems that have a counterpart in **V**, **A** is much closer to this treatise than **L** and **C** (while sharing the title with **L**); it has all of Jacopo’s examples with identical parameters, deviating mainly at the level of orthography; however, where Jacopo left spaces open in example (4b) in order to insert later the result of $4\cdot\sqrt{54}$ (cf. note 6), **A** has the correct result “radicie di 864”. As we see, it even agrees strictly with **V** in the decision whether to divide *in* or *per*; both must hence descend by careful copying from a common archetype.

With a single exception, however – *viz* Gherardi’s only four-member case – **A** has all those examples for the higher-degree cases which we find in Gherardi, including his false rules for irreducible cases; but the agreement is not verbatim as with Jacopo. There also is a rule and an example for the reducible case $\alpha K+\gamma t = \beta C$, which **A** distinguishes from its mirror image $\beta C = \alpha K+\gamma t$; only the latter and not the former shape is present in **V**. Those higher-degree rules that are found in **V** but not in **G** (including the just-mentioned $\beta C = \alpha K + \gamma t$) follow **V** and are equally devoid of examples.

So far, only the middle part of the tripartite *Trattato dell'alcibra amuchabile* was spoken of. The first part starts by presenting the sign rules ("più via più fa più e meno via meno fa più ...") and then goes on to teach operations with roots – number times root, root times root, products of binomials containing roots and the division of a number or one such binomial by another binomial. For the product of binomial by binomial, a diagram is introduced to illustrate the procedure – for instance, for $(5+\sqrt{20})\cdot(5-\sqrt{20})$:

```
5 e  piu  ℞ di 20
via
5 e  meno ℞ di 20   .
```

As was usual in algebraic manuscripts from the Maghreb [Abdeljaouad 2002], the diagram stands outside the running text and recapitulates what is done by rhetorical means in the text. For the division of a number by a binomial, for instance 100 by $10+\sqrt{20}$, we find the similar diagram

```
10  piu  ℞ di 20
10  meno ℞ di 20   ,
```

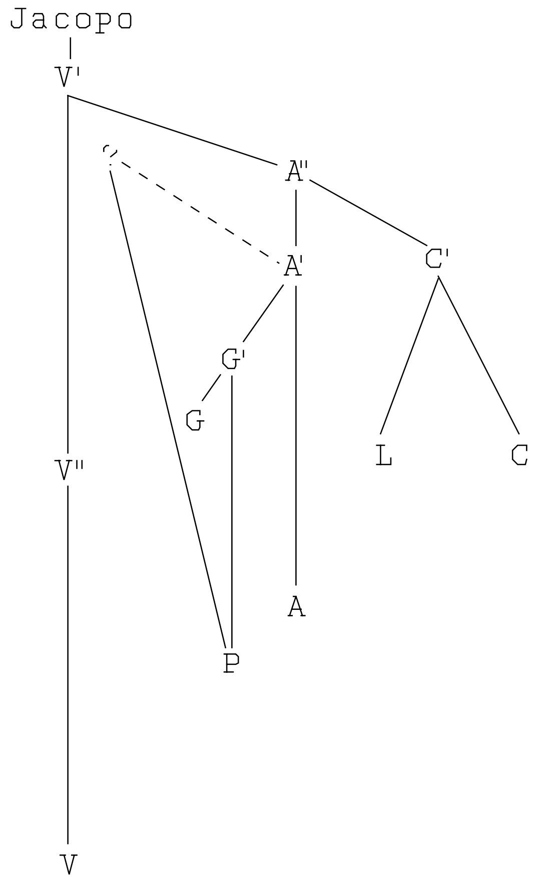
which serves to illustrate that both dividend and divisor are to be multiplied by $1-\sqrt{20}$. Whether the writer thinks in terms of formal fractions is not clear at this point.

However, in the third part [ed. Simi 1994: 41*f*] we find Gherardi's illustration to the sixth case; in **A** it is stated in direct words that the addition $\frac{100}{t}+\frac{100}{t+5}$ is to be performed "in the mode of a fraction", explained with the parallel $\frac{24}{4}+\frac{24}{6}$.

The Parma manuscript

The algebra section of **P**, the *Libro di conti e mercatanzie*, is closer to **G**, also in the treatment of those cases that had been dealt with by Jacopo. But in the illustration of the case $\alpha C = \beta t+n$ (still the problem $\frac{100}{t}+\frac{100}{t+5} = 20$) it has the explicit formal fractions of **A** (distorted in the beginning in a way that suggests the writer did not understand) and not Gherardi's diagram. It also has the case $\alpha K+\gamma t = \beta C$ that was absent from **G** but present in **A**, with the same example as **A** – but the mirror case $\beta C = \alpha K+\gamma t$ is absent from **P** though present in **A**. Gherardi's only four-member problem ($\alpha K = \gamma t+\beta C+n$), absent from *A*, is present in **P**.

P also provides examples to four of those fourth-degree rules which had none in **A**; three of these are of the usual facile pure-number type, but one ($\alpha CC = n$) is illustrated by a geometric question – to find the side of an equilateral triangle with given area. Further we find a biquadratic that was omitted in **V** (and **A**), and more examples involving roots of numbers ($\alpha C = n+\sqrt{\nu}$ being solved by taking the root of the right-hand members separately!). The four-member problem and the three problems involving roots of numbers all normalize by dividing *in*, where all other normalizations are *per*.



The two cases $\alpha K+\beta C+\gamma t = n$ and $\alpha CC+\beta K+\gamma C+\delta t = n$ are of a new kind. The rules are still false, but they are not copied from rules for second-degree cases – and they work for the examples that are given. The former example coincides with Jacopo's example (4a), with the difference that the 100 *libre* are lent for three, not two years – but the capital still grows to 150 *libre*, which leaves little doubt about the inspiration. In the latter example, 100 *libre* are lent for four years and grow to 160 *libre*. The rules, (complicated as they look because the *thing* is put equal to the interest in *denari* per month of one *libra*) appear to be constructed from the solutions that may be found from $\sqrt[3]{150/100}$ and $\sqrt[4]{160/100}$. The fraud is certainly more intelligent than that behind Gherardi's formulae – but it remains a fraud, and was probably recognized as such by its inventor (who was *not* the compiler of **P**).[23]

[23] There is not much reason to wonder that mathematicians would invent and publicize wrong formulae. As a rule, the authors of the *abbaco* texts were not "mathematicians" but teachers advertising and selling their proficiency in a free market, where cheating the customers (parents of future students or communal councils) successfully was just as efficient as convincing them honestly. The condition for successful fraud was not mathematical truth but the inability of competitors to unmask the deceit (whence the usefulness of solutions containing roots). Tartaglia's fortunes and misfortunes illustrate the point well.

Compilers of texts like **P** were probably quite unaware of the fraud; they merely repeated what they believed to be good algebra.

Lines of ancestry and descent

We have now come to the point where it is possible to construct an approximate stemma showing the connections between the various treatises discussed so far (the vertical axis corresponds to time, Jacopo writing in 1307, **G** being from 1328 and **V** from c. 1450). On top, we have Jacopo's original writing. **V´** is the hypothetical archetype for all the actual manuscripts – perhaps identical with Jacopo's original work.[24] **V´´** is the faithful copy from which **V** is made (cf. note 5 and preceding text). **A´´** is the common archetype for **A**, **L** and **C**, which must still have been very faithful to **V´** and can have contained none of the false rules, nor examples for the higher-degree rules. **C´** must be the common ancestor of **L** and **C** (since everything that is in **C** is also in **L** they are likely to have a common ancestor not very different from **C** but already free with respect to **A´´**). **A´** is a common ancestor to **A** and **G**, faithful to **V´** in the parts coming from Jacopo but already provided with examples for some of the higher-degree cases and false rules for some irreducible cases. **G´** is an ancestor to **G** from which **P** descends (the agreement of **P** and **A** in the case $\alpha K+\gamma t = \beta C$ appears to exclude direct descent of **P** from **G**). The extra cases in **P** involving square roots of numbers (and the striking agreement in their choice of division preposition) suggests that these has been borrowed from an unidentified source or area labelled **"?"** and not created between **G´** and **P** as generalizations of the case $\alpha K = \sqrt{n}$. It is likely that the latter problem (shared by **G** and **A**) has been adopted into **A´** from the same area.

Crosswise contamination is not to be totally excluded, but the distribution of shared versus particular features in the various treatises makes more than minimal importance of such influences unlikely. The stemma suggested here should hence be close to the truth.

This means, firstly, that everything written on algebra in Italian vernacular in the first generation after Jacopo depended on his work, with only a marginal influence from the "area **?**". This excludes the existence of an Italian environment practising algebra before his times. Jacopo must have gone abroad in order to find the discipline – and his whole treatise indeed suggests that he was very conscious of presenting knowledge that was *new* to his public. Secondly, since **A**, **L** and **C** are all written in Tuscan with no traces of non-Tuscan orthography,

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[24] But probably not if the hypothesis formulated in note 13 is correct, and the beginning of the algebra chapter has disappeared in transmission: **A** starts the chapter in question exactly as **V**.

even **A´´** and **A´** are likely to have been written in Tuscan area; if this is so, then Paolo Gherardi must have sought his inspiration in Italian writings[25] and found little or nothing of algebraic interest in Montpellier.[26] But if there was no environment practising algebra in Montpellier in 1328, there can hardly have been any in 1307.

This gives us no direct answer to the question concerning the localization of that Romance-speaking area from which Jacopo drew his knowledge of algebra. Indirectly, however, things begin to narrow down: if Italy and Provence are excluded, little beyond Catalonia remains – easily attained from Montpellier, and at the time involved in intense trading relations with the Arabic world as far as Egypt, and also an obvious channel for Ibero-Islamic influences.[27]

Maestro Dardi da Pisa

Dardi's *Aliabraa argibra*, apparently from 1344, is the first vernacular algebra that does *not* depend on Jacopo. It is also the earliest extant vernacular work devoted solely to algebra – and it is more than four times as long as the *Trattato dell'Alcibra amuchabile* from c. 1365.[28] Like Jacopo's treatise, it contains no single Arabism (unless we count the word "algebra" of the title as one).

Its structure is fairly similar to the first two sections of **A**. However, at first

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25 In the introductory passage [ed. Arrighi 1987: 15] he also presents himself as being from Florence.

26 Pure veneration for Jacopo can be excluded, since his name does not appear in Gherardi's treatise. Since Jacopo knows none of the false rules (according to the style of his work he would have mentioned it if he knew about them and understood them to be false), even they cannot come from Montpellier. Only the so far enigmatic "area **?**" could perhaps be Montpellier (but see below, p. 31).

27 It is worth observing in this connection that the semantic distinction between "partire in" and "partire per" (see note 21) is still fairly present in Francesc Santcliment's Catalan *Summa de l'art d'aritmètica* from 1482 [ed. Malet 1998]. Thus, fol. 27v, "digues: que partisses 589 en 6 parts", versus fol. 32r, "no es nenguna altra cosa partir per 25, ho per 35 ho 57 ho 77 [...] sino partir per 12 ho per 19".

28 I used the Vatican manuscript Chigi M.VIII.170 from c. 1395 (**D$_1$**), together with Raffaella Franci's edition [2001] of the Siena manuscript I.VII.17 from c. 1470 (**D$_2$**) – both datings based on watermarks and according to [Van Egmond 1980]; the former is in Venetian, the latter in a Tuscan dialect. See also the description in [Van Egmond 1983], and Hughes' account of a newly discovered manuscript [1987]. I thank Raffaella Franci for supplementary information on **D$_2$** and for discussions.

comes an introduction and an index listing all 194+4 cases to be dealt with.[29] The sign rules of **A** are missing – but Dardi proves[30] when arriving to the point where it is first needed that "meno via meno fa più" (using the example (10–2)·(10–2)). Instead the index is followed directly by a "Treatise on the rules which belong to the multiplications, the divisions, the summations and the subtractions of roots".[31] Then comes a presentation of the six fundamental cases, with geometric demonstrations (**A** has nothing similar), and finally a presentation of 194 "regular" and 4 "irregular" cases, all with rule and example (at times several examples). The distinction regular / irregular is made in the introduction; a note to the index instead distinguishes between cases governed by general respectively non-general rules.

In $\mathbf{D_1}$, the following abbreviations are made use of consistently: *ç* for *censo*, *c* for *cosa*, *nũo* for numero, ℞ for *radice*, *m̃* for *meno*; the notation for multiples of *ç* and *c* emulates that for fractions, writing the "denominator" below the "numerator" with a stroke in between – for instance, $\frac{10}{c}$ for "10 things". *ç*, *c* and ℞ are also used in $\mathbf{D_2}$, but the fraction-like notation not; whether it was used in Dardi's original or introduced by a copyist thus remains an open question.[32]

Chapter 1: calculating with roots

In the chapter on roots, we find diagrams illustrating the multiplication of binomials similar to those in **A** – for instance, for (3–√5)·(3–√5),[33]

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[29] The index is absent from $\mathbf{D_2}$, but the introduction promises to bring it and leaves three empty pages – the obvious intention being to insert it once the equally promised corresponding folio numbers were known. In $\mathbf{D_1}$, the introduction and the first page of the index is missing, and the first folio number is 2.

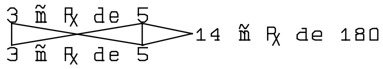
[30] $\mathbf{D_2}$ p. 44; $\mathbf{D_1}$ fol. 5^v.

[31] $\mathbf{D_2}$ p. 38; $\mathbf{D_1}$ fol. 3^v.

[32] In general, however, $\mathbf{D_1}$ is not only much closer to Dardi in time than $\mathbf{D_2}$ but also closer to their common archetype in various respects (apart from its Venetian dialect). One example is the reference to the rule of three in the passage of $\mathbf{D_1}$ quoted in note 34 and the absence of the reference in $\mathbf{D_2}$; since $\mathbf{D_2}$ cites it when referring backwards to the passage, it must have been present in the original. Another is the use of the term *adequation* in $\mathbf{D_1}$, corresponding to *dequazione* in $\mathbf{D_2}$; they are indistinguishable in the definite form *ladequation* / *ladequazione*, which explains that one of the manuscript has misunderstood the intended term of the original; but in one place (p. 77) $\mathbf{D_2}$ has an unexpected and indubitable *adequazione*.

Globally, the differences between the two manuscripts are modest.

[33] $\mathbf{D_2}$ p. 45; $\mathbf{D_1}$ fol. 6^r.



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We notice that Dardi's diagram is fuller than that of **A**, which makes it implausible that **A** should have simply borrowed from **D**.

When looking at the explanation of how to divide a number by a binomial we find greater differences.[34] In order to divide 8 by 3+√4, Dardi first makes the calculation (3+√4)·(3–√4) = 5 and concludes that 5 divided by *3+√4* gives *3–√4*. *What*, he next asks, will result if *8* is divided similarly and finds the answer by application of the rule of three.

Chapter 2: the six fundamental cases

The chapter proving the correctness of the second-degree rules has no counterpart in **A**, nor in any of the other Italian treatises discussed so far. The demonstrations descend from those found in al-Khwārizmī's algebra, but their style is as different as it would be if somebody not versed in the received conventions governing the use of letters in geometric diagrams were to relate al-Khwārizmī's proofs from memory to somebody not too well versed in geometry. As an illustration (which should speak for itself as soon as it is confronted with any version of al-Khwārizmī's text) I translate the beginning

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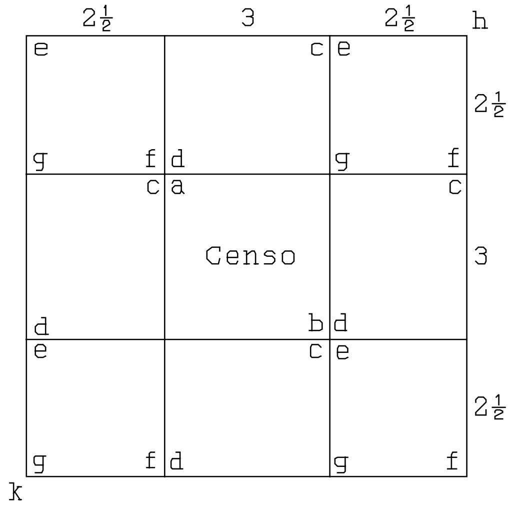
34 I render the text of $\mathbf{D_1}$ (fol. 12v); punctuation and diacritics have been adjusted / added; words in ⟨ ⟩ are corrections of copyist's omissions inserted between the lines in a different hand (as is evident from the presence of the same words in $\mathbf{D_2}$):

> Se tu volessi partir nũo in ℞ e nũo, serave a partir 8 in 3 e ℞ de 4, tu die moltiplicar 3 e ℞ de 4, che monterà 5. Adonqua a partir 5 in 3 e ℞ de 4 te ne vien 3 m̃ ℞ de 4 perché ogne nũo moltiplicado per un'altro nũo, la moltiplication che ne vien partida per quel nũo si ne vien l'altro nũo moltiplicado per quello. Adunqua partando 5 in 3 e ℞ de 4 si ne vien 3 m̃ ℞ de 4, e partando 5 in 3 m̃ ℞ de 4 si ne vien l'altra parte, zoè 3 e ℞ de 4, e inperzò diremo che questo 5 sia partidor, e metteremo questo partimento alla regla del 3, e diremo, se 5, a partir in 3 e ℞ de 4, ne ven 3 m̃ ℞ de 4, che ne vegnirà de 8, e moltiplica 3 m̃ ℞ de 4 via 8, che monta 24 m̃ ℞ de 256, la qual moltiplication parti in 5, che ne vien $4\frac{4}{5}$ per lo nũo. Ora resta a partir ℞ de 256 ⟨meno⟩ in 5, che ne vien ℞ de $10\frac{6}{25}$, che a partir ℞ in nũo el se die redur lo nũo a ℞ , zoè lo 5 redutto in ℞ monta ℞ de 25. E così avemo che a partir 8 in 3 e ℞ de 4 si ne vien $4\frac{4}{5}$ men ℞ de $10\frac{6}{25}$.

$\mathbf{D_2}$ omits the explicit mentioning of the rule of three, but it turns up in a later backward reference to the calculation (p. 62, corresponding to $\mathbf{D_1}$ fol. 14r); it thus belongs to the common archetype.

of the first proof verbatim (repeating the grammatical inconsistencies of the text),[35] reproducing also the first diagram :



> How 1 *ç* and 10 *c* are proved to be equal to 39. Since the *c*, which is said to be ℞ of the *ç*, the *ç* now comes to be a quadrangular and equilateral surface, that is, with 4 corners and four equal and straight sides. Now we shall make a square with equal sides and right corners, and we shall say that the *ç* is its surface, which is *ab*, and since the *c* is the ℞ of the *ç*, it comes to be the sides of the said square, and since to the *ç* $\frac{10}{c}$ are added, we divide this $\frac{10}{c}$ in 4 parts, which comes to be $\frac{2}{c}\frac{1}{2}$ each, and since the *c* comes to be the sides of the *ç*, we shall place each of these four parts along *ç*, each along its own side of *ç*, the surface of each being *cd*, and outside each of the corners of *ç* falls an equilateral quadrangle with right corners, which as side will have the breadth of the *c*, that is, 2½, which breadth, or length, multiplied by itself amounts to 6¼, that is, *ef*, [...].

A closer look at some textual details reveals that the chapter has been adopted from the same environment as Jacopo's algebra (which was not a priori to be expected, given that Jacopo brings no geometric proofs). Dardi's rule for the fifth case runs as follows in $\mathbf{D_1}$ (fol. 16r, emphasis added; $\mathbf{D_2}$ similarly, p. 66):

> Quando li *ç* e'l numero e equali ala *c*, el se die partir tutta l'adequation per la quantità dei *ç*, e pò partir le *c* in 2, e una de queste mità, zoè la quantità de una de queste parte, moltiplica in si medesima, e de quella moltiplication trazi lo numero e la ℞ de quello che roman zonzi all'altra mità dela quantità dele *c*, e tanto vegnirà a valer la *c*, e *sappi che in algune raxon te convegnirà responder esser la c per lo primo modo, zoè la mità dela quantità dele c più ℞ de quello che roman, e algun fiade per lo secondo modo, zoè la mità dela quantità dele c m̃ la ℞ de quello che roman, e algune se pò responder per tutte e 2 li modi, com'io te mostrerò.*

Jacopo's corresponding rule (fol. 39v) is not very similar (except, by necessity, in mathematical substance):

> Quando le cose sonno oguali ali censi et al numero, se vole partire nelli censi, et poi dimezzare le cose et multiprichare per se medesimo et cavare el numero, et la radice de quello che romane, et poi el dimezzamento dele cose vale la cosa. Overo el dimezzamento dele chose meno la radice de quello che remane.

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35 $\mathbf{D_2}$ pp. 68*f*; $\mathbf{D_1}$ fols 16v–17r.

However, when Jacopo comes to present the double solution of example (5b), we find the following passage (fol. $40^{\text{r–v}}$, emphasis added):

> Siché tu vedi che all'uno modo et all'altro sta bene. Et però quella così facta regola è molto da lodare, che ce dà doi responsioni et così sta bene all'una come all'altro. Ma *abbi a mente che tucte le ragioni che reduchono a questa regola non si possono respondere per doi responsioni se non ad certe. Et tali sonno che te conviene pigliare l'una responsione, et tale l'altra. Cioè a dire che a tali ragioni te converà rispondere che vaglia la cosa el dimezzamento dele cose meno la radice de rimanente. Et a tale te converrà dire la radice de remanente e più el dimezzamento dele cose.* Onde ogni volta che te venisse questo co'tale raoguaglamento, trova in prima l'una responsione. Et se non te venisse vera, de certo si piglia l'altra senza dubio. Et averai la vera responsione.

The similarities between the two italicized passages are too particular to allow explanation merely from shared general vocabulary and style. However, several reasons speak against Dardi copying directly from Jacopo's text: not least the total absence of shared examples and of anything similar to the Jacopo's *fondaco* problems from the *Aliabraa argibra*. Moreover, *if* Dardi had found the italicized passage in Jacopo interesting and moved it to the rule (because the examples he promises only come in the following chapter), he would not have changed its finer texture as seen in the excerpt;[36] nor would he have had any reason to invent the term *adequazione* in replacement of *raoguaglamento* if using Jacopo's treatise. In consequence, Dardi must have drawn his inspiration for this chapter from the very environment which Jacopo had once drawn on. And he must have kept fairly close to his direct source: only too faithful copying explains the sudden appearance of "78 dramme, zoè numeri" in the example illustrating the fourth case ($\mathbf{D}_1$, fol. 16^{r}, similarly $\mathbf{D}_2$, p. 65) – up to this point, all numbers have been nothing but *numeri*.

Chapter 3: 194+4 regular and irregular cases

As mentioned, the final chapter presents 194 "regular" cases with rules, only a small selection of which are listed in the scheme on p. 18. A very large part of them involve radicals, not only roots of numbers but also of *things*, *censi*, *cubi* and *censi di censo* – thus, for instance, no. 59, $\alpha t = \sqrt[3]{\beta C}$, and no. 123, $\sqrt{\beta t} + \sqrt[3]{n} = \alpha t$ (notation as in the scheme). All are solved correctly (apart from two slips,

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36 When we are able to compare Dardi's text with another one deriving from the same source, as Dardi's first irregular case with the corresponding case in **P** (see presently), Dardi can be seen to change at most the wording of the single phrases while conserving their order and mutual relation (but since **P** is later and hence more likely than **D** to have changed with regard to the original source, Dardi may well be even more faithful).

convincingly explained in [Van Egmond 1983: 417], and all provided with an illustrative example (at times two or, with rules allowing a double solution, three examples). All are pure-number problems, almost half of them of the fraudulently complicated type asking for two or three numbers in given proportion; a good fourth asks for a single number fulfilling conditions fashioned in agreement with the equation type, some 15 percent deal with a divided ten. The order of the six fundamental cases is the same as in the other treatises we have looked at, which is likely to be significant. Even the order of the next three cases coincides with that of Jacopo – but since these are simply the simplest higher-degree cases (cubes equal to number / *things* / *censo*), this agreement is not significant. After that, Dardi's order is wholly his own.

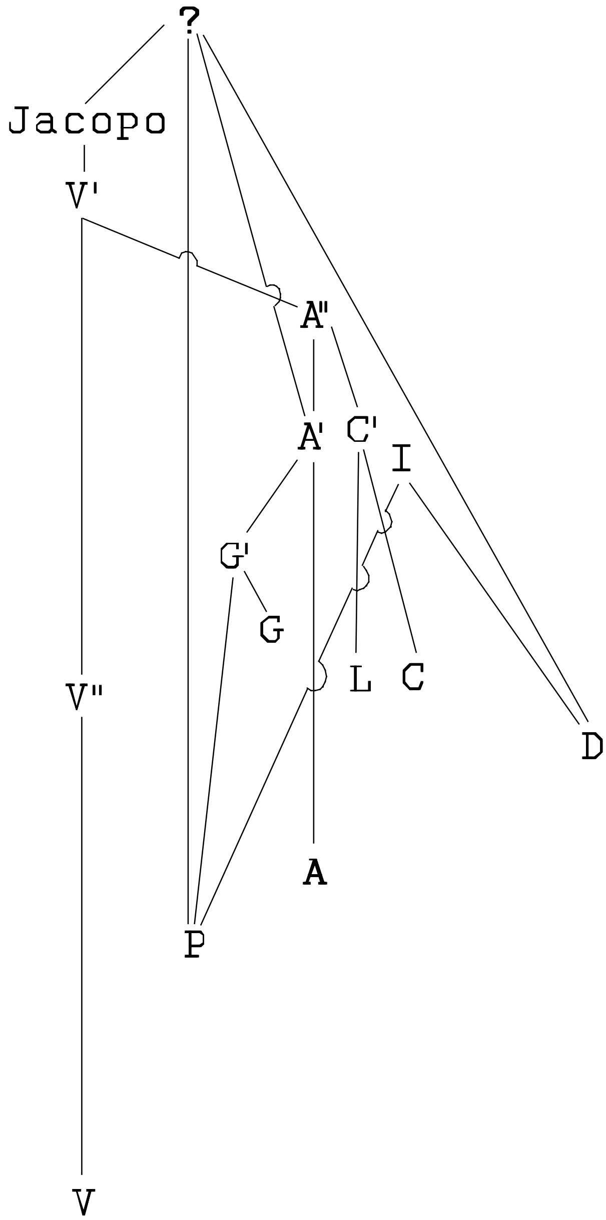
The four "irregular" cases are inserted between the regular cases no. 182 and 183, after a note pointing out that all equations up to this point contain no more than three members. In contrast, the regular cases from no. 183 onwards all correspond to four-member equations. The irregular cases are presented at this point as "adapted solely to their problems, and with the properties these possess"[37] but included all the same because they may turn up in certain problems. This, and their separate numbering, suggests that Dardi has adopted the group wholesale and inserted it into the main body of his treatise. The character of the examples supports this inference. Two of them (no. 1 and no. 2) are strictly identical with examples (24) and (25) from **P**, which means that they are the only problems in Dardi's treatise that do not treat of pure numbers (but of lending with interest, as we remember), and that they are directly inspired by Jacopo's example (4a). The other two, $\alpha t+\beta C+\gamma CC = n+\delta K$ and $\alpha t+\gamma CC = n+\beta C+\delta K$, are based on the divided ten; had it not been for their constituting a closed group together with the former two, they could have been Dardi's invention; as things actually stand, this is unlikely.

Dependency or independence

Dardi's many rules involving radical and roots of numbers shows him to share in the inspiration coming from "area **?**". They do not tell whether he only received general inspiration and used that as a starting point for something going far beyond what his source tradition had made, or he borrowed in large scale.

[37] "[...] reghulati solamente alle loro ragione, e di quelle proprietà delle quale elle sono ordinate" ($\mathbf{D_2}$ p. 269; similarly $\mathbf{D_1}$, fol. 102[r]).

Some details in the chapter on roots suggests dependency on a model,[38] and the importance of a model for several features of the presentation of the 6 fundamental cases was already discussed. But the main body of the last chapter, the regular cases 1–194, may still have been structured by Dardi. Of the single cases, quite a few had been dealt with before, as we have seen, and Dardi may plausibly have known about that, just as he knew about the way to construct pseudo-complex examples by asking for numbers in given proportion (while copying no examples directly from the known predecessors, neither Jacopo nor Gherardi); yet no evidence contradicts a conjecture that most were devised by Dardi.



The principle of creating new algebraic cases involving roots, as argued, was inspired from the unidentified "area **?**". For the use of diagrams, Dardi seems to have shared a common inspiration with **A**; **A** and **G** (and hence their shared archetype **A´**) make use of the related calculations with formal fractions. Finally, the order

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[38] Thus, a number of procedures are illustrated by polynomials containing rational roots (e.g., 36/(√4+√9+√16), treating them *as if* they were surds ("intendando de queste ℞ discrete como s'elle fosse indiscrete" – $\mathbf{D_1}$, fol. 3^v, similarly $\mathbf{D_2}$ p. 62), the obvious point being that this allows control of the correctness of the result; however, no proof is made, nor is any other advantage taken of the choice of rational roots, except an unproven statement that the result coming from the calculation (in the exemple $\sqrt{40\frac{24}{25}}+\sqrt{92\frac{4}{25}}+\sqrt{5\frac{19}{25}}-\sqrt{163\frac{21}{25}}-\sqrt{10\frac{6}{25}}$) *can* be reduced. Skipping a proof when copying (or using a model where it has been lost in transmission) may easily happen; but that the author prepares it repeatedly and then himself omits it each time is less likely.

of the fundamental cases, the discussion of the double solution to the fifth case and the use of the rule of three as an algebraic tool shows affinity with Jacopo, while, as we have seen, the details of Dardi's text speak against direct borrowing; even Jacopo and Dardi hence share a source of inspiration.

Occam's razor is a dangerous weapon – wielding it was what led to the assumption that *abbaco* algebra had to come from Fibonacci. But ad hoc multiplication of explanatory entities beyond what is needed remains gratuitous, and a reasonable working hypothesis is that all these unidentifiable sources of shared inspiration belong to the same area – that is, our "area **?**" (in which case this area can hardly be Montpellier itself). The only extra entity that we are forced to accept appears to be the one which, in the wake of the success of Jacopo's higher-degree cases, invented **P**'s and Dardi's irregular cases – which we may designate **I**. These various observations cause the addition of new elements and links to our stemma, without changing anything (except the age ascribed to "area **?**") in what was already drawn up.

Summing up

The existence of the "area **?**" followed from indirect arguments and, as far as its being a single area is concerned, from plying Occam's razor. However, the fact that several of the lines connecting **"?"** with known Italian writings in the revised stemma represent multiple inspirations (for instance, **V´** and **D** having in common the order of the basic cases, the way the double solution to the fifth case is spoken of, and the use of the rule of three as an algebraic method), rejection of the assumption of one unitary area of inspiration would force us to accept that each author belonging to the first generation of Italian vernacular algebra was inspired by several or all of a multiplicity of direct sources – a multiplicity of Romance-speaking sources, moreover, given the absence of Arabisms in the texts.

Since the only Romance-speaking area outside Italy where the next 150 years offers any evidence of algebraic interest is the Provençal-Catalan region, and since Montpellier itself appears not to have been a rich source, it seems reasonable to conclude that the "area **?**" was indeed one area, to be identified with or located in the Catalan region (see also note 27 and preceding text).

Within this area, most of that by which the first generation of Italian algebra goes beyond al-Khwārizmī will already have been known either fully unfolded or in germ: polynomial algebra, the use of diagrams, the beginnings of formal computations. The easy way to create problems looking more complex than they are may have originated here together with the interest in equations involving roots of numbers and perhaps other radicals. The carrying environment is likely to have been close to the teaching of commercial mathematics, given the generalized use of the rule of three and of the partnership structure and the preponderance of *muʿāmalāt* problems in **V**. Only the invention of false rules for the irreducible higher-degree problems seems to be a local Italian development (the cheap imitations of the second-degree rules as well as the rules valid for special cases only).

Quite independently of this we may notice that the points where the first generation of Italian vernacular algebras goes beyond al-Khwārizmī were to become centrally important when, in Karpinski's words, two centuries of *abbaco* algebra "bore fruit in the sixteenth century in the achievements of Scipione del Ferro, Ferrari, Tartaglia, Cardan and Bombelli": *viz* polynomial algebra, schematic number diagrams, the use of standard abbreviations in formal operations preparing the genuine symbolic operations of Descartes – and even the thirst for solutions to irreducible higher-degree problems notwithstanding the fraud it had led to. The mathematical competence of a Jacopo and a Paolo Gherardi and even a Dardi will plausibly have been well below that of Fibonacci, and many of the *abbaco* teachers may hardly deserve a characterization as "mathematicians"; but collectively they were the ones who prepared the algebraic take-off of the sixteenth century and that whole transformation of the mathematical enterprise which it brought about in the seventeenth and eighteenth centuries.

Sigla

A: Florence, Riccardiana Ms. 2263, fols 24^{r}–50^{v}. Anon., *Trattato dell'Alcibra amuchabile*. [Ed. Simi 1994].

C: Lucca, Biblioteca Statale, Ms. 1754, fols 50^{r}–52^{r}. Anon., "Le reghole della cosa". [Ed. Arrighi 1973].

D^{1}: Vatican Library, Chigi M.VIII.170, fols 2^{r}–114^{r} (original foliation). Dardi da Pisa, *Aliabraa argibra*.

D^{2}: Siena, Biblioteca Comunale, I.VII.17. Dardi da Pisa, *Aliabraa argibra*. [Ed. Franci 2001].

F: Florence, Riccardiana Ms. 2236. Jacobo da Firenze, *Tractatus algorismi* (abridged). [Ed. Simi 1995].

G: Florence, Magliabechiana Cl. XI, 87, Paolo Gherardi, *Libro di ragioni*. [Ed. Arrighi 1987; Van Egmond 1978 (partial)].

L: Lucca, Biblioteca Statale, Ms. 1754, fols 81^{r}–82^{v}. Anon., "Le reghole dell'aligibra amichabile". [Ed. Arrighi 1973].

M: Milan, Trivulziana Ms. 90. Jacobo da Firenze, *Tractatus algorismi* (abridged).

P: Parma, Biblioteca Palatina, Ms. Pal. 312. Anon., *Libro di conti e mercatanzie*. [Ed. Gregori & Grugnetti 1998].

V: Vatican Library, Vat. Lat. 4826. Jacobo da Firenze, *Tractatus algorismi*. [Ed. Høyrup 2000 (partial)].

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