# Bronze Age formal science?

## With additional remarks on the historiography of distant mathematics

Høyrup, Jens

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# BRONZE AGE FORMAL SCIENCE?
## *With additional remarks on the historiography of distant mathematics*

JENS HØYRUP

**FILOSOFI OG VIDENSKABSTEORI PÅ ROSKILDE UNIVERSITETSCENTER**

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In memoriam Robert Merton
1910–2003

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The paper was prepared during a stay at the Max-Planck-Institut für Wissenschaftsgeschichte,Berlin. I use the opportunity to express my sincere gratitude for the hospitality I received.

Referee: Aksel Haaning

My talk falls in two unequally long parts, each turning around a particular permutation of the same three key words:*

– The first, longer and main part treats of *past understandings* of *mathematics*.
– The second, shorter part takes up *understandings* of *past mathematics*.

In both parts, the "past" spoken of focuses on the Near Eastern Bronze Age, but other pre-Modern mathematical cultures also enter the argument.

## *I. Past understandings of mathematics*

In any proper sense, a "formal science" is a science which does not positively tell us anything about the world, a glove that fits any possible hand. Understood thus, the conception of mathematics as formal science was not urgent before, say, the acceptance of non-Euclidean geometry or the reception of Hilbert's *Grundlagen*, and hardly possible before Kant's definitive formulation of the distinction between analytic and synthetic propositions.

Etymology, of course, only tells us what words do not mean any longer. In the present case, however, the derivation of "formal" from "form" (and ultimately from εἶδος, of which this Latin term is a loan-translation) is worth taking into account. This allows us to trace explicit formulations of precursor ideas back to classical Antiquity.

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## *Aristotle and others*

According to Aristotle, “the mathematicals” (τὰ μαθηματικά – i.e., geometrical shapes, numbers, and ratios) are *properties* of real-world objects or collections of objects. These properties are isolated through a process of “abstraction” or “removal” of other aspects of the same objects or collections.[1] In *Metaphysics* M, chapters I–III, the term used is ἀφαίρεσις, “taking away”, and it depends on the discipline which applies its perspective whether physical or mathematical properties are the essential ones (the others being “accidents”); from the point of view of geometry, the process that brings forth the sphere from the bronze sphere is thus (in modern terms) an objective one. In *Physics* II, the term is χωρίζω, “to separate”, and the process is performed “according to thought” (τῇ νοήσει), the physicality of the object remaining in any case essential – perhaps because the topic of the whole work is, exactly, *physics*.

Whether objective or subjective, this bringing-forth the mathematicals by a process of abstraction does not suggest that mathematics – posterior, not prior to actual reality – be in any way a “formal” science. But there is more to it. As pointed out in *Metaphysics* B, 998[a]2–5 [trans. Tredennick 1933: I, 115] “the [sensible] circle touches the ruler not at a point but [along a line] as Protagoras used to say in refuting the geometricians)”; if the mathematicals were mere properties as we understand that term, it seems strange that a sensible line and a sensible circle should touch each other along a (sensible) line but their corresponding mathematical properties only in a point (which then, seemingly, should be the mathematical property of the sensible line of touching). Moreover, in *De caelo* II.14 (297[b] 21–23, ed. [Guthrie 1939: 250][2]) we find that

> the earth is either spherical, or it is spherical according to nature, and one should say each thing to be such as it professes to be according to nature and subsistence, not however as it is by violence or against nature.

Sphericality – clearly a mathematical property – is thus “by nature”. A nature, however, is what we might term an *ideal* (and essential) property, an aim or τέλος which may not be achieved completely (or not at all). As explained in *Physics* II (199[a]32-199b4, trans. [Charlton 1970: 41], with a slight correction in ⟨ ⟩):

> Mistakes occur even in that which is in accordance with art. Men who possess the art of writing have written incorrectly, doctors have administered the wrong medicine.

[1] See [Høyrup 2002b] for substantiation of the following condensed observations and for references to earlier work.

[2] As everywhere in the following where no translator is indicated, the translation is mine.

> So clearly the same is possible also in that which is in accordance with nature. If it sometimes happens over things which are in accordance with art, that that which goes right is for something, and that which goes wrong is attempted for something but miscarries, it may be the same with things which are natural, and monsters may be ⟨failures⟩ at that which is for something. [...]

Finally, there is an interesting passage in the *Posterior Analytics* I, 79ª6–10 (trans. [Tredennick & Forster 1960: 91], corrections in ⟨ ⟩):

> Of this kind [viz, studied by more than one science] are all objects which, while having a separate substantial existence, yet exhibit certain specific forms. For the mathematical sciences are concerned with forms; they do not confine their demonstrations to a particular substrate. Even if geometrical problems ⟨treat of⟩ a particular substrate, at least they do ⟨not do so *qua* treating of a substrate⟩.

This passage, like *Metaphysics* M, presupposes that it depends on the discipline investigating an object what constitutes its *form*; if geometry is that discipline, then its sphericality may be its form *even if* (according to the *De Caelo*-passage, and like "substantial" forms) *that form is not achieved completely*. The same mathematical form, on its part, may apply (together with everything that holds for a geometrical sphere) to all spherical objects irrespective of their remaining properties (their form or nature according to the perspective of physics, as well as their accidents). All in all, we are not far away from what could reasonably be described as a formal view of mathematics.

This view remained familiar to all those who grew up intellectually with Aristotelianism between the twelfth and the seventeenth century, and has to do with the Scholastic endorsement of the Plotinian just-mentioned "substantial forms"[3] – which presupposes that objects may possess other forms that do not define them as substances, e.g., the mathematical form of a bronze sphere, the eighty-ness of a flock of sheep.[4] The view is more unexpected for those who have grown up with the modern standard view of Platonism and Aristotelianism, whom it might give the impression that Aristotle's view of mathematics is, after all, closer to Platonism than we would expect; if we take a closer view of Aristotle's various refutations of what he claims to be Plato's

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[3] As discussed by Anderson [1969], Thomas Aquinas takes the existence of mathematical form (and mathematical matter) quite seriously.

[4] Mathematical forms, however, were not the only non-substantial forms discussed by St. Thomas and other schoolmen.

view of numbers (ideal as well as mathematical and sensible[5]) and accept that Aristotle was better informed than we poor readers of his dialogues will ever be, the reverse conclusion may seem more plausible: that the supposedly "Platonist" view of geometrical shapes etc. as ideals which the sensible world emulates imperfectly is a post-Renaissance construction resulting from the superimposition of the Aristotelian view on Plato's reminiscence theory as it can be read from the dialogues mixed up with the "Platonism" of Plotinus and Porphyrios, already formulated in Aristotelian terms. Genuine Platonism may have been further removed from regarding mathematics as a formal science – but whether it really was is difficult to know.

## *Scribal cultures I: Middle Kingdom Egypt*

It may seem even more difficult to grasp what the anonymous scribes of the Near Eastern Bronze Age thought about the mathematical techniques they were using. Yet whereas Plato left no mathematical writings (and no Greek mathematical writings survive that can be unquestionably dated to his or earlier epochs), then we know some of the materials by which the scribes were taught.

Let us first examine the situation in Middle Kingdom Egypt. The most important mathematical source is without doubt the Rhind Mathematical papyrus – according to its colophon a copy of an original written under Amenemhet III (c. 1844–1797 BCE). Its introductory passage [trans. Clagett 1999: 122] promises nothing like a distinction between mathematical and other kinds of knowledge:

> Accurate reckoning [or Rules for reckoning] for inquiring into things, and the knowledge of all things, mysteries ... all secrets.

What follows, however, treats of neither "mysteries" nor "all things"[6] but only mathematical calculation.[7] At first comes a tabulation of the solutions to the

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5 E.g., *Metaphysics* N, 1090$^b$27-35 [trans. Tredennick 1933: II, 281], "no mathematical theorem applies to [ideal numbers], unless one tries to interfere with the principles of mathematics and invent particular theories of one's own"; further, those who invented "two kinds of number, the Ideal and the mathematical as well, neither have explained nor can explain in any way how mathematical number will exist and of what it will be composed".

6 Admittedly, the words of the introductory phrase can be translated in different ways – according to various translators it speaks of "secrets", "obscurities" or "mysteries"; but there is no doubt as to the general tenor.

7 A few non-mathematical entries in end of the copy, counted as #85–87, did not belong in the original – cf. [Peet 1923: 128*f*].

problem to express 2 as a sum of aliquot parts of *n*, *n* being an odd number between 3 and 101 – a table which serves as a calculational aid for all that follows. Then comes a tabulation of the numbers 1–9 divided by 10, also (as all fractional numbers in Middle Kingdom and later Pharaonic mathematics) expressed as sums of aliquot parts; problems applying this table to the division of *n* loaves among 10 men, *n*=1,2,6, 7, 8 and 9; and a sequence of problems dealing with pure numbers or indeterminate "quantities". #35–84 treats of concrete computations involving the various metrological systems, a geometry section (#41–60) presenting also the standard rules for determining triangular, rectangular and circular areas and the volumes of rectangular and circular prisms. Sacred or magical uses of numbers, on the other hand, are excluded, though certainly not absent from Egyptian scribal wisdom in general.

The territory that is covered is thus the whole of mathematics understood as the determination of concrete quantities together with the purely arithmetical and geometrical tools for this.[8] Expressed in global terms, the cognitive territory delimited by the papyrus corresponds to the Hellenistic-Biblical phrase "in number, measure, and weight" (which, through Augustine and Isidore, was to remain the most common characterization of the mathematization of the world until the sixteenth century). The organization of the text – first all-purpose computational aids, then preparatory pure-number- and pure-quantity-problems, then problems dealing with measurable entities) shows that arithmetic was really understood as a glove that could be fitted unto any determination of a quantity.

If we believe the introductory phrase to be sincere when claiming to deal with "all things", then we may conclude that the author-scribe assumed *everything* to be made according to "measure, number, and weight", and that his text was meant to present in the beginning something like a body of formal knowledge which was next to be applied to everything – or at least to everything falling within a scribe's horizon. We know, however, that a professional scribe (but not necessarily a teacher of scribal mathematics) was supposed to deal with matters that could not be reduced to quantitative measure; if not necessarily for the author of the text then at least for scribes in general we must therefore assume that the body of "formal" knowledge encompassing arithmetic and geometric rules was only supposed to be generally applicable to a restricted area. This area

[8] This conclusion also fits the Moscow Mathematical Papyrus [ed. Struve 1930], although this second major source for Middle Kingdom mathematics (a copy of a collection of problems with solutions, apparently some kind of corrected examination paper) does not on its own entail quite as far-reaching conclusions.

constitutes what we usually refer to as “Egyptian” (or better, “Pharaonic” mathematics). We do not know how it was spoken about (though “reckoning” / *ḥsb* is a fair guess); but there is no doubt that it was an Egyptian concept, not a splitting of the Egyptian cognitive world imposed artificially from outside. Students of ethnomathematics may be right that mathematics is *our* concept when applied to non-literate cultures – but it covers the thinking of Middle-Kingdom scribes quite well.

### *Scribal cultures II: Old Babylonian epoch*

From Old Babylonian Iraq (2000–1600 BCE) we possess no global presentation of mathematics similar to the Rhind Papyrus – Old Babylonian mathematics was much too extensive to be contained in a single or a couple of clay tablets, and so far archaeologists have not stumbled on an undisturbed library which might inform us in a similar way.

The sources we have fall in three groups:

1. Table texts: tables of reciprocals and multiplication (and squares etc.); tables for metrological conversion; and tables of technical constants.
2. Tablets for rough mathematical work – at least as a rule student exercises.[9]
3. Problem texts.

The problem texts can be grouped in two intersecting ways. Firstly, they may tell the procedure to be used (“procedure texts”), or they may give the statement only and perhaps the solution (“catalogue texts”); on the other hand, they may contain a collection of sundry problems (“anthology texts”); a sequence of systematically organized problems belonging to a single domain (“theme texts”); or a single or a couple of problems. Only the second division concerns us here (but catalogue texts are always theme texts).

The table texts contain nothing but a single or combined table, a fact from which little can be concluded.[10] The tablets for rough mathematical work

[9] That these have to be singled out as a separate group was pointed out in recent years by Eleanor Robson – see, e.g., her [1999: 246–251] and [2000: 23–30]. The tablets from one sub-group contain pure numerical computations, the others a geometric figure (a triangle, a square with diagonals, etc.), with some pertinent numbers; specimens belonging to the last group are to be found in [Neugebauer & Sachs 1945: 42–44].

[10] In contrast, however, a Kassite or Middle Babylonian table (perhaps c. 1200 BCE) of technical constants occurs together with musical matters [Kilmer 1960], and a late Babylonian text (W 23273, perhaps c. 500 BCE) combines a metrological table with a list of the sacred numbers of the gods [Friberg 1993: 400].

regularly carry numbers on one face and a writing exercise (a proverb) on the other; this corresponds to a modern student's use of a single notebook for all subjects, and tells us something about school organization: firstly, that the same students were learning to write Sumerian and to make mathematical computations; secondly, that basic arithmetic was learned at a moment when literary education was quite advanced.

For the moment we shall leave out of the picture the contents of the problem texts, and hence also the single-problem texts as a group. Most informative in the first instance is the organization of anthology- and theme texts.

Anthology texts, as explained, contain mathematical problems from different mathematical domains, in which respect they are similar to the Rhind and Moscow papyri. Like these, moreover, they contain *nothing but* mathematics (understood as the determination of numerical or measurable quantities). Theme texts were evidently even more restricted in scope. Mathematics *in toto* was thus a self-contained cognitive territory.

The anthology texts contain nothing but problems, nothing like the initial table and pure-number introduction of the Rhind Papyrus – tables were produced and arithmetical exercises were performed separately. However, the corpus of mathematical problem texts (anthology texts, theme texts and single-problem texts together) demonstrates to be linked specifically to the corpus of tables in a different way, namely through its number notation. All mathematical texts employ the same sexagesimal place value system as the tables. This was a floating-point system, and thus only of any use in contexts where the order of magnitude could be presupposed. It could not and did not serve in accounting or other economical texts – it would not allow a judge to decide whether a debt was $^1/_6$ shekel of silver or 360 shekel. Its original practical use will have been for intermediate calculations (just like the engineers' use of the slide rule, also floating-point); but it is also employed in all mathematical problem texts.[11] Together with the stock of formulas for geometric calculation, sexagesimal place-value arithmetic was thus a framework of formal knowledge that held together the cognitive territory of "Old Babylonian mathematics".

All this is not very different from what we could derive from the organization of the Egyptian mathematical papyri. But in the present case a look at what has been called "Old Babylonian algebra" may bring to light another level.

[11] Some of the texts, in order to emphasized their (often sham) real-world character may express a datum or translate a final result into non–place-value notation; but such isolated instances of "real-life" notation are nothing but token references.

The traditional reason for using this expression is that a certain class of problems and problem solutions can be translated more or less homomorphically into modern equation algebra (at least as well as the theorems of *Elements* II.1–10 can be translated into algebraic identities), and that the technique they make use of was therefore believed initially to *be* a numerical technique of the same kind. This turns out on closer inspection to be a misconception.

Actually, the technique is geometrical – more precisely, based on what we might characterize as a geometry of measured and measurable lines in a rectangular grid.[12] As a first step in the argument we may look at a very simple problem, BM 13901 #1:[13]

**Obv. I**

**1.** The surface and my confrontation[14] I have accumulated:[15] 45´ is it. 1, the projection,[16]

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12 See, for example, [Høyrup 2002a].

13 Translation and discussion [Høyrup 2002a: 50–52]. I use a "conformal translation", firstly, in order to avoid a reading through modern mathematical concepts, secondly, so as to distinguish operations or concepts which were kept apart by the Babylonians even though the traditional arithmetical interpretation conflates them (the two different "additions" found in the present text, two different "subtractions", four different "multiplications", and two different "halves"). Numbers are transliterated according to Thureau-Dangin's generalization of the degree-minute-second system, ´, ˝, ... indicating decreasing, `, ``, ... increasing sexagesimal order of magnitude and ° "order zero" (1°15´ thus stands for 1¼, 1` for 60). In the interest of readability I have omitted the square brackets indicating damages to the text – no reconstructions are subject to doubt in the texts I quote.

14 As seen by the Babylonians, a square configuration (*mitḫartum*, literally "[situation characterized by a] confrontation [of equals]") was numerically parametrized by and hence identified with its side – it "was" its side and "had" an area, whereas our "has" a side and "is" an area. The "confrontation" thus stands for the configuration as well as for the length of its side.

15 "To accumulate" is an additive operation which concerns or may concern the *measuring numbers* only of the quantities to be added. It thus allows the addition of (the measuring numbers of) lengths and areas, of areas and volumes, or of bricks, men and working days.

Another addition ("appending") is concrete and therefore by necessity homogeneous. It serves when a quantity *a* is joined to another quantity *A*, augmenting thereby the measure of the latter without changing its identity (as when interest, in Babylonian "the appended", is joined to *my* bank account while leaving it as mine).

16 The "projection" (*wāsītum*, literally something which protrudes or sticks out) designates a line of length 1 which, when applied to another line *L* as width, transforms it into a rectangle ▭(*L*,1) without changing its measure.

2. you posit. The moiety[17] of 1 you break, 30´ and 30´ you make hold.
3. 15´ to 45´ you append: by 1, 1 is equal. 30´ which you have made hold
4. in the inside of 1 you tear out: 30´ the confrontation.

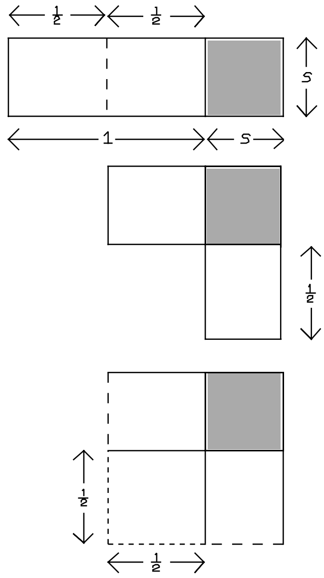


**Figure 1**. The procedure of BM 13901 #1, in slightly distorted proportions.

The problem deals with a "confrontation", a square configuration identified by its side *s* and possessing an area. The sum of the measuring numbers of these is told to be 45´ (=¾). The procedure can be followed in Figure 1: The left side *s* of the shaded square is provided with a "projection" (line 1), which creates a rectangle ⊏⊐(*s*,1) whose area equals the length of the side *s*; this rectangle, together with the shaded square area, must therefore also equal 45´. "Breaking" the "projection 1" (together with the adjacent rectangle) and moving the outer "moiety" so as to "hold" a small square □(30´) = 15´ does not change the area (line 2), but completing the resulting gnomon by "appending" the small square results in a large square, whose area must be 45´+15´ = 1 (line 3). Therefore, the side of the large square – that which is "equal by" (the square area) 1 – must also be 1 (line 3). "Tearing out" that part of the rectangle which was moved so as to "hold" leaves 1–30´ = 30´ for the "confrontation", [the side of] the square configuration.

To a first view, this does not look much as anything we know as "algebra". However, if we move to the level of principles, the text has much in common with equation algebra as trained in a modern school.

Firstly, the method is *analytical:* It presupposes that the solution exists, and moves stepwise from the (moderately) complex relation that is given until the unknown side of the square has been isolated. The steps are also quite similar to those through which *we* would solve the problem $x^2 + 1\cdot x = {}^3/_4$:

$$x^2 + 1\cdot x = {}^3/_4 \quad\Leftrightarrow\quad x^2 + 2\cdot ½\cdot x + ({}^1/_2)^2 = {}^3/_4 + ({}^1/_2)^2$$
$$\Leftrightarrow\quad x^2 + 2\cdot ½\cdot x + ({}^1/_2)^2 = {}^3/_4 + {}^1/_4 = 1$$
$$\Leftrightarrow\quad (x + {}^1/_2)^2 = 1$$

17 The "moiety" of an entity is its "necessary" or "natural" half, a half that could be no other fraction – as the circular radius is by necessity the exact half of the diameter, and the area of a triangle is found by raising exactly the half of the base to the height. It is found by "breaking", a term which is used in no other function in the mathematical texts.

$$\Leftrightarrow \quad x + \tfrac{1}{2} = \sqrt{1} = 1$$
$$\Leftrightarrow \quad x = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

Finally, the Old Babylonian procedure is reasoned but "naive", just as the sequence of algebraic operations – once the meaning of the terms and the nature of the operations is understood, no explanation beyond the indication of the steps themselves seems to be needed for the correctness of the procedure to be evident. In contrast, an exposition which made explicit the reasons for the validity of each steps could be termed "critical", in the Kantian sense (asking for the possibility and limits of their validity).

Modern equation algebra, of course, has further characteristics. It is used to solve structurally similar problems belonging to many different ontological domains, but it is based on a neutral representation – the realm of pure numbers. In this sense, even equation algebra is a self-contained body of formal knowledge. Can anything similar be said about "Old Babylonian algebra"?

It is tempting to give a negative answer at least inasmuch as the neutral representation is concerned, accustomed as we are (Hilbert notwithstanding) to seeing geometrical entities as more ontologically loaded than numbers. As a matter of fact, however, the answer should be affirmative. This becomes clear if we investigate the terminological usage of the "algebra" texts.

The basic entities they deal with are uš, sag̃, a.šà and *mitḫartum*. uš and sag̃ stand, respectively, for the length and width (more properly "front") of a rectangle, a.šà for the area of a rectangle or square (or other geometric figure). All three are Sumerian terms, and they are consistently written in Sumerian even though they were pronounced in Akkadian (Babylonian), as *šiddum*, *pūtum* and *eqlum*. However, if the length of a wall or a carrying distance is referred to, *šiddum* is written in syllabic Akkadian; moreover, even though the non-technical meaning of a.šà / *eqlum* is "field", mathematical problems which take real fields as a pretext use a different and less adequate word, seemingly in order not to interfere with the technical word for the area. *mitḫartum*, the term designating the square configuration parametrized by the side, is Akkadian. This does not imply, however, that it was the word employed in Akkadian-speaking practical surveying; here, a square field was supposed to possess 4 *fronts* (*pāt*, plural of *pūtum*).

We may conclude that the rectangles and squares of the "algebra" texts were seen as belonging to a different category than the rectangular and quadratic pieces of real land dealt with, for instance, by surveyors or architects. If the numbers of modern algebra are abstract, namely by being distinguished from *numbers of something*, then the geometry of the Old Babylonian "algebra" is also abstract. Remains the question whether this “algebra” was also “used to solve structurally similar problems belonging to many different ontological domains”.

The answer, one again, is positive. Let us first look at problem 12 from the tablet (BM 13901) whose first problem we have just discussed:[18]

**Obv. II**

**27.** The surfaces of my two confrontations I have accumulated: 21´40˝.
**28.** My confrontations I have made hold: 10´.
**29.** The moiety of 21´40˝ you break: 10´50˝ and 10´50˝ you make hold,
**30.** 1´57˝21‴40⁗ is it. 10´ and 10´ you make hold, 1´40˝
**31.** inside 1´57˝21‴40⁗ you tear out: by 17˝21‴40⁗, 4´10˝ is equal.
**32.** 4´10˝ to one 10´50˝ you append: by 15´, 30´ is equal.
**33.** 30´ the first confrontation.
**34.** 4´10˝ inside the second 10´50˝ you tear out: by 6´40˝, 20´ is equal.
**35.** 20´ the second confrontation.

We are thus told that the sum $\square(s_1)+\square(s_2)$ of two square areas is 21´40˝, while the area $\sqsubset\!\sqsupset(s_1,s_2)$ of the rectangle contained by the two sides is 10´ (Figure 2, left). Formally, the problem is of the fourth degree, but it is easily solved as a biquadratic in several ways. The text chooses to represent the *surfaces* $\square(s_1)$ and $\square(s_2)$ by the *sides* (say, $L$ and $W$) of a rectangle, whose surface $\sqsubset\!\sqsupset(L,W)$ is then found as $\sqsubset\!\sqsupset\big(\square(s_1),\square(s_2)\big) = \square\big(\sqsubset\!\sqsupset(s_1,s_2)\big) = \square(10´) = 1´40˝$, while the sum $L+W$ of the sides is known to be 21´40˝ – see Figure 2. This is a standard problem and solved according to the standard, as shown in the lower right part of the diagram: we may imagine the rectangle to be prolonged with the width, in such a way that the total length equals the known magnitude $L+W$ = 21´40˝. This segment is bisected and “made hold”, which produces a square with side 10´50˝ and surface 1´57˝21‴40⁗. Part of this square is identical with the original rectangle $\sqsubset\!\sqsupset(L,W)$, which is “torn out”. The dotted remainder is a square with surface 17˝21‴40⁗ and hence side 4´10˝. Adding this to the side of the square gives us the length $L$ of the rectangle; “tearing it out” from the other side gives $W$. $s_1$ and $s_2$ are then found as the “equalsides” of $L$ and $W$.

This is a case of representation *inside* the standard representation – areas being represented by lines. Another, more complex case is found in the problem TMS XIX #2 [Høyrup 2002a: 195–200]: It deals with a rectangle for which is given, beyond the area, the area of another rectangle whose length is the cube erected on the original length, and whose width is the original diagonal; this is a bi-biquadratic problem (thus of the eighth degree), and it is solved as a cascade

[18] From [Høyrup 2002a: 71]. I tacitly correct some erroneous numbers in the text.

of quadratic problems.

Much simpler is the problem YBC 6967 [Høyrup 2002a: 55–58], which deals with two numbers (*igûm*, meaning “the reciprocal”, and *igibûm*, “its reciprocal”) belonging together in the table of reciprocals; their product must hence be 1 or, as supposed in the actual case, 60.[19] Since this number is spoken of as a “surface” (and since the operations are the usual geometric ones), the numbers are represented by the sides of a rectangle. The text runs as follows:

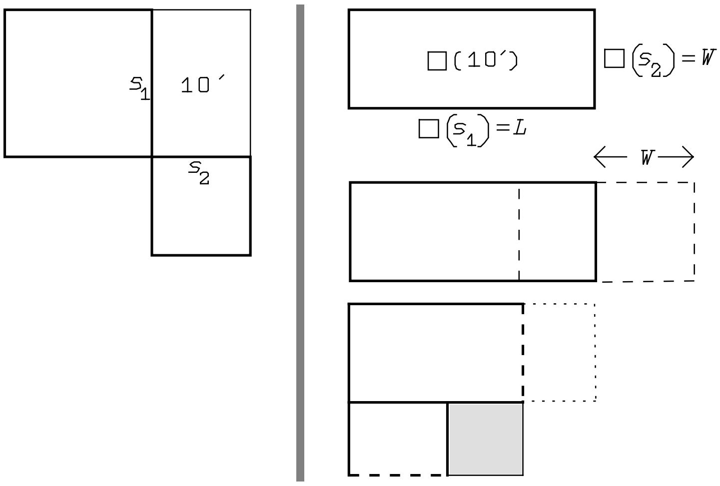


**Figure 2**. The procedure of BM 13901 #12.

**Obv.**

**1.** The *igibûm* over the *igûm*, 7 it goes beyond
**2.** *igûm* and *igibûm* what?
**3.** You, 7 which the *igibûm*
**4.** over the *igûm* goes beyond
**5.** to two break: 3°30´;
**6.** 3°30´ together with 3°30´
**7.** make hold: 12°15´.
**8.** To 12°15´ which comes up for you
**9.** 1` the surface append: 1`12°15´.
**10.** The equalside of 1`12°15´ what? 8°30´.
**11.** 8°30´ and 8°30´, its counterpart, lay down.

**Rev.**

**1.** 3°30´, the made-hold,
**2.** from one tear out,
**3.** to one append.
**4.** The first is 12, the second is 5.
**5.** 12 is the *igibûm*, 5 is the *igûm*.

The procedure can be followed in Figure 3; as in Figure 1, the excess of the length over the width is bisected and the outer moiety moved so as to hold together with the inner moiety a quadratic complement. By joining the gnomon

[19] In principle, any power of 60 is possible; but due to the floating-point character of the notation, only the values 1 and 60 can be distinguished.

to this complement we get a larger square with surface 72¼, whose sides (the "equalside" and its "counterpart") are found to be 8½; removing the moiety 3½ of the excess that was joined to the width leaves the width (the *igûm*); joining it where it originally belonged gives the length (the *igibûm*). What we have is thus a perfect example of algebraic *representation*, only inverted in respect of what we are accustomed to: geometric entities represent numbers, not *vice versa*.

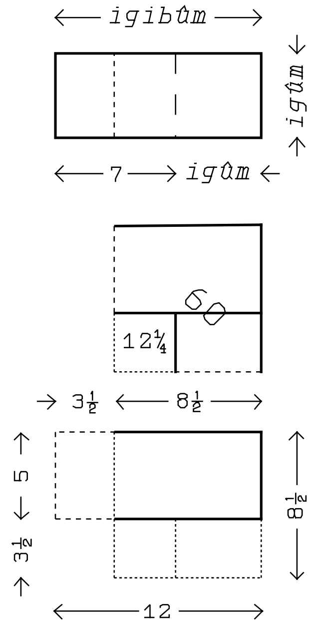

**Figure 3**. The procedure of YBC 6967.

Entities belonging more definitely to "real life" are also represented. In the problem contained in the text TMS XIII [Høyrup 2002a: 206–209], we are told the difference between the rates (volume units per shekel) at which a given quantity of oil was bought and sold, together with the total profit resulting from the transaction. This is another problem of the second degree – the area of the rectangle contained by the two rates turns out to be their difference divided by the total profit and multiplied by the total amount of oil. In AO 8862 #8 we know the sum of a number of workers, the number of days they work, and the number of bricks they produce according to a fixed rate per man-day (the number of bricks being thus proportional to the number of man-days). The text contains no description of the procedure, but there is no doubt that the numbers of men and days are to be represented by the sides of a rectangle, whose area is proportional to the number of bricks.

All in all we thus see that Old Babylonian "algebra", just as modern equation algebra, was "used to solve structurally similar problems belonging to many different ontological domains [while being] based on a neutral representation". It was hence in itself a body of formal knowledge, nested within the larger body of "Old Babylonian mathematics".[20]

One related question then presents itself: did the authors of the Old Babylonian texts see this body of "algebraic" knowledge as a distinct body?

This question can be answered from the theme texts – these, indeed, single

[20] It should be noticed that the problems about rectangles that turn up in Seleucid and other Late Babylonian texts do not serve in a similar way to represent entities belonging to a variety of categories; they ar formulated within the same kind of geometry as their Old Babylonian counterparts, and they are solved by a analytic procedure – but on this essential account they differ from modern equation algebra as well as Old Babylonian "algebra".

out groups of problems which the authors saw as belonging together. On the whole, the answer turns out to be affirmative.

Let us first look at the text BM 13901. It contains 24 problems, all of which deal with "algebraic" problems about one or more squares (although the biquadratic #12, as we have seen, is reduced to an "algebraic" problem about a rectangle). This we may compare, on one hand, with the text TMS V, on the other with BM 15285.

TMS V[21] is a catalogue text, also dealing with squares. Unlike BM 13901, however, it includes two types of linear problems (a multiple of the side being given; and the difference between the areas of two squares being given together with either the sum of the sides or their difference). It thus does not go beyond the "algebraic" realm.

BM 15285[22] is another catalogue dealing with squares – more precisely, with a square of given side which is subdivided in various ways into smaller figures (squares, right isosceles triangles, circles and circular segments. No single problem of algebraic character is treated. Together with BM 13901 and TMS V it thus suggests that "algebra" was clearly distinguished from "non-algebra". As a matter of fact, however, things are slightly more complicated.

This is demonstrated by BM 85200+VAT 6599, a procedure text whose formal theme is (rectangular prismatic) *excavations*.[23] In most problems, one or two dimensions can be eliminated, and these problems are then solved by means of the usual algebraic second- and first-degree techniques. Some, however, are genuine irreducible cubic problems; their solution is obtained by means of factorization,[24] or through use of a table referred to as "equilateral, one appended" and listing the numbers $n^2 \cdot \tilde{n} \cdot n$ = 12¼+60($n$+1). These solutions remain analytical inasmuch as they start from the presupposition that the solution *exists*; but they come from systematic trial-and-error,[25] not from a string of

21 Published in [Bruins & Rutten 1961: 35–49].

22 Now published together with a newly found fragment in [Robson 1999: 208–217].

23 See [Høyrup 2002a: 137–162].

24 For instance, finding three factors $p$, $q$ and $r$ of the number 10´4˝48˝´ which fulfil the conditions $p$+$q$=1, $r$=$p$+6´. Starting from simple factors, the solution can be found fairly quickly by systematic trial and error – but only because an exact solution exists.

25 Or pretended trial-and-error; since the author constructed the problems backward from a known solution, he would know the correct factorization in advance. The text only gives this correct factorization and does not tell how it is arrived at.

successive operations; we would hardly consider them "algebraic", even though we might translate the *problems* into third-degree equations. The author of the text, however, appears to have regarded them as members of the same group as the indubitably "algebraic" first- and second-degree problems. That self-contained body of knowledge within which the "algebraic" problems of BM 13901 and TMS V belonged was thus not made up exactly as we would expect it to be – but none the less such a self-contained body existed. Not only Old Babylonian mathematics as a whole but also the "algebraic" domain (delimited in this particular way) was thus treated in practice (and, who knows, perhaps also somehow spoken of) as a closed body of ("formal") knowledge applicable to a variety of particular concrete domains.

### *Riddle collections – and a hypothesis*

Does this mean that mathematics (at least the mathematics of literate cultures) was always or inherently treated as a formal science? The answer is no.

This is illustrated, for instance, by two well-known mathematical riddle collections: book XIV of the *Greek Anthology* [ed. Paton 1918], and the *Propositiones ad acuendos iuvenes* [ed. Folkerts 1978] ascribed in some manuscripts to Alcuin of York. Both combine properly arithmetical problems with other matters – the former with non-mathematical riddles and with oracles, the second with non-mathematical riddles and riddles with mock solutions.

Such collections, of course, were not put together by any kind of professional mathematicians, nor as the basis for systematic mathematics teaching; they represent the view of the more or less informed *outsider*. The fact that they constitute the most obvious exceptions to the assumption that mathematics was always inherently a *practically formal* science (and the fact that mathematics begins to be mixed up with other matters in Babylonia when the scribe school had disappeared[26]) suggests a more restricted hypothesis; that *mathematics when taught as an autonomous subject within an institutionalized school system* tends to be treated as practically formal knowledge.[27] For this view, direct reasons

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[26] Cf. note 10; the scribe school disappeared at the end of the Old Babylonian period, the education of scribes taking henceforth place within "scribal families" which as a rule were *real* families.

[27] Beyond the Middle Kingdom Pharaonic and the Old Babylonian examples, one may refer to Chinese mandarin education of the Han epoch, whose mathematics we find presented in the *Nine Chapters on Arithmetic* [ed., trans. Vogel 1968], or of the Indian Jaina tradition, whose mathematics we find in Mahāvīra's *Ganita-sāra-sangraha* [ed., trans.

can be given: only if mathematical knowledge can be applied to several distinct domains is there any reason to teach it autonomously; but *if* it is taught under such conditions by teachers whose primary practice is to teach mathematics, it is almost unavoidable that such principles as *can* be applied to several domains are presented in a relatively abstract form, that is, as practically formal knowledge.

## *II. Understandings of past mathematics*

Let us now change the perspective and take it for granted that mathematics *is* a formal science. 2+2=4, independently of the kind of units we are counting and adding, whether sheep or galaxies. The mathematician, therefore, tends to see professionally *the same* proposition 2+2=4 as what is *essentially* at stake in both situations – similarly to Aristotle's geometer considering a bronze sphere.

What happens when this principle is applied to the historical sources? Let us look at proposition II.6 from Euclid's *Elements*:[28]

> "If a straight line be bisected and a straight line be added to it in a straight line, the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half is equal to the square on the straight line made up of the half and the added straight line".

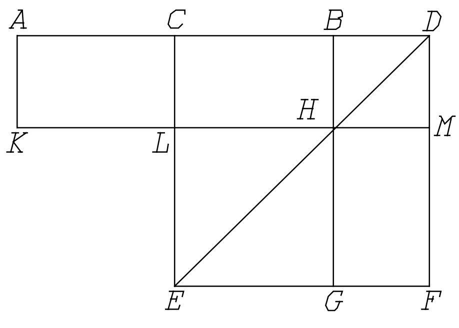


**Figure 4**. The diagram of *Elements* II.6.

This may not be quite easy to grasp, but a diagram helps – see Figure 4. The straight line that is bisected is *AB*, the added line is *BD*.

Raṅgācārya 1912]. Other surviving Indian medieval sources of importance are more closely linked to astronomy, but their contents is so much broader than astronomical computation that it points to the existence of a structure similar to what we find in Mahāvīra.

[28] Translation [Heath 1926: I, 385].

Euclid's proof makes use of the diagram and shows in this way that the proposition holds true; however, for a modern mind trained in symbolic algebra it follows much easier from the identity

$$(2a+e)\cdot e+a^2 = (a+e)^2$$

if $AC = CB = a$, $BD = DM = e$.

If sheep or galaxies do not matter for the truth of 2+2=4, does it then matter whether we think of geometry or algebra in the Euclidean case? Is it really *the same thing* mathematically speaking?

The dilemma can be sharpened by a look at Old Babylonian mathematics. Let us consider the *igûm-igibûm*-problem of YBC 6967, discussed above.

*We* would solve this problem by means of a system of two algebraic equations, for instance like this:

$$\tilde{n}\cdot n = 60 \quad , \quad \tilde{n}-n = 7$$

$$\frac{\tilde{n}-n}{2} = 3\tfrac{1}{2}$$

$$\left(\frac{\tilde{n}-n}{2}\right)^2 = 12\tfrac{1}{4}$$

$$\left(\frac{\tilde{n}+n}{2}\right)^2 = \left(\frac{\tilde{n}-n}{2}\right)^2+\tilde{n}\cdot n = 12\tfrac{1}{4}+60 = 72\tfrac{1}{4}$$

$$\frac{\tilde{n}+n}{2} = \sqrt{72\tfrac{1}{4}} = 8\tfrac{1}{2}$$

$$\tilde{n} = \frac{\tilde{n}+n}{2}+\frac{\tilde{n}-n}{2} = 8\tfrac{1}{2}+3\tfrac{1}{2} = 12$$

$$n = \frac{n+\tilde{n}}{2}-\frac{\tilde{n}-n}{2} = 8\tfrac{1}{2}-3\tfrac{1}{2} = 5$$

We might also have made a substitution,

$$\tilde{n} = n+7$$
$$(n+7)\cdot n = n^2+7n = 60$$
$$n^2+2\cdot 3\tfrac{1}{2}\cdot n+3\tfrac{1}{2}^2 = 60+12\tfrac{1}{4}$$
etc.

but the symmetric scheme has the advantage to correspond exactly to the Old Babylonian procedure, which however was geometric (and which has the same structure as the proof of *Elements* II.6), the two numbers and their product being represented by the side and the area of the rectangle ⊏⊐($\tilde{n}$,$n$).

The numerical steps in the tablet coincide with those of the symbolic-algebraic solutions, and the underlying principle that is made use of coincides with the identity expressed in *Elements* II.6. Does this mean that it is "the same" mathematics which we find in all cases, even though Euclid proves geometrical identities which *we* may translate into algebraic identities, whereas the Babylonian mathematics teacher asks for and shows the solution of a problem?

Mathematicians, we might expect, trained as they are as practitioners of a formal science, would tend to give an affirmative answer. Historians, including historians of ideas and independently of whether they come from the Aristotelian-idiographic[29] or the hermeneutic tradition (searching either for the particular or trying to penetrate the conceptual world of the source material "from within") should tend to react in the opposite way. In real life, this may be an oversimplification. What we find is, however, that those who have claimed explicitly that the history of mathematics can only be written by mathematicians – first and foremost, H. G. Zeuthen[30] and André Weil [1978] – are also among those who most outspokenly have argued in favour of identification. Claiming that only mathematicians are competent to write the history of mathematics thus in practice amounts to a claim that the history of mathematics should be treated as a formal science.

### *Second (didactical) thoughts*

Before leaving this discussion we may give it a didactical twist. If we are to make students understand some general principle, we present it through a variety of concrete representations – no elementary school trains addition *only* on cows, and engineering students who have trained differentiation with respect exclusively to a variable called $x$ will mostly be at a loss when the physics teacher asks them for a differentiation with respect to a variable called $t$ and standing for time.[31] *Transfer* is only achieved when students encounter the same underlying structure in many different situations.

The question about mathematical sameness can therefore be turned into a didactical problem: Would it facilitate the practical insight into what algebra is and can be used for if students encountered algebraic reasoning and structures

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29 The "statements [of poetry] are of the nature rather of universals, whereas those of history are singulars" – Aristotle, *De poetica* 1451$^{b}$7, trans. [Bywater 1924].

30 See [Lützen & Purkert 1994].

31 I speak from personal experience.

in several different versions – modern symbolic, Babylonian "cut-and-paste" procedures, and perhaps Euclidean proofs of identities? Would it be worth trying? If the answer is yes, then history serving didactics has to find a middle ground between the formal and the hermeneutic understanding of its role: it must show, *both* the difference between the two or three approaches *and* the underlying sameness. Omitting sameness means that the historical material becomes irrelevant for the purpose of mathematics teaching; omitting difference reduces the same material to a mirror in which we see nothing but our own face (a mirror from which, as Georg Lichtenberg once pointed out, "no saint will look out if a monkey looks into it").

Yet another twist is possible. So far this was formulated as if it dealt with the teaching of mathematical structure only, but we need not stop at that. Algebra when used by others than pure mathematicians is a technique for solving problems dealing with real-life entities – prices, income distributions, acceleration of charged particles in electric fields, etc. Essential for this is *representation*. As long as we only train it as a representation by means of pure numbers, students (and former students having become mature physicists, economists, and so on) tend to forget that the representations *are* representations and not the entities themselves. Working also (early in their educational career) on cases where numbers or prices are represented by geometric magnitudes they might get a better insight in what goes on when mathematical techniques are used in practice, and perhaps get slightly better at avoiding the pitfalls that inhere in the process.

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Author’s address:

Jens Høyrup
Roskilde University
Section for Philosophy and Science Studies
P.O. Box 260
DK-4000  Roskilde
Denmark
email: jensh@ruc.dk

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