On the algebraic method of quotient constructions

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Parts of My Talk

1. Intro: Methods and philosophy of mathematics
2. Constructions in group theory and generally in algebra
3. Category theoretical analysis of the construction procedure
4. Conclusions

This is Work Mainly Within the Philosophy of Mathematics

- The differences between philosophy of mathematics and history of mathematics.
- Epistemic techniques / methods that shape (epistemology — methodology — ontology).
- Impossibility / possibility relatively to construction method.

Motivation for Studying Methods in the Philosophy of Mathematics

- General question in philosophy of science: Examine the methods science uses for obtaining *knowledge*.
- Mathematics is a special science; much focus has been on proof-methods.
- But there are other methods.

Plato on Matematical Constructions

"[. . .] no one who has even a slight acquaintance with geometry will deny that the nature of this science is in flat contradiction with the absurd language used by mathematicians, for want of better terms. They constantly talk of "operations" like "squaring", "applying", "adding", and so on, as if the object were to *do* something, whereas the true purpose of the whole subject is knowledge – knowledge, moreover, of what eternally exists, not of anything that comes to be this or that at some time and ceases to be." (Republic 527a)

We will in this talk, however, consider elements of a generalized version of constructivism.

Kant on Mathematical Constructions

Based on a reading of Euclid Kant talks about *construction in intuition*.

We will try to generalize this notion.

Towards the constructions of quotients

Groups and Homomorphisms

Let G be a set with an operation $\cdot : G \times G \to G$. This is called a *group* if

1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, for all $x, y, z \in G$
2. There is a neutral element e such that in G:
   i) $e \cdot x = x$, for all x
   ii) For every x there is x^{-1} such that $x \cdot x^{-1} = e$.

A mapping $f : G \to H$ is called a *homomorphism* if,

$$f(xy) = f(x)f(y).$$

Subgroups, Cosets and Normal Subgroups

Suppose U is a subgroup of G.

1. If $x \in G$, then

   $$xU = \{xu \mid u \in U\}$$

   is a *left coset of U in G*. (Similiarly for *right*).

2. $G/U = \{xU \mid x \in G\}$.

Observe that $xU = yU \Leftrightarrow x^{-1}y \in U$. Then $x \sim y :\Leftrightarrow x^{-1}y \in U$ defines an equivalence relation on G.

A subgroup N of G is a *normal subgroup* if $xN = Nx$, for all $x \in G$.

Kernel and Construction of Factor Group

1. The *kernel* of a homomorphism is a normal subgroup.
Thus, the kernel defines an equivalence relation.

2. *Factor group*. Given a group G with a normal subgroup N we *construct* G/N and define

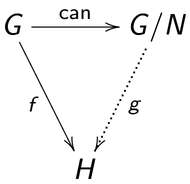
$$\text{can} : G \to G/N, \ \text{ by } x \mapsto xN.$$

There is precisely one structure on G/N such that can is a homomorphism. Kernel of can is N.

3. N is normal *iff* N is kernel of some homomorphism.
"$\Rightarrow$" by 2, and "$\Leftarrow$" by 1.

The Universal Property of the Factor Group and the Homomorphism Theorem

If $f : G \to H$ is a homomorphism with kernel N, then there exists uniquely a homomorphism $g : G/N \to H$, such that f (epi-mono-) factorizes through can and g.



Homomorphism Theorem. If $f : G \to H$ is a surjective homomorphism with kernel N, then

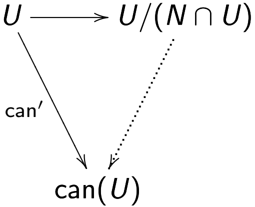
$$G/N \cong H$$

Noether's First Isomophism Theorem

G is a group with subgroups U and N, N normal. Construct:

$$G \xrightarrow{\text{can}} G/N$$

Restrict can to U and obtain can′:



$\text{can}(U) = \{uN \mid u \in U\} = \{xN \mid x \in UN\} = UN/N$

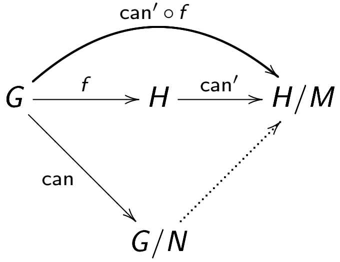
Noether's First. If G is a group with subgroup U and normal subgroup N, then $U \cap N$ is a normal subgroup of U and

$$U/(U \cap N) \cong UN/N.$$

Noether's Second Isomophism Theorem

Noether's Second. If $f : G \rightarrow H$ is a surjective homomorphism, M is a normal subgroup of H and $N := f^{-1}(M)$ then N is a normal subgroup and

$$G/N \cong H/M$$



Essence of the Factor Group Construction Method

Suppose we are given a homomorphism $f : G \rightarrow H$.

1. First we extract the kernel N of f. The kernel defines an equivalence relation $x \sim_N y$. The relation holds if and only if $f(x) = f(y)$.
2. Then we collapse G with respect to kernel, i.e., we form G/N (in set theoretic notation this is $G/\sim_N$) and define the canonical homomorphism from G to G/N. The factor group has the universal property.
3. As the factor group has the universal property f factorizes uniquely through G/N in an, epi-, respectively, monomorphism.

The Method is Canonical

Colin McLarty : Artin, Noether and van der Waerden founded modern algebra on this method. Quotient constructions and factorization theorems are very central to the study of groups, rings, fields, vector spaces, modules and so on.

The idea of a “structure preserving mapping” is a unifying concept (method) for investigating different algebraic domains in a uniform way.

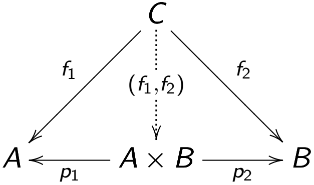
What is interesting is not the theorems, nor the proofs (as verifications) but the general construction method.

Category Theoretical Analysis: Back to Methods

Epistemologically a set-theoretic product of two sets A and B is not just the ordinary Cartesian product:

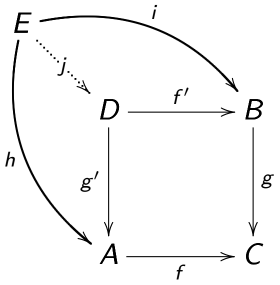
$$A \times B = \{(x, y) \,|\, x \in A \;\&\; y \in B\}.$$

If we are talking about our *understanding* and ability to *use*, *manipulate* and *examine* a product, then we need to take the projection functions and the univeral property into consideration.



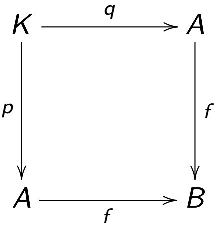
Pullbacks

A *pullback* of two morphisms $f : A \rightarrow C$ and $g : B \rightarrow C$ is an object D and a pair of morphisms $g' : D \rightarrow A$ and $f' : D \rightarrow B$ satisfying two conditions, a) $f \circ g' = g \circ f'$ and b) the universal property.



Kernel pairs

Definition of kernel pair. The *kernel pair* of a morphism $f : A \to B$ in a category with pullbacks is the universal object K and the two morphisms p and q with domain K such that the following diagram is a pullback:

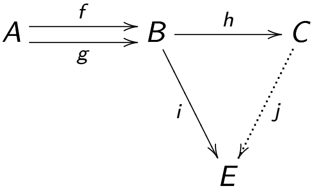


Coequalizer

A *coequalizer* of two parallel morphisms $A \overset{f}{\underset{g}{\rightrightarrows}} B$ is a morphism

$$h : B \to C$$

satisfying the following two conditions: a) $h \circ f = h \circ g$ and b) the universal property.



Category Theoretic Analysis (1/2)

Suppose we in **Grp**, the category of groups, are given a homomorphism $f : G \to H$.

1. By pulling f back along itself we obtain the kernel pair consisting of the two projection morphisms p and q having the 'equivalence'-group as domain:

$$\begin{array}{ccc} \sim_N & \xrightarrow{q} & G \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ G & \xrightarrow[f]{} & H \end{array}$$

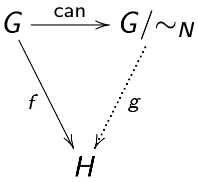
2. Then we construct the factor group and the canonical homomorphism by co-equalizing the kernel pair:

$$\sim_N \underset{q}{\overset{p}{\rightrightarrows}} G \xrightarrow{\text{can}} G/{\sim_N}$$

$G/{\sim_N}$; can has the category theoretic (co-)universal property.

Category Theoretic Analysis (2/2)

3. As f also respects the equivalence relation on the set G i.e., $f \circ p = f \circ q$, we apply the (co-)universal property of $\text{can} : G \to G/{\sim_N}$ and obtain the unique factorization of f through the factor group:



It can be argued that g is a monomorphism and we know that any coequalizer is epic, thus we have our epi-mono-factorization of f.

Conclusions: Formal Philosophy

What does the category theoretic analysis with us?

- It is a conceptualization of a very general methodology for constructing mathematical objects.
- The methodology goes from being in general very abstract to being concrete in particular categories.
- Symmetry is involved: it is dual of image-factorization (shown in category theory, for instance)
- The method is generalisable to all toposes, for instance.
- The method can in category theory be abstracted to all exact categories.

Seemingly there is a strong methodological component that partially determines the subject matter.