Indexical Hybrid Tense Logic

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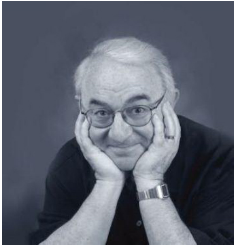
The Logic of Indexicals: Background

Hans Kamp:
“Formal Properties of ‘Now’ ” (1971)

“I learned last week that there would now be an earthquake”

David Kaplan:
“On the Logic of Demonstratives” (1978)

“I am here now”



Kamp-Kaplan Semantics Meets Hybrid Logic

Hybrid Logic started in the context of tense logic (of course). We want to be able to refer to points in time

"It is five o'clock April 21 1973"

and to define, say, irreflexivity.

Hybrid Logic is a natural context for analysing indexicals like *now*, *yesterday*, *today* and *tomorrow*. The project started here:

Patrick Blackburn: "Tense, Temporal Reference and Tense Logic", *Journal of Semantics:* **11** (1994).

Basic Hybrid Tense Logic (1/2)

Syntax. We work with a set $\{p, q, r, \ldots\}$ of propositional symbols and a set $\{i, j, k, \ldots\}$ of nominals and diamonds P and F and $@$-operators.

$$\phi ::= i \mid p \mid \neg\phi \mid \phi \wedge \psi \mid P\phi \mid F\phi \mid @_i\phi$$

We call this language $\mathcal{L}$.

Basic Hybrid Tense Logic (2/2)

Semantics. Models are given by a frame (T, R) together with a valuation function V. Nominals name points in time. Therefore:

- $V(p)$ is a subset of T, for propositional symbols.
- $V(i)$ is a singleton subset of T, for nominals.

$$
\begin{array}{lll}
\mathfrak{M}, t \models a & \text{iff}, & a \text{ is atomic and } t \in V(a) \\
\mathfrak{M}, t \models \neg\phi & \text{iff}, & \mathfrak{M}, t \not\models \phi \\
\mathfrak{M}, t \models \phi \wedge \psi & \text{iff}, & \mathfrak{M}, t \models \phi \text{ and } \mathfrak{M}, t \models \psi \\
\mathfrak{M}, t \models P\phi & \text{iff}, & \text{for some } t',\ t'Rt \text{ and } \mathfrak{M}, t' \models \phi \\
\mathfrak{M}, t \models F\phi & \text{iff}, & \text{for some } t',\ tRt' \text{ and } \mathfrak{M}, t' \models \phi \\
\mathfrak{M}, t \models @_i\phi & \text{iff}, & \mathfrak{M}, t' \models \phi \text{ and } t' \in V(i)
\end{array}
$$

Adding *Now*

We add the special atomic formula *now* and obtain the language $\mathcal{L}(now)$. Syntactically *now* behaves like a nominal; the following are formulas in the extended language.

$$\phi ::= now \mid i \mid p \mid \neg\phi \mid \phi \wedge \psi \mid P\phi \mid F\phi \mid @_{now}\phi \mid @_i\phi$$

Kamp's Idea: A Second Index

We want $\phi \leftrightarrow @_{now}\phi$ to come out as a *kind of* a validity. If we work only with one index, then

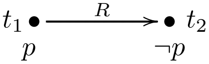
$$\mathfrak{M}, t \models @_{now}\phi \quad \text{iff,} \quad > \text{something} <$$

This won't work, as we also want

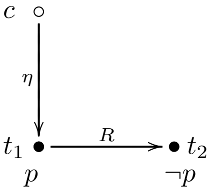
$$\phi \rightarrow G@_{now}\phi$$

to be a validity of this kind.

$\mathfrak{M}_1$



$\mathfrak{M}_2$



Semantics for *now* (1/2)

A contextual model $\mathfrak{M}$ is a 5-tuple

$$\mathfrak{M} = (T, R, V, C, \eta)$$

where η is a mapping from contexts to points in T. The function η is crucial: it specifies, for any context $c \in C$, what the utterance time (or temporal location) of that context is.

Semantics for *now* (2/2)

The valuation function:

- $V(c, p)$ is a subset of T, (propositional symbols)
- $V(c, i)$ is a singleton subset of T, (nominals)
- $V(c, now) = \{\eta(c)\}$.

Some of the clauses for *satisfiability* in a contextual model $\mathfrak{M} = (T, R, V, C, \eta)$ are:

$$
\begin{array}{lll}
\mathfrak{M}, c, t \models now & \text{iff,} & t = \eta(c) \\
\mathfrak{M}, c, t \models \phi \wedge \psi & \text{iff,} & \mathfrak{M}, c, t \models \phi \text{ and } \mathfrak{M}, c, t \models \psi \\
\mathfrak{M}, c, t \models P\phi & \text{iff,} & \text{for some } t',\ t'Rt \text{ and } \mathfrak{M}, c, t' \models \phi \\
\mathfrak{M}, c, t \models @_{now}\phi & \text{iff,} & \mathfrak{M}, c, \eta(c) \models \phi
\end{array}
$$

Remember, η specifies for any context the moment of that context. Thus, η is what Kaplan calls the *character* of *now*.

Two Kinds of Validities

Logical validity. ϕ is *logically valid*, if ϕ is satisfied in any model $\mathfrak{M}$ at any pair c, t, i.e., if for any $\mathfrak{M}, c, t$

$$\mathfrak{M}, c, t \models \phi.$$

Contextual validity. ϕ is *contextually valid*, if ϕ is satisfied in any model $\mathfrak{M}$ in any context c (evaluated at the moment of utterance), i.e, if for any $\mathfrak{M}, c$

$$\mathfrak{M}, c, \eta(c) \models \phi.$$

Kamp's Elminability Result

With regard to contextual satisfiability the $@_{now}$-operator can be eliminated: For any ϕ there is an $@_{now}$-free ϕ' such that ϕ is satisfiable at $(c, \eta(c))$ iff ϕ' is satisfiable at that point.

Idea: Suppose $@_{now}\psi$ is some subformula occurrence in ϕ. Then (following ten Cate) we observe that:

$$\left(@_{now}\psi \land \phi[@_{now}\psi \leftarrow \top]\right) \lor \left(\neg @_{now}\psi \land \phi[@_{now}\psi \leftarrow \bot]\right)$$

is equivalent to ϕ.

That basically means, we can pull $@_{now}$ out to the front.

Towards Completeness: Reduction to 1D

Lemma (Reduction to 1D). Let ϕ be a formula in $\mathcal{L}(\mathit{now})$ and j a nominal not occurring in ϕ, then $\phi[\mathit{now} \leftarrow j]$ is satisfiable in an ordinary one-dimensional model if and only if ϕ satisfiable is in a contextual model with just one context.

Logical Completeness

Let K_h^t be some standard complete axiom system for minimal hybrid tense logic formulated in $\mathcal{L}$. And let $K_h^t(now)$ be the same axiom system in $\mathcal{L}(now)$ where *now* is treated as just another nominal.

Theorem (Logical Completeness). $K_h^t(now)$ is complete with respect to the logically valid formulas in contextual models.

Contextual Completeness (1/2)

Extend $K_h^t(now)$ with:

Kamp's Rule (KR). If we have proved $@_{now}\phi$ then we have a proof of ϕ; in other words:

$$\text{If } \vdash @_{now}\phi \text{ then } \vdash \phi.$$

Restriction: Kamp's Rule can only be used once in any proof and only as the very last step.

Example:

$$\begin{array}{lll} 1. & @_{now} now & (\text{Standard axiom, instance of } @_i i.) \\ 2. & now & (\text{Kamp's Rule}) \end{array}$$

Contextual Completeness (2/2)

We observe that, $@_{now}\phi$ is a LV iff ϕ is a CV.

Theorem (Contextual Completeness). $K_h^t(now) + \text{KR}$ is complete with respect to the contextually valid formulas in $\mathcal{L}(now)$.

Proof. Suppose ϕ is contextually valid. Then by the previous observation $@_{now}\phi$ is logically valid. By logical completeness $@_{now}\phi$ is provable in $K_h^t(now)$. Take this proof and apply Kamp's Rule and we have the required proof of ϕ.

Note our simple reduction strategy

Some Results for *now*

Theorem. Let Λ be K_h^t extended with pure axioms, and $\Lambda + \textit{now}$ be its contextualized counterpart. Then:

1. $\Lambda + \textit{now}$ is contextually complete with respect to the same classes of models as that Λ is logically complete for.
2. Λ-satisfiability has the same complexity as $\Lambda + \textit{now}$ satisfiability.
3. There is a terminating tableau system for $\Lambda + \textit{now}$ iff there is a terminating tableau system for Λ.