Mathematical applications and modelling in the teaching andlearning of mathematics

Proceedings from Topic Study Group 21 at the 11th International Congress on Mathematical ducation in Monterrey, Mexico, July 6-13, 2008

Blomhøj, Morten; Carreira, Susana

IMFUFA tekst

- I, OM OG MED MATEMATIK OG FYSIK

Mathematical applications and modelling in the teaching and learning of mathematics

Proceedings from Topic Study Group 21 at the 11th International Congress on Mathematical ducation in Monterrey, Mexico, July 6-13, 2008

Editors:

Morten Blomhøj, NSM, Roskilde University, Denmark

Susana Carreira, University of Algave, Portugal

June 2009

nr. 461 - 2009

Roskilde University,
Department of Science, Systems and Models, IMFUFA
P.O. Box 260, DK - 4000 Roskilde
Tel: 4674 2263   Fax: 4674 3020

Mathematical applications and modelling in the teaching and learning of mathematics

Proceedings from Topic Study Group 21
at the 11th International Congress on Mathematical Education
in Monterrey, Mexico, July 6-13, 2008

IMFUFA tekst nr. 461/ 2009 – 252 pages – ISSN: 0106-6242

These proceedings contain the papers reviewed and accepted for Topic Study Group 21 (TSG21) at ICME-11. Prior to acceptance all papers were reviewed by at least two reviewers and revised on the basis of the review reports. Preliminary versions of all the papers were published at the congress web-site before the congress so as to form a common basis for the work of the TSG. The papers presented during the TSG sessions are all marked with a '*' in the table of contents found below. After the congress revised versions of the papers have been submitted for the proceedings.

During the TSG21 sessions the papers were presented and discussed according to a thematic organisation in three themes, namely: Theme 1: Different perspectives on mathematical modelling in educational research; Theme 2: Challenges in international collaboration on the teaching of mathematical modelling, and Theme 3: Didactical reflections on the teaching of mathematical modelling. The intentions of theme 1 were to present and discuss an overall view on different perspectives found in the field of educational research on the teaching and learning of mathematical modelling, and to use this overview to characterise and discuss the research presented in the TSG. The two other themes are more particular foci on the teaching and learning of mathematical modelling dealt with in the papers submitted for TSG21. In these proceedings the papers are presented according to the theme under which they were presented or discussed during the congress.

Morten Blomhøj, July 2009

Roskilde University,
Department of Science, Systems and Models, IMFUFA
P.O. Box 260, DK - 4000 Roskilde
Tel: 4674 2263   Fax: 4674 3020

Table of contents

Theme 1: Different perspectives on mathematical modelling in educational research

Theme 2: Challenges in international collaboration on the teaching of mathematical modelling

Theme 3: Didactical reflections on the teaching of mathematical modelling

*) Papers presented orally during the TSG21 sessions at ICME-11.

DIFFERENT PERSPECTIVES IN RESEARCH ON THE TEACHING AND LEARNING MATHEMATICAL MODELLING
- CATEGORISING THE TSG21 PAPERS

Morten Blomhøj
NSM, Roskilde University, Denmark

Introduction

We have a large and growing collection of didactical research on mathematical modelling. Moreover this research even seems to have had a serious impact on the practices of mathematics teaching at least on curricula level. During the last couple of decades the introduction of mathematical modelling and applications is probably - together with the introduction of information technology - the most prominent common features in mathematics curricula reforms around the world (Kaiser, Blomhøj and Sriraman, 2006, p. 82). Curricula reforms in many western countries, especially at secondary level have emphasised mathematical modelling as an important element in an up-to date mathematics secondary curricula preparing generally for further education. Didactical research has undoubtedly played an important role in this development. The fundamental goals in the teaching of mathematical modelling and the reasons for pursuing these goals developed and analysed in research can be pinpointed in the guidelines for mathematics teaching in many countries. Also, the general understanding of the model concept and of a modelling process expressed in many mathematics curricula is clearly influenced by didactical research. (Blum et al, 2007) and (Haines et al, 2006).

However, the way mathematical modelling and applications is organised in curricula and, especially, how these parts of the curricula are assessed reveal only a very limited influence from research. And when it comes to the level of teaching practice in the classroom it is still a pending question to which degree the many developmental modelling projects carried out and analysed in research have actually influenced the practices of teaching mathematical modelling.

Influencing practices of mathematics teaching are not the only criteria for progress in the didactical research on mathematical modelling. It is also relevant to try to evaluate the coherency of the theories developed. In the editorial *Towards a didactical theory for mathematical modelling* of ZDM (2, vol. 38), we argued that at a general level it is possible to identify in the field of research

> *... a global theory for teaching and learning mathematical modelling, in the sense of a system of connected viewpoints covering all didactical levels: learning goals, fundamental reasons for pursuing these goals at different levels of the educational systems, tested ideas about how to support teachers in implementing learning goals and recognised didactical challenges and dilemmas related to different ways of organising*

> *the teaching, theoretically and empirically based analyses of learning difficulties connected to modelling, and ideas about different ways of assessing students' learning in modelling activities and related pitfalls.* (Kaiser, Blomhøj and Sriraman, 2006, p. 82)

However, this "global theory" is not based on a single strong research paradigm. On the contrary, in fact, it is possible to identify a number of different approaches and perspectives in mathematics education research on the teaching and learning of modelling. This is, precisely, the reason for choosing *Conceptualizations of mathematical modelling in different theoretical frameworks and for different purposes* as one of the themes for the Topic Study Group on Mathematical applications and modelling in the teaching and learning of mathematics at ICME-11 (TSG21). We intended to provide a background for in-depth discussions of the theoretical basis of the different approaches within the field.

Kaiser & Sriraman (2006) report about the historical development of different research perspectives and identify seven main perspectives describing the current trends in the research field.

These perspectives may have overlaps and also they do not necessarily cover the entire research area. Nevertheless, they all represent distinctive perspectives of research on the teaching and learning of mathematical modelling, and they have been developed in particular research milieus over a long period of time and all of them have produced a considerable number of research publications. The main rationale for developing a categorisation of research perspectives is of course to deepen our mutual understanding of the individual perspective and to recognise similarities and differences amongst these. The idea is not to try to judge about their relevance or their relative importance.

Five of the research perspectives pinpointed by Kaiser & Sriraman (2006) are – according to my analysis – represented among the sixteen papers accepted for TSG21. As an introduction to our work in Topic Study Group it is therefore relevant to characterise and briefly discuss these five research perspectives. For each perspective I give a short presentation of the TSG21 papers that I found can be said to representing the perspective. The aim is to provide a background for discussing the many interesting papers of TSG21 in relation to their research perspective and theoretical foundation. Hopefully, the categorisation can also facilitate discussions of similarities and differences among the perspectives. It goes without saying that the categorisation in itself should made object for discussion and debated. At the end of the paper, I summarise, in the form of a template, the descriptions of the perspectives in few words together with a list of the TSG-papers, which I have categorised under the individual perspectives. In the following I refer to the TSG21 papers included in the proceedings by the authors' names and (TSG21).

The realistic perspective

The realistic perspective on the teaching and learning of mathematical modelling takes its point of departure in the fact that mathematical models are being extensively used in very many different scientific and technological disciplines and in many societal contexts. In this perspective, mathematical modelling is viewed as applied problem solving and a strong emphasis is put on the real life situation to be modelled and on the interdisciplinary approaches.

According to this perspective, in order to really support the students' development of a mathematical modelling competence that is relevant for their further education and for their subsequent professions, it is essential that the students work with realistic and authentic real life modelling. The students' modelling work should be supported by the use of relevant technology, such as for example advanced computer programmes for setting up and analysing mathematical models. The modelling process and the model results should be assessed through validation against real or realistic data. Therefore, in this perspective it is important to study in depth mathematical modelling processes in different professions and areas of societal applications of mathematical models. Such studies should inform the design of modelling courses in schools in order for the teaching in modelling to as realistic as possible.

The main criterion for progress in the students' learning is the students' success with solving real life problems by the means of mathematical modelling. Pollak (1969) can be regarded as a prototype of the realistic perspective.

Often physicists and sometimes also researchers from other natural sciences argue that what we in mathematics education calls mathematical modelling in their subject area should be thought of as physics modelling (or just physics – because modelling is what physicists do all the time – they say) or biological modelling. Nevertheless as mathematics educators we focus on the general elements in the teaching and learning and not on the differences of modelling in different areas. Should we need to defend ourselves, we could argue that so far not much educational attention or research has been directed towards (mathematical) modelling outside mathematics education research. However, the realistic perspective is really taking the subject area of the application of mathematics very seriously, and actually in this perspective is seen as an interdisciplinary problem solving activity in which, of course, mathematics is playing a very important role.

The paper by Rodríguez (TSG21) is an example of how the conceptualisation of a mathematical modelling process may be influenced by the subject area in which the modelling takes place. The paper reports from a developmental project carried out at a French University context where mathematical modelling was used as a didactical means for supporting the students' learning of mathematics and physics in a calculus and a physics course, respectively. A physics domain was "inserted" in a six phased model of the modelling process in order to distinguish the physical elements in the systematisation and mathematisation processes and in the interpretation of the model results. In the physics course the modelling process was conceptualised as illustrated in figure 1. However, since the learning goal in this project was to support the students' learning of mathematics and physics by means of mathematical modelling and not to model real life situations, in my opinion, this paper does not belong to the realistic perspective, but rather to the educational perspective discussed below.

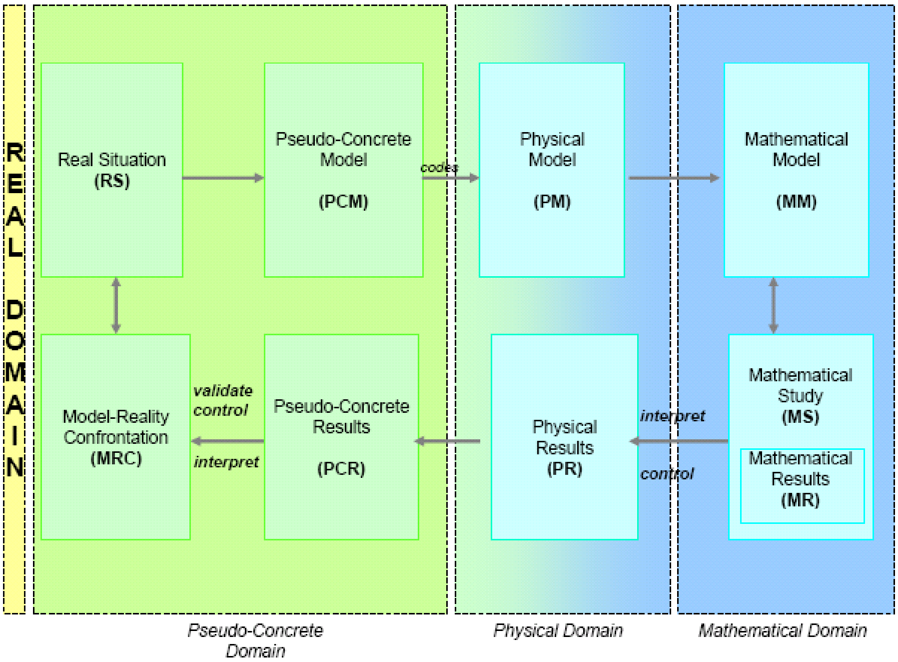


Figure 1: The modelling process in a physics course (Rodríguez, TSG21).

The paper by Kadijevich (TSG21) is properly the closest we get to a paper within the realistic perspective among the TSG21 papers. The paper reports and analyses the experiences from a developmental project for undergraduate business students in Serbia. The students were to build and analyse a total financial balance model in the form of a spreadsheet for a business activity of their own choice. The use of technology in the form of a spreadsheet is an important and integrated element in this approach. The success criterion for the students' modelling work was to apply the model for deciding whether or not their business activity was a profitable one and to make suggestions on how to make more profit. This type of pragmatic criteria for solving authentic real life problems or realistic problems by means of mathematical modelling is, I think, characteristics for the realistic perspective. I consider it also as a characteristic element in this approach that Kadijevich in his design builds on the heuristics for technology-supported modelling of real life situations developed by Galbraith & Stillman (2006).

The contextual perspective

This perspective has developed primarily on North American grounds, and is based on extensive research on problem solving and the role of word problems – in mathematics teaching called contextual modelling. In the last decade the modelling eliciting perspective has been further developed by deepening the philosophical background as well the connection to general physiological learning theories.

First of all this research perspective focuses on developing and testing designs for modelling eliciting activities, which are guided by six principles: (1) the reality principle – the situation must appear meaningful to the students, and connect to their former experiences; (2) the model construction principle – the situation should create a need for the students to develop significant mathematical constructs; (3) the self-evaluation principle – the situation should allow students to assess their elicited models; (4) the construct documentation principle – the situation and context should require the students to express their thinking while solving the problem; (5) the construct generalization principle – it should be possible to generalize the elicited model to other similar situations; and (6) the simplicity principle – the problem situation should be simple. (Lesh & Doerr, 2003)

It is the clear focus on the didactical design of modelling eliciting activities with situations carefully structured to support the students' learning that distinguishes the contextual perspective from the realistic perspective. It can be argued that this perspective could therefore be thought of as part of the educational perspective described below. However, the modelling eliciting perspective insists on seeing mathematical modelling as a special type of problem solving, and therefore the psychological aspects of problem solving are conceived a basis for understanding the learning difficulties related to mathematical modelling and for teaching mathematical modelling under the contextual perspective. And moreover in this perspective mathematical modelling is not conceived as a specific competency as is the case in the educational perspective.

Maybe surprisingly – Mexico being so close to the US – we did not for TSG21 receive any papers within the contextual perspective.

The educational perspective on mathematical modelling

The main idea of the educational perspective is to integrate models and modelling in the teaching of mathematics both as means for learning mathematics and as an important competency in its own right. Accordingly classical didactical questions about educational goals and related justifications for teaching mathematical modelling at various levels and branches of the educational system, ways to organise mathematical modelling activities in different types of mathematics curricula, problems related to the implementation of modelling in school culture and teaching practices, and problems related to assessing the students' modelling activities are all been addressed under this research perspective.

Niss (1987, 1989) and Blum & Niss (1991) are classical references to this research perspective, which has been quite dominant in Western Europe in the last three decades. Defining and discussing the basic notions in the field – such as: model, modelling, the modelling cycle or modelling cycles, modelling applications and competency– and the meaning of these notions in relation to mathematics teaching at different educational levels is an important element in the research under the educational perspective. The introduction to the ICME-14 study volume gives an overview of the concept clarifications and the history of the field. (Niss, Blum & Galbraith, 2007)

In my interpretation (Blomhøj, 2004), the three main arguments for teaching mathematical modelling as an integrated element in mathematics in general education especially at secondary level, which be identified in research under the educational perspective are the following:

(1) Mathematical modelling bridges the gab between students' real life experiences and mathematics. It motivates the students' learning of mathematics, gives direct cognitive support for the students' conceptions, and it places mathematics in the culture as a means for describing and understanding real life situations.

(2) In the development of highly technological societies, competences for setting up, analysing, and criticising mathematical models are of crucial importance. This is the case both from an individual perspective in relation to opportunities and challenges in education and work-life, and from a societal perspective in relation to the need for an adequately educated workforce.

(3) Mathematical models of different kinds and complexity are playing important roles in the functioning and forming of societies based on high technologies. Therefore, the development of an expert as well as a layman competence to critique mathematical models and the way models and model results are used in decision making, are becoming imperatives for the maintaining and further development of democracy.

The third argument is also part of the basis of the socio-critical perspective dealt with below, where it is further developed. However, it is important to recognise that a critical perspective on mathematical modelling and the use of mathematical models in society are also included in the educational perspective.

It is within the educational perspective that we find most of the TSG21 papers, and the research in these papers reflects modelling both as a means for learning mathematics and as an educational goal. Therefore, I do not distinguish between these two types of research in my listing of the TSG21 papers.

In the paper by Lambardo & Jacobini (TSG21) the authors are reporting from their developmental project in Brazil with teaching Linear Programming and mathematical modelling to students employed in various businesses and industries, and who are taking a college degree. Working in pairs, the students were challenged to find problems from their own working life that could be addressed by means of mathematical modelling and Linear Programming. The experiment involved both a mathematics course and a course in data processing in which the students were introduced to software for optimisation. The clear connection to the students' working life created a strong motivation for learning the "mathematics behind" and for learning how to use the software in order to reach an optimal solution to a LP problem but the students did not, by themselves, engage in reflections about model assumptions, the stability of their optimal solution or the general validity of the model, and the possible implementation of the model results in real life. So, authenticity and close connection to real life experiences do not ensure the occurrence of relevant and critical reflections among the students.

The paper by vom Hofe et al. (TSG21) reports on an extensive German research project and places itself clearly within the educational perspective. The research project has the double focus characteristic for the educational perspective. On the one hand, mathematical modelling is seen as a means to challenge and develop the students' mathematical understanding and especially their basic mathematical beliefs (Grundvorstellungen, GV), and, on the other hand, mathematical modelling is seen as an educational goal in its own right. The research is based on comprehensive data material from a longitudinal study, yearly assessing grade 5 to 10 students' performance solving mathematical modelling tasks. The findings concerning the development of the students' modelling competency from grade to grade in the three different school branches in the German system are presented and discussed. However, the data are also intended for pinpointing weak spots in the students' mathematical understanding and beliefs (their GVs) at the different levels and in the different school branches, with the intention of forming a basis for designing teaching material that could help overcome identified learning difficulties in the future. The connection between the students' mathematical beliefs (GVs) and their performance in modelling task is illustrated in figure 2. According the authors, it is in the processes of mathematization and interpretation that the students' basic mathematical beliefs (GVs) can be unveiled.

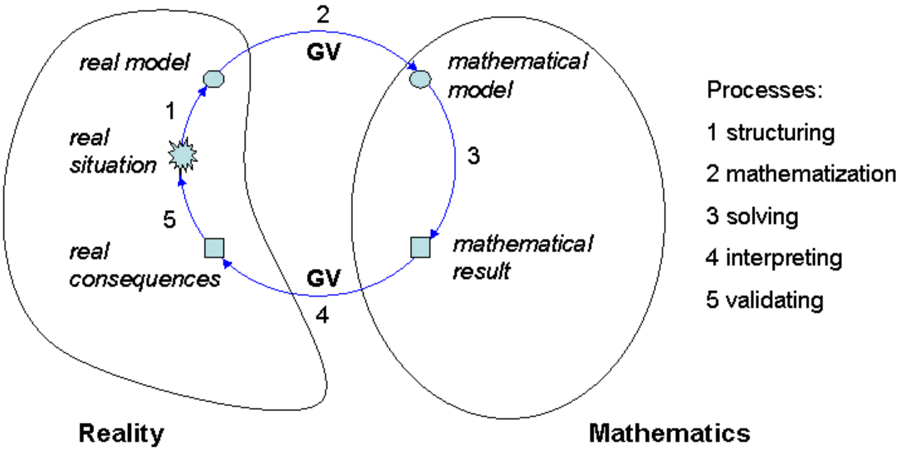


Figure 2: The modelling process. The students' basic mathematical beliefs (GVs) are activated in particular in relation to the processes of mathematization and interpretation.

Ludwig & Xu (TSG21) report on a comparative study on the development of mathematical modelling competency in upper secondary students in Germany and China. Building on the conceptualisation of the mathematical modelling process by Blum & Leiss (2005), the authors define five levels of mathematical modelling competency, which they use to measure the students' performance in different modelling tasks in the two countries. This research lies within the educational perspective with a clear focus on mathematical modelling competency as an educational goal.

The paper by Meier (TSG21) reports on a comprehensive developmental European project supported by the European Union, where mathematics teachers, primarily of the secondary level, and mathematics education researchers from eleven countries work together in developing and testing mathematical modelling tasks. One of the main research questions in the project is "What is a good modelling task?", and so far a template for assessing modelling tasks with respect to particular learning objectives has been developed by the project. The template is intended to functioning as a tool for teachers for selecting and reflecting on modelling tasks and, in the paper, template is explained and illustrated through the analysis of a particular task. The project clearly lies within the educational perspective and the research characterising good modelling tasks tries to take both types of goals into account.

Oliveira & Barbosa (TSG21) have investigated tensions that elementary Brazilian teachers experience when teaching mathematical modelling. This research also seems to be within the educational perspective. However, it is not stated in the paper whether the teaching was focusing on modelling as a means for learning mathematics or as a goal. Teachers might experience different types and degrees of tensions in these two cases. However, this is not investigated in this paper and it may need further research to analyse such possible relations between tension in the practice of teaching and goal of teaching mathematical modelling.

The paper by Rodríguez belongs as already mentioned to the educational perspective and this research also have a clear dual focus on mathematical modelling as a goal and as a means for learning mathematics and, in this case, also physics. Likewise the paper by Kadijevich could also be considered to belong to the educational perspective. In this case, however, the project is focusing on creating a didactical setting in which the students' can work with a problem that they conceive as a realistic problem. Moreover the main criterion for success of the students' work is the solution of the business problem and not the development of mathematical modelling competency or the students' learning of some particular mathematical concepts or methods. Therefore, I have placed this paper under the realistic perspective.

The epistemological perspective

Under the epistemological perspective mathematical modelling is subordinated the development of more general theories for the teaching and learning of mathematics. Two very different examples of such theories are the Realistic Mathematic Education theory (RME) (see Treffers (1987), and Gravemeijer & Doorman (1999)) and the theory of mathematical praxeologies developed by Chevallard (see Garcia et al, 2006).

The paper by Andresen (TSG21) is the closest we get to research with a particular reference to one of these two general theories. In this paper, the author presents and discusses a model (in a different sense) for teaching mathematical modelling, which is based on her combining the four level model of mathematical activities developed in RME and another model, also with four level, for different types of mathematical reflections taken from the philosophy of mathematics. The teaching model is illustrated by a number of questions, which refer to specific mathematical tasks, and which are intended to prompt the students' reflections on each of the four levels. The model is meant to become a tool for upper secondary teachers to balance, on the one hand, the instrumental aspects of the students' work with solving problems related to modelling and models using advanced CAS calculators and, on the other hand, the reflections related to the modelling process and the use of the model results. The model has not yet been tested in teaching practice. In my view, this research falls under the epistemological perspective, since the main research interest seems to be to understand and describe the nature of the mathematical activities and related reflections involved in CAS-supported mathematical modelling.

Of course, one also finds research within mathematical modelling with the aim of supporting the development of theories for other types and nature. According to my reading the paper by Tarp and the paper by Siller are both examples hereof.

In his paper Tarp (TSG21) analyses the epistemological basis of mathematical concepts in arithmetic, algebra and analysis. His analysis is general but based on many years experience with teaching mathematics at upper secondary level in Denmark. He argues that traditional mathematics teaching, which he labels "pastoral metamatism" disregards important aspects of the epistemology of the concepts, and that this leads to serious learning difficulties when the concepts are activated in modelling reality. The basic claim in the paper is that fundamental mathematical concepts should be re-invented in mathematics teaching by working with modelling real phenomena, without losing important aspects of the concepts' epistemology.

The work by Siller (TSG21) analyses the potentials of a particular type of software, Prograph, as a tool for teaching functional modelling. I would also consider this investigation as belonging to the epistemological perspective. The aim is to develop and investigate a technological tool for supporting the students' modelling of stochastic phenomena, and the focus is on how certain mathematical concepts and relations can be represented in a algorithmic form so as to be represented in the particular software Prograph.

The cognitive perspective

Within the cognitive perspective the main interest is to understand which cognitive functions are activated in the individual student's mathematical modelling activities. To that end, students' particular modelling processes are analysed and the students are interviewed with the purpose of reconstructing their individual routes through the modelling process in specific modelling situations. The aim is to identify (types of) individual cognitive barriers in the modelling processes.

This perspective is, of course, closely related to the educational perspective and the goal of developing mathematical modelling competency. However, it could also be considered as basic research on the learning of modelling competency, while research within the educational perspective might be characterised rather as applied science. The work by Boromeo Ferri (2006) is a good example of research within the cognitive perspective on mathematical modelling.

I only find one TSG21 paper clearly belonging to the cognitive perspective. That is the paper Camarena (TSG21). She investigates the cognitive skills activated in the modelling of an engineering problem. In this work a conceptualisation of a mathematical modelling process in engineering is used to structure the analysis. The cognitive elements are mainly related to (1) the mathematical conceptions that are needed in the mathematization process, (2) the more general cognitive skills in engineering related to the different phases of the modelling process, and (3) the specific cognitive elements concerning particular types of engineering models and problems. I regard this research as belonging primarily to the cognitive perspective. However, the main objective behind the research is to place the mathematical modelling competency as an important part of mathematics in a science and engineering context. The research, therefore, also has an educational perspective in relation to modelling competency as a goal.

The socio-critical perspective

Mathematical models of different kinds and complexity are playing important and growing roles in the functioning and forming of societies both in developing countries and in more developed countries. Mathematical models are used to define and describe social and economic inequality; both micro and macro economics are built on mathematical models of different kinds – interest rates in micro loans and mortgages for financing real estate, population and epidemic forecasts and control policies among others are based on mathematical models, while taxation and election systems are prescribed by mathematical models, and healthcare data and crime rates are described and discussed by means of statistical models. These and many more important aspects of societal life are being transformed and formatted through mathematical modelling and applications of mathematical models. Therefore, the development of an expert as well as a layman competence in the general population to critique mathematical models and the ways in which they are used in decision making is becoming imperative for the developing and maintaining of societies based on equality and democracy. Skovsmose has analysed the formatting power of mathematical modelling in detail and discussed its consequences for mathematics education in several papers and books, see Skovsmose (2005, part 2). This analysis forms an important part of the basis of the socio-critical perspective on mathematical modelling in mathematics education and is used as a reference in many of the TSG21 papers that I categorise as belonging to the socio-critical perspective.

The extensive use of mathematical modelling in society contributes to establishing mathematics as a language of power. Therefore, mathematics education, and especially the teaching of mathematical modelling and applications, possesses a potential for empowering students as autonomous and independent citizens in society. This phenomena is captured and analysed by D'Ambrosio (1999) using the concept *mathemacy* as a parallel to *literacy*. Moreover, the uncovering of the societal role and function of mathematical modelling in the teaching of mathematics can create an important motivation for learning mathematics and modelling among students. This seems to be especially evident in areas and countries with much poverty and inequality.

Empowering students to use mathematical modelling to reflect critically on societal issues and to criticize specific mathematical modelling processes and authentic applications of mathematical models in real life situations, is therefore pinpointed as an important goal for teaching mathematical modelling and mathematics in general under the socio-cultural perspective.

This leads to teaching practices where students are working with mathematical modelling in relation to real life problems – often of a social, medical and environmental nature. In this work particular emphasis is placed on the development among the students of a reflexive discourse on the modelling process. Quite a few of the TSG21 papers belong to the socio-critical perspective, and this perspective seems to be developing fast especially in some Latin American countries.

The paper by Barbosa (TSG21) is an example of this approach from Brazil. In this study the question is how to support the development of a reflexive discourse related to the students' mathematical modelling. The point of departure is the students' modelling of an environmental problem concerning the level of water in a lake used for power production. From the findings it seems that in this educational setting it was possible to create a reflexive discourse among the students concerning the criteria for a good model in the given situation and in relation to the comparison of the different models developed by the students.

Araújo's (TSG21) paper is another example of Brazilian research adhering to the socio-critical perspective. Here the thesis of the formatting power of mathematics is addressed directly. The paper analyses and discusses to what extent undergraduate students in a project where they work with modelling the yearly variation of rainfall in a particular region based on authentic data, actually realise the formatting power of their own modelling activity.

Caldeira (TSG21) is jet another interesting of developmental research project from Brazil within the socio-critical perspective. The paper describes and analyses a particular form of teaching practice implemented in teacher education up till middle school. The approach is called Participatory Environmental Diagnosis (PED) and the main idea is that groups of teacher students take as their point of departure their local district and identify a number of serious environmental problems in that district. In this practice, it seems as if mathematical modelling naturally becomes part of the teacher students' work when they describe, analyse and forecast the seriousness of the particular problems. In this work the objects of the teacher students' reflections and critique are the environmental problems and the societal handling of these problems, and here mathematical modelling becomes a means for such reflections, rather than a goal in itself.

The paper by Aravena & Caamaño (TSG21) reports on a developmental project at secondary level in Chile. The research in this project is focusing on the development of the students' mathematical modelling competency through project work. The students in the project were from a non-privileged area, and a decisive element in the design of course in the project was that it was up to the students themselves to decide on the theme of their modelling work. After preliminary investigation and serious discussions the students decided to work with the prevalence of breast cancer in the female population. The main educational focus in the project is on the formative impact on the students of working in project groups using mathematical modelling to analyse an important societal problem. I therefore consider this project as belonging to the socio-critical perspective rather than to the educational perspective.

Generally, within the socio-critical perspective on teaching mathematical modelling, reflection and critique play a dominant role. However, the object of such reflections and critique can be of different types and, from a research point of view, this is important to analyse more deeply in order to better understand how the students' development of critical reflections can be supported and also way this is important as a goal for mathematics education.

In relation to mathematical modelling and the applications of models critical reflections can focus either on (1) the modelling process or selected sub-processes, (2) the actual applications of a mathematical model or (3) the societal issues where mathematical modelling and models are used as means for analyses and critique of political decisions or societal phenomena. Of course, often there will be a interplay between reflections and critique concerning the three objects.

However, from a didactical point of view I ague that the case of (2) is the most complex one. Therefore, in the following I analyse a bit more deeply this case. A situation where a mathematical model is developed and – maybe later and in another context – applied as a basis for a technological and/or political decision can be depicted as in figure 3. A mathematical model has been developed for intended applications under the influence of certain interests held by the acting subjects. The modelling process that lies behind the model could be theoretically and/or empirically well or less well, poorly or even badly founded. However, when the model is applied as part of a political and/or a technological investigation, the model is often separated from the modelling process and from the possible critical reflections connected to the modelling process behind the model. The purpose of a specific application of a model may be different from what the model was originally constructed for, and in a new context the application of the model may be influenced by other interests. Therefore, the process of applying a mathematical model in general tends to cause the following types of side effects:

(a) Reformulation of the problem in hand in order for it to be appropriate for investigation by means of mathematical modelling

(b) Changes in the public discourse on the problem – issues that can be investigated by means of the model are becoming dominant in the discourse

(c) Delimitation of the possible actions or solutions taken into consideration to those, which can easily be evaluated in the model

(d) Delimitation of the group of people that takes part in the discussions and act as a basis of critique of the model and decisions taken based on the model.

Awareness of and experiences with such general phenomena can provide a strong basis for the students' external reflections concerning the use of mathematical models, and it is precisely one of the main concerns within the socio-critical perspective on the teaching and learning of mathematical modelling and applications.

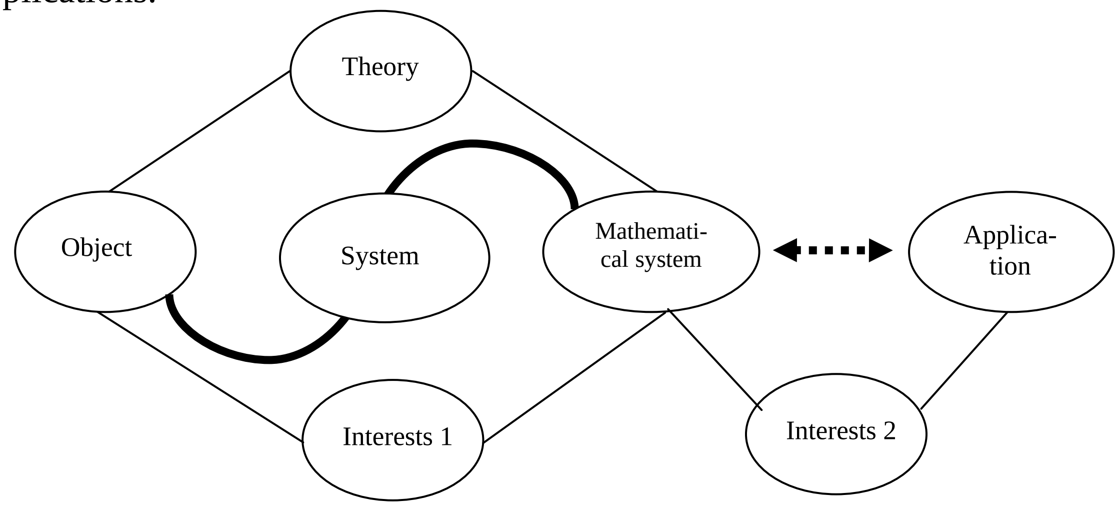


Figure 3: The application of a mathematical model to a real life problem is a process more or less independent of the modelling process. The application process often causes changes in the context of the real life problem as well as in the mathematical system when the model is adjusted to the problem. Moreover different types of interests acted out by different types of professionals such as researchers, technical experts, politicians or official administrators may influence the modelling process and the application process. Reflections connected to the scientific investigation related to the modelling process and the validation of the model tend to be disregarded in a technological and political decision making process.

Rounding off

As a closing remark I would like to repeat that the different perspectives in research on the teaching and learning of mathematical modelling are taken from Kaiser & Sriraman (2006) and used in this paper to categorise the TSG21 papers. At the TSG21 session at ICME-11 the focus was on the presentation and discussion of the individual papers but also on the characterisation of the papers within the six different perspectives. Based on the discussions the categorisation appeared to be meaningful and relevant for understanding and discussing the theoretical foundation of the individual research projects presented at TSG21.

The identification and characterisation of the perspectives, in itself, could and should of course be debated. However, this did not happen at the TSG21 sessions during the ICME-11, and is therefore left to be dealt with in coming activities in the field of research on the teaching and learning of mathematical modelling.

Template overview of different perspectives in research on the teaching and learning of mathematical modelling

Perspective	Aims	Background	TSG21 papers	Research question or aim	Role of modelling cycle
Realistic	Pragmatic-utilitarian goals	Pollak (1969)	Kadijevich (Lombardo & Jacobini)	What conditions and support (in form of IT) are needed to model a particular real problem?	Used for analysing a real life practice or problem situation.
Contextual	Subject related and psychological goals	Lesh & Doerr (2003) Lesh & Caylor (2007)		How to design contexts for students' meaningful modelling activities?	The modelling process is not in focus - Modelling Eliciting Activities are.
Educational – learning mathe-matics	Modelling as a means of learning mathematics	Niss (1987,1989) Blum & Niss (1991) Blum & Leiss (2005) Blum, Niss et al. (2006)	Lombardo & Jacobini vom Hofe et al. Ludwig & Xu; Meier Aravena & Caamaño Oliveira & Barbosa Rodríguez (Kadijevich)	How to challenge the students' mathematical conceptions (GVs) and how to support their mathematical learning?	Used for designing and analysing modelling tasks with respect to particular intentions for the students' learning.
Educational – learning modelling	Modelling competency as an educational goal			What is a good modelling task? What specific learning difficulties can be detected in the different phases in modelling	Used for defining mathematical modelling competency as a learning goal.
Epistemo-logical	Re-constructing mathematics through modelling, RME. Math. praxeologies	Freudenthal (1983) Traffers (1987) Chevallard	Andresen Tarp Siller	How can modelling be used to re-construct the function concept for learning?	Emphasising mathema-tisation and the "model of – model for" transition. Used to characterise a modelling praxeology.
Cognitive	Analyses of cognitive processes involved in mathematical modelling	Piaget, Skemp Boromeo Ferri (2006)	(Tarp) (Ludwig & Xu) Gamarena	Which cognitive structures are involved in modelling competency and which cognitive skills are related to the different phases in the modelling cycle?	Used to structure the modelling process so as to identify the cognitive skills needed to model a given situation
Socio-critical	Critical and reflexive understanding of reality and the use of mathematical modelling	Skovsmose (1994, 2005) D'Ambrosio (1999)	Barbosa Araújo Caldeira	Uncover the formatting power of math. modelling. How to create reflexive discourses among students?	To structure the critique and reflections in relation to the modelling process and the application process.

References

Blomhøj, M. (2004). Mathematical Modelling – A Theory for Practice. In Clarke, B. et al. (eds.), *International Perspectives on Learning and Teaching Mathematics.* Göteborg: National Center for Mathematics Education, pp. 145-159.

Blum, W. & Leiss, D. (2005). How do students and teachers deal with mathematical modelling problems? The example “Sugarloaf”. In Haines, C., Galbraith, P., Blum, W. and Khan, S. (2006), *Mathematical modelling (ICTMA 12): Education, engineering and economics*. Chichester: Horwood Publishing, pp. 222-231.

Blum, W., Galbraith, P. L., Henn, H.-W. & Niss, M. (eds.) (2007). *Modelling and applications in mathematics education*, The 14th ICMI-study 14. New York: Springer-Verlag.

Blum, W. & Niss, M. (1991). Applied mathematical problem solving, modelling, applications, and Links to Other Subjects. *Educational Studies in Mathematics*, 22, pp. 37-68.

Boromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process, *ZDM*, 38 (2). pp. 86-95.

D’Ambrosio, U. (1999). Literacy, mathemacy and technocracy: a trivium for today. *Mathematical thinking and learning*, 1 (2), pp.131-153.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures.* Dordrecht: Reidel.

García, J., Gascón, J., Higueras, L.R. & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics. *ZDM*, 38 (3). pp. 226-246.

Galbraith, P. & Stillman, G. (2006). A framework for identifying student blockages during transitions in the Modelling process. *ZDM*, 38 (2), pp. 143-162.

Gravemeijer, K. & Doorman, M. (1999). Context problems in realistic mathematics education: A calculus course as an example. *Educational Studies in Mathematics*, 39, 111-129.

Haines, C., Galbraith P., Blum, W. and Khan, S. (2006). *Mathematical modelling (ICTMA 12): Education, engineering and economics.* Chichester: Horwood Publishing.

Kaiser, G., Blomhøj, M. and Sriraman, B. (2006). Towards a didactical theory for mathematical modelling. *ZDM*, 38 (2), pp. 82-85.

Kaiser, G. and Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM,* 38 (3), pp. 302-310.

Lesh, R. & Doerr, H. (eds.) (2003). *Beyond Constructivism – Models and Modeling Perspectives on Mathematics Problem Solving, Learning and Teaching*. Mahwah: Lawrence Erlbaum.

Lesh, R. & Caylor, B. (2007). Introduction to the special issue: Modeling as application versus modeling as a way to create mathematics. *International Journal of computers for mathematical Learning*, 12, pp. 173-194.

Lesh, R. & Sriraman, B. (2005). Mathematics Education as Design Science. *ZDM*, 37 (6), pp. 490-505.

Niss, M. (1987). Application and modelling in mathematics curricula – state and trends. *International Journal for Mathematical Education in Science and Technology*, 18, pp. 487-505.

Niss, M. (1989). Aims and scope of applications and modelling in mathematics curricula. In Blum, W. et al. (eds.), *Application and modelling in learning and teaching mathematics.* Chichester: Ellis Horwood, pp. 22-31.

Niss, M., Blum, W. & Galbraith, P. (2007). Introduction. In Blum, W., Galbraith, P. L., Henn, H.-W. & Niss, M. (2007) (Eds.), *Modelling and applications in mathematics education*, The 14th ICMI-study 14. New York: Springer-Verlag, pp. 3-32.

Pollak, H. (1969). How can we teach application of mathematics? *Educational Studies in Mathematics*, 2, pp. 393-404.

Sánchez, M. (To appear, *c*). Uso crítico de los índices y modelos matemáticos gubernamentales en el desarrollo de profesores en servicio. *Educación Matemática*.

Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education*. Dordrecht: Kluwer.

Skovsmose, O. (2005). *Travelling through education. Uncertainty, mathematics, responsibility*. Rotterdam: Sense Publishers.

Treffers, A. (1987). *Three dimensions: a model of goal and theory descriptions in mathematics instruction – the Wiskobas Project*. Dordrecht: Kluwer.

DIFFERENTIAL EQUATIONS AS A TOOL FOR MATHEMATICAL MODELING IN PHYSICS AND MATHEMATICS COURSES:

A STUDY OF HIGH SCHOOL TEXTBOOKS AND THE MODELING PROCESSES OF SENIOR HIGH STUDENTS

Ruth Rodríguez Gallegos

ruthrdz@itesm.mx
ITESM Campus Monterrey (MEXICO),
Joseph Fourier University (FRANCE)

Abstract

This paper proposes a study which deals with the learning and teaching of mathematical modeling in physics and mathematics courses. It was oriented specifically for the senior high school students in France. In 2002, the new syllabi for the Physics and Mathematics courses emphasized the role of mathematics as a tool for modeling in other sciences. Firstly, a description of the modeling process was established for this work. Secondly, the textbooks commonly used in the Physics and Mathematics courses were analyzed. These analyses revealed the transposition process of the "modeling process" practiced by the experts into a different process adapted for school. The setting up of an experimental situation including some unusual tasks (out of the scope of the common didactic contract) for senior high school students allowed the identification of the influence of the existing "praxeologies" in these classes when students were subjected to problem-solving situations. Some of the difficulties linked to the setting up of this transposition process ware analyzed and are presented in the study.

1. Introduction

Nowadays, society has new expectations about the skills of the young individuals. Particularly, some international studies have established the importance of the development of individual skills to model and solve real life problems (OCDE, 2003). In one hand, the Programme for International Students Assessment (PISA), a triennial survey of the knowledge and skills of 15-years-olds, looks in detail at mathematics performance. "PISA uses a concept of mathematical literacy that is concerned with the capacity of students to analyze, reason and communicate effectively as they pose, solve and interpret mathematical problems in a variety of situations involving quantitative, spatial, probabilistic or other mathematical concepts" (OCDE 2007). In the other hand, in 2002, the new syllabi for the Physics and Mathematics courses emphasized the role of mathematics as a tool for modeling in other sciences.

This article proposes a study that deals with the learning and teaching of modeling, specifically in senior high school Math and Physics courses in France. The aims of this work were to study how the modeling processes "becomes alive" in French schools and to identify the students' difficulties in modeling a real life problem.

2. The framework in use

2.1 Modeling a mathematical modeling

The first step was to establish a description of the "modeling" process in this work. To establish this definition, a review of several research works about the subject was necessary; for instance, Study 14 of the International Commission on Mathematical Instruction (ICMI) and the memories of some international conferences, such as International Congress on Mathematical Education (ICME) and the International Community of Teachers of Mathematical Modelling and Applications (ICTMA). Finally, this description was built considering the definitions used by Blum and Niss (1991), Kaiser (1995) and Henry's (2001) works. The final definition to be used in this work is represented in Figure 1 (for a detailed description of this seven-stage process see Rodriguez 2007):

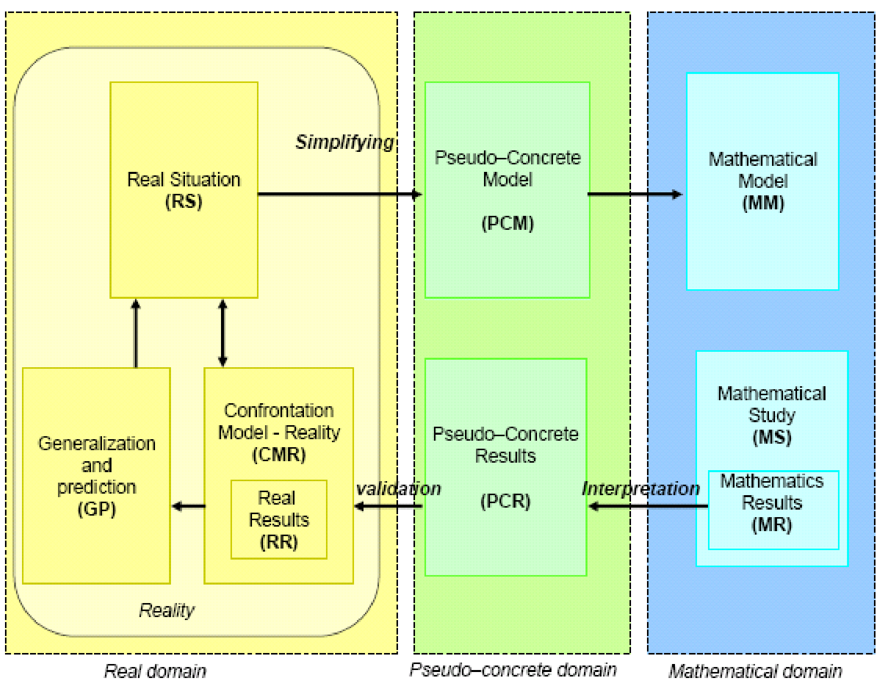


Figure I: Description of the Modeling Process

Based on the description of "mathematical modeling" established before, some of the textbooks commonly used in the Physics and Mathematics courses were analyzed. The results of these analyses allowed the characterization of the proposed modeling process "to be taught" in the senior year of high school. The methodology followed to analyze the textbooks is described in the next section.

2.2 The didactical transposition

This study used the notion of *praxeologies* as a useful tool to analyze textbooks. This notion was taken from Chevallard's anthropological theory (1999) which has been used extensively and further developed by many French researchers such as Artaud (2007). She says that "in this approach, two main aspects are considered. The first one regards what is learnt and taught and is modeled in terms of mathematical praxeologies. The second one is concerned with the learning and teaching activities as such and is modeled in terms of didactic praxeologies" (Artaud, 2007, page 373). A praxeology has four components:

- a number of types of tasks T, which refers to what one has to do
- a technique τ , which provides a way to achieve tasks of the given type
- a technology θ, for every technique, which is the "discourse" that justifies and explains the technique
- a theory Θ, which is the "discourse" that justifies and explains the technology

In order to carry out this analysis and taking into account a first review of the French syllabi and textbooks, the notion of "differential equation" was focused on a modeling tool. This notion is first taught in high school. The methodology followed in textbooks was identified in a first review. From the analysis of the chapter of "Differential Equations", it was determined and classified the kind of tasks students are commonly demanded to do. In the second part of the analysis, this classification of tasks was validated by analyzing the content of the chapter. If, in this part, there was some reference to the task, this kind of task was kept since it was representative of a task demanded from students. When the list of tasks was validated, a second detailed analysis took place to identify the techniques utilized for each task, and to, eventually, find out if a technology or theory had been considered in the chapter.

3. The first results: the analysis of textbooks

The analysis of the math textbooks allowed the identification of other types of tasks to be practiced by students in the course:

Type of task	*Transition between phases in the modeling process*	*Description of the task*
T_{DE}	Pseudo-Concrete Model → Mathematical Model	Set up a differential equation which models a real situation in pseudo-concrete terms (found in the explanatory texts of each exercise) that will lead to a Mathematical Model
T_{GS}	Mathematical Model → Mathematical Study	Find a general solution of the differential equation
T_{PS}	Mathematical Model → Mathematical Study	Find a particular solution using an initial condition (given in the explanatory text of the exercise)
T_{AQ}	Pseudo Concrete Domain → Mathematical Model	Answer a question, formulated in pseudo-concrete terms, based on the mathematical results obtained

The task T_{DE} to "set up a differential equation" to model a real situation is rarely demanded from students. Most of the times this equation is given by the exercise; and sometimes this kind of tasks are reduced to "justifying" that the model is an equation given on the statement. It was also observed that in a Mathematics class, "to solve" a differential equation means to use a theorem previously demonstrated by the teacher in class. Writing a mathematical model (in this case, a differential equation) is an important stage in the modeling process, but this study confirmed the absence of this in-class task, so it was decided to extend the domain of the study to the Physics class. Three textbooks of the Physics class were analyzed, in particular the chapters of "Circuit RC" by using the same methodology followed for the math course. The types of tasks identified in this class are shown in the following table:

Type of task	*Transition between phases in the modeling process*	*Description of the task*
T_{EC}	Pseudo-Concrete Model → Physical Model	Represent an electric circuit scheme diagram (a resistor-capacitor circuit, RC circuit)

T_{DE}	Physical Model → Mathematical Model	Set up a differential equation which models the tension capacitor $U_C(t)$ present in the circuit.
T_{PS}	Mathematical Model → Mathematical Study	Find a particular solution (verifying that a given function in the exercises is the solution of the differential equation
T_I	Pseudo-Concrete Domain ↔ Physical Domain ↔ Mathematical Domain	Determine the intensity of the electric current i(t) in the circuit using the function $U_C(t)$

Therefore, it was evident the need to insert a Physical Domain in the modeling process of reference in order to describe the modeling process that took place in the Physics course. The original scheme was modified; thus, a "new" description was devised for the next part in this paper, as shown in Figure II.

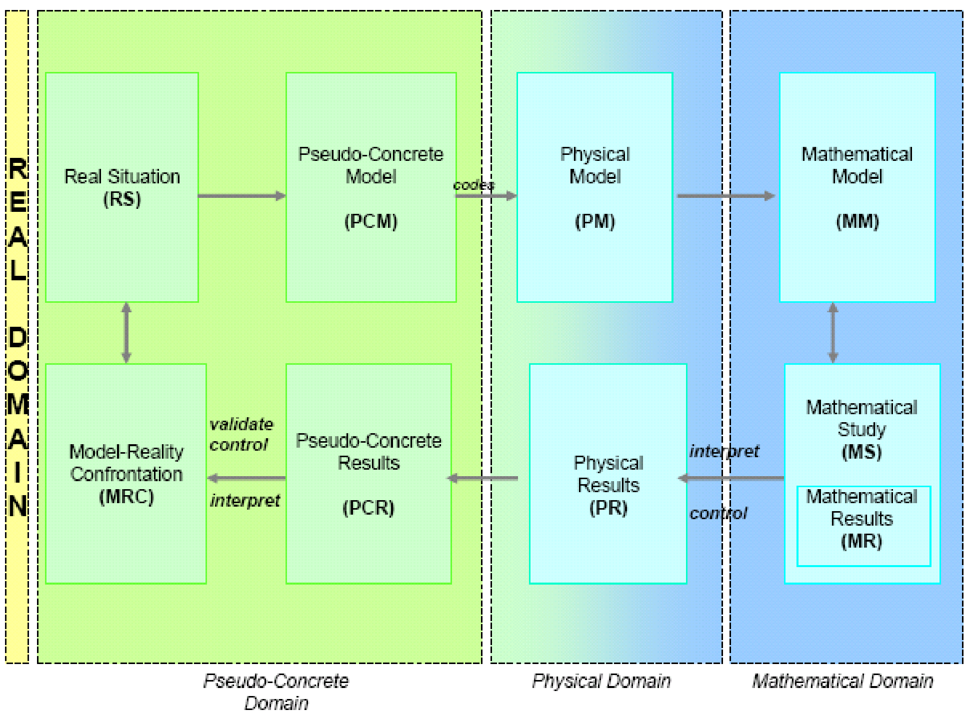


Figure II: Modeling process in a Physics course

It is important to indicate the insertion of the Physical Domain in the original process of modeling. Two stages were included in this domain. The first stage was concerned with the physical model. In the case of the electric circuit, a physical device or a circuit diagram in paper were included. The second stage was related to the mathematical treatment of the differential equation, which modeled the intensity in the circuit. Consequently, some physical results could be obtained. Physics teachers might find this process significant when working with mathematical modeling.

As a conclusion for the part regarding the textbook analyses, these revealed the transposition process (Chevallard, 1991) of the experts' modeling processes into a different process used in schools. Even when the modeling process is practiced more often in the Physics class than in the Math class, some important shortcomings were observed and these are commented below:

*Even if the type of task T_{EC} "Represent an electric circuit scheme" appears in the textbooks, it is a task not commonly assigned.

*The task T_{DE} "Set up a mathematical model" (a differential equation) is usually assigned to the students, but the steps to do it are given in the exercise

*In the Mathematics and Physics classes, there was a lack of exercises to make the student face the transition from the stages of Pseudo-Concrete Results to those of Model-Reality Confrontation. Henry (2001), among other researchers, has considered this transition important from a didactic point of view.

The lack of tasks to make students face modeling situation along with the experts' processes led to the design of a new experimental situation.

4. A brief description of the experimental situation

The second aim of this work was to identify the students' difficulties to model a real life problem built around the results of the textbooks' analysis. Special attention was paid to the kind of tasks and techniques available in the textbooks and usually presented to students in class. An experimental situation was set up to include some unusual tasks (out of the scope of the common didactic contract) for the senior high-school students. Following are the three characteristics chosen to design this experimental situation.

1. To confront students with the transition from Real Situation + Pseudo-Concrete Model towards the construction of the Physical Model. This confrontation was normally absent according to the textbook analysis of the physics class.

2. To provide no directions to students regarding the writing of the Mathematical Model (transition from Physical $\rightarrow$ Mathematical Model). A guide was usually observed to establish a differential equation such as a mathematical model of a particular situation in math and physics class.

3. To confront the students to the transition from Pseudo-Concrete Results → Model-Reality Confrontation. This confrontation was also absent in both of the analyzed classes.

In the experimental situation, the students were proposed to do the modeling of the functioning of a defibrillator. This electronic device applies an electric shock to restore the rhythm of a fibrillating heart. A description about the functioning of this device was introduced in a text. It explained to the students the mechanism of operation in physical (electric) terms.

After the introductory text, students are given a key question. Then, with a set of five tasks, students are guided to reach the answer for the original question:

"What is the probability of survival of a man who has a cardiac problem in the street and is assisted with a defibrillator?"

Out of five tasks, tasks A and B are provided as examples of the following:

Task A: *We need to model the defibrillator with an electric circuit, like those studied in class. Draw an electric circuit diagram and justify your choice.*

A possible (and correct) answer for this question was an RC circuit diagram like the one shown in Figure III.

Figure III: Possible answer to question A

It is worthwhile noting the presence of the photo in the figure, showing the use of the defibrillator. The photo might have influenced the students' answers.

This task is related to the type of task T_{EC}: Represent an electric circuit scheme diagram (a resistor-capacitor circuit, RC circuit) observed in the physics ´textbooks.

Task B: *Set up a model (a differential equation) for the tension in the defibrillator. Justify the laws that were used to establish the model.*

A possible (and correct) answer for this question was $\frac{dU_C}{dt}+\frac{1}{RC}U_C=0$.

This task is related to the type of task T_{DE}: Set up a differential equation which models the tension capacitor $U_C(t)$ present in the circuit.

There were three other verification tasks proposed by the activity:

Task C: *Verify that $Ae^{-t/RC}$ is solution of this differential equation. Determine the value of the constant A using an initial condition.*

In this case, the initial condition is given implicitly by the statement of the activity. The student needs to consider that Uc (0)= E, E is the tension in the capacitor at time t = 0. Next task (D) is related to the type of task T_{PS} *Find a particular solution*.

Task D: *We know that only 4% of the electric current is received by the patient. You need to precise the maximal current received by the patient´s heart.*

The students need to establish that i(t) =Uc(t) / R using the particular solution of the differential equation founded in task C. This task is related to the type of task T_I: Determine the intensity of the electric current i(t) using the function $U_C(t)$.

Task E: *Compare the result of the task D with the data of chart I. In base of this comparison, determine if the patient has (or not) possibilities to survive.*

In the chart (given to the students), they can observe the different reactions of the human body to the electric current.

In the next section, the results for tasks A and B are analyzed (see Chapter VI in Rodriguez -2007- for a more detailed analysis of the students' productions).

5. Experimentation settings

The experimentation was done with 25 students in their senior year of three different French high-schools. The setting of the experimentation was done after teaching the topic of "Electrics Circuits". The students worked in pairs and had one hour to solve the problem. Taking into account the type of tasks usually demanded form the students in the physics class and some elements of the techniques found in the textbooks, the answers to the experimental situation provided by the pairs were analyzed. The analysis also included the final response concerned with had the patient been or not been able to survive. It was of particular interest to determine what difficulties students had in this kind of tasks and if the textbooks presented some problem-solving technique. Also, it was studied how the students solved the problem. Finally, it was interesting to specify what stages (and transitions in between) of the modeling process were more challenging for the students.

6. Results: the modeling activity of the students

The students' answers for questions A and B allowed the identification of the influence of the existing *praxeologies* in the Mathematics courses, and mainly in the Physics ones, in the students' solving processes.

About Task A: A difficulty observed in question A was that students found it hard to draw a diagram with all the physical elements of the circuit such as resistances, capacitors, etc. Some “hybrid” configurations appeared. The difficulty to insert a resistance as part of the electric circuit (replacing the patient’s thorax) was recognized from the students’ solutions. Even if the word “resistance” appeared in the introductory of the activity, this word did not make specific reference to a physical term. This could be observed in the answers of two pairs of students:

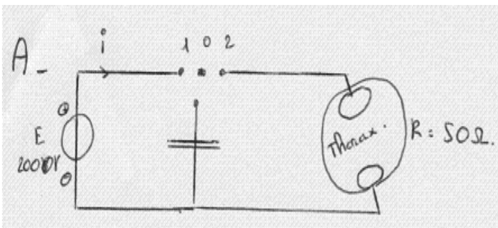


Figure IV

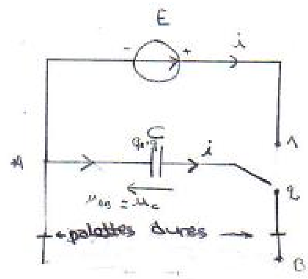


Figure V

In figure IV the thorax is represented by a “circle” and the electrodes are also represented in the diagram. Even if the legend “R = 50 Ω” referred to a resistance, it was difficult for these students to use the “correct” electrical symbol as observed. It is possible the image present in the activity can have an influence to the answer of these students to task B. It seems that these students can represent the defibrillator as a electrical devices (such as a generator and a capacitor) but it is difficult to them to represent a human body (the patient) as a electrical components and for this reason they decided to use a symbols non conventional as a circles to represent the patient (thorax). As well, in figure V, the electrodes were drawn by the students (“*palettes dures*” in French) but in this case, the students did not include any reference to a resistance in their circuit. This pair ignored the place of the patient. Students might not have considered the patient as part of the circuit since the patient had no electrical nature regardless of his importance in this scheme.

Some other students proposed a scheme from a “physical model” such as the one shown in Figure VI:

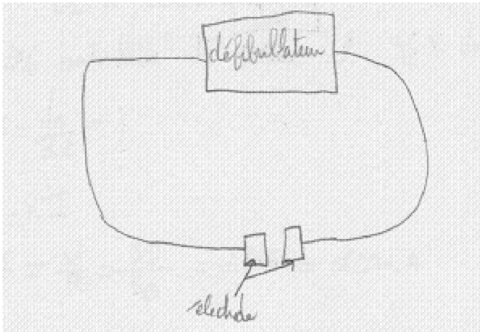


Figure VI

In figure VI, the students were completely in a Pseudo-Concrete Domain. The elements shown in the diagram were not physical elements. This answer could not be considered at all. It made us wonder about the students' comprehension of the task assigned. This difficulty could be located in the transition from Real Situation + Pseudo-Concrete Model (text of the activity) to the writing of a Physical Model. In reference to the observed praxeologies in the analyzed textbooks from the first part, no technique was observed in these manuals to perform this kind of task, even if it was an important step from a modeling point of view.

About Task B: For question B, several difficulties were identified to propose a differential equation to model the capacitor tension in the circuit. Some students forgot the Physique laws to do that as well as how to establish a relation between the magnitudes involved. However, they sometimes used incorrect physical laws or principles. Some pairs of students established a differential equation for the charge capacitor q(t): $\frac{dq}{dt}+\frac{1}{RC}q=0$ (figure VII):

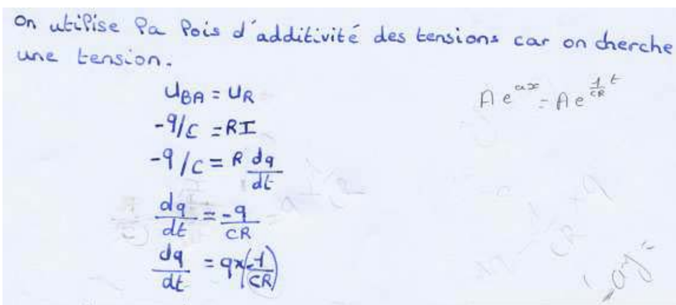


Figure VII

It is worthwhile mentioning that this kind of answer in the students' productions was observed since there was no technique in class or in the textbooks suggesting this procedure. It seemed "natural" for the students to establish this differential equation. Once again, no technique was observed in the analyzed textbooks.

Another difficulty observed was the explicit explanation (or its absence) about the selection of the study between the charge or discharge of the capacitor and the resistance. The interest in this case was to study the discharge from the defibrillator to the patient, but an important number of students apparently “forgot” that fact. In many cases, question A worked as the validation for the model proposed and it helped the students “to correct” the differential equation proposed. An example of this is illustrated in Figure VIII.

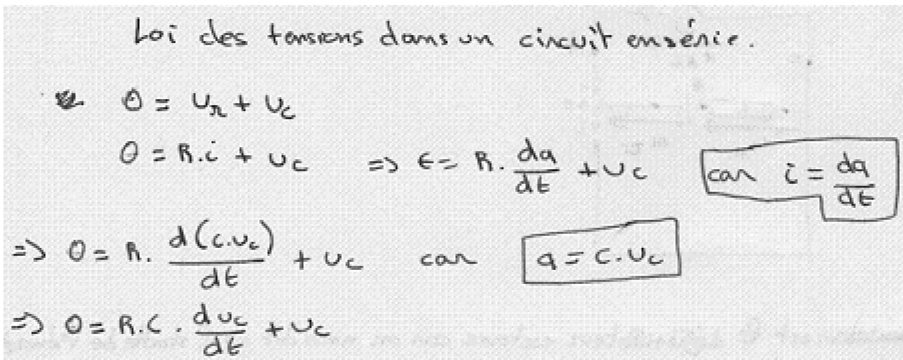


Figure VIII

As it can be seen, symbol “E” was erased from the last two equations because these students had written an “E” instead of a “0” (zero) which corresponded to the generator charge from the circuit if the discharge was studied. This validation occurs after realize the task C. The task C seems to be a form of validation not only for the answer to task B even to answer given to task A (the electrical diagram). A more detailed description will be given in the next section.

About Task C: This task asked for verification of the solution of the differential equation for a given function. The feedback given for task C allowed the validation or/and correction of the model proposed in task B. This difficulty showed the absence of relationships between the physical phenomena to study (real situation described) and their modeling (discharge of a RC circuit + differential equation).

For example, a group of students (see figure IX) take in account that the value of E need to be zero because the verification of a function is a solution of the DE, they arrived to the conclusion E = 0. After this process, these students reformulate the particular solution established in task B and also about the condition around the phenomenon taken in account for the task A.

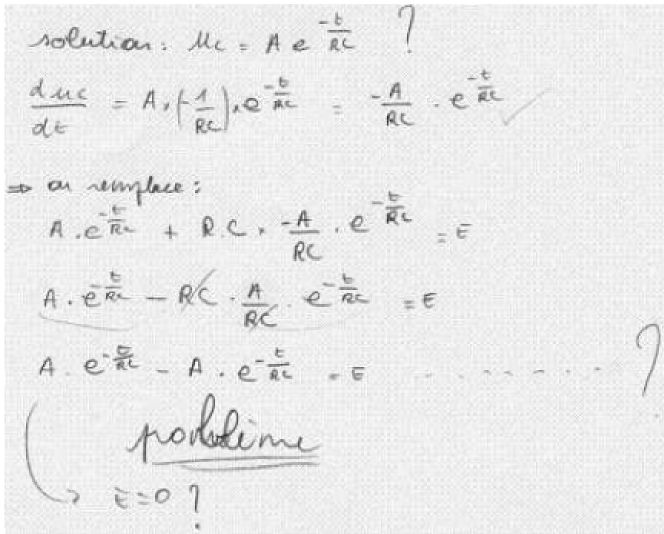


Figure IX

The students' solution of this experimental situation showed evidence of the role of the « pseudo-concrete » model for both, the initial real situation and the students' physical model, which they built by using the modeling approach. The results can illustrate the importance of having a clear understanding of the situation to be modeled as a consequence, establishing a pseudo-concrete model to build a correct mathematical model. The benefits of the transition between the "objects and world events" (called the real domain in this work) and the "models and world theories" (called the physical and mathematical domains) had already been observed and documented by other researchers like Thiberghien and Vince (2004). The conclusions of this study agree with these researchers particularly in the importance of students' self practice of the task "Real Situation→Physical Model→Mathematical Model," which are crucial for the acquisition of the modeling process. However, the exploration of the study revealed that it can be difficult to attain. The difficulties more frequently observed in the students' answers are showed below and also some of them are situated related to the stages of the modeling process of reference in this study and also respect the transition between these stages:

Macro-level ***(stage or transitions between stages into the modeling process)***	***Task of the Experimental Situation***	***Micro-level*** ***(praxeologies observed in the textbooks)***	***Description of the difficulty identified***
Transition *SR* $\rightarrow MPC \rightarrow MP$	*Task A : configuration du circuit*	*No technique related to* T_{CIR}	*1.The students don´t need necessary to include a resistance in the electric circuit to represent the thorax of the patient.*
MM	*Task B*	**Praxis* (T_{ME}, τ_{ME}) ** No technique taught in class about this* **Not link between the Physical Domain and the Mathematical Domain*	*2. The students forget some physical relations and laws and/or how related them to set up a differential equation.* *Sometimes, they use an incorrect way of these laws or principles.* *2'. The students establish a DE in terms of the charge q in the circuit and not to the tension Uc in the capacitor.* *2'' The students don´t establish the value E = 0, corresponding to a initial condition so important to solve the DE. That correspond to an inadequate comprehension to the situation to model*
$MM \rightarrow EM$	*Task C* $\rightarrow B$	**Praxis* $(\mathrm{T}_{RED}, \tau_{RED/D})$	*3.Not defined a value to E permit a validation for the answer given in task A and also for the task B*

EM→SR→EM	*Task C*	*No technique related to this kind of sub-task*	*4. An important difficulty to establish the initial condition from the introductory text in the experimental situation..*
RM→RP→RPC	*Task D*	*Praxis* $(T_{Ia}, \tau_{Ia} / \tau_{Ia'})$ $(T_{Ib}, \tau_{Ib} / \tau_{Ib'})$	*5.The students don´t consider the electric current i as a function of temps t.*

In some occasions, the non existence of a praxeology (in general a praxis) related a specific kind of task can explain the difficulties of the students to solve a task. There are other examples where praxis is given by the textbooks implicitly or explicitly but the students have several difficulties to accomplish the task correctly. In several of these important difficulties identified are related to several transitions between stages into the modeling process and even between transitions between the Pseudo-Concrete Domain, the Physical Domain and the Mathematical Domain.

6. Conclusions

To conclude, this study agrees with Chevallard’s work (1991) about the type of modeling that is finally taught (“taught” knowledge) in the Physics and Mathematics courses. It also stands for an important gap in regards to the experts’ modeling processes (the “wise” knowledge). An important future objective in this work is to propose a general characteristic to design scholar activities to taught and learn the modeling process. The process of modeling defined as a point of departure in this work can be modified according to some issues observed in the student’s activities. Similar to research of Borromeo (2006 and 2007), an empirical differentiation to the stages in the modeling process can be formulated in base of the procedures realized by the students during the resolution of the activity proposed. In particular, for future works a cognitive perspective will be considered because a praxeology approach can permit to understand a side of the phenomenon about the learning and teaching modeling but not consider understanding the cognitive processes involved in the subject. For example, include the stage “mental representation of the situation” (MRS) as a Borromeo (2006, 2007) is an interesting way to study the mathematical modeling.

It is also important to emphasize how to define or where to find definitions for the experts’ practice, topic which is commonly debated amongst the mathematical modeling community members. The definition can be customized depending on the scientific field where a specific situation needs to be modeled. This must be an interesting subject to discuss and develop in future modeling works. Another important issue in this work is the importance of both, the construction of an adequate pseudo-concrete model and the physics model which leads students to model it themselves. This study finds that it is absolutely necessary that the aforementioned be done in school. Secondly, giving feedback from one task onto another develops a proper solution of each task. This is an important feature to take into account in the design of future modeling activities. In future papers, the relevance of teachers' external interventions to help students overcome their difficulties could also be discussed. Learning acquisition is learning through practice. Therefore this study recommends the design of activities with all possible modeling phases as well as teacher training courses in this topic.

References

Artaud, M. (2007). Some conditions for modeling to exist in mathematics classrooms. *Modeling and Applications in Mathematics Education. The 14th ICMI Study* (371-378). New York: International Commission on Mathematical Instruction ICMI.

Blum, W. & Niss, M. (1991). Applied mathematical problem solving, modeling, applications, and links to other subjects – State, trends and issues in mathematics instruction. *Educational Studies in Mathematics 22(1)*, 37-68.

Borromeo Ferri, R. (2006). Theoretical and empirical differentiations of phases in the modelling process. Zentralblatt für Didaktik der Mathematik 38(2),86-95.

Borromeo, Ferri, R. (2007). Modelling problems from a cognitive perspective. Editado por Haines, Galbraith, Blum and Khan. Mathematical Modelling (ICTMA 12): Education, Engineering and Economics (260-270). Horwood Publishing. Chichester, UK.

Chevallard, Y. (1991). *La transposition didactique - Du savoir savant au savoir enseigné*, deuxième édition. Grenoble : La Pensée Sauvage éditions.

Chevallard, Y. (1999). L'analyse de pratiques d'enseignantes en théorie anthropologique du didactique. *Recherche en Didactique des Mathématiques 19(2)*, 221-266.

Henry, M. (2001). Notion de modèle et modélisation dans l'enseignement. In Henry, M. (Ed.), *Autour de la modélisation en probabilités* (149-159). Besançon : Commission Inter-IREM Statistique et Probabilités.

Kaiser, G. (2005). *Mathematical Modeling in school – Examples and Experiences*. (Article issued at a seminary of the Education Department, University of Hamburg, Hamburg, Germany).

Organisation for Economic Co-operation and Development [OCDE] (2003). *The PISA 2003 Assessment Framework-Mathematics, Reading, Science and Problem Solving Knowledge and Skills*, 194. Retrieved October 13, 2008, from http://www.oecd.org/dataoecd/46/14/33694881.pdf

Organisation for Economic Co-operation and Development [OCDE] (2007). *PISA 2006: Science Competencies for Tomorrow´sWorld Executive Summary*, 55. Retrieved October 13, 2008, from http://www.pisa.oecd.org/dataoecd/15/13/39725224.pdf

Rodriguez, R. (2007). *Les équations différentielles comme outil de modélisation en Classe de Physique et des Mathématiques au lycée : une étude de manuels et de processus de modélisation en Terminale S*. Doctoral dissertation, University Joseph Fourier, Grenoble, France.

Tiberghien, A., & Vince, J. (2004). Études de l'activité des élèves de lycée en situation d'enseignement de la physique. In Pugibet, V. et Gettliffe-Grant, N. (Eds.), *Cahiers du Français Contemporain (*10). ENS Editions.

MATHEMATICAL MODELING: FROM CLASSROOM TO THE REAL WORLD

Denise Helena Lombardo Ferreira
Otavio Roberto Jacobini
Pontifícia Universidade Católica de Campinas

Abstract: *In this article we approach the mathematical modeling as a methodological option for the Linear Programming discipline. The mathematical modeling activities allowed for establishing a relationship between mathematical contents approached in the discipline with some problems related to the students' reality. In this article we have, as an objective, to analyze the possibility of pedagogical contributions when using such association between curricular contents and the application of mathematical modeling, with the support of technology, in daily life student situation, especially when such situations are related to their current or future professional activities. As main results we highlight the perceptions by the students about the relevance of the discipline, being it for their intellectual formation or for their professional valorization and for the technology applicability, and the collaborative environment built among the students which contributed for their interaction in the group assignments and for academic and professional experiences exchange.*

Keywords: *Mathematical Modeling, Linear Programming, Collaboration.*

1. Introduction

It is quite usual for college students to show difficulties when it comes to learning disciplines like Mathematics, even for those who attended exact sciences courses. Such difficulties, on one hand, may become worse if the students are not able to envision an usage for what they are studying and, usually, they study only for getting ahead on the course. On the other hand the student involvement may occur in a deeper way if the studied contents in the subjects linked to mathematics were directly connected to the subjects of the chosen course. We have noticed the existence of these two situations in the disciplines that we teach.

Most of the students would rather attend mathematic classes which do present some level of connection to reality. For such students this connection would allow for more meaningful and less stressing learning. By approaching the perspectives that guide the concepts of mathematic literacy extrapolating the traditional concept related to the abilities for calculation and problems solution and enlarging the horizons of its meaning, Jablonka (2003) says that bringing to the classroom the mathematic used on working environment is one of the ways for associating the out of school mathematic with the curricular contents and, consequently, of showing the mathematical practical utility. This association, by contributing for the meaningfulness of the teaching activities for the students, on top of reducing what is known as mathematical anxiety towards learning the concepts and handling numbers and algorithms, allows for the relationship between the academic and professional learning, valorizing the diversity of culture (mathematic) present on the working places.

However, many times, relating mathematical techniques used routinely in the work place to curricular mathematic is quite hard because, on top of the rigidity that usually characterizes the courses curriculum, the meaningful concept, in this case, must be relativized in a way that, what may be meaningful for one student may not be for another. Besides, most of the time, the association between mathematic and reality demands more effort and commitment from the students than traditional classes centered on teacher lectures. It also demands available time from the students for research and for other tasks away from the classroom. This situation is worse for night class, where most of the students do work during day time and do not have available time for out of class activities, required for tasks of such nature.

With the challenge of helping students on the understanding of the Linear Programming discipline, the first author of this article has conducted a pedagogical experience using mathematical modeling mathematical modeling on her classes for the Information Systems Course students on the night shift of a Campinas City private school in Brazil. These students, usually, work professionally in activities related to the data processing field.

So, based on such experience, we have as the objective, to evaluate the possibilities for the teaching and the learning the mathematical contents in graduation courses, when the Mathematical modeling mathematical modeling is used with the support of technology, based on problems related to the students daily life, mainly when such situations are related to their professional activities, current or future.

Next, on this article, right after some reflections over Mathematical Modeling, we approach the methodology used in the study, we present the built environment and discuss the results reached.

2. Some reflections over mathematical modeling in the classroom

The idea of creating models for understanding and studying a wide variety of phenomenon is very old, since the man across time has used real world representations in order to get a solution for the built model. The validation of such models is done through analysis, reflections and discussions over the reached results.

Mathematical models are representations, in mathematical terms, of the aspects of interest from the problem being studied and can be formulated "[…] using numerical expressions or formulas, diagrams, graphs or geometric representations, algebraic equations, tables etc" (BIEMBENGUT; HEIN, 2000, p. 12). We highlight that one single model, with minor adaptations, can represent many applications. This is very useful in the professional modeling as well as in the classroom modeling, because it allows for the utilization of a single model to solve different situations.

The mathematic and the reality can be connected through modeling. This interactive connection is done by using known mathematical process, with the objective of studying, analyzing, explaining, forecast real daily life situations around us (CAMPOS, 2007).

The mathematical modeling, in a direction, on top of being an important applied mathematical tool for solving real problems, also creates the need for data gathering and for simplification of real situations. In this same direction Mathematical modeling mathematical modeling favors the construction of an environment where the students can perform simulations and analogies, considering that the same model can be useful in the representation of many different situations, helping the students in the identification of applications in other areas of knowledge and under different environments.

In a second direction, the Mathematical modeling Mathematical modeling identify itself with a pedagogical perspective focused in the citizenship construction and a social/political conscience of the student, who aims at valorizing their individual abilities required for an effective participation in a democratic society and, similarly to the thinking of Skovsmose (1996), emphasizing the critic evaluation of the practices that involve mathematic, taking into consideration the cultural environment to which all the students belong. This identification comprehend the central core of a mathematical literacy turned towards social changes, as proposes Jablonka (2003), aimed at the formation of a critical citizen, with a power of arguing and, as said by Jacobini and Wodewotzki (2006) interested in questioning the social issues relevant to the community.

In a third direction, we see the role of data processing technology as an indispensable actor for working with mathematical modeling, being it as a operational supporting tool or as a instrument that comes for contributing for overcoming many challenges frequently found in traditional classroom, such as students lack of interest or lack of required abilities for the working environment. We highlight that we are not alone in valorizing data processing technology role considering that currently, most of the researchers interested in the mathematical modeling considers indispensable, in their studies, the presence of such technology.

3. Methodology

Aiming at creating conditions for analyzing the association among curricular contents and application of mathematical modeling with the support of technology in the students daily life situations, the first author of this paper conducted, during the second semester of 2007, a pedagogical experiment on the Linear Program discipline, which is part of the Information Systems Course at a Private College in Campinas, Brazil, where the students were encouraged to work on real situations. In this experience, as previously said, we have tried to emphasize the construction of knowledge and to make the students more critical and with a higher arguing skill.

Linear Programming discipline is taught on the third year of Information Systems Course and, at that point, the students usually are already working, not as interns, but rather as regular employees. This way, the time spent with school activities is very low, mainly for mathematical disciplines which are considered as supporting subjects to the ones specific to the students' formation. This situation gets worse as the students are not able to visualize immediate usage for the activities currently being developed at the Companies they are working for. We believe that this is the main reason for the difficulties the students face at the discipline of Linear Programming and such situation generate discomfort for the students as well as for the teacher. Many times the students are dependent only on this discipline to graduate from the Course and, consequently, get better opportunities at the Company. On top of this, the requirement for linear algebra knowledge do established discomfort moments for many students that, by the time they attended such discipline, not only presented difficulties in understanding the concepts, which are in general abstract, but also, some of them, have failed in this discipline.

Lately it has been part of the evaluation process of this discipline, the development of assignments involving practical applications of the subjects studied. At such assignments the students (groups of two) chose problems related to what is being taught at the discipline, search for data, model the problems, that is, try to mathematically represent it, solve the problems using software required that are available, analyze and validate, whenever possible, the solution found. Since most of the students work professionally it is quite usual that they gather data from the Companies they are working for and, next step, present the mathematical formulation in the Linear Programming representation. However, we have also found students not interested in working with real problems from their daily life due to, mainly, the complexity demanded for data and information gathering, as well as the mathematical representation, they would rather work with the problems available on the text books.

In any situation, gathering real data or using data from text books, it is required from the students to use the software to solve the problem represented as Linear Programming. They may ask for a license from a software supplier for using a specific tool, such as LINGO – Language for Interactive Optimizer or use the resources available at the Microsoft Office Excel.

By the end of the semester we request the students to present the assignment developed to their classmates. The doubles present the problem, the Linear Programming formulation, justifying the variables, the objective function and the restrictions. In the sequence, they present the solution they arrived at and its interpretation and, at last, some simulations related to the theory taught in classroom. In some cases, simplifications and analogies are presented. It is worthwhile to highlight that some of the assignments envision Integer Programming or Non-linear Programming for treating its applications.

The project evaluation represents 20% of the final students' grade and, for such evaluation, it's taken into consideration: a) the complexity of the problem chosen; b) the mathematical formulation as well as the justification for the objective function, the restrictions and the variables; c) software simulations for theory checking and sensitivity analysis.

4. The modeling environment built during the second semester of 2007

We have considered the learning environment as an educational space built by the teacher aiming at developing its pedagogical activities. The expositive class at which the teacher centers at herself the teaching task, the working with cooperative learning or in small groups based on situations brought by the teacher, exploratory activities through data processing technology, the learning based on problems solution, through ethno-mathematics, the mathematical modeling mathematical modeling or the working with projects are some of the examples of learning. (JACOBINI; WODEWOTZKI, 2006).

From the perspective of mathematical modeling mathematical modeling as a teaching methodology, we have considered adequate to conceptualize it as a learning environment (to be built in the classroom) at which the students are invited (by the teacher) to investigate, through mathematic and with the support of technology, situation-problem brought from the students daily life, specially when such situations are related to their professional activities, current or future.

The creation of a pedagogical scenario where the students are incited to investigate actual situations, connected to a Company daily life and, based on the curricular content studied in classroom, search for solutions for the problems derived from these situations is, therefore, a learning environment. At the scenario we have built at the Information Systems course, named modeling environment, we have considered the mathematical modeling mathematical modeling pedagogical assumptions. We saw similarities between such environment and the scenarios for investigation proposed by Skovsmose (2008).

In this environment, at the beginning of the classes, the students were informed that they should perform a practical assignment involving real problem, whose solution should be obtained based on the contents studied within the Linear Programming discipline. They were also informed that such assignment would be required by the end of the course and presented to their peers. We have left for the students the decision to choose the problem. Picking up the problem is in general a difficult that must be overcome, because the students are not familiar with the idea of creating their own problems, considering that usually they are formulated and proposed by the teacher.

Furthermore, as shown on some studies (CROUGH; HAINES (2004); GALGRAITH; STILMAN (2006)), the transition of real problems to mathematical models is another difficulty for the students. Crouch and Haines (2004), for instance, state that engineering students, science and technology, in general, carry in their resume, activities performed through investigations and projects and, although used to work with mathematical models, in courses of such nature, the students presented serious difficulties in performing both transitions, from real world to the mathematical model and from the mathematical solution found back to the real situation from where the problem was extracted from. To deal with such difficulties is another task for the teacher who chooses modeling as a pedagogical action.

In this phase of the assignment, the students find themselves lost as to which path to take and they, most of the times, start searching using the internet and/or through text books recommended by the teacher. This phase is very interesting because it entices the need for researching through many sources and not just the traditional didactic material provided by the teacher.

Modeling environment activities

In the modeling environment, referred to by this article, the students had the opportunity to know many Linear Programming application examples in real situations, with the proper mathematical formulation. Many of these examples illustrated applications in the industry and, some of them referred to situations experienced by the teacher as a consultant in a Optimization Consulting Company. Such problems were related to optimized forest planning, optimized poultry production integrated planning. During the presentation of such problems it was highlighted that simplifications were required in order to reach a solution through mathematical treatment. We took advantage of this moment to show the students that, many times, the same mathematical tool is used to solve different problems; for instance, the model used to solve the feed formulation problem can be adapted for solving mixing problems for juice, steel, etc.

During the subject for the assignments choosing phase, many students have opted for working on projects related to real problems, most of them directly related to situations originated within their working places. Usually, in the companies, the professional activities related to optimization problems, solved through Linear Programming resources, are found in departments involving production control and planning. So, within the working environment, the requirement for the knowledge of such resources, although superficial, are justified, mainly due to the application of theoretical concepts in the decision making process. Reciprocally, at school, the possibility of practical application constitutes itself in an important motivational factor in the process of teaching and learning. About the relationship between professional mathematic and classroom mathematic D'Ambrósio (2002), properly, reminds us that the real life facts help us on the knowledge acquisition.

However, due to the complexity involving these problems, some students would soon give up such choice, and rather follow the constructions of simpler problems which would require less effort from them. The students that kept their interest in the problems from their working places, in order to present them as Linear Programming, had to do simplifications in the data gathering phase, as well as during the restrictions formulation for the construction of the mathematical model. This was meaningful for them since they could realize the real requirement to do simplifications on the original conditions that surround the problem of interest in order to obtain a possible and viable solution.

The discussions in the learning environment, build within the classroom, produced, in most cases, simplifications and reformulations and, in some other cases, abandonment of the subject chosen and exchanging for another. Later the students were oriented on using a software for solving the problems. Due to its data loading simplicity, part of the students chose the software named LINGO – Language for Interactive General Optimizer. Due to acquaintance the remaining students chose the Microsoft Spreadsheet Software Excel.

The students, in general, presented some difficulty during the results interpretation provided by the software, as in identifying the optimal solution and the value of the objective function, in the understanding of the meaning of the slack and surplus variables, as well as the dual price. This problem was partially solved when, based on the classroom discussions, the students listed the solutions representations found with the concepts studied during the course. As an example we can mention the pedagogical moment when the students noticed the application of many concepts seen in classroom, as those related to the slack and surplus variables and also the Sensitivity Analysis Theory. The usage of software packages to perform many simulations which contributed to the understanding of the theory studied on top of allowing for software packages comparison.

The difficulties in understanding the concepts related to the Sensitivity Analysis Theory were partially solved as the students, based on the simulations performed with the support of LINGO or Excel, could perceive the practicality of such theory. This was thoroughly discussed during the presentations from the doubles, when the results from the simulations could be compared to those found in the theory.

As mentioned before, by the end of the course the groups (two students) presented in the classroom the results found. This procedure was very significant because it allowed, from one side, each group to present to the class all the phases of the problem directly related to Linear Programming. And, on the other side, that all could visualize many application on different areas. In many occasions the teacher had to intermediate the discussion, relating the students presentations to what had been taught in the classroom. We can mention, as an example, the moment when some of the doubles, in their presentations, approached the necessity of doing simplification in the original problem such that it could be formulated as Linear Programming.

We have noticed in these presentations, as well as reading over the text provided, that few of the doubles went back to the real situation in order to validate the solution found. To be able to reach to a solution was already considered satisfactory in the student’s point of view. As it happens in many modeling assignments, reaching to a result is the objective to be met. In those, the discussions, being it related to the reliability and the adaptability of the results obtained or related to the its meaning and its consequences (social, cultural, economical, environmental etc) to the daily live from where these problems were originated are, most of the times, put aside[1]. During the assignments presentations we have tried to approach such aspects.

Among all the assignments accomplished in the environment, we highlight initially what has been developed individually by a student, who had previously failed this discipline, and grabbed this opportunity, chose for the solution in a problem that he was facing at the Company he was working for. This problem was related to the bi-dimensional wood boards cutting process for furniture manufacturing. Using the subjects studied in the classroom and also some additional help from the teacher and counting on Microsoft Excel support he could manage to reduce the loss on the wood boards cutting. The accomplishment of such practical assignment, beyond providing for the applicability of the school learning within his working environment, also contributed favorably for the student to overcome his difficulties with the discipline and, as a consequence, get approved in it. On top of that, the presentation of better solutions in his work place contributed to his professional growing.

Other doubles have also chosen their assignment based on the requirements of the company where one of the participants worked. The assignments from these doubles related to (1) the optimization of the production line on a car plant; (2) the expansion study on a textile company; (3) the optimization on the employee allocation on a call center company and (4) the optimization on the steel roll cutting process.

On top of theses students that have chosen for the development of projects directly related to their working environment, some others created fictitious problems, but related to their professional fields. We included in these cases the projects related to (1) the optimization on the number of internet users in order to reach the advertisement of a product; (2) the optimization on the allocation of projects from a software company and (3) the optimization on the employees hiring for a data processing company.

Other doubles would rather create models similar to the examples presented in classroom, but in a certain way related to their interest such as, for instance, the projects related to (1) the optimization on the manufacturing of a chemical fertilizer; (2) the optimization on the manufacturing of chocolate; (3) the dieting problem; (4) the optimization on the resources used in a farm and (5) the optimization on choosing a car based on its fuel consumption.

5. Results

The assignments developed allowed for minimizing the felling of lack of relevance with respect to this discipline, for they could visualize many applications where

[1] These questions appropriately are argued in the scope of the Critical Mathematical Education. (SKOVSMOSE, 1996, 2007; JACOBINI, 2007).

Linear Programming can help in the problem solving and in the decision making process for the problems from their work places or in their daily lives. Beyond that, there was an intense collaboration among the students allowing for a better interaction which is highly significant since meetings for exchange of ideas and experiences are very difficult considering they are already working professionally. The debates in this environment were about what the students have found in the text books and about their difficulties in the software results interpretation.

The usage of software packages in the solution of the problems brought about by the students allowed for a greater interaction among them, generated more knowledge e showed them the possibility of visualize the relationship on between mathematic (through the contents related to Linear Programming), real problems and technology. We have considered meaningful the perception of such visualization since the students, usually, complain exactly about the class taught in the traditional form where the relation between what they learn and their real professional lives, in the data processing field, are not perceivable. We saw the presence of the technology as an important tool for collaboration in the mathematical classroom, for it allows the treatment of real situations that involve different levels of algebraic complexity, mainly for the Information Systems Course students, for whom technology is common-place in the school and/or professional daily life. Text books, such as Winston et al (1997), Hillier e Lieberman (2006), Colin (2007), in their approach to Linear Programming associate examples of real applications and the usage of computational resources.

The mathematical modeling mathematical modeling presents itself as a pedagogical strategy that complements this association between real application and the utilization of computational resources, as it provides for the construction of favorable environments for the students to choose their problems of interest, gather their own data, and participate on the investigations, analysis, discussions and reflections.

Blum (1995) shows five arguments favoring the inclusion of the usage of such strategy in the school environment: motivation, learning facilitation, preparation for the usage of mathematic in different areas, development of general abilities for exploration and understanding of the mathematic role in the society. In this same line, Zbiek and Conner (2006), highlight some objectives to be reached when working with the Mathematical modeling Mathematical modeling in the classroom such as, prepare the students for working professionally with the modeling, motivate the students by showing them the applicability of the mathematical ideas in the real world and provide opportunities for the students to integrate mathematics with other areas of the knowledge.

The student's statements, some of them shown below, valorize the professional opportunities provided by the work performed. They also show the importance, for teaching and for learning, of the learning environments construction based on this association between mathematical modeling mathematical modeling centered on real problems related to the working world and the utilization of computational resources.

Such statements also confirm that the curricular content involvement with a daily life mathematic (through mathematical modeling) helps not only to show the practical utility of mathematics and the relevance of its learning but, likewise, to reduce stress feelings and fear towards it.

> "I have found very interesting the software Excel application for solving the problem, I have learned a little more about this software, I could not even think that it had such a tool".
>
> "This discipline was very important to me, for solving problems from my daily life and I am also foreseeing the application of its contents in many situations at the Company I work for".
>
> "I have been applying Linear Programming for getting a better solution for the problem at the Company I work for".
>
> "Among all of the Course subjects, this was the one that was worthwhile".
>
> "I was very satisfied when I could solve my problem, I was afraid of this subject because everybody say it was very hard to get approved".
>
> "Currently I make money out of Linear Programming".

6. Final Considerations

We have evaluated, from this experience, that Mathematical modeling Mathematical modeling by providing the students with opportunities for identifying and studying problem-situation from their professional realities or interests and by creating opportunities for the construction of a more critical and reflexive knowledge, presents itself as an adequate pedagogical way for teaching and learning contents related to Linear Programming. We have also evaluated that experiences with Mathematical modeling Mathematical modeling favor collaboration, between the participants of the groups (or doubles in this case) and among all the students when questions related to the software utilization arise or in the interpretation of the results obtained.

We also highlight the importance, in the created environment, of the interaction among the students and the teacher, either through e-mails exchange or through classroom talks, since this, by providing a closer proximity among all players contributes, on one hand, to the expectations and questions from the students to be readily debated and clarified. And, on the other hand, to facilitate the assignment execution and experiences exchange. As the student sees the teacher and his peers as collaborators, he (she) can see the classroom and the working place closing in and, as a consequence, he (she) associates the knowledge resulting from the pedagogical process with his (her) professional requirements. In summary, the student can see a practical meaning to what he (she) learns in school.

Finally we highlight the advances related to the knowledge of available resources on the software used (Excel and LINGO). Such advances are equally mentioned by researchers interested in Mathematical modeling Mathematical modeling as a pedagogical strategy. At the work referred here, the students presented initially, some difficulties towards the software usage, since not specific activity had been previously taken place. As time went by, com classes being taught and mainly with the collaboration among them, with some helping others, the difficulties were overcome. It has also contributed to the enhancement on the software usage the results from many researches over the internet, where user manuals were found by the students and made available to all participants. Borba et al (2007) highlighted the collaboration as part of the interactive process where teacher and students act as partners in the learning process.

We finalize this article with two considerations related to the Mathematical Modeling, environment built at the Linear Programming discipline. On the first we highlight, on one side, the perceptions by the students of the relevance of the discipline, being it as much for their intellectual formation as for their professional valorization, and the technology applicability, through the usage of specific software packages, in a mathematic discipline taught within a Information System field course. We believe that the course has awaken in the students the interest for learning and, despite the short contact with theory e with the applications presented in the classroom, has also collaborated in such a way that they can continue by themselves in the application and solution of other problems, from the simple daily life problem to more complex ones at their working places. And, on the other hand, the environment built among the students has contributed for the interaction in the assignments in a group and in the moments of professional and academic experiences exchange.

On the second, we highlight that the option for practices that differentiate themselves from the common way in a classroom, characterized mainly by predictable actions and done with the single intention of transmitting information intrinsic to the programmed content, requires a lot of effort and dedication from the teacher. So, jobs of such nature are incompatible with a docent agenda full of classes or many activities.

References

BIEMBENGUT, M. S.; HEIN, N. *Modelagem matemática no ensino*. S.P.: Contexto, 2000.

BLUM, W. Applications and Modeling in mathematics teaching and mathematics education – some important aspects of practice and of research. In: SLOYER, C. et al (Eds.). *Advances and perspectives in the teaching of Mathematical modeling and Applications*. Yorklyn, DE: Water Street Mathematics, 1995, p. 1-20.

BORBA, M. C.; MALHEIROS, A. P. S.; ZULATTO, R. B. *A. Educação a Distância online*. 1 ed. Belo Horizonte: Autêntica, v. 1, 2007.

CAMPOS C. R. *A Educação Estatística*: uma investigação acerca dos aspectos relevantes à didática da Estatística em cursos de graduação. Tese (Doutorado em Educação Matemática). 242 f. Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Rio Claro, 2007.

COLIN, E. C**. *Pesquisa Operacional***, Rio de Janeiro: LTC. 2007.

CROUCH, R.; HAINES, C. Mathematical modeling: transitions between the real world and the mathematical model. *International Journal of Mathematical Education in Science and Technology*; v. 35 (2), p. 197-206, 2004.

D'AMBRÓSIO, U. Matemática nas escolas. *Educação Matemática em Revista*, São Paulo, ano 9, n. 11, p. 29-33, 2002.

GALGRAITH, P.; STILMAN, G. A framework for identifying student blockages during transitions in the modeling process. *Zentralblatt für Didaktik der Mathematik,* v. 38 (2), p. 143-162, 2006.

HILLIER, F. S., LIEBERMAN, G. J. *Introdução à Pesquisa Operacional.* Tradução A. Griesi; Revisão técnica J. Chang Junior. São Paulo: McGraw-Hill, 8ª ed. 2006.

JACOBINI, O. R. *A modelagem matemática em sua dimensão crítica: novos caminhos para conscientização e ação políticas*. V Conferência Nacional sobre Modelagem e Educação Matemática. Ouro Preto. Brasil. 2007.

JACOBINI, O. R.; WODEWOTZKI, M. L. L. *Mathematical modelling:* a path to political reflection in the mathematics class**.** Teaching Mathematics and Its Applications. Oxford Journals. v. 35, p. 33 a 42. Publicação on line. 2006.

JABLONKA, E. *Mathematical Literacy*. In**:** Second International Handbook of Mathematics Education**.** Dordrecht, NL: Kluber Academic Publishers, 2003.

SKOVSMOSE, O. Critical mathematics education: some philosophical remarks. In: INTERNATIONAL CONGRESS ON MATHEMATICS EDUCATION, 8., 1996, Selected lectures. Sevilha: S. A. E. M., 1996. p. 413 – 425.

WINSTON, W. L., ALBRIGHT, S. C., BROADIE, M. *Management Science: spreadsheet modeling and applications*. USA: Wadsworth Publishing Company. 1997. 796 p.

ZBIEK, R. M., CONNER, A. Beyond Motivation: exploring mathematical modeling mathematical modeling as a context for deepening students' understanding of curricular mathematics. *Educational Studies in Mathematics***.** v. 63, n. 1, p. 89-112, 2006.

On the Development of Mathematical Modelling Competencies The PALMA Longitudinal Study

Rudolf vom Hofe[1], Alexander Jordan[1], Thomas Hafner[1], Pascal Stölting[1], Werner Blum[2] & Reinhard Pekrun[3]

[1] University of Bielefeld, Department of Mathematics – Institute of Didactics of Mathematics, D-33501 Bielefeld, Email: vomhofe@math.uni-bielefeld.de
[2] University of Kassel, Department of Mathematics, D-34109 Kassel
[3] University of Munich, Department of Psychology, D-80802 Munich

Abstract

TIMSS and PISA have documented considerable differences in students' achievements, including mathematics, which led to lively discussions concerning the effectiveness of the teaching of mathematics in school. However, these studies have some serious deficiencies which only partly explain the causes for differences in achievement. Studies like these are essentially producing a descriptive system monitoring concerning specific measuring moments and age groups, but due to their descriptive, cross-sectional design, they cannot provide insights into the achievement development which has led to the stated results, nor into the impact of corresponding mathematical modelling competencies and the required mental models of mathematical concepts (that we call "Grundvorstellungen"). Especially these points, however, are important for providing causes for the identified achievement deficits and evidence for possibilities for the improvement of classroom practice. Thus the aim of the research project PALMA is to pursue longitudinally students' mathematical achievement and its conditions. The essential aims are (1) the analysis of mathematical achievement development as well as corresponding modelling competencies and Grundvorstellungen from grade 5 to 10, (2) the analysis of causes of this development, and (3) providing hints for the improvement of teaching and learning of mathematics in each age group. In this paper we will present some selected aspects of our study.

1. Introduction of PALMA

The **P**roject for the **A**nalysis of **L**earning and Achievement in **MA**thematics (PALMA) analyzes students' development in the domain of mathematics during secondary school (grades five to ten, 11- to 16-year-olds). Using a longitudinal design involving annual assessments, the main part of the project investigates the development of mathematical modelling competencies (Blum, Galbraith, Henn & Niss, 2007) across these years. The first survey took place in summer 2002. In 2006, our student population was equivalent to the third PISA wave (see *figure 1*).

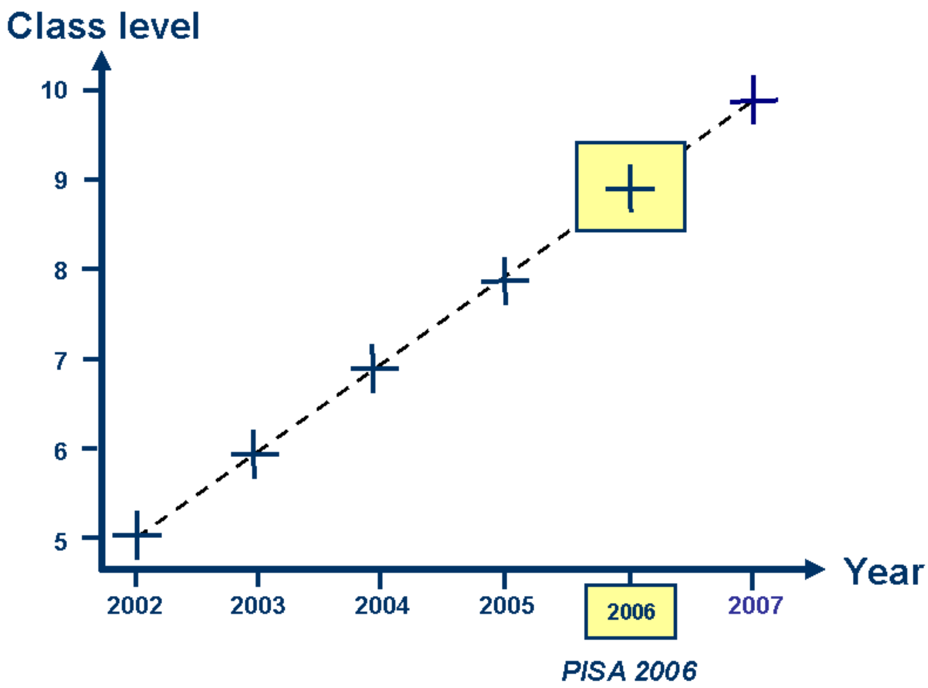


Figure 1. Development of mathematical competence from grade 5 to 10

The theoretical framework of the project employs a conception of mathematical competencies that is consistent with the notion of mathematical literacy used in the OECD’s Programme for International Student Assessment (PISA; see section 2 of this paper). In particular, the differentiation between competencies for mathematical modelling (in a broad sense), on the one hand, and for performing algorithmic operations, on the other hand, is of fundamental importance for this project. Furthermore the framework emphasizes the role of basic concept images (*Grundvorstellungen*; see vom Hofe, 1998; vom Hofe, Kleine, Blum & Pekrun, 2005) for mathematical modelling. The final goal of the project is to develop materials that can be used in educational practice.

In this paper, the conceptual basis (section 2), the methodology (section 3) and some exemplary results (section 4) of the longitudinal study will be presented. At present, data are available for all six annual assessments, so that findings reported here pertain to grade levels 5 to 10. At the end of the paper, a short summary of conclusions and an outlook of still upcoming work will be presented (section 5).

2. Conceptual Basis: Mathematical Modelling and Grundvorstellungen

Experiences both with lessons and empirical research show that many problems in mathematical thinking are caused by conflicts concerning the intuitive level (cf. Fischbein, 1987, 1989). Essential reasons for these problems are due to the fact that often mathematical concepts and symbols are filled by students with a totally different meaning from what is intended by the teacher. In order to counteract these problems, different concepts of the generation of "mental models" have been developed which emphasise the constitution of meaning as a central aim of mathematical teaching. In Germany mental models which bear the meaning of mathematical concepts or procedures are called *Grundvorstellungen*, abbreviated to GVs.

Concepts of GVs have a long tradition in the history of mathematical education in Germany. There is also currently a lot of research on GVs concerning all school grades (see for more details vom Hofe, 1998; Blum, 1998). Naturally, such concepts are not only restricted to Germany, but can be found in many other countries as well; cp. for example the concept of “intuitive meaning” (Fischbein, 1987), “use meaning” (Usiskin, 1991) or “inherent meaning” (Noss, 1994) in English-spoken countries.

In mathematical education research the term GV is used both in a prescriptive and a descriptive way: GVs as a *prescriptive notion* describe adequate interpretations of the core of the respective mathematical contents which are intended by the teacher in order to combine the level of formal calculating with corresponding real live situations. In contrast, the term GV in descriptive empirical studies is used also as a *descriptive notion* to describe ideas and images which students actually have and which usually more or less differ from the GVs intended by mathematical instruction.

Examples of elementary GVs:

- *Subtracting* as (a) taking away, or (b) supplementing, or (c) comparing.
- *Dividing* as (a) splitting up, or (b) sharing out.
- *Fractional number* as (a) part of a whole, or (b) operator, or (c) ratio.

GVs can be interpreted as elements of connection or as objects of transition between the world of mathematics and the world of real live situations. In this context “generation of GVs” does not mean sampling a collection of static mental models which are valid forever. Quite the reverse, in the long run the generation of GVs will be a dynamic process in which there are changes, reinterpretations and substantial modifications. Especially if the individual is going to be involved with new mathematical subjects, he or she will have to modify and extend his or her system of mental models. Otherwise GVs which have been successful for so long could become misleading “tacit models” (referring to Fischbein, 1989) when one is dealing with new mathematical subjects.

The generation of GVs is especially important for the mathematical concept development, characterising three aspects of this process:

- *Constitution of meaning of mathematical concepts* based on familiar contexts and experiences,
- *generation of generalised mental representations* of the concept which enable operative thinking (in the Piagetian sense),
- *ability to apply a concept to reality* by recognising the respective structure in real life contexts or by modelling a real life situation with the aid of the mathematical structure.

In PISA, basic mathematical competence is described as *mathematical literacy*

(OECD, 1999 & 2003) which emphasises the role of conceptual understanding and meaningful application of mathematics in contrast to mere algorithmic calculating and formula manipulating. Dealing with mathematics in that way stresses the importance of *modelling* as a major mathematical competency. The following figure simplistically illustrates the typical steps of a modelling process and shows its cycle character (see *figure 2*); for more details see Blum et al., 2002.

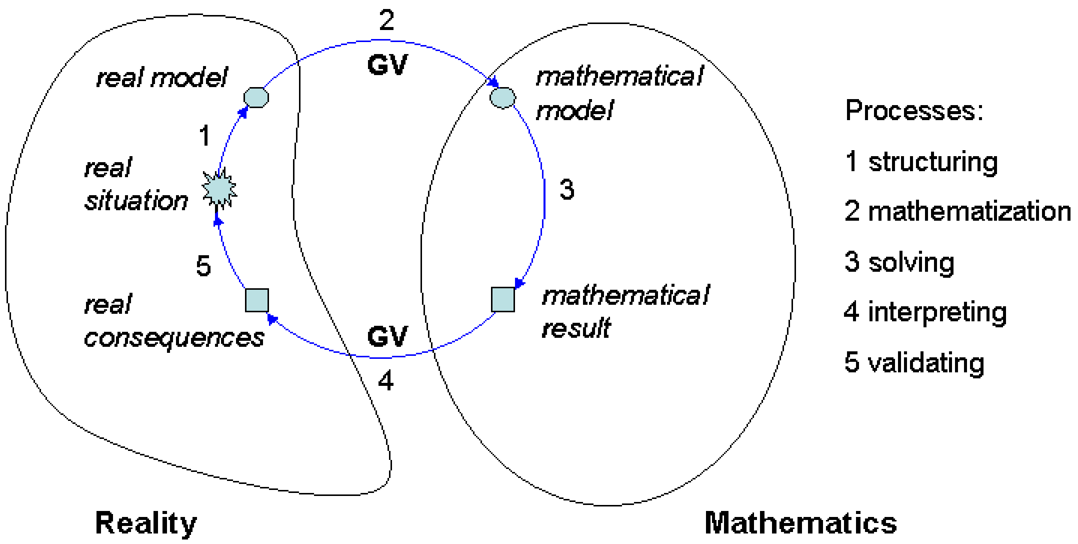


Figure 2: Modelling Process

When carrying out this process, *translating between the real world and mathematics* is a main mathematical activity, for example finding mathematical concepts or procedures which represent a given real life context on the mathematical level or interpreting what a mathematical solution means for the given real world situation. Therefore mental models respectively Grundvorstellungen are needed which carry the meaning of mathematical notions and procedures and so enable the student to move mentally between mathematics and reality (Freudenthal, 1983, vom Hofe, 1998). Thus the generation of a dynamic network of GVs is an important prerequisite for the development of mathematical modelling competence as a whole, and especially for the activities of mathematization and interpreting where mental translations between mathematics and real live contexts are required. Without GVs however, mathematical operation becomes a lifeless formalism which is taken away from the areas of application and reality.

Due to their mental nature, it is difficult to explore students' GVs by empirical research; and there are many theoretical and methodological problems when gaining an insight into students' mental models. However, regardless of these problems the development of working methods for empirical based analysing of mathematical thinking is an important issue in mathematical education.

We are convinced that serious problems of mathematical achievement development are caused by an insufficient growth of modelling competency and corresponding GVs during secondary school. We furthermore assume that many students turn too frequently to formula calculating which is one of the reasons for difficulties with applied mathematical problems. In the following sections we will elaborate a little more the role of GVs for mathematical modelling.

For our long term study we designed test instruments which are conceptually based on the idea of mathematical modelling and the involved activation of GVs. To get insights into the development of the modelling competency of students, we constructed series of items which progressively require modelling activities concerning the topics of arithmetic, algebra, elementary functions and geometry. In addition, we also included series of items which can be solved by mere formula calculating without any GVs or thinking about the meaning of the involved concepts or procedures ("technical items" in the sense of Neubrand et. al. 2001).

Two elementary examples of *grade 6*, one of each kind:

(1) Kevin wants to buy new sport shoes for 80 €. He has already saved 3/10 of the price. How much more money does he still need to buy the shoes?

(2) Calculate: $\frac{1}{3} \cdot \left(\frac{1}{2} - \frac{2}{5} \right)$

In example (1), GVs of subtraction (most likely supplementing) and fractions (as operator) are needed. In example (2), mere algorithmic knowledge of multiplying and subtracting of fractions is sufficient.

Beside elementary items which only require single parts of the modelling process we also use series of complex items to measure different levels of mathematical modelling and the interplay of different GVs. A typical example regarding *grade 9* is "Online Services" (see page 5):

For a successful solution of item a) simple translation processes between reality and mathematics by elementary GVs of functions are needed (discrete mapping). However, item b) demands higher requirements, because the linear growth of both services must be compared with each other. For this situation elaborated GVs of functions (covariation of two variables) are necessary. Furthermore, in item c), GVs of variables must be activated to find out the correct formula.

Online Services

The following overview shows the conditions of the contract of two online services:

	e-online	online-pro
Monthly basic charge	7 €	7 €
Price per hour	3 €	4 €
Free hours per month	None	3

a) Compare the monthly total expenses of both online services with an existing contract. *Complement the following table.*

Useful life	total expenses e-online	total expenses online-pro
0 hours	7 €	
1 hour		7 €
5 hours	22 €	

b) Which online service is cheaper for frequent use?

☐ e-online ☐ online-pro

Found your answer.

c) *Give a formula for the total expenses for e-online as a function of the useful life.*

3. Methodology

To provide detailed development data, the PALMA longitudinal study includes annual assessments from grades 5 to 10. Our samples consist of students, their mathematics teachers, and their parents. To make it possible to analyze also classroom instruction and the classroom context, the student samples primarily comprise students from whole classes. Samples are drawn from Bavarian schools so as to be sufficiently representative for the student population of the state of Bavaria, thus including students from all three school types of the Bavarian school system. The three school types differ in academic demands and students' entry level of academic ability. As in most German states, these school types consist of a low-ability track (Hauptschule), a medium-ability track (Realschule), and a high-ability track (Gymnasium). In the first year (grade 5), the student sample comprised 2,070 students (1.043/1.027 male/female students; mean age 11.7 years) from 83 classrooms and 42 schools. The sample of participating parents included 1,977 parents, and the teacher sample all of the 83 mathematics teachers of the participating classes. At each grade level, the assessment took place towards the end of the school year (May and June). The student assessment comprised the following main variables and instruments:

(1) The Regensburg Mathematical Achievement Test. Using Rasch-scaled scores, this test measures students' modelling competencies and algorithmic competencies in the fields of arithmetics, algebra, and geometry. It also comprises subscales pertaining to more specific contents (e.g., fractions; vom Hofe et al., 2005). The test was designed so that it allows to assess students'competencies across all grade levels of secondary school, and it takes students' different abilities within grade levels into account.

(2) Student and teacher questionnaires including variables of mathematics classroom instruction and of the social composition of the classroom context.

These instruments are administered by trained external test administrators in the students' classrooms. Total testing time is 180 minutes at each grade level. The parent and teacher assessments comprise questionnaires that were administered individually. The German Data Processing Center (DPC) of the International Association for the Evaluation of Educational Achievement (IEA) is responsible for drawing the student, parent, and teacher samples, and for organizing the annual assessments.

In addition to the quantitative annual assessments, we also conducted qualitative interviews in sub-samples of students. The interviews aimed at analyzing in more detail the cognitive strategies students use to solve mathematical problems. They also included think-aloud procedures of assessing problem solving strategies. All interviews were videotaped and transcribed before being analyzed.

4. Results and Discussion

In the following sections we present some selected findings of our study. These findings pertain to (1) the development of mathematical achievement, (2) corresponding modelling and algorithmic competencies, and (3) the role of Grundvorstellungen.

Global findings: Development of Mathematical Competencies

Using our longitudinal data, we aimed at investigating how students' mathematical competencies develop over the schoolyears. As mentioned before, we moreover wanted to analyze whether there are differential developments for different types of competencies, or for different groups of students. Our analyses pertained to the main topics of the German mathematics curriculum for the grade levels considered. One important question was whether students' difficulties are due to inadequate mathematical concept images.

The development of competence scores was analyzed by using the longitudinal sample of students who had participated at least in five of the six annual assessments from grades 5 to 10. The scaling of competence scores also was based on this sample (standardized to M = 1,000, SD = 100 at grade 9). As figure 3 shows, there was a substantial increase of overall competence scores. However, there are clear school type differences: Indeed, the development of the mathematical competence of students from the high-ability-track schools and of students from the medium-ability-track schools is widely parallel. As expected the high-ability-track lies continuously over the medium ability-track. Moreover, the low-ability-track has a larger distance to the medium-ability-track and the development of the low-ability-track runs clearly much more moderate than in the other two school types. It is astonishing that the average achievement of students from the low-ability-track schools at the end of time 5 (grade 9) lies only briefly over the average achievement of students from the medium-ability-track schools at the end of time 1 (grade 5) and even shortly under the average achievement of students from the high-ability-track schools at that time.

Figure 3. Development of mathematical competence, grade 5 to 10

We will now have a more detailed look at the development of modelling and algorithmic competence. *Figure 4* shows that we have an increasing development in both competences from grade 5 to 10 at most time, but we also see that there are phases of stagnation of modelling competencies at the end of grade 9 in the low-ability-track schools (*Hauptschule*). In this period no advancement of modelling competencies takes place. Furthermore, there are even phases of decline of algorithmic competencies at the end of grade 5 in this school type. Both results lead to the question whether the system of the low-ability-track schools promotes the students reasonably and appropriately.

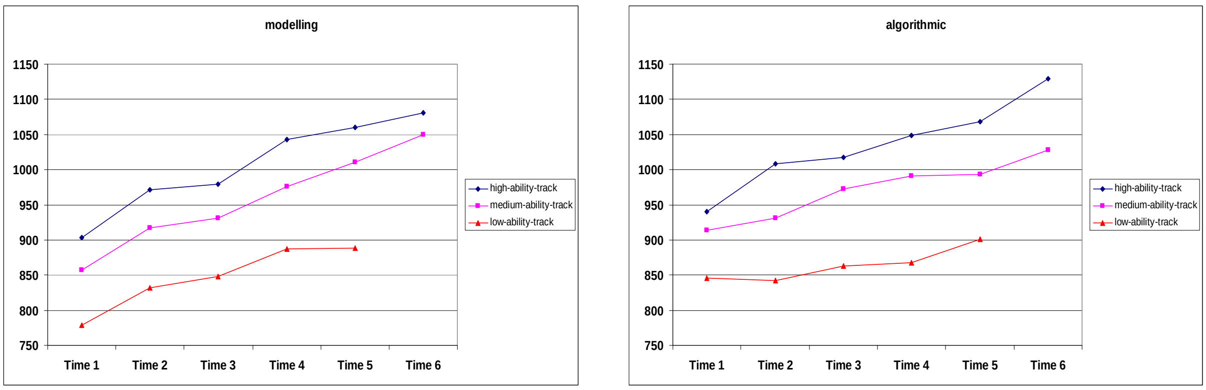


Figure 4. Mean scores for modelling and algoritmic competencies, grade 5-10

In connection with these considerable differences in the average development of modelling and algorithmic competence in the three school types it is interesting to look at the achievement distributions and the variance within the school forms (see *figure 5*). An important question, in Germany lively discussed, is whether these achievement distributions justify the system of separated and mostly impermeable school forms. Also here an astonishing result appears: There are high overlappings between the school forms in time 1, which are similar from time 2 to time 4. Even at the last common measuring time 5 considerable overlappings still exist between the school forms which make clear that a lot of students from the low-ability-track schools reach the level of students from the medium- and the high-ability-track schools. These results let it seem dubious that an adequate support is guaranteed by the early selection from 10-year-olds in three different school forms.

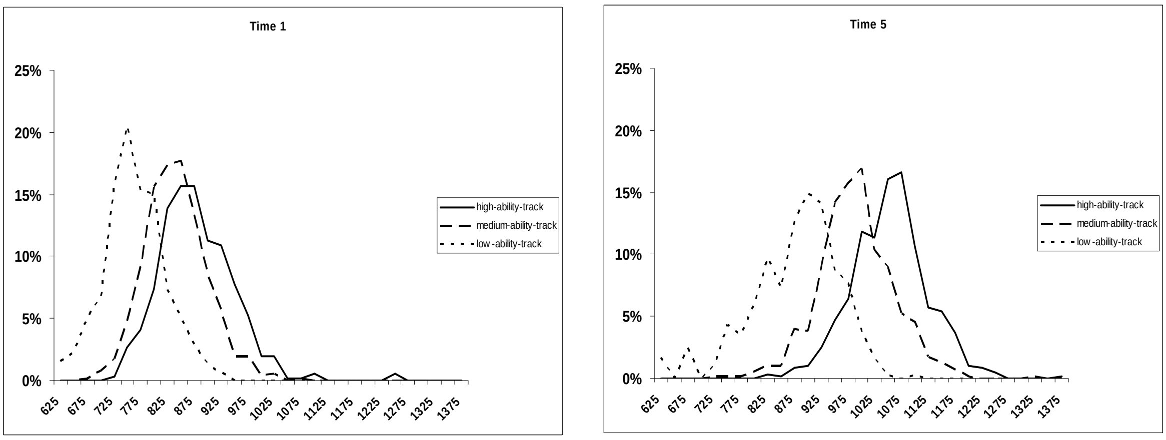


Figure 5. Development of mathematical competence at time 5 and 9

Even more clearly these findings appear in the following comparison of the high-ability-track schools with different branches of the medium-ability-track schools. In Bavaria students are split in the medium-ability-track schools after class 6 in three different branches: (1) mathematical branch, (2) business branch, (3) socio-scientific branch.

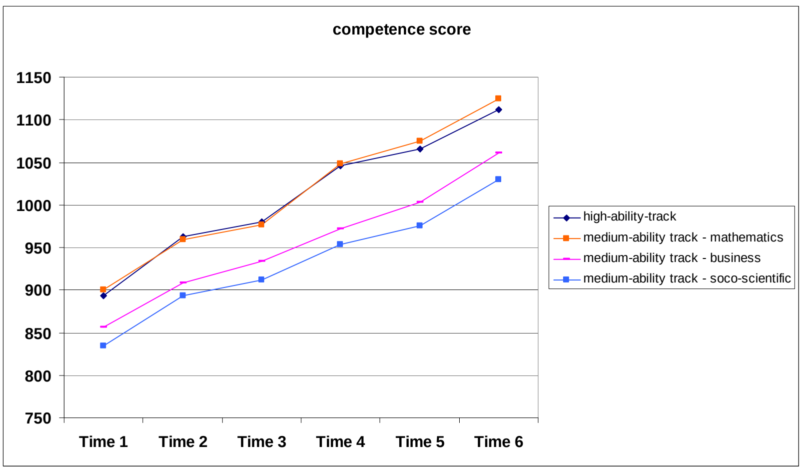


Figure 6. Development of mathematical competence in different branches of medium-ability-track and high-ability-track

As can be seen from *figure 6* the students of the mathematical branch reach analogous scores of mathematical competence as students of the high-ability-track schools from time 1 to time 4. At time 5 and 6 the scores of these students lie even above the scores of the students from the high-ability-track schools. If one thinks, however, that on account of the different curricula in the high-ability-track schools and in the medium-ability-track schools, nevertheless, for the students no chance exists to change between these school forms, then the question must be put once more whether this division between the school forms is reasonable.

Detail findings: The role of basic mathematical concept images

The quantitative studies described in the previous section serve for the analysis of the development of the different school forms at a global level. So far we reported the aggregated development on the class level of individual learning histories. In particular, specific problems of mathematical modelling and involved basic mathematical concept images (*Grundvorstellungen*) cannot be explained by these analyses. That is why we have carried out other detailed explorations: (1) analysis of typical students' mistakes on the basis of selected items of the quantitative test, (2) interviews with subgroups of students (N = 36 each year) solving these items. The interviews are related to main topics of the curriculum of grades five to ten (grades 5 and 6: fractions and proportionality; grade 7: fractions and negative numbers; grades 8, 9 and 10: equations and functions). The analyses of these documents are not finished yet. In the following, we will present two typical examples to illustrate difficulties in the modelling process and, besides, relevant GVs. The following example is taken from an interview with a *sixth-grade* student.

Chocolate

Lily takes half of the bar of chocolate depicted at the right. She eats 3/5 of what she took. *How many pieces did she eat?*

Contrary to expectations (10 pieces · 3/5), the student did not use a part-whole operation to solve this task, but made an attempt to solve it by performing a division (10 pieces : 0,6). Her answers given in the interview made clear why she did so:

I: *Why divided by zero point six?*
S: *Yes, ehm, yeah.*
Because ten is equal to zero point five, and this is half of all of it.
And the zero point six are the three fifths.
And that's what you need of the half, since it's only a half, and not all of it.
I: *My question related more to the arithmetic operation. Why divided by?*
And not times or plus or minus?
S: *Because, ehm, by doing that, it would become more, but it has to become less and less, because she doesn't eat more than the bar, but less than the bar.*
S performs the division 10 : 0,6 (26 seconds)

As illustrated by this example, the primary reason given by students for performing this kind of an arithmetic operation was that they could not conceive of any other way to obtain a diminution of the starting value. This misconception is based on a mathematical concept image of multiplication that “multiplication always makes bigger and division always makes smaller” (Wartha, 2007). Of course, while a multiplication implying natural numbers always leads to an augmentation, it can lead to a diminution when using fractions.

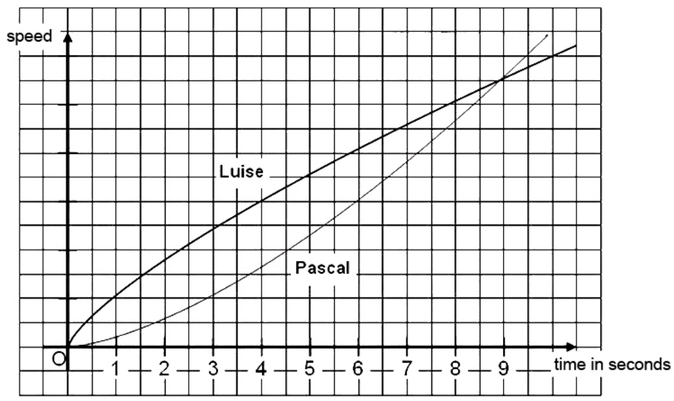
Such problems by the translation between reality and mathematics which exist because of a non-adequate development of GVs are quite typical for fractions which are treated in grade 6. It is disconcerting that such problems are not only typical at the beginning of the secondary school, they still exist till the end of this school form (Jordan, 2006; Stölting, 2008). Thus a sensible interpretation of functional connections which are also extremely relevant for successful accomplishment of everyday problems is extremely problematic for a huge number of students at the end of the secondary school. For example, a lot of students of grade 9 cannot solve the following item:

Sprinter

Luise and Pascal do a race by bicycle.

In the chart their speed is shown

as a function of the time.



a) Who goes 7 seconds after the start faster?

O	Pascal
O	Luise
O	this question cannot be answered by the chart

b) What happens between 5th and 8th second?

O	The distance between Luise und Pascal becomes lower
O	The distance between Luise und Pascal becomes larger
O	The distance between Luise und Pascal stays the same
O	this question cannot be answered by the chart

Found your answer:

In item b) the students must recognise that between the 5th and 8th second Luise goes faster than Pascal and therefore the distance becomes larger between them and decreases only from the intersection of the graph. A typical error of this example which is classical for functions lies in a non-adequate development of elaborated GVs of functions (see section 1). This error leads to an interpretation of the graph as a way of real life, as the following sequence taken from an interview with a *ninth-grade* student shows:

S: *Yes, Luise becomes slower and Pascal becomes quicker.*
I: *How do you see that Luise becomes slower?*
S: *Because the straight rises no more so strongly.*

The student founds his answer about the gradient of the given graphs. To the question of the interviewer what she should say about the distance between Luise and Pascal he gives the following answer:

S: *The distance becomes also less and less when Luise becomes slower and Pascal becomes faster ...*
I: *But now you conclude this, or do you also see in the graph that the distance becomes less?*
S: *I see this also, because of both straight lines of Luise and Pascal …*
I: *OK. What happened then in the ninth second?*
S: *... they meet then.*
I: *Does this mean, they drive side by side, or what is called, they meet?*
S: *Yes, side by side they go .... But Pascal goes faster, and then he overtakes Luise.*

Those and other problems concerning the translation between reality and mathematics are typical – as said above – from grade 5 to grade 10. These findings of our interviews thus corroborated the importance of basic mathematical concept images for solving mathematical problems, and of inadequate images for errors (see also Fischbein, Tirosh, Stavy & Oster, 1990). Overall, approximately half of the students' errors that were analyzed in our interviews were found to be due to inadequate basic concept images.

5. Summary and Outlook

In this paper we presented some selected results of the project PALMA. An important result is (1) that the competency scores of the three German school types overlap with each other to a large extend even at the end of secondary school and (2) that many students are not supported adequately by the current German school system which devides 10 year olds in separated school types. Detailed analyses from grade 5 to 10 show that there are phases which are dominated by an increase of algorithmic competencies while modelling competencies are hardly increasing or even stagnating and vice versa. Findings from the qualitative interviews corroborated that inadequate mathematical concept images play a major role for the difficulties many students have with mathematical tasks. We assume that supporting students to develop adequate concept images could substantially help preventing these problems.
At the moment the analyses of our data are still going on. To get a deeper insight into

the teaching and learning in Bavarian classes, the following further analyses will be carried out in the future: (1) Connection of the achievement data with psychosocial data, (2) Continuation of the detailed studies, in particular evaluation of all interviews.

We expect our work to lead to different perspectives concerning learning and teaching mathematics at school. Especially teachers need more diagnostic competence to promote the development of GVs and to thwart misconceptions which constrain the further progress in applied mathematics. On the basis of our data we are developing special modules for teacher education which can support competence in analysing students' strategies and mistakes.

References

Blum, W. (1998): 'On the Role of 'Grundvorstellungen' for reality-related Proofs – Examples and Reflections'. In P. Galbraith, W. Blum, G. Booker & I.D. Huntley (Eds.): *Mathematical Modelling – Teaching and Assessment in a Technology-Rich World*, Horwood, Chichester, p. 63 – 74.

Blum, W. et al. (2002): ICMI Study 14 on Applications and Modelling in Mathematics education – Discussion Document. *Educational Studies in Mathematics*, 51, p. 149 – 171.

Blum, W., Galbraith, P., Henn, H.-W. & Niss, M. (2007): *Modelling and Applications in Mathematics Education*. New York: Springer.

Fischbein, E. (1987): *Intuition in Science and Mathematics: An Educational Approach.* Dodrecht: D. Reidel

Fischbein, E. (1989): Tacit Models and Mathematik Reasoning. In: *For the Learning of Mathematics*, 9, 9 – 14.

Fischbein, E., Tirosh, D., Stavy, R. & Oster, A. (1990). The autonomy of mental models. *For the Learning of Mathematics*, 10, 23-30.

Freudenthal, H. (1983): *Didactical Phenomenology of Mathematical Structures.* Dordrecht: Reidel.

Hofe, R. vom (1998): On the Generation of Basic Ideas and Individual Images: Normative, Descriptive and Constructive Aspects. In: J. Kilpatrick & A. Sierpinska (Eds.): *Mathematics Education as a Research Domain: A Search for Identity*. Kluver Academic Publishers, p. 317 – 331.

Hofe, R. vom, Kleine, M., Blum, W. & Pekrun, R (2005): On the Role of Grundvorstellungen" for the Development of Mathematical Literacy - First Results of the Longitudinal Study PALMA. *Mediterranean Journal for Research in Mathematics Education*, 4, 67 – 84.

Jordan, A. (2006): Mathematische Bildung von Schülern am Ende der Sekundarstufe I. Analysen und empirische Untersuchungen [*Mathematical Literacy at the end of secondary school. Analyses and empiric studies*]. Hildesheim: Franzbecker.

Neubrand, M., Biehler, R., Blum, W., Cohors-Fresenborg, E., Flade, L., Knoche, N., Lind, D., Löding, W., Möller, G. & Wynands, A., (2001): Grundlagen der Ergänzung des internationalen PISA-Mathematiktests in der deutschen Zusatzerhebung [Foundations of the Supplement of the International PISA Mathematics Test in the German National Study]. *Zentralblatt für Didaktik der Mathematik*, 33 (2), p. 45 – 49

Noss, Richard (1994): Structure or ideology in the mathematics curriculum. *For the Learning of Mathematics*, 14(1), 2 – 10.

OECD (1999): *Measuring Student Knowledge and Skills. A New Framework for Assessment*. Paris: OECD Publication Service.

OECD (2003): *The PISA 2003 Assessment Framework – Mathematics, Reading, Science and Problem Solving Knowledge and Skills*. Paris: OECD Publication Service.

Pekrun, R., Hofe, R. vom, Blum, W., Frenzel, A., Götz, T. & Wartha, S (2007). Development of mathematical competencies in adolescence – The PALMA longitudinal study. In M. Prenzel (Ed.): *Studies on the educational quality of schools*. Münster, New York, München, Berlin: Waxmann.

Stölting, P. (2008): Die Entwicklung funktionalen Denkens in der Sekundarstufe I [*The development of functional thinking in secondary schools*]. Phd-Thesis, University of Paris 7/ University of Regensburg.

Usiskin, Z. (1991): Building Mathematics Curricula with Applications and Modeling. In M. Niss, W. Blum & I.D. Huntley (Eds.): *Mathematical Modeling and Applications*, Chichester: Horwood, p. 30 – 45.

Wartha, S. (2006): Längsschnittliche Untersuchungen zur Entwicklung des Bruchzahlbegriffs [*Longitudinal Studies about the development of the fraction concept*]. Hildesheim: Franzbecker.

THE TEACHERS' TENSIONS IN MATHEMATICAL MODELLING PRACTICE

Andréia Maria Pereira de Oliveira
State University of Feira de Santana, Brazil
ampodeinha@uol.com.br

Jonei Cerqueira Barbosa
State University of Feira de Santana, Brazil
joneicb@uol.com.br

Abstract *- We present partial results of an empirical study on the tensions that teachers experience when doing mathematical modelling. The context of this research was taken from the first experiences with mathematical modelling of three lower secondary school teachers from public schools. The data was collected through observation accomplished through filming of lessons, interviews after each lesson and narrative about the lessons done by each teacher. The analysis of the data suggests the following tensions in the teachers' practices: the tension of the students' engagement, the tension of the students' comprehension of mathematical content, the tension of understanding of the modelling task by students and the tension of conducting a modelling task.*

1. Introduction

The debate on the insertion of mathematical modelling in the school curriculum has been gaining visibility in last decades. By modelling, we understand a learning environment in which the students are invited to investigate, through mathematics, problems from other disciplines or daily situations (Barbosa, 2003, 2006). One of the questions has been the teacher's role in this enterprise, just as, for example, it was approached in the ICMI Study 14 (Blum et al., 2007).

Niss, Blum e Galbraith (2007) have pointed out that the teachers need to have opportunities to use modelling during their pre-service education and through regular tasks in their professional development. On light of this, the teachers need to develop several modelling tasks, so that they can experience a variety of situations and discuss their pedagogical implications.

Empirical studies have discussed the interventions (Leiß, 2005), content knowledge and pedagogical content knowledge (Doerr & English, 2006), dilemmas (Blomhøj & Kjeldsen, 2006) and strategies (Chapman, 2007) that are constituted when teachers are developing modelling in their classrooms. These studies make clear that the modelling demands the configuration of situated actions in the school culture for the development of this environment, instead of only the knowledge of doing modelling. Thus, understanding these aspects refer to conducting a learning environment in the school that challenges the traditional practices, can support teachers in implementing modelling in their classrooms.

Also, the aforementioned studies suggest the notion of "tension" as a starting point to capture the insecurities, difficulties, concerns and dilemmas in the development of modelling. In this proposal, we will present part of a wider research on teachers' tensions in the implementation of modelling in their practices. In particular, we delimited our focus with the following question: *Which tensions in the teachers' discourses are constituted when developing mathematical modelling in their practice?* In order to address this question, the first author followed the teachers doing modelling in their classrooms. We mean discourse as an oral or written text that is produced by an individual that belongs to a specific social context (Bernstein, 2000). By this way, we hope to produce initial theoretical understandings that can help us to understand the teacher's role in developing modelling. In addition, the found results could support programs of teachers' professional development as well as other actions that focus on the insertion of modelling in the educational systems.

In the study carried out by Evans, Morgan and Tsatsaroni (2006), the authors used the category "emotions" to understand the students' mathematics practice. The authors considered emotions as socially discursive phenomena, being formed by relationships of power and constitutive of the social identity. In a similar way, we can say that the tensions in discourses are socially organized, because they have a more interpersonal origin than an individual one. Following this point, the tension in discourses does not represent the manifestation of some inner psychological instance, but it is constituted through the contact among discourses that circulate and are legitimized (or not) in the social environment. However, the production of these discourses does not happen freely, being socially positioned and being part of their production conditions.

In the study conducted by Doerr and English (2006), the teacher manifested uncertainty in relation to which strategy the students could use to solve the problem during the modelling task. This uncertainty refers to the legitimacy of her action in this context, as discussed by the authors, constituting the tension between the implementation of the task and the uncertainty about which solutions the students could develop.

Thus, the tension in discourses was constituted in the contact among them. Without discourse, we cannot talk about the tension among them, because it is discourse that gives meaning to the situation that the teacher experiences. The tensions can only be produced when the subject identifies – by using words, images, symbols – different possibilities of action. Therefore, we assume that the tension in the teacher's discourses is manifested through concerns, dilemmas and uncertainties, referring to the possibilities of actions, being identified as discursive phenomena.

2. Context and Methodology

The context was taken from the first experiences with mathematical modelling of three lower secondary school teachers at public schools in the Northeast of Brazil. In the period in which the data were collected, these teachers were finishing a training program for non-certified teachers at the State University of Feira de Santana. In this training course, the teachers solved problems from other disciplines or daily situations by themselves as well as they applied modelling in their classes.

As the purpose of the research was to understand which tensions are manifested in the modelling practice, the nature of the research follows a qualitative perspective (Denzin & Lincoln, 2005). The data was collected through observation of the lessons (through filming), interviews after each lesson and narrative about the lessons, with each teacher. The data analysis was inspired by grounded theory (Charmaz, 2006), with the intention of producing theoretical understandings based on the collected data and orientated by the aim of the research. This analysis occurred in two phases: the first phase involved the codification of the teachers' tensions in each lesson, and the second phase consisted of classifying the codes into more general categories. After that, we were able to produce an understanding for the research problem by integrating the results in the literature.

3. The tensions in the teachers' discourses in modelling environment

In this section, we present the tensions experienced by teachers when developing modelling-based lessons. Then, we will bring illustrative extracts of the teachers' discourses in the lessons to present the tensions. The participants were three experienced lower secondary school teachers, called over by pseudonyms, Vitoria, Maria and Boli. Each teacher had been teaching for over 14 years when we collected our data. They organized the modelling environment according to what Barbosa (2003) calls Case 2; in other words, the teacher presents a problem and students should collect data and investigate it. They organized the lessons into groups for the development of modelling tasks. Following, we are going to discuss each tension that was constituted in implementing modelling.

3.1 Vitoria

Vitoria was the teacher who worked with the modelling task entitled "The minimum wage and the cost of living of a family" in a class (8^{th} grade) of the lower secondary school. She developed the modelling task during seven weeks between August and November 2006. Vitoria explained that she used modelling in order to make students learn mathematics: *"the students understand how mathematical contents are applied in daily life"* (FROM NARRATIVE). She planned the modelling task in some phases such as discussing the theme, getting information about families' expenses, establishing the products and quantities of a basic basket of goods, collecting data, making calculations and comparisons, elaborating tables.

Vitoria was concerned whether the chosen theme would interest her students. She suggested that there would be some themes more difficult and other easier to work with them.

"First, I thought a project about building a football field. Then, I thought that it would be very difficult. My concern was whether the students would be able to develop the task. After that, I thought a project about public transportation, but I thought it was too difficult. Next, I thought a modelling task about a vegetable garden, but I didn't know if I would be able to manage it. However, my biggest concern was with the class. Then, I began to think about a theme that would be easy for them" (FROM INTERVIEW).

In this extract, Vitoria commented on her concern in choosing a theme that would make sure that her students would participate effectively in running the task. In the second lesson, she was concerned, because they did not have brought the required information about the family' expenses.

"I was quite in panic, because the majority of students hadn't brought the information. I asked for them to sit in groups and they resisted. When the lesson started, some students were outside the classroom, not engaging in the task. They resisted in working together on the task" (FROM NARRATIVE).

In this extract, she seemed frustrated, because her students did not bring the information about the family' expenses. They did not want to sit in groups and they showed some resistance in taking part in the task. Thus, as she was worried about which theme to choose as well as she was trying to make students engaged in the task, we call such tension of *the students' engagement*.

In the third lesson, after Vitoria announced the task, a student asked: *"Teacher, how do I make this calculation?"* She was concerned with this situation, because she noticed that her students had not understood the modelling task.

"'Teacher, what is to do?' Or, 'Teacher, I don't know how to do that'. These questions have left me frustrated, because a few days ago we had worked with many percentage problems. And they, except a few, went very well. Then, I asked: How come? We did several percentage problems and you did very well, didn't you?" (FROM NARRATIVE)

As her students did not understand what they would have to do in the task, then it constituted a tension for her. We call this tension of the *understanding of the modelling task by students,* because they were not able to know what to do and which mathematical content to use to solve the task.

Vitoria was concerned if the students could develop the tasks, because they had difficulties in relation to the mathematical content. Thus, she had difficulties in how to teach the contents during the development of the task.

"Now, my concern is if they will develop the tasks. They have a lot of difficulties with the contents, for example, graphs. They don't know what the graph is. I would like to teach graphs and other mathematical contents. So far, I approached the operations and percentage. I have difficulties of teaching other contents. I don't know how to teach this, because they don't have knowledge of the other contents" (FROM INTERVIEW).

That situation constituted a tension for her. We call this tension of the *students' comprehension of mathematical content*, because her students had difficulties in relation to the mathematical content. Thus, she had difficulties in how to teach the contents during the task. Vitoria commented on the difficulties during the task. The difficulties were related to how to conduct a modelling task.

"I was concerned with the task. I had difficulties during the task. I think: how do I do? I am always reading the articles on mathematical modelling. I am reading them again. [...] *I didn't know how to conduct if I get in the class without anything. What will I do? I will ask something, they will answer. What do I do? I finish the lesson, don't I? I elaborated the tasks, because I would like to know how to run the lesson. I wasn't confident"* (FROM INTERVIEW).

These difficulties generated the tension of *conducting a modelling task*, because Vitoria had difficulties to decide which actions to accomplish first during the modelling task.

3. 2 Maria

Maria was the teacher who worked with the modelling task entitled "Analyzing the water bill" in a class (7th grade) of the lower secondary school. She developed the modelling task during ten weeks between August and October 2006. Maria explained that she used modelling as a way to make students learn mathematics: *"It is important that the teachers introduce modelling mathematics in their pedagogic practices to facilitate the learning mathematics for their students"* (FROM NARRATIVE). She planned modelling task in some phases such as discussing the theme, getting information, analyzing tables, making calculations and comparisons.

She spoke to the researcher (first author) during one of the lessons that the groups were not able to make the calculations, because they did not understand the modelling task. *"Most of the students aren't able to understand the worksheet of the task"* (FROM OBSERVATION). In this extract, she explained what happened: *"They asked me for explanations about the task. Some students called me and they said that they didn't have any idea how to make the calculations. One student asked me: 'Which calculations shall I make?'"*

(FROM NARRATIVE) That situation constituted a tension for her, which we name tension of the *understanding of the modelling task by students*, because they were not able to know which mathematical content to use to solve it and what to do.

In the interview, she related that she had come across unexpected situations during the development of the modelling task: "*Now, what am I going to do? Therefore, I spoke to you* (first author). *At first, I thought about water consumption, but the concept of parameters and treatment came up. Several new things appeared that weren't in the original script. So, I had that to give attention to the students. I didn't want that they lose the enthusiasm. I had to give attention to them. Really, at every moment that I had doubts, I thought: Now, where do I go next?*" (FROM INTERVIEW)

In this extract, Maria referred to the moment when she presented the task to her students. She asked them that they chose three items of the water bill to discuss with her. She did not imagine that they would choose other items beyond the consumption of water. Thus, she had difficulties about how to work with the unexpected options presented by them.

These difficulties generated the tension of *conducting a modelling task*, because Maria had difficulties to make decisions when unexpected situations came up. Due to that, she asked the researcher for advice after each lesson. She spoke on the difficulties and concerns that appeared during the lessons. In the below extract, she was concerned, because some groups did not bring the collected information. She was wondering that they were not interested in the modelling task.

"The information was not collected by two groups. In a group, three students didn't attend the previous lesson and they did not know what to do. Another group said that they had lost the information. The goal is to make sure everybody engaged. When I saw that a group had not collected the information and another group had lost, I got worried. I thought that perhaps the invitation had missed out on something. Then, I was concerned. How do I develop modelling task?" (FROM INTERVIEW)

Maria wanted that all students to take part in the task. When she saw that some groups had not brought the information for the lesson, she thought that she had not motivated them successful to develop the modelling task. These concerns constituted the tension of *the students' engagement*. She noticed that her students stopped developing the task, because they did not know how to make the calculations.

"Running task, a deadlock happened: they didn't get to convert liters to cubic meters. At that moment, I finished the lesson. I would make the questions for the coming lesson" (FROM NARRATIVE). In such extract, she commented on the deadlock that happened at the end of the lesson: *"At the end of the lesson, the last question was to convert liters to cubic meters, but no group did it. Then, that was a deadlock for me. They couldn't consult the table, because consumption*

was not converted liters to cubic meters. They could not do the task" (FROM INTERVIEW).

As her students did not know convert liters to cubic meters, the last question of the worksheet was not completed by them. That situation constituted a tension for her, which we call tension of the *students' comprehension of mathematical content*, because they had difficulties in relation to the mathematical content.

3. 3 Boli

Boli was the teacher who worked with the modelling task entitled "Basic food for all" in two classes (9th grade) of the lower secondary school. He developed the modelling task during eight weeks between August and October 2006. He explained that he used modelling as a way to make students *"understand the presence of the mathematics in daily situations"* (FROM NARRATIVE). He planned modelling task in some phases such as discussing the theme, defining the family' expenses, defining the products and quantities of a basic basket of goods, collecting information, making calculations and comparisons, drawing graphs.

Boli's concerns referred to the comprehension that the students would have about mathematical contents. Most of the time, they demonstrated gaps in the expected mathematical content knowledge for their grade level, frequently interrupting the development of the modelling tasks. During a lesson, when he was circulating around the groups that were developing the percentage calculations for the family's expenses, he was confronted with the fact that some students were not able to accomplish the task.

In view of that, he asked his students: *"Did you study proportion and percentage in the 6th grade, didn't you?"* (FROM OBSERVATION) In this lesson, he commented on what he observed: *"I verified that quite all of the students didn't have any notion of percentage or the rule of three"* (FROM NARRATIVE). That situation constituted a tension for him. We call this tension of the *students' comprehension of mathematical content*, because he did not expect that they would not know the content. Thus, the tasks could not be developed. Because of that, he used part of the lessons destined for the modelling task to deal with the students' difficulties in relation to previous content.

Boli commented on his frustration due to the some students' indifference at the modelling task. Despite the task has been of interest to them, he noticed that not always the task mattered to all.

"It had a group that wasn't making the task anymore. Sometimes, a task could interest some students and not interest others. I was disappointed. They might take part in, because it is a task that interest me and also to them. [...] *I was concerned with the participation of some students who hadn't much interest"* (FROM INTERVIEW).

In this extract, he noticed that the task will not interest all students. There was a group that was not taking part in the task. He was frustrated with this situation, because he would hope that the theme would interest all. These concerns constituted the tension of *the students*' *engagement*. Boli commented the unexpected situations that have happened in the development of the modelling task. In each time it, he was concerned about what to do to develop it.

"A new thing always happened. I am always concerned with the task. Some problems always appear. I thought: how do I do that?"(FROM OBSERVATION) *"My concern was with the first time even, because I never know which it could occur on the day following.* [...] *What will I do tomorrow?* [...] *Then, the concern was the following: what do I do in the next meeting? I felt this concern"* (FROM INTERVIEW).

These concerns generated the tension of *conducting a modelling task*, because Boli did not know what to do during the task when it appeared unexpected situations. When he was circulating around of the groups, he noticed that the calculations made by the students were incorrect. *"I noticed that the calculations were wrong. They have made them again. I was concerned with values so high that they had found. I was concerned, because the difference was very large that they had found from the calculations"* (FROM INTERVIEW).

Boli spoke to the researcher that he asked his students to make the calculations again, because the values were not correct. Thus, he was concerned with the students' incorrect calculations, because this situation indicated that they didn't know how to accomplish the task. That situation constituted a tension for him, called of the *understanding of the modelling task by students*.

4. Discussion

These tensions presented through the teachers' discourses were constituted by the difficulties, concerns and dilemmas involved in the development of the modelling tasks. These aspects seem to refer to the legitimacy of the actions in the practices these teachers while they are implementing modelling, constituting the following tensions:

- *The tension of the students' engagement,* referring to concern and difficulties related to the students' participation in the tasks. The teachers expect that students be involved in the task, but sometimes they seem indifferent. Thus, teachers were concern how to stimulate the students to run the modelling task;
- *The tension of understanding of the modelling task by students,* referring to concern in relation to how the students understand the task. The students did not understand what they would have to do in the task and they did not know which mathematical content to use to solve it;

– *The tension of the students' comprehension of mathematical content*, referring to a concern in relation to what they know about mathematical ideas and algorithms. The teachers expect that students have some mathematical content knowledge about previous content, but they seem to have many difficulties to use it during the development of the modelling task.
– *The tension of conducting a modelling task*, referring to concerns and difficulties in relation to what to do during the task and to a dilemma in relation to what to do at any given instant, for example, when any unexpected situations come up. The teachers were undecided about different actions to accomplish in the development of lesson.

As the teachers developed a different practice in their lessons, they found new challenges to manage. These challenges refer to how to deal with the students, how to teach mathematical content, how to approach daily situations, how to develop the task. Thus, these challenges refer to how to implement pedagogic innovations, in this case, mathematical modelling, in classroom practice.

The identified tensions are likely found in other types of pedagogic innovations (and not only in maths) as well. The tensions *understanding of the modelling task by students* and *students' comprehension of mathematical content* describe the teachers' concerns about the students' interest in the task, mainly the comprehension about what is to do and the application of previous contents.

On the other hand, we noticed some concerns seemly associated to modelling such as the teacher choosing real subject for modelling and discussing the parameters for a situation. The data illustrated that as tensions of the *students' engagement* and *conducting a modelling task,* which addressed the presence of "real" situations in classroom.

5. Final remarks

The analysis of the data suggests a discursive dimension for the tensions in the teachers' discourses in relation to unexpected situations that happen in the mathematical modelling practice. That discursive dimension manifests itself through the concerns, dilemmas and uncertainties referring to the possibilities of actions being socially organized and constituted through the contact between discourses that circulate and those that are legitimated in the social environment.

As in the study of Blomhøj and Kjeldsen (2006), the teachers from our study faced dilemmas that constituted specific tensions in relation to the modelling process. Understanding the aspects that constitute tensions in the teachers' discourses in the modelling practice can provide indications about how teacher develops the pedagogical knowledge to deal with them (Doerr, 2007) and strategies used for the modelling practice in the classroom (Chapman, 2007).

In our study, we observed that the teachers had to accomplish actions in order to work with the situations that arose during the modelling task. In this way, the tensions can help the teachers in their professional development, since they can produce actions, strategies and pedagogical knowledge in the accomplishment of a new practice in their classrooms.

As implications for teacher education in modelling, it is important to approach the specific nature of the modelling and to bring the discussion on the possible difficulties, dilemmas and tensions that can happen when the teacher works with modelling by the first time.

6. Acknowledgments

We thank the reviewers for their comments on an earlier version of this paper. Our thanks to the State of Bahia Research Foundation (FAPESB) for supporting research with grants (BOL 1786/2006 and APR0059/2007).

References

Barbosa, J. C. (2006). Mathematical modelling in classroom: a critical and discursive perspective. *ZDM – The International Journal on Mathematics Education*, *38* (3), 293-301.

Barbosa, J. C. (2003). What is Mathematical Modelling? In S. Lamon, W. Parker, & S. Houston (Eds.) *Mathematical Modelling: a way of life ICTMA 11* (pp. 227-234). Chichester: Horwood Publishing, 227-234.

Bernstein, B. (2000). *Pedagogy, symbolic control and identify: theory, research, critique*. Lanham: Rowman & Littlefield Publishers.

Blomhøj, M., & Kjeldsen, T. (2006). Teaching mathematical modelling through project work. *ZDM – The International Journal on Mathematics Education*, *38* (2), 163-177.

Blum, W. et al. (2007). *Modelling and Applications in Mathematics Education*: *the 14th ICMI study*. New York: Springer.

Chapman, O. (2007). Mathematical modelling in high school mathematics: teachers' thinking and practice. In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.) *Modelling and Applications in Mathematics Education*: *the 14th ICMI study* (pp. 325-332). New York: Springer.

Charmaz, K. (2006). *Constructing Grounded Theory*: *a practical guide through qualitative analysis*. Thousand Oaks: Sage.

Denzin, N., & Lincoln, Y. (2005). Introduction: the discipline and practice of qualitative research. In N. Denzin, & Y. Lincoln (Eds.) *Handbook of qualitative research* (3rd ed.) (pp. 1-32). Thousand Oaks: Sage.

Doerr, H. (2007). What Knowledge do teachers need for teaching mathematics through applications and modelling? In W. Blum, P. Galbraith, H. Henn, & M. Niss (Eds.) *Modelling and Applications in Mathematics Education*: *the 14th ICMI study* (pp. 69-78). New York: Springer.

Doerr, H. M., & English, L. D. (2006). Middle grade teachers' learning through students' engagement with modelling tasks. *Journal of Mathematics Teacher Education*, *9,* 5–32.

Evans, J., Morgan, C., & Tsatsaroni, A. (2006). Discursive positioning and emotion in school mathematics practices. *Educational Studies in Mathematics*, *63* (2), 209-226.

Leiß, D. (2005). Teacher intervention versus self-regulated learning? *Teaching Mathematics and its Applications*, *24* (2-3), 75-89.

Niss, M., Blum, W., & Galbraith, P. L. (2007). Introduction. In W. Blum, P., Galbraith, H. Henn, & M. Niss (Eds.) *Modelling and Applications in Mathematics Education*: *the 14th ICMI study* (pp. 3-32). New York: Springer.

TEACHING TO REINFORCE THE BONDS BETWEEN MODELLING AND REFLECTING

Mette Andresen, *head of NAVIMAT,*
University College Copenhagen, Denmark,
mea@cvukbh.dk

Abstract: When students engage in mathematical modelling in problem solving settings, they tend to focus on the application aspects and on the results of the modelling process. The introduction of compulsory use of CAS may amplify this risk, so that students develop a bare technical view on mathematical modelling, at the expense of a profound understanding of the nature of mathematical thinking. This paper advocates a teaching, which balances out the 'technical-application's view by explicit reflections upon the use of models and upon the modelling process. A new model is suggested, based on the idea of combining levels of mathematical activities, with levels of reflections, presented in a recent work on philosophy of mathematics (Prediger 2007). The design of a teaching sequence is outlined in the paper.

1. BACKGROUND FOR THE STUDY

Growing interest in models and modelling in Danish upper secondary

Danish upper secondary school was subject to a reform in 2006. One element of the reform was the introduction of multi-disciplinary projects, which also involved mathematics. Mathematical models and modelling were explicitly mentioned as part of the curriculum in mathematics, although there were no formal requests for specific modelling sequences or themes; the teachers autonomously decide about the planning at that level of details. Consequently, the scene was set for the teaching of models and modelling in close connection with other subjects like physics, chemistry, social sciences, economy etc.: teachers who wants to, and who feels competent to, can carry out teaching sequences on modelling and/or on the investigation of authentic mathematical models. To some degree, this was also the case before the reform. The new point is that since all teachers are now obliged to carry out multi-disciplinary projects, a growing number of teachers, apparently, turn their interest to modelling issues (Andresen and Lindenskov 2007).

A bare technical view on modelling as a potential drawback of CAS

The 2006-reform also implied the introduction of compulsory use of computer algebra systems (CAS) in mathematics. This introduction will, obviously, cause comprehensive changes now and in the future. For the teaching of models and modelling, the use of CAS opens up for a wider range of topics, and for numerical treatment of a variety of models, like for example in the case of differential equations models. It has potentials for a huge extension and development of the teaching of models and technical modelling in the sense of comparing a number of models and fitting them with a set of data (Andresen 2007 p5). It also has potentials to support students' model recognition and capability to understand and criticize authentic use of ready-made models in different contexts.

Results from our previous research, though, show that in general, the use of CAS tends to change focus of attention into technical and practical aspects of upper secondary school mathematics. This tendency results from the individual teachers' choices based on preferences, habits and CAS competencies. In general, teaching with computer is centred upon solving tasks, whereas the reading of proofs and theoretical treatments in general are carried out without use of computer (Andresen 2006 p 28). Thus, there is a potential danger, that the same trend might direct the teaching of 'models and modelling' into a bare 'application' view on mathematics by the students, at the expense of giving the students a more profound insight into mathematical activities, theory and knowledge.

To avoid this, the students' more technical and practical view on models and modelling, partly caused by the introduction of CAS, can and should be balanced out by explicit reflections upon the use of models and upon the modelling process. In the following, Gravemeijer's four level model of mathematical activity (Fig.1) is combined with a four level stratification model of mathematical reflections, discussed in (Prediger 2007). Combination of the two models serves as a basis for a teaching model that aims at such balance.

2. FOUNDATIONS

The role of reflections in learning mathematics

The role of reflections in learning mathematics is apparent in the domain-specific instruction theory for realistic mathematics education (RME). RME is rooted in Hans Freudenthal's idea of '*mathematics as a human activity*'. According to this theory, students should be given the opportunity to reinvent mathematics by mathematizing; mathematizing subject matter from reality

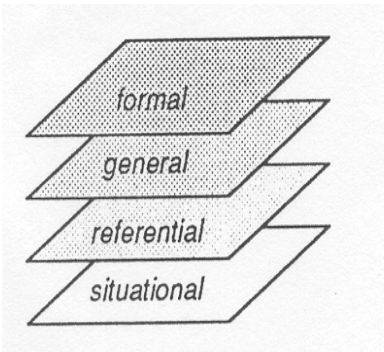


Figure 1: Levels of activity. Gravemeijer, K. & Stephan, M. (2002). p 159

(horizontal mathematizing) and mathematizing mathematical matter (vertical mathematizing). This implies that the students develop a high level of intellectual autonomy. Hence, the core principle is that mathematics can and should be learned on one's own authority, through one's own mental activities. Horizontal and vertical mathematizing may be modelled by the passing of four levels of activity (fig.1). A new mathematical reality is created at each level. Reflections substantiate the progressive mathematizing (Gravemeijer, 2002, p.147 ff).

Stratification of mathematical reflections combined with levels of activity

The use of philosophical reflections as a tool for mathematical reasoning was recently discussed (Prediger 2007). Prediger's discussion was based on the stratification in (Neubrand 2000) of reflective practice in mathematics into four levels:

1) The level of the mathematician
2) The level of the deliberately working mathematician
3) The level of the philosopher of mathematics
4) The level of the epistemologist.

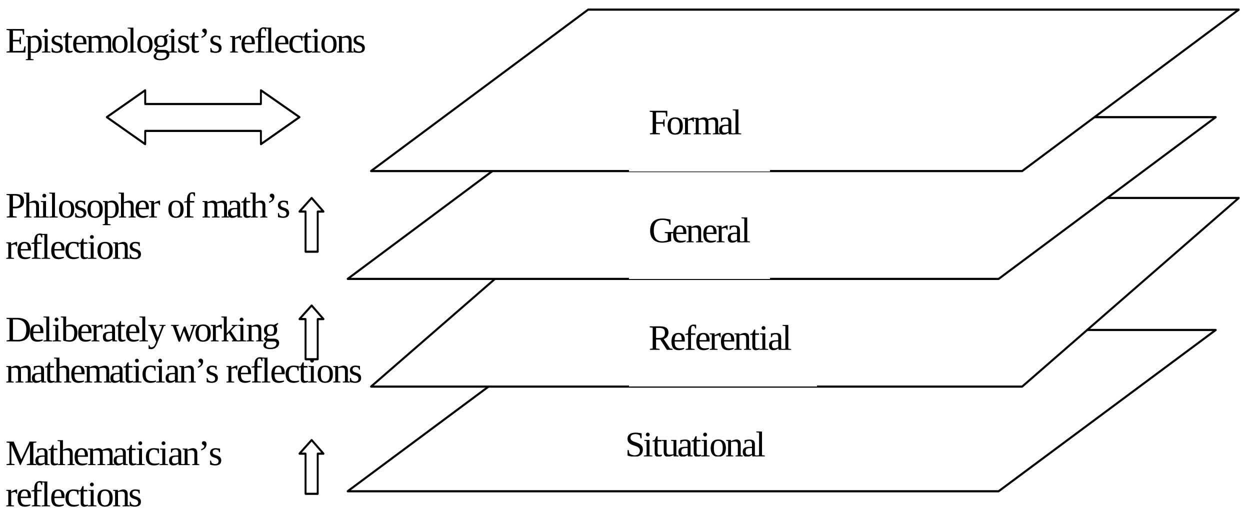


Figure 2: The combined model

The teaching model, which was presented in (Andresen and Froelund 2008), involves the preparation of a reflection guide. Basic to the preparation of the guide is the combination of Neubrand's four levels of reflection with the four levels of activity in Gravemeijer's model, as it is illustrated in (Fig 2.): the reflections should initiate or support students' activities to pass from one level to the next in Gravemeijer's model. Thus, reflections at level 1 in Neubrand's model bring the student from the situational to the referential level. Next, reflections at level 2 bring the student from the referential to the general level. Finally, reflections at level 3 bring the student from the general to the formal level. It should be noticed, that (Andresen and Froelund 2008) does not suggest a fifth level on top of the 'formal' level in our combined model. Rather, the epistemological reflections are considered to widen the formal level in its horizontal dimensions. This attests the authors' view, that epistemological reflections do not and should not represent an external or additional level to adequate mathematical reflections (see also Prediger 2007 p 45).

3. **REALISATION**

Use of the combined model

Use of the combined model is not restricted to teaching sequences on modelling in the 'applications' sense of the concept. According to RME, mathematics is a human activity and since the vertical and horizontal mathematising are main heuristics for learning mathematics, all mathematical activities imply some sorts of modelling. The combined model, therefore, may be used in all teaching sequences. To illustrate this point, the following example of using it takes a rather traditional series of tasks as its starting point. We will demonstrate how it may be used to stress the modelling aspects by explicit reflections upon traditional mathematical activities.

The basic idea for the teacher, during the design of the students' learning trajectory, is to pick out moments of interest for making the reflections explicit. The term 'moments of interest', here, relates to the levels in Gravemeijer's four-level model or, more precisely, to situations with special potentials for the students to rise or descend from one level to another. Stimulation of reflections relevant to the level, and the efforts of making them explicit, will not only support the students' concept formation and learning, but also aim to enhance the students' awareness, knowledge and consciousness about mathematical activity as such.

Since the students' reflections should be regarded as intellectual activities, carried out autonomously by the single individual, we intend to stimulate the reflections by making the teacher ask thought-provoking questions. The questions form a reflection-guide, following our combined model. When designing the teaching sequence, the teacher prepares the reflection guide.

4. Preparation of a reflection guide

A thorough analysis of the teaching materials in case must precede formulation of questions for the reflection guide. The analysis aims to identify potential levels of students' mathematical activity as they are illustrated in (Fig1.). We take as our starting point that the reflection guide should be tailored to fit the teaching materials, not vice versa. Hence, the task to identify potential activities may in some cases become an issue of interpretation, to discern the textbook's rationale. Or it may start with an imagination of the students' hypothetical learning trajectory. In the following, we give an example of preparation of a guide based on a calculus worksheet with tasks, picked out from (Christiansen et al. 2006), translated from Danish in Encl1.

Identification of potential activity and formulation of thought-provoking questions

Thought-provoking questions at all four levels of reflection are based on the potential activity levels. In the following, four groups of thought-provoking questions, one group for each of the levels in Prediger/Neubrands model, are formulated in relation to the levels of activity, identified and picked out from Encl1's text.

The tasks in Encl1 appear like traditional word problems and concern with optimisation. So, some of the questions, especially at the two first levels, may appear traditional, plain and naïve. In this demonstration, we consider working with the whole worksheet as being one, overarching activity. According to its authors, the worksheet was meant for students' training (Crhistiansen et al. 2006, the website). A realistic estimation of the duration of the students' work with accomplishing the worksheet would be two lessons and the rest as homework – depending, of course, of the group of students.

What we find interesting here is that the questions to the students can give rise to reflections upon the worksheet's tasks as a whole, as well as reflections upon the single tasks, elements of solving them etc. Therefore, the scene is set for reflections at all four levels in Prediger/Neubrand's model, even if the worksheet deals with short, traditional word problems and exercises rather than a full modelling cycle.

The students' knowledge about the levels of reflections, their consciousness about their own reflections and the thinking they initiate, are key points for the outcome.

Questions at the level of the mathematician.

To deepen the students' understanding of the rise from a situational to a referential model, questions at first level in Prediger's model should be asked. In terms of RME, this rise is horizontal mathematising, where the model emerges.

Examples of *activities* at the situational level in Encl1:

- In task 1; talking about the total of two numbers, the cube of each of the numbers and the total of the cubes of the numbers
- In task 2; talking about the boat situated 10 km from the coast, the house 12 km along the coast, the distances along the beach, across the water

- In task 3; talking about a certain point of the parabola and its distance to another point.
- In task 6; talking about a box and its length, width, surface and about how much it contains

In this context, *questions* that stimulate reflections at the level of the mathematician, which means stimulate rising to the referential level, could be like these:

- Concerning task 1: what could we call the two numbers? How could the cube of the first number be denoted? The cube of the second? How can we express that the total of the two numbers is 12?
- Concerning task two: how could we denote the distance he has to go, crossing the water? How can we express the time this part of the tour it takes?

To deepen the students' understanding of the mathematising, *questions* might be asked that stimulate descending from referential level to situated, like for example:

- Concerning task 4: What is the shape and orientation of the parabola? How is the parabola situated relative to the point?
- Concerning task 5: what does A mean? What would happen in line 21 if *h* equals zero?

Questions at the level of the deliberately working mathematician.

To identify activities at the referential level, generated by the worksheet, the teacher has to build on his or her imagined learning trajectory. In our example, a plain interpretation of the intended trajectory involves traditional activities at this level. Examples of such *activities* are:

- In task 1; putting up the equations $a+b=12; s=a^3+b^3$ where a and b are still interpreted as the two, unknown numbers and s is thought of as the total of the numbers' cubes
- In task 2; making a drawing like Fig.3 and put up expressions like

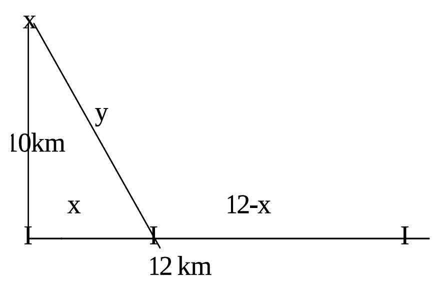


Figure: 3

$$y=\sqrt{x^2+100};t_1=\frac{\sqrt{x^2+100}}{4};t_2=\frac{12-x}{8};t_1+t_2=\frac{12-x+2\sqrt{x^2+100}}{8}$$

- Where, for instance, t_1 is still thought of as the time spent on crossing the water and t_{1+} t_2 is interpreted by the students as the total time spent on the tour
- In task 5; finding the derivative of A, substituting 500 for V and isolating r^2in the equation in line 23

In this context, *questions* that stimulate reflections at the level of the deliberate mathematician, which means stimulate rising to the general level, could be like these:

- Concerning task 1: Which one is the variable, if we see *s* as a function? That is, what does *s* depend on?
- Concerning task 2: How can we determine the derivative of t_{1+} t_2?
- Concerning task 7: How could *f(v)* be maximised? What are the restrictions on the variable *v*, if any?

Again, *questions* might also be asked, which stimulate descending from general to referential level, and maybe descending further to the situational level, like for example:

- Concerning task 2: Can t_1 be negative? What would the maximum value be for x?
- Concerning task 3: How many solutions can there possibly be?
- Concerning task 5: What are the restrictions on *r* and *h*?

Discussions of how to discern between the three levels may support the individual student’s deliberate changes between all three levels, and prepare for the next level of reflections.

Questions at the level of the philosopher of mathematics

Rise from general to formal model tends to happen over time, sometimes in a somehow subtle way. By solving the tasks in Encl1, certain routines are carried out repeatedly. The overarching activity to accomplish the worksheet’s tasks, therefore, gives insight into corresponding methods. Reflections upon the emergence of a method from carrying out a routine a number of times, give insight into mathematics at formal level. So, questions that stimulate reflections upon methodological issues such as how to identify a method and how to distinguish between different methods support the students’ rise from activity at the general level to activity at the formal level.

In our case, *questions* at the level of the philosopher of mathematics, therefore, could be like:

- What do you need to know, to be able to minimise an unknown quantity?
- What kind of expressions can be maximised?
- How do you state the assumptions and put up the conditions for optimisation?
- Is it always possible to put up an expression and find the derivative? What exceptions can you imagine?
- Do you know any analogous to this way of problem solving?

Questions at the level of the epistemologist

Activities at the formal level may be widened by further reflections. The characteristics of mathematics and related issues can be enlightened by classroom discussions of questions like:

- Do you know any examples of modelling from other subjects than mathematics?
- What are the aims of models in other subjects like physics, chemistry, economy, and text analysis, social sciences?
- Who decide about the validity of a model in each of these subjects? What are the 'rules'?
- How can you decide whether a model belongs to mathematics? What are the characteristics?
- Who make the models, what are they made for?

Design of teaching with the reflection guide

The detailed design has to be carried out by the teacher, since the teaching must be tailored to fit the actual group of students in its unique situation and context. One of the main questions the teacher has to decide about with regard to the structure of the sequence is: *when* are the students supposed to become aware of the reflection process - under the process or after it is accomplished? The teacher's choice must be conditional on the complexity of the sequence, the level of difficulties with the content and the expected outcome for the students. Different designs of the teaching sequences give weight to different levels of reflections. Hence, the weighting of reflections is another important question that regards the learning goals. For example, reflections at the first two levels may serve to develop technical aspects of the 'mathematical modelling competency' whereas reflections at the other two levels may serve to throw light on modelling processes. In general, reflections at the higher levels tend to be more general than the others since the very notion of reflect means '*transcending the immediate object of the present consciousness in a learning subject. The outcome of a reflection is a consciousness at a more general level. At this level the object at the first level is situated in a broader context. Consequently knowledge of the external relations of the first object to similar objects in the same class is now produced.*' (Andresen and Froelund 2008 p3).

In our example, the teacher has to decide when the students are supposed to answer the guide's questions, and how to organise this part of the teaching sequence. One way to do it could be to arrange a classroom discussion, introduced by the teachers' presentation of an adapted version of the basic idea. Another way could be to let the students write an essay, if the formal regulations include that sort of activity in mathematics. The preparation of a reflection guide based on our model helps the teacher to clarify the learning goals and to be aware of and realise hidden potentials for the students' learning. In line with the above discussion of the need for a more balanced teaching, the model can serve to draw students' attention to what might be called core mathematical activities. Nevertheless, there are no obvious reasons to assume that the model would not work in other parts of the modelling processes too.

The questions in the first group, supposedly, resemble 'normal' questions, asked by any math teacher. None of the other questions in the reflection guide have to be a stroke of genius, neither. The point is for the teacher to make the students aware of their own thinking, and to encourage them to develop new insight during discussion in the classroom.

5. CONCLUSION

Naturally, use of the combined model to prepare a reflection guide, should and will be followed by a try out in the classroom. The first try out of the model is planned to take place in spring 2008, in connection with a research project with focus on the teaching of authentic models in upper secondary. The actual case concentrates on probability and estimation in relation to statistical models. Another try out is planned in the same period, in an experimental, multi-disciplinary project involving mathematics and philosophy. The philosophy class will concentrate on mathematical knowledge, reflections and philosophy, whereas the mathematics class will base some of the lessons on guided reflections in accordance with the combined model. These two settings seem obvious for reinforced bonds between modelling and reflecting. The above example, though, demonstrates that it is possible to accentuate the modelling aspect of mathematical activities even in a prearranged 'exercise-setting'. In this case, the reflection guide's questions deliberately stress the student's experience of underlying ideas, basic principles and generalisation. The guided reflections in this case intend to enhance and focus those experiences, which the students in all cases might gain at random, when making series of exercises.

REFERENCES

Andresen, M. (2006). Taking advantage of computer use for increased flexibility of mathematical conceptions. Danish University of Education.

Andresen, M. (2007). Modelling with the Software 'Derive' to Support a Constructivist

Approach to Teaching.15 pages. I: *International Electronic Journal of Mathematics Education,* 2(1). ISSN: 1306-3030 **http://www.iejme.com**

Andresen, M. and Froelund, S. (2008). *Philosophical reflections made explicit as a tool for mathematical reasoning.* Ten pages paper, will be presented at '5th International Colloquium on the Didactics of Mathematics', 17. - 19. April 2008, University of Crete, GREECE

Andresen, M og Lindenskov, L. (2007). *Multi-disciplinary projects in upper secondary school – new roles for mathematics?*. 9 pages. MACAS2 (The Second Internationale Symposium of Mathematics and its connection to the Arts and Sciences), 29.31. maj 2007, Odense

Christiansen, O., Dybkjær, F., Pedersen, L. and Sørensen, S. K. (2006). *Origo. Matematik 2 til gymnasiet.* Malling Beck, Denmark (www.origo.mb.dk)

François, K. & van Bendegem, J.- P. (Eds.)(2007): *Philosophical Dimensions in Mathematics Education*, Dordrecht.

Gravemeijer, K. and Stephan, M. (2002) Emergent models as an instructional design heuristic. In Gravemeijer et al. pp145-169

Neubrand, Michael (2000). Reflecting as a Didaktik Construction, in: Westbury, Hopmann & Riquarts *Teaching as a Reflective Practice. The German Didaktik Tradition,* New Jersey.

Prediger, S. (2007). Philosophical reflections in mathematics classrooms. In: François, K. and van Bendegem, J.- P. pp 43-58

Calculus 4

Task 1:

The total of two numbers is 12. Determine the two numbers under the assumption that the total of their cubes is smallest possible.

Task 2:

A man sits in a boat 10 km from the coast. He wants to reach a house, situated 12 km along the coast, compared to his actual position. How far along the beach should he land, assumed that he can walk 8 km/h and row 4 km/h, to arrive the fastest possible, and how long time does it take?

Task 3:

Find that point of the parabola $f(x) = 2x^2$, which is closest to the point (2; -1)

Task 4:

Find that point of the parabola $f(x) = -x^2 + 2x$, which is closest to the point (3; -8)

Task 5:

The area of a cone's surface may be determined by:

$$A = \pi \cdot r \cdot \sqrt{r^2 + h^2}$$

and the cone's volume by

$$V = \frac{1}{3} \cdot \pi \cdot r^2 \cdot h$$

Determine radius and height of a cone with the smallest possible surface, given that the volume is 500.

Task 6:

A box, containing 2 L, without lid must have a length, double of the width. Determine the width so that the area of the box's surface is smallest possible.

Task 7:

The number of cars passing a hindrance is given by:

$$f(v) = \frac{500v}{0{,}007v^2 + 0{,}2v + 5}$$

where *v* is the speed in km/h. Determine the speed which allows the largest number of cars to pass the hindrance.

Results:

1: 6 and 6	2: 5,77 km and 3,66 hour	3: (0,338; 0,228)	4: (3,97; -7,84)
5: 6,96 and 9,86	6: 11,4 cm	7: 26,7 km/h	

APPLYING PASTORAL METAMATISM OR RE-APPLYING GROUNDED MATHEMATICS

Allan Tarp, *MATHeCADEMY.net, Denmark*

When an application-based mathematics curriculum supposed to improve learning fails to do so, a question may be raised: Is 'applying mathematics' what it claims to be, or something else? Skepticism towards wordings leading to anti-pastoral sophist research, identifying hidden alternatives to pastoral choices presented as nature, uncovers two kinds of mathematics: a grounded mathematics enlightening the physical world, and a pastoral self-referring mathematics wanting to 'save' humans through 'metamatism', a mixture of 'metamatics' presenting concepts as examples of abstractions instead of as abstractions from examples; and 'mathematism' true in the library, but seldom in the laboratory.

Applying Mathematics Improves Learning – or Does it?

The background of this study is the worldwide enrolment problem in mathematical based educations (Jensen et al 1998), and 'the relevance paradox formed by the simultaneous objective relevance and subjective irrelevance of mathematics' (Niss in Biehler et al 1994: 371). To improve learning it has been suggested that applications and modelling should play a more central role in mathematics education. However, when tested in the classroom the result is not always positive: 30 years ago the pre-calculus course at the Danish second-chance high school changed from being application-free to being application-based by replacing e.g. quadratic functions with exponential functions. Still student performance deteriorated to such a degree that at the 2005 reform the teacher union and the headmasters suggested that pre-calculus should no more be a compulsory subject. Thus a question may be raised: Is 'applying mathematics' what it claims to be, or something else? Skepticism towards wordings leads to postmodern thinking that, dating back to the ancient Greek sophist warning against choices presented as nature, can find hidden meanings behind the term 'applying mathematics' by deconstructing the term using concept-archeology.

Anti-Pastoral Sophist Research

Ancient Greece saw a struggle between the sophists and the philosophers as to the nature of knowledge. The sophists warned that to practice democracy people should be enlightened to tell choice from nature in order to prevent patronization presenting its choices as nature. To the philosophers, seeing everything physical as examples of meta-physical structures only visible to them, patronization was a natural order if left to the philosophers (Russell 1945).

The Greek democracy vanished with the Greek silver bringing wealth by financing trade with Far-East luxury goods as silk and spices. Later this trade was reopened by German silver financing the Italian Renaissance; and by silver found in America. Robbing the slow Spanish silver ships returning on the Atlantic was no problem to the English; finding a route to India on open sea to avoid Portuguese forts was. Until Newton found out that when the moon falls to the earth as does the apple, it is not obeying the unpredictable will of a meta-physical patronizor only attainable through faith, praying and church attendance; instead it is following its own predictable physical will attainable through knowledge, calculations and school attendance.

This insight created the Enlightenment period: when an apple obeys its own will, people should do the same and replace patronization with democracy. Two democracies were installed, one in the US, and one in France. The US still has its first republic; France now has its fifth.

The German autocracy tried to stop the French democracy by sending in an army. However, the German mercenaries were no matches to the French conscripts only too aware of the feudal consequences of loosing. So the French stopped the Germans, and later occupied Germany.

Unable to use the army, the German autocracy used the school to stop the enlightenment spreading from France. Humboldt was asked to create an elite school, and used Bildung as counter-enlightenment to create the self-referring Humboldt University (Denzin et al 2000: 85).

In the EU the sophist warning against structuralism is kept alive in the French post-structural or postmodern thinking of Derrida, Lyotard and Foucault warning against patronizing categories, discourses and institutions presenting their choices as nature (Tarp 2004).

Derrida recommends that patronizing categories, called logocentrism, be 'deconstructed':

> Derrida encourages us to be especially wary of the notion of the centre. We cannot get by without a concept of the centre, perhaps, but if one were looking for a single 'central idea' for Derrida's work it might be that of decentring. It is in this very general context that we might situate the significance of 'poststructuralism' and 'deconstruction': in other words, in terms of a decentring, starting with a decentring of the human subject, a decentring of institutions, a decentring of the logos. (Logos is ancient Greek for 'word', with all its connotations of the authority of 'truth', 'meaning', etc.) (..) It is a question of the deconstruction of logocentrism, then, in other words of 'the centrism of language in general'. (Royle 2003: 15-16)

As to discourses Lyotard coins the term 'postmodern' when describing 'the crisis of narratives':

> I will use the term modern to designate any science that legitimates itself with reference to a metadiscourse (..) making an explicit appeal to some grand narrative (..) Simplifying to the extreme, I define postmodern as incredulity towards meta-narratives. (Lyotard 1984: xxiii, xxiv)

Foucault calls institutional patronization for ‘pastoral power’:

> The modern Western state has integrated in a new political shape, an old power technique which originated in Christian institutions. We call this power technique the pastoral power. (..) It was no longer a question of leading people to their salvation in the next world, but rather ensuring it in this world. And in this context, the word salvation takes on different meanings: health, well-being (..) And this implies that power of pastoral type, which over centuries (..) had been linked to a defined religious institution, suddenly spread out into the whole social body; it found support in a multitude of institutions (..) those of the family, medicine, psychiatry, education, and employers. (Foucault in Dreyfus et al 1982: 213, 215)

In this way Foucault opens our eyes to the salvation promise of the generalized church: ‘you are un-saved, un-educated, un-social, un-healthy! But do not fear, for we the saved, educated, social, healthy will save you. All you have to do is: repent and come to our institution, i.e. the church, the school, the correction center, the hospital, and accept becoming a docile lackey’.

To Foucault, institutions building on discourses building on categories build upon choice, so they all have a history that can be uncovered by ‘knowledge archeology’.

The French skepticism towards words, our most fundamental institution, is validated by a ‘number&word observation’: Placed between a ruler and a dictionary a so-called ’17 cm long stick’ can point to ‘15’, but not to ‘stick’; thus it can itself falsify its number but not its word, which makes numbers nature and words choices becoming pastoral if hiding their alternatives.

On this basis a research paradigm can be created called ‘anti-pastoral sophist research’ deconstructing pastoral choices presented as nature by discovering hidden alternatives. Anti-pastoral sophist research doesn’t refer to but deconstruct existing research by asking ‘in this case, what is nature and what is pastoral choice presented as nature, thus covering alternatives to be uncovered by anti-pastoral sophist research?’ To make categories, discourses and institutions anti-pastoral they are grounded in nature using Grounded Theory (Glaser et al 1967), the natural research method developed in the American enlightenment democracy and resonating with Piaget’s principles of natural learning (Piaget 1970).

Using Concept Archaeology on Mathematics

The natural fact many provoked the creation of mathematics as a natural science addressing the two fundamental human questions ‘how to divide the earth and what it produces?’ Distinguishing the different degrees of many leads to counting that leads to numbers and to operations predicting counting results.

1.order counting counts in 1s and creates number-icons by rearranging the sticks so that there are five sticks in the five-icon 5 etc. if written in a less sloppy way.

2.order counting counts in bundles using bundle-sizes with a name and an icon, resulting in a double stack of bundled and unbundled, e.g. T = 3 5s + 2 1s = 3)2) =

3.2 5s = 3.2*5 if using cup-writing and decimal-writing separating the left bundle-cup from the right single-cup. The result can be predicted by the 'recount-formula' T = (T/b)*b iconizing that counting in bs means taking away bs T/b times, and the 'restack-formula' T = (T-b)+b iconizing that a bundle can always move from the top of a stack to a position next to the stack. Thus recounting 4 5s in 7s is predicted as:

T = (4*5)/7*7 = 2.?*7 = (4*5-2*7) + 2*7 = 6 + 2*7 = 2)6) = 2.6*7.

3.order counting counts in tens, the only number having a name but not an icon since the bundle-icon is never used: counting in 5s, T = 5 1s = 1 5s = 1.0 bundle = 10 if leaving out the decimal and the unit.

In Greek, mathematics means knowledge, i.e. what can be used for prediction, making mathematics a language for number-prediction: The calculation '2+3 = 5' predicts that repeating counting 3 times from 2 will give 5. '2*3 = 6' predicts that repeating adding 2 3 times will give 6. '2^3 = 8' predicts that repeating multiplying with 2 3 times will give 8. Also, any calculation can be turned around and become a reversed calculation predicted by the reversed operation: In the question '3+x = 7' the answer is predicted by the calculation x = 7–3, etc.

Thus the natural way to solve an equation is to move a number across the equation sign from the left forward- to the right backward-calculation side, reversing its calculation sign.

3+x = 7 x = 7–3	3+x = 7 x = 7–3	x^3 = 7 x = 7^(1/3)	3^x = 7 x = $\log_3(7)$

In Arabic, algebra means reuniting, i.e. splitting a total in parts and (re)uniting parts into a total. The operations + and * unite variable and constant unit-numbers; ∫ and ^ (to the power of) unite variable and constant per-numbers. The inverse operations – and / split a total into variable and constant unit-numbers; and d/dx and √ & log split a total into variable and constant per-numbers:

Totals unite/split into	Variable	Constant
Unit-numbers $, m, s, ...	T = a + n *T – n = a*	T = a * n *T/n = a*
Per-numbers $/m, m/s, m/100m = %, ...	ΔT = ∫ f dx *dT/dx = f*	T = a ^ n $\sqrt[n]{T} = a$, $\log_a T = n$

In Greek, geometry means earth measuring. Earth is measured by being divided into triangles, again being divided into right-angled triangles, each seen as a rectangle halved by a diagonal.

Recounting the height h and base b in the diagonal d produces three per-numbers:

sinA = height/diagonal = h/d, tanA = height/base = h/b, cosA = base/diagonal = b/d.

Also a circle can be divided into many right-angled triangles whose heights add up to the circumference C of the circle: $C = 2 * r * (n*\sin(180/n)) = 2 * r * \pi$ for n big.

However without the Arabic numbers, Greek geometry turned into Euclidean geometry, freezing the development of mathematics until the Enlightenment century:

> The enthusiasm of the mathematicians was almost unbounded. They had glimpses of a promised land and were eager to push forward. They were, moreover, able to work in an atmosphere far more suitable for creation than at any time since 300 B.C. Classical Greek geometry had not only imposed restrictions on the domain of mathematics but had impressed a level of rigor for acceptable mathematics that hampered creativity. The seventeen-century men had broken both of these bonds. Progress in mathematics almost demands a complete disregard of logical scruples; and, fortunately, the mathematicians now dared to place their confidence in intuitions and physical insights. (Kline 1972: 399)

The success was so overwhelming that mathematicians feared that mathematics (called geometry at that time) had come to a standstill at the end of the 18th century:

> Physics and chemistry now offer the most brilliant riches and easier exploitation; also our century's taste appears to be entirely in this direction and it is not impossible that the chairs of geometry in the Academy will one day become what the chairs of Arabic presently are in the universities. (Lagrange in Kline 1972: 623)

But in spite of the fact that calculus and its applications had been developed without it, logical scruples soon were reintroduced arguing that both calculus and the real numbers needed a rigorous foundation. So in the 1870s the concept 'set' once again forced rigor upon mathematics.

Mathematics Versus Metamatics

Using sets, a function is defined 'from above' as a set of ordered pairs where first-component identity implies second-component identity; or phrased differently, as a rule assigning exactly one number in a range-set to each number in a domain-set. The Enlightenment defined function 'from below' as an abstraction from calculations containing a variable quantity:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities. (Euler 1748: 3)

So where the Enlightenment defined a concept as an abstraction from examples, the modern set-based definition does the opposite; it defines a concept as an example of an abstraction. To tell theses alternatives apart we can introduce the notions 'grounded mathematics' abstracting from examples versus 'set-based metamatics' exemplifying from abstractions, and that by proving its statements as deductions from meta-physical axioms becomes entirely self-referring needing no outside world. However, a self-referring mathematics soon turned out to be an impossible dream.

With his paradox about the set of sets not being a member of itself, Russell proved that using sets implies self-reference and self-contradiction known from the classical liar-paradox ‘this statement is false’ being false when true and true when false:

Definition $M = \{A \mid A \notin A\}$, statement $M \in M \Leftrightarrow M \notin M$.

Likewise, without using self-reference it is impossible to prove that a proof is a proof; a proof must be defined. And Gödel soon showed that theories couldn’t be consistent since they will always contain statements that can neither be proved nor disproved.

Still, set-based mathematics soon found its way to the school in spite of syntax errors:

A formula containing two variables becomes a function, e.g. y = 2*x+3 = f(x) where f(x) = 2*x+3 means that 2*x+3 is a formula containing x as the variable number. A function can be tabled and graphed, both describing if-then scenarios ‘if x = 6 then y = 15’. But writing f(6) = 15 means that 15 is a calculation containing 6 as the variable number. This is a syntax error since 15 is a number, not a calculation, and since 6 is a number, not a variable. Functions can be linear or quadratic, not numbers. So saying that at function increases is a syntax error. Numbers can increase, words can’t.

Set-based metamatics defines a fraction as an equivalence set in a product set of two sets of numbers such that the pair (a,b) is equivalent to the pair (c,d) if a*d = b*c, which makes e.g. (2,4) and (3,6) represent then same fraction ½. However, this definition conflicts with Russell’s set paradox, solved by Russell by introducing a type-theory stating that a given type can only be a member of (i.e. described by) types from a higher level. Thus a fraction that is defined as a set of numbers is not a number itself, making additions as ‘2+3/4’ meaningless.

Wanting fractions to be ‘rational’ numbers, set-based mathematics has chosen to neglect Russell’s type-theory by accepting the Zermelo-Fraenkel axiom system making self-reference legal by not distinguishing between an element of a set and the set itself. This removes the distinction between examples and abstractions and between different abstraction levels thus hiding that historically mathematics developed through layers of abstractions; and that mathematics can be defined through abstractions in a meaningful and uncontroversial way.

Mathematics Versus Mathematism

Traditionally, both 2+3 = 5 and 2*3 = 6 are considered universal true statements. The latter is grounded in the fact that 2 3s can be recounted as 6 1s. The first, however, is an example of ‘mathematism’ true in a library, but not in a laboratory where countless counter-examples exist: 2weeks + 3 days = 17 days, 2m + 3cm = 203cm etc. Thus addition only holds inside a bracket assuring that the units are the same: 2m + 3cm = 2*100cm + 3cm = (200 + 3)cm = 203cm.

Adding fractions without units is another example of mathematism:

Inside the classroom	20% (20/100) + 10% (10/100)	= 30% (30/100)
Outside the classroom e.g. in the laboratory	20% + 10%	= 32% in the case of compound interest = b% (10<b<20) as a weighted average

Mathematics Modelling in Primary School

Having learned how to assign numbers to totals by counting in bundles, a real-world question as ‘what is the total of 2 fours and 3 fives’ can lead to two different models.

Model 1 says: This question is an application of addition. The mathematical problem is to find the total of 2 and 3. Applying simple addition, the mathematical solution is 2 + 3 = 5, leading to the real-world solution ‘the total is 5’. An answer that is invalid because it has left out the unit.

Model 2 says: This question is a re-application of recounting. The mathematical problem is to find the total of 2 4s and 3 5s. Re-applying recounting to find a mathematical solution, the units must be the same before adding so we recount the 2 4s in 5s, predicted by the recount-formula T = (2*4)/5*5 = 1 5s + 3 1s, giving the total of T = 4 5s + 3 1s = 4.3 5s, which leads to the real-world solution ‘the total is 4 fives and 3 ones’. A prediction that holds when tested:

IIII IIII + IIIII IIIII IIIII -> IIIII III + IIIII IIIII IIIII -> IIIII IIIII IIIII IIIII III = 4)3) = 4.3 5s

This example shows that applying mathematism may lead to incorrect solutions when modelling addition problems. Whereas re-applying grounded mathematics creates the categories ‘stack’ and ‘recounting’, and allows practicing recounting by asking e.g. 2 fours = ? fives.

Recounting a stack in a different bundle-size is a brilliant example of a modelling process using mathematics to predict a real-world soultion:

	Problem		Solution
Mathematics	T = 2*4 = ?*5	→	T = (2*4)/5*5 = 1.3 5s
	↑		↓
Real-world	IIII IIII -> ? 5s		IIII IIII -> IIIII III

Rephrasing the addition-problem to ‘what is the total of nines in 2 fours and 3 fives’ introduces integration already in primary school. As a matter of fact, the core of mathematics can be introduced as re-applications of recounting, using 1digit numbers alone (Zybartas et al 2005).

However, this is impossible in a ‘10=ten’-curriculum that by presenting 10 as the follower of nine introduces at once the number ten as the standard bundle-size, a pastoral choice hiding that also other numbers can be used as bundle-size. 10 simply means bundle, i.e. 1.0 bundle if not excluding the unit. Thus counting in 7s, 10 is the follower of 6, and the follower of nine is 13.

With ten as bundle-size, recounting-problems disappear, and all numbers loose their units, which creates the basis for mathematism where 3 + 2 IS 5 without discussion.

Thus in primary school an application-based curriculum using recounting to learn the modelling process is prevented by a pastoral choice, ‘10=ten’-centrism, hiding that also other numbers can be used when counting in bundles. And prevented by mathematism claiming that 3 + 2 IS 5 without regarding the units, in spite of countless counter-examples.

Mathematical Modelling in Middle School

Middle school introduces fractions as rational numbers and to be added without units in spite of the fact that fractions are multipliers carrying units: 1/3 of 6 = 1/3*6.

The real-world question ‘what is the total of 1 coke among 2 bottles and 2 cokes among 3 bottles?’ can lead to two different models.

Model 1 says: This question is an application of adding fractions. The mathematical problem is to find the total of 1/2 and 2/3. Applying simple addition of fractions, the mathematical solution is 1/2 + 2/3 = 3/6 + 4/6 = 7/6, leading to the real-world solution ‘7 out of the 6 bottles are cokes’. An answer that is meaningless and invalid because it has left out the unit: we cannot have 7 cokes if we only have 6 bottles; and we do not have 6 bottles, we only have 5.

Model 2 says: This question is a re-application of adding stacks by integrating their bundles. The mathematical problem is to find the total of 1/2 of 2 and 2/3 of 3. Re-applying integration, the mathematical solution is T = 1/2 * 2 + 2/3 * 3 = 3 = 3/5 * 5, giving the real world solution ‘the total is 3 cokes of 5 bottles’. A prediction that holds when tested on a lever carrying to the left 1/2 unit in the distance 2 and 2/3 units in distance 3, and to the right 3/5 units in distance 5.

Sharing-problems asking ‘the boys A, B and C paid $1, $2 and $3 to a pool buying a lottery ticket. How should they share a 300$ win?’ can lead to two different models.

Model 1 says: This question is an application of fractions. The mathematical problem is to split a total of 300 in the proportions 1:2:3. Applying simple addition of fractions gives the answer: since boy A paid 1/(1+2+3) = 1/6 of the ticket he should receive 1/6 of the win, i.e. 1/6 of $300 = 1/6*300 = 50; likewise with the other boys: boy B will get 2/6 and boy C 3/6 of 300$. So the real-world solution is: boy A $50, boy B $100, and boy C $150. As applications such questions can only be answered after fractions and its algebra has been taught and learned.

Model 2 says: This question is simply a re-application of recounting. The mathematical problem is to recount the win in pools, i.e. in 6s, which then can be paid back to the boys a certain number of times. Since 300 = (300/6)*6 = 50*6, the boys are paid back 50 times. So the real world solution is A: $1*50 = $50, B: $2*50 = $100, and C: $3*50 = $150.

Trade-problems as 'if the cost is 2$ for 5kg, what then is the cost for 14kg, and how much can I buy for 6$?' can lead to three different models.

Model 1 says: This question is an application of proportionality, fractions and equations. The mathematical problem is to set up an equation relating the unknown to the 3 known numbers. Applying proportionality, fractions and equations, we can set up a fraction-equation expressing that the cost c and the volume v is proportional, c/v = k. Hence c1/v1 = c2/v2, or 2/5 = x/14 and 2/5 = 6/x. Now the x can be found by solving the equations, or by cross-multiplication. So the real-world solution is 5.6$ and 15 kg. As applications such questions can only be answered after fractions and proportionality and equations has been taught and learned.

Model 2 says: This question is an application of linear functions. The mathematical problem is to set up a linear function expressing the price y as a function of the volume x, y = f(x) = m*x + c, given that the points (0,0) and (5,2) belongs to the graph of the function. The mathematical solution first finds c by inserting the point (0,0) in the formula: f(0) = m*0+c = 0, so c = 0; then we find m by inserting the point (5,2) in the formula: f(5) = m*5 = 2, so m = 2/5 = 0.4. Hence the linear formula is f(x) = 0.4*x. To answer the questions we insert the points (14,y) and (x,6) into the function: f(14) = 0.4*14 = y, and f(x) = 0.4*x = 6. Solving these equations give y = 5.6 and x = 15. So the real-world solution is 5.6$ and 15 kg. As applications such questions can only be answered when general and linear functions and equations has been taught and learned.

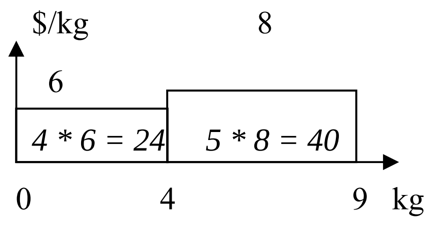
Model 3 says: This question is a re-application of recounting. The mathematical problem is to recount the 14kg in 5s and the 6$ in 2s since the cost is 2$ per 5kg. Thus the mathematical solution is 14kg = (14/5) * 5kg = (14/5) * 2$ = 5.6$, and 6$ = (6/2) * 2$ = (6/2) * 5kg = 15kg.

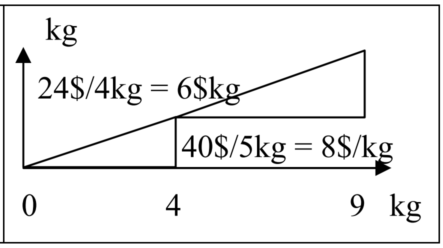
The examples show that many problems in middle school are re-applications of recounting from primary school; instead they are presented as applications of fractions, proportionality and equations, forcing modelling to be postponed until these subjects has been taught and learned, which excludes students unable to learn ungrounded mathematics.

Again, unreflectively applying mathematism may lead to invalid solutions. Whereas using grounded mathematics replaces fractions with the category per-numbers coming from double-counting in two different units, in 1s and 5s: 3 1s = (3/5)*5, or in $ and kg: 2$/5kg = 2/5 $/kg.

Adding numbers with units also occurs when modelling mixture situations, generalizing primary school's integrating stacks to middle school integral and differential calculus. Thus asking 4 kg at 6$kg + 5kg at 8$/kg = 9 kg at ? $/kg can be answered by using a table or a graph, realizing that integration means finding the area under the per-number graph; and vice versa, that the per-number is found as the gradient on the total-graph:

4kg at 6$/kg = 4 * 6 = 24	
5kg at 8$/kg = 5 * 8 = 40	
9kg at ?$/kg = 9 * ? = 64	
	? = 64/9





Mathematical Modelling in High School

High school claims set-based functions as its basis: a quantity growing by a constant number IS an example of a linear function; and a quantity growing by a constant percent IS an example of an exponential function; and both ARE examples of the set-based function concept. A grounded alternative will place formulas as its basis, rooted in calculations as 3+5=8 becoming equations if containing one unknown, and functions if containing two unknowns, and formulas if containing only unknowns.

Thus the real-world problem '200$ + ? days at 5$/day is 300$' leads to two different models: model 1 seeing the question as an application of linear functions; and model 2 seeing the question as a re-application of a formula stating that with constant change, the terminal number T is the initial number b added with the change m x times: T = b + m*x. Inserting T = 300, b = 200 and m = 5 and using the Math Solver on a graphical display calculator, the solution is found as x = 20. This prediction can be tested by graphing the function y = 200 + 5*x, and by observing that tracing x = 20 gives y = 300.

Cumulating a capital C by a yearly deposit p and interest rate r leads to two different models: model 1 seeing the question as an application of a geometric series; and model 2 setting up two accounts, one with the amount p/r from which the yearly interest p/r*r = p is transferred to the other, which after n years contains the cumulated interest p/r * R, where by 1+R = (1+r)^n, as well as the generated capital C. And C = p/r * R gives a beautiful a simple formula: C/p = R/r.

Also two different models come out of the real-world problem 'Out driving, Peter observed the speed to be 6, 18, 11, 12 m/s after 5, 10, 15 and 20 seconds. What was the speed after 6 seconds? When was the speed 15m/s? When did he stop accelerating? When did he begin to accelerate again? What was the total distance traveled from 7 to 12 seconds?

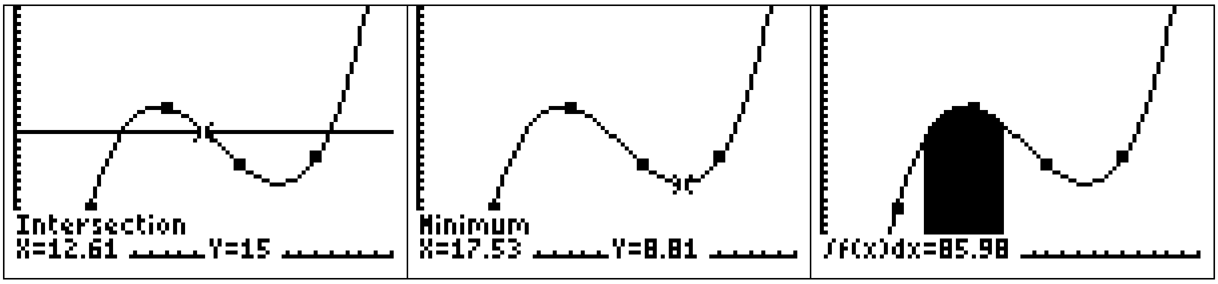
Model 1 says: This question is an application of matrices and differential and integral calculus. The mathematical problem is to set up a function expressing the distance y as a function of the time x, given that the function's graph contains the points (5,6), (10,18), (15,11), and (20,12). Applying matrices to solve 4 equations with 4 unknowns, the mathematical solution is y = 0.036 x^3 – 1.46 x^2 + 18 x – 52. Now the point (6,0) is inserted in the function to find y = 11.22. Inserting the point (x,15) in the function leads to a 3^{rd} degree equation.

To solve this equation we guess a solution in order to factorize the 3rd degree polynomial to $y = 0.036(x - 7.07)(x - 12.61)(x - 20.87)$. To find the turning points we must find the zeros of the derivative $y' = 0.108 x^2 - 2.92 x + 18$, i.e. $x = 9.51$ and $x = 17.53$, as well as the signs of the double-derivative $y'' = 0.216 x - 2.92$ changing sign from minus to plus in $x = 13.52$. Finally the distance traveled from 7 seconds to 12 seconds comes from the integral:

$$\int_7^{12} (0.036 x^3 - 1.46 x^2 + 18 x - 52)\, dx = \left[0.009 x^4 - 0.49 x^3 + 9 x^2 - 52x \right]_7^{12} = 85.98.$$

As applications this must wait till matrices and calculus are taught and learned.

Model 2 says: This question is a re-application of per-numbers. The math problem is to find a per-number formula f(x) from a table of 4 data sets. On a graphical display calculator Lists and CubicRegression do the job. Tracing x = 6 gives y = 11.22. Finding the intersection points with the line y = 15 using Calc Intersection gives x = 7.07, 12.61 and 20.87. Finding the turning points using Calc Minimum and Calc Maximum gives a local maximum at x = 9.51 and y = 18.10, and a local minimum at x = 17.53 and y = 8.81. The total meter-number from 7 to 12 seconds is found by summing up the m/s*s, i.e. by using Calc ∫f(x)*dx, which gives 85.98.



Change Equations

Solving any change-equation dy/dx = f(x,y) is easy when using technology. The change-equation calculates the change dy that added to the initial y-value gives the terminal y-value, becoming the initial y-value in the next period. Thus is dy = r*y, r = ro*(1 – y/M) is the change-equation if a population y grows with a rate r decreasing in a linear way with the population having M as its maximum. A spreadsheet can keep on calculating the formula y + dy -> y.

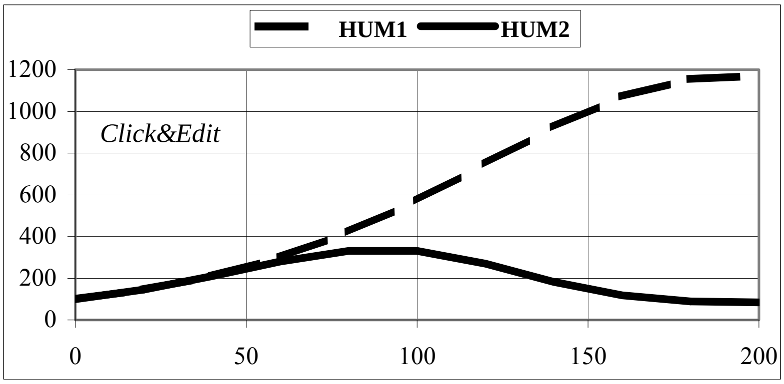
The Grand Narratives of the Quantitative Literature

Literature is narratives about real-world persons, actions or phenomena. Quantitative literature also has its grand narratives. That an infinity of numbers can be added by only one difference if the numbers can be written as change-numbers is a grand narrative: If y-changes dy are recounted in x-changes dx: dy = (dy/dx)*dx = y'*dx, then the infinite sum $\int y'dx = \int dy = \Delta y = y2 - y1$:

Since $x^2 = (x^3/3)'$, $\int x^2\, dx = \int (x^3/3)'\, dx = 7^3/3 - 2^3/3$ if summing from 2 to 7.

In physics, grand narratives can be found among those telling about the effect of forces, e.g. gravity, producing parabola orbits on earth, and circular or ellipse orbits in space. Jumping from a swing is a simple example of a complicated model. Physics' grand narratives enabled the rise of the Enlightenment and the modern democracy replacing religion with science.

Time	HUM1	HUM2
0	100	100
20	145	145
40	210	210
60	301	280
80	421	331
100	572	330
120	747	269
140	925	182
160	1071	118
180	1156	89
200	1171	84



In economics, an example of a grand narrative is Malthus' 'principle of population' comparing the linear growth of food production with the exponential growth of the population; and Keynes' model relating demand and employment creating the modern welfare society. As are the macroeconomic models predicting effects of taxation and reallocation policies. Also limit-to-growth models constitute grand narratives predicting the future population depending on different assumptions as to e.g. food and pollution: leaving out pollution only food will restrict human growth, but including pollution the population level might be different.

A Bourdieu Perspective on Applying Mathematics

Bourdieu says about contemporary society that below the economical welfare society a class society is hidden where education uses 'symbolic violence' to install inequality as to the distribution of knowledge capital also called cultural capital:

> All pedagogic action is, objectively, symbolic violence insofar as it is the imposition of a cultural arbitrary by an arbitrary power (Bourdieu 1970: 5).

Since mathematics and its applications are part of education, we could ask: what is their contribution to the symbolic violence producing the inequalities of the distribution of the knowledge capital, becoming more important than economical capital after the industrial society has changed into a knowledge society?

Bourdieu sees society as divided into different fields all producing economical and cultural capital. However, the share depends upon the individual's habitus, i.e. habits developed as a practitioner in the field. In many social fields its habitus involves applying mathematics. So to be more successful in such fields, this is made part of another field, education, demanding its own habitus to be successful and also having its own 'doxa' or political correctness to support its practice.

As to education in general two different forms of doxa exist. The EU Humboldt Bildung education is organized in lines forcing the students to stay in a fixed class waiting many years before being allowed to take an exam, that cannot be retaken. Outside the EU, the international standard is set by the American enlightenment schools organized in coordinated modules allowing the students to choose their own combinations of modules basing marks on frequent tests that might be retaken. In both cases the habitus brought from the home determines the knowledge capital acquired, making Bildung-schools elitist and enlightenment-schools egalitarian.

Bildung and enlightenment education share the doxa that ‘of course mathematics must be learned before it can be applied’, which turns mathematics upside down by postponing its real-world examples till after their abstraction has been taught.

To this orthodoxy a heterodoxy presents itself: Since mathematics is learned to be applied, the habitus of applying mathematics is best learned in a field where this takes place, and being historically developed as abstractions from examples, mathematics should be learned not before but through its examples, meaning that the wording ‘applying mathematics’ should be renamed to ‘re-applying mathematics’.

As to economical capital the main question is: who should own the machines producing the capital? Likewise, as to knowledge capital the main question is: who should determine the stories that produce knowledge through textbooks and schools? This question goes back to the ancient Greek controversy between the sophists and the philosophers as to the nature of knowledge resulting in two different forms of stories: enlightening stories about the physical world bringing knowledge capital to the many; and patronizing stories about a meta-physical world bringing knowledge capital to the few. Also mathematics education has two stories: an enlightening story about counting and adding many predicting the result by operations in formulas, and a pastoral story about a meta-physical concept set and its network of examples.

So to make its knowledge capital more widespread, mathematics should be rooted in its generating examples; its education should be divided into half-year modules marked using frequent tests; and Bildung should be replaced by enlightenment.

Using Applied Ethnography to Test a Grounded Set-free Curriculum

Working together with Mogens Niss 30 years ago, we discussed ways to include applications and modelling in the Danish mathematics curriculum. Niss stayed at the university, I chose to become an applied ethnographer in the classroom. By replacing set-based functions with formulas at the pre-calculus level, I designed and tested an application and modelling based curriculum that made all students learn everything.

However, Danish education is the only in the world emphasizing oral exams in spite of the fact that talking instead of doing mathematics makes the student’s knowledge capital deteriorate to a degree, where the passing level at a written mathematics exam has been lowered from the international level at 60% correctness to 40% in the Danish upper secondary school, and to 20% at the Danish lower secondary school.

Also Denmark holds on to its strong Central Administration created in its autocratic past and protecting the oral Humboldt Bildung system sorting out the elite for its offices by sending out external examiners to all oral exams. And this Central Administration didn't like a set-free curriculum unable to sort out the elite. Likewise, countless articles to the mathematics teachers' journal on the advantages of a set-free formula-based curriculum were neglected. It took 30 years before the Central Administration finally followed my advice: to improve learning, replace functions with formulas and make the curriculum application and modelling based. So now I try to persuade the Central Administration to follow the international standard by replacing oral exams with written, a project that probably will take another 30 years.

Factors Preventing Success of an Application & Modelling Based Curriculum

Four preventing factors have been identified in this paper:

1. When applying mathematism instead of mathematics, answers proven correct in a library may not hold in a laboratory. This makes mathematics totally self-referential and impossible to use as a prediction of real-world situations. Adding numbers without units in primary school and adding fractions without units in middle school are examples of mathematism.

Different examples of 'centrism' claiming to have monopoly in certain application situations prevent or postpone many fruitful modelling situations, and exclude many potential learners thus producing an unequal distribution of knowledge capital.

2. In primary school, '10=ten'-centrism conceals that '10 IS ten' is a pastoral choice hiding its alternatives, e.g. 10=five in the case of counting in 5-bundles. This prevents a modelling of changing units through recounting as e.g. 4 5s = ? 7s, predicted by the recount-formula T = (T/b)*b and the restack-formula T = (T-b)+b.

3. In middle school, fraction-centrism presenting fractions without units is a pastoral choice hiding its alternative, per-numbers. This forces proportionality to become an application of fractions, and prevents it from being a re-application of recounting. Being defined as sets, fractions become an example of 'metamatism' merging metamatics with mathematism.

4. In high school, set-centrism demands all concepts be defined as examples of the concept set. Solving equations by the set-based neutralizing method prevents equations from being solved by reversed calculation. Set-based functions and calculus prevent change situations from being modeled by re-applying per-numbers and using a graphical display calculator.

Factors Entailing Success of an Application & Modelling Based Curriculum

Two entailing factors have been identified in this paper.

First, changing or deconstructing ‘applying mathematics’ to ‘re-applying mathematics’ will signal that historically mathematics was created as an application modelling real-world problems, i.e. as a language describing and predicting the natural fact many. This distinction is useful when answering the question ‘does ‘applying mathematics’ mean applying pastoral metamatics and mathematism, or re-applying grounded mathematics?’

Second, banning from school set-based metamatics and mathematism will bring back a new Enlightenment period with grounded mathematics. Thus in primary school ‘10=ten’-centrism will be prevented by practicing counting in 5-bundels, 7-bundles etc. before finally choosing ten as the standard bundle-size. In middle school fractions will be presented as per-numbers, and will always carry units; and equations will be introduced as reversed calculations. In high school equations and functions will be presented as formulas with one and two variables treated on a graphical display calculator using regression to produce formulas that describe per-numbers to be integrated to totals, or totals to be differentiated to per-numbers.

Conclusion

So, to answer the initial question: What is called ‘applying mathematics’ might turn out to be instead ‘applying metamatism’, i.e. applying a mixture of metamatics presenting concepts as examples of abstractions instead of as abstractions from examples, and mathematism true in a library but not in a laboratory and therefore unable to predict real-world situations; and hiding its natural alternative ‘re-applying mathematics’ rooted in real-world examples.

Applying metamatism forces three cases of centrism upon mathematics as pastoral choices hiding their alternatives. The use of ‘10=ten’-centrism hides that also other numbers than ten can be used as bundle-size when counting in bundles. Fraction-centrism hides that proportionality and many other applications of fractions can also be presented as re-applying recounting. And set-centrism hides that modelling change can take place without the use of set-based concepts as functions and limits. Finally the wording ‘apply mathematics’ installs as self-evident that ‘of-course mathematics must be taught and learned before it can be applied’, thus hiding that historically mathematics is rooted in the real world as a model.

A grounded approach will respect the historical nature of mathematics as a natural science rooted in the physical fact many. Here mathematics is created through its real-world roots and then re-applied to similar situations. To avoid ‘10=ten’-centrism in primary school, before introducing 3.order counting installing ten as the only bundle-size, 2.order counting is used to emphasize that mathematics is a language for predicting real-world numbers, and to allow the learning of 1digit mathematics. To avoid fraction-centrism in middle school, proportionality is based upon recounting and per-numbers, and fractions always carry units when added. To avoid set-centrism in high school, the graphical display calculator is used when modelling change, both the linear, exponential and polynomial models and more complicated models.

Applying pastoral metamatism means excluding the grand narratives and bringing the minor narratives to a halt until the metamatism applied is taught and learned. Re-applying grounded mathematics invites the grand narratives of the quantitative literature into the mathematics curriculum, and allows the minor narratives to be introduced at an early stage. Does modelling want to be a docile lackey of pastoral metamatism presented as a meta-physical science, or to respect the historical fact that mathematics is a grounded natural science investigating the natural fact many? Does modelling want a mathematical knowledge capital only for privileged elite or for all?

References

Biehler, R., Scholz, R. W., Strässer, R. & Winkelmann, B. (1994). *Didactics of Mathematics as a Scientific Discipline.* Dordrecht: Kluwer Academic Press.

Bourdieu, P. (1977). *Reproduction in Education, Society and Culture*. London: Sage.

Denzin. N. & Lincoln, Y. (2000). *Handbook of Qualitative Research 2nd ed.* London: Sage.

Dreyfus, H. L. & Rabinow, P. (1982). 2. ed. *Michel Foucault, beyond structuralism and hermeneutics.* Chicago: University of Chicago Press.

Euler, L. (1988). *Introduction to Analysis of the Infinite*. New York: Springer Verlag.

Glaser, B. G. & Strauss, A. L. (1967). *The Discovery of Grounded Theory.* NY: Aldine de Gruyter.

Jensen, J. H, Niss, M. & Wedege, T. *(1998): Justification and Enrolment Problems in Education Involving Mathematics or Physics.* Roskilde: Roskilde University Press.

Kline, M. (1972). *Mathematical Thoughts from Ancient to Modern Times*. NY: Oxford Univ. Press.

Lyotard, J. (1984). *The postmodern Condition.* Manchester: Manchester University Press.

Piaget, J. (1970). *Science of Education of the Psychology of the Child*. New York: Viking.

Royle, N. (2003). *Jaques Derrida.* London: Routledge.

Russell, B. (1945). *A History of Western Philosophy.* New York: A Touchstone Book.

Tarp, A. (2004). *Pastoral Power in Mathematics Education.* Paper accepted for presentation at the Topic Study Group 25, ICME, 2004. http://mathecademy.net/Papers.htm.

Zybartas, S. & Tarp, A. (2005). *One Digit Mathematics.* Pedagogika (78/2005). Vilnius, Lithuania.

A 'NEW' TYPE OF DIAGRAM TO SUPPORT FUNCTIONAL MODELING – PROGRAPH DIAGRAMS

Hans-Stefan Siller

University of Salzburg, Department of Mathematics and Informatics Education

This paper shows the basic concept of Functional Modeling in Mathematics Education which gets more and more important. Hence of this significance it is necessary to think about adequate graphical methods to explain the fundamental idea of a function and its influence to values and other functions. PROGRAPH diagrams are a very good possibility to explain these combinations effectively. The basic idea of a function, composition of functions or returning values of a function can be shown impressively. It is also possible to find easy nested diagrams for recursive functions. So the mathematical sight to topics which can be handled through functions gets opened and the possibilities for working with functions get more voluminous.

1. Functional Modeling

One of the basic ideas of functional modeling is the idea of functional thinking which has been published by Hans-Joachim Vollrath (1989) but can also be found as projection a little bit earlier in the dissertation of Karl Josef Fuchs (1988). If functional modeling is done with students it will be necessary that they will be able to practice functional thinking. For this reason anyone could explain "Functional Thinking" as the pattern for mental activity in the process of "Functional Modeling".

Following the definition of functional thinking given by Vollrath we have to focus on a:

- Methodological aspect,
    - Dependency of parameters,
    - Idea of systematic – dynamical variation,
- Phenomenological aspect,
- Quantitive aspect,
- Input-Output aspect.

And as K. Fuchs writes in his paper ‘Functional Thinking – a fundamental idea in teaching Computer Algebra Systems’ it is necessary to add one more aspect for a better understanding (2007).

- The Algorithmical Aspect

Another fundamental idea which has to be followed is the idea of modeling. There exists a lot of literature for this idea like my dissertation or my paper in 2007. So I want to characterize it shortly. Models in mathematics can be seen as additives for the array, appliance and advancement of theories. Models provide a basis for demonstration and description of contents.

Through the combination of these two fundamental ideas it is possible to create the idea of functional modeling which is a deeply and pivotal idea in Mathematics and Computer Science. Thereby it is obvious to work with CAS where the order of execution of functions and composed functions can be seen as a program. Models should be facile, clear and intuitional on one hand meaningful on the other hand. Hence we have to split the construction of a functional model in two phases:

- **Black-Box-Phase**
  We look at the function as a whole. The formal description is done through a consequent use of data-

flow-diagrams, the so called PROGRAPH diagrams but the function stands for a “machine” where values are shoveled in from one side and tumbling out from the other side. The process which is going on inside this machine is of no interest in this phase. This can be found in the Input-Output aspect of Vollrath.

- **White-Box-Phase**
  It is profitable for exploring the inner structure of a model. Through working and acting automatically the interactions and references of the components of a model can be explored.

These both principles have been described by B. Buchberger (1992) in conjunction with CAS for the first time.

2. What are PROGRAPH diagrams?

For the process of implementation it is absolutely necessary to use diagrams as mentioned above. Every function can be described in a data-flow diagram like:

- What is the aim of the function?
- How is it implemented?

A very suitable way to see these aspects evidently is PROGRAPH. Arguments for using this type of diagrams are:

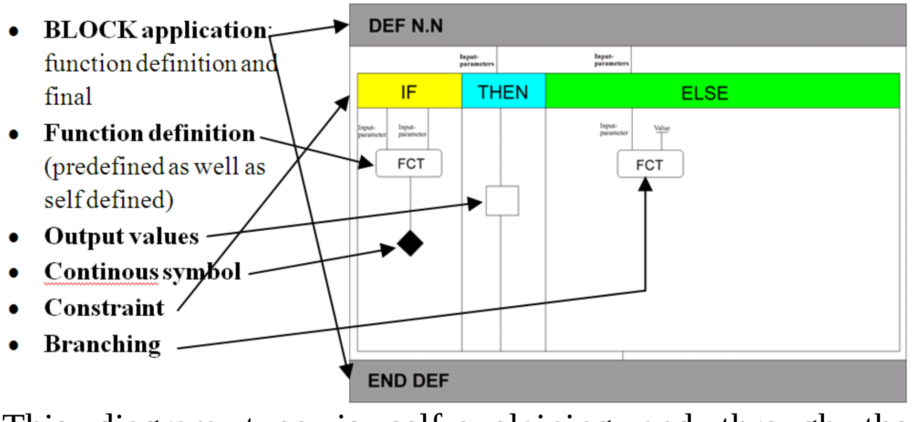
- The input parameters process continuously through the application of functions. Matwin (1985) calls it ‘principle of single assignment’. Fuchs (2007) writes in this case: ‘You can haunt a once determined value staying unchanged through the rest of the functional system.’
- Each function returns only one value. You may say it meets the ‘definition of a function in Mathematics’.

- Recursion can be done easily because - as Fuchs writes - ’You can experience the ‘Picture within a Picture’ – structure as a visual metaphor for recursion easily`.

Another point I want to add is that you can recognize the function declaration:

Definiendum	Definition Mark	Definiens

All the PROGRAPH diagrams have the following style. In the given picture you can see a special PROGRAPH diagram which I have designed to show the prototypical elements of this type of diagram:



This diagram type is self-explaining and through the following examples the structure will become clear at all.

With the help of such PROGRAPH-diagrams it is possible to simplify the step of “Mathematization”. So students get an additive for this step. They have to find the variables and the functional parts in the “Real Model” so that they can formulate and prepare a “Mathematical Model”. The structuring through abstraction with the help of flow-chart-diagrams so that proper functions can be found is a very helpful way for students. I want to show this fact first with the help of two examples “Flipping a Coin” and “Expectancy and Variance”. Then I want to show some results of my two courses where students were confronted with such diagrams.

3. Examples for working with PROGRAPH and CAS

The treatment of examples of probability is often a stepchild in computer aided mathematics education. When they are encouraged through the usage of a computer the programs for experimenting are already prepared and the pupils don’t have to think about the mathematical backgrounds. They use such programs as Black-Boxes and the White-Box sight on topics in probability gets lost completely. For a better understanding of activities in probability it’s necessary to discuss certain topics like Laplace-contribution or coin-flipping intensively.

a. Flipping a Coin

If the learners are well experienced in these fundamental themes they can solve the following example.

> A small field of a board has the shape of a circle with radius R. A coin is thrown on this field. Think about a program which is able to determine whether the coin is located within or on the boarder (outside) of the inner circle field.

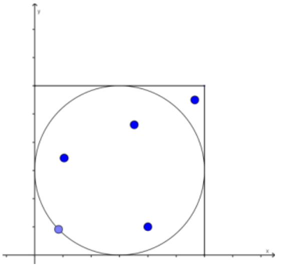
The first two steps of the process of modeling and the part of functional thinking in this process should be shown with the help of this example. Therefore I take the graphic from above and show the steps of the process explicitly:

1st Step: Reality/Environment:

In the first step the students have to understand the situation given in the example. It is obvious to recognize that there is no given specific description for a situation in reality. But this is not very problematic because through an easy modification the example could look like the following:

A small dart field has the shape of a circle with radius R. An (dart) arrow is thrown on this field. Think about an implementation that helps you to analyze whether the coin is located within or on the boarder (outside) of the inner circle field.

For realizing the solution of this problem the students have to think about the opportunities of such a coin. Maybe they will draw a picture like the following – see the picture beside.



After thinking about this fact they can go to the second step with the help of simplifying the facts. In case of the dart field it would mean that they do not think about "special inner fields" of the dart field.

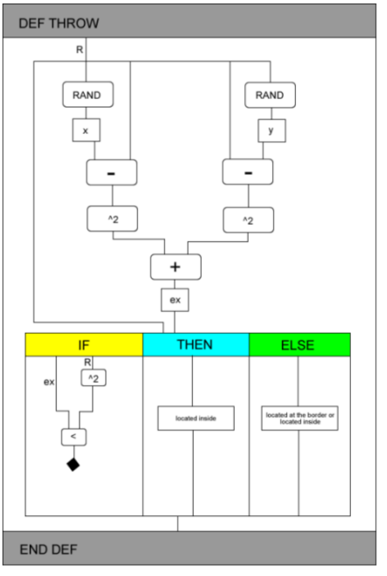
2^{nd} Step: Real Model

The realization of such an implementation is in the first instance a brainteaser. So the students have to think about the location of the coin at the field and its description through coordinates. Depending to the input-parameter it is able to modify the example in the way to find the coin outside the field. For the first consideration this arrangement is too difficult. So the learners should think about the case whether the coin can be located inside or at the border of the board.

With the help of PROGRAPH diagrams the mathematical model is created in the next steps respectively the students switch with the help of these diagrams from the "Real Model" to the "Mathematical Model". This part is shown in the third step.

3rd Step: Mathematical Model

Hence a circle board was chosen for this example it is possible to get the equation for the circle board as $(x - R)^2 + (y - R)^2 = R^2$ if the circle is positioned like shown in the graphic.

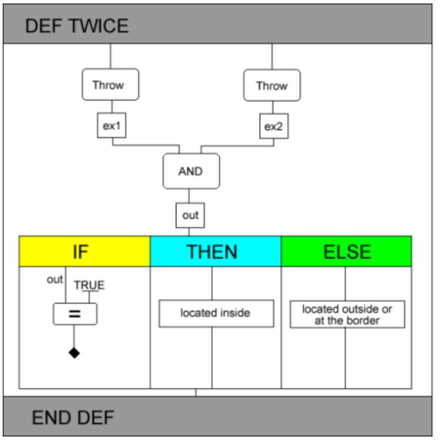


The left part of this equation is substituted with the expression ‘ex’. For the output it is obvious that we get Boolean values, because if the coin is located in the board, TRUE will be the common output otherwise FALSE. Such an example could be described very easy through a PROGRAPH-diagram in the first step. This leads to an understanding White-Box-sight on such procedures. The implementation is specific to the used program or CAS. So we have another argument to draw a PROGRAPH-diagram, because of the general sight to such functional topics. The diagram could look like the one drawn on the right margin.

As said before the output for this module is a Boolean data type {TRUE, FALSE}. After realizing this part it is possible to expand this example because it is not necessary to have a look at only one coin. An expansion of this example would be the view on two coins which are thrown to the circle board. The possibilities we are able to scan for are:

- 2 coins located inside the field,
- 1 coin located inside, 1 coin located outside the field,
- 2 coins located outside the field.

If we are interested in the coins located inside the circle board the case differentiation will get easier because if TRUE is unreturned in at least one of the cases it will be sure that not less than one coin is located outside the circle field. The PROGRAPH diagram is a following diagram of the one yet given and looks like the given.



With the help of these diagrams the implementation is possible in every CAS, because the structure of these examples does not change.

So this type of diagram is also well adapted for the third step, because the functional structure and the functions are becoming obvious for students. In the 4th step they only have to interpret their solutions.

Another very important point which can be seen evidently in this example is the fact of modularization. It means dividing a problem into little parts which can be handled separately. Here I have chosen the other way. We have created a second part out of a main part. These two composed parts can be fit together and if you look at these parts as a joint problem it will be easy to see the main problem of throwing two coins. This modular construction system could be implemented in lot problems given in

Mathematics or Informatics. Especially difficult problems in mathematical education can be handled more easily. Functions also can be handled easier if an informatical view is allowed to them in mathematical education. It is obvious to see that the functions in this view have an explicit module character.

b. Expectancy and Variance

Another topic of probability is the topic of distributions. There certain parameters have to be calculated very often because they are of immanent importance for the interpretation of these distributions. Such parameter values are expectancy and variance.

Expectancy and variance of well known distribution (binomial distribution, standardized normal distribution) can be calculated very easy through the input of the known formulas. But if an empirical distribution is given, it will be necessary to assess the values of expectancy and variance with the help of the given definitions for $p_1 = P(X=x_1)$, $p_2 = P(X=x_2)$, …, $p_n = P(X=x_n)$ and $X = \{x_1, x_2 \dots x_n\}$:

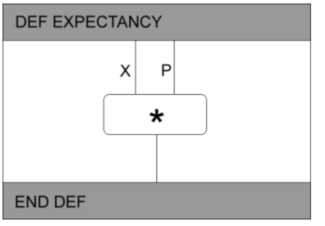
$$\mu = E(X) = x_1 p_1 + x_2 p_2 + \cdots + x_n p_n,$$

$$V(X) = (x_1 - \mu)^2 p_1 + (x_2 - \mu)^2 p_2 + \cdots + (x_n - \mu)^2 p_n$$

When E(X) is specified it is obvious that this is an inner product of the two lists $\{x_1, x_2 \dots x_n\}$ and $\{p_1, p_2 \dots p_n\}$. If we understand the two given lists as two vectors and define it like vectors with a CAS it is to calculate the solution. For the variance it is necessary to formulate a Sum-function like the following:

$$V(X) = \mathrm{Sum}((x_k - \mu)^2 p_k, k, 1, \mathrm{Dimension}(X))$$

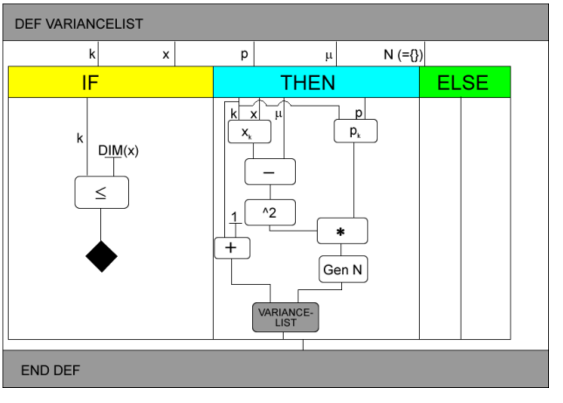
If this fundament is created it is possible to have a look at the PROGRAPH diagrams. First we look at the expectancy.



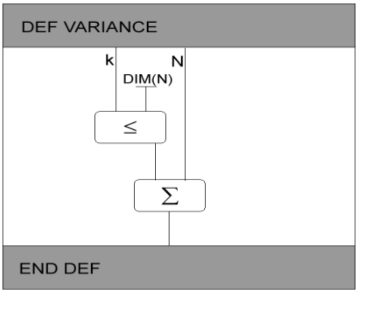
The PROGRAPH diagram for the implementation of the variance is a little bit more complicated. It is necessary to split it into two parts. The first part creates a list of n elements

$$\{(x_1-\mu)^2 p_1, (x_2-\mu)^2 p_2, \dots, (x_n-\mu)^2 p_n\}$$

out of a firstly empty set N. This list is filled consecutively through the function **Gen N**. The filled list can be called N again. The function $\mathbf{x_k}$ and $\mathbf{p_k}$ extracts the element in position k of the two input-lists x = $\{x_1, x_2, \dots, x_n\}$ and p = $\{p_1, p_2, \dots, p_n\}$. After composing this list in the second part it is possible to sum up the elements of list N. The PROGRAPH diagram for the first part could look like the following:



The second part of adding the elements is the easy part of calculating the variance. It is represented by the shown PROGRAPH-diagram besides.

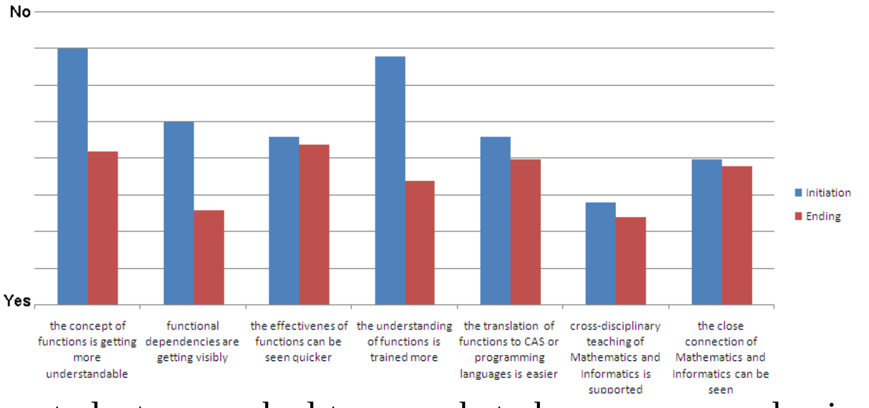


After realizing these diagrams it is very easy to implement these functions in any CAS used in education. Only the specific notation in the different programs has to be observed. But if the learners are common with one system it won’t be a problem to implement the functions.

c. Students opinion

The work with PROGRAPH diagrams in the course “Functions, Calculus and Models” was completely new for students who attended it. Their first impressions about discussing functions when using computers with the help of PROGRAPH diagrams were very skeptical. They could not imagine the powerfulness of such diagrams although they had been confronted with the use of Nassi-Shneiderman diagrams in earlier courses. Before they had to work with such diagrams on their own, they got an introduction. Simple functions had to be analyzed with the help of PROGRAPH diagrams. After a while they had to work on their own. Before introducing PROGRAPH diagrams the students were asked about their opinion on the use of diagrams for introducing functions. After the students have finished their work on PROGRAPH diagrams they were asked about their opinion again. 19 students were present in both interviews comparably. Every question (shown in the diagram) had four possible answers: yes, rather yes, rather no, no. In certain cases the beliefs using diagrams for analyzing functions changed after attending this course.

The interviews show a possible change of beliefs while working on topics with computers in class. Some of the positive results concerning computer use can be seen in the figure, whereas a small bar is closer to the answer “Yes” a bigger bar closer to “No”.



The students are asked to say what changes occurred using PROGRAPH diagrams in mathematics classes.

This first result shows that it could be interesting to have a closer look for using PROGRAPH diagrams when modeling with functions is discussed. For this research it is necessary to find more examples with relevance to Functional Modeling or modeling with functions.

Recapitulatory the use of diagrams in mathematics education can support and create understanding for functions, in order to improve motivation. The role of diagrams in education while discussing a modeling cycle has to be strengthened and examples in education have to be adapted and even created. To implement these points more research in this field needs to be done.

4. Epilogue

Through the division of the modeling process into a Black-Box- and a White-Box-Phase it is possible to get a clear, intuitive and on the first sight understanding description of this process which is meaningful and detailed.

In the first part, the Black-Box-Phase, the question ‘Which information is for which component?’ and ‘Which information is necessary for another part of the process?’

can be answered. The inner structure of the used components is not answered yet. This is an exercise for the second part, the White-Box-Phase, where the functions are implemented as a source-code. The special use of PROGRAPH diagrams is very efficient in the first part because the structure can be seen on the first sight. The functional character of the involved parts appears because it is easy to see that each function is fed through a unique assignment of input parameters. The advantages for using PROGRAPH diagrams are:

- Easy to learn because there are less basic concepts. Especially there are no allocations, no loops or no skips.
- Higher efficiency because the source code which should be implemented is very short compared to an imperative program code.
- Higher trustiness because considerations or proofs of the correctness of the source code is easier because of the mathematical background.

Like Nassi & Shneiderman (1973) diagrams for imperative modeling PROGRAPH diagrams return a very bright picture of the processes which should be described. The structure of the drawn diagrams can be implemented 1:1 in a CAS. It doesn’t depend on the system.

Another very important point which has to be articulated is that students or pupils are highly motivated and challenged by Functional Modeling also problems can be very difficult in description, shown in 3.b. in implementation.

Through the combining of diagram construction and Computer Algebra Systems the teaching with the help of computers or graphical calculators gets more exciting again. For students it is more efficient because mathematical facts will be understood through informatical handling. The interdisciplinary aspect of Mathematics and Informatics gets starched and new forms of education are possible.

Another important point of Functional Modeling is delivered by the implementation of Standards in mathematical education which is done right now in Austria and has been done in other countries like Germany.

If we have a closer look at those we can find a lot of interesting points at the website of the NCTM (www.nctm.org) it is obvious to see the listed points which K. Fuchs (2007) has mentioned in his paper:

- Students should learn an ambitious common foundation of mathematical ideas and applications,
- Students need to understand the mathematical concepts of the function,
- Students should be adept visualizing, describing and analyzing situations in mathematical terms.

Through the attendance of Functional Modeling I am sure that these Standards can be achieved and more understanding for mathematical circumstances can be created.

References

Buchberger, B. (1992). Teaching Math by Math software: The White Box / Black Box Principle, Paper of the RISC-Institute of the Johannes Kepler University Linz.

Fuchs, K.J. (1988). Projektion, EDV – Nutzung – Zwei fundamentale Ideen und deren Bedeutung für den Geometrisch – Zeichenunterricht (Engl.: Projection, E(lectronic)D(ata)P(rocessing) – Use – Two

Fundamental Ideas and their Weight in Geometric Drawing). Disseration, University of Salzburg.

Fuchs, K.J. (2007). Functional Thinking - A fundamental idea in teaching Computer Algebra Systems, ICT-Conference, Boston.

Fuchs, K.J.; Siller, H.-St., Vásárhelyi, E. (2008). Basics in Functional Modeling, Casio Europe GmbH, Budapest.

Matwin, S. & Pietrzykowski, T. (1985). The Programming Language PROGRAPH: A Preliminary Report. In: Computer Languages, 10:2, pp. 91 - 125.

Nassi, I. & Shneidermann B. (1973). "Flowchart techniques for structured programming", ACM SIGPLAN Notices, vol. 8, pp. 12-26.

Nickerson, J. V. (1994). Visual Programming, Chapter 2: A Survey of Visual Programming. Ph.D. dissertation, New York University.

Schwill, A.; Schubert, S. (2004). Didaktik der Informatik, Spektrum Akademischer Verlag, München.

Siller, H.-St. (2006). Modellbilden – eine zentrale Leitidee der Mathematik (Engl.: Modeling – A Pivotal idea in Mathematics). Dissertation, University of Salzburg.

Siller, H.-St. (2007). Modeling - a pivotal idea of interdisciplinary teaching in mathematics and computer science, CADGME-Conference, Pecs.

Vollrath, H. J. (1989). Funktionales Denken (Engl.: Functional Thinking). In: Journal für Mathematikdidaktik, pp. 3 – 37.

MATHEMATICAL MODELS IN THE CONTEXT OF SCIENCES

Patricia Camarena Gallardo
Instituto Politécnico Nacional, México

Abstract. *This paper shows a research which determines a classification and characterization of mathematical models in the engineering careers, as well as, a definition of "a mathematical model" and "mathematical modeling" in an engineering context.*

Introduction. This research influences university studies where mathematics are not a goal themselves. Due to the mathematical richness that prevails in engineering, the project comes particularly to these studying areas. Mathematization of the phenomena and problems that come up in the working field of the future engineer is a cognitive conflict point since he studied mathematics and engineering separately, so that when he uses both knowledge areas, these cognitive areas are separated and he has to integrate them in order to mathematize the problem to be resolved (Camarena, 1984). The intention is that the students build mathematics for their future life. On the other side, the mathematical model is one of the themes that appear in the hidden curriculum of university careers, since it is supposed that the graduate must know how to model with mathematics, but in many study plans and programs the term "mathematical modeling" is not mentioned at all. Inside the goals of study programs of other curricula, it is said that the student must know how to model problems from other areas of knowledge, and this term is included in the subject programs in very few curricula. But, in no case it is said how to include mathematical modeling in mathematics courses, neither how to make students model situations of other areas or problems from the daily life (Camarena, 1987; Camarena, 1993). In fact, there is no engineering subject that comes to work mathematical models. Besides, the mathematics teachers feel that this point is concerned by the teachers of engineering courses, while the latest presuppose that the mathematics teachers are the ones who have to teach the students how to model engineering phenomena or problems (Camarena, 1990).

The research problem. I want to classify and characterize the mathematical models in engineering. To deal with the research problem, we have the following research questions (Camarena, 2000): What is a mathematical model?, What is mathematical modeling?, How are the mathematical models characterized and classified?

The objective. The objective of the research is to determine how to classify and characterize mathematical models in the engineering careers, as well as, how to define the "mathematical model" and the "mathematical modeling".

The theoretic framework. The theory in which this research is based is *Mathematics in the Sciences Context* (Camarena, 1984; Camarena, 1993; Camarena, 1995; Camarena, 1999; Camarena, 2002). It is necessary to explain briefly what *Mathematics in the Sciences Context* is. This theory takes mathematics learning and teaching in engineering careers as a system which includes the student, the teacher and the mathematical knowledge, considering the interactions among the student, the teacher and the mathematics knowledge, all included in the learning environment where there are social, economical, political and human relations aspects. This systemic look makes five phases of the *Mathematics in the Sciences Context* theory, as we will see later. *Mathematics in the Sciences Context* is based in three paradigms: Mathematics is a supporting tool and an educational subject. Mathematics has a specific function in each educational level. Knowledge is born integrated.

The educational philosophic assumption of this theory is that the student is trained to transfer mathematics knowledge to the areas which require it, so that they develop competences for their working and professional life.

The five phases of the theory are: Curricular, developed since 1984. Didactic, started since 1987. Epistemological, tackled in 1988. Teachers Training, defined in 1990. Cognitive, studied since 1992.

This research presents incidence in the didactic phase which has a didactic strategy which is called *Mathematics in Context*, through which contextualized events are worked, that is, problems and projects in the context of other knowledge areas of the student, in the future professional and working activities and, in the daily life.

Mathematics in Context considers nine steps (Camarena, 1984; Camarena, 1987): 1. Analyze the text books of other subjects studied by the student. 2. Establish the event of the context. 3. Identify the variables and constants of the event. 4. Include the mathematical topics and concepts necessary for development of the mathematical model and its solution. 5. Determine the mathematical model. 6. Give the mathematical solution of the event. 7. Determine the solution required by the event in the context disciplines. 8. Interpret the solution in the event terms and the context disciplines. 9. Present the descontextualized mathematics in the classroom, so that the student knows that it is applied in other knowledge fields and he develops the skills given by the formal mathematics.

When *Mathematics in Context* is used, the teacher works with groups of three students in the classroom, taking into account the Vygotsky (1978) socialization knowledge. To undertake the contextualized events, Polya (1976) heuristics are considered, as well as, metacognitive elements, thinking abilities and beliefs of the students (De Bono, 1997; Santos, 1997).

Through *Mathematics in Context*, the teaching traditional paradigm is changed. Now we are speaking about teaching with integrated knowledge, linking mathematics concepts with the other subjects studied and presenting them at the rhythm and times required by the student (Camarena, 1987; Camarena 1993). With *Mathematics in Context* the student builds integrated not fragmented knowledge, meaningful learning and knowledge that lasts, which is not volatile.

Working methodology. Since, it is desired to characterize and classify mathematical models used in engineering, the methodology used will be the engineering text analysis, such as the analysis of some engineering research projects and engineering text books. It is necessary to work with some particular engineering areas, so, this text analysis comes to electronic engineering and its adjacent fields particularly. As known, the text analysis is a methodology which works to detect certain elements related with the teaching and learning of sciences (Camarena, 1984). It depends on what is persecute, to look in the correct way those texts. The text analysis will be done in an implicit and explicit way. Thus, for engineering mathematical models, we mainly look for: Establishing events to be tackled. How the established events are represented mathematically. What are the concepts of engineering subjects which are described mathematically?

The text book sample. For the text book analysis it was considered the classification established by the "Asociación Nacional de Universidades e Instituciones de Educación Superior en México" about the engineering careers subjects. This classification defines 5 subject blocks: the basic sciences, the engineering basic sciences, the engineering specialization sciences, the social and humanistic sciences and, the economical and administrative sciences. It is clear that the first three blocks are the important ones for the present research. Inside the basic sciences are physics and chemistry as engineering foundations, while mathematics are supporting tools for them, without forgetting the thinking skills development that mathematics offer to the future engineer (Camarena, 1984; Camarena, 1990; Camarena, 1995). The electric circuits, electromagnetism, computation, basic electronics and basic communication subjects are the engineering basic sciences. Communications, electronics, control, acoustic, robotics, telephony, and computation are the engineering specialization sciences. This classification, for electronic engineering and its adjacent fields, is called engineering cognitive stages; as you can see in table 1. The first column includes these cognitive stages and their subjects are in the corresponding second column.

Mathematical Models Characterization.

To start, mathematics in engineering is a language, since almost everything said in engineering can be represented through mathematical symbols (Camarena, 1984; Camarena, 1987; Camarena, 1993). Even more, representing it through mathematical terminology and since mathematics are used in engineering, it helps the engineering to have a scientific character from one side, and from the other, it facilitates its communication with the engineering scientific community (Camarena, 1984; Camarena, 1987; Camarena, 1993).

Engineering Cognitive Stages	***Knowledge Areas***
Basic Sciences	Physic, Chemistry
Engineering Basic Sciences	Electric circuits, Electromagnetic, Computation, Basic Electronic, Basic Communication
Engineering Specialization Sciences	Electronic, Communications, Control, Acoustic, Robotic, Telephony, Computation

TABLE 1. ENGINEERING COGNITIVE STAGES

Inside the engineering knowledge, there are engineering problems and projects, there are engineering objects that are represented mathematically for a better use or reference, and there are also situations that can be described through the mathematical symbols. These cases will permit to characterize the mathematical models. It is necessary to mention that a project is a complex problem which can be solved by parts, where each part is a problem by its self, so, we included just problems in the next analysis.

Examples of each case are presented bellow, problem, object and situation.

a) Problems. We want to know how a condenser (capacitor) is charched, which capacity is C. This is connected in series with a resistor with resistance R to battery terminals which provide a constant tension V, this layout can be represented by the following linear differential equation: $Rq'(t)+(1/c)q(t)=V$. That is to mention that under the term problem are included the phenomena presented in engineering as the charge of a condenser, the free fall of a body, the movement of a pendulum, etc.

b) Objects. Consider an electrical signal of the sinusoidal alternating type, the signal is the engineering object which is represented through the function: $f(t) = A \sin (t+\propto)$.

c) Situations. The charge condenser $q=q(t)$ is totally discharged at the beginning of the problem. This situation may be represented mathematically, taking into account that at the beginning of the problem $t=0$ and the charge is a time function, as: $q(0)=0$.

The mathematical model concept. From the three mentioned cases, the ones that characterize the models are the objects and the problems or projects, so the definition is: *A mathematical model is a mathematical relation that describes engineering objects or problems.*

The mathematical relations may be from an equation, equations system, to a distribution probability, etc. (Camarena, 2000).

Mathematical models classification.

In the same way that any object can be classified in different ways, it has been detected, through the analysis that there are at least two classifications for engineering mathematical models. The first is structured according to the use of the given model by engineering, while the second classification is done according to the knowledge blocks that the student has to study.

I. Engineering objects.

When the mathematical models describe *engineering objects*, the engineer sometimes has to do mathematical operations and other times he does not have to do mathematical operations, so these originate dynamic or static type models (Camarena, 2000).

a) The ***dynamic models*** are mathematical relations that due to the engineering requirements need mathematical modifications constantly, thus mathematical operations are done with them. An example of a dynamic model is an electrical signal of the sinusoidal alternating type, which is modeled mathematically by a real function of a real variable: f(t)=Asin(at). If its amplitude wave and frequency are changed, the new function is g(t)=Bsin(bt), where B=kA and b=ca, then, the original function was modified by multiplying it by a constant "k" and its composition with a linear function: h(t)=ct. So, (k)[f(t)] = kA sin at = B sin at and $(f\circ h)(t) = f[h(t)] = A \sin a[h(t)] = A \sin act = A \sin bt$. Then, $(k)[f\circ h](t) = (k)f[h(t)] = kA \sin a[h(t)] = kA \sin act = B \sin bt = g(t)$

b) The ***static models*** are mathematical relations which describe an engineering object as if it was a "nickname", that is, nothing else is done mathematically. An example of a dynamic model is the impulse function in electronic engineering, which is modeled mathematically by the Dirac

delta. $\delta(t) = \begin{cases} 0, t \neq 0 \\ \infty, t = 0 \end{cases}$ and $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

The engineers only use this definition and the ones of displacement

$$\delta(t-a) = \begin{cases} 0, t \neq a \\ \infty, t = a \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t-a)\, dt = 1$$

and they do not do any mathematical operation.

As it can be observed from static and dynamic models classification, it is according to the use given in engineering, because it is obvious that a given model could be dynamic in any engineering specialty, while in another context it could be static. So, from this point, it is important to know which type of engineering the teacher is working on.

II. Engineering problems.

When mathematical models describe *engineering problems*, the difficulty of each problem is different. It is not the same if we have a problem that includes a data collection, a problem which includes single elements, a problem that has many elements where each one is a combination of single elements (which are called complex elements), a problem which includes many complex elements, a problem that includes many single and complex elements and combinations of these, etc. These characteristics help us to do the next classification (Camarena, 2000).

a) If we are in a laboratory and we want to know what is the relation among the experimental data of resistance (R), voltage (V) and current (I), we establish a mathematical relation which is the Ohm law, V=RI. This is a problem which includes single elements, which are: voltage, current and resistance, because they do not need another element to be determined. The Ohm law is a complex element because needs single elements. Another examples of these kind of problems are the engineering phenomena as the condenser charge, the free fall of a body, a pendulum movement, etc. In general, these are mathematical relations that originate laws or theorems of the physic and chemistry, which are a foundation of electronic engineering and its adjacent fields and which are basic sciences, as we saw en table 1. We classified these as ***first generation models***, they are the most simple models.

b) If we have a problem which includes single element combinations, that is to say, first generation models, and we have to relate them to build a new mathematical relation which modeled the problem, we called it ***second generation model.*** It is showing an example from a text book about it (Huelsman, 1988).

> *Como ejemplo de un circuito RLC de segundo orden, considérese el circuito de una malla que se muestra en la figura 6-1.1. Aplicando la LVK (ley de voltaje de Kirchhoff) a este circuito, podemos escribir* $Li'(t) + Ri(t) + (1/c)\int_{-\infty}^{t} i(\tau)d\tau = 0$ *Si diferenciamos esta ecuación y dividimos todos los términos entre L, de manera que el coeficiente del término de la*

derivada de mayor orden se haga igual a la unidad, obtenemos $\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i(t)=0$

As we said, the text book analysis can be done explicitly and implicitly. So, in implicit way we can see several first generation models, as the Ohm law, Kirchhoff laws, the current in a condenser and a bobbin, and condenser voltage: $v(t)=Ri(t)$, $v_L(t)+v_R(t)+v_C(t)=0$, $i_L(t)+i_R(t)+i_C(t)=i$, $i_c(t)=q'(t)$, $Li_L'(t)=v_L(t)$, and $v_c(t)=q(t)/c$.

These generate the integrated-differential equation, where each element added is a first generation model, so the equation is second generation model. These kind of problems are studied in the electric circuits subjects which are the engineering basic sciences, as shown in table 1.

c) In the text books analysis we detected problems like the next one.

Empezaremos notando que los circuitos RL y RC de segundo orden deben contar con más de una malla o más de un par de nodos. Si éste no es el caso, habrá dos elementos que almacenen energía conectados en paralelo o en serie en el circuito. Éstos pueden obviamente sustituirse por un solo elemento que almacena energía equivalente; el circuito resultante puede tratarse entonces como uno de primer orden. Un ejemplo de un circuito que contiene dos inductores, en el que éstos no pueden reducirse en un solo inductor, se muestra en la figura 6-3.1. Para caracterizar este circuito, es necesario escribir dos ecuaciones diferenciales. Éstas se encuentran aplicando la LVK (ley de voltaje de Kirchhoff) alrededor de las dos mallas del circuito. De este modo, obtenemos

$R_1i_1(t)+L_1\frac{d(i_1-i_2)}{dt}=0$ *and* $R_2i_2(t)+L_1\frac{d(i_2-i_1)}{dt}+L_2\frac{di_2}{dt}=0$ *(Huelsman, 1988).*

As you see, in this problem all the elements are included in letter b), but it also includes other elements which make to combine some of the second generation models. In this case, it was detected implicitly that we have an equations system; in fact, we have two electrical net works and they are modeled by equations system.

$$\begin{cases} R_1 i_1(t) + L_1 \dfrac{d(i_1 - i_2)}{dt} = 0 \\ R_2 i_2(t) + L_1 \dfrac{d(i_2 - i_1)}{dt} + L_2 \dfrac{di_2}{dt} = 0 \end{cases}$$

This mathematical model is formed by two models of second generation, so it is called **third generation model.** These kind of problems are worked in electrical circuits subjects, which are in the engineering basic sciences group, but they are also used and worked in the control theory subjects which belongs to the group of engineering specialization sciences. We found this kind of models in engineering specialization subjects.

d) When the student is working with professional problems, as well as, his engineering specialization subjects, he requires models that describe problems where third generation models are included and other kind of complex elements which has to combine and to build new mathematical relations, these relations are named as **fourth generation models**. Sometimes, due to the problem complexity it is difficult to tackle it, so it requires to be modeled by simulation in the computer, so a mathematical models family is built on the same problem. These kinds of problems are more common in professional field, that is to say, in applied engineering.

From the aforementioned, the existing correlation between the classification of these models and the cognitive engineering stages are observed, as demonstrated in Table 2.

COGNITIVE ENGINEERING STAGES	MODELS TYPES
Basic Sciences	First generation models
Engineering Basic Sciences	Second generation models
Engineering Specialization Sciences	Third generation models
Applied Engineering	Fourth generation models

TABLE 2. CORRELATION BETWEEN COGNITIVE STAGES AND MODELS TYPES

A summary of a mathematical models classification according to their characterization is presented in table 3.

MATHEMATICAL MODELS CHARACTERIZATION					
Engineering Objects Modeling		Engineering Problems Modeling			
The **classification** is in terms of the use given by engineering		The **classification** is in terms of the engineering cognitive stages			
Static Models	Dynamic Models	First G. Models	Second G. Models	Third G. Models	Fourth G. Models

TABLE 3. CLASSIFICATION OF THE MATHEMATICAL MODELS ACCORDING TO THEIR CHARACTERIZATION

Mathematical modeling concept.

From *Mathematics in Context* steps and the detections made in the analysis of the problems studied for this research, we defined a mathematical modeling concept. *Mathematical modeling is conceived as the cognitive process that has to be carried out to build the mathematical model of a problem or an object of the context area.*

This cognitive process consists of three moments, which constitute the mathematical modeling indicators:

- Identify variables and constants of the problem, it includes the identification of what changes and what remains constant.
- Establish relationships between these, through the involved concepts of the problem, either implicitly or explicitly, whether they are from the mathematical or from the context area.
- Validate the ”mathematical model” that models the problem, which is made through going back and verifying that it involves all the data, variables and concepts of the problem; depending on the problem, some times the mathematical model can be validated through seeing if the mathematical expression predicts the experimental given information; in other cases, to validate the model, it is necessary to give a mathematical solution to see that the involved elements are predicted.

An important point to mention is that the mathematical model is not unique, there are several mathematical representations that describe the same problem, and this is the reason why its validation is necessary (third moment). The way to treat mathematically the mathematical model, is neither unique, this element permits to verify the mathematics versatility as well as its consistency.

Conclusions. The mathematical models are a fundamental part of *Mathematics in the Sciences Context,* the classification and characterization of mathematical models, as well as, the cognitive elements and thinking skills that had been detected give a knowledge source to teach models in the mathematics classroom, and to fortify the *Mathematics in Context*.

References.

Camarena, G. P. (1984). El currículo de las matemáticas en ingeniería. *Memorias de las Mesas Redondas sobre Definición de Líneas de Investigación en el IPN* (pp. 21-25). México, México: Instituto Politécnico Nacional.

Camarena, G. P. (1987). *Diseño de un curso de ecuaciones diferenciales en el contexto del análisis de circuitos eléctricos*. Tesis de Maestría no publicada, Centro de Investigación y Estudios avanzados del Instituto Politécnico Nacional, México, México.

Camarena, G. P. (1990). *Especialidad en docencia de la ingeniería matemática en electrónica*. Manuscrito no publicado, Instituto Politécnico Nacional en México, México.

Camarena, G. P. (1993). *Curso de análisis de Fourier en el contexto del análisis de señales eléctricas*.

Manuscrito no publicado, Instituto Politécnico Nacional en México, México.

Camarena, G. P. (1995, octubre). La enseñanza de las matemáticas en el contexto de la ingeniería. *XXVIII Congreso Nacional de la Sociedad Matemática Mexicana*. Colima, México.

Camarena, G. P. (1999, noviembre). Hacia la integración del conocimiento: Matemáticas e ingeniería. *Memorias del 2º Congreso Internacional de Ingeniería Electromecánica y de Sistemas.* México, México.

Camarena, G. P. (2000). *Los modelos matemáticos como etapa de la matemática en el contexto de la ingeniería*. (Reporte técnico de investigación CGPI 2000731). Instituto Politécnico Nacional, México, México: ESIME-IPN.

Camarena, G. P. (2001). *Las Funciones Generalizadas en Ingeniería, construcción de una alternativa didáctica.* México, México: ANUIES.

Camarena, G. P. (2002, noviembre). El proceso enseñanza aprendizaje de las ciencias básicas. *Memorias del Foro de Desarrollo Curricular e Investigación Educativa para la Enseñanza de la Ingeniería.* México, México.

De Bono, E. (1997). *El pensamiento lateral, manual de creatividad*. Buenos Aires, Argentina: Paidós Empresa 5.

Duval R. (1993). Registres de représentation sémiotique et fonctionnement cognitif de la pensée. *Annales de Didactique et des Sciences Cognitives*, 5. IREM Strasbourg, Istra.

Huelsman L. (1988). *Teoría de Circuitos*. México: Prentice Hall.

Nickerson, R. & Perkins, D. (1994). *Enseñar a pensar, aspectos de la aptitud intelectual*. Madrid, España: Paidós.

Polya G. (1976). *Cómo plantear y resolver problemas*. México: Trillas.

Santos T. L. M. (1997). *Principios y métodos de la resolución de problemas en el aprendizaje de las matemáticas*. México: Iberoamérica.

Vigotsky L. S. (1978). *Mind in Society. The development in higher psychological processes*. Harvard University Press.

MATHEMATICAL MODELLING, THE SOCIO-CRITICAL PERSPECTIVE AND THE REFLEXIVE DISCUSSIONS

Jonei Cerqueira Barbosa
State University of Feira de Santana, Brazil
http://www.uefs.br/nupemm
joneicb@uol.com.br

Abstract: This paper is part of a wider study on the students' practice at the micro-social level of modelling-based classrooms by seeking to gain insights about their discursive practices. In particular, the aim of this study was to analyse the production of reflexive discussions in a mathematical modelling environment. The importance of this topic is related to the possibility of understanding how the socio-critical perspective works in action. Following the qualitative approach, a group of students was filmed during a modelling task. The findings suggest two ways of producing reflexive discussions: debating the influence of the criteria in building the mathematical models, and comparing the different models constructed by students. These results provide subsides for teachers who want to follow the socio-critical perspective.

Introduction

The mathematical modelling environment may carry out different goals (Niss, Blum & Galbraith, 2007). It's possible to associate them with the perspective debate, which represents a wider way of conceptualizing the modelling, including the didactical interests. In Kaiser and Sriraman (2006) and Kaiser, Sriraman, Blonhoj and Garcia (2007), there is a clarification of the perspectives presented in the literature. One of them is the "so-called" socio-critical modelling, according to what is addressed in Barbosa (2006).

The formulation of the socio-critical perspective is based on the implications from the studies about the relation among mathematics, mathematical models and society. For example, Skovsmose (1994, 2006) has pointed out how mathematics has become a strong argument for making decisions in society, and the mathematical processes used in this are hidden. As the mathematical models are not neutral, they depend on their building processes, which include how the modeller understands the problem-situation, and how he/she projects mathematical conceptions into the situation. Following this argument, the biased nature of the mathematical models brings a closer relationship between the mathematical model and the criteria used to construct it.

In society, the mathematical results are usually presented as neutral descriptions from reality. This has increased in the age of technology, which has reduced the importance of mathematical skills and knowledge for people in general. According to Jablonka and Gellert (2007), the mathematisation of society and the demathematisation of people are two connected processes. As people may not be skilled enough to discuss a mathematical argument, then they have to accept it. It means that the social groups which control the mathematical arguments could have more power than those who don't.

This analysis brings implications for modelling in mathematics education. If we assume that education should go beyond the work training (D'Ambrósio, 2007), then the modelling environment should address the nature and the role of mathematical models in society. In Barbosa (2006), I have pointed out the presence of studies in literature which have this focus for modelling.

Formulating the socio-critical perspective for modelling is useful to support classroom experiences and vice-versa. One focus of the research is to study how this perspective works at the classroom level in relation to teachers' and students' actions. The main point in the classroom is that socio-critical perspective has to address the nature and the role of mathematical models in society.

In Barbosa (2006), I have developed an approach to analyse the students' modelling. By focusing on discursive activity, we can understand how people participate in social settings through discourses. When invited to solve a modelling problem, students may develop many kind of discourses. In Barbosa (2006), taking Skovsmose (1990) in consideration, I have examined three kinds of discussions:

- mathematical: refers to the ideas belonging to the pure mathematics field.
- technological: refers to the techniques of building the mathematical model.
- reflexive: refers to the nature of the mathematical model, the criteria used in its construction and the consequences of this criteria.

The reflexive discussions may be associated with interests in socio-critical perspective. It doesn't mean that the other discussions don't have a place in the modelling environment, but they are re-positioned. The reflexive discussions involve the other discussions, since one should always analyse criteria used in the model.

However, students can't be "compelled" to produce reflexive discussions, but their production has to make sense to the students. In particular, *this paper is meant to examine the situations in modelling-based lessons which bring up the reflexive discussions*. This study calls for theorizing the socio-critical perspective for modelling at the social-micro level, besides providing subsidies for teachers who want to put this perspective in action inside the classroom.

Methodology

The research setting was a modelling course for future mathematics teachers at the State University of Feira de Santana, in the Brazilian Northeast. The students were invited to solve some modelling problems in groups. One of them was based on a newspaper article that informed them about the possibility of having electrical energy rationed, because the level of the water reservoir was low at the hydroelectric power station. In this paper, I am going to focus on the students' group composed of Ana, Beatriz, Maria and Tereza.

The study followed a qualitative approach, because this research methodology is more appropriate for capturing the meanings of discourses produced by students (Bodgan & Biklen, 1998). The data was collected through observation by a research assistant, and filming was a form of registration. The data was analysed and inspired by the Grounded Theory (Charmaz, 2006).

Data

The students were invited to analyse a newspaper article published in November, 2007, which reported on a reduction in volume of the Sobradinho Lake, due to the lack of rain. The water of this lake is used to produce electrical energy for 75% of the population of the Brazilian Northeast.

I would like to present some information from the article in a few words. At that time, the volume of water represented 15% of the capacity of the lake, which produced 450 megawatts/hour. According to the newspaper, the expectation was to operate at capacity the 13% next month. The lake produces 1,050 megawatts/hour at its fullest capacity.

Part 1

The students were invited to anticipate when the Sobradinho Lake would have zero capacity. The group composed of Ana, Beatriz, Maria and Tereza started by trying to structure the situation and listing the variables which interfere in the lake's volume: the use of the water by the population which lives near the lake, evaporation, and water provided by tributary rivers. However, they quickly noticed that data was not available in the newspaper article that considered these variables.

The students pointed out to the teacher that the results would be a bit different, because they couldn't collect important information. They had to use the variables provided by the article. The students' point was that different variables would bring different results.

Thus, students noticed that the results depended on the considered variables. This episode suggests that the students developed their own perceptions about the close relationship between chosen variables and models. This sort of discussion may be seen as reflexive, because it shows the interference of criteria – in this case, variables – in models.

Part 2

At the end of the task, every group was invited to present their results to the whole class. That group composed of Ana, Beatriz, Maria and Tereza showed their results in the form of a table (see table 1), which enabled them to conclude that the necessary time for the lake to become empty would be 8.5 months, and so the production of electrical energy would be interrupted.

Time (month)	Useful volume
1	15%
2	13%
3	11%
4	9%
5	7%
6	5%
7	3%
8	1%
9	-1%

Table 1. Tabular data

The criteria used by the students were that the volume of the lake was changing constantly 2% of its full capacity per month. In other words, the variation was constant.

However, another group found a different result. This one took the full volume of the Sobradinho Lake, which is 28,669 Hm^3 and calculated 15%, so that the students found the volume at that moment, 4,300.35 Hm^3. Then, they considered the variation as proportional to the volume each month. As they fitted an exponential equation, from a mathematical point of view, they found that the lake would never become empty. However this would happen within 40 months, in practice.

Comparing the students' strategies, it is supposed that they framed the problem-situation with previous mathematical experiences. In this case, they used proportion and exponential function for that. In doing so they established real criteria as tied to the mathematical criteria. However, I am not going to expand this point here, because it is out of the focus.

The different results made the students surprised and they went to review the procedures used by them. From comparing their models, they re-examined the strategies used for building the results, which enabled them to notice the different criteria used by them. While the first group considered a monthly variation of lake volume as a constant, the second group took it as depending on the monthly volume.

In this case, students produced what we may recognize as reflexive discussions, because they related mathematical models to the criteria used to produce them. By working in different groups, the students had the opportunity to use different criteria and so produce different models to compare. The presentation of the results to whole class seems a crucial moment for the generation of this sort of discussion.

Discussion

The results found suggest that the students produced reflexive discussions through two situations: analysing the criteria to be used in building mathematical models; and comparing the models produced from different student groups. In the first situation, when students were discussing how to simplify the problem by defining hypothesis and choosing variables, they anticipated the influence of these criteria in the model. In the second situation, the students' group compared the results with those found from other groups, and they verified different mathematical models. Debating why different results were produced, they perceived how different models were established according to the way that problem-situation was simplified. These findings have also been observed in a recent study conducted by Santos (2007).

The reflexive discussions are not clearly connected to the validation. This one seeks to verify the models in terms of the real data (Edwards & Hamson, 1990). In fact, the latter is more related to what I named previously technical discussions, because it addresses the building of a model whereas the reflexive discussions do the relation between criteria and results.

In episodes above, considering the concept of reflexive discussions found in Barbosa (2006), the data suggests that the genesis of reflexive discussions is related to the presence of the criteria and results (or mathematical models) in students' debates. In this way, the students had the opportunity to challenge the view that mathematical models are neutral descriptions of reality, and so put the socio-critical perspective in action at the classroom level.

Final remarks

These results suggest evidence regarding the production of reflexive discussions: analysing the influence of criteria in building mathematical models, and comparing different models. In both cases, there is the opportunity to analyse the relation between criteria and results, bringing to the classroom the implications from socio-critical perspective at a discursive level (Barbosa, 2006).

The focus of reflexive discussions is more on the link between criteria and results than validity. This is more appropriate to be found in technological discussions. The former is defined in terms of analysing how different criteria produce different results. Once every model is biased, then reflexive discussions might address the biases.

This study also brings a demand for a particular research agenda to the socio-critical perspective. For example, questions about the relationship between reflexive discussions and other aspects in school settings need more attention. From the point of view of teachers interested in a socio-critical perspective, this study allows us to infer strategies to motivate the reflexive discussions in modelling environment, such as calling students to anticipate results according to the criteria used and to examine the different models produced in the classroom.

Acknowledgments

I thank the reviewers for their comments on an earlier version of this paper.

References

Barbosa, J. C. (2006). Mathematical modelling in classroom: a critical and discursive perspective. *ZDM – The International Journal on Mathematics Education*, 38 (3), 293-301.

Bogdan, R. C. & Biklen, S. K. (1998). *Qualitative research for education: an introduction to theory and methods*. Needham Heights, MA: Ally & Bacon.

Charmaz, K. (2006). *Constructing grounded theory: a practical guide through qualitative analysis*. London: Sage.

D'Ambrósio, U. (2007). The role of mathematics in educational systems. ZDM – The *International Journal on Mathematics Education*, 39 (1-2), 173-181.

Edwards, D. & Hamson, M. (1990). Guide to mathematical modelling. Boca Raton: CRC Press, 1990.

Jablonka, E. & Gellert, U. (2007). Mathematisation - demathematisation. In: U. Gellert & E. Jablonka (Eds.). *Mathematisation and demathematisation: social, philosophical and educational ramifications*. Rotterdam: Sense Publishers, 1-18.

Kaiser, G. & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM – The International Journal on Mathematics Education*, 38 (3), 302-310.

Kaiser, G.; Sriraman, B.; Blomhøj, M. & Garcia, F. J. (2007). Report from the working group modelling and applications – diferentiating perspectives and delineating commonalties. Retrieved 5th November 2007 at http://www.erzwiss.uni-hamburg.de/Personal/Gkaiser/pdf-publist/CERME5WG13_Introduction.pdf.

Niss, M., Blum, W. & Galbraith, P. (2007). Introduction. In: W. Blum, P. Galbraith, H. Henn & N. Mogens (Eds.). *Modelling and applications in mathematics education – the 14th ICMI Study*. New York: Springer, 3-32.

Santos, M. A. (2007). *A produção de discussões reflexivas em um ambiente de modelagem matemática.* Unpublished master dissertation. Federal University of Bahia and State University of Feira de Santana

Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education*. Dordrecht: Kluwer.

Skovsmose, O. (2006). Challenges for Mathematics Education Research. In: J. Maasz & Schloeglmann, W. (Eds.) *New mathematics education research and practice*. Rotterdam: Sense Publishers, 33-50.

MATHEMATICAL MODELING AND ENVIRONMENTAL EDUCATION[1]

Ademir Donizeti Caldeira

Federal University of Santa Catarina – Brazil

Abstract:

The objective of this research was to identify possibilities for using mathematical modeling as a resource in the education of future elementary and middle school teachers. The study was carried out in four schools: two rural schools, one urban school, and a fourth located on an island of fishermen, located on the southern coast of Brazil. It was carried out in two stages: the first consisted of a Participatory Environmental Diagnosis (PED) of local environmental problems, and the second involved qualitative and quantitative discussions of a specific problem identified in the PED. The results indicate that it is possible, in the process of mathematical modeling of environmental issues, to provoke critical reflections among teachers regarding the role of mathematics and its uses in environmental debates.

Keywords: mathematics education, mathematical modeling, environmental education, teacher education.

1. Introduction:

The objective of this research was to identify possibilities, through the elaboration of projects and use of mathematical modeling, for discussing the uses of mathematics to understand local environmental problems.

It was developed in two stages: the first was composed of a Participatory Environmental Diagnosis (PED); and the second involved the elaboration of mathematical models and quantitative/qualitative discussions focused on a specific problem identified in the diagnosis and selected by the teachers in each locale.

Four groups were formed, with a total of nineteen teachers: the first group was composed of seven teachers from an urban school in Guaraqueçaba; the second consisted of four teachers from a rural school in Tagaçaba; the third included three teachers from a rural school in Serra Negra. All of these are located in the

[1] This research was supported by CAPES/PRODOC

municipality of Guaraqueçaba. The fourth group was composed of five teachers from a school located on an island of fisherman known as Ilha das Peças.

2. The first stage: Participatory Environmental Diagnosis (PED)

The objective of the PED was to encourage the participation of teachers in local environmental issues. Ten one-day meetings were held in each locale at intervals of at least 15 days. The meetings included procedures of information and reflection outlined in IBAMA/SMA/UNICAMP (1998) and included the following types of activities:

a) *Conceptual presentations:* aimed to discuss basic concepts from the different fields of knowledge associated with environmental issues, such as: social sciences, biodiversity, and sustainable development, among others. This activity was carried out by the researcher.

b) *Group activities:* with the objective of integrating the theoretical base with information collected about the local reality, in order to share, deepen, and reflect on the information, creating a process of construction of a local vision. In this process, the teachers recognized the need to understand their own reality in their own depositions.

c) *Empirical research:* allowed the teachers to experience every stage of the participatory diagnosis, including collection of relevant information to characterize local, and occasionally regional, problems.

The work began in the four sites with a survey of the environmental problems carried out by the teachers in their locales, in a process of unveiling that included identification of the actors involved, reflection on the characteristics of a participatory environmental analysis, and the necessary steps to carry it out.

These steps were used with the aim of reflecting on, organizing, ordering, and systematizing the construction of a basic instrument to guide the empirical research in each locale. The first step was to identify difficulties encountered by the teachers in the elaboration of the DEP, and afterwards, offer guidance by responding to the questions: How is a participatory environmental diagnosis carried out? What does it contain?

Through the activities and discussions carried out in groups, the teachers began to reflect on what such a diagnosis consists of and identify steps, which included:

1st step: identify an environmental problem or problems in your locale. The main questions were:

a) What can be considered a problem?

b) How to proceed to seek consensus regarding the definition (or not) of a problem?

2nd step: consult the local population about the main problems; why they are considered problems, and what can be done to resolve them. The main questions were:

a) Who should be consulted?

b) What are the best methodological procedures for this consultation?

3rd step: relate the available information about the various aspects and points of view regarding the problems raised.

4th step: faced with the information obtained, identify the blanks remaining to be filled to completely unveil the problem.

The methodology for consultation adopted by the teachers for the survey was *informal conversations*, i.e., without the use of an audio-tape recorder, but based on a previously established interview guide. Basically, they sought answers to the following questions: What is a problem? Why is it a problem? What solutions are proposed? The number of people interviewed depended on the time the teachers had available to carry out the survey.

3. Results of the first stage:

As a result of the first stage of this study, in which the teachers carried out the DEP, the following environmental problems were identified in each locale:

Table 1. Survey of environmental problems in the study sites.

Problems/locales	Ilha das Peças	Guaraqueçaba	Serra Negra	Tagaçaba
Garbage	X	X		X
Sewage treatment	X	X	X	X
Poaching of wild animals		X	X	
Palm-heart cutting		X	X	
Stationary telephone poles	X	X		X
Transportation/roads		X	X	X

Flooding	X			
Burning		X	X	
Crabs	X	X		
Hygiene		X	X	
Leisure	X			X
Water	X	X		
Predatory fishing	X	X		
Education	X			X
Health	X	X		X
Loose animals	X	X		
River pollution		X		X
Snails	X	X	X	X
Deforestation		X		
Sanitary inspection		X		X
Security	X	X		
Urban inspection		X		

Following the completion of the DEP, with the local environmental data in hand, the second stage of the research began. Mathematical modeling processes were carried out aiming to construct the instruments of qualitative/quantitative understanding of the environmental realities, experienced through mathematical concepts, resulting from the search for solutions to the questions reflected in the themes. The following themes were selected:

Locales/ Projects	Water	Garbag e	Hygiene
Ilha Da Peças	X		
Guaraqueçaba		X	
Serra Negra			X
Tagaçaba		X	

Table 2. Themes selected by the teachers.

4. Second stage of the research: Mathematical Modeling

The work was carried out by the same groups in the second stage of the research, using pedagogical assumptions based in mathematical modeling.

According to Meyer & Caldeira (2001), although many approaches are referred to as mathematical modeling, most of them include:

1st stage: the *formulation* of the question, in which the critical attitude is revealed at the moment the essential aspects of each problem are selected for inclusion in the mathematical model. This formulation includes the establishment of the question itself, but also its expression in a language from the mathematical universe, i.e, a mathematical problem.

The following mathematical questions were constructed by the teachers in each locale:

- Ilha das Peças: What would be the capacity of a water deposit capable of meeting the needs of the island for one day?
- Guaraqueçaba**:** What would be the quantity and weight of the paper, plastic, and pencil shavings discarded during one week (Monday to Friday) in the following classrooms: pre-school – 1st grade; 2nd grade; 3rd grade; 4th grade in the municipal school?
- Tagaçaba: How much garbage is produced in Tagaçaba?
- Serra Negra: What is the hygiene situation of the 1st, 2nd, and 3rd grade students in the Serra Negra Municipal School?

Collection of data to try to answer these questions was carried out entirely by the teachers, in some cases with the participation of their students and, in the case of Tagaçaba, with the participation of the community in the collection and selection of garbage.

2nd stage: *resolution*. The resolution of the problem that was expressed mathematically was, obviously, only approximate. Here a critical view of the adequacy of the mathematical instrument is also necessary, given the use of a mathematical instrument for non-mathematical ends: mathematics as a means of understanding reality. The same critical attitude is also necessary in the evaluation of the precision of the response acquired, as well as the evaluation of the results.

To find solutions to the questions, the mathematical concepts used were basically operations with natural numbers (addition, subtraction, multiplication, division), measures (perimeter, area, and volume), geometry (plane and spatial figures), and information handling (graphs and tables).

3rd stage: *evaluation.* In addition to evaluating the mathematical results for the problem studied, it is also necessary to critically analyze the adequacy of this solution as a response to the concerns of the community: its problems, life, quality of life, and environment. While objective characteristics were highlighted in the process of evaluation, subjective aspects were also raised, as the evaluation process is composed not only of the mathematical evaluation, but can also include evaluation processes that are important for the community problem – and may even involve the solution of the problem by the community itself. This assumes taking a position, making a commitment, and critical engagement, which led us back to the beginning of the process, given the context of the starting point, which is necessarily inserted in a dynamic environment and can, therefore, lead to problems that are studied and abandoned, or problems that continue to be restudied. In the case of Ilha das Peças, for example, the work continued after the project ended, and the water deposit was built by the inhabitants of the island.

In summary, based on the construction of the diagnosis, we began with an environmental problem selected by the teachers, which was then modeled mathematically and, as such, understood in a new way. In the attempt to resolve the problem proposed by the model, the mathematical contents became mobilized and were used as mathematical tools, as means for achieving a larger goal: quality of life.

Each teacher's experience with learning mathematics was a determining factor in the establishment of the mathematical contents and methods needed to determine the solutions to the mathematical questions. On the other hand, these solutions were obviously not the only ones, given the fact that they were determined by the themes and contents chosen by each group.

4.1 Development of the group work.

4.1.1 Ilha das Peças

The theme chosen by the teachers on Ilha das Peças for the mathematical modeling project, as shown in Table 2, was "Water". This choice was motivated by the experience of the island's inhabitants, where potable water is often in short supply or unavailable. Due to the many septic tanks installed on the island, the quality of the potable water has become totally compromised. The alternative sought by inhabitants of the island, despite their location in the middle of the Atlantic Forest, is to pipe drinking water from the continent.

This situation led us to reflect on the possibility of having a water cistern on the island. Thus, the mathematical problem was to find an answer to the following question: *What capacity would the cistern need to have to meet water supply needs on the island for one day?*

The first answer was to determine the number of people that use water in the island. This led to a discussion regarding the flow of people present on the island – local residents and tourists. Thus, we initially raised the following questions:

1. *What is the number of houses of residents and of tourists on the island?;*
2. *What is the flow if people during high and low tourist seasons?;*
3. *What is the total number of people who use potable water on the island in one year?;*
4. *How much water does each person use per day?;*
5. *How much water is consumed per day in the maintenance of a home?*

The first answers began to emerge according to the teachers' knowledge regarding the locale, as a result of intuition and experience acquired in some activities. The opinions of children and students were also taken into account in some cases. Based on the responses, some tables were created to provide us with an overview of the data.

Faced with these data, we carried out various simulations, for example:

1. Quantity of water used per day by local residents for personal hygiene and home maintenance;
2. Quantity of water consumed on weekends;
3. Quantity of water used on a festival on the island, when approximately five thousand people gather.

Thus, we constructed various simulations and always interpreted them in terms of real situations.

The first idea was to work with the arithmetic (number concept and basic operations) needed to arrive at a better understanding of the phenomenon, as well as to show that the mathematics taught in the early grades is necessary to achieve this understanding.

The second step was to seek an answer to the initial question: *What size of cistern is needed to meet the water supply needs on the island for one day?*

It was decided to take into consideration only the local residents. This led us to build a model (using data collected by the teachers) of a cistern with an approximately 90,000 liter capacity.

At that moment, we began to notice that arithmetic alone would not suffice. It was necessary to introduce concepts of geometry (perimeter, area, volume). In the end, we carried out some simulations with cisterns or various shapes and sizes, and concluded that it should measure 3m x 5m x 6m.

4.1.2 Guaraqueçaba

As shown in Table 2, the theme chosen by the teachers was garbage in the classroom. The procedure followed for the research was as follows: at the end of every class, the teachers responsible for the class swept the floor, collected the garbage, counted it, weighed it using a letter scale, and wrote the information down in a table, thus creating tables and graphs.

This was done for all the classes involved in the research. All together, the following tables and graphs were made:

1. quantity and weight of paper collected per day in each classroom;
2. quantity and weight of plastic collected per day in each class;
3. weight of pencil shavings of each grade per day;

All graphs were created without using any technology. We had no computers in the school, so they were created using graph paper, and the results were discussed with the teachers.

The results showed a significant amount of garbage produced in the classrooms, and led to a discussion about the importance of minimizing garbage production in and out of the classroom.

4.1.3 Tagaçaba

As presented in Table 2, the problem chosen by this group was garbage in the locale. The teachers of this group were interested in addressing the following question: *How much recyclable garbage is produced in Tagaçaba?*

Other questions arose from this initial question, such as:

How much money would be raised by selling the recyclable garbage produced in one year?

How much would it cost to buy the school?

The development of the modeling process involved a sample of ten houses in Tabaçaba, and for 30 days, the teachers gathered garbage, separated the paper, plastic, glass, and aluminum, and weighed the material, creating tables with the data.

After the table was created, we discovered the price per kilo of each type of material: plastics, paper, glass, and aluminum, and then proceeded to answer the following question: How much would the teachers earn if they sold the garbage collected in one year?

Based in the results, it was concluded that, despite the very low value, it was nonetheless worthwhile to recycle garbage as a way to preserve the environment and reuse materials that become garbage in the community.

4.1.4. Serra Negra

Among the environmental problems detected in Table 2, the teachers of Serra Negra chose the theme of personal hygiene among students.

This choice of theme was due to the teachers' perception that many students were coming to class without meeting conventional standards for hygiene: soiled clothing, unbathed, hair uncombed, smelling of urine, among other things. There was a certain resistance in the group to working with this theme as we agreed that it was not very related to mathematical issues. We nonetheless insisted on working with the theme, although we did encounter difficulties.

To obtain the data, the following steps were followed:

1. Elaboration of a close-ended questionnaire about various forms of hygiene for the students to fill out;
2. Tabulation of the data and transformation into graphs.

The questionnaire addressed the following themes about the students: sanitary conditions in their homes; bathing conditions; clothing; towel use; use of toothbrush; use of toilet paper; among others. The results pointed to possibilities for discussing the importance of personal hygiene with the students.

The data were tabulated and then transformed into tables and graphs. The next step was to carry out some mathematical simulations with the period of dental treatment of children. In this way, the teachers identified and resolved the following problems:

1. Knowing that the dentist is available three times per week and treats three children per day, how long would the last child have to wait to be treated by the dentist?
2. How many children would the dentist have to see per day in order to see all the children in one month?

After resolving all the problems, we discussed the viability of more timely dental treatment for the children.

5. Discussion of the Results:

The themes raised in the PED increased the teachers awareness regarding the indispensable partnership of the school in the search for solutions to questions faced by the community. The choice of the theme by the teachers was unanimous, as they recognized the urgent need to discuss and resolve the problem identified.

This approximates us to the ideas of Skovsmose (2001) when he calls attention to the role of mathematics in society, and shows us the need to bring mathematics education and critical education closer together. Such an education is characterized by the critical engagement of teachers (and students) in the process, with dialogue and the teacher-student relationship occurring in a democratic process.

The study also showed that it is possible, in the process of learning about mathematics and environmental issues, for teachers to develop a critical competence, without it being imposed, but rather constructed through the experiences of the actors themselves; and that it is also possible to discuss a curriculum that questions the functions of school contents, the interests and assumptions reflected in these contents, and finally, the possible social interventions they can catalyze (or slow down), and under what circumstances.

The approach adopted for the study, based on the theoretical-methodological assumptions of mathematical modeling, showed us that it is possible for teachers to become involved in situations in their social context, and that the problems encountered there, when perceived as relevant, and because they are objectively existent social problems, lead to the critical engagement of the teachers in their communities.

From this perspective, teaching would no longer be developed through lectures and repetitive classroom activities, in the insulation of school buildings, but would situate the teacher in a process of reflection-formulation-action that leaves behind individuality in favor of the action of studies (Caldeira, 1998; Barbosa, 2001; Monteiro, 1991; Borssoi, & Almeida, 2002).

Modifying the teachers’ customary daily work habits of class preparation based on textbooks, this process required them to identify specific environmental phenomena that could inform their practice, and work not only with environmental concepts, but also mathematics, to interpret the situations presented to them.

The result was that teachers no longer tried only to respond to questions in the textbooks for their students, but constructed their own questions, as well as attempting to respond to them. Herein lies the critical-creative seed defended by D' Ambrósio (1996).

For this to occur, a proper environment, where the focus was on the research, was necessary. The teachers were researchers of the environmental problems as well as the mathematics needed to understand them in a more meaningful way. This process of curiosity and challenge is what motivated them to do the work.

In the sphere of relations between the School and Society, we perceived, through the study, that the school is neither the neutral institution depicted by the ideas of functionalism, defended by Talcott Parsons, in which the school is capable of bringing an end to social inequality through the transmission of norms, values, and knowledge that assure social integration; nor does it play the role characterized by reproductionism, based on the theoretical studies of Bourdieu and Passeron, Althusser, Baudelot, and Establet, and of Bowles and Gintes (Moyses, 1997), who see the school as reproducing social inequalities and serving as ideological apparatus for the State, destined to perpetuate the system. In our case, the function of the school more closely approximated the ideas put forth by the so-called *critical theories*, which emerged in the 1980s, and sought to recover the positivity of the aforementioned trends, seeking to overcome the innocent fragility of functionalism as well as the immobility present in the reproductionist theories (Giroux, 1986; 1997).

Within this conception, it is possible to verify the value that the school should have without falling into the notion of neutrality or uselessness for social transformation. As stated by Cortella (2001; p.136), “yes, the School can serve to reproduce injustices, but concomitantly, it is also capable of functioning as an instrument for change; the elite use it to guarantee their power, but because they are not aseptic, it can also serve to confront them”.

In the context of the present study, school education and teachers have relative autonomy; it was possible to see the relation between School and Society as two-way; not completely independent, as seen by the functionalists, nor entirely dominant, as seen by the reproductionists. For this reason, the study made it possible to journey through this contradiction, always creating opportunities for building effective spaces for innovation in the educational practice developed by each teacher in his/her classroom.

Regarding the pedagogical aspects, in addition to communing with the ideas of mathematical modeling, we also identified very much with what is being denominated “Ecopedagogy” (Gutiérrez & Prado, 1999; Gadotti, 2000; Padilha, 2004), whose main theoretical basis lies in the struggle for a culture of sustainability, promotion of meaningful learning, and attributing meaning to everyday actions. The study closely approximated the work of those authors who defend a pedagogy of questioning, democratic and committed to solidarity, that invites teachers and students to guarantee the sustainability of each of our everyday acts as human beings who share life on this planet with other living beings.

6. References

Barbosa, J.C. (2001) Modelagem matemática: concepções e experiências de futuros professores. *Tese de Doutorado*. Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista. Rio Claro, SP.

Borssoi, A. E Almeida, L. M. W. (2002) O Processo de ensino e aprendizagem acontecendo num Ambiente de modelagem matemática e tecnologias informáticas: buscando uma aprendizagem significativa. *Anais eletrônicos do VII EPREM – Encontro Paranaense de Educação Matemática- Foz do Iguaçu, PR*.

Caldeira, A D. (1998) Educação Matemática e Ambiental: um contexto de mudanças, *Tese de doutorado*. Faculdade de Educação da Universidade Estadual de Campinas. Campinas-SP.

Cortella, M. S. (2001) *A Escola e o Conhecimento: fundamentos epistemológicos e políticos*. São Paulo: Cortez.

D' Ambrosio, U. (1996) *Educação Matemática: da teoria à prática.* Campinas: Papirus.

Gadotti, M. (2000) *Pedagogia da Terra*. São Paulo: Fundação Peirópolis.

Giroux, H. (1997) *Os Professores como Intelectuais: Rumo a uma Pedagogia Crítica da Aprendizagem*. Porto Alegre: Artes Médicas.

Giroux, H. (1986) *Teoria Critica e Resistência em educação: para além das teorias de reprodução*. Petrópolis: Vozes.

Gutiérrez, F.& Prado, C.(1999) *Ecopedagogia e cidadania planetária*. São Paulo: Cortez.

IBAMA/SMA/UNICAMP.(1998) *Diagnóstico ambiental participativo do Vale do Ribeira e Litoral Sul de São Paulo: subsídios para a discussão do plano de ação governamental para o desenvolvimento sustentável.* São Paulo: Secretaria de Estado do Meio Ambiente.

Meyer, J.F.C. & Caldeira, A.D.(2001, jan/dez) Educação Matemática e Ambiental: uma proposta de formação continuada – e de mudanças. *Zetetiké – CEMPEM – FE/UNICAMP,* 9, 155-170.

Monteiro, A. (1991) O ensino de matemática para adultos através do método Modelagem Matemática. *Dissertação de Mestrado* – Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Rio Claro

Moyses, L. (1997) *Aplicação de Vygostsky à educação matemática*. Campinas: Papirus.

Padilha, P. R.(2004) *Currículo intertranscultural: novos itinerários para a educação*. São Paulo: Cortez.

Skovsmose, O (2001). *Educação Matemática Crítica: A questão da democracia*. Campinas: Papirus. Trad. Abgail Lins & Jussara Loiola Araújo. Coleção Perspectiva em Educação Matemática. 2001.

MATHEMATICAL MODELS IN THE SECONDARY CHILEAN EDUCATION

Aravena D., María & Caamaño E., Carlos
Basic Sciences Institute
Catholic University of the Maule. Talca-Chile

SUMMARY.

This work is a part of a wider research accomplished in the Secondary Chilean Education, at Statal schools of Talca commune, it was financed by the National Fund for Scientific & Technological Development (FONDECYT 1030122, years 2003-2006), a proposal based on the modelling through the project work was done. A methodology of qualitative type was used, by means of an interpretative analysis which enabled a thorough study of the productions of the groups during the development of the work. At conclusions arrival, it is evidenced the development of cognitive and meta-cognitive skills, and of transversal formation as well. Also, an efficient performance in the use of concepts and mathematical and non mathematical processes is evidenced.

I. OVERVIEW OF THE PROBLEM.

The study discussed a problem related to the difficulties that students have in the learning of Mathematics in the Chilean Secondary Education, especially at stat schools of the country, where the most disfavoured socio-cultural strata are assisted. As a fact, from decades, the mathematical formation at Chile presents serious difficulties in the form as the matters are articulated, the work in the classroom is guided, preferentially in the exercising and the handling of algorithms with scarce linkage with the other areas of knowledge even with mathematics itself, leaving aside its application in authentic contexts. This means, the majority of the themes are disconnected from real and scientific world, what has as a result, the fact that students can not perceive the benefit that mathematics has in their formation. This mistake is increased at schools that assist students of medium to low socioeconomic level strata and even more, to rural marginal areas (Aravena & Caamaño, 2007).

In addition, the researches on Didactics of Mathematics evidence that one of the themes that caught their interest is the design of activities based on Modelling. Also stating that if it is worked out through projects, it turns to be into a promising way, either to tackle with difficulties and deficiencies or to raise the quality of the mathematical learning (Abrantes, 1994; Aravena, 2001). In different countries and conditions, its inclusion in the curriculum, has enabled to develop of cognitive, meta-cognitive skills and of a transversal sort. Thus, evidencing that, modelling approaches favour the understanding of the concepts and mathematical methods and enables a comprehensive overall view of it. These are outstanding issues for a society where the citizens are going to be confronted to solve problems or to propose new problems in (Niss, 1992; Keitel, 1993; Matos**,** et.al, 1996; William & Ahmed, 1997; Blomhøj, 2000; Aravena, 2001; Gómez, 2007, Aravena, Caamaño &Giménez, 2008; Blomhøj, 2008). Among those that catch our attention, we highlight: organizing and interpreting information, the matematization of situations (De Lange, 1998; Niss, 2001), the creativity, the interest for discoveries, the capacity for analyzing and interpreting present examples through mathematics. Besides, it also helps to develop communicative skills through the development of ideas, the communication of methods and justification of processes (Alsina, 1998; Aravena & Gimènez, 2002; Aravena, Caamaño &Giménez, 2008).

At the same time, in Chile there is a neglectance on the work on models, applications and projects at all levels of education (Aravena, Caamaño & Giménez, 2008), which justifies this type of works, Implementing this type of work can be a powerful means in order to overcome difficulties as citizens so as to face a changing society, particularly for the most socioeconomic disadvantaged students (Abrantes, 1994, Alsina, 1998, Niss, 2001, Aravena, 2001).

From the situation described above, we set ourselves the following objectives:

1) Analyzing the proposals on the work of modelling and projects. Recognizing those that highlight an adequate working procedure for students of secondary education.
2) Organizing an integrative proposal, that relates mathematics to the different areas of knowledge, by means of activities where students may give sense to the mathematical and no mathematical concepts.
3) Introducing secondary students of stat schools of the commune of Talca, from medium-low and low socioeconomic strata into tasks of modelling and projects.
4) Analyzing the process of work on modelling by means of projects, selecting three groups for the case study. Recognizing skills that are developed.

II. THEORETICAL BACKGROUND

Historical evidence from the most ancient civilizations has showed the use of mathematics to describe important mankind phenomena, especially between the years 1500 and 1770, it helped establishing laws to explain and predict the behaviors of many phenomena, which has characterized all the sciences up to the present days ((Kline, 1992; Markarian, 2000). Indeed, the current applied mathematics and Mathematization of situations have had an accelerated growth in all areas of knowledge; Therefore, it becomes imperative to handle mathematical concepts related to daily life in order to understand the different social phenomena (Gómez, 1998; Aravena, 2001). In this context, it is important positioning theoretical modeling in the teaching of mathematics, and the integration of projects through modeling situations.

2.1. An Approach based on Modelling in the Teaching of Mathematics.

Despite that doing a work based on mathematical modelling and its applications is not entirely new, different authors justify its importance and its incorporation in the activities of mathematics curriculum (Blomhøj, 2008). From the ideas of Niss (1989), incorporating such problems in the classroom enables students to enter into a systematic work, appreciate the applicability of the concepts and their practical use (Swetz & Hartzler,1996). From the learning point of view, teaching through the actions of modeling is more convenient for the better performance of mathematics. Since it is based on the development of specific problems complemented with a theoretical treatment., where new mathematical objects are modeled, enabling in time, the introduction to increasingly abstract situations (Niss, 1989; Gomez, 1998; Aravena, 2001; Niss, 2001; Gómez, 2007; Aravena, Caamaño & Giménez, 2008).

2.2. Integration Projects through Modelling.

Though in recent years the idea of looking at applications through problem solving has been developed, several researchers have thought about the use of projects in the mathematical learning. This type of work is a "powerful strategy" for the teaching of Mathematics and presents many advantages with regards to other types of activities. Modelling and organization of projects, in terms of Oliveras (1996), have enabled breaking the atomization of the traditional mathematics curriculum and building a new mathematical approach; a more global and more contextualized one (Abrantes, 1994; Matos, 1996; Alsina, 1998; William & Ahmed, 1997; Gomez, 1998; Aravena & Gimenez, 2002).

III. METHODOLOGY.

The focus of investigation was in a qualitative modality, for such purpose, it was designed and implemented in the classroom: (1) Work in group workshops, in such a way of introducing students in processes of modelling, incorporating situations, that connect mathematics with different fields of knowledge and of reality. (2) Work projects, where students select a situation of the real world that is in close relationship with the matter of study, susceptible to be formulated in terms of a polynomial function. The projects of work were developed during four months and included: studies of the state of development and advance, written report and oral presentation. We justify these stages, since the trend that a project of work follows is too complex; also, because it deals with work done outside the classroom. Such activities take time and effort, in particular in the formulation of the model (Aravena & Giménez, 2002; Aravena, Caamaño & Giménez, 2008).

Methods and instruments of analysis: The sample consisted of 98 students from three stat schools of secondary education, 3 groups were selected for the study case. We present in this report one of the four groups of study cases. We chose this group because it belongs to one of the schools with the highest level of economic and social vulnerability; it is the school with the highest social risk which is included within the lowest one in the National Tests for Measuring the Quality of Education (SIMCE, 2004). We have chosen for the analysis a group of 3 students due to the special personal conditions they show. Besides studying, they also work. Moreover, they do not have any way to accede to any system of communication, being this one their first experience with computational tasks.

Categories of Analysis: An interpretative analysis of the contents was done, which enabled us to recognize the real progress in each one of the stages of the project. For that purpose, the following categories of analysis were designed beforehand, forming a second level of analysis: ***Cognitive aspects****:* (1) Organization and interpretation of the problem: Identifying data and conditions. (2) Mathematizing the problem: Description of mathematical relationships that interpret the process. (3)Verification and validation of the model; Relationship between the mathematical data and the real problem. (4) Mathematical communication: Giving an interpretation of the data and of the concepts from the real problem point of view. ***Meta-cognitive aspects****:* (5) Creativity: The contribution of new ideas in its solution. (6) Strategic thinking: Vision of the future and the way of tackling the problem. ***Aspects of Transversal Formation***. (7) Mathematical attitude: Valuation of the use they give to the concepts and recognition of their usefulness. (8) Autonomy: Responsibility in work, adaptation to the context, to the external environment and decision making. (9) Self evaluation: evaluation the student makes of his quality of work (Aravena & Caamaño, 2007; Aravena, Caamaño & Giménez, 2008).

Reliability and Validation: A triangular methodology was used for the reliability and validation of the work of projects. The same method of analysis in different occasions was used, and a triangulation of subjects was also used, and the transcriptions were analyzed regarding to the performance in the work of projects that depict the management in the different moments.

IV. DESIGN OF WORK PROJECTS.

General description and planning: The design of project work, three stages were considered to be essential in this type of work. (1) Initial stage corresponding to the presentation and rules of the game, the formation of the groups and the election of the subject work by the groups. (2) Development stage, which consists of two sessions of advances and a written report. Due to the complexity of this type of work, the stages of advance are very important for the teacher to give guidance to students in view to the formulation of the model, as well as for providing guidance for the written report (Abrantes, 1994; Alsina, 1998; Gomez, 1998; Aravena, 2001; Aravena & Gimenez, 2002, Aravena, Caamaño & Giménez, 2008) and (3) Final stage, which consists of presenting an oral summary of the work by emphatizing on results and mathematical work being done (Aravena, 2001, Aravena, Caamaño & Giménez, 2008).

STAGES	CONTENT	ACTIVITY	METHODOLOG	REGULATION
Directed by the teacher	Presentation Development of a project Characteristics: • Progress reports • Written Report • Speaking • Oral presentation	Project Objectives Report Oral presentation Group formation Suggestions and ideas of items to work. *Guidelines presentation*	Exposition by the teacher on the conditions of the project.	*Teacher`s daily journal*.
Of elaboration Advance meeting 1	Background collected Organization Information Generation of ideas. Formulation of the problem. Planning Decision making. Communication.	Report and develop ideas through questions designed by means of an Interview, in order to know the state of the project. Theoretical summary of the chosen topic.	Interview with each group of work. Following a record and a pattern.	*Evaluation chart* *Journal with the teacher`s report*
Advance meeting II	Net of information relationships Management and self-control Mathematical Knowledge Communication Progress.	Report on variations detected in advance I. Report and describe and report ideas no this stage.	Interview with each group following a record Using chart with guidelines.	*Evaluation chart. teacher´s daily journal.*
Written Report	Fundamentals and definition of problem Objectives, Hypotheses. Methodology Results. Model of Analysis Conclusions, suggestions, presentation, bibliography, annexes.	Designing a report containing the elements described above, according to a format. Presentation of the report at the required time.	Group work throughout the lasting of the project.	*Evaluation chart for the reports of the investigation.*
Final stage Oral presentation	Introduction of the theme under study. Introduction of the theme and justification. Objectives, hypotheses. Results. Description of the model Conclusions. Comments.	Oral presentation by each group. Evaluation of each group according to evaluation chart.	Exposition of each group, in a time of 15 to 20 minutes max. Questions from the audience or the evaluators. Comments from the teacher.	*Final Evaluation Chart.*

Table 1: Stages and tasks planned for the Project Work.

V. RESULTS AND ANALYSIS.

This section describes the experience with a case study group, after having chosen the theme "Breast Cancer". For this purpose, the stages of progress and the oral presentation were considered. We will show how they tackled the work. This article describes a detailed analysis of the productions of students in different times and stages according to the categories of analysis. We want to show how they related the mathematical and non mathematical aspects when facing a real problem.

5.1. First stage of Progress.

In this first meeting the group has identified the problem they want to study, describing the importance of the theme under study. They present the information they have gathered, where they have been able to formulate the problem. This stage is very important, since the group is at the beginning of the investigation, so the teacher must be very careful in planning on how to direct questions. Next we show how the work was developed according to the categories of analysis.

(1) Organization of information and interpretation of the problem: Exploration of the information that has been collected. In the first stage, there is recognition of the problem they want to study, but only from the real point of view. The interest is focused on explaining what they intend to do, in order to know the percentage of cancer patients.

The extract of the interview shows us how they to deal with the problem.

[E: Well, let's see what kind of work is being done. What is it?

Marianela: we are interested to know about breast cancer, the percentage of….

Ruth: how many women suffer from breast cancer?

Valeria: if cancer has been increasing

Ruth: It is a daily problem in today's society?

Marianela: is a big problem of women in Chile, many die from breast cancer

Ruth: It needs determine the percentages of women suffering from breast cancer, in different years.

Valeria: for example how many women have died, how many have exceeded it.

Ruth: move and move into mathematics, it was determined through a mathematical analysis and graphics that is a model for how has been on the rise or decline]

From this episode it is concluded that the group has an idea of the problematic to be worked out. They clearly manifest what they want to investigate and the importance of the problem. They relate some mathematical elements to the real problem, only at the level of ideas. They emphasize theoretical studies developed from the mathematical point of view.

(2) Mathematization: Identifying data and conditions, in view of the projection of the problem and of the intended mathematical elements.

[E Let`s see …how`s that? What do you want to model?

Marianela: with the data we have.

E: Do you have any data? What do think to do with the data?

Ruth: yes, we have got the data for some years.

Marianela: we must draw the graph and a table of values

Valeria: we are going to model the percentage of sick or deaths yearly?

We have to see that.

E: Do you have a clear view of the variables under study? What are they?

Valeria: the independent one …well (…) years and the dependent one..….

Ruth: well that is the percentage of sick or dead, it is what we are seeing, we have not decided yet.]

The problem focuses on the need to find data in order to find a model that represents the situation. There is an interest in graphing the collected data. Mathematical variables are recognized, which will be used to draw graphs, the variables of the study are not. There is an explicit recognition of the independent and dependent variables and of what they pretend to model.

(3) Implementing and verification: In view of the projection of data problems.

E: What is the importance of solving this problem?

Marianela: because the more years elapse breast cancer will increase in its rate, moiré years, older age the highest the rate…., eh?

Valeria: it increases more and more

Marianela (…) that is what we need to do, to see how it behaves.

At this level the students give some projections of what will happen over time, with regards to breast cancer, but only at a speculation level and from what they have read about the problem under study.

(4) Mathematical Communication: Giving an interpretation of the data and concepts from real problem point o: view. Difficulties in the first advances.

[E: well, what do you believe?, as far as projecting, which might be the difficulties in this work?

Marianela: yes, it is going to be difficult to estimate the percentage of patients

Ruth: the model in the end

Marianela: this stuff of the graphics and their function, comes to be very difficult for us.

Ruth: Yes, in the field of mathematics we are weak. Because we already have the information

E: but with what you have been seeing in the functions. Does it make it clear for you?

Valeria: yes of course, but if we were wrong?

E: what might be wrong with that?

Ruth: Yes, we learn more and more each time.]

Regarding to the concepts, students evidence that they have difficulties in the field of calculations and on the contents, their main concern is finding the model that approximates the situation. They feel weak in managing concepts and in applying them.

5.2. Second Progress

At this stage students should already have the data and have converted the information into the graph, they must transfer the concepts discussed in class, analyzing the variables involved, using a more technical and accurate language. There must be clarity in the mathematical aspects.

(1) Regarding to the organization and interpretation of information, there have been made changes according to suggestions made at the first stage. They have focused their work only on what they have defined it is their problem and their goals.

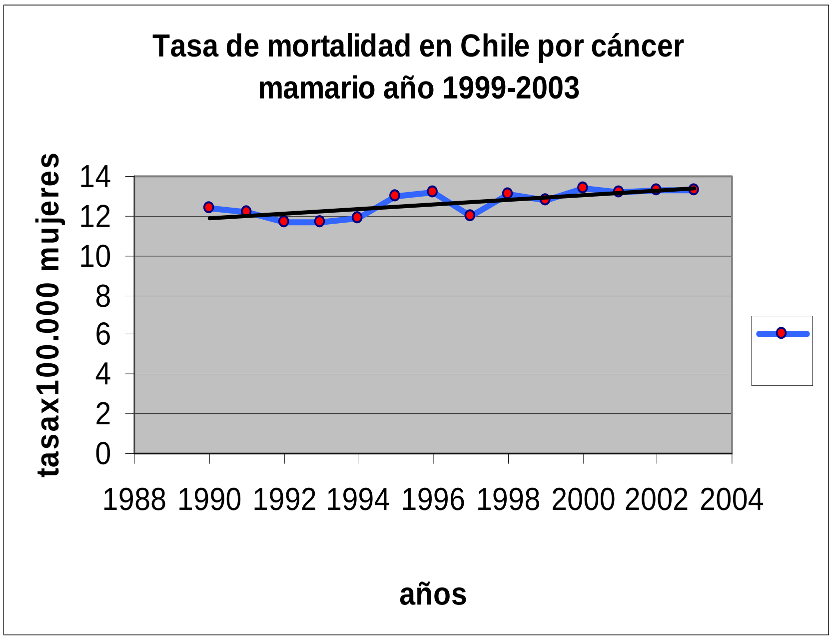
The next episode gives an account of the adequacy they have had to do:

[E: Have you made any change in its project?

Marianela: Yes, although we were not able to get all the data, we got some help in graphing and modeling.

Valeria: we have focused on the data we have collected and we will analyze the issue regarding cancer deaths. Here's the table of values and the graph of the mortality rate in Chile.

Marianela: We made this graph according to the years as independent variables and the dependant one corresponds to the mortality rate.]



Picture 1. Graphic produced by the

They take account of the type of analysis carried out, the graphics that have developed and the mathematical work which stresses adjustment of data and the recognition of the mathematical variables. In Table 4, we can see one of the charts used in the workplace.

(2) Mathematizing the problem: mathematical description of the mathematical relationships that interpret the process. This stage is very important, because at this stage they should have the model so as to make the necessary adjustments for the final report and the oral presentation. The mathematical work and relationships found must be explained. The next episode shows the relationships they made:

[E: What are the mathematical aspects worked?

Marianela: We could do a regression.

Ruth A linear approximation, use that formula of sumatories.

E: Good and Do they know how to use the formula?

Ruth: Of course, Web learned how to use it after you explained it to us (…) because they have the same form (..) data, are as scattered.

Marianela: For our model we based on the statistics of the region. It is what matters us(..)

Ruth. ss ..., we are interested in what is happening here, we might be able to see what happens if this trend continues

E: So, Are the models ready?

Yes. But it is what we did, and we are not sure if it is right:

Ruth: We found the mathematical model making an adjustment of data. (..) though it was a linear one, but at replacing the data the margin of error was larger with respect to the actual data.]

From the analysis, the idea of approximating and fixing of data must be highligted. Among the concepts displayed, the ideas of regression and lineal function are strongly empathized.

(3) Verification and Validation. Regarding to determining whether the mathematical model satisfies the initial conditions of the problem, the model has been evaluated with the mastery (domain) data; proving that it meets the requirements but only for that data. The next episode gives an account of the situation.

[Ruth: We evaluated the data in the model, but we are unable to follow this project in the future. We have placed some information but we can not say what will happen in other years.

Marianela. Yes, because at the beginning we thought that there was a linear function, that it could give a decrease or increase in other years as well. We can not predict the future.

Ruth: we can get the data back,, but for a certain period only.]

From the episode we can outstand that, its regression model, at a certain interval, satisfies the initial conditions, checking through the assessment of data domain. But they emphasized that this does not enable them to establish future icons. This idea is very important, since in processes of modelling, the central idea is to recognize if the model is useful to predict, thus, proving its validation.

5.3. Oral presentation.

The oral presentation is the culmination of work projects. It is a very important step because it is here where the students put all their capabilities to the test. In addition, for us it represents the reference and the key nucleus of how they organized mathematical knowledge.

(1) Organization of the information. We emphasize that the group presents a planning showing the steps followed in its investigation and this is detailed and developed in depth. In the following episode the exhibition is presented.

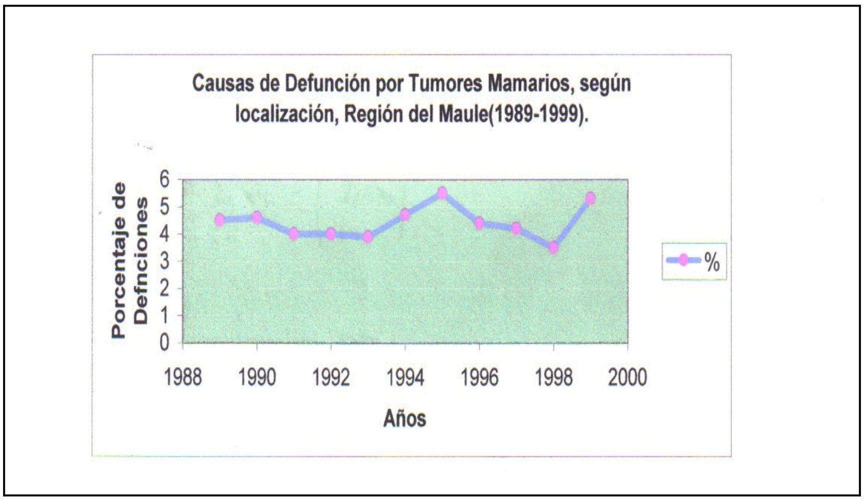
[Marianela: "Well, we are interested in breast cancer research because it is an important problem in our country, and in the region (..........).It was designed as a plan to organize everything in the work (…), and you know that was being studied what exists on the subject, (….) ". We want to create awareness on the subject, and that breast cancer is a public health problem. It is the most common cancer in women (….) It is the leading cause of cancer death in women between 25-45 years . (….…). The objectives of our study were (….) to describe

mathematically the increase or decrease of breast cancer in the region, projecting their behaviour through a mathematical model. (……)"]

It is important to highlight, according to the scheme of analysis, that the students have taken self concern of the problem of study, to give a complete picture of what is expected of their job, especially in its objectives, as for the analysis made from the mathematical point of view.

(2) Mathematizing:. They recognize the mathematical concepts involved and they are completely familiar with the concepts seen in class, they have a proper handling of graphs interpretation, using algorithms and properties involved and evaluated new data in the mastery of the model described.

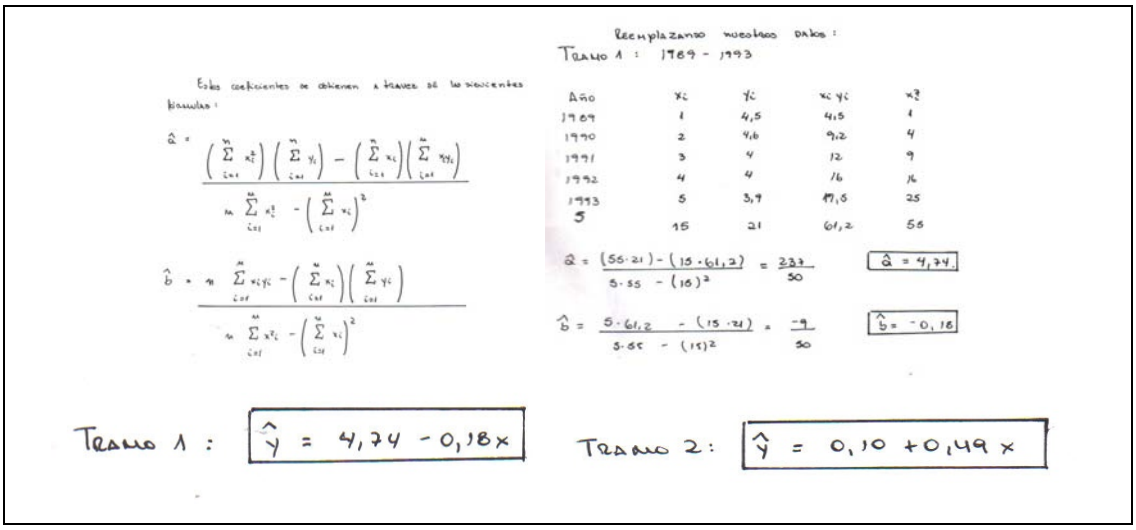
The next episode describes one of the situations that occurred in the exposition:



Picture 2. Graphic produced by

[Ruth. "(… ..) First, I will present the value table and graph, and then I will explain how we developed the model. At observing the figure in the Axis, we can clearly see that is an oscillation between a 3.5 and 5.5 of all deaths caused by tumors, it means that in the year 1989 of 863 deaths, a 4.5% of the tumors was due to breast cancer. In this chart in ax X we can find are the years and in the axis Y we can find the percentages of deaths from breast cancer, in respect of total deaths from tumors. (….) There is a more or less linear growth, in some parts, then, what we did was to take from the year 89 to 93 a linear adjustment. With that, through the method of least squares, or line of the minimum squares of Y on X. And also we did it for years 95 to 98. For the others, we had a linear model (….)"]

It presents the work done in order to describe concepts and processes, calculations and algorithms used. It explains the mathematical work of how each one of the equations was reached, up to arriving to the description of the model. We will show the prediction done by the group through the formulation of the model.



Picture 3. Mathematical Production of the Group.

In Picture 3, mathematical relationships that are established, where *a* and *b* parameters are calculated. Thus, finding the mathematical equation that answers to data from the actual values described in the table.

(3) Communication of Mathematics and Validation. From the model to the conclusions. They present the conclusions of the study, by establishing a relationship between the mathematical data and the actual problem.

[Valeria: (…) according to our mathematical model and the polls we took, we demonstrate our hypothesis: Lack of education and lack of knowledge that exists on this subject, reflected in the mortality rates that have been kety in recent years, with some variations. There is not a tendency to a decrease (…). Although our model does not allow us to predict what will happen in the next few years we should do something in order to reduce mortality breast cancer, which is among the highest in Chile, and it is in second place in the country and sixth place the causes of death in Talca comune.]

The group responded to the objectives and assumptions. They have clarity in the meaning of modeling, to clearly justify the validity of this model as an approximation of reality. They explain the strengths and limitations of the model; things that can not be predicted.

VI. CONCLUSIONS ABOUT THE PROCESSING OF THE PROJECTS.

As far as the category of the cognitive aspects is concerned, the study case groups has had a relevant progress. In addition to the mathematical concepts that they have considered from the advance 1 stage up to the oral exposition. Strengthening concepts and processes involved in their problem of study. It is verified that they do an interpretation of the problem, identifying data and conditions, establishing restrictions, using different systems of representation.

From the mathematization point of view, they recognize mathematical and non mathematical variables that intervene in the problem; they describe mathematical relationships that interpret the process, presenting the formulation of the mathematical model that approaches to the situation. Regarding to ***verification and validation of the model***, they could determine if this fulfills initial conditions, submitting the variables from the model to data of reality, by means of the evaluation of this, with new data of the domain, concluding the validity of this.

Regarding to the category: ***meta-cognitive aspects***, a development of the strategic and creative thought stands out in their behaviour, by facing the problem from different perspectives and by projecting data constantly. In the ***transversal point of view,*** relevant skills are manifested, by recognizing mathematics as a useful process to solve problems of the real world. There is an evident development of the mathematical attitude, of autonomy and of intrapersonal and interpersonal relationships generated by themselves in this type of work. In connection with the performance at work, it is evident that advances are fundamental to verify the progress of the group, as well as for detecting difficulties and opportunities. It is verified that, they learn to plan, to select information, to formulate a research problem, to express ideas both written and orally. Also, the students' learn how to use the mathematical knowledge in different situations.

VII. FINAL CONCLUSIONS AND DIDACTIC IMPLICATIONS.

It is our concern to deal with some points which we think are fundamental to synthesize as a result of the experience, and which have been deeply analyzed in the previous chapter.

7.1. The Importance of the Project Work Approach

We have proven that the modeling work approach not only enables to appreciate the use of these concepts, but also, enables students to discover the meaning of the mathematical concepts. In this case the students think about the problem put forward, discussing in group, and formulate explanations about the concepts that are beyond the solutions. This point is very important for a work based on activity, since it means that the student analyzes the conditions for an adequate performance, monitoring his study habits and self regulation of his own knowledge at the same time.

We would like to outstand in this proposal, that projects are developed outside the classroom. Powering in this way autonomy, also developing strategic thinking and generating a major capacity for exploring the information available. In this way, we coincide with the proposal of Gómez (1998), Aravena, Caamaño & Giménez (2008) and , who state as fundamental that students must learn to recognize the difference between having the information and being able to get to it when it is needed; between knowing techniques and methods, and knowing when, where, how, and why to apply them..

We have proven that a Project work approach performed outside the classroom, where the students themselves carry out the complete research on a theme taken from reality, enables them not only to be able to appreciate the use of mathematics, but also, to develop metacognitive skills, such as: the reflection, creativity, analysis, originality, and strategic thinking ability development (Abrantes, 1994).

7.2 The Importance of the Advance States

(1) Through the analysis of the activity theory, we recognize the importance of the instances and moments we have established for the project work. Especially, the importance of advances is recognized, since it is through these two moments we have been able to monitor group work, recognize weaknesses and strengths in general terms of the investigation and in specific terms of the mathematical concepts as well.

(2) From the pedagogical point of view, the advances are very important, since a closer relationship is created between the teacher and the groups. That motivates a more positive attitude towards learning, a disposition of the group in order to accept orientations, to organize their information, for the decision taking, to think about on what they are doing and recognize weaknesses, strengths and difficulties.

(3) From the mathematical concepts point of view, advances are vital, since one can appreciate in each member the aspects that are put into risk in order to solve a mathematical problem.. In this case it enables the group to take decisions on what concepts were implied in their problem, and it enabled the teacher to detect how concepts were being assimilated, the properties, the language, how they were facing the problems treated in the classes, and also to detect, whether there were study habits or they just left everything for the last minute. In the particular case of the group we have analyzed, we have detected, for instance, an important issue: that the students manage themselves quite well with the contents in oral production. But afterwards, they really have problems in organizing their ideas in order to answer the problem from an analysis of variables implied in the study point of view. That is why we recommend making a little written report for each advance they make. In addition, it is not detected in the reports of the researches that advances are being considered for the work of projects, neither any proposal for regulating this moment.

7.3 Oral Report

Lastly, we underline the importance of the oral report, since it favors communicative skills, originality, responsibility in the exposition, the developing of the strategic thinking, and scene mastery when presenting the theme. The following reflection comes out from the analysis and our own experience:

(1) These encounters are a *reflection space.* Burton (1999) considers it to be vital in the work with modeling problems. It is important from the point of view of the context, new ideas are presented. It is important from the knowledge construction, the research of information is remarked, the generalization of results, the mathematical communication. It is important from the widening of knowledge point of view, since it points out to show how each group tackled the problem, the relationship it established between the mathematical concepts, the procedures, the structure that was given to the mathematical concept, and the systematization of the content.

(2) C*ommunicational spaces*. For the students it is very important, since it enables to see what other groups have done, explaining how they have tackled the problem from other views, generating learning in the others. From the mathematical learning point of view, it allows to internalize contents either in the group itself or in the others. The strategies used enable the other groups to take some ideas and widen their resources. Besides, it enables students to recognize other aspects that they have not been considered, thus enriching their work through the comments of others. *New ideas generation and strengthening of the contents.*

(3) Critical Analysis of the Information. This is an essential point, because at present, there exists an explosion and a big accumulation of information. We must promote a culture of information in students, as it is stated by Alsina (1998), learning through the situations modeling approach develops a critical analysis of it. Regarding this point, the experience that has been developed encourages the use and analysis of the information which enables the different groups of work, especially to the group being analyzed. To analize the available information, looking for and selecting the data, analyzing the data, establishing relationships among the variables and using the knowledge in the formulation of the mathematical modelling approach that answers the situation. All that has enabled the groups to solve the problem from the available knowledge basis coinciding with Alsina (1998), on the idea that the modeling approach provokes an investigative approximation of the problematic under study

(4) Development of Autonomy. Among the main actitudinal skills, we coincide with Abrantes (1994) that through projects, students develop a critical and communicational attitude, they develop their own opinions, back up their

opinions, counterargument with basis. All these manifestations were observed in the group under study.

(5) Mathematical attitude. The valoration towards the use of mathematics is an issue that Alsina (1998), considers as an important endeavor to accomplish in students. From this view point, the mathematical work through the model of situations approach clearly develops a positive attitude towards mathematics in students. From the group under study, we might like to highlight that during the whole experience, students valued the use of modeling as a mathematical process, quite useful for the problem solving. One important point to highlight is the accurate use of the mathematical and technical language they showed in the oral report.

Lastly, we justify the importance of this kind of work, since, as Niss (1989) states, it prepares students for a better insertion in society where, they, as citizens, will have to face problem solving, do some estimations, take some decisions, assessing a real importance for the present mathematical formation. From the methodological, social and cultural point of view, it World be of great importance to apply this experiences in similar situations in Chile as well as in other Latino American countries, so as to verify if the same capacities or skills we have detected in low and extreme socio cultural levels of poverty. Finally, we can conclude that, this type of work is a promising endeavor in order to overcome obstacles and deficiencies in the teaching of mathematics in the Latino American Region to a significant level. In addition, it is convenient and important to test modeling experiences in similar contexts, as well as projects that enable to validate this kind of studies, and present innovating proposals that can answer to the present demands in order to raise the quality of the learning skills in developing countries.

REFERENCES.

Abrantes, P. (1994). *O Trabalho de Projecto e a Relaçao dos Alunos com a Matemática a experiência do Projecto. MAT789.* Tesis doctoral. Lisboa, Portugal.

Alsina, C. (1998). *Neither a microscope nor a telescope, just a mathscope*. Proceed. ICTMA-1997

Aravena, M. (2001). Evaluación de proyectos para un curso de álgebra universitaria. Un estudio basado en la modelización polinómica. Tesis Doctoral no publicada. Universidad de Barcelona, España.

Aravena, M. ; Giménez, J. (2002). Evaluación de procesos de modelización polinómica mediante proyectos. Monografía modelización y matemáticas. Revista UNO. Didáctica de las Matemáticas. 31, 44-56.

Aravena, M.; Caamaño, C. (2007). Modelización matemática con estudiantes de secundaria de la comuna de Talca - Chile. *Revista Estudios Pedagógicos*. *33*, 7-25.

Aravena, M; Caamaño, C. & Giménez (2008): Modelos matemáticos a través de proyectos., *Revista Latinoamericana de Investigación en Matemática Educativa.V11, número 1*

Blomhøj, M. (2000). Developing modelling competence: The different roles of modelling and problem solving. Roskilde University, Denmark.

Blomhøj, M. (2008). Different perspectives on mathematical modelling in mathematics educational research. *Proceedings of the 11th Congreso of Mathematics Education ICME. Monterrey, México.*

Burton, L. (1999). Why is Intuition so important to mathematics but Missing from mathematics Educations. *For the Learning of mathematics*, *19* (3), 27-32

Davydov, V. (1998). The Concept of Development Teaching . *En Journal of Russian and East European Psychology*. July-August, M.E. Sharpe Inc, New York.

Dubinsky, E. (1996). Aplicación de la perspectiva piagetiana a la educación matemática Universitaria. *Revista Educación Matemática 8* (3), 24-41

De Lange, J. (1996). Real problems with real world mathematics. , in Alsina C. et. Al (Eds): *Proceedings of the 8th Congreso of Mathematics Education ICME* (pp. 83-110). Sevilla, España.

Giménez, J. (1997). *Evaluación en Matemáticas. Una Integración de Perspectivas*. Editorial Síntesis. Madrid.

Gómez, J. (1998). *Contribució a l'estudi dels processos de modelització a l'ensenyament/aprenentatge de les matemàtiques a nivel universitari*. Tesis publicada en http://www.tesisenxarxa.net/TDX-0920105-165302/ Departament de Didáctica de la Matemàtica i de les Ciéncies Experimetals. Universitat Autónoma de Barcelona. España.

Gómez, J. (2007). *La Matemática. Reflejo de la realidad. La modelización como herramienta para la enseñanza/aprendizaje de la matemática*. FESPM, Badajoz. España.

Jorba, J., Casellas, E. (1997). *La regulación y la autorregulación de los aprendizajes*. Madrid: Síntesis Ediciones.

Keitel, C. (1993). Implicit Mathematical Models in Social Practice and Explicit Mathematics Teaching by Aplications. En De Lange, J. and Keitel, C. Hunthey, I. Niss, M. (Ed) *Innovations in Maths Education by Medelling an Aplications*. Chichister, Ellis Horwood Limited.

Kline, M. (1994). *El Pensamiento Matemático de la Antigüedad a nuestros días.* (Vols. I, II y III.) Trad. Martínez y otros. Madrid: Alianza Editorial.

Leontiev, A. N. (1993). *"Actividad, Conciencia y Personalidad"*. Editorial Cartago. México.

Matos, J.F; Blum, W.; Houston, S.K. & Carrera, S.P. (1996). *Modelling and mathematics education*. Chichester. Horwoord Publishing

Markarian R. (2000). "¿Por qué los matemáticos estudiamos el Caos".Cubo Matemática Educacional. Universidad de la Frontera. Vol 2. Temuco, Chile.

Ministerio de Educación de Chile. (2004). Prueba SIMCE 2° Medio. Análisis de Resultados. Obtenido en Septiembre 20, de 2006, de http://.www.biblioteca.mineduc.cl. pdf.

Niss, M. (1989). Aim and scope of applications and modelling in mathematics Curricula. In W. Blum et al. (Eds), *Applications and Modelling in Learning and Teaching Mathematics* (pp. 22-32). Norwoord, Chichester:UK.

Niss, M. (1992). Applications and Modeling in school mathematics-directions for Future Development. In I. Wrszup i Steint (Ed) *Developement in school mathematics around the world* V. 3 NCTM. Reston

Niss, M. (2001). Issues and problems of research on the teaching and learning of applications and Modelling. In Matos, J.F. ; Blum,W. ; Houston,S.K. ; Carrera,S.P.(Ed.) *Modelling and mathematics education.* (pp. 73-88) Chichester. Horwoord Publishing.

Oliveras (1996). Etnomatemáticas. Formación de profesores e innovación curricular. Granada: Comares.

Swetz, F. & Hartzler, J.S. (eds.). (1996). *Mathematical modeling in the secondary school curriculum: A resource guide of classroom exercises.* Reston, VA: National Council of Teachers of Mathematics

Williams, H. & Ahmed, A. (1997). Applications, modelling and communication in secondary school mathematics. (Ed) *Modelling and Mathematics Educations*.(pp.11-21) Chichester. Horwood Publications.

CHALLENGES WITH INTERNATIONAL COLLABORATION REGARDING TEACHING OF MATHEMATICAL MODELING

Thomas Lingefjärd
University of Gothenburg, Sweden

Introduction

There is no doubt that there is a strong relation between different teachers' teaching styles and their students' learning in mathematics. Teachers are influenced by their theoretical knowledge from a didactical and subject content perspective, but maybe even more influenced by local, cultural matters. Obviously teaching and learning can not be conducted or organized in exactly the same way around the globe. In fact there are huge differences even among countries in Europe. In November 2007, I and other members in the DQME II project had the opportunity to experience mathematics teaching in a gymnasium in Bratislava, Slovakia. We, the observers, were seated in the back of the classroom. About 15 students came in and sat down at their desks, silent small talk could be heard. When the teacher arrived, they all raised into an almost military "attention", standing silently beside their desk, all staring straight forward. The teacher walked up to her desk and called out: "God morning, students". The students, about 18 years old and studying in a natural science program, answered in chorus: "God morning, teacher!"

The lesson started with carefully, well prepared exercises -- the content were geometrical series, both finite and infinite. The teacher set the agenda and the students immediately reacted when the teachers gave them a sign to step up front and do some calculations at the black board. No discussions among students were heard. The students were allowed to borrow modern calculators from a mutual cabinet, but as far I as could see no student actually used the calculator. The students seemed rather contented in the silent classroom experience we took part of.

Two week lather I visited a "normal" lesson at a Swedish gymnasium, with students in the same age and also in a natural science program. The teacher went in first into the classroom to make sure that everything was in order. Then she opened the door and allowed the students to come in, looking them in their eyes and exchanging some small talk to each and every one when they passed her. When the class was in order after some few minutes, the teacher and the students started a negotiating discussion. What should we discuss today? What did we do the last time we met? Do you think you can handle geometrical series now? (The content was more or less the same as in the school in Bratislava). After some discussions, the teacher clarified some basic concepts with regard to geometrical series before the students started to work on a small project task, organized in small groups. The teacher walked around in the classroom, assisting the groups when they asked for help, but not really teaching. She seemed to be listening more than talking. All students had their own calculator, which they used frequently together with different paper and pencil techniques. The level of noise when the students expressed opinions in the discussions was sometimes very high.

Is it really self evident that I saw the facilitation and growth of the same mathematical knowledge taking place in these two different classrooms? Is it possible to use exactly the same tasks in these two different classroom situations? Would it be possible to just switch the two teachers? Would it be possible to neglect the social behavior of students and teachers when comparing results? Do the students in these two different classrooms have about the same mathematical platform to stand on when they are challenged with a new problem? No student acquires exactly the same toolbox of mathematical tolls as any other student, but it is important to know that students who are expected to tackle the same modeling problem also have about the same preparation. It is likely that the teaching situation in different countries might become even more different when mathematical modeling is the focus of the teaching.

Classroom practice

During the last ten years or so, a rather intensive debate has been going on regarding the relation between teacher's teaching and the learning among students that eventually becomes the result of that teaching. TIMSS video studies has among other things clearly illustrated that teaching traditions are heavily depended on cultural and national background (Straesser, 1998).

Lipings (1999) comparisons of teaching practice in China and USA together with Stiglers & Hieberts book The Teaching Gap (2000) has caused a rather extensive public discussion in the USA regarding teaching practice. The Learner's Perspective Study is examining the patterns of participation in competently-taught eighth grade mathematics classrooms in fifteen countries spread over Europe, Asia and in Australia and in the USA.

There is also a growing concern that of all the research that is done regarding teaching practice, not much actually reaches the classroom floor and the practicing teachers. Therefore it is of utterly importance that collaboration regarding the practice of teaching mathematical modeling involves teachers from the very start and at all levels of the intended collaboration. The classroom is a very complex milieu, almost impossible to interpret, analyze and understand from mainly a distant researcher's perspective. It is a fact that learning always take place in a complex web of contexts – any student is affected by many different factors, depending on internal as well as external circumstances.

> With this theoretical insight in hand, the field is now able to grasp the idea that interaction in any mathematical classroom is not merely the projection of what an

> individual teacher, individual students, or the discipline of mathematics itself brings to the encounter, but rather the joint construction of a viable response to the conditions and constraints that make possible as well as affect their encounter (Herbst & Silver 2007, p. 63).

Regardless of how we for instance interpret and understand the statement, so very ambiguous and full of nuances, *learning of mathematics in a classroom*, we probably would agree that learning itself has not only a cognitive dimension, but obvious also emotional and social dimensions (Illeris, 2004). The social dimension could further be divided into two sub-dimensions: one focused on the spontaneous social interactions, and the other focused on interactions that are structured by an organization. Hopefully we could also agree that there is a distinct difference between spontaneous and often instant learning in the so-called “everyday life” and the more well-defined goal oriented long-term learning in school. Quite different subjects or concepts are learned and different ways of thinking are developed. Learning mathematics in a mathematics course may also be quite different from learning mathematics in the workplace; in financial or engineering institutions, or in technical positions in factories or offices, where it is mostly tacit knowledge that is necessary. Each of these domains has their individual and very specific learning discourses. So any theoretical framework connected to any of these various discourses, would have different aims.

> Theories generated by research *in* mathematics education have various functions. They can have a *descriptive* function, providing a language to frame a way of seeing, and in this sense they affect an ideology. They may offer an explanation of how or why something happened, thus relating what has been observed to the past, whether through statistical correlation, cause-and-effect-analysis, influence, or co-evolving mutuality. They may attempt to *predict* what will happen in similar situations through stating necessary and appropriate conditions (and for this they need to specify what constitutes 'similar' and 'situation'). They may serve to *inform* practice by sharpening or heightening sensitivity to notice and act in future. Which ever of these functions a theory contributes to, it comes from, belongs to, even constitutes, a *weltenschauung*, and communication between different world-views is at best problematic. (Mason and Waywood, 1996, p. 1060, italics in original)

Any classroom in mathematics has its boundaries and constraints, in terms of curriculum, content in text books and number of mathematics hours per week, teacher attitudes, beliefs, and competencies, outer resources, technology, and so forth. The list of frame factors may easily be longer. In a project where we are sharing materials across geographical and cultural borders, it is important to acknowledge that learning always take place in a social and cultural framework.

Sharing teaching materials

To share educational materials, courses or even full programs across national or cultural boundaries is becoming more and more common. One may even say that it is a clear trend at present time to share educational experiences in a global world. A number of factors are related to the growth in institutions offering courses to distant students in other countries, whether to one other institution half way around the world, or to dispersed students in many distant countries, or to their own citizens that are resident in other countries. Some factors are financial, other factors could be technological, and some might be mainly practical. Sometimes researchers and/or teachers enjoy the idea to share educational resources across borders for a variety of reasons. In the European project Developing Quality in Mathematics Educations II it is a main objective to share materials since the project has several common goals across national and cultural borders.

It is in the nature of mankind to compare things and if we share we also start to compare. There has been a huge expansion of major comparisons with focus on the learning of mathematics during the last decade or so; TIMSS, PISA, and so on. In general these studies may be seen as rather high scaled studies with main focus of the learning outcome among the students and not so much on the actual teachers work. One nice example of a relatively high scaled but nevertheless more direct and personal study is the study reported by Ludwig and Xu (2009) in this publication. It seems that it is easier to relate to one or at least just a few problems that the students are supposed to solve and thereby in some sense understand the comparison.

Nevertheless, the teachers are often omitted and unseen. There are good reasons for this. One major problem is to compare the teaching as such, what instruments can possibly be used to do the comparisons between different schools in different countries? A regular international comparison is defined by standardized tests at a certain time and date; school coordinators; test administrators, and national and international monitors. A modeling project is stretched out over time; it includes extensive group work and the use of advanced technology. Our project has no school coordinators, no test administrators, and no national or international monitors.

Nevertheless, as pointed out in the beginning in this article, there are many good reasons why one should also observe and document what teachers do with the students in their class and reality enables us to use what we can; video cameras and notebooks. A small scale comparison study creates reasons for other differences. The teachers in different countries and different schools are involved in both the teaching and the documentation of the event. The researchers in different countries and different schools are involved in both the documentation and in some events also in the teaching. In any such low scaled and low budget endeavor, there are major difficulties when trying to make sure that the different classrooms activities in different countries will be videotaped in about the same way. That is connected to the fact that the researchers and teachers in the different countries do not have exactly the same idea about what it is that is most important to document.

Collaboration takes place

Already in Bratislava in November 2007, some of the researchers and teachers from Germany and I decided that we should collaborate around a teaching experiment in a German school. The class was a class in grade 8, which means that the students are around 15 years old. For a Swedish point of view, it is impossible to be a student in the gymnasium when you are 15, since the Swedish gymnasium is for grades 10 – 12. Before that we have a compulsory school with the same curricula across Sweden. Germany contains at least 16 different curricula and modes of school culture. In addition to that, Germany has three different secondary school forms called Hauptschule, Realschule and Gymnasium, while Sweden in present time has one unified Gymnasium with all different programs covering all sorts of vocational and theoretical programs. In Sweden the pupils enter their 3-year Gymnasium program after 9 years in compulsory school, in Germany pupils are divided into different school forms at the age of ten or eleven. The Gymnasium in Germany is organized over 8 years, and each state has its own mandate to organize it, which has led to a great variation.

Nevertheless, we decided that I should communicate a “typical” task regarding the concept of volume from a Swedish textbook and that the German teacher would see if it was useful for his intentions. In fact the concept of volume is taught in grade 9 in Sweden so I translated a text from a grade 9 book and mailed it to Germany in early December. See Figure 1.

> Despite the rather cool weather during winter, small outdoor swimming pools are popular among private house owners in Sweden. Imagine a swimming pool that is circular with a radius of 2.75 meters and a depth of 1.18 meters. The distance between the water surface and the pool edge is 0.06 meters. Every spring the pool is filled through two water pipes, each of them delivering 20 liters of water per minute. The water cost 2 Euro per cubic meter.
>
> Questions:
>
> a. How much water is here in the swimming pool? Answer in the unit cubic meters.
>
> b. How much does it cost to fill the swimming pool?
>
> c. How long does it take to fill the swimming pool?
>
> d. How many humans should be in the swimming pool at the same time in order for the water to pour over the edge? Find out the average volume for an average person yourself.
>
> From Matte Direkt /Mathematics Directly (Grade 9, 2003, p. 53)
>
> Translation: Thomas Lingefjärd

Figure 1: A mathematical task regarding volume from a Swedish textbook.

Within one week I received an answer from the German teacher that he would be happy use this task, but that he would not use the questions a – d (see Figure 1). Instead he would prefer to organize his students in groups and ask the group members to invent their own questions. See there for a difference in teaching style that probably is more related to personal attitude towards teaching and not so much about cultural difference! The classroom experience during one lesson was filmed and three minutes of the film was cut out and subtitles in English were added, and then this three minute cut was sent to me. At the end of January I was presenting the project and our cross national collaboration at a mathematics education conference in Stockholm where it was very convenient for me to actually show the film as a genuine proof of our collaboration. That short film, and the description of how our collaboration changed the task, seemed to draw much more attention than I had expected. Overall, it tuned out that the challenges and opportunities within this collaborative project were larger than I thought at first hand, but the outcome turned out to be more spectacular than I could imagine. See Stefanie Meier (2009) in this publication for an extensive description of the Swimming pool problem.

The Medicine project

The next endeavor in the DQME II project was that the Danish and Swedish members of the DQME II research group decided to use a mathematical modeling problem from medicine (see Appendix 1 & 2) as their next mutual project. It is a fact that many students seem to appreciate modeling problems from the area of medicine as more real and more interesting than others (Lingefjärd 2006, p. 111). After some negotiating, also the Hungarian and Romanian colleagues in the DQME II project agreed to use the same problem in their schools. One main concern was what the gymnasium (upper secondary) teachers in respectively country would think about the problem? Would they approve to the level of mathematical thinking? Would they approve to the directions of inquiry? Mutual teaching experiments across cultural and national borders require a lot of work and concern some very delicate issues. It is a fact that no one in the research group knows much about the actual teaching that takes place in Danish, Hungarian, Romanian or Swedish classrooms as existing apart from their own countries. The research group agreed about the intention to film the experiments in the classrooms. Would that become an obstacle or an opportunity to run the project even better? What about the actual teaching and learning that will take place? In what way will that effect the facilitation of mathematical thinking in the classrooms?

Research questions:

- What can pass as a good mathematical modeling project in four different countries?

- How will mathematics teachers in upper secondary schools in four different countries actually teach the same mathematical modeling project?
- How will mathematics teachers in upper secondary schools in four different countries appreciate the same mathematical modeling project?

It would be naïve to believe that one just can take a modeling problem and send it to different teachers with different classes and to expect that they would teach it just like that, in the same way. Our first backlash was the message that came from Hungary in the beginning of the project. When I was a student, Hungary was well-known for the situation that many families went for small mathematics contests during the weekends, walking around in the hometown, looking at various mathematical tasks that someone put up on public places, and solving them together family wise. I know from the Hungarian curriculum that in Hungary some students are taught advanced mathematics about functions already in grade 8. The very same content will occur in courses at the gymnasium in Sweden, approximately two years later. So I was surprised of the content in the message from the project member in Hungary.

> *Dear Thomas*
>
> *I have sad news. Unfortunately two of the three teachers who I asked about the Medicine project have stepped back from the project. They told me after long hesitation that the project is too hard for their students.*
>
> *I am sorry but the Medicine problem is very alien for Hungarian pupils since there is nothing about so deep statistics and calculus in Hungarian curriculum. I thought it would be suitable for special mathematics classes but they are currently working hard on more elementary problems and preparing for international Olympiads.*
>
> *Best wishes*

As I understand it, the Hungarian teachers were reluctant to engage their students in an open ended modeling task without prescribed methods since that was uncommon in Hungary. The second problem occurred when we found out that the classes our teachers were teaching were different in age and mathematical preparation in the three remaining countries. The students in Sweden were in grade 11 and 18 years old, while the students in Denmark and Romania were in grade 12 and 19 years old. Therefore one modeling problem finally became two similar but different modeling problems. The first mathematical modeling task is labeled the Asthma problem and the second mathematical modeling task I have labeled the Medicine problem. The reader should know that the findings from the teaching experiments in four different schools in three different countries are based on four different video films and on the author's presence in two of the schools and in the authors reading of student's solutions.

The underlying mathematics in the Asthma and the Medicine project

Both the Asthma and the Medicine project (see Appendix) deals with the dosage of a drug that is expected to be sustained in the body, a rather complicated situation. For a drug to be used safely two important pieces of information are crucial. One is the dosage level that normally will vary according to the required effect and at the same time is bounded by a maximum safe dose. The other important issue is the time interval between doses. Once the required start dose has been ascertained by research and experimentation, the doctor's or the pharmacologist's mission is to instruct the user what the frequency of the maintaining dose must be. In order to solve this problem, we need to consider a model over how the body deals with a drug, i.e.:

(1) The absorption of the drug.

(2) Its excretion via various organs of the body.

In order to develop a model for this situation, many assumptions have to be made. Every drug and every recipient has its particulars, so in order to obtain a model involving the available mathematics I will simplify the real situation a lot and not all mathematics that is involved will be explicitly explained. I recommend a standard book in Calculus in general or perhaps *Motivating A-level Mathematics* from 1986, pp. 45-48 in particular (The Spade Group, 1986). Nevertheless the book we chose and even if we oversimplify the human body as a one-compartment model, two important concepts are at play here, namely *Absorption* and *Drug Clearance*.

Absorption

Human absorption is in general defined as the process of a drug being distributed through the tissues of the body. Normally the process is accomplished rather quickly, because of the following main facts.

a. The estimated blood volume of an average and healthy 70 kg man is around 6 liters while the heart circulates around 5 liters of blood per minute, a measurement of the so called Cardiac Output (se Lingefjard 2002 for an extended discussion of the concept of Cardiac Output). Thus, in slightly more than a minute the entire blood volume in the body has circulated once.
b. Drugs are normally required to be absorbed into the blood as quickly as possible and in order to facilitate fast action drugs are often taken through intravenous injections or orally.

It is important to acknowledge that we also have the fact that the blood is only about 7 per cent of the total body liquid which yields that a complete absorption of a drug must take much longer than the minute intimated above. Right here we sense the problem with a one-compartment model; obviously blood is not just blood. Nevertheless, if we carry on with our model we can conclude that drug absorption in general is relatively fast, especially when compared with the time it takes for the body to clear itself from the drug.

Drug Clearance

Once a drug is absorbed by the body, the process of removing the drug from the body tissues begins. The process of drug clearance is accomplished by a variety of organs in the body, although foremost the kidneys. The renal clearance performed by the kidney releases the drug via the bladder into the urine, and urine tests are often used in order to determine the rate of renal clearance. It is notable that some drugs cannot be excreted by renal clearance at all, while some drugs only can be excreted this way. In order to create a mathematical model over drug clearance I will use well known data from research on renal excretion and build my reasoning on the assumption that the renal clearance rate is proportional to the quantity of the drug in the body tissues. I will further assume that this is true for drug clearance in general. Since the amount of blood plasma flowing past the kidneys is about 0.65 liters per minute, we have a range of 0 – 0.65 liters per minute for the clearance rate.

When a drug is administered its toxicity will immediately start to decline due to clearance and if a constant level is to be maintained, frequent further doses will be required. The problem is thus to administer the drug in such a way that the concentration of the drug in the body of the recipient remains within the required therapeutic range. That, in turn, leads us to the mathematical challenge of finding an equation that describes the concentration of a drug at time t. If we continue with our simplified one-compartment model of the body and call the quantity of a drug in the blood stream and body tissue at the time t for $y(t)$, the discussion earlier will lead us to the following mathematical expression:

$$\frac{dy}{dt} \propto y \qquad \text{i.e.} \qquad \frac{dy}{dt} = -ky \qquad (1)$$

where k is a positive constant. Standard procedures for differential equations yield the general solution:

$$y = y_0 e^{-kt} \qquad (2)$$

Here y_0 is the initial dose of the drug given at time $t = 0$. In the Asthma project (see Appendix for details) we want to maintain a certain level of the drug concentration in the blood. In order to do so, assume that we at a time $t = T$ want to give an identical dose of the given drug, and then consequentially at times $2T$, $3T$, etcetera. I chose to denote the time just before doses are given as T^-, $2T^-$, $3T^-$ and the instants just after doses are given as T^+, $2T^+$, $3T^+$, etcetera. The sequence of quantities of the drug in the bloodstream at times $t = 0$, T^-, T^+, $2T^-$, $2T^+$, etcetera thus are:

$$
\begin{aligned}
&y(0) = y_0 \\
&y(T^-) = y_0 e^{-kT} && \text{(from equation (2))} \\
&y(T^+) = y_0 + y_0 e^{-kT} && \text{(with dose of } y_0 \text{ given)} \\
&\quad = y_0(1 + e^{-kT}) \\
&y(2T^-) = y_0(1 + e^{-kT})e^{-kT} \\
&y(2T^+) = y_0 + y_0(1 + e^{-kT})e^{-kT} && \text{(another dose of } y_0 \text{ given)} \\
&\quad = y_0(1 + e^{-kT} + e^{-2kT})
\end{aligned}
$$

Thus, a continuation of this procedure yields the result that at the time $t = nT^{+}$ we will have

$$y(nT^{+}) = y_0(1 + e^{-kT} + e^{-2kT} + \ldots + e^{-nkT}) \qquad (3)$$

What we have here is partly a geometric progression with a first term of 1 and a common ratio of e^{-kT}. The sum of such a geometrical progression is

(4) $\dfrac{1-e^{-(n+1)kT}}{1-e^{-kT}}$. Hence we have $y(nT^{+}) = \dfrac{y_0\left(1-e^{-(n+1)kT}\right)}{1-e^{-kT}}$

If the expected drug dose will be administered many times, then as n becomes very large, it is obvious that $e^{-(n+1)kT} \to 0$.

Thus we have that $y(nT^{+}) \to \dfrac{y_0}{1-e^{-kT}}$

The right part of this expression is a constant and represents the maximum quantity of the drug which can be accumulated in the body from a succession of doses of quantity y_0 at intervals of time T. This amount is in general called the saturation level since the ratio can not exceeds this level and consequently we denote the level y_s. Now we have the possibility that for a given saturation level of the drug in the bloodstream, we can choose either the dosage or the period between the doses respectively and calculate the other. What we have left is to find a measurement of the constant k, pretty much analogous to the decay constant in radioactive decay.

From equation (2) we have that $y / y_0 = e^{-kT}$, so that $\ln(y / y_0) = -kt$. If we let $t = 0$ when $y = y_0$, and $t = \tau$ when $y = ½ y_0$; then at $t = \tau$ we get that $\ln(½ y_0 / y_0) = -k\tau$. Simplifying this equation yields

$$\tau = -\frac{\ln(1/2)}{k} \quad \Leftrightarrow \quad k = \frac{\ln 2}{\tau} \qquad \text{(equation 5)}$$

This yields a new concept in our model, namely τ which is the time required for the quantity of the drug to reduce from y to $y_0/2$ when no further drug is administered and, as can be seen from equation 5 above, is a constant. In this setting, the constant τ is called the half-time of elimination. In radioactivity decay it is in general called the half-life of an element. So in order to find k in equation 5 we need to find τ which in general is done experimentally.

Comparing methods

In practice, the method of administering a fixed dose at placed intervals of time leads to a very slow build-up to the saturation level, the level we denoted y_s. One way to solve this problem is to start with one large dose and then proceed with smaller doses. One natural and obvious choice for this large dose is the y_s, and then the next dose y_d brings the level in the blood up to y_s again so we have

$y_s = y(T^+) = y_s\, e^{-kt} + y_d \leftrightarrow y_d = y_s\,(1 - e^{-kt})$. Thus $y_d = y_0$ since we know that $y_s = y_0/(1 - e^{-kt})$.

I have tried to explain that there is mathematics involved here that guides us to a significant descriptive plan for medication. Obviously a saturation level of y_s can be achieved by an initial dose of y_s and then successive doses of y_0. This method of administering a drug and the previous method with giving successive equal doses of y_0 are illustrated in figure 2. Clearly the second method is an improvement for the receptor and the involved body mechanisms.

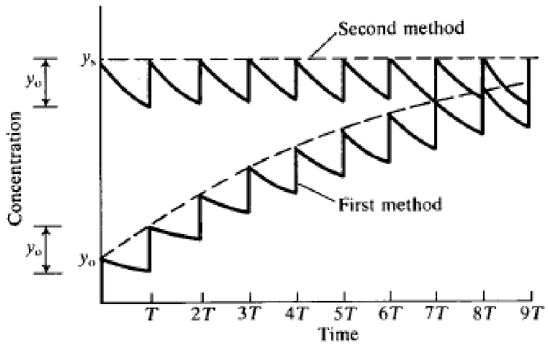


Figure 2: A visualization of the difference between the two methods (The Spode group, 1986, p. 48)

We can use these mathematical models to answer the questions asked to the students in the project. From the table with measured values in the task sheet, we can see that the half time of elimination for the drug is 4 hours. The level to search for is 5 - 15 mg/L. Let us say that we do not want the receptor of this treatment to wake up too frequently to take hers or his medicine, so we settle for a six hour treatment to start with (equal to 360 minutes).

The first dose to be administered should be 6 ·15 = 90 mg, and then we need to calculate y_0 which is the continuing dose. Recalling the symbolism earlier in the paper, it seems that we know τ, y_s, and T. First we determine k via equation 5.

$k = \frac{\ln 2}{\tau} = \frac{\ln 2}{240}$. From equation 4 we get $y_s \le \frac{y_0}{(1 - e^{-kt})}$. Hence $y_0 \approx y_s\left(1 - e^{-kt}\right)$.

With $k = \ln 2/(240)$; $y_s = 15$; $t = 360$ minutes, we get $y_0 \approx 90\left(1 - e^{-\frac{\ln 2}{240}\cdot 360}\right) \approx 58.2$.

We divide by 6 in order to get the value for mg/L and se that the value 9.7 mg/L is close to the bottom of the advised interval at 5 – 15 mg/L and the receptor will experience a huge imbalance of theophylline in hers or his tissues. Is the only solution to this is a more frequent medication plan? The most important question here is however how the students and their teacher will treat this complicated problem. Not all students involved in the project have studied mathematics up to this level. So how will the teachers in the different classrooms handle that situation?

The Medicine problem

What we have done so far relates chiefly to the view of the body as a one-compartment model. The body is a very complex system and a drug undergoes many steps as it is being absorbed, distributed through the body, metabolized or excreted. Even a one-compartment model is far more complicated than we have acknowledged. So called Multi-compartment models have during the last decades logically been widely applied to more complex systems in biology, ecology, engineering, and medicine, just to mention a few areas, whereas material is transferred in complex ways and over time between compartments. One major difficulty with Multi-compartment models is that they quickly become very complex and difficult to handle. Figure 3 is an attempt to describe this complexity in a four-compartment model:

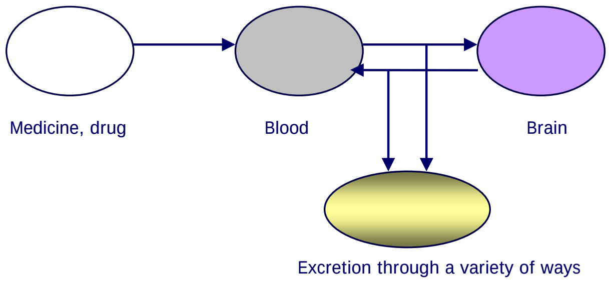


Figure 3: An attempt to visualize the complexity in a four-compartment model

It is easy to justify a multiple compartment model by raising questions like how to accurately describe the absorption through the wall of intestine into the blood or if the medicine will distribute itself into one or several compartments? And how do we really describe the elimination of the medication?

Remarks on the Asthma problem

The material for analyze consists of video documentation of students solving the Asthma problem in two gymnasiums in Sweden and of students solving the medicine problem in Denmark and Romania. There also exist student's solutions in the countries respectively native language. It is interesting that the two

Swedish gymnasiums (in Gothenburg) only are about 5 kilometers from each other in a geographical sense, but the way the two Swedish teachers handled the Asthma project were quite different.

One of the Swedish teachers gave her students the Asthma problem and allotted them a couple of weeks to work on the various tasks beside the regular teaching in class, and the I was invited to come and film the presentations by the different groups in the class. Her students were organized in groups of five or six students in every group and almost all students in her class showed up. The whole event was taking place within one lesson in April 2008 (45 minutes) and was regarded as an important mathematical application within the course the students were studying: Mathematics D. The students also gave me group wise written solutions afterward, and it seems evident both from the presentations and from their written solutions that they had worked carefully and mathematically satisfactory on the problem.

The other Swedish teacher organized a workshop between 08.30 and 12 on a day in June 2008 when the school spring semester actually was over. His students were studying mathematics D over two semesters, spring and fall, and were asked to come in to school for an extra enterprise they would benefit from. The students did not know anything about the problem when they came and had no chance to really select groups. The teacher organized them in groups of two and two (and one group with three students) from how they were seated in class. Altogether seven students showed up at 08.30, several other students came around 10.00 but they had no chance to really accomplish anything with such a short time to work on the task and consequently refused to present anything. For me it was quite a surprise that the students who came in at 08.30 actually could accomplish anything, with such a short notice and when being forced to work under the presence (and pressure) of a camera. One group gave me a report of the conclusions they had reached during that morning. The most striking observation though, is from the film and shows how one of the students at IHGR is unsatisfied with a one-compartment model and argues for other parameters to be part of the model. She asks for parameters such as heart-minute volume, age, weight, and the general condition of the receptor to be part of the model.

Remarks on the Medicine problem

The Danish team decided to focus the teaching experiment on creating a situation where the students could experience that the same data can be describe and explained "equally well" by different mathematical models. The idea was that the students should decide on which model to use for a particular purpose after that different groups of students have set up models of different complexity and estimated parameters for their model to fit the data best possible. In the Danish experiment the point of departure for the students' modeling work was authentic data for the concentration of a psychopharmacological drug. The experiment was documented by video film of one class during three different occasions, namely January 3, January 7 and January 14 (2008). There are altogether ten different video cuts from these three occasions. It seems as if the first session consists of three lessons, the second session consists of four lessons, while the final session consists of three lessons. Each lesson is 45 minuets long. Although I not can follow the Danish easily, it seems obvious to me that the teacher in Denmark carefully introduces and tries to problematize the Medicine project for the students during the session at January 3. The teacher teaches the modeling process to the students and among other things, he introduces the concept of compartment models in relation to the Medicine project. The students were also encouraged to ask questions during that session. The students are mainly scattered around the computers in what seems to be a computer laboratory, although the film cuts sometimes shows an interview with just a few students where they can explain how they think about the Medicine project.

The next session, at January 7, starts with the teacher who asks the students if there are any new questions since the last time. The teacher emphasizes that the task at hand is to select the suitable model out of several different models. Then the researcher also becomes more and more involved in the students work, asking probing questions and giving support to several groups. The fact that the researcher takes active part in the modeling process the students are going through unfortunately also affects the quality of the filming in a negative way. It is hard to be a film producer, a researcher, and an involved teacher at the same time.

The final session, January 14, is a session totally devoted to group wise presentations of solutions the different tasks within the Medicine project. The students show over head images and write on the black board. Other students are encouraged to ask questions and some interesting discussions seem to take place. My Danish is poor but my Danish research colleagues have ensured me that in last session the students' actually presented and discussed two different models – a one and a two compartment model - with different parameters. In the closing discussion it became clear that it is possible fit the pharmacological data with different models, that testing the models against new data is imperial and that a more complex model does not need to better for at particular purpose than a simple model. The Danish films will be subtitled as part of the DQME-II project.

The Romanian film is subtitled, especially in the beginning. It starts with explaining that the students at the gymnasium were reinforced with students from the university, the students were organized in four different groups and each group had to work with their own compartment model compared to the others. Each group consisted of both university and gymnasium students. The Romanian teacher group also consisted of one gymnasium teacher and one university teacher in mathematics. It seems as if they decided to teach mathematics before the modeling activities took place, the mathematician from the university taught the concepts he considered important to have learnt when solving the Medicine problem. They also had to teach the students about how to handle a graphing calculator or Excel, since Romanian mathematics at the gymnasium or at the university not include the use of technology. Further more, the techniques attached to curve fitting or regression analysis is not known by the students in Romania. Just as in Denmark, the main activities by the students were to maneuver in Excel and to explore how different numbers affected the model compared to the data.

A look back at the research question I put forward in the beginning of this paper might be useful.

- It seems as if the Asthma project will pass as a good mathematical modeling project in the two Swedish gymnasiums.
- Even with the same problem in Sweden and in Denmark and Romania we can conclude that the two Swedish teachers and the Danish teacher compared to the Romanian teachers all taught the Asthma and the Medicine project quite differently.
- It is hard to exactly define what the students in the three countries learned, but at the same time it is evident from the video films that they did learn something. Hopefully the teachers will bear that in mind when grading her or his students.
- I have been looking through the four video films and I have read through the solutions from students that I have and it seems to me that the students at Kita’s in Gothenburg did the most mathematical modeling, while maybe the Danish students experienced more modeling competencies. Yet, another observer might reach another conclusion.

Here are the comments from the different teachers regarding the Asthma project in their teaching:

Mikael: The students were generally positive working with the Asthma. They seemed to appreciate the fact that this was “a real” problem, a problem they could actually see the use of trying to solve. Examples of negative aspects that were brought up was that they were given to little time to work on the problem and that they didn't understand that this was actually a part of the mathematics course (which would have made them take the task more seriously). It seemed that the few negative comments were usually related to lack of proper communication regarding the importance and the seriousness of the problem than the actual problem itself.

One difficulty for me, as a teacher, when working with this kind of problems, is to know how much help I should to give to my students. A certain amount of help is necessary to deepen their understanding of the problem. But I also believe it is very easy to "give away to much help" which results in that it is more the teacher than the student that solves the problem. I've experienced that I need to decide for myself if I want the students to arrive at a "mathematically correct solution" or if I am more focused towards trying to develop their problem solving capabilities. These two perspectives open up for two very different types of methodologies when working with the cases.

Tatiana: It feels rather important to choose the right time for working with new problems or in a new manner. To solve the Asthma problem was not easy at all for my students. It was exiting to see the several different suggestions to solutions and the many different constraints which the students acknowledged such as: not wakening the patient at night, not expose the patient against to large fluctuations in medication, and so forth. The students we surprisingly enthusiastic and they appreciated that data was authentic; they asked me a lot about measuring methods and about the accuracy of the measuring instruments that were used. It is my opinion that they worked much more seriously with the Asthma problem than with nay of the problems they find in the text book. And when the students finally got a good grip around the large amount of mathematical concepts that actually play a part in the Asthma problem, then they really acknowledge deep Aha feelings.

I, as teacher, consider the lessons within the DQME II project as rewarding. To solve the Asthma problem has been difficult for many students, but it has also given many students the possibility to surpass their own expectations. That is perhaps the most beneficial with using this kind of working method.

Peter: The translation from differential equations to spreadsheet layout caused the students some trouble, which we found was due to conceptual difficulties concerning the concept “rate of change” rather than technical problems with the spreadsheet “language”. The oral presentations were repeated during a seminar open to the public (“The Day of Research”), and were followed by a good discussion between the students and the audience (upper secondary teachers, university teachers, upper secondary students). The discussion showed that the students really had considered details of the situation where two different models can fit the same data. The authentic data did in fact enhance the motivation, which was notably higher than normal. This project did not train the early part of the modeling process (translation from “the real situation” to “mathematical model”), nor did it focus upon the use of modeling as a means of learning mathematical concepts, which therefore might be worthwhile to dwell more upon in a future project.

Final remarks

It is extremely beneficial as a researcher to have this opportunity to actually be part of what takes place in the class room where I do my video documentation. It is also exceptionally beneficial to be allowed to see how different teachers handle the same problem quite differently and in harmony with what they consider possible to achieve in their class. In one of the Swedish classes the students used graphing calculators as the main technology but mainly worked with paper and pencil techniques, nevertheless they got involved in quite advanced mathematical discussions. In the other Swedish class they chiefly used a computer based tool by name Graphmatica and although restricted by a tight time frame they also did a nice mathematical work.

The Danish teaching experiment were much more ambitious with ten video recorded lesson over the occasions, the class was also more mathematically matured and could therefore use more sophisticated tools in their modeling process and reach more throughout results. Nevertheless, it is important to acknowledge that none of the results in the four classes could answer to the request from two girls at IHGR: where are the specifics for each receptor in the medication schema? What doctor could simply neglect age, sex, general conditions, and such factors?

The main technological tool in the Danish class room was Excel, although it is evident that all Danish and Swedish students have access to advanced handheld technology in their calculators. Excel was also used in the class of Romania, although their teacher told me that they never use technology when solving mathematical problems. As a consequence, it seems from the Romanian video film as if the Romanian students had to learn more facts about technology before they could start the modeling process. Nevertheless, it is a diligently conclusion that all students in these four classes learned a lot of mathematics in connection to their work with the Asthma and the Medicine problem all though not necessary exactly the same mathematics. That is something we must accept and be happy with when working with comparisons of the teaching and learning of mathematical modeling over national and cultural borders. Now there is in fact that we from table 1 know that if the half-time of elimination of the Asthma medicine is four hours and if we by some way observe the level of 15 mg/l at T = 0 within the receptor. If so, well then only 7.5 mg/L will remain after 4 hours and only 3.75 mg/L remains after 8 hours and since we were asked to sustain the lever above 5 mg/L, we will probably end up at a six hour medication schema after all. But it would be hard for the students to maintain such a cool common sense with all this modeling going on, wouldn’t it?

References

Bennet, N. (1999). *Research on Teaching-Learning Processes. Theory into practice: Practice into Theory*. Paper presented at the Earli conference in Gothenburg 24-28 Aug.

Carlsson, S., Öberg, B., & Hake K. B. (2003). Matte Direkt [Mathematics Directly]. Stockholm: Bonnier Publishing.

Herbst, P. & Silver, E. (2007). Method. In F. Lester (Ed.). The second handbook on research in mathematics education. Chapter 2: pp. 39 – 67. Reston, VA: National Council of Teachers of Mathematics.

Illeris, K. 2004. *Learning in working life*. Roskilde University Press.

Liping, M. (1999). *Knowing and Teaching elementary mathematics*. Mahwah, NJ: Lawrence Erlbaum.

Ludwig, M. & Xu, B. (2008). *A comparative study on mathematical modelling competences with German and Chinese students.* Paper submitted to the TSG 21 at ICME 11.

Lingefjärd, T. (2002). Mathematical modeling for preservice teachers: A problem from anesthesiology. *The International Journal of Computers for Mathematical Learning* 7(2), pp. 117–143.

Lingefjärd, T (2006). Faces of mathematical modeling. *Zentralblatt für Didaktik der Mathematik*,. Volume 38, Number 2, pp. 96-112.

Mason, J. & Waywood, A. (1996). The Role of Theory in Mathematics Education and Research. In Bishop A. J., Clements, K., Keitel, C., Kilpatrick, J. & Laborde, C. (Eds.), International Handbook of Mathematics Education (pp. 1055-1089). Dordrecht: Kluwer Academic Publishers.

Meier, S. (2008). *Mathematical modelling in a European context – A European network*. Paper submitted to the TSG 21 at ICME 11.

Stigler & Hiebert (2000). *The teaching gap*. New York: The Free Press.

Straesser, R. (1998). The TIMSS Video Study: Teaching mathematics differently*? EMS Newsletter (28),* 23-25.

The Spode Group. (1986). Motivating A-Level Mathematics. Oxford: Oxford Science Publications.

Appendix 1

Humans that suffer from Asthma are often treated with the medicine theophylline. Theophylline, also known as dimethylxanthine, is a methylxanthine drug used in therapy for respiratory diseases such as asthma under a variety of brand names. Patients are often treated with an equally large dose, D mg, over equally large time intervals, T hours. A doctor has measured how the concentration of theophylline in the blood of one patient varies after the patient has being injected with a dose of 60 mg. The results from this measurement are the basic data for the project. See table 1.

Table 1: Accumulation of theophylline in the blood

Time (hours)	Concentration mg/liter (mg/L)
0	10.0
2	7.0
4	5.0
6	3.5
8	2.5
10	1.9
12	1.3
14	0.9
16	0.6
18	0.5

The students were asked to construct a mathematical model for the situation and to write a report to the doctor that addressed the following questions:

1. How will the concentration of theophylline in the blood decrease over time?
2. How can we plan a continuously medication schema with a fixed dose D over a fixed time interval T, so that the concentration after a couple of injections is in the interval 5-15 mg/L?
3. How can we plan a continuously medication schema with a start dose and thereafter a fixed dose D over fixed time interval T, so that the concentration **directly** will be within the interval 5-15 mg/L?
4. What considerations must be taken into account before one use this medication plan for a patient?

Appendix 2

The students were given two sets of data (table 2 & 3), consisting of observations of the concentration in a person after oral intake of medication ”X” and ”Y”. Both medications are anti-depressive agents in pills. Several mathematical models can be used to fit the data. In connection to models based on differential equations the student should consider:

1) How to describe the absorption through the wall of intestine into the blood?

2) Will the medicine distribute itself into one or several compartments?

3) How to describe the elimination of the medication?

A common model of the rate of absorption in oral medication can be written in the form: $dx/dt = k_1 \cdot e^{(-k_2 \cdot t)}$, where x refers to the amount of drug. You will get a good fit to data-sample 1 by combining this absorption model with a one-compartment model. In the same way you will get a good fit to data-sample 2 by using a two-compartment model. Try this out at first and experiment further. Good Luck!

Table 2

Typical 1-compartment

TIME (h)	CONC (ng/mL)
0	0
1	6,54
2	12,5
3	17,1
4	27,2
6	27,8
8	26,9
12	25
24	19,6
36	21,4
48	16,9
72	15,5
96	10,8
120	8,66
168	6,15
216	4,9

Table 3

Typical 2-compartment

TIME (h)	CONC (ng/mL)
0	0
1	3,99
2	13,8
3	19,4
4	19,6
6	20,9
8	16,2
12	14
24	8,87
36	5,9
48	4,58
72	2,42
96	1,39
120	0,779

Specific details concerning the relations between the compartments in the multiple compartment models were given in terms of differential equations and the students were explicitly asked to find the parameters by doing experimental work in the spreadsheet Excel.

Example from the four-compartment model task (the figure is excluded here):

We want to follow the concentration, $y(t)$, of the medication in the blood [ng/ml] as a function of time, t [h], elapsed after the intake of the pill. The pill, the blood and the brain are considered to be compartments, each containing well mixed material. Compartments are represented by boxes and the connections between the compartments are represented by arrows. Every compartment (that is every box) has a number of connections leading to the box (inflows) and a number of arrows leading from the box (outflows). Material (medication) can either flow from one compartment to another, it can be added from the outside through a source, or it can be removed through a drain or a sink. We wish to account for the medication in each compartment by the concentration (p, y and z).

The rate of in/outflow depends on the concentration of the medication already present in the compartments and the flow-constants k_0, k_1, k_2, k_3, k_4, k_5 and k_6 as can be seen in the equations (1) and (2). (Please notice that $k_5 = k_1 - k_3$ and $k_6 = k_2 - k_4$)

:

p : the concentration of medication in the stomach/lower gastrointestinal tract (the pill) [ng/ml]
y : the concentration of medication in the blood [ng/ml]
z : the concentration of medication in the brain [ng/ml]

(1) $y' = k_0 \cdot p - k_1 y + k_4 z$

(2) $z' = k_3 y - k_2 z$

Task: Set up a spreadsheet with cells for $p(0)$, $y(0)$, $z(0)$, , k_0, k_1, k_2, k_3, and k_4 and columns for the relevant variables (t, p, y and z) and a column for the observed data. Change the values of the constants k_0, k_1, k_2, k_3, k_4 and the initial size of the pill, $p(0)$, until the calculated values fits the observed data. Observe that $y(0) = 0$ and $z(0) = 0$ (there is no medication in blood and brain when the treatment start). Do you consider the resultant model to be a good model? Explain your case in an oral presentation.

A Comparative Study on Mathematical Modelling Competences with German and Chinese Students

Matthias Ludwig, Department of Mathematics, University of Education Weingarten, Germany, ludwig@ph-weingarten.de
Binyan Xu, Institute of Curriculum and Instruction, East China Normal University, China,xubinyan650@hotmail.com

Abstract: During the last ten years both in Germany and in China mathematic education focuses on promoting students to develop their mathematic modelling competences. In our study we have investigated modelling competence level of students from grade 9 to 11 (15- 17 years old) while dealing with the special real situation "peeling a pineapple". This study analyzed which level related to the modelling cycle the German and Chinese students have reached, and if there are some gender differences during doing mathematic modelling process. Moreover this study explored the mathematic modelling competences related to different grades, and some diversity between German and Chinese students in relation to the modelling competences.

1. Introduction

During the last ten years it has been a central aim for mathematic education to help students to develop abilities to recognize the relation between the real world and mathematics, and to identify and understand the role that mathematics plays in the world. (KMK, 2003; YZZ, 2003, NCTM 2006) Such abilities are related to the development of mathematic modelling competences, such as analyzing, assimilating, interpreting and validating a problem. In our study we have investigated modelling competencies activated by student when dealing with an authentic problem from the real world. (Maass, 2006)

2. Theoretical Framework

Following several research works "mathematical modelling competencies" have been defined as "the ability to identify relevant questions, variables, relations or assumptions in a given real world situation, to translate these into mathematics and to interpret and validate the solution of the resulting mathematical problem in relation to the given situation."(Werner Blum, Peter L. Galbraith, Hans-Wolfgang Henn & Morgens Niss, 2007, p.12) Similarly, Blum (2002) defined modelling competence as the ability to structure, mathematize, interpret and solve problems and, in addition, the ability to analyse or compare models by investigating the assumptions being made, checking properties and scope of models etc. In our study we have divided mathematical modelling competences into different levels as follows:

- Level 0: The student has not understood the situation and is not able to do sketch or write anything concrete about the problem.
- Level 1: The student only understands the given real situation, but is not able to structure and simplify the situation or cannot find connection to any mathematic ideas.
- Level 2: After investigating the given real situation, the student finds a real model through structuring and simplifying, but does not know how to transfer this into a mathematical problem (word problem).
- Level 3: The student is able to not only find a real model, but also translate it into a proper mathematic problem, but cannot work with it clearly in the mathematic world.
- Level 4: The student is able to pick up a mathematic problem from the real situation, work with this mathematic problem in mathematic world, and have results.
- Level 5: The student is able to experience the mathematic modelling process and validate the solution of a mathematic problem in relation to the given situation.

Based on a concrete problem situation we will show different actions in relation to different levels. The levels defined above are basing on the traditional modelling cycle which is used e.g. by Blum and Leiß (Blum/Leiß, 2005). We can link the five levels above with the steps in this modelling cycle. Level 0 corresponds the Situation before step1(Understandig the task). Level 1 is between Step one and two. Level 2 corresponds to Step 2. Level 3 corresponds to the mathematical modell. Level 4 is comparable to the "mathematical result" and level 5 means that the student has make a full modelling cycle comparable to Step 6.

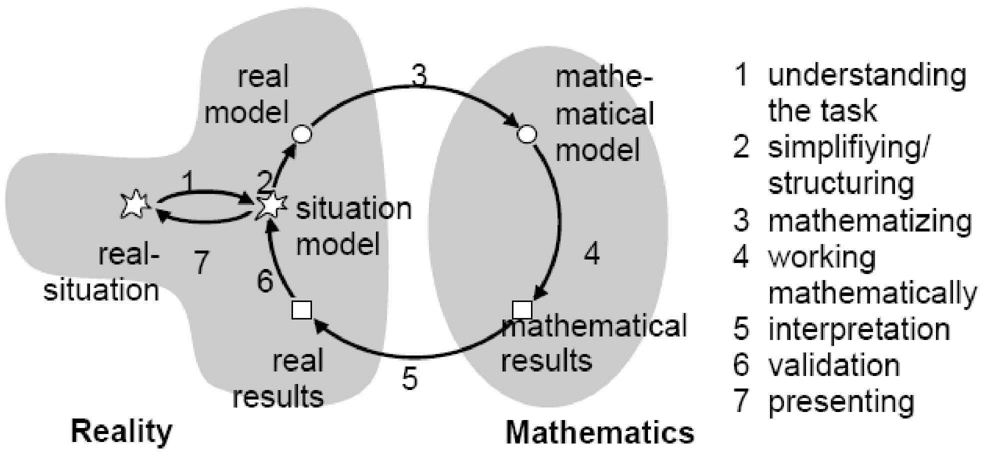


Fig.1 Taken from Blum/ Leiß 2005

Of course we know by the reserach of Borromeo-Ferri (Borromeo-Ferri, 2006), that most of the students do not pass the circle consecutive, but they jump e.g. directly from the real situation to the mathematical model. But we assume, that the farther you will end in the modelling cycle you have to negotiate more cognitive obstacles.

3. A Comparative Study

3.1 Study Design

A Real Situation: We give the following real situation (peeling a pineapple):

Every April is season for pineapples. While we buy a pineapple, sometimes the salesman will peel it for us, especially in China. This is an artistic peeling process, after peeling very nice spirals will leave behind.

Please think about it, why does the salesman peel the pineapple in this way? Is it easy for peeling? Or do they aovid lossing too much pulp? Or are there some other reasons? Please explain it mathematically.

Testee Students: We have chosen more than 1,000 students respectively from grade 9 to grade 11 in Germany and China. The distribution of the testee students is as follows:

(Total Number)		Grade 9	Grade 10	Grade 11
German testee students (from South Germany)	Female	64	76	53
	Male	81	71	83
Chinese testee students (from Shanghai)	Female	103	124	106
	Male	103	129	115

3.2 Study Implementation

The testee students have been shown a 90-second video about peeling a pineapple. Then they received a work sheet with explanation of peeling pineapples and some questions like:

"Please think about it mathematically, why does the salesman peel the pineapple in this way?"

The students could work for 25 minutes to 35 minutes. At the end of the lesson the students should present their results to their classmates.

3.3 Result Analyzing

According to level criterion on mathematic modelling competence, the students' solutions have been carefully rated, classified and registered. Using some statistical tools the classified information has been analyzed in response to the following research questions:

> Which level related to the modelling cycle the German and Chinese students have reached? Does the mathematic modelling competence relate to different grades? Are there some gender differences when doing mathematic modelling process? Are there some diversities between German and Chinese students in relation to the modelling competences?

4. Mathematics behind the problem

4.1. A first solution

Before we present some students solution which are correspondent to our levels we want to show one possible mathematical modelling of pineapple peeling. The question was why the fruit seller peels the pineapple in this special way. A first approach is to consider the pineapple as a cylinder. The black dots are very regular, see fig. 2 left. You can see the spirals in a good way. The cylinder on the right side looks nearly like the peeled pineapple.

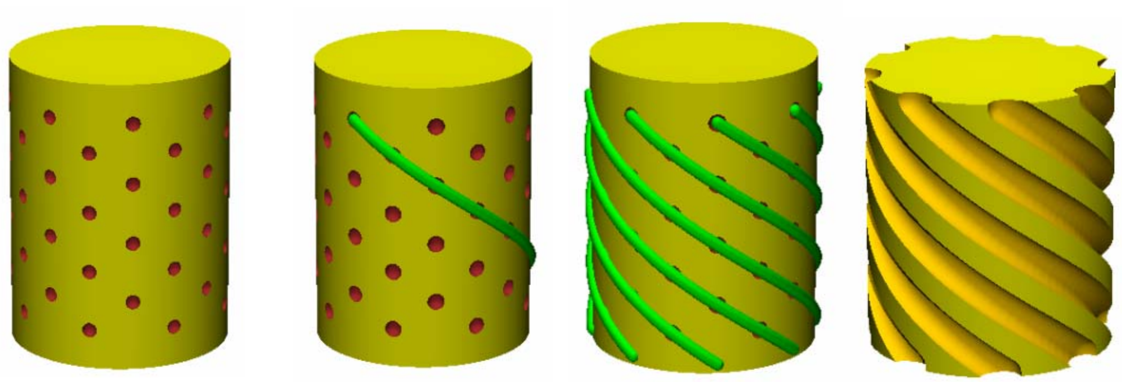


Fig. 2 the developement of the mathematical model

In figure 2 we have transformed the real situation into a mathematical situation. Our next step is only mathematically: we spread the curved wall to a rectangle. And now we see that the spirals transformed into straight lines. We know that a straight line is the shortes conjunction between two points. But we have to make clear while in this case the diagonal lines are the optimum.We assume that the dots are building a square grid. So we can calculate that the length of the diagonal d is $\sqrt{2}$ s. This means that the length of the peeling way is more than 40% longer if you take the vertical way or the horizontal way instead the diagonal way. (see Figure 3, left)

4.2. A students solution

Here we want to show you two students solutions the left one is related to level 4 and the right one to level Zero. The student is able to translate the real model into a mathematical model. He found out that the diagonal way is longer and he calculated this and got a mathematical result but finally there is no feedback to the real world. So the student misses level 5.

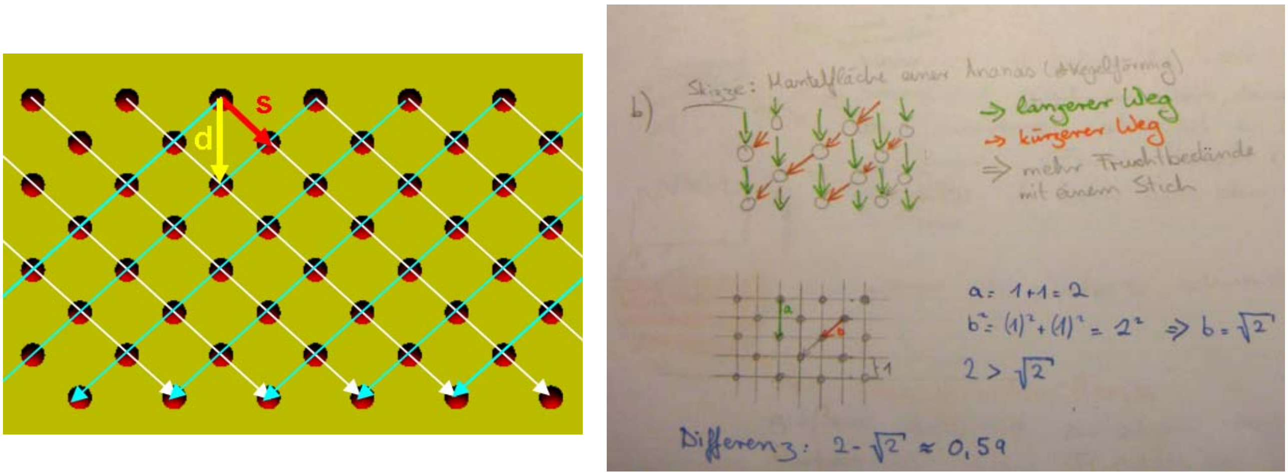


Fig.3: The grid made on the spread wall (left) , a students solution corresponds to level 4 (right)

5. Discussion

5.1. First results

We found some interesting results. First the general performance of german and chinese students is nearly the same, but for chinese students, the development of mathematic modelling competence depends strongly on different grades. From grade 9 to grade 11 the students have developed their competences better and better, while among German students there have not been obvious differences between grade 9 and 10, but in grade 11 there is a big leap. In addition we have found gender differences especially for chinese students when dealing with mathematic modelling problem. For German students we found that the boys will develop their mathematic modelling abilities very strong in the 11th grade. (see table 1)

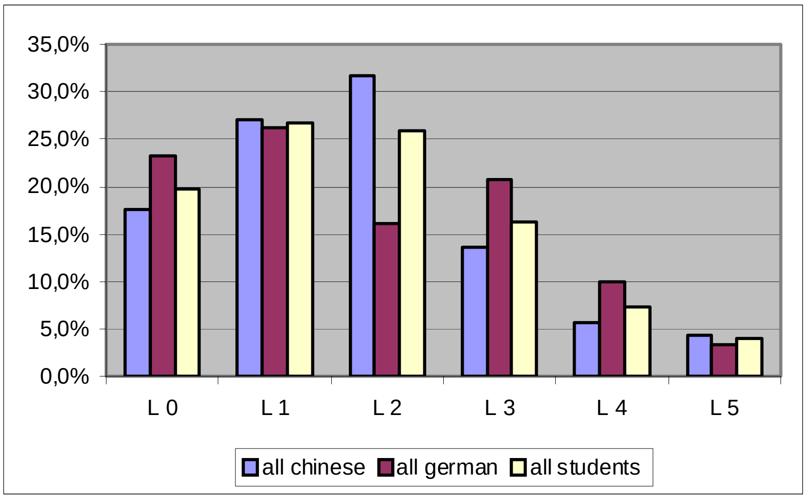


Fig. 4a: Percentage of students who reach level x

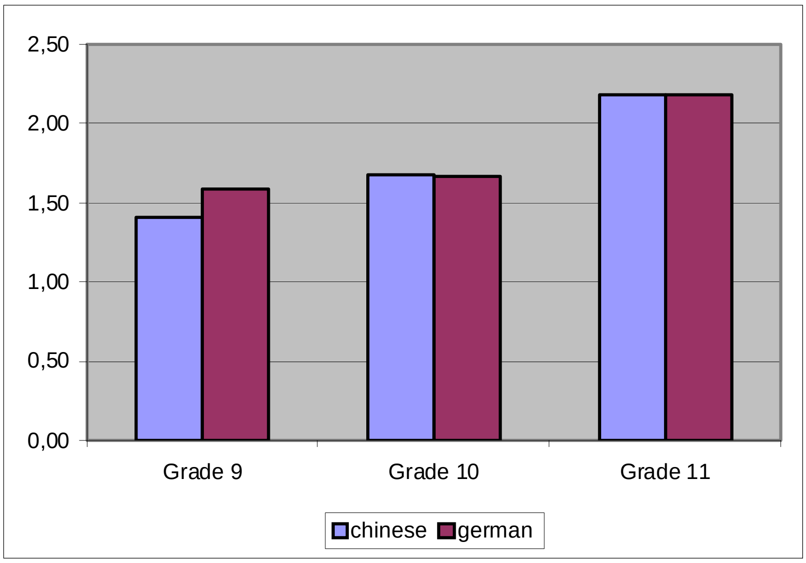


Fig 4b: The average level the students reach.

Table 1 shows the results which depended on gender and grade.

		Grade 9	SD		Grade 10	SD		Grade 11	SD
	N	Mean	SD	N	Mean	SD	N	Mean	SD
Chinese girls	**103**	**1,63**	1,27	**124**	**1,71**	1,09	**106**	**2,30**	1,30
Chinese boys	**103**	**1,18**	1,20	**129**	**1,64**	1,15	**115**	**2,07**	1,49

German girls	**64**	**1,78**	1,34	**76**	**1,83**	1,46	**53**	**2,09**	1,18
German boys	**81**	**1,43**	1,45	**71**	**1,49**	1,37	**83**	**2,21**	1,50

Table 1

5. 2 Have a closer look

Let us have a look on the performance of each country.

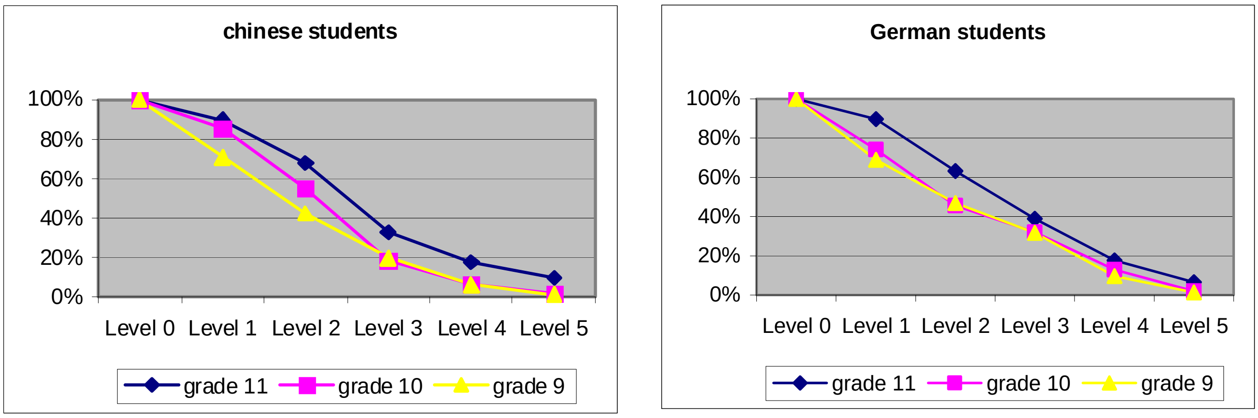


Fig. 5. :Percentage of Chinese students who reach level x and percentage of German students who reach level x

These two diagrams in figure 5 look nearly similar. This seems also a signal that between the Chinese and German students performance is not a high difference. For example we see that the 11th grade students (♦- line) have a higher performance in each country. To proofing this we made also an unpaired t-test.

Table 2

		Df	t	Two tail P-value	
Grade 9	German-chinese	349	.441	.223	n.s.
Grade 10	German-chinese	394	.058	.954	n.s.
Grade 11	German-chinese	355	.013	.899	n.s.
all	German-chinese	1102	.445	.657	n.s.

So let us have a closer look to the male and female students of each country.

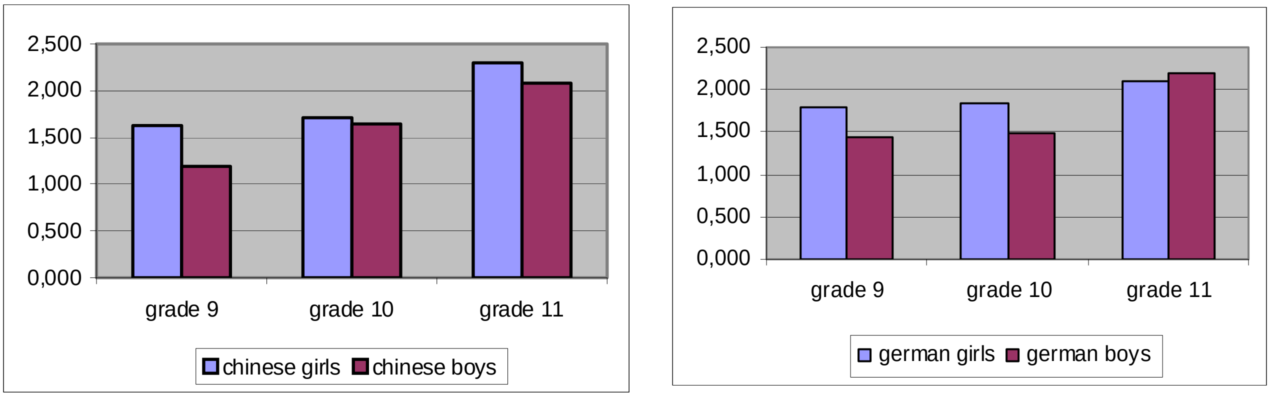


Fig. 6: The average level the Chinese students reach and the average level the German students reach

Also these two diagrams in figure 6 look nearly the same. We can see that in the 9th grade and the 10 th grade the girls perform better in each country (of course not significantly) but in the 11th grade the male German student perform better than the female.

In the diagrams we see in a good way that every year the students perform better and better. Before we have a special look on the development of the performance in the different grades we want to have a deeper look to the different levels.

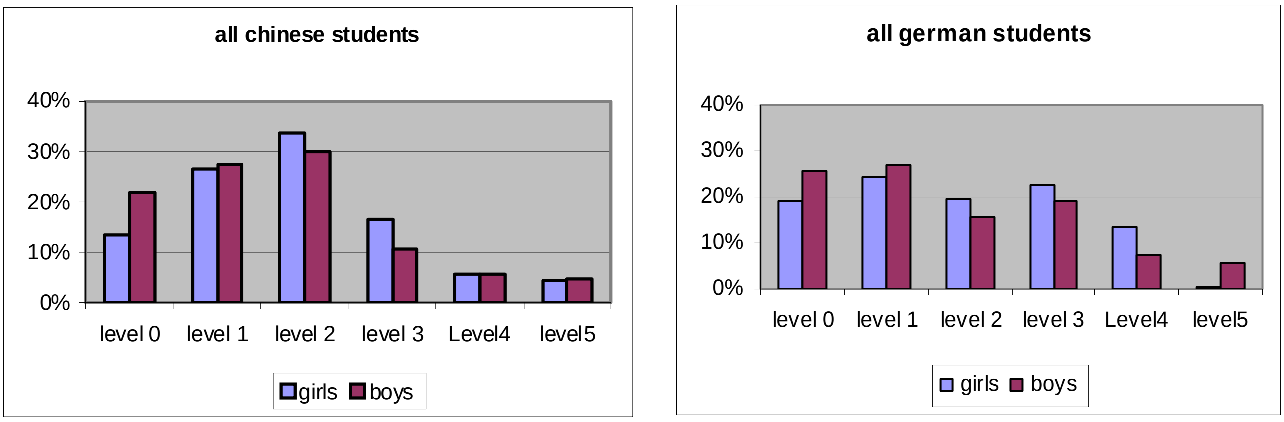


Fig. 7: Percentage of students who reach level x

First we see, that the German girls do not reach the level 5. But in China there are nearly no differences between the boys and girls. We see very clearly, that for the Chinese students the level 3 is a barrier. Only 42 % of the students who reach Level 2 will reach level 3 or more. In Germany more than 66% of the students who reached level 2 will reach level 3 or more. For the German students level 4 is the barrier. Remember level 3 means, that the student can calculate in the mathematical model found by himself, Level 4 means in addition that he can reasoning his calculation and find a result. This means for the shanghai students that they cannot orient themselves in the mathematical model they found by themselves. The German students have problems to interpret their mathematical results and give them a meaning in the real world.

Table 3

		Df	t	Two tail P-value	
Chinese Girls	11th vs.10th	226	3.64	.00003	v.s.
	11th vs.9th	207	3.75	.00002	v.s.
	10th vs. 9th	223	0.51	.609	n.s.
Chinese boys	11th vs.10th	240	2.50	.013	s.s.
	11th vs.9th	216	4.83	$2.5x10^{-6}$	v.s.
	10th vs. 9th	228	2.90	.004	v.s.
German girls	11th vs.10th	127	1.13	.262	n.s.
	11th vs.9th	115	1.33	.186	n.s.
	10th vs. 9th	138	.200	.842	n.s.
German boys	11th vs.10th	152	3.05	.003	v.s.
	11th vs.9th	162	3.33	.001	v.s.
	10th vs. 9th	150	0.26	.792	n.s.

Of course the Chinese boys perform in the 9th and 10th grade very poor but they perform every year better and better. The differences between all grades are statistically significant and we have an effect from about 0.38. For the German boys and Chinese girls we found that there is no significant difference between the 9th grade and the 10th grade but a very statistically significant difference between the 11th grade and the 10th grade ($p<0,01$) with an effect size from nearly 0.5. Very interesting is also the situation of the German girls. Although they perform better and better every year the difference is not statistically significant.

Over all we see, that in grade 11 the students' performance make a great leap forward. Although the mathematical knowledge to solve the pineapple tasks (e.g. Theorem of Pythagoras) was taught in grade 9 they cannot use this knowledge.

6. Conclusions

To use modelling tasks for a comparative study is a very fruitfully thing. And the results are really interesting. For example, we do not yet have an idea while the results of the chinese and german students are nearly the same. You must know that they got an completly different math education. The results show us also that chinese students have difficulties to work clearly in the mathematic world, although chinese math education traditional would be regarded that is good at training mathematics knowledge and skill. It's also interesting why there are gender differences during working with mathematical modelling tasks. To investigate these could be the next project.

Reference

Blum, W., Galbraith, P.L., Henn, H-W. & Niss, M. (2007). *Modelling and applications in mathematics education. The 14th ICMI Study*. Springer.

Blum, W. & Leiss, D. (2005): How do students and teachers deal with mathematical modelling problems? The example "Sugarloaf". In: ICTMA 12 Proceedings, S. 222-231

Blum, W.et al. (2002). ICMI Study 14: Application and modelling in mathematics education – Disscussion document. *Educational Studies in Matematics* 51(1-2), 149-171.

Boromeo Ferri, R.(2006). Theoretical and empirical differentiations of phases in the modelling process, ZDM 2006 Vol.38 (2). S. 86-95

KMK (Kultusministerkonferenz) (2003). *Bildungsstandards im Fach Mathematik fuer den mittleren Bildungsschnitt.* Retrieved January 20, 2004, from http://www.kmk.org/schul/bildungsstandards/mathematik_msa_bs_04-12-2003.pdf

Maass, K. (2006). What are modelling competencies? *ZDM* 2006 Vol. 38 (2)

NCTM (2006). *Principles and Standards for School Mathematics*. National Council of Teachers of Mathematics, Reston.

YZZ (Research group for constituting mathematic curriculum standards) (2003). Mathematic curriculum standards for high school in China. People Education Press.

MATHEMATICAL MODELLING IN A EUROPEAN CONTEXT – A EUROPEAN NETWORK PROJECT

Stefanie Meier
Technische Universität Dortmund, Germany
Stefanie.meier@mathematik.uni-dortmund.de

The study reported in this presentation is part of the research in the Comenius-Network-Project "Developing Quality in Mathematics Education II". One main focus of this network-project is to develop learning materials for maths education containing realistic and perhaps "European" contents. The solving of these tasks shall, aside from encouraging other competencies, also further pupils' modelling activities, so that the main focus of the research group is to analyse modelling processes in maths education in different European countries. The first step is to develop a checklist which helps teachers to identify good modelling tasks. This presentation reports about a preliminary individual study, which shall test this checklist for teachers.

Introduction

"Developing Quality in Mathematics Education II" (DQME II) is a Comenius-Network, funded by the EU (LifeLongLearning Programme). It is a continuation and expansion of the associated project "Developing Quality in Mathematics Education". In each of the no longer four but now eleven participating countries, there are groups consisting of members of universities, teacher education institutions and schools. This is a special feature of this project and promotes a strong connection between theory and practice, as well as between the research on and development of mathematics education. One main area of the work in the project is the development and evaluation of learning material by the research group and the implementation of it in the classroom by the teachers.

For the evaluation of the learning material the research group decided to focus on mathematical modelling processes. One main focus is to develop a checklist which shall help teachers to identify "good" modelling tasks. This resulted from the request of the participating teachers, to find out how they can identify a good modelling task. This will be detailed in this paper.

Integrated in this description will be the results of testing the checklist at a teacher training conference. Those teachers who tested it think that such a checklist may be helpful, especially for teachers at the very beginning of their teaching careers.

Identifying a "good" modelling task? – Theoretical Basis

The theoretical basis of the work will be mathematical modelling processes. The three objectives the research group will deal with are the following:

1) Clarifying learning objectives, which are necessary for the development of mathematical modelling competence,
2) Identifying which quality criteria need to be fulfilled for learning materials and teaching methods that further mathematical modelling competence,
3) Developing and running a pilot study for cross-cultural comparison of exemplary learning materials.

The first two objectives of the research group will lead to a checklist, which shall help teachers to identify good modelling tasks and lessons for classroom practice. The third objective includes the studies which will be done during the project.

The basis of our first discussion during the first meeting was the complex model of the mathematical modelling process by Blømhoj & Jensen (2006)

The model shows one ideal way of how to work on a real life situation mathematically. It should be a motivating situation that leads to a "Domain of inquiry". This has to be systemised, idealised and simplified so that a mathematical view of this "System" is possible. This can now be translated into mathematical representations so that calculations can be done (Mathematical system).

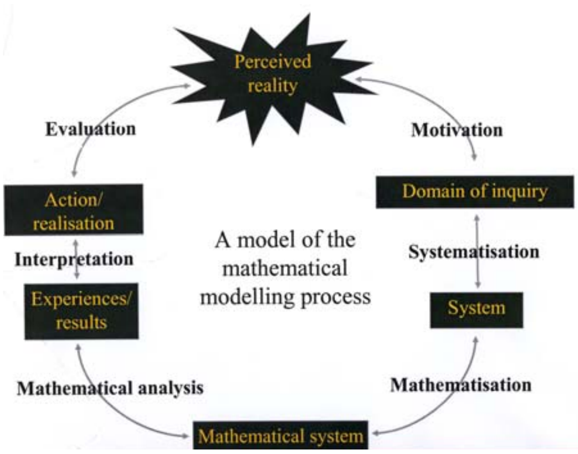


Figure 1: A model of the mathematical modelling process

Analysing the mathematical results with mathematical methods leads to "Results", which can be interpreted according to the "Domain of inquiry". This "Realisation" or insight can now be evaluated and validated in accordance with the perceived reality, which was the starting point of the whole process. If the results are satisfying, the process is finished; if not, you have to start over again. According to the results of Borromeo Ferri (2006), students do not follow this modelling process in a linear way, but you can find all stages in a complete and finished modelling process. Therefore, by analysing videos of pupils doing maths, we can observe the stages of the modelling process, but not necessarily in the order the modelling circle shows.

There are two different purposes for using modelling tasks in maths lessons. The first purpose is to impart knowledge and competence about mathematical modelling. The second purpose is to impart knowledge about mathematical contents. These two purposes can, of course, not be divided, but the teacher can focus on one. Both purposes should have the background of integrating mathematics with reality, as well as getting the pupils involved in and motivated to deal with the real life situation. Especially this last point is important: at a first glance, to motivate the pupils to deal with the real life situation mathematically and, at a second glance, to make them work on these questions independently. This follows the idea of the constructive learning theory: "Learning is an active, autonomous construction of knowledge" (cp. Leuders, 2001). Leuders concludes different criteria for a constructive learning ambiance from this idea.

- The pupils have the opportunity to work on their own and to think independently from teachers and other pupils.
- The pupils have the opportunity to use their previous knowledge and experiences and to connect these with the new knowledge. (prior knowledge)
- There are possibilities for the pupils to interact, to argue and to agree on one solution. (negotiation)
- There are opportunities for pupils to realise that learning is a process, which makes it possible to solve problems. (student centeredness)

(My translation based on Leuders 2001)

These four points fit the modelling process very well. Being involved in a real life subject motivates you to work and think independently. Further, the pupils have to use their prior knowledge about mathematical and real life contents. In order to convince others of their mathematical model and their solutions, it is essential to let the pupils argue. After that they can reflect on the whole process of solving a modelling problem as a learning process.

To make the ideal model of a mathematical modelling process a bit clearer for teachers, it was simplified into four categories:

- Motivation,
- Systematisation and Mathematisation,
- Doing the mathematics and
- Interpretation and Validation.

These resulting topics were then filled with criteria (descriptors), which describe what the learning objectives mean in detail. I added a third column, which describes possible activities of pupils according to the descriptors. The first two columns of the following table contain the first proposal for a checklist.

Learning objectives	**Descriptors**	**Possible pupils' activities relating to the descriptors**
Motivation	• Engagement (personal and societal)	• They develop their own questions • They can obviously identify with the task/ the real life content of the task

	• Teaching purpose	• Not really identifiable by pupils' activities. • Control by the teacher at the end of the lesson: Did the pupils learn mathematical contents? And/or did they improve their mathematical modelling competence?!
	• Authenticity	• The given situation is really a real life situation
	• Linking existing math. knowledge	• They use, e.g. mathematical formulas, without looking them up in the maths book
	• Challenging	• They do not find a solution in five minutes but go on trying to find a solution
Systematisation & Mathematisation	• Is data needed?	• Pupils estimate or look in a book or the internet for data
	• Abstraction	• Pupils create a mathematical model for the real life situation
	• Assigning variables	• Self-explanatory
	• Making assumptions	• Self-explanatory
	• Simplifying	• They simplify the real life situation to create a mathematical model • They simplify their mathematical model to make the handling easier
	• Representation(s)	• They use mathematical symbols etc. to describe the real life situation
Doing the mathematics	• Formalizing and analyzing the math problem	• Self-explanatory
	• Using data	• Self-explanatory
	• Approximation and estimation	• They have to approximate and estimate data for their calculation
	• Use of Information and Communication Technology (ICT)	• Self-explanatory
	• Use known algorithms	• They use, e.g. mathematical formulas, without looking them up in the maths book
	• Mathematical common sense	• Self-explanatory
	• Proof (validation of the math used)	• Self-explanatory
	• Use of math. representation(s)	• They use mathematical symbols etc. to describe the real life situation
Interpretation &	• Validation of the	• They question if their solution is

Validation	solution mathematically	possible • They recalculate to make sure that the solution is correct
	• Validation of the solution in the 'real world'	• They question, if their mathematical solution fits to the real world • They search for other possibilities to otherwise depict the situation in the real world
	• Are the results good enough?	• Pupil question: Are the results good enough in relation to the real life situation?
	• Or is another cycle needed?	• Do they modify their first mathematical model?

Table1: Results of the first meeting of the research group

The questions to be answered are:

- Are the criteria manageable for the teacher in terms of formulation and expenditure of time?
- Is it possible to identify a "good" modelling task with these criteria?

The approach for a first answer to these questions was as follows: The task the teacher wanted to use was analysed by the teacher with the help of the developed descriptors before giving a lesson to find out if the task is (in his opinion) a modelling task. Then a lesson with this task was held by the same teacher and analysed by me according to the whole modelling circle.
The result is a comparison of what the teacher decided about the task according to the developed criteria and with what the pupils really did according to the model of the mathematical modelling process. This approach was developed after what Wittmann (1995) calls "design science".
According to this theoretical part a first definition of a "good" modelling task in this paper could be: If it is possible to identify nearly all modelling activities described in the modelling circle (according to the descriptors and possible pupil activities) in the pupils' work with one special task, this task is a "good" modelling task.

Identifying a "good" modelling task? – Practical realisation

Modelling Task- Swimming Pool

The original version of the used task was as follows:
Despite the rather cool weather during winter, small outdoor swimming pools are popular among private house owners in Sweden. Imagine a swimming pool that is circular with a radius of 2.75 meters and a depth of 1.18 meters. The distance between the water surface and the pool edge is 0.06 meters. Every spring the pool is filled through two water pipes, each of them delivering 20 liters of water per minute. The water cost 2 Euro per cubic meter.

Questions

- How much water is here in the swimming pool? Answer in the unit cubic meters.
- How much does it cost to fill the swimming pool?
- How long does it take to fill the swimming pool?
- How many humans should be in the swimming pool at the same time in order for the water to pour over the edge? Find out the average volume for an average person yourself.

(From Matte Direkt (Mathematics Directly) Grade 9, 2003, p. 53; translated by Thomas Lingefjärd)

In the study, this task was used in class 8 of a German Gymnasium. They had dealt with calculating volumes during the previous lessons, so the tasks fit to the current content. The teacher decided to delete the questions, add a picture (Figure 1) and led the pupils find their own questions corresponding to the real life situation he gave them. His tasks for the pupils were the following:

- Think out at least 2 meaningful mathematical questions to the text and answer them using a calculation!
- Reach an agreement in your group, which one of your questions should be written on the blackboard!
- Hence solve the questions of the other groups!

Figure 2: www.badepool.eu

Using the checklist

Upon first looking at the task, including the mathematical questions, the teacher decided that this is a good modelling task. Then he used the checklist to evaluate this first thought. For this we translated the English version into German and added a scale: 1 for not existent and 4 for existent. I have chosen the scale from one to four to make it clearer to the teacher that a "good" modelling task is if more items are answered with 4. After testing the checklist this one time, we realised that this is not a good idea, because it is not clear what 2 or 3 means. Some of the teachers at the mentioned teacher training conference think that this scale might be helpful to identify single modelling competences. But on the whole, it is perhaps better to add a scale from 1 to 2 so that a descriptor can be found or not; or to leave this scale out and let the teacher answer questions.

You find the answers of the teacher in the following table.

Learning objectives	**Descriptors**	**Answers of the teacher**
Motivation	• Engagement (personal) • Engagement (societal) • Teaching purpose	• 2 • 2 • -

	• Authenticity • Linking existing math. knowledge • Challenging	• 4 • 4 • 3
Systematisation & Mathematisation	• Is data needed? • Abstraction • Assigning variables • Making assumptions • Simplifying • Representation(s)	• 3 • 1 • 1 • 3 • - • -
Doing the mathematics	• Formalizing and analyzing the math problem • Using data • Approximation and estimation • Use of Information and Communications Technology (ICT) • Use known algorithms • Mathematical common sense • Proof (validation of the math used) • Use of math. representation(s)	• 1 • 4 • 4 • 1 • 1 • 4 • 1 • -
Interpretation & Validation	• Validation of the solution mathematically • Validation of the solution in the 'real world' • Are the results good enough? • Or is another cycle needed?	• 2 • 4 • yes • no

Table 2: Answers of the teacher

The reason the teacher did not answer all of the questions is that he did not understand all of the questions. On the whole, he answered 17 questions. For 8 of them he decided that the asked descriptor is existent in the task (answer: 3 or 4). For 9 descriptors he decided that they are not existent in the task (answer: 1 or 2). As a result of the answers on the checklist, it is not really clear now, if the task is a “good” modelling task or not. Is it good, since half of the answered questions are positive, or is it not good, since the other half of the answered questions are negative. How many answers have to be positive to decide that the task is a good modelling task? Or: How many descriptors for each learning objective have to be answered positive?

Before modifying the checklist, I tested it at a teacher training conference with 37 teachers. I gave them some modelling tasks, which they had to evaluate with the checklist. Some results from the questionnaire the teachers filled out which were important for me, are that they think a checklist to identify modelling tasks is helpful, especially for beginners and that they would prefer a list with criteria describing modelling tasks.

So in terms of formulation the checklist has to be modified as well as the evaluation mode for the checklist.

What did the pupils do with that task?
In groups, the pupils developed the following questions concerning the above mentioned task[1]:

- *How much does it cost if you fill up the whole pool?*
- *How much water (in cubic metres) can you fill in the pool?*
- *How long does it take to fill up the whole pool?*
- *How much water are you allowed to fill in the pool?*
- *How much canvas cover do you need for the pool, if it is cylindrical?*
- *How much does it cost if you fill 170 litres in the pool?*
- *How long does it take to clean the pool if one pump can pump 175 litres per hour?*
- *How many people have to go into the pool so that it overflows?*

The videotaped group of pupils, consisting of five boys, started with question 2: How much water can you fill in the pool?
To evaluate if the pupils are doing mathematical modelling, we tried to find a relation between the actions and discussions of the pupils and the descriptors in the way the description of the pupils' activities in table 1 show.
The authenticity of this task can be evaluated by having a look at advertisements for such swimming pools. The producers of such pools do not tell the user how much water has to be filled into them (Modelling Circle → Perceived reality). By letting the pupils develop their own questions, they got personally involved in the task. (Descriptors → Engagement). This task is challenging, because especially this group tried to find a "difficult" question, which not only the other pupils, but also they themselves had to answer. (Modelling Circle → Motivation)
They simplified the pool by assuming that the "sides of the pool are straight." (Modelling Circle → Systemisation; Descriptors → Simplifying) Then they began discussing the formula to calculate the volume of the pool for the highest possible water level: "R to the power of 2 times pi times 1.18-0.06." (Descriptors → Linking existing mathematical knowledge; Modelling Circle → Motivation) And they also started mathematising because of calculating the volume using the formula. They used their calculator to get a result, which was in cubic metres (Modelling Circle → Mathematical System). The scale unit "cubic metres" was not helpful for them to validate their result, so they calculated how many litres their cubic metres are. Then, they compared it with a cube with a 1m edge length and

[1] The teacher did not comment on or change the questions before the pupils worked on them.

decided that their solution is possible. (Modelling Circle → Interpretation and Validation; Descriptors → Validation of the solution mathematically and in the "real world")
Thus we find Motivation, a Domain of Inquiry, Systematisation, a System, Mathematisation, a Mathematical System and Analysis as well as the Validation of the results in the work of the students. According to this analysis the task is a "good modelling task" because all stages of the modelling circle can be found.

Conclusions/ Outlook
When we now compare what the teacher said before using the task and what the pupils really did with this task, we can answer the questions:

- Are the criteria manageable for the teacher in terms of formulation and expenditure of time?

According to the questionnaires the teachers at the teacher training conference filled out, the checklist is too long and time consuming, therefore, it has to be shortened.

- Is it possible to identify a good modelling task with these criteria?

It is obvious that the first proposal of the checklist is not that helpful for identifying a "good" modelling task, because according to the analysis of what the pupils did the task is a "good" modelling task.
I already discussed the problem of the scale from 1 to 4 in the above text. Another view on the checklist must include the difference of what the teacher said about the task and of what the pupils really did. One reason could be that some descriptors describe common mathematical competencies like the ones marked in red in the following table. They are not explicitly indicators for modelling tasks.

Learning objectives	**Descriptors**	**Answers of the teacher**
Motivation	• Engagement (personal) • Engagement (societal) • Teaching purpose • Authenticity • Linking existing math. knowledge • Challenging	• 2 • 2 • - • 4 • 4 • 3
Systematisation & Mathematisation	• Is data needed? • *Abstraction* • Assigning variables • Making assumptions • Simplifying • Representation(s)	• 3 • *1* • 1 • 3 • - • -
Doing the mathematics	• *Formalizing and analyzing the math problem* • Using data	• *1* • 4

	• Approximation and estimation • Use of Information and Communications Technology (ICT) • Use known algorithms • Mathematical common sense • Proof (validation of the math used) • Use of math. representation(s)	• 4 • 1 • 1 • 4 • 1 • -
Interpretation & Validation	• *Validation of the solution mathematically* • Validation of the solution in the 'real world' • Are the results good enough? • Or is another cycle needed?	• *2* • 4 • yes • no

Table 3: Analysis of the answers of the teacher

If you work on a mathematical problem not connected to real life, you also use your mathematical competencies like using known algorithms or using ICT. Leaving aside these non-explicit descriptors leads to the relation that 7 are judged positive and 5 negative by the teacher. This is also not convincing that this task is a good modelling task. So we should have a look at the negatively judged descriptors marked in italics in the table. The main questions resulting are: Why does the teacher think that the given task did not include "Abstraction", "Formalizing and analyzing the mathematical problem" and "Validation of the solution mathematically"? My assumption is that this is connected to the group of pupils you want to present the task to. How good is this group in abstracting or structurising real life situations to make them fit into a mathematical model? Is it then necessary for this group to formulate a mathematical problem? And is a deeper analysis of the solution necessary? This has to be discussed in detail during the oral presentation of this paper.

On the whole some of the descriptors must be formulated more explicitly and an evaluation tool for the usage of the table has to be developed. So the next steps will be modifying and the checklist, testing the modified checklist with the same teacher as well as using the same task and the modified checklist with different teachers.

References

- Blomhøj, M.; Jensen, T.H. (2006): What's all the fuss about competencies? In: Blum, W.; Galbraith, P.L.; Henn, H.-W.; Niss, M. (Eds.): Applications and Modelling in Mathematics Education. New York: Springer, (p. 45-56).
- Borromeo Ferri, R.; Leiss, D.; Blum, W. (2006): Der Modellierungskreislauf unter kognitionspsychologischer Perspektive. In: Beiträge zum Mathematikunterricht. Hildesheim: Franzbecker
- Leuders, T. (2001): Qualität im Mathematikunterricht der Sekundarstufe I und II. Berlin: Cornelsen Scriptor
- Wittmann, E. Ch. (1995): Mathematics Education as a „Design Science". In: Educational Studies in Mathematics (29), (p. 355-374).

Didactical Reflections on the teaching of mathematical modelling – Suggestions from concepts of “time” and “place”

Toshikazu Ikeda, *Yokohama National University*

The characteristics of mathematics in a society seem to differ according to the “time” and “place” which the mathematical model has constructed. This paper discusses that concepts about “time” and “place” have an important role in the practical teaching and learning of applications and modelling, for the following three aspects: (1) Problem situations which people are interested in, (2) Purposes for using mathematics in a society, (3) Existence of mathematical models, methods, etc. embedded in a society.

Introduction

When we consider the teaching and learning of mathematical modelling in school mathematics, it is meaningful to analyze both the characteristics of mathematics outside of school - in the society, and obstacles or challenges in school mathematics. Then, by contrasting both aspects, practical issues or suggestions might be derived. When we analyze the characteristics of mathematics outside of school, we focus on the following facts noted by Niss (2008). Namely any application of mathematics relies on the introduction of a mathematical model, whether explicitly or implicitly. Thus any mathematical model must be constructed by someone. Therefore, when and where was each mathematical model constructed by someone? The characteristics of mathematics in a society seem to differ according to the “time” and “place” in which the mathematical model was constructed. It is suggested that concepts about “time” and “place” have an important role in the teaching and learning of applications and modelling. The following three points are discussed, that differ according to “time” and “place”.

(1) Problem situations which people are interested in

(2) Purposes for using mathematics in a society

(3) Existence of mathematical models, methods, etc. embedded in a society

Problem situations which people are interested in

Concerning “place”, problem situations which people are interested in differ according to the place where people are living, such as country, area, etc. It is obvious that the problem situations people face or are interested in differ between developing countries and developed countries. Further, even in the

For example, the following problem situation is relatively familiar for students living in an oil producing country. On the other hand, it is not familiar for students living in a country which does not produce oil.

> A pipeline is to be constructed connecting an oil rig at sea to a nearby refinery on the coast. Find the cheapest method of constructing a pipeline taking into account the difference in cost of construction for ‘on land’ versus ‘under water’ sections of the pipeline. The ‘under water’ cost can be expected to be greater than the ‘on land’ cost. Because of the difference in cost of ‘on land’ and ‘under water’ sections, the cheapest cost will not automatically correspond to the shortest length of pipeline. (Board of Studies, 1997)

Concerning “time”, it is also said that problem situations which people are interested in differ between past society and present society. For example, constructing a figure to measure the length or angle was important in past society, but as we now have convenient instruments to measure, it is not important at present.

Further, as Jablonka (2007) noted, “different purposes may result in different mathematical models of the ‘same’ reality.” An example is explained as follows. “For the problem of a bank employee (aided by a software package), who must advise a client in the comparison of financing offers for a mortgage, for the manager of the bank this is a problem of profitability, and for the customer it is one of planning her personal finance.”

Before thinking about the teaching and learning of applications and modelling, it is suggested for the teacher to understand the differences of problem situations according to the place, time, and person. Further, the teacher should understand that problem situations for people in a society and problem situations for students are also different. The teacher should select an appropriate modelling task so that students can make sense from the problem situation proposed by the teacher. This suggestion raised practical questions, such as “What is an appropriate modelling task?” Galbraith (2007) notes the following two meaningful points regarding this question.

> (T1) Consistency with avowed purpose
>
> “For example, if applications and modelling is included in mathematics education to attain goals such as ‘students will experience school mathematics as useful for solving problems in real life outside the classroom’ then students, to some extent, need to encounter tasks that are close parallels to comparable problem situations encountered outside the mathematics classroom.”

(T2) Introducing real world modelling tasks

The following two aspects are raised; "(a) the importance of using models based on experience (influenced by student background)", "(b) motivation e.g. looking to the world and other disciplines for knowledge and problems."

(T1) is a basic important issue which may be neglected by some teachers in practical teaching. Regarding (T2), more detail discussion is required regarding practical teaching. For example, the following points should be discussed regarding (T2)(a).

T2(a1): Does the problem situation concern the surroundings of students at present, or in the future?

(a2): If the problem situation concerns the present surroundings of students, is it concerned with most students or a few students? For example, mirror problems and "rock-paper-scissors" problems etc. are familiar with most Japanese students. But problems about soccer, tennis, rugby, etc. are only familiar with students who are interested in these sports.

(a3): If the problem situation concerns the surroundings of students in the future, is it concerned with the situation confronted as citizens, as individuals or in their profession/vocation? The former two situations concern many students. But profession/vocation situations only concern the particular students who want to work in that direction.

Regarding (T2)(b), it was suggested that students' motivations should be considered. However, we must notice that students' motivations differ according to each student's interests and experience. The following two points should be considered. First is the reason why someone had to solve the problem. This concerns the background or goals for developing the mathematical model. It is not clear for others why someone had to solve the particular problem. As a result, it becomes very hard for others to develop the mathematical model. Therefore, students should know the reasons before solving a real world problem. For example, real world problems about fitting functions to the given data are often used in teaching and learning modelling.

When Japanese forecast when cherry trees will bloom, the reason for fitting the functions to the given data is obvious for the particular people, because they need to determine the date beforehand to do a party on the weekend for watching the cherry blossoms. However, when we forecast the cooling rate of coffee, what are the reasons for fitting the function to the given data? In general, a mathematical model is developed to attain a particular purpose. It is meaningless to fit a function without a purpose. In the case of the cooling rate of coffee, it requires a problematic situation so that students can derive questions

Second, it is important for the teacher to evaluate whether or not students can accept the problem posed by someone as their own problem. If students can accept the problem posed by someone as their own problem, it is an ideal setting for the teacher to treat modelling activity in the classroom. One of the methods is for the teacher to let students select an interesting task among several modelling tasks presented by teacher, and tackle it individually or in a group. Otherwise, the teacher proposes a series of observations or actions, so that students can use these to derive similar problems. This approach begins by observing or analyzing the phenomenon or action by students. Students are expected to derive key questions which will be solved by using mathematics.

For example, the teacher asks students “What shapes of cans are used in a supermarket? Let’s examine them this weekend.” Through this activity, students find out that most cans are cylinders, though some are not cylinders. Further, considering the relationships between the shapes and contents of cans, they will find that there are two types. In the first type, the shape is affected by the can’s contents, while the second type is not affected by the can’s contents. Then students find that the shapes of cans which are not affected by the contents are generally cylinders. Through these activities, students gradually formulate the real problem like “Why are the shapes of cans generally cylinders, not cubes?”

Let’s pick one more example, the problem about baton passing in a relay (Osawa, 2004). Students discuss how they can win in a relay in school sports. Several issues are derived from students to win a relay, such as the order of runners, how to pass the baton, etc. When focusing on the baton pass, the following problem is formulated in verbal terms “When does the next runner begin to run to get the baton from the previous runner, for the shortest baton pass time? In other words, what is the best distance between the previous runner and the next runner, so the next runner can get the baton with the least time loss?” In this example, the problem is derived by students themselves. The reason why they need to solve this problem is obvious for students.

For teaching modelling, it is crucial to propose appropriate observations or actions involving discussion between the teacher and students, so the students can accept the proposed problem as their own.

Purposes for using mathematics in a society

The purposes for using mathematics differ according the topic people are working on. Niss (2008) identified three different kinds of purposes for using mathematics in other disciplines or areas of practice:

(P1) in order to *understand* (represent, explain, predict) parts of the world,

(P2) in order to subject parts of the world to some kind of *action* (including making decisions, solving problems),

(P3) in order to *design* parts or aspects of the extra-mathematical world (creating or shaping artifacts, i.e. objects, systems, structures).

I think these three purposes are important for the following three points.

(1) Educational goals students are expected to attain

(2) Understanding of modelling process for the beginner

(3) Appreciation of the usefulness of mathematics in society

(1) Educational goals for what students are expected to acquire

This first point is characterized by the question "What kinds of educational goals are emphasized in teaching and learning mathematical modelling?" Modelling is treated for a variety of educational goals, such as foundations of science, critical citizenship, professional/vocational preparation, way of living, etc. There seems to be a strong connection between purposes for using mathematics and educational goals. So, what are the relationships between the three purposes and educational goals?

In the case of "(P1) *understand*", parts of the world are considered to be phenomenon of extra-mathematical domains such as natural or social phenomenon. The mathematical model is verified by contrasting it with real data taken from the phenomenon. Therefore, the aims such as foundation of science, professional/vocational preparation are emphasized more when we treat mathematical models which aim to "understand". In the case of "(P2) *action*", parts of the world are considered problem situations, in which people have to make a decision or solve a problem. Two types of mathematical model seem to exist. First is a social system model which is developed to make an objective and safe decision among people in a society, such as taxi prices or railway schedules. These models concern all citizens. After this mathematical model is embedded in a society, it becomes a main source for the reconstruction of reality (Skovsmose, 1994). The second is developed with personal purposes, such as planning a family trip, or planning for family savings or loans. However, we must again note that "different purposes may result in different mathematical models of the "same" reality (Jablonka, 2007)." For example, trip planning may become part of a tour conductor's job. The mathematical model developed is effectively validated by developing another model to compare it with. Therefore, aims such as critical preparation for citizens and professions/vocations are emphasized more when we treat mathematical models which have the purpose of "action". In the case of "(P3) *design*", the focus is on objects which make our life more comfortable, such as furniture, architecture and designs using tessellation. This type of object is evaluated by individual sense of value. Therefore, the aim of professional/vocational preparation is emphasized more when we treat a mathematical model which has a purpose to "design". When we consider the teaching of modelling, we should examine the relation between the purpose for using mathematics and educational goals.

(2) Understanding of modelling process for the beginner

Second, these three purposes imply that the modelling process depends on the purpose or area of other disciplines. For example, when we "understand" the natural/social phenomenon, the mathematical model is abstracted from the real world phenomenon, and the mathematical model is verified by contrasting it with the real world phenomena. However, when we make an "action" or "design", multiple mathematical models are developed to make a decision, and the appropriate mathematical model is selected among several models according to the aim.

When we introduce mathematical modelling for students, a particular diagram of modelling process is often used to let students understand roughly what modelling is. We have to pay more attention to the fact that the modelling process differs according to the purpose for using the mathematics or area of other disciplines, and we need to consider the reason why the teacher introduces the particular modelling diagram for students.

(3) Appreciation of the usefulness of mathematics in society

Third, these three kinds of purposes are also useful when we teach the usefulness of mathematics to students. When we teach how mathematics is used in a real world situation, one of the methods is to identify purposes for using mathematics in the real world. By tackling a series of modelling tasks, students are expected to reflect and find out the purposes for using mathematics in a variety of cases studied before. For example, one of the methods is for the teacher to assess students' appreciation of the usefulness of mathematics by asking "How is mathematics useful when we see real world situations from a variety of viewpoints?", before and after modelling teaching. The teacher can assess how students deepened their appreciation of the usefulness of mathematics in society, by comparing their writing before and after teaching modelling. For example, regarding the criteria to assess students' writing, one of the following criteria can be set as the viewpoint:

Criteria 1: From only students' personal perspectives, not from social perspectives

Criteria 2: From social perspectives, which are not clear or only special cases

Criteria 3: From social perspectives which are clear and integrated, such as the three different kinds of purposes identified by Niss

For example, the following writings are the results of student A before and after the experimental teaching. The experimental teaching was done for Japanese 9th grade students, composed of 18 sessions of 50 minute classroom teaching (Ikeda, 2002). The writing of student A before the teaching is assessed as criteria 2, however, student A's writing after the teaching can be assessed as criteria 3. We can see that student A made progress in appreciation of the usefulness of mathematics.

[Writing of student A before the experimental teaching]

We can acquire mathematical thinking and judgment from mathematics, but most people don't use mathematics in their daily lives. So it is not meaningful to consider in school how to use mathematics in daily life.

[Writing of student A after the experimental teaching]

Mathematics is useful to establish criteria or theory in a real world situation so that everyone can agree on it. Mathematics is useful to consider before doing something. We can use mathematics to predict a solution, without actually doing the task.

Existence of mathematical models, methods, etc. embedded in a society

The last aspect is existence of mathematical models, methods, concepts, etc. embedded in daily life or society. The quantities and qualities of mathematical models, methods, concepts, etc. embedded in daily life or society differ between past and present society. Niss (2008) noted that in the past, the mathematical methods which we used were searched for by building up a mathematical model, but at present the mathematical methods are used with cultural techniques. Further, as Skovsmose (1994) noted, "Mathematics not only creates ways of describing and handling problems, it also becomes a main source for the reconstruction of reality." In particular, we need to pay attention to the existence of a variety of matters which people have developed to solve real world problems in the past, which are familiar and embedded in daily life or society at present, such as objects, systems, and structures which were built through full modelling processes in the past (e.g. taxi prices, railway schedules, savings and loans, bicycle reflectors, suspension bridges and methods to find an earthquake location, etc.).

When we think about the teaching and learning of applications and modelling, we need to first pay attention to the existence of a variety of matters familiar to students, which were developed using mathematics. This fact leads to two quite different approaches to teaching mathematical modelling. The first approach is to present to students a matter which someone else has developed in the past, It is important for the teacher to let students observe the problem situation before formulating the real problem. Observing or analyzing the phenomenon or action by students themselves plays an important role in this point. The following communication is expected between the teacher and students, so that students can clarify the aim of observation.

Teacher: Do you know what a reflector is? What is its function?

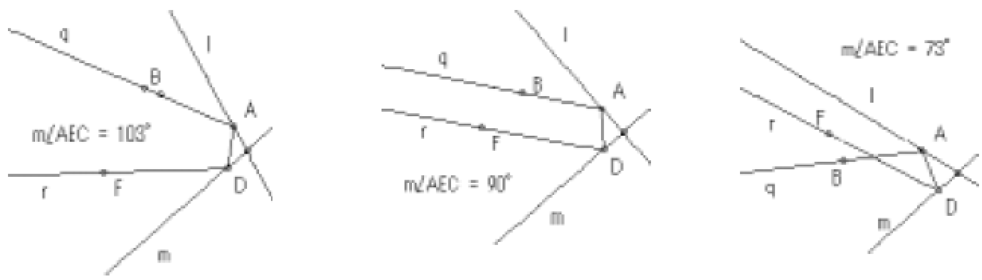
Student A: It makes the rider safer! It reflects light brightly.

Student B: Is that true? Let’s examine it!

As a result of experiment and observation, the following fact can be derived.

> “When we conduct an experiment in a dark room using an object from daily life like a penlight, it seems that the bicycle reflector reflects light in the direction exactly opposite to which it came from.”

By recalling the principle of a mirror, it is understood by students that the structure of a reflector is different from a simple mirror. So they focus on a question like “What kind of structure does the reflector have?” Then, by observing the reflector’s structure, students are expected to find that the angles with which each reflector mirror is constructed are right angles. Therefore, the reason why the angles are right angles is investigated. This means that students interpret the reason why someone has made the mirror angles to be right angles. As the situation is 3-dimensional, it is difficult for students to consider. Students are expected to consider the simple case, namely a 2-dimensional situation as follows. Students are expected to find out and prove the fact “When the angle between two mirrors is 90 degrees, the incident light and reflected light are parallel”.



Then, students consider the 3-dimensional situation. Like this example, when the object or system developed to solve a real world problem in the past and which is embedded in daily life or society at present is considered in the classroom, we expect the following teaching flow for students to follow.

(1) To clarify why and for whom the object or system has been developed.

(2) To find out the characteristics of the object or system by observing and analyzing it.

(3) Considering the intention or role why the object or system is used in a dairy life or society, the mathematical structure of the characteristics found in (2) is investigated. Mathematical analysis is done in this stage.

(4) Appreciating the wisdom of ancestors. Then, by analyzing implicit assumptions established in this situation, further modifications are considered.

This teaching flow is different from the full modelling process which is generally recognized. The following is the full modelling process by Pollak (1997).

(1) We identify something we want to know, do, or understand. The result is a question in the real world.

(2) We select "objects" that seem important in the real-world question and study the relations among them. The result is the identification of key concepts.

(3) We decide what we will examine and what we will ignore about the objects and their interrelations. You simply cannot take everything into account. The result is an idealized version of the original question.

(4) We translate this idealized version into mathematical terms, and obtain a mathematical formulation of it.

(5) We identify the field(s) of mathematics that are needed, and bring to bear the instincts and knowledge of those fields.

(6) We use mathematical methods and insight, and get results. Out of this step come techniques, interesting examples, solutions, theorems, algorithms.

(7) We translate back to the original field and obtain a theory of the idealized question.

(8) Now we come back to the reality. Do we believe what is being said? Are the results practical, the answers reasonable, and the consequences acceptable?

(a) If yes, the real-world problem solving has been successful, and our next job-both difficult and extraordinarily important- is to communicate with potential users.

(b) If no, we go back to the beginning. Why are the results impractical, or the answers unreasonable, or the consequences unacceptable? Because the model was not right. We examine what went wrong, try to see what caused it, and start again.

In addition, it is also important for the teacher to understand that the teaching flow slightly differs according to the character of matters which are embedded in daily life or society.

The second approach is to present a real world problem for students and let them solve the problem. The intention is for the students themselves to construct the mathematical model. The teaching flow almost follows the full modelling process generally recognized as mentioned directly above. The teacher is expected to set a series of observations or actions by students themselves, then let students derive key questions which will become the intended problem presented by the teacher

When a teacher presents the modelling task couched in verbal terms for students, we need to pay attention to the fact that the formulated problems couched in verbal terms are also regarded as matters which someone has developed in the past, while we should exclude word problems which are “dressing up” of purely mathematical problems in words referring to a segment of the real world. Therefore, the teacher should take enough time in teaching modelling so that students can interpret why and how the problem situation was generated, before tackling how to solve the problem. The problem is derived by others, not by students themselves. Interpreting formulated problems couched in verbal terms can be considered one of the teaching aims, in addition to meaningful attempts to solve the problem.

It is the teacher’s role to consider which approach is appropriate for students, by taking account of both the educational goals and students’ surroundings.

Summary

This paper discussed that concepts about “time” and “place” have an important role in the practical teaching and learning of applications and modelling, for the following three aspects: (1) Problem situations which people are interested in, (2) Purposes for using mathematics in a society, (3) Existence of mathematical models, methods, etc. embedded in a society.

Regarding (1), it is suggested for the teacher to understand that problem situations for people in a society and problem situations for students are different. Therefore, the following two points should be especially stressed in teaching. First is to examine the relation between the problem situation and students from perspectives such as “at present or in the future”, “for all students or for a few students” and “for citizens, for individuals, or for profession/vocation”. Second is how to introduce modelling tasks for students. It is suggested for the teacher to clarify the reason why students must solve the problem or to set a series of observations or actions so that students can accept the problem posed by someone as it was their own problem developed themselves.

Regarding (2), it is noted that the purposes for using mathematics in other disciplines or areas of practice are important for the following three points: (a) Educational goals students are expected to attain, (b) Understanding of modelling process for the beginner, (c) Appreciation of the usefulness of mathematics in a society. For example, in (c), it is possible for the teacher to assess how students can deepen their appreciation about the usefulness of mathematics by setting the criteria based on the purposes to use mathematics: *understand*, *action* and *design*.

Regarding (3), we need to pay attention to the existence of a variety of matters familiar to students in daily life which were developed using mathematics. This fact leads to two quite different approaches to teaching mathematical modelling. The first approach is to present a matter which someone else has developed in the past to students and let them interpret it. The second is to present a real world problem for students and let them solve the problem. It is the teacher's role to consider which approach is appropriate for students by taking into account both the educational goals and students' surroundings.

Reference

Board of Studies (1997). Mathematical Methods Common Assessment Task (CAT): Investigative project - Student booklet. Carlton, Australia.

Galbraith, Peter (2007). Authenticity and Goals –overview. In Werner Blum, Peter L. Galbraith, Hans-Wolfgang Henn and Mogens Niss, *Modelling and Applications in Mathematics Education, The 14th ICMI Study*, Springer, 181-184.

Ikeda, Toshikazu (2002). The curriculum implementation focused on mathematical modelling in a lower secondary school mathematics, *The proceedings of 35th paper presented of mathematical education,* Japan Society of Mathematical Education, pp.103-108 (in Japanese).

Jablonka, Eva (2007). The Relevance of Modelling and Applications: Relevant to whom and for what purpose? In Werner Blum, Peter L. Galbraith, Hans-Wolfgang Henn and Mogens Niss, *Modelling and Applications in Mathematics Education, The 14th ICMI Study,* Springer, 193-200.

Niss, Mogens (2008). Perspectives on the balance between applications & modelling and 'pure' mathematics in the teaching and learning of mathematics. *Plenary lecture at the First Century of the International Commission on Mathematical Instruction*, Accademia Nazionale dei Lincei, Roma.

Osawa, Hironori (2004). Development of Applications and modelling by Action Research in Japanese Secondary School. In Hans-Wolfgang Henn and Werner Blum, *ICMI Study 14: Applications and Modelling in Mathematics Education, Study Conference in Dortmund*, 199-204.

Pollak, Henry O. (1997). Solving Problems in the Real World. In Lynn Arthur Steen, *Why Numbers Count, Quantitative Literacy for Tomorrow's America,* College Entrance Examination Board, New York, 91-105.

Skovsmose, Ole (1994). *Towards a Philosophy of Critical Mathematics Education,* Dordrecht: Kluwer Academic Publishers.

FORMATTING REAL DATA IN MATHEMATICAL MODELLING PROJECTS[1]

Jussara de Loiola Araújo[2]

Universidade Federal de Minas Gerais – UFMG – Brazil

Abstract:

Skovsmose (1994) proposes the thesis of the formatting power of mathematics, according to which part of our reality is projected by means of mathematical models. If we intend to develop mathematical modelling projects in our classrooms, within the Critical Mathematics Education perspective, it is important to bring the concepts discussed by this theoretical perspective to the educational practice. In particular, it is desirable to bring the "formatting power of mathematics" thesis to educational practice. In this paper, I present an experience that took place during a Mathematics course proffered to undergraduate geography students at the Federal University of Minas Gerais (UFMG), Brazil. I focus on the presentation of a mathematical modelling project developed by a group of students enrolled in the course. The objective is to analyse students' handling of the data gathered during the development of the project. The group decided, without much justification, to adopt a periodic function to model the data. However, the data did not seem to fit a periodic function. The students then proceeded to re-group the data in such a way that they seemed to fit a mathematical model represented by a periodic function. I analyse whether or not students' procedure can be understood as an example of the formatting power of mathematics.

1. MATHEMATICAL MODELLING PROJECTS AND THE FORMATTING POWER OF MATHEMATICS: INTRODUCING THE PROBLEM

Mathematical modelling has stood out among current perspectives in mathematics education. In general terms, it can be understood as the utilization of mathematics to resolve real problems. When applied in the classroom, this approach takes on special forms, depending on the educational context, the professionals involved, and the profile of the students, among other factors.

---

[1] This paper is a result of the research project "Communication in milieus of learning with computers" developed with the support of the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and the Fundação de Amparo à Pesquisa do Estado de Minas Gerais (FAPEMIG), both Brazilian governmental agencies that support scientific research.

[2] Web site: www.mat.ufmg.br/~jussara. E-mail: jussara@mat.ufmg.br.

Bassanezi (2002), for example, understands mathematical modelling – whether as a scientific method or a teaching and learning strategy – as the “art of transforming problems from reality into mathematical problems and resolving them through interpretation of their solutions in the language of the real world” (p. 16). For Barbosa (2001), “modelling is a milieu of learning in which students are invited to question and/or investigate, by means of mathematics, situations with reference in reality” (p. 31).

In my classes, when I am the teacher and propose the development of mathematical modelling projects, I seek to put into practice an understanding of mathematical modelling as

> an approach, by means of mathematics, to a non-mathematical problem based in reality, or to a non-mathematical situation based in reality, chosen by groups of students in such a way that questions of Critical Mathematics Education form the basis for the development of the work (Araújo, 2002, p. 39).

Within this perspective, there are some explicit characteristics of the milieu of learning that I seek to put into effect when I propose the development of mathematical modelling projects, including working in groups, and basing the work on Critical Mathematics Education.

According to Skovsmose (1994), the main concern of Critical Mathematics Education is the development of *mathemacy*, which is an extension to mathematics of the problematizing and liberating conception of education proposed by Freire (1970). A similar concept – *matheracy* – has also been discussed by D’Ambrosio (1999). In *mathemacy*, the objective is not to merely develop the ability to carry out mathematical calculations, but also to promote the critical participation of students/citizens in society, discussing political, economic, and environmental issues in which mathematics serves as a technological support. In this case, critique is directed to mathematics itself, as well as to its use in society, the concern thus extending beyond the teaching and learning of mathematics.

Taking account of mathematical modelling, within the Critical Mathematics Education theoretical perspective, demands special care with respect to the role of mathematics (or the role of mathematical models) in society. According to Skovsmose (1994), “mathematics not only creates ways of describing and handling problems, it also becomes a main source for the reconstruction of reality.” (p. 52). The author defends the thesis that mathematics is used to format reality: *the formatting power of mathematics*. According to this thesis, part of our reality is projected by means of mathematical models. One example of this is the Human Development Index (HDI): based on mathematical models, a number from 0 to 1 is associated with every city, country or any other locale. Based on this index, governments or international institutions, for example, decide how to distribute funds to achieve a given objective. A city with an HDI near 1, for example, because of their relatively high rating, might not be selected to receive funds that could resolve some of their problems. Thus, mathematical models are used to create a “real situation” that did not exist before. Critical Mathematics Education questions this power with which mathematics is imbued.

This discussion has been carried out by Skovsmose (1994) in social terms; that is, he discusses using mathematical models to build part of reality, but does not elaborate on how the thesis of the formatting power of mathematics can be discussed in educational contexts. However, since I intend to develop mathematical modelling projects in my classrooms within the Critical Mathematics Education perspective, I believe it is important to bring the “formatting power of mathematics” thesis to educational practice.

Milanezi (2007) tried to extend the formatting power thesis to the educational context in a study of the power of mathematics in decision-making in Military Schools. However, rather than restricting her discussion to the use of mathematical models to project part of reality, she also considered the formatting power of the discipline of mathematics itself to analyse “situations in which mathematics, in some way, influences the school reality and the relationships underlying it.” (p. 42).

Skovsmose (1994) himself acknowledges that this thesis, as well as others concepts presented in his work, “are not immediately operational in relation to particular educational situations.” (p. 74).

In this paper, I intend to discuss the formatting power of mathematics in an educational context: I present an experience that took place during a mathematics course proffered to undergraduate geography students at the Federal University of Minas Gerais (UFMG), Brazil. I focus on the presentation of a mathematical modelling project developed by a group of students enrolled in the course. The objective is to analyse students’ handling of the data gathered during the development of the project. I analyse whether or not students’ procedure may be understood as an example of what Skovsmose (1994) calls the *formatting power of mathematics*.

In the next section, I describe the context in which the mathematical modelling project was developed.

2. CONTEXT OF THE STUDY

The experience I describe here took place in the first semester of 2006 (March to July) in a Mathematics course offered to undergraduate students of the geography program at UFMG. I was assigned by the Mathematics Department to teach the course. The mathematics contents planned for the course included functions, derivatives, and notions of integral. However, in the interest of developing the classes in accordance with the pace of the students, not all of this content was covered in 2006.

The main activities that took place during that semester were lectures, activities with computers, and the development of mathematical modelling projects. These activities were carried out in such a way that they were not entirely separate and disconnected from one another, and I, as the teacher, tried to stimulate students to establish relationships among them. To address the objective of this paper, I will focus on the discussion about the modelling projects.

The development of the modelling project in the mathematics course began with the discussion of a text (Araújo, 2006). In this text, I presented my understanding of mathematical modelling and suggestions for topics that should be considered in the “research proposal” to be written by the groups. At the same time, students were asked to think about themes for their projects and about the formation of groups to develop them.

In the following class, themes and groups were defined through a long process of negotiation. In the first semester of 2006, each group ended up with approximately seven members, and the themes chosen were the following: the transposition of the São Francisco River (two groups formed, one to address physical aspects and the other social aspects); physical impacts of the implantation of hydroelectric dams; socio-economic aspects of the Linha Verde (Green Line) freeway construction project in Belo Horizonte; Campus 2000: consequences for transportation in the UFMG; climate myths; and solar energy.

Once the themes had been defined, each group elaborated a work plan, which I evaluated and returned to the group. In this evaluation, I encouraged them to describe in detail all the steps to be followed during the development of the project, as well as the definition of the focus of the research. I also sought to raise questions regarding how mathematics would be used in the project.

After the projects had been approved, the groups began to carry them out, holding meetings during and outside of class. They presented partial progress reports each month, and based on these reports, each group received guidance and suggestions - my own as well as from the entire class - regarding how to proceed. During each of these advisory sessions, I sought to take into account the concerns of Critical Mathematics Education.

At the end of the semester, all the groups made an oral presentation of their project to the class (which were videotaped), and handed in a written version of the project. One project, in particular, attracted my attention because of the group’s careful treatment of the mathematical information. This project is considered in greater detail in the section that follows.

3. THE PROJECT "TRANSPOSITION OF THE SÃO FRANCISCO RIVER: PHYSICAL ASPECTS"

The theme of the group's project was "physical aspects of the transposition of the São Francisco River". The group's choice of theme portrays, at the same time, the relation with their field of interest, geography, and their interest in a controversial subject, the transposition of the São Francisco River[3]. The objective of the project was to analyse whether or not the rainfall in a given region along the course of the river would be sufficient to compensate for the amount of water that would be diverted as a result of the transposition.

This small report demonstrates the possibility for using mathematics (**quantity** of rainfall and diverted water) to discuss a problem from geography (quantity of **rainfall**) in a critical manner (questioning the environmental consequences). Thus, a mathematical modelling project was proposed that could be approached from a Critical Mathematics Education perspective.

In order to present the group's procedure for handling the data gathered during their project, in the next sub-section, I will first introduce the methodological approach used in the present study.

3.1. Methodological Aspects

The research was developed through a qualitative approach, which is characterized by: the natural environment as the source of data; being descriptive; greater interest in the process than in the final results; inductive data analysis; and the attribution of vital importance to the meaning given to the facts (Bogdan & Biklen, 1994).

The methodological procedures included observation of some meetings of one of the groups (socio-economic aspects of the Green Line project); observation of every group during the partial reports and the final presentations; and analysis of the written version of the project handed in by the groups. All the observations were videotaped.

During data collection, I had no pre-established categories to be used and/or verified. I did not seek data that confirmed (or refuted) any *a priori* established theory. Indeed, I wanted to understand the facts in the way that they occurred. Lincoln & Guba (1995) introduce, therein, the idea of inductive data analysis. According to the authors,

---

[3] The São Francisco River is the most important river in the Brazilian Northeast, the driest region in the country. The fertile areas along the river contrast with the rest of the region, dominated by caatinga. For years, there has been talk in Brazil of diverting the waters of the river to other areas of the northeast. However, the river has been suffering from pollution and silting, and it is not known whether it would withstand such a procedure. It is also known that there are political interests involved. In summary, it is a very controversial topic.

> the investigator typically does *not* work with either a priori theory or variables; these are expected to emerge from the inquiry. Data accumulated in the field thus must be analyzed *inductively* (that is, from specific, raw units of information to subsuming categories of information) in order to define local working hypotheses or questions that can be followed up. (p. 203). (Emphasis in the original).

Taking this into account, during analysis of data derived from the videotapes, I selected *episodes*, which are small “clips” of video of the groups while they were developing activities. I identified episodes that I judged to be appropriate for contributing to understanding the questions that had arisen from the research. From the written work, I selected sections or excerpts that helped to deepen my understanding of the episodes.

In the following sub-section, I present two excerpts and one episode. They show how the group that studied “physical aspects of the transposition of the São Francisco River” proceeded in their treatment of the data they collected for their project.

3.2. Excerpts and one episode: some data

At the end of the written work, the group presented a report of the steps followed to develop the project. On this report, they asserted that, after they had defined the objective of their project, they began to seek a mathematical model to represent the quantity of water in the river in the region they had chosen (the region near the cities of Juazeiro, Petrolina, Cabrobó and Penedo). Their concern was made explicit in the following question, taken from the written report[4]: “*Which mathematical model to use?*”. The group provided the following answer to this question:

Excerpt 1:

“Laura had the idea to use an exponential function. However, the group analysed the idea and observed that it wouldn’t be possible to apply it, since we didn’t have data on how much water the pump would take from the river. To solve this problem, Bernardo and Elton had the idea to use a periodic function with constant [sic] *to calculate how much the flow of the river would decrease at some points. Everyone in the group liked the idea and we decided to put it into practice.”*

[4] The complete reference of the group’s written work is not presented here to preserve their identity.

To put this idea into practice, the group searched for data on the Internet regarding the daily rainfall from 2000 to 2002, in the cities chosen. They used Excel software to organize and to manage data.

In their oral presentation of the project to the class, the group provided more details about how they treated data, as can be observed in the following episode:

Episode:

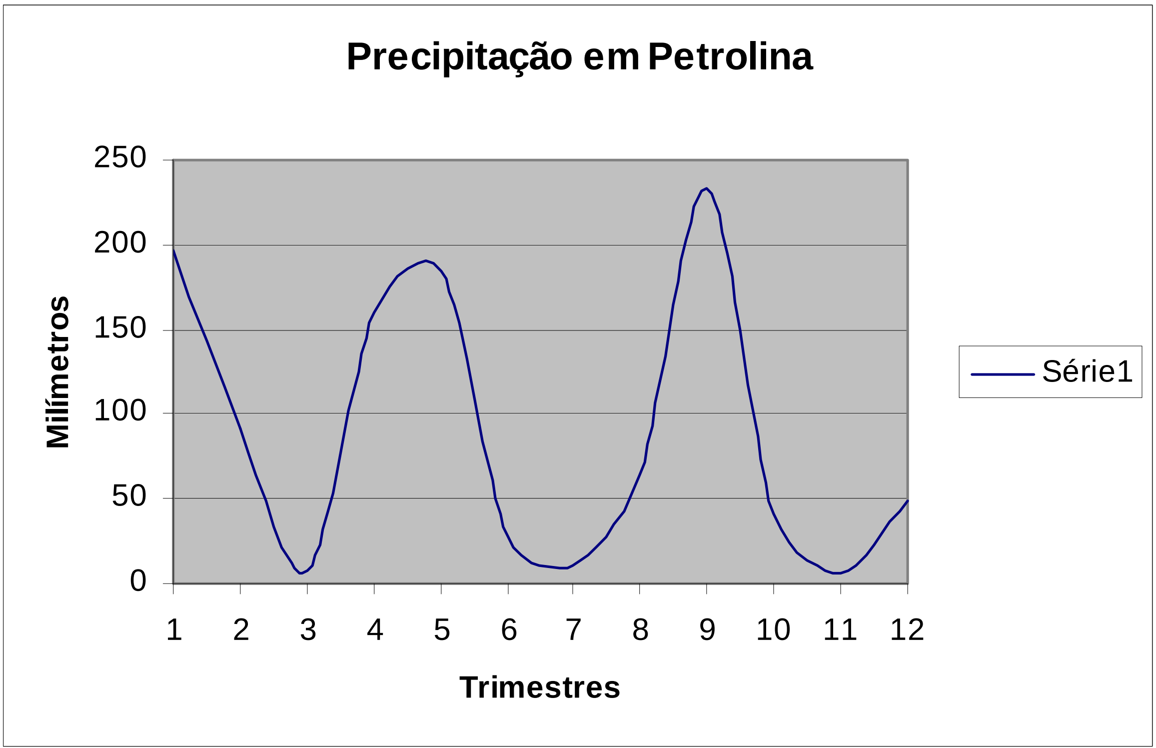
7:32 – 8:20[5]

[Elton summarises Israel's talk, explaining that the soil in the region is crystalline, which makes it difficult for the water to penetrate the soil and form aquifers. Because of this, the flow of the São Francisco River is determined mainly by the amount of rainfall; that is, it is the precipitation that will determine the flow. The group wanted to analyse the quantity of rainfall (which would determine the flow of the river) and predict what would happen with this flow following the transposition.]

8:22 – 9:30

Elton: *Then, we went and took here ... We have ... when we take a rainfall indicator of some city, as in this case, we have the general rainfall indicator. For example, 1000 mm per year, and so on. Then we took the rainfall indicator and divided it trimester by trimester, to know the influence that it had in each trimester. That is, how much it rained during each trimester. Then, doing like this, we took here, put, then ... added up... Here are ... At each four annn ... here* [pointing to the graph (Graph 1)] *it would be one trimester, two trimesters, three, four, that is, there are four trimesters here: it adds up to one year. Here we have three years represented on each one of these graphs. Then what did we do? Because if we had represented it monthly, then, during one year, we wouldn't have had a function ... Then, this function that seems ... that looks like a* [pause] *sine function, a periodic function ... If we had put just one year, then it wouldn't have had this aspect. Then, we did, we considered by trimester, so it would have this aspect.* [Pointing to the graph.].

[5] These times are relative to the video length of time.



Graph 1: amount of rainfall in the region of Petrolina from 2000 to 2002, represented trimester by trimester.

9:31 [Some comments and laughs from the audience.]

9:31 – 9:45

Elton: [laughing] *I mean ... not "so it would have" this aspect! We didn't force it ... But, doing it this way, it was easier to visualize. That is ... it was possible, but it was a bit strange! This way, it was much easier to observe.*

This procedure is also described on the written version of the work:

Excerpt 2:

"[ ] *these daily rainfall indicators were added and grouped trimester by trimester, adding up to twelve trimesters over these three years, as we can observe on the X axis of the graphs* […]. *The Y axis, in turn, represents the absolute value of precipitation for each trimester.* "

4. DISCUSSION

In summary, the group reported that, after agreeing on the objective of the research (to analyse whether the quantity of rainfall in a given region along the course of the river would be sufficient to compensate for the amount of water that would be diverted as a result of the transposition), they began to consider what mathematical model to use, and decided, without much justification, to adopt a periodic function. This choice may have resulted from the students' knowledge regarding the behaviour of rainfall, but it may also have been influenced by the subject discussed in mathematics class, which would exemplify what Araújo & Barbosa (2005) call the inverse strategy in the modelling process. This possibility is reinforced by the fact that the group did not use this part of the study to arrive at their conclusions. In the written work, they just say that "from data about the annual flow of the São Francisco River, it is possible to conclude that, if the transposition were in fact carried out, according to data available from the Integration Ministry, it could cause environmental impacts, positive as well as negative", without to present any argument based on graphics nor on mathematical calculations.

However, data that the group had gathered, related to the rainfall in a given region along the course of the river, did not seem to fit a periodic function. Thus, the solution that the group found was to re-group data in such a way that they seemed to fit a mathematical model represented by a periodic function.

Can we conclude that the group's procedure for handling data gathered during the mathematical modelling project is an example of the thesis of the formatting power of mathematics? Taking into account that no part of reality was projected or reconstructed by the group when they treated the data as they did, we can readily conclude that the group's procedure is not an example of this thesis.

However, a mathematical model was used to format the data gathered by the group. That is, the periodic model, chosen beforehand, formatted a set of information which, in principle, did not have a familiar form, if compared with the models that we had been studying in mathematics classes. Elton's disconcertion as he unveiled the group's procedure to his classmates, explaining that the group had not forced the results, shows that he had perceived some problem in their treatment of the data. I, as the teacher, could have taken advantage of this moment to exercise a *social imagination* (Negt, 1964[6] *apud* Skovsmose, 1994): what would happen, in the real transposition of the São Francisco River, if data about precipitation in the region chosen by the group were really treated in the same way by the geographers who work on the project?

The *social imagination* is related to the concept of *exemplarity* presented by Skovsmose (1994). The author uses this concept with the intention of applying sociological discussions of Critical Mathematics Education to educational contexts. Based on Negt's ideas, Skovsmose (1994) presents three theses to explain exemplarity:

1) A small phenomenon can be a reflection of a larger complexity.

[6] NEGT, O. (1964) *Soziologische Phantasie und Exemplarisches Lernen*. Europäische Verlagsanstalt, Frankfurt am Main.

2) “It is possible to understand a social complexity by concentrating on a particular event.” (p. 77).

3) The objective of education is not to transfer information to the students, but to stimulate them to try to change their own situation.

I believe that the discussion about the group’s procedure while treating data gathered during the mathematical modelling project could be broadened to the social phenomenon “Transposition of the São Francisco River”. In other words, the exemplarity of the project developed by the group could provoke a discussion among all the students regarding the formatting power of mathematics in society. Although this did not occur, I think that the three theses of exemplarity, in this situation, could potentially be:

1) The procedure of the geography students, using a mathematical model to format rainfall data in a given region, can be a reflection of data handling in larger social situations.

2) While discussing and understanding the formatting power of mathematics in the project developed by the group, the entire class could come to understand how this thesis could be applied in more complex situations.

3) This discussion could lead students to re-think their responsibility as future geographers.

Therefore, although the treatment given by the group to the modelling project data is not an example of the general thesis of the formatting power of mathematics, it represents one possible way of approaching the thesis in a specific educational situation.

REFERENCES

Araújo, J. L. (2002). *Cálculo, tecnologias e modelagem matemática: As discussões dos alunos*. Unpublished Doctoral Thesis. Rio Claro: Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista.

Araújo, J. L. (2006). Projeto de modelagem matemática. Unpublished text. Belo Horizonte: Departamento de Matemática, Universidade Federal de Minas Gerais.

Araújo, J. L., & Barbosa, J. C. (2005). Face a face com a modelagem matemática: Como os alunos interpretam essa atividade?. *Boletim de Educação Matemática (Bolema)*, 23, 79-95.

Bassanezi, R. C. (2002). *Ensino-aprendizagem com modelagem matemática*. São Paulo: Editora Contexto.

Barbosa, J. C. (2001). *Modelagem matemática: Concepções e experiências de futuros professores*. Unpublished Doctoral Thesis. Rio Claro: Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista.

Bogdan, R. C., & Biklen, S. K. (1994). *Investigação qualitativa em educação: Uma introdução à teoria e aos métodos*. (M. J. Alvarez, S. B. Santos & T. M. Baptista, Trans.). Porto: Porto Editora.

D'Ambrosio, U. (1999). Literacy, matheracy and technocracy: A trivium for today. *Mathematical Thinking and Learning*, 1(2), 131-153.

Freire, P. (1970). *Pedagogia do oprimido*. Rio de Janeiro: Editora Paz e Terra.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. California: Sage Publications.

Milanezi, P. L. (2007). O poder formatador da matemática, a ideologia da certeza e a educação matemática. In J. L. Araújo (Org.), *Educação matemática crítica: Reflexões e diálogos.* (pp. 39-47). Belo Horizonte: Argvmentvm Editora Ltda.

Skovsmose, O. (1994). *Towards a philosophy of critical mathematics education*. Dordrecht: Kluwer Academic Publishers.

SIMPLE SPREADSHEET MODELING BY FIRST-YEAR BUSINESS UNDERGRADUATE STUDENTS: DIFFICULTIES IN THE TRANSITION FROM REAL WORLD PROBLEM STATEMENT TO MATHEMATICAL MODEL

Djordje Kadijevich
Mathematical Institute SANU and Megatrend University, Serbia
Email: djkadijevic@megatrend-edu.net

Abstract. The study used a sample of first-year business undergraduate students who developed simple (deterministic & non-optimization) models with Microsoft Excel. The students were required to choose business of their choice and give recommendations to have it profitable or more profitable. To help them achieve this end, the teacher presented to the students his solution regarding a taxi business within a local area, having previously explained to the students various ways to do "what-if" analyses. By referring to the framework of Galbraith and Stillman (2006), this paper mainly presents some of frequent modelers' difficulties regarding the transition from real world problem statement to mathematical model. These difficulties deal with selecting variables, initializing variables, and relating variables. Possible reasons for such difficulties and suggestions for further research are included.

Introduction

In general, many students experience difficulties in moving between the real and the mathematical world (Crouch & Haines, 2004). This is particularly true for genuine modeling tasks (e.g. Choose business of your choice and give recommendations to have it profitable or more profitable) where modelers have to formulate problems to solve in mathematical terms. It is thus important to realize and appropriately deal with the role of context in the modeling process (Galbraith & Stillman, 2001). The use of technology may improve the matters (see Keune & Nenning, 2003), enabling us to concentrate on subtasks causing the most difficulties in moving between the real and the mathematical world. To achieve this end, Microsoft Excel can be used. As regards business applications, this software can be used for "what-if" and optimization analyses (Conway & Ragsdale, 1997; Teo & Tan, 1999). Moreover, it can be used as a DSS (Decision Support System) tool (Coles & Rowley, 1996; Heys, 2008), especially when various add-ins are utilized (e.g. SimTools for simulations, RiskOptimizer for simulations with optimizations, and XLMiner for data mining). It is important to underline that although utilizing powerful technology can promote better understanding of mathematics (Kadijevich, Haapasalo & Hvorecky, 2005), students may find challenging to develop a technology-based solutions to problems whose underlying mathematics is known to them (see Parramore, 2007).

Through searching for critical aspects relevant to transitions between main stages in the modeling process (messy real world situation, real word problem statement, mathematical model, mathematical solution, real world meaning of mathematical solution, and evaluation—revise model or accept solution), Galbraith and Stillman (2006) find that the transition from real world problem statement to mathematical model is one of the most difficult part of the modeling cycle. As regards this transition, being concerned with technology-supported modeling, these researchers recognize the following nine critical activities:

1. Identifying dependent and independent variables for inclusion in algebraic model,
2. Realizing that independent variable must be uniquely defined,
3. Representing elements mathematically so formulae can be applied,
4. Making relevant assumptions,
5. Choosing technology/mathematical tables to enable calculation,
6. Choosing technology to automate application of formulae to multiple cases,
7. Choosing technology to produce graphical representation of model,
8. Choosing to use technology to verify algebraic equation,
9. Perceiving a graph can be used on function graphers but not data plotters to verify an algebraic equation. (see p. 147)

By using a sample of first-year business undergraduate students who developed simple (deterministic & non-optimization) models with Microsoft Excel, we analyzed the shortcomings of the developed models. This analysis evidenced that many modelers are likely to fail in selecting variables, initializing variables, and relating variables. These three areas of shortcomings, which are respectively related to the above-mentioned activities 1&2, 4, and 3, are exemplified in a section to follow. Before this section we describe the learning task to be done and the help offered to the modelers.

Learning task and help offered

Learning task

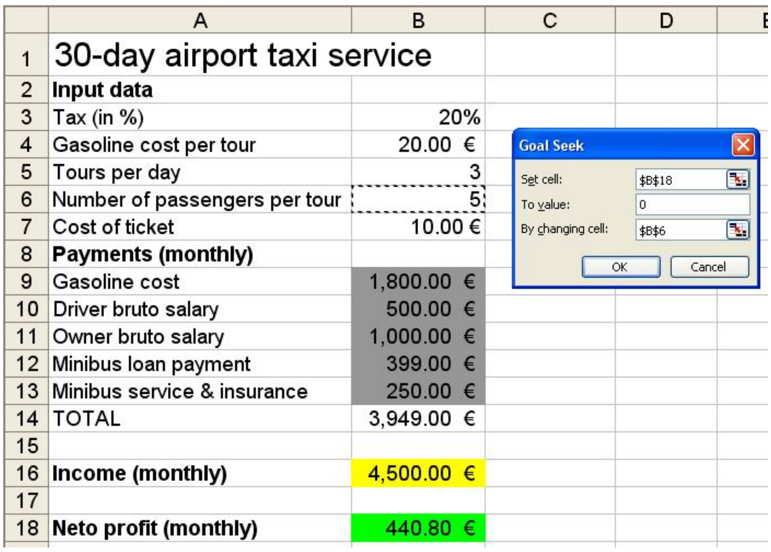
The students were required to choose business of their choice and give recommendations to have it profitable or more profitable.

Bearing in mind that prior modelers' competences of applicable mathematics and technology are to be ensured (Galbraith & Stillman, 2006), we tried to make both mathematical and technological prerequisites as simple as possible. Indeed, the developed models just reflected simple deterministic and non-optimization business situations, as will be exemplified by spreadsheet screenshots given below. Furthermore, required Excel tools (or commands to more precise) only dealt with "what-if" analyses.

Because of such prerequisites, learning challenges were mostly related to three transitions: from messy real world solution to real word problem statement, from real world problem statement to mathematical model, and from evaluation to report. Contrary to students in Galbraith and Stillman (2006) who, for example, used video animations to clarify situations to be modeled, our students had themselves to cope with all these learning challenges. However, a modest help was offered to the students.

Help offered

Having previously explained to the students various ways to achieve "what-if" analyses within Excel by using its commands Tools/Goal Seek, Data/Table, and Tools/Scenarios, the teacher (the author of this paper) presented to the students his solution regarding a taxi business within a local area. The goal (How to make business profitable?) was achieved through answering the following question: How many passengers, on average, are required by a profitable taxi business? This question was answered with the Goal Seek command that found the number of passengers when the sum of required payments and an expected income equaled zero. This is represented on Screenshot 1.



	A	B
1	30-day airport taxi service	
2	**Input data**	
3	Tax (in %)	20%
4	Gasoline cost per tour	20.00 €
5	Tours per day	3
6	Number of passengers per tour	5
7	Cost of ticket	10.00 €
8	**Payments (monthly)**	
9	Gasoline cost	1,800.00 €
10	Driver bruto salary	500.00 €
11	Owner bruto salary	1,000.00 €
12	Minibus loan payment	399.00 €
13	Minibus service & insurance	250.00 €
14	TOTAL	3,949.00 €
15		
16	**Income (monthly)**	4,500.00 €
17		
18	**Neto profit (monthly)**	440.80 €

Screenshot 1. Help provided

For these data, the Goal Seek returns 4,39 passengers. A model-grounded business recommendation would be: “For 3 tours per day and a 10-EUR ticket, the number of passengers should at least be 4 or 5, with 5 present less often!” or “Very rarely have tours with less than 4 passengers!”

Findings

The modelers worked on a modeling project that lasted 3-4 week. They mostly worked in groups with 2 or 3 students. As this task was optional, just 20-30% of all students chose to work on it. About fifteen solutions to this task were analyzed for each of the two last academic years. The subsections to follow summarize main shortcomings regarding selecting, initializing, and relating variables. Note that these tree types of shortcomings, which emerged from an informal and explorative study, usually influence each other.

Selecting variables

This shortcoming frequently occurs when modeler fails to view the costs of a production or a service through its fixed and variable parts. Consider, for example, celebrating an anniversary. The costs for a music band and a place to be rented do not depend on the number of participants, whereas the costs for food and drink to be served do so. Screenshot 2 illustrates this shortcoming. Although the modelers made the distinction between fixed and variable costs, these costs were in fact all fixed as the datum for the number of guests was not used.

It is important to underline that specifying fixed and variable costs in an appropriate and exhaustive way is a key step in developing a good business plan. While term *exhaustive* is related to selecting variables, term *appropriate* relates to initializing variables.

Initializing variables

This shortcoming deals with assigning inappropriate values to (some of) selected variables. For example, the values of payments, costs and income may not be realistic or even wrong, especially if the modelers are not familiar with the context of the analyzed business situation. Needless to say, such initializations would yield business recommendations that are not context-grounded.

Initializing variables is, for example, related to selecting variables (directly) and relating variables (indirectly). Ask, for example, weather the value of the fixed cost is appropriate in terms of its underlying fixed costs.

Ticket price (EUR)	30		
No of tables	22	?	
No of guests	132		
No of waithers	4	?	
Payments			
for restorant	1000		
for waithers	100		
for organizers	200		
for decoration	100		
for music	200		The number of waithers depends on the number of guests!
for food	800		For example, one waither per 20 guests
for drink	500		Payments for food and drink depend on the number of guests
for security	50		
Fixed costs	1350		
Variable costs	1600		Data in B22 and B24 are not used!
Total	**2950**		
Income			
from tickets	3960		
Profit	1010		

Screenshot 2. Fixed or variable costs

Relating variables

This shortcoming occurs when variables are wrongly or inappropriately related. In Screenshot 2, the variable cost is not expressed in terms of the number of participants. Another example, presented on Screenshot 3, is related to critical activity 2 mentioned in the Introduction. If students do not diversify different services offered (i.e. cleaning car, washing car, and cleaning & washing car) with respect to the number of served customers, and also combine such diversified payments and incomes, the analysis of the profit will be wrong or incomplete.

Profitability of car washing service	
PAYMENTS	
for running busines	1,000.00 €
for investment	300.00 €
for flat TAX rate	200.00 €
for workers	1,000.00 €
for water	200.00 €
for electicity	250.00 €
for marketing	150.00 €
TOTAL	**3,100.00 €**
Services	
S1: Cleaning inside	15.00 €
S2: Washing outside	10.00 €
S1 & S2	22.00 €

ELEMENTS	*VALUES*	
Service 1 (S1)	15.00 €	
Number of days per month	30	
Number of cars per day	4	PROFIT
INCOME	1,800.00 €	(1,300.00) €

ELEMENTS	*VALUES*	
Service 2 (S1)	10.00 €	
Number of days per month	25	
Number of cars per day	5	PROFIT
INCOME	1,250.00 €	(1,850.00) €

ELEMENTS	*VALUES*	
Service 2 (S1)	22.00 €	
Number of days per month	30	
Number of cars per day	5	PROFIT
INCOME	3,300.00 €	200.00 €

Screenshot 3. Relating variables

Discussion

Initializing variables reconsidered

The question of initializing variables again appears when several business scenarios are to be generated and compared (e.g. "What would the outcome under optimal, favorable and unfavorable market conditions be?"). The modelers thus should not only know what input variables are critical to their output variables, but also what values of these critical input variables should be used for different scenarios. These issues, which are relevant to the transition from evaluation to report, are connected with selecting, initializing and relating variables in the transition from real world problem statement to mathematical model. An example of the use of scenarios is given on Screenshot 4. Because the modeler's approach to her problem was disintegrated with respect to different services offered (see Screenshot 3), such a use of scenarios, though context-grounded, was useless concerning a business recommendation to propose.

Scenario Summary	Current Values:	bad	acceptable	good
Changing Cells:				
Service cost	15.00 €	20.00 €	15.00 €	12.00 €
No. of service provided	5	10	15	20
Result Cells:				
Income	75.00 €	200.00 €	225.00 €	240.00 €

Screenshot 4. Initializing variables

Reasons for the three shortcomings

Possible grounds for shortcomings in selecting and relating variables can be extrapolated from the literature. First, technology perceived as a master (see Galbraith, 2002) does not require everything to be specified. In other words, for some modelers technology may act in a smart way (even if the things are not clear to modelers who use it and they do not realize that). Second, because in the SOLO model (Biggs & Collins, 1982) person's understanding of a task progresses from single aspect (uni-structural response) to several, but disjoint, aspects (multi-structural response) to several, integrated aspects (relational response), some modelers may give multi-structural responses as evidenced on Screenshots 3 and 4.

The modelers' presentations of their solutions revealed that most of them had problems to understand the analyzed business context. Because of such problems, many students were concerned with the limitations of their models with respect to the detail richness and appropriateness of their input variables as well as the values assigned to them, which influenced the quality of selecting, initializing and relating variables.

Suggestions for further research

The modelers analyzed business situations with deterministic, non-optimization models, which required repeated calculations. But this was just a part of the landscape involving four types of models: deterministic with no optimization, deterministic with optimization, stochastic with no optimization, and stochastic with optimization. For this landscape with calculations, simulations, optimizations, and simulations with optimizations, selecting, initializing and relating variables may become selecting, initializing and relating modeling objects. Further studies may examine the following questions: "What are these modeling objects?" and "What reasons are likely to cause shortcomings in selecting, initializing and relating them?"

When using models, their results may be clear, but it may not be clear what initialization of variables to apply, what to infer when using models with such (repeatedly) changed data, and what to require for the analyzed business to improve it. Because of such metacognitive decisions, it was not surprising that making business recommendations that are both model- grounded and context-grounded was simply out of reach of most students in this study. Having realized that, the framework of Galbraith and Stillman (2006) may be extended with critical activities concerning the transition from evaluation to report. A refinement of this framework may also be needed for the above-listed activities regarding the transition from real world problem statement to mathematical model (see the Introduction). In the context of this study, choosing technology to automate application of formulae to multiple cases (Activity 7) is relevant to the former transition not the latter one. In general, a refinement of this framework may be undertaken to reflect different types of models and versatile technologies like Microsoft Excel and its add-ins.

Acknowledgement

This contribution resulted from the author's work on projects 144032D and 144050A funded by the Serbian Ministry of Science.

References

Biggs, J. B. & Collins, K. F. (1982). *Evaluating the Quality of Learning: The SOLO Taxonomy*. New York: Academic Press.

Coles, S. & Rowley, J. (1996). Spreadsheet modelling for management decision making. *Industrial Management & Data Systems*, **96**, 7, 17-23.

Conway, D. & Ragsdale, C. (1997). Modeling optimization problems in the unstructured world of spreadsheets. *Omega*, **25**, 3, 313-322.

Crouch, R. & Haines, C. (2004). Mathematical modelling: transitions between the real world and the mathematical model. *International Journal of Mathematical Education in Science and Technology*, **35**, 2, 197-206.

Galbraith, P. (2002). Life wasn't meant to be easy: separating wheat from chaff in technology aided learning. *Proceedings of the 2nd International Conference on the Teaching of Mathematics at the Undergraduate Level.* Retrieved 1 April 2006 from: www.math.uoc.gr/~ictm2/Proceedings/invGal.pdf

Galbraith, P. & Stillman, G. (2001). Assumptions and context: Pursuing their role in modelling activity. In J. F. Matos, S. K. Houston, W. Blum, & S. P. Carreira (Eds.), *Mathematical modelling and mathematics education: Applications in science and technology* (pp. 317-327). Chichester: Horwood Publishing.

Galbraith, P. & Stillman, G. (2006). A framework for identifying student blockages during transitions in the modeling process. *Zentralblatt für Didaktik der Mathematik*, **38**, 2, 143-162.

Heys, C. (2008). Getting the best out of Excel. *Teaching Mathematics and its Applications*, **24**, 1, 48-52.

Kadijevich, Dj., Haapasalo, L. & Hvorecky, J. (2005). Using technology in applications and modeling. *Teaching Mathematics and its Applications,* **24**, 2/3, 114-122.

Keune, M. & Nenning, H. (2003). Modelling and spreadsheet calculation. In Q-X. Ye, W. Blum, K. Houston & Q. Jiang (Eds.), *Mathematical modelling in education and culture: ICTMA10* (pp. 101-110). Chichester, England: Horwood Publishing.

Parramore, K. (2007). Enlisting Excel**.** *Teaching Mathematics and its Applications*, **26**, 2, 73-78.

Teo, T. S. H. & Tan, M. (1999). Spreadsheet development and ‘what-if’ analysis: quantitative versus qualitative errors. *Accounting, Management & Information Technologies*, **9**, 3, 141-160.