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The Maslov index in weak symplectic functional analysis

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Abstract We recall the Chernoff-Marsden definition of weak symplectic structure and give a rigorous treatment of the functional analysis and geometry of weak symplectic Banach spaces. We define the Maslov index of a continuous path of Fredholm pairs of Lagrangian subspaces in continuously varying Banach spaces. We derive basic properties of this Maslov index and emphasize the new features appearing.

Keywords Closed relations, Fredholm pairs of Lagrangians, Maslov index, spectral flow, symplectic splitting, weak symplectic structure

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1 Introduction

1.1 Our setting and goals

First, we recall the main features of finite-dimensional and infinite-dimensional strong symplectic analysis and geometry and argue for the need to generalize from strong to weak assumptions.

1.1.1 The finite-dimensional case

The study of dynamical systems and the variational calculus of N-particle classical mechanics automatically lead to a symplectic structure in the phase space $X = \mathbb{R}^{6N}$ of position and

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impulse variables: when we trace the motion of N particles in 3-dimensional space, we deal with a bilinear (in the complex case sesquilinear) anti-symmetric (in the complex case skew-symmetric) and non-degenerate form $\omega : X \times X \to \mathbb{R}$. The reason for the skew-symmetry is the asymmetry between position and impulse variables corresponding to the asymmetry of differentiation. To carry out the often quite delicate calculations of mechanics, the usual trick is to replace the skew-symmetric form ω by a skew-symmetric matrix J with $J^2 = -I$ such that

$$\omega(x,y) \;=\; \langle Jx, y\rangle \quad \text{ for all } x,y \in X, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in X.

For geometric investigations, the key concept is a Lagrangian subspace of the phase space. For two continuous paths of Lagrangian subspaces, an intersection index, the *Maslov index* is well defined. It can be considered a re-formulation or generalization of counting conjugate points on a geodesic. In Morse Theory, this number equals the classical Morse index, i.e., the number of negative eigenvalues of the Morse index form. This Morse Index Theorem (cf. M. Morse [30]) for geodesics on Riemannian manifolds was extended by W. Ambrose [1], J.J. Duistermaat [22], P. Piccione and D.V. Tausk [33,34], and the second author [41,42].

For a systematic review of the basic vector analysis and geometry and for the physics background, we refer to V.I. Arnold [2] and M. de Gosson [25].

1.1.2 The strong symplectic infinite-dimensional case

As shown by K. Furutani and the first author in [7], the finite-dimensional approach of the Morse Index Theorem can be generalized to a separable Hilbert space when we assume that the form ω is bounded and can be expressed by (1) with a bounded operator J, which is skew-self-adjoint (i.e., $J^* = -J$) and not only injective but invertible. The invertibility of J is the whole point of *strong* symplectic structure. Then, without loss of generality, one can assume $J^2 = -I$ like in the finite-dimensional case (see also Lemma 1 below), and many calculations of the finite-dimensional case can be preserved with only slight modification. The model space for strong symplectic Hilbert spaces is the von Neumann space $\beta(A) := \operatorname{dom}(A^*)/\operatorname{dom}(A)$ of *natural* boundary values of a closed symmetric operator A in a Hilbert space X with symplectic form given by Green's form

$$\omega(\gamma(u),\gamma(v)) :=\; \langle A^*u, v\rangle - \langle u, A^*v\rangle \quad \text{ for all } u,v \in \operatorname{dom}(A^*), \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in X and $\gamma : \operatorname{dom}(A^*) \to \beta(A)$ denotes the trace map. A typical example is provided by a linear symmetric differential operator A of first order over a manifold M with boundary Σ. Here we have the minimal domain $\operatorname{dom}(A) = H^1_0(M)$ and the maximal domain $\operatorname{dom}(A^*) \supset H^1(M)$. Note that the inclusion is strict for $\dim M > 1$. Recall that $H^1_0(M)$ denotes the closure of $C^\infty_0(M\setminus\Sigma)$ in $H^1(M)$. For better reading we don't mention the corresponding vector bundles in the notation of the Sobolev spaces of vector bundle sections.

As in the finite-dimensional case, the basic geometric concept in infinite-dimensional strong symplectic analysis is the Lagrangian subspace, i.e., a subspace which is isotropic and co-isotropic at the same time. Contrary to the finite-dimensional case, however, the common definition of a Lagrangian as a *maximal* isotropic space or an isotropic space of *half* dimension becomes desolate.

To define the Maslov index in the infinite-dimensional case as intersection number of two continuous paths of Lagrangian subspaces, one has to make the additional assumption that corresponding Lagrangians make a Fredholm pair so that, in particular, we have finite intersection dimensions.

Following a suggestion by A. Floer [23], a multitude of formulae was achieved of varying generality to express the spectral flow of a curve of self-adjoint extensions of a fixed or of a curve of symmetric operators by the Maslov index of corresponding curves of Lagrangians, see T. Yoshida [39], L. Nicolaescu [31], S. E. Cappell, R. Lee, and E. Y. Miller [18], the first author, jointly with K. Furutani and N. Otsuki [8,9] and P. Kirk and M. Lesch [27]. See also the results by the present authors in [13] for varying boundary conditions but fixed maximal domain and in [14] (in preparation) also for varying maximal domain. Recently, M. Prokhorova [35] considered a path of Dirac operators on a two-dimensional disk with a finite number of holes subjected to local elliptic boundary conditions and obtained a beautiful explicit formula for the spectral flow (respectively, the Maslov index).

1.1.3 Beyond the limits of the strong symplectic assumption

Weak (i.e., not necessarily strong) symplectic structures are met on the way to a spectral flow formula in the full generality wanted: for continuous curves of, say linear formally self-adjoint elliptic differential operators of first order over a compact manifold of dimension ≥ 2 with boundary and with varying maximal domain (i.e., admitting arbitrary continuous variation of the coefficients of first order) and with continuously varying regular (elliptic) boundary conditions, see [14]. An interesting new feature for the comprehensive generalization is the following "technical" problem: For regular (elliptic) boundary value problems (say for a linear formally self-adjoint elliptic differential operator A of first order on a compact smooth manifold M with boundary Σ), there are three canonical spaces of boundary values: the above mentioned von Neumann space $\beta(A) = \operatorname{dom}(A^*)/\operatorname{dom}(A)$, which is a subspace of the distributional Sobolev space $H^{-1/2}(\Sigma)$; the space of boundary values $H^{1/2}(\Sigma) \simeq H^1(M)/H^1_0(M)$ of the operator domain $H^1(M)$; and the most familiar and basic $L^2(\Sigma)$.[1] As in (2), Green's form induces symplectic forms on all three section spaces which are mutually compatible.

More precisely, Green's form yields a strong symplectic structure not only on $\beta(A)$, but also on $L^2(\Sigma)$ by

$$\omega(x,y) := -\langle Jx, y\rangle_{L^2(\Sigma)}\,.$$

Here J denotes the principal symbol of the operator A over the boundary in inner normal direction. It is invertible (= injective and surjective, i.e., with bounded inverse) since A is elliptic. For the induced symplectic structure on the Sobolev space $H^{1/2}(\Sigma)$ the corresponding operator J' is *not* invertible for $\dim\Sigma \geq 1$, see Remark 2b in Section 2.1 below. So, for $\dim\Sigma \geq 1$ the space $H^{1/2}(\Sigma)$ becomes an only *weak* symplectic Hilbert space, to use a notion introduced by Chernoff and Marsden [19, Section 1.2, pp. 4-5].

An additional incitement to investigate weak symplectic structures comes from the stunning observation of E. Witten (explained by M.F. Atiyah in [3] in a heuristic way). He considered a weak (and degenerate) symplectic form on the loop space $\operatorname{Map}(S^1, M)$ of a finite-dimensional closed orientable Riemannian manifold M and noticed that a (future) thorough understanding of the infinite-dimensional symplectic geometry of that loop space "should

[1] In the tradition of geometry inspired analysis, we think mostly of *homogeneous* systems when talking of elliptic boundary value problems. Our key reference is the monograph [11] by K. P. Wojciechowski and the first author and the supplementary elaborations by J. Brüning and M. Lesch in [16]. For a more comprehensive treatment, emphasizing *non-homogeneous* boundary value problems and assembling all relevant section spaces in a huge algebra, we refer to the more recent [37] by B.-W. Schulze.

lead rather directly to the index theorem for Dirac operators" (l.c., p. 43). Of course, restricting ourselves to the linear case, i.e. to the geometry of Lagrangian subspaces instead of Lagrangian manifolds, we can only marginally contribute to that program in this paper.

1.2 Main results and plan of the paper

In this paper our goal is to deal with the preceding "technical" problem. To do that, we generalize the results of J. Robbin and D. Salamon [36], S.E. Cappell, R. Lee, and E.Y. Miller [17], K. Furutani, N. Otsuki and the first author in [8,9] and of P. Kirk and M. Lesch in [27], i.e., we shall give a rigorous definition of the Maslov index for continuous curves of Fredholm pairs of Lagrangian subspaces in varying weak symplectic Banach spaces and derive basic properties. We also want to find a method to treat the case of singular manifolds. Consequently, part of our results will be formulated and proved for relations instead of operators.

At the whole, we aim for a "clean" presentation in the sense that results are proved in the affordable generality. We shall, e.g., prove purely algebraic results algebraically in symplectic vector spaces and purely topological results in Banach spaces when ever possible - in spite of the fact that we shall deal with symplectic Hilbert spaces in most applications.

The routes of [8,9] and [27] are barred to us because they rely on the concept of strong symplectic Hilbert space. Consequently, we have to replace some of the familiar arguments of symplectic analysis by new arguments. A few of the most elegant lemmata of strong symplectic analysis can not be retained, but, luckily, the new arguments will show a considerable strength that is illustrative and applicable also in the conventional strong case.

In Section 2, we give a thorough presentation of weak symplectic functional analysis. Basic concepts are defined in Subsection 2.1. A new feature of weak symplectic analysis is the lack of a canonical symplectic splitting: for strong symplectic Hilbert space, we can assume $J^2 = -I$ by smooth deformation of the metric, and obtain the canonical splitting $X = X^+ \oplus X^-$ into mutually orthogonal closed subspaces $X^\pm := \ker(J \mp iI)$ which are both invariant under J. That permits the representation of all Lagrangian subspaces as graph of unitary operators from X^+ to X^- (see Lemma 2), which yields a transfer of contractibility from the unitary group to the space of Lagrangian subspaces. Moreover, that representation is the basis for a functional analytical definition of the Maslov index. For weak symplectic Hilbert or Banach space, the preceding construction doesn't work any longer and we must assume that a symplectic splitting is given and fixed (its existence follows, e.g., from Zorn's Lemma).

For applications to an elliptic differential operator A of first order, acting on sections of a Hermitian vector bundle E over the Riemannian manifold M with boundary Σ, we note that the symplectic Hilbert space structures of $H^{1/2}(\Sigma;E|_\Sigma)$ and $L^2(\Sigma;E|_\Sigma)$ are compatible and their symplectic splitting is defined by the bundle endomorphism (the principal symbol of A in inner normal direction) $J : E|_\Sigma \to E|_\Sigma$ in the following way:

$$
\begin{aligned}
H^\pm \;:=\; H^{1/2}(\Sigma;E^\pm|_\Sigma) \quad \text{and} \quad L^\pm \;:=\; L^2(\Sigma;E^\pm|_\Sigma)& \\
\text{with } E^\pm|_\Sigma \;:=\; \text{lin. span of} \begin{Bmatrix} \text{positive} \\ \text{negative} \end{Bmatrix} \text{eigenspaces of } iJ. & \qquad (3)
\end{aligned}
$$

Note that L^+, L^- change continuously if J changes continuously.

In Subsection 2.2, we turn to Fredholm pairs of Lagrangian subspaces to prepare for the counting of intersection dimensions in the definition of the Maslov index. Here another new feature of weak symplectic analysis is that the Fredholm index of a Fredholm pair of Lagrangian subspaces need not vanish. On the one hand, this opens the gate to new interesting theorems. On the other hand, the re-formulation of well-known definitions and lemmata in the weak symplectic setting becomes rather heavy since we have to add the vanishing of the Fredholm index as an explicit assumption.

As a side effect of our weak symplectic investigation, we hope to enrich the classical literature with our new purely algebraic conditions for isotropic subspaces becoming Lagrangians, see Lemma 4 and Propositions 1 and 2.

At present, the homotopy types of the full Lagrangian Grassmannian and of the Fredholm Lagrangian Grassmannian remain unknown for weak symplectic structures. As a service to the reader, we give a list of related open problems in Subsection 2.3 below. To us, however, it seems remarkable that a wide range of familiar geometric features can be regained in weak symplectic functional analysis – in spite of the incomprehensibility of the basic topology!

In Subsection 2.4, we lay the next foundation for a rigorous definition of the Maslov index by investigating continuous curves of operators and relations that generate Lagrangians in the new wider setting. Referring to the concepts of our Appendix, we define the spectral flow of such curves.

In Section 3 we finally come to the intersection geometry. In Subsection 3.1, we show how to treat continuously varying weak symplectic structures and define the Maslov index in fixed weak symplectic Banach space with continuously varying symplectic splitting. We obtain the full list of basic properties of the Maslov index as listed by S.E. Cappell, R. Lee, and E.Y. Miller in [17]. We can not claim that this new Maslov index is always independent of the splitting projections. However, for strong symplectic Banach space the independence will be proved in Proposition 6. That establishes the coincidence with the common definition of the Maslov index.

In Subsection 3.2, in our general context, we establish the relation between real symplectic analysis (in the tradition of classical mechanics) on the one side, and the more elegant complex symplectic analysis (as founded by J. Leray in [28]) on the other side.

In Subsection 3.3, we pay special attention to questions related to the embedding of symplectic spaces, Lagrangian subspaces and curves into larger symplectic spaces. Our investigations are inspired by the extremely delicate embedding questions between the two strong symplectic Hilbert spaces $\beta(A)$ and $L^2(\Sigma)$ as studied by K. Furutani, N. Otsuki and the first author in [9]. One additional reason for our interest in embedding problems is our observation of Remark 2c, that each weak symplectic Hilbert space can naturally be embedded in a strong symplectic Hilbert space, imitating the embedding of $H^{1/2}(\Sigma)$ into $L^2(\Sigma)$.

In Appendix A, we give a rigorous definition of the spectral flow for admissible families of closed relations. Our discussion of continuous operator families in Subsection 2.4 and the whole of Section 3 is based on that definition.

The main results of this paper were achieved many years ago by the authors and informally disseminated in [12]. Through all the years, our goal was to establish a truly general spectral flow formula by applying the weak symplectic functional analysis. But here we met a technical gap in the argumentation: Only recently we found the correct sufficient conditions for continuous variation of the Cauchy data spaces (or, alternatively put, the continuous variation of the pseudo-differential Calderón projection) for curves of elliptic operators in joint work with G. Chen and M. Lesch [6]. Now that gap is bridged and a full general spectral flow formula is obtained in [14]. We have a full picture of the meaning of weak symplectic functional analysis and consider that the time has come for regular publication of our results.

2 Weak symplectic functional analysis

2.1 Basic symplectic functional analysis

We fix our notation. To keep track of the required assumptions we shall not always assume that the underlying space is a Hilbert space but permit Banach spaces and – for some concepts – even just vector spaces. For easier presentation and greater generality, we begin with *complex* symplectic spaces.

Definition 1 Let X be a complex Banach space. A mapping

$$\omega : X \times X \longrightarrow \mathbb{C}$$

is called a (weak) symplectic form on X, if it is sesquilinear, bounded, skew-symmetric, and non-degenerate, i.e.,
(i) $\omega(x,y)$ is linear in x and conjugate linear in y;
(ii) $|\omega(x,y)| \leq C\|x\|\|y\|$ for all $x,y \in X$;
(iii) $\omega(y,x) = -\overline{\omega(x,y)}$;
(iv) $X^{\omega} := \{x \in X \mid \omega(x,y) = 0 \text{ for all } y \in X\} = \{0\}$.
Then we call (X,ω) a *(weak) symplectic Banach space*.

There is a purely algebraic concept, as well.

Definition 2 Let X be a complex vector space and ω a form which satisfies all the assumptions of Definition 1 except (ii). Then we call (X,ω) a *complex symplectic vector space*.

Definition 3 Let (X,ω) be a complex symplectic vector space.
(a) The *annihilator* of a subspace λ of X is defined by

$$\lambda^{\omega} := \{y \in X \mid \omega(x,y) = 0 \quad \text{for all } x \in \lambda\}.$$

(b) A subspace λ is called *symplectic*, *isotropic*, *co-isotropic*, or *Lagrangian* if

$$\lambda \cap \lambda^{\omega} = \{0\}, \quad \lambda \subset \lambda^{\omega}, \quad \lambda \supset \lambda^{\omega}, \quad \lambda = \lambda^{\omega},$$

respectively.
(c) The *Lagrangian Grassmannian* $\mathscr{L}(X,\omega)$ consists of all Lagrangian subspaces of (X,ω).

Definition 4 Let (X,ω) be a symplectic vector space and X^+, X^- be linear subspaces. We call (X, X^+, X^-) a *symplectic splitting* of X, if $X = X^+ \oplus X^-$, the quadratic form $-i\omega$ is positive definite on X^+ and negative definite on X^-, and

$$\omega(x,y) = 0 \quad \text{for all } x \in X^+ \text{ and } y \in X^-. \tag{4}$$

Remark 1 (a) By definition, each 1-dimensional subspace in real symplectic space is isotropic, and there always exists a Lagrangian subspace. However, there are complex symplectic Hilbert spaces without any Lagrangian subspace. That is, in particular, the case if $\dim X^+ \neq \dim X^-$ in $\mathbb{N} \cup \{\infty\}$ for a single (and hence for all) symplectic splittings.
(b) If $\dim X$ is finite, a subspace λ is Lagrangian if and only if it is isotropic with $\dim \lambda = \frac{1}{2}\dim X$.
(c) In symplectic Banach space, the annihilator λ^{ω} is closed for any subspace λ. In particular, all Lagrangian subspaces are closed, and we have for any subspace λ the inclusion

$$\lambda^{\omega\omega} \supset \overline{\lambda}. \tag{5}$$

(d) Let X be a vector space and denote its (algebraic) dual space by X'. Then each symplectic form ω induces a uniquely defined injective mapping $J : X \to X'$ such that

$$\omega(x,y) \ = \ (Jx,y) \quad \text{ for all } x,y \in X, \tag{6}$$

where we set $(Jx,y) := (Jx)(y)$.

If (X,ω) is a symplectic Banach space, then the induced mapping J is a bounded, injective mapping $J : X \to X^*$ where X^* denotes the (topological) dual space. If J is also surjective (so, invertible), the pair (X,ω) is called a *strong symplectic Banach space*. As mentioned in the Introduction, we have taken the distinction between *weak* and *strong* symplectic structures from Chernoff and Marsden [19, Section 1.2, pp. 4-5].

If X is a Hilbert space with symplectic form ω, then the induced mapping J is a bounded, skew-self-adjoint operator (i.e., $J^* = -J$) on X with $\ker J = \{0\}$ and can be written in the form $J = \begin{pmatrix} iA_+ & 0 \\ 0 & -iA_- \end{pmatrix}$ with $A_\pm > 0$ bounded self-adjoint (but not necessarily invertible, i.e., $A_\pm^{-1}$ not necessarily bounded). As in the strong symplectic case, we then have that $\lambda \subset X$ is Lagrangian if and only if $\lambda^\perp = J\lambda$.

The proof of the following lemma is straightforward and is omitted.

Lemma 1 *Any strong symplectic Hilbert space $(X,\langle\cdot,\cdot\rangle,\omega)$ (i.e., with invertible J) can be made into a strong symplectic Hilbert space $(X,\langle\cdot,\cdot\rangle',\omega)$ with $J'^2 = -I$ by smooth deformation of the inner product of X into*

$$\langle x,y\rangle' \ := \ \langle \sqrt{J^*J}x,y\rangle$$

without changing ω.

Remark 2 (a) In a strong symplectic Hilbert space many calculations become quite easy. E.g., the inclusion (5) becomes an equality, and all Fredholm pairs of Lagrangian subspaces have vanishing index.
(b) From the Introduction, we recall an important example of a weak symplectic Hilbert space: Let A be a formally self-adjoint linear elliptic differential operators of first order over a smooth compact Riemannian manifold M with boundary Σ. As mentioned in the Introduction, we have (we suppress mentioning the vector bundle)

$$H^{1/2}(\Sigma) \simeq H^1(M)/H^1_0(M)$$

with uniformly equivalent norms. Green's form yields a strong symplectic structure on $L^2(\Sigma)$ by

$$\{x,y\} \ := \ -\langle Jx,y\rangle_{L^2(\Sigma)} \,.$$

Here J denotes the principal symbol of the operator A over the boundary in inner normal direction. It is invertible since A is elliptic. For the induced symplectic structure on $H^{1/2}(\Sigma)$ we define J' by

$$\{x,y\} \ = \ -\langle J'x,y\rangle_{H^{1/2}(\Sigma)} \quad \text{ for } x,y \in H^{1/2}(\Sigma).$$

Let B be a formally self-adjoint elliptic operator B of first order on Σ. By Gårding's inequality, the $H^{1/2}$ norm is equivalent to the induced graph norm. This yields $J' = (I+|B|)^{-1}J$. Since B is elliptic, it has compact resolvent. So, $(I+|B|)^{-1}$ is compact in $L^2(\Sigma)$; and so is J'. Hence J' is not invertible. In the same way, any dense subspace of $L^2(\Sigma)$ inherits a weak symplectic structure from $L^2(\Sigma)$.

(c) Each weak symplectic Hilbert space $(X, \langle\cdot,\cdot\rangle, \omega)$ with induced injective skew-self-adjoint J can naturally be embedded in a strong symplectic Hilbert space $X', \langle\cdot,\cdot\rangle', \omega')$ with invertible induced J' by setting $\langle x,y\rangle' := \langle |J|x,y\rangle$ as in Lemma 1 and then completing the space. This imitates the situation of the embedding of $H^{1/2}(\Sigma)$ into $L^2(\Sigma)$. It shows that the weak symplectic Hilbert space $H^{1/2}(\Sigma)$ with its embedding into $L^2(\Sigma)$ yields a model for all weak symplectic Hilbert spaces. In Section 3.3, we shall elaborate on the embedding weak $\hookrightarrow$ strong a little further.

A key result in symplectic analysis is the following lemma. The representation of Lagrangian subspaces as graphs of unitary mappings from one component X^+ to the complementary component X^- of the underlying symplectic vector space (to be considered as the induced complex space in classical real symplectic analysis, see, e.g., K. Furutani and the first author [7, Section 1.1]) goes back to J. Leray [28]. We give a simplification for complex vector spaces, first announced in [41]. Of course, the main ideas were already contained in the real case.

Lemma 2 *Let (X,ω) be a strong symplectic Hilbert space with $J^2 = -I$. Then*

(i) the space X splits into the direct sum of mutually orthogonal closed subspaces

$$X \;=\; \ker(J-iI)\oplus\ker(J+iI),$$

which are both invariant under J;

(ii) there is a 1-1 correspondence between the space $\mathscr{U}^J$ of unitary operators from $\ker(J-iI)$ *to* $\ker(J+iI)$ *and $\mathscr{L}(X,\omega)$ under the mapping $U\mapsto\lambda := \mathfrak{G}(U)$ (= graph of U);*

(iii) if $U,V\in\mathscr{U}^J$ and $\lambda := \mathfrak{G}(U)$, $\mu := \mathfrak{G}(V)$, then (λ,μ) is a Fredholm pair (see Definition 5b) if and only if $U-V$, or, equivalently, $UV^{-1}-I_{\ker(J+iI)}$ is Fredholm. Moreover, we have a natural isomorphism

$$\ker(UV^{-1}-I_{\ker(J+iI)})\simeq\lambda\cap\mu\,. \tag{7}$$

The proof of (i) is clear; (ii) will follow from Lemma 3; and (iii) from Proposition 2 below.

The preceding method to characterize Lagrangian subspaces and to determine the dimension of the intersection of a Fredholm pair of Lagrangian subspaces provides the basis for defining the Maslov index in strong symplectic spaces of infinite dimensions (see, in different formulations and different settings, the quoted references [7], [9], [24], [27], and Zhu and Long [43]).

Surprisingly, it can be generalized to weak symplectic Banach spaces in the following way.

Lemma 3 *Let (X,ω) be a symplectic vector space with a symplectic splitting (X,X^+,X^-). (a) Each isotropic subspace λ can be written as the graph*

$$\lambda \;=\; \mathfrak{G}(U)$$

of a uniquely determined injective operator

$$U:\mathrm{dom}(U)\longrightarrow X^-$$

with $\mathrm{dom}(U)\subset X^+$. *Moreover, we have*

$$\omega(x,y) \;=\; -\omega(Ux,Uy) \quad \textit{for all } x,y\in\mathrm{dom}(U). \tag{8}$$

(b) If X is a Banach space, then $X^{\pm}$ are always closed and the operator U, defined by a Lagrangian subspace λ is closed as an operator from X^+ to X^- (not necessarily densely defined).
(c) For a closed isotropic subspace λ in a strong symplectic Banach space X, we have $\operatorname{dom}(U)$ *and* $\operatorname{im} U$ *are closed. In particular, if λ is Lagrangian, then* $\operatorname{dom}(U) = X^+$ *and* $\operatorname{im} U = X^-$*; i.e., the generating U is bounded and surjective with bounded inverse.*

Proof a. Let $\lambda \subset X$ be isotropic and $v_+ + v_-, w_+ + w_- \in \lambda$ with $v_\pm, w_\pm \in X^\pm$. By the isotropic property of λ and our assumption about the splitting $X = X^+ \oplus X^-$ we have

$$0 \;=\; \omega(v_+ + v_-, w_+ + w_-) \;=\; \omega(v_+, w_+) + \omega(v_-, w_-). \tag{9}$$

In particular, we have

$$\omega(v_+ + v_-, v_+ + v_-) \;=\; \omega(v_+, v_+) + \omega(v_-, v_-) \;=\; 0$$

and so $v_- = 0$ if and only if $v_+ = 0$. So, if the first (respectively the second) components of two points $v_+ + v_-, w_+ + w_- \in \lambda$ coincide, then also the second (respectively the first) components must coincide.

Now we set

$$\operatorname{dom}(U) \;:=\; \{x \in X^+ \mid \exists y \in X^- \text{ such that } x + y \in \lambda\}.$$

By the preceding argument, y is uniquely determined, and we can define $Ux := y$. By construction, the operator U is an injective linear mapping, and property (8) follows from (9).
b. One checks easily that $X^\pm = (X^\mp)^\omega$. Annihilators are always closed. This proves the first part of (*b*). Now let λ be a Lagrangian subspace, i.e., $\lambda = \lambda^\omega$. So, λ is closed. It is the graph of U. So U is closed.
c. Let $\lambda = \mathfrak{G}(U)$. Let $\{x_n\}$ be a sequence in X^+ convergent to $x \in X^+$. Since X is strong, we see from (8) that the sequence $\{Ux_n\}$ is a Cauchy sequence and therefore is also convergent. Denote by y the limit of $\{Ux_n\}$. Since λ is closed, we have $x \in \operatorname{dom} U$ and $y = Ux$. Thus $\operatorname{dom}(U)$ is closed. We apply the same argument to $\operatorname{dom}(U^{-1}) \subset X^-$, relative to the inner product $i\omega$ and obtain that $\operatorname{im} U$ is closed.

Now assume that λ is a Lagrangian subspace. Firstly we show that U is densely defined in X^+. Indeed, if $\overline{\operatorname{dom}(U)} \neq X^+$, there would be a $v \in V$ where V denotes the orthogonal complement of $\operatorname{dom}(U)$ in X^+ with respect to the inner product on X^+ defined by $-i\omega$. Clearly $(\operatorname{dom}(U))^\omega = V + X^-$. So, $V = (\operatorname{dom}(U))^\omega \cap X^+$. Then $v + 0 \in \lambda^\omega \setminus \lambda$. That contradicts the Lagrangian property of λ. So, we have $\overline{\operatorname{dom}(U)} = X^+$.

Since U has a closed graph, it follows that $\operatorname{dom}(U) = X^+$ and U is bounded. Applying the same arguments to $\operatorname{dom}(U^{-1}) \subset X^-$, relative to the inner product $i\omega$ yields $\operatorname{im} U = \operatorname{dom}(U^{-1}) = X^-$ and U^{-1} is bounded.

Remark 3 (a) Note that the symplectic splitting is not unique. Its existence can be proved by Zorn's Lemma. In our applications, the geometric background provides natural splittings (see Equation 3). For varying splittings see also the discussion below in Section 3.
(b) The symplectic splitting and the corresponding *graph* representation of isotropic and Lagrangian subspaces must be distinguished from the splitting in complementary Lagrangian subspaces which yields the common representation of Lagrangian subspaces as *images* in the real category (see Lemma 9 below).

2.2 Fredholm pairs of Lagrangian subspaces

A main feature of symplectic analysis is the study of the *Maslov index*. It is an intersection index between a path of Lagrangian subspaces with the *Maslov cycle*, or, more generally, with another path of Lagrangian subspaces.

Before giving a rigorous definition of the Maslov index in weak symplectic functional analysis (see below Section 3) we fix the terminology and give several simple criteria for a pair of isotropic subspaces to be Lagrangian.

We recall:

Definition 5 (a) The space of (algebraic) *Fredholm pairs* of linear subspaces of a vector space X is defined by

$$\mathscr{F}^2_{\mathrm{alg}}(X) := \{(\lambda,\mu) \mid \dim\lambda\cap\mu < +\infty \text{ and } \dim X/(\lambda+\mu) < +\infty\} \tag{10}$$

with

$$\operatorname{index}(\lambda,\mu) := \dim\lambda\cap\mu - \dim X/(\lambda+\mu). \tag{11}$$

(b) In a Banach space X, the space of (topological) *Fredholm pairs* is defined by

$$\mathscr{F}^2(X) := \{(\lambda,\mu)\in\mathscr{F}^2_{\mathrm{alg}}(X) \mid \lambda,\mu \text{ and } \lambda+\mu\subset X \text{ closed}\}. \tag{12}$$

Remark 4 Actually, in Banach space the closedness of $\lambda+\mu$ follows from its finite codimension in X in combination with the closedness of λ,μ (see [8, Remark A.1] and [26, Problem 4.4.7]). So, the set of algebraic Fredholm pairs of Lagrangian subspaces of a symplectic Banach space X coincides with the set $\mathscr{F}\mathscr{L}^2(X)$ of topological Fredholm pairs of Lagrangian subspaces of X.

We begin with a simple algebraic observation.

Lemma 4 *Let (X,ω) be a symplectic vector space with transversal subspaces λ,μ. If λ,μ are isotropic subspaces, then they are Lagrangian subspaces.*

Proof From linear algebra we have

$$\lambda^\omega\cap\mu^\omega = (\lambda+\mu)^\omega = \{0\},$$

since $\lambda+\mu=X$. From

$$\lambda\subset\lambda^\omega, \mu\subset\mu^\omega \tag{13}$$

we get

$$X = \lambda^\omega\oplus\mu^\omega. \tag{14}$$

To prove $\lambda^\omega=\lambda$ (and similarly for μ), we consider a $x\in\lambda^\omega$. It can be written in the form $x=y+z$ with $y\in\lambda$ and $z\in\mu$ because of the splitting $X=\lambda\oplus\mu$. Applying (13) and the splitting (14) we get $y=x$ and so $z=0$, hence $x\in\lambda$.

With a little work, the preceding lemma can be generalized from direct sum decomposition to (algebraic) Fredholm pairs. Firstly we have

Lemma 5 *Let V,W be two vector spaces and $f: V\times W\to\mathbb{C}$ be a sesquilinear mapping. Assume that* $\dim W<+\infty$. *If for each $v\in V$, the condition $f(v,w)=0$ for all $w\in V$ implies $v=0$, then we have* $\dim V\leq\dim W$.

Proof Let $\tilde{W}$ be the space of conjugate linear functionals on W. Let $\tilde{f} : V \to \tilde{W}$ be the induced map of f defined by $(\tilde{f}(v))(w) = f(v,w)$. Then $\tilde{f}$ is linear. Our condition is $\tilde{f}$ is injective. Thus we have $\dim V \le \dim \tilde{W} = \dim W$.

Corollary 1 *Let (X,ω) denote a symplectic vector space.*
(a) For any finite-codimensional linear subspace λ, we have $\dim \lambda^{\omega} \le \dim X/\lambda$.
(b) For any finite dimensional linear subspace μ, we have $\mu^{\omega\omega} = \mu$ and $\dim \mu = \dim X/\mu^{\omega}$.

Proof a. Define $f : \lambda^{\omega} \times (X/\lambda) \to \mathbb{C}$ by $f(x, y+\lambda) = \omega(x,y)$ for all $x \in \lambda^{\omega}$ and $y \in X$. Then f satisfies the condition in Lemma 5. So our result follows.
b. Define $g : (X/\mu^{\omega}) \times \mu \to \mathbb{C}$ by $g(x+\mu^{\omega}, y) = \overline{\omega(x,y)}$ for all $x \in \mu$ and $y \in H$. Then g satisfies the condition in Lemma 5. So we have $\dim X/\cap \mu^{\omega} \le \dim \mu$. By (a) we have $\dim \mu^{\omega\omega} \le \dim X/\cap \mu^{\omega}$. Since $\mu \subset \mu^{\omega\omega}$, our result follows.

Proposition 1 *Let (X,ω) be a symplectic vector space and $(\lambda,\mu) \in \mathscr{F}^2_{\mathrm{alg}}(X)$. If λ,μ are isotropic subspaces with* $\operatorname{index}(\lambda,\mu) \ge 0$*, then λ and μ are Lagrangian subspaces of X,*

$$\operatorname{index}(\lambda,\mu) = 0, \quad (\lambda+\mu)^{\omega} = \lambda \cap \mu, \quad \text{and } (\lambda+\mu)^{\omega\omega} = \lambda+\mu.$$

Proof Firstly we show that $\widetilde{X} := (\lambda+\mu)/(\lambda\cap\mu)$ is a symplectic vector space with the induced form

$$\widetilde{\omega}([x+y],[\xi+\eta]) := \omega(x+y,\xi+\eta) \quad \text{for } x,\xi \in \lambda \quad \text{and} \quad y,\eta \in \mu,$$

where $[x+y] := x+y+\lambda\cap\mu$ denotes the class of $x+y$ in $\frac{\lambda+\mu}{\lambda\cap\mu}$. Since λ,μ are isotropic, we have $\omega(x+y+z,\xi+\eta+\zeta) = \omega(x+y,\xi+\eta)$ for any $z,\zeta \in \lambda\cap\mu$. So $\widetilde{\omega}$ is well defined and inherits the algebraic properties from ω.

To show that $(\widetilde{X})^{\widetilde{\omega}} = \{0\}$, we observe

$$(\lambda+\mu)^{\omega} = \lambda^{\omega} \cap \mu^{\omega} \supset \lambda \cap \mu. \tag{15}$$

By Corollary 1a, we have

$$\dim(\lambda+\mu)^{\omega} \le \dim X/(\lambda+\mu) \le \dim(\lambda\cap\mu).$$

Here the last inequality is just the non-negativity of the Fredholm index as defined in (11). This proves

$$\dim(\lambda+\mu)^{\omega} = \dim X/(\lambda+\mu) = \dim(\lambda\cap\mu). \tag{16}$$

Combining (16) with (15) yields

$$\lambda\cap\mu = \lambda^{\omega}\cap\mu^{\omega} = (\lambda+\mu)^{\omega}. \tag{17}$$

By Corollary 1b, we have

$$\dim X/(\lambda+\mu) = \dim \lambda\cap\mu = \dim X/(\lambda\cap\mu)^{\omega\omega}.$$

Thus $\lambda+\mu = (\lambda+\mu)^{\omega\omega}$.

Moreover, one checks that

$$\Big(\frac{\lambda+\mu}{\lambda\cap\mu}\Big)^{\widetilde{\omega}} = \frac{(\lambda+\mu)^{\omega}}{\lambda\cap\mu}. \tag{18}$$

With (17) that proves that $\frac{\lambda+\mu}{\lambda\cap\mu}$ is a true symplectic vector space for the induced form $\widetilde{\omega}$ which is spanned by the transversal isotropic subspaces

$$\frac{\lambda+\mu}{\lambda\cap\mu} \;=\; \frac{\lambda}{\lambda\cap\mu}\oplus\frac{\mu}{\lambda\cap\mu}\,.$$

By Lemma 4, the spaces $\frac{\lambda}{\lambda\cap\mu}$, $\frac{\mu}{\lambda\cap\mu}$ are Lagrangian subspaces.

Clearly $\lambda\subset\lambda^{\omega}\cap(\lambda+\mu)$. Now consider $x\in\lambda$ and $y\in\mu$ with $x+y\in\lambda^{\omega}$. Then

$$[x+y]\in\Big(\frac{\lambda}{\lambda\cap\mu}\Big)^{\widetilde{\omega}} \;=\; \frac{\lambda}{\lambda\cap\mu}$$

by the Lagrangian property of $\frac{\lambda}{\lambda\cap\mu}$. It follows that $x+y\in\lambda$, hence

$$\lambda^{\omega}\cap(\lambda+\mu) \;=\; \lambda \text{ and similarly } \mu^{\omega}\cap(\lambda+\mu) \;=\; \mu\,. \tag{19}$$

Combined with the fact

$$\lambda^{\omega}\subset(\lambda\cap\mu)^{\omega} \;=\; (\lambda+\mu)^{\omega\omega} \;=\; \lambda+\mu,$$

the inclusion $\lambda\supset\lambda^{\omega}$ follows and so the Lagrangian property of λ (and similarly of μ).

Remark 5 For related topological (unsolved) puzzles see below Subsection2.3.

We close this subsection with the following characterization of Fredholm pairs.

Proposition 2 *Let (X,ω) be a symplectic Banach space and let (X,X^+,X^-) be a symplectic splitting. Let λ,μ be isotropic subspaces. Let U,V denote the generating operators for λ,μ in the sense of Lemma 3. We assume that V is bounded and bounded invertible. Then*
(a) The space μ is a Lagrangian subspace of X.
(b) Moreover,

$$(\lambda,\mu)\in\mathscr{F}^2(X) \iff UV^{-1}-I_{X^-} \textit{ Fredholm operator with domain } V(\operatorname{dom}U).$$

(c) In this case, $U-V$ closed Fredholm operator with domain $\operatorname{dom}U$ *and*

$$\operatorname{index}(\lambda,\mu) \;=\; \operatorname{index}(UV^{-1}-I_{X^-}).$$

Proof a. Since $\mu=\mathfrak{G}(V)$ is an isotropic subspace of X with $V:X^+\to X^-$ bounded and bounded invertible, the space $\mu':=\mathfrak{G}(-V)$ is also isotropic. We show that μ,μ' are transversal in X. Then by Lemma 4, μ (and μ') are Lagrangian subspaces. First, from the uniqueness of defining V (see, e.g., the proof of Lemma 3a), we have $\mu\cap\mu'=\{0\}$.

Next, let $x+y$, or, more suggestively, $\begin{pmatrix}x\\y\end{pmatrix}$ denote any arbitrary point in X with $x\in X^+$ and $y\in X^-$. We set

$$z \;:=\; \frac{x+V^{-1}y}{2} \text{ and } w \;:=\; \frac{x-V^{-1}y}{2}\,.$$

Then $z+w=x$ and $z-w=V^{-1}y$, so

$$\begin{pmatrix}x\\y\end{pmatrix} \;=\; \begin{pmatrix}z\\Vz\end{pmatrix}+\begin{pmatrix}w\\-Vw\end{pmatrix}.$$

This proves $X = \mu \oplus \mu'$.
b and c. Let $\lambda = \mathfrak{G}(U)$ and $\mu = \mathfrak{G}(V)$ with V bounded and bounded invertible. Let P_+, respectively P_- , denote the projection of $X = X^+ \oplus X^-$ onto the first, respectively, the second factor. Then

$$\lambda \cap \mu \;=\; \left\{ \begin{pmatrix} x \\ Vx \end{pmatrix} \mid x \in \mathrm{dom}(U) \text{ and } Ux \;=\; Vx \right\}.$$

So, P_- induces an algebraic and topological isomorphism between $\lambda \cap \mu$ and $\ker\big(UV^{-1} - I_{X^-}\big)$.

Now we determine

$$\begin{aligned}
\lambda + \mu \;&=\; \left\{ \begin{pmatrix} x \\ Ux \end{pmatrix} + \begin{pmatrix} y \\ Vy \end{pmatrix} \mid x, y \in X^+ \right\} \\
&=\; \left\{ \begin{pmatrix} x' \\ Vx' \end{pmatrix} + \begin{pmatrix} 0 \\ z \end{pmatrix} \mid x' \in X^+ \text{ and } z \in \mathrm{im}(UV^{-1} - I_{X^-}) \right\} \\
&=\; \mu \oplus \mathrm{im}(UV^{-1} - I_{X^-}).
\end{aligned}$$

The last direct sum sign comes from the invertibility of V which induces $\mu \cap X^- = \{0\}$ and, similarly, $\mu + X^- = X$, and so finally the direct sum decomposition $X = \mu \oplus X^-$ with projections Π_μ and Π_- onto the components. So, Π_- yields an algebraic and topological isomorphism of $\lambda + \mu$ onto $\mathrm{im}(UV^{-1} - I_{X^-})$. In particular, we have $\lambda + \mu$ closed in X if and only if $\mathrm{im}(UV^{-1} - I_{X^-})$ closed in X^- and

$$X/(\lambda + \mu) \simeq X^-/\mathrm{im}(UV^{-1} - I_{X^-})$$

with coincidence of the codimensions.

2.3 Open topological problems

2.3.1 Fredholm pairs of Lagrangian subspaces with negative index?

Corollary 1a shows that Fredholm pairs of Lagrangian subspaces in symplectic vector space can not have positive index. In contrast to the strong case, one may expect that we have pairs with negative index in weak symplectic Hilbert space. By now, however, this is an open problem.

2.3.2 Characterization of Lagrangian subspaces by canonical symmetry property of the projections?

The delicacy of Lagrangian analysis in weak symplectic Hilbert space may also be illuminated by addressing the orthogonal projection onto a Lagrangian subspace. In strong symplectic Hilbert space with unitary J, the range of an orthogonal projection is Lagrangian if and only if the projections P and $I - P$ are conjugated by the J operator in the way

$$I - P \;=\; JPJ^*,$$

which is familiar from characterizing elliptic self-adjoint pseudo-differential boundary conditions for elliptic differential of first order, see [11, Proposition 20.3]. In weak symplectic analysis, J maps the range $\mathrm{im}\, P$ onto a dense subset of $\ker P$, but there the argument stops.

2.3.3 Contractibility of the space of Lagrangian subspaces?

There are two more potential differences between the weak and the strong case, namely regarding the topology: while the Lagrangian Grassmannian $\mathscr{L}(X,\omega)$ inherits contractibility from the space of unitary operators in separable Hilbert space by Lemma 2(ii), more refined arguments will be needed to prove the contractibility in the weak case, if at all it is true.

2.3.4 Bott periodicity of the homotopy type of the space of Fredholm pairs of Lagrangian subspaces?

Next, consider the space $\mathscr{F}\mathscr{L}_\lambda(X)$ of all Lagrangian subspaces which form a Fredholm pair with a given Lagrangian subspace λ. Its topology is presently also unknown in the weak case, whereas we have

$$\pi_1\left(\mathscr{F}\mathscr{L}_\lambda(X)\right) \cong \mathbb{Z}$$

in strong symplectic Hilbert space X (see [8, Corollary 4.3] and the generalization to Bott periodicity in [27, Equation (6.2) with Lemma 6.1 and Proposition 6.5]).

2.4 Spectral flow for curves of "unitary" operators

We begin with some observations on inner product space.

Lemma 6 *Let (X,h_X), (Y,h_Y), (Z,h_Z) denote three inner product spaces, A, B linear relations between X and Y, and C a linear relation between X and Z.*
(a) Assume that C is a linear operator, $\operatorname{dom}(A) \subset \operatorname{dom}(C)$*, and* $h_Y(y,y) \le h_Z(Cx,Cx)$ *for all* $(x,y) \in A$*. Then A is a linear operator.*
(b)Assume that B is a linear operator, $\operatorname{dom}(A) = \operatorname{dom}(C) \subset \operatorname{dom}(B)$*, and*

$$h_Y(y,y) + h_Z(z,z) \le h_Y(Bx,Bx) \tag{20}$$

for all $(x,y) \in A$ *and* $(x,z) \in C$*. Then A and C are linear operators and* $\ker(B-A) \subset \ker C$.

Proof *a*. Let $y \in \ker A$, i.e. $(0,y) \in A$. By our assumption we have $h_Y(y,y) \le h_Z(C0,C0) = 0$. Since h_Y is positive definite, we have $y = 0$.
b. By (a) A and C are linear operators. Let $x \in \ker(B-A)$. Then $Bx = Ax$. By (20) we have $h_Z(Cx,Cx) \le 0$. Since h_Z is positive definite, we have $Cx = 0$, i.e. $x \in \ker C$.

Let X be a complex Banach space. We introduce some notations for various spaces of operators in X:

$$\begin{aligned}
\mathscr{C}(X) &:= \text{closed operators on } X,\\
\mathscr{B}(X) &:= \text{bounded linear operators } X \to X,\\
\mathscr{K}(X) &:= \text{compact linear operators } X \to X,\\
\mathscr{F}(X) &:= \text{bounded Fredholm operators } X \to X,\\
\mathscr{C}\mathscr{F}(X) &:= \text{closed Fredholm operators on } X.
\end{aligned}$$

If no confusion is possible we will omit "(X)" and write $\mathscr{C}$, $\mathscr{B}, \mathscr{K}$, etc.. By $\mathscr{C}^{\mathrm{sa}}, \mathscr{B}^{\mathrm{sa}}$ etc., we denote the set of self–adjoint elements in $\mathscr{C}$, $\mathscr{B}$, etc. in the case that X has an inner product.

We assume that X is a inner product space with a fixed inner product (i.e., a sesquilinear, self-adjoint positive definite form) $h : X \times X \to \mathbb{C}$ which is bounded

$$|h(x,y)| \le c\|x\|\|y\| \quad \text{for all } x,y \in X.$$

Definition 6 An operator $A \in \mathscr{C}(X)$ will be called *unitary* with respect to h, if

$$h(Ax,Ay) \;=\; h(x,y) \quad \text{ for all } x,y \in \operatorname{dom}(A).$$

Remark 6 (a) Note that h induces a uniformly smaller norm on X which makes X into a Hilbert space if and only if X becomes complete for this h-induced norm.
(b) The concept of h-unitary extends trivially to closed operators with dense domain in one Banach space, equipped with an inner product, and range in a second Banach space, possibly with a different inner product. Exactly in this sense, for any Lagrangian subspace the generating operator $U \in \mathscr{C}(X^+,X^-)$ (established in Lemma 3) is h-unitary with $h(x,y) = \mp i\omega(x,y)$ on $X^\pm$.

Like for unitary operators in Hilbert space, the following lemma shows that a unitary operator with respect to h has no eigenvalues outside the unit circle.

Lemma 7 *Let $A \in \mathscr{C}(X)$ be unitary with respect to h and $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$. Then* $\ker(A - \lambda I) = \{0\}$.

Proof Let $x \in \ker(A - \lambda I)$, so $Ax = \lambda x$ and

$$h(x,x) \;=\; h(Ax,Ax) \;=\; |\lambda|^2 h(x,x).$$

Since $|\lambda| \neq 1$, we get $h(x,x) = 0$ and so $x = 0$ by h positive definite.

For a certain subclass of unitary operators with respect to h we show that they have discrete spectrum close to 1. Consequently, they are admissible with respect to the positive half-line ℓ (in the sense of Definition 11 of our Appendix) and so permit the definition of spectral flow through ℓ for continuous families (same Appendix).

Proposition 3 *(a) Let X be a Banach space with bounded inner product h. Let $A \in \mathscr{C}(X)$ be an operator satisfying*

$$h(Ax,Ay) \;\leq\; h(x,y) \quad \textit{for all } x,y \in \operatorname{dom} A.$$

We assume $A - I \in \mathscr{CF}(X)$ of index 0. If either A is h-unitary or A is bounded, then there is a bounded neighborhood $N \subset \mathbb{C}$ of 1 such that

$$\sigma(A) \cap \bar{N} \subset \{1\}, \quad \dim P_N(A) \;=\; \dim\ker(A - I).$$

(b) Let $\{h_s\}$ be a continuous family of inner products for X. Let $A_s \in \mathscr{C}(X)$ be unitary with respect to h_s. We assume that the family $\{A_s\}$ is continuous. We denote $h_0 =: h$ and $A_0 =: A$ and choose N like in (a). Then for $s \ll 1$ the spectrum part $\sigma(A_s) \cap N$ has finite algebraic multiplicity and we have

$$\sigma(A_s) \cap N \subset S^1 .$$

Proof *a*. Since $\ker(A - I)$ is finite-dimensional, we have an h-orthogonal splitting

$$X \;=\; \ker(A - I) \oplus X_1$$

with closed X_1 . (Take $X_1 := \Pi(X)$ with $\Pi(x) := x - \sum_{j=1}^n h(x,e_j)e_j$, where $\{e_j\}$ is an h-orthonormal basis of $\ker(A - I)$). We notice that $\ker(A - I) \subset \operatorname{dom}(A)$, so

$$\operatorname{dom}(A) \;=\; \ker(A - I) \oplus (\operatorname{dom}(A) \cap X_1). \tag{21}$$

Then the operator A can be written in block form

$$A \;=\; \begin{pmatrix} I_0 & A_{01} \\ 0 & A_{11} \end{pmatrix}, \tag{22}$$

where I_0 denotes the identity operator on $\ker(A-I)$.

Since A is h-unitary, by Lemma 6b we have $\ker(A_{11}-I) \subset \ker A_{01}$. So we have

$$\ker(A_{11}-I) \subset \ker(A_{11}-I) \cap \ker A_{01} \cap X_1 \;=\; \ker(A-I) \cap X_1 \;=\; \{0\}.$$

If A is h-unitary, let $y \in \mathrm{dom}(A) \cap X_1$ and $x \in \ker(A-I)$. Then

$$h(x,Ay) \;=\; h(Ax,Ay) \;=\; h(x,y) \;=\; 0$$

by (21). So, the range $\mathrm{im}(A|_{\mathrm{dom}(A)\cap X_1})$ is h-orthogonal to $\ker(A-I)$ and, hence, contained in X_1 . Hence $A_{01} = 0$. We observe that $A-I$ is closed as bounded perturbation of the closed operator A; it follows that the component A_{11} and the operator $A_{11}-I_1$ are closed in X_1 .

If A is bounded, then both A_{01} and A_{11} are bounded. Denote by I_1 the identity operator on X_1 . Then we have

$$\begin{aligned} \mathrm{index}\,(A_{11}-I) \;&=\; \mathrm{index}\,(\mathrm{diag}(0,A_{11}-I) \\ &=\; \mathrm{index}\,((A-I)\,\mathrm{diag}(0,I_1)) \\ &=\; \mathrm{index}\,(A-I) + \mathrm{index}\,(\mathrm{diag}(0,I_1)) \\ &=\; 0. \end{aligned}$$

By $\ker(A_{11}-I_1) = \{0\}$ we have $A_{11}-I_1$ surjective. By the Closed Graph Theorem, it follows that $(A_{11}-I_1)^{-1}$ is bounded and so $A_{11}-I_1$ has bounded inverse. Then A_1 has no spectrum near 1. From the decomposition (22) we get $\sigma(A) = \sigma(I_0) \cup \sigma(A_1)$ with $\sigma(I_0) = \{1\}$. So, if $1 \in \sigma(A)$ it is an isolated point of $\sigma(A)$ of multiplicity $\dim\ker(A-I)$.

b. From our assumption it follows that $\sigma(A) \cap \partial N = \emptyset$, and, actually, $\sigma(A_s) \cap \partial N = \emptyset$ for s sufficiently small. Then

$$P_N(A_s) \;:=\; -\frac{1}{2\pi i}\int_{\partial N}(A-\lambda I)^{-1}d\lambda$$

is a continuous family of projections. From T. Kato [26, Lemma I.4.10] we get

$$\dim\mathrm{im}\,P_N(A_s) \;=\; \dim\mathrm{im}\,P_N(A) \;<\; +\infty \quad \text{ and } \quad P_N(A_s)A_s \;\subset\; A_sP_N(A_s),$$

and from [26, Lemma III.6.17] we get $\sigma(A_s) \cap N = \sigma(P_N(A_s)A_sP_N(A_s))$. Since all operators $P_N(A_s)A_sP_N(A_s)$ are unitary with respect to $h_s|_{\mathrm{im}P_N(A_s)}$, it follows $\sigma(P_N(A_s)A_sP_N(A_s)) \subset S^1$.

Thus, it follows that any h-unitary operator A with $A-I$ Fredholm of index 0 has the same spectral properties near $|\lambda| = 1$ as unitary operators in Hilbert space with the additional property that 1 is an isolated point of the spectrum of finite multiplicity. This now permits us to define the Maslov index in weak symplectic analysis.

3 Maslov index in weak symplectic analysis

Now we turn to the geometry of curves of Fredholm pairs of Lagrangian subspaces in weak symplectic Banach space. We show how the usual definition of the Maslov index can be suitably extended and derive basic and more intricate properties.

3.1 Definition and basic properties of the Maslov index

Our data for defining the Maslov index are a *continuous* family $\{(X,\omega_s,X_s^+,X_s^-)\}$ of weak symplectic Banach spaces with *continuous* splitting and a *continuous* family $\{(\lambda_s,\mu_s)\}$ of Fredholm pairs of Lagrangian subspaces of $\{(X,\omega_s)\}$ of index 0. Our main task is defining the involved "continuity".

Definition 7 Let X be a fixed complex Banach space and $\{\omega_s\}$ a family of weak symplectic forms for X. Let (X,ω_s,X_s^+,X_s^-) be a family of symplectic splittings of (X,ω_s) in the sense of Definition 4.
(a) The family $\{(X,\omega_s,X_s^+,X_s^-)\}$ will be called *continuous* if the induced injective mappings $J_s: X\to X^*$ are continuous as bounded operators, and the families $\{X_s^\pm\}$ are continuous as closed subspaces of X in the gap topology. Equivalently, we may demand that the family $\{P_s\}$ of projections

$$P_s: x+y \;\mapsto\; x, \quad \text{for } x\in X_s^+ \text{ and } y\in X_s^-,$$

is continuous.
(b) Let $\{(X,\omega_s,X_s^+,X_s^-)\}$ be a continuous family of symplectic splittings and $\{(\lambda_s,\mu_s)\}$ a continuous curve of Fredholm pairs of Lagrangian subspaces of index 0. Let $U_s:\operatorname{dom}(U_s)\to X_s^-$, resp. $V_s:\operatorname{dom}(V_s)\to X_s^-$ be closed h_s-unitary operators with $\mathfrak{G}(U_s)=\lambda_s$ and $\mathfrak{G}(V_s)=\mu_s$. We define the *Maslov index* of the curve $\{\lambda_s,\mu_s\}$ with respect to P_s by

$$\operatorname{Mas}\{\lambda_s,\mu_s;P_s\} \;:=\; \operatorname{sf}_\ell\Big\{\begin{pmatrix} 0 & U_s\\ V_s^{-1} & 0\end{pmatrix}\Big\}, \tag{23}$$

where V^{-1} denotes the algebraic inverse of the closed injective operator V and $\ell:=(1-\varepsilon,1+\varepsilon)$ with suitable real $\varepsilon>0$ and with upward co-orientation. The discussion around Lemma 8 below shows that the spectral flow on the right side of (23) is always well-defined.

Remark 7 Let $\{(X,\omega_s,X_s^+,X_s^-)\}$ be a continuous family. A curve $\{\lambda_s\}$ of Lagrangian subspaces is continuous (i.e., $\{\lambda_s=\mathfrak{G}(U_s)\}$ is continuous as a curve of closed subspaces of X), if and only if the family $\{S_{s,s_0}\circ U_s\circ S_{s,s_0}^{-1}\}$ is continuous as a family of closed, generally unbounded operators in the space $\operatorname{im}P_{s_0}$. Here U_s denotes the generating operator $U_s:\operatorname{dom}U_s\to X_s^-$ with $\mathfrak{G}(U_s)=\lambda_s$ (see Lemma 3); $s_0\in[0,1]$ is chosen arbitrarily to fix the domain of the family; and

$$S_{s,s_0}:\operatorname{im}P_s\longrightarrow \operatorname{im}P_{s_0}$$

is a bounded operator with bounded inverse which is defined in the following way (see also [26, Section I.4.6, pp. 33-34]):

$$S_{s,s_0} \;:=\; S'_{s,s_0}(I-R)^{-1/2} \;=\; (I-R)^{-1/2}S'_{s,s_0},$$

where

$$R \;:=\; (P_s-P_{s_0})^2 \qquad \text{and} \qquad S'_{s,s_0} \;:=\; P_{s_0}P_s+(I-P_{s_0})(I-P_s).$$

We have the following lemma:

Lemma 8 *Let (X,ω) be a weak symplectic Banach space. Let Δ denote the diagonal (i.e., the canonical Lagrangian) in the product symplectic space $X\boxplus X:=(X,\omega)\oplus(X,-\omega)$, and λ,μ are Lagrangian subspaces of (X,ω). Then*

$$(\lambda,\mu)\in\mathscr{F}\mathscr{L}^2(X) \quad\Longleftrightarrow\quad (\lambda\boxplus\mu,\Delta)\in\mathscr{F}\mathscr{L}^2\big(X\boxplus X\big)$$

and

$$\operatorname{index}(\lambda,\mu) \;=\; \operatorname{index}(\lambda\boxplus\mu,\Delta),$$

where $\lambda\boxplus\mu := \{(x,y)\mid x\in\lambda, y\in\mu\}$.

Proof Clearly $(\lambda\boxplus\mu)\cap\Delta\simeq\lambda\cap\mu$, and $\lambda\boxplus\mu$, Δ are Lagrangian subspaces of $X\boxplus X$. Since

$$(\lambda\boxplus\mu+\Delta)\,\cap\,(\{0\}\boxplus X) \quad\simeq\quad \{0\}\boxplus(\lambda+\mu),$$

we have $\lambda+\mu$ closed, if $\lambda\boxplus\mu+\Delta$ is closed. Re-arranging

$$\lambda\boxplus\mu+\Delta \;=\; \{(x,y)+(\xi,\xi)\mid x\in\lambda, y\in\mu, \xi\in X\} \;=\; \{(x,y)\mid x-y\in\lambda+\mu\}$$

proves the opposite implication. Moreover, we obtain $\lambda\boxplus\mu+\Delta=\Delta+\Delta'_{\lambda+\mu}$ with $\Delta'_{\lambda+\mu}:=\{(x,-x)\mid x\in\lambda+\mu\}$. So $\lambda\boxplus\mu+\Delta$ is closed, if $\lambda+\mu$ is closed.

Setting, similarly, $\Delta':=\{(x,-x)\mid x\in X\}$ yields

$$\frac{X\boxplus X}{(\lambda\boxplus\mu)+\Delta} \;=\; \frac{\Delta\oplus\Delta'}{\Delta\oplus\Delta'_{\lambda+\mu}} \;\simeq\; \frac{\Delta'}{\Delta'_{\lambda+\mu}} \;\simeq\; \frac{X}{\lambda+\mu}.$$

This proves our assertion.

Let (X,ω) be a weak symplectic Banach space with a symplectic splitting (X,ω,X^+,X^-) and a corresponding projection $P:X\to X^+$. Let $(\lambda,\mu)\in\mathscr{F}\mathscr{L}^2(X)$. We denote the generating operators by U, respectively V. Then we have

$$\begin{pmatrix}0 & U\\ V^{-1} & 0\end{pmatrix} \;=\; \begin{pmatrix}U & 0\\ 0 & V^{-1}\end{pmatrix}\begin{pmatrix}0 & I_{X^+}\\ I_{X^-} & 0\end{pmatrix},$$

and

$$\widetilde{\mathfrak{G}}\begin{pmatrix}U & 0\\ 0 & V^{-1}\end{pmatrix} \;=\; \lambda\boxplus\mu, \quad\text{ and }\quad \widetilde{\mathfrak{G}}\begin{pmatrix}0 & I_{X^-}\\ I_{X^+} & 0\end{pmatrix} \;=\; \Delta,$$

where $\widetilde{\mathfrak{G}}$ denotes the graph of closed operators from $\operatorname{im}\mathscr{P}$ to $\operatorname{im}(I-\mathscr{P})$ with $\mathscr{P}:=P\boxplus(I-P)$.

This leads to the following important result.

Proposition 4 *Let $\{(X,\omega_s)\}$ be a continuous family of symplectic space for X with a continuous family of symplectic splittings (X,ω_s,X_s^+,X_s^-) in the sense of Definition 7a and a corresponding family of projections $\{P_s:X\to X_s^+\}$. Let $\{(\lambda_s,\mu_s)\}$ be a continuous curve in $\mathscr{F}\mathscr{L}^2(X)$. We denote the generating operators by U_s, respectively V_s.*

(a) If V_s is bounded and has bounded inverse for each $s\in[0,1]$, then we have

$$\operatorname{Mas}\{\lambda_s,\mu_s;P_s\} \;=\; \operatorname{sf}_\ell\{U_sV_s^{-1}\}, \tag{24}$$

where $\ell:=(1-\varepsilon,1+\varepsilon)$ with suitable real $\varepsilon>0$ and with upward co-orientation.

(b) We have

$$\begin{aligned}\operatorname{Mas}\{\lambda_s\boxplus\mu_s,\Delta;\mathscr{P}_s\} \;&=\; \operatorname{Mas}\{\lambda_s,\mu_s;P_s\} && (25)\\ &=\; \operatorname{Mas}\{\mu_s,\lambda_s;I-P_s\} \qquad \textit{in } (X,-\omega_s) && (26)\\ &=\; \operatorname{Mas}\{\Delta,\lambda_s\boxplus\mu_s;I-\mathscr{P}_s\} \quad \textit{in } (X,-\omega_s)\boxplus(X,\omega_s), && (27)\end{aligned}$$

where $\mathscr{P}_s:=P_s\boxplus(I-P_s)$.

Proof By our assumption, we have

$$\dim\ker(z^2 I - U_s V_s^{-1}) \;=\; \dim\ker\begin{pmatrix} zI & U_s \\ V_s^{-1} & zI \end{pmatrix},$$

for all $z \in \mathbb{C}$. By definition, we have

$$\mathrm{Mas}\{\lambda_s, \mu_s; P_s\} \;=\; \mathrm{sf}_\ell\Big\{\begin{pmatrix} 0 & U_s \\ V_s^{-1} & 0 \end{pmatrix}\Big\} \;=\; \mathrm{sf}_\ell\{U_s V_s^{-1}\}.$$

b. Let $\widetilde{\mathfrak{G}}$ denote the graph of closed operators from $\mathrm{im}\,\mathscr{P}_s$ to $\mathrm{im}(I-\mathscr{P}_s)$. By (a), (b) and c(i) we have

$$\begin{aligned}\mathrm{Mas}\{\lambda_s \boxplus \mu_s, \Delta; \mathscr{P}_s\} \;&=\; \mathrm{Mas}\Big\{\widetilde{\mathfrak{G}}\begin{pmatrix} U_s & 0 \\ 0 & V_s^{-1} \end{pmatrix}, \widetilde{\mathfrak{G}}\begin{pmatrix} 0 & I_{X_s^-} \\ I_{X_s^+} & 0 \end{pmatrix}; \mathscr{P}_s\Big\} \\ &=\; \mathrm{sf}_\ell\Big\{\begin{pmatrix} U_s & 0 \\ 0 & V_s^{-1} \end{pmatrix}\begin{pmatrix} 0 & I_{X_s^+} \\ I_{X_s^-} & 0 \end{pmatrix}\Big\} \\ &=\; \mathrm{sf}_\ell\Big\{\begin{pmatrix} 0 & U_s \\ V_s^{-1} & 0 \end{pmatrix}\Big\} \;=\; \mathrm{Mas}\{\lambda_s, \mu_s; P_s\}.\end{aligned}$$

So (25) is proved. By the definition of the Maslov index we have (26). (27) follows from (26) and (25).

From the properties of our general spectral flow, as observed at the end of our Appendix, we get all the basic properties of the Maslov index (see S. E. Cappell, R. Lee, and E. Y. Miller [17, Section 1] for a more comprehensive list).

Proposition 5 *(a) The Maslov index is* invariant under homotopies *of curves of Fredholm pairs of Lagrangian subspaces with fixed endpoints. In particular, the Maslov index is invariant under* re-parametrization *of paths.*
(b) The Maslov index is additive under catenation*, i.e.*

$$\mathrm{Mas}\big\{\lambda_1 * \lambda_2, \mu_1 * \mu_2; P_s * Q_s\big\} \;=\; \mathrm{Mas}\big\{\lambda_1, \mu_1; P_s\big\} + \mathrm{Mas}\big\{\lambda_2, \mu_2; Q_s\big\},$$

where $\{\lambda_i(s)\}, \{\mu_i(s)\}$, $i=1,2$ *are continuous paths with* $\lambda_1(1) = \lambda_2(0)$, $\mu_1(1) = \mu_2(0)$ *and*

$$(\lambda_1 * \lambda_2)(s) \;:=\; \begin{cases} \lambda_1(2s), & 0 \le s \le \frac{1}{2}, \\ \lambda_2(2s-1), & \frac{1}{2} < s \le 1, \end{cases}$$

and similarly $\mu_1 * \mu_2$ *and* $\{P_s\} * \{Q_s\}$.
(c) The Maslov index is natural *under symplectic action: let* $\{(X', \omega_s')\}$ *be a second family of symplectic Banach spaces and let*

$$L_s \in \mathrm{Sp}(X, \omega_s; X', \omega_s') \;\;:=\;\; \{L \in \mathscr{B}(X, X') \mid L \text{ invertible and } \omega_s'(Lx, Ly) = \omega_s(x,y)\},$$

such that $\{L_s\}$ *is a continuous family as bounded operators. Then, clearly,* $\{X' = L_s(X_s^+) \oplus L_s(X_s^-)\}$ *is a continuous family of symplectic splittings of* $\{(X', \omega_s')\}$ *inducing projections* $\{Q_s\}$*, and we have*

$$\mathrm{Mas}\{\lambda_s, \mu_s; P_s\} \;=\; \mathrm{Mas}\{L_s\lambda_s, L_s\mu_s; Q_s\}.$$

(d) The Maslov index vanishes*, if* $\dim \lambda_s \cap \mu_s$ *constant for all* $s \in [0,1]$.
(e) Flipping. *We have*

$$\mathrm{Mas}\{\lambda_s, \mu_s; P_s\} + \mathrm{Mas}\{\mu_s, \lambda_s; P_s\} \;=\; \dim \lambda_0 \cap \mu_0 - \dim \lambda_1 \cap \mu_1 .$$

We can not claim that the Maslov index, $\mathrm{Mas}\{\lambda_s,\mu_s;P_s\}$ is always independent of the splitting projection P_s in general Banach space. However, we have the following result.

Proposition 6 *Let $\{(X,\omega_s)\}$ be a continuous family of* strong *symplectic Banach spaces (with fixed underlying Banach space X) and let $\{X = X^+_{s,t} \oplus X^-_{s,t}\}$ be two continuous families of symplectic splittings in the sense of Definition 7a with projections $P_{s,t} : X \to X^+_{s,t}$ for $s \in [0,1]$ and $t = 0,1$. Let $\{(\lambda_s,\mu_s)\}$ be a continuous curve of Fredholm pairs of Lagrangian subspaces of $\{(X,\omega_s)\}$. Then*
(a) index $(\lambda_s,\mu_s) = 0$ *for all $s \in [0,1]$; and*
(b) $\mathrm{Mas}\{\lambda_s,\mu_s;P_{s,0}\} = \mathrm{Mas}\{\lambda_s,\mu_s;P_{s,1}\}$.

Note 1 Commonly, one assumes $J^2 = -I$ in strong symplectic analysis and defines the Maslov index with respect to the induced decomposition. In view of Lemma 1, the point of the preceding proposition is that the Maslov index is independent of the choice of the metrics.

Proof a. Using $-i\omega_s$, we make (X,ω_s) into a symplectic Hilbert space and deform the metric such that $J_s^2 = -I$. Clearly, the dimensions entering into the definition of the Fredholm index do not change under the deformation. So, we are in the well-studied standard case.
b. We recall that our two families of symplectic splitting define two families of Hilbert structures for X defined by

$$\begin{aligned}\langle x,y\rangle_{s,t} \;:=\; & -i\omega_s(x^+_{s,t},y^+_{s,t}) + i\omega_s(x^-_{s,t},y^-_{s,t}) \\ & \text{for } x = x^+_{s,t}+x^-_{s,t}, y = y^+_{s,t}+y^-_{s,t}, x^+_{s,t}, y^+_{s,t} \in H^+_{s,t}, x^-_{s,t}, y^-_{s,t} \in H^-_{s,t}, t = 0,1.\end{aligned}$$

For any $t \in [0,1]$ we define

$$\langle x,y\rangle_{s,t} \;:=\; (1-t)\langle x,y\rangle_{s,0} + t\langle x,y\rangle_{s,1}.$$

Then all $(X,\langle\cdot,\cdot\rangle_{s,t})$ are Hilbert spaces.

Define $J_{s,t}$ by $\omega_s(x,y) = \langle J_{s,t}x,y\rangle_{s,t}$ and let $X^\pm_{s,t}$ denote the positive (negative) space of $-iJ_{s,t}$ and $P_{s,t}$ the orthogonal projection of X onto $X^+_{s,t}$.

Then the two-parameter family $\{J_{s,t}\}$ is a continuous family of invertible operators; $\{P_{s,t}\}$ is continuous; and $\{H^+_{s,t}\}$ is continuous. So $\mathrm{Mas}\{\lambda_s,\mu_s;P_{s,t}\}$ is well defined. So, by homotopy invariance and additivity under catenation we obtain

$$\mathrm{Mas}\{\lambda_s,\mu_s;P_{s,0}\} \;=\; \mathrm{Mas}\{\lambda_s,\mu_s;P_{s,1}\}.$$

3.2 Comparison with the real (and strong) category

For fixed strong symplectic Hilbert space X, choosing one single Lagrangian subspace λ yields a decomposition $X = \lambda \oplus J\lambda$. This decomposition was used in [7, Definition 1.5] (see also [9, Theorem 3.1] and [24, Proposition 2.14]) to give the first functional analytic definition of the Maslov index, though under the somewhat restrictive (and notationally quite demanding) assumption of *real* symplectic structure. Up to the sign, our Definition 7b is a true generalization of that previous definition. More precisely:

Let (H,ω) be a real symplectic Hilbert space with

$$\omega(x,y) \;=\; \langle Jx,y\rangle,\; J^2 \;=\; -I,\; J^t \;=\; -J.$$

Clearly, we obtain a symplectic decomposition $H^+ \oplus H^- = H \otimes \mathbb{C}$ with the induced complex strong symplectic form $\omega_{\mathbb{C}}$ by

$$H^{\pm} := \{(I \mp iJ)\zeta \mid \zeta \in H\}.$$

Now we fix one (real) Lagrangian subspace $\lambda \subset H$. Then there is a real linear isomorphism $\varphi : H \cong \lambda \otimes \mathbb{C}$ defined by $\varphi(x+Jy) = x+iy$ for all $x, y \in \lambda$. For $A = X + JY : H \to H$ with $X, Y : H \to H$ real linear and

$$X(\lambda) \subset \lambda,\ Y(\lambda) \subset \lambda, \quad \text{and} \quad XJ = JX,\ YJ = JY, \tag{28}$$

we define

$$\varphi_*(A) := \varphi \circ A \circ \varphi^{-1} = X + iY, \overline{A}_\lambda := X - JY, A^{t_\lambda} := X^t + JY^t,$$

where X^t, Y^t denotes the real transposed operators.

Lemma 9 *Let (λ, μ) be any pair of Lagrangian subspaces of H (in the real category). Let $\widetilde{V} : H \to H$ with $\widetilde{V}J = J\widetilde{V}$ be a real generating operator for μ with respect to the orthogonal splitting $H = \lambda \oplus J\lambda$, i.e., $\mu = \widetilde{V}(J\lambda)$ and $\varphi_*(\widetilde{V})$ is unitary. Let $U, V : H^+ \to H^-$ denote the unitary generating operators for $\lambda \otimes \mathbb{C}$ and $\mu \otimes \mathbb{C}$, i.e., we have*

$$\lambda \otimes \mathbb{C} = \mathfrak{G}(U) \ \text{and} \ \mu \otimes \mathbb{C} = \mathfrak{G}(V).$$

Then we have $VU^{-1} = -\overline{S_\lambda(\widetilde{V})}$, where $S_\lambda(\widetilde{V}) := \varphi_*(\widetilde{V})\varphi_*\big(\widetilde{V}^{t_\lambda}\big)$ is the complex generating operator for $\mu \otimes \mathbb{C}$ with respect to λ, as defined by J. Leray in [28, Section I.2.2, Lemma 2.1] and elaborated in the preceding references.

Proof We firstly give some notations used later. For $\zeta = x + Jy \in H$ with $x, y \in \lambda$ we define $\overline{\zeta}_\lambda := \varphi^{-1}\big(\overline{\varphi(\zeta)}\big) = x - Jy$. Moreover, for $A = X + JY : H \to H$ with $X, Y : H \to H$ real linear with (28), we define $\widetilde{S}_\lambda(A) := AA^{t_\lambda}$. Then we have $S_\lambda(A) = \varphi_*\big(\widetilde{S}_\lambda(A)\big)$.

Now we give explicit descriptions of U and V. It is immediate that U takes the form

$$\begin{array}{cccc} U: & H^+ & \longrightarrow & H^- \\ & (I - iJ)\zeta & \mapsto & (I + iJ)\overline{\zeta}_\lambda \end{array}.$$

By the definition of $\widetilde{V}$, we have

$$\mu = \widetilde{V}(J\lambda) = \{2\widetilde{V}Jx + 2i\widetilde{V}Jy \mid x, y \in \lambda\}.$$

We shall find $V : (I - iJ)\zeta \mapsto (I + iJ)\zeta_1$ with $\zeta, \zeta_1 \in H$ such that $\mathfrak{G}(V) = \mu \otimes \mathbb{C}$, i.e., we shall find ζ_1 to $\zeta = x + Jy$ such that

$$(I - iJ)\zeta + (I + iJ)\zeta_1 = 2\widetilde{V}Jx + 2i\widetilde{V}Jy \quad \text{for all } x, y \in \lambda. \tag{29}$$

Comparing real and imaginary part of (29) yields $\zeta + \zeta_1 = 2\widetilde{V}Jx$ and $-iJ(\zeta - \zeta_1) = -i\widetilde{V}Jy$, so

$$\zeta = \widetilde{V}(Jx - y) \quad \text{and} \quad \zeta_1 = \widetilde{V}(Jx + y).$$

From the left equation we obtain $\overline{\zeta}_\lambda = -\overline{\widetilde{V}}_\lambda(Jx + y)$. Since $\varphi_*(\widetilde{V})$ is unitary, we obtain from the right side

$$\zeta_1 = \widetilde{V}(Jx + y) = -\widetilde{V}\overline{\widetilde{V}}_\lambda^{-1}\overline{\zeta}_\lambda = -\widetilde{V}\widetilde{V}^t\overline{\zeta}_\lambda = -\widetilde{S}_\lambda(\widetilde{V})\overline{\zeta}_\lambda.$$

This gives

$$\begin{array}{rccc} V: & H^+ & \longrightarrow & H^- \\ & (I-iJ)\zeta & \mapsto & -(I+iJ)\widetilde{S}_\lambda(\widetilde{V})\overline{\zeta}_\lambda . \end{array}$$

So for all $z_1 := (I+iJ)\zeta_1$ with $\zeta_1 \in H$, we have

$$\begin{aligned} VU^{-1}z_1 &= -(I+iJ)\widetilde{S}_\lambda(\widetilde{V})\zeta_1 \\ &= -\widetilde{S}_\lambda(\widetilde{V})(I+iJ)\zeta_1 \\ &= -\widetilde{S}_\lambda(\widetilde{V})(I-iJ)\overline{\varphi(\zeta)} \\ &= -\overline{\varphi_*(\widetilde{S}_\lambda(\widetilde{V}))}(I-iJ)\overline{\varphi(\zeta)} \\ &= -\overline{S_\lambda(\widetilde{V})}(I+iJ)\zeta_1 \\ &= -\overline{S_\lambda(\widetilde{V})}z_1 . \end{aligned}$$

That is, $VU^{-1} = -\overline{S_\lambda(\widetilde{V})}$.

With the preceding notation, we recall from [7, Definition 1.5] the definition of the Maslov index

$$\mathrm{Mas}_{\mathrm{BF}}\{\mu_s, \lambda\} := \mathrm{sf}_{\ell'}\{S_\lambda(\widetilde{V}_s)\} \tag{30}$$

of a continuous curve $\{\mu_s\}$ of Lagrangian subspaces in real symplectic Hilbert space H which make Fredholm pairs with one fixed Lagrangian subspace λ. Here $\ell' := (-1-\varepsilon, -1+\varepsilon)$ with downward orientation.

Corollary 2

$$\mathrm{Mas}\{\lambda\otimes\mathbb{C}, \mu_s\otimes\mathbb{C}\} = -\mathrm{Mas}_{\mathrm{BF}}\{\mu_s, \lambda\}.$$

Proof Let ℓ, ℓ' denote small intervals on the real line close to 1, respectively -1 and give ℓ the co-orientation from $-i$ to $+i$ and ℓ' vice versa. We denote by ℓ^- the interval ℓ with reversed co-orientation. Then by our definition in 23, elementary transformations, the preceding lemma, and the definition recalled in (30):

$$\begin{aligned} \mathrm{Mas}\{\lambda\otimes\mathbb{C}, \mu_s\otimes\mathbb{C}\} &= -\mathrm{sf}_\ell\{UV_s^{-1}\} = -\mathrm{sf}_{\ell^-}\{V_sU^{-1}\} \\ &= -\mathrm{sf}_{\ell^-}\{-\overline{S_\lambda(\widetilde{V}_s)}\} = -\mathrm{sf}_\ell\{-S_\lambda(\widetilde{V}_s)\} \\ &= -\mathrm{sf}_{\ell'}\{S_\lambda(\widetilde{V}_s)\} = -\mathrm{Mas}_{\mathrm{BF}}\{\mu_s, \lambda\}. \end{aligned}$$

3.3 Invariance of the Maslov index under embedding

We close this subsection by discussing the invariance of the Maslov index under embedding in a larger symplectic space, assuming a simple regularity condition.

Lemma 10 *Let $\{(X, \omega_s, X_s^+, X_s^-)\}$ be a continuous family of symplectic splittings for a fixed (complex) Banach space X and $\{(\lambda_s, \mu_s) \in \mathscr{FL}^2(X, \omega_s)\}$ a continuous curve with* index $(\lambda_s, \mu_s) = 0$ *for all $s \in [0,1]$. Let Y be a second Banach space with a linear embedding $Y \hookrightarrow X$ (in general neither continuous nor dense). We assume that*

$$\widetilde{\omega}_s := \omega_s|_{Y\times Y} \quad \text{and} \quad Y_s^\pm := X_s^\pm \cap Y.$$

yields also a continuous family $\{(Y,\widetilde{\omega}_s,Y_s^+,Y_s^-)\}$ *of symplectic splittings. Moreover, we assume that* $\dim\lambda_s\cap\mu_s-\dim\lambda_s\cap\mu_s\cap Y$ *is constant and* $(\lambda_s\cap Y,\mu_s\cap Y)\in\mathscr{F}\mathscr{L}^2(Y,\widetilde{\omega}_s)$ *of index* 0 *for all s, and that the pairs make also a continuous curve in* Y*. Then we have*

$$\mathrm{Mas}\{\lambda_s,\mu_s;P_s\} \;=\; \mathrm{Mas}\{\lambda_s\cap Y,\mu_s\cap Y;\widetilde{P}_s\},$$

where P_s *and* $\widetilde{P}_s$ *denote the projections of* X *onto* X_s^+ *along* X_s^- *and the projections of* Y *onto* Y_s^+ *along* Y_s^- *respectively.*

The lemma is an immediate consequence of Lemma 17 of the Appendix.

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A Spectral flow

The spectral flow for a one parameter family of linear self-adjoint Fredholm operators was introduced by M. Atiyah, V. Patodi, and I. Singer [4] in their study of index theory on manifolds with boundary. Since then other significant applications have been found. Later this notion was made rigorous for curves of bounded self-adjoint Fredholm operators in J. Phillips [32] and for continuous curves of self-adjoint (generally unbounded) Fredholm operators in Hilbert space in [10] by Cayley transform. The notion was generalized to higher dimensional case in X. Dai and W. Zhang [21], and to more general operators in [38,41,43].

For manifolds with singular metric, there may appear linear relations (cf. C. Bennewitz [5] and M. Lesch and M. Malamud [29]). It is well known that many statements on *relations* can be translated into those on the resolvents in the realm of *operator* theory, see, e.g., B. M. Brown, G. Grubb, and I. G. Wood [15]. It seems to us, however, that this translation can not always be made globally, i.e., not for a whole curve of relations.

In this Appendix we shall provide a rigorous definition of the *spectral flow* of *spectral-continuous* curves of *admissible* closed linear relations in Banach space relative to a co-oriented real curve $\ell\subset\mathbb{C}$. (All the preceding terms will be explained).

A.1 Gap between subspaces

Let $\mathscr{S}(X)$ denote the set of all closed subspaces of a Banach space X.

The gap topology

The *gap* between subspaces $M,N\in\mathscr{S}(X)$ is defined in [26, Section IV.2.1]:

$$\hat{\delta}(M,N) \;:=\; \max\{\delta(M,N),\delta(N,M)\}, \tag{1}$$

where $\delta(M,N):=\sup\{\mathrm{dist}(x,N)\mid x\in M,\|x\|=1\}$, $\delta(M,\{0\}):=1$ for $M\neq\{0\}$, and $\delta(\{0\},N):=0$. The sets $U(M,\varepsilon)=\{N\in\mathscr{S}(X)\mid\hat{\delta}(M,N)<\varepsilon\}$, where $M\in\mathscr{S}(X)$ and $\varepsilon>0$, form a basis for the so-called *gap topology* on $\mathscr{S}(X)$. This is a complete metrizable topology on $\mathscr{S}(X)$ [26, Section IV.2.1].

Let X be a Hilbert space. Then the gap between closed subspace M,N is a metric for $\mathscr{S}(X)$ and can be calculated by

$$\hat{\delta}(M,N) \;=\; \|P_M-P_N\|, \tag{2}$$

where P_M,P_N denote the orthogonal projections of X onto M,N respectively, [26, Theorem I.6.34].

We have the following lemmata.

Lemma 11 *Let* X *be a Hilbert space, and* Y *be a closed linear subspace of* X*. Then the mapping* $M\mapsto M+Y$ *induces a bijection from the space* $\mathscr{S}(X,Y)$ *of closed linear subspaces of* X *containing* Y *onto the space* $\mathscr{S}(X/Y)=\mathscr{S}(Y^\perp)$ *of closed linear subspaces of* X/Y *, which preserves the metric.*

Proof We view X/Y as $Y^\perp$. Let $M,N \subset Y^\perp$ be two closed subspaces and P_M, P_N be the orthogonal projections onto M, N respectively. Then we have

$$\hat{\delta}(M+Y,N+Y) \;=\; \|P_{M+Y}-P_{N+Y}\| \;=\; \|P_M-P_N\| \;=\; \hat{\delta}(M,N).$$

Uniform properties

In general, the distances $\delta(M,N)$ and $\delta(N,M)$ can be very different and, worse, behave very differently under small perturbations. However, for finite-dimensional subspaces of equal dimension (and in Hilbert space) we can estimate $\delta(M,N)$ by $\delta(N,M)$ in a uniform way.

Lemma 12 *Let X be a Hilbert space and M,N be two subspaces with* $\dim M = \dim N = n \in \mathbb{N}$. *If $\delta(N,M) < \frac{1}{\sqrt{n}}$, then we have*

$$\delta(M,N) \le \frac{\sqrt{n}\,\delta(N,M)}{1-\sqrt{n}\,\delta(N,M)}. \tag{3}$$

Proof Let $y_1,\dots,y_n$ be an orthonormal basis of N. Let $x_k \in M$ denote the vectors with $\|x_k-y_k\| = \operatorname{dist}(y_k,M)$. Then $\|x_k-y_k\| \le \delta(N,M)$.

For any $a_1,\dots,a_n \in \mathbb{C}$, set $x=\sum_{k=1}^n a_k x_k$. Then we have

$$\begin{aligned}\|x\| \;&=\; \|\sum_{k=1}^n a_k y_k + \sum_{k=1}^n a_k(x_k-y_k)\| \ge \|\sum_{k=1}^n a_k y_k\| - \sum_{k=1}^n |a_k|\|x_k-y_k\| \\ &\ge\; (\sum_{k=1}^n a_k^2)^{\frac12} - \sum_{k=1}^n |a_k|\,\delta(N,M) \ge (1-\sqrt{n}\,\delta(N,M))(\sum_{k=1}^n a_k^2)^{\frac12}.\end{aligned} \tag{4}$$

If $x=0$, by (4) we have $a_k=0$. Thus $x_1,\dots,x_n$ are linearly independent and therefore they form a basis of M.

For any $x \in M$ with $\|x\|=1$, let $y=\sum_{k=1}^n a_k y_k$. By (4) we have

$$\|x-y\| \;=\; \|\sum_{k=1}^n a_k(x_k-y_k)\| \le \sum_{k=1}^n |a_k|\,\delta(N,M) \le \frac{\sqrt{n}\,\delta(N,M)}{1-\sqrt{n}\,\delta(N,M)}.$$

Hence we have (3).

Clearly, taking the sum of two closed subspaces is not a continuous operation, in general, but becomes continuous when fixing the dimension of the intersection and keeping the sum closed.

The following Lemma is well-known and the proof is omitted.

Lemma 13 *Let X,Y be two Hilbert space and $A_s \in \mathscr{B}(X,Y)$ be a continuous family of semi-Fredholm bounded operators (continuous in the operator norm). If* $\dim\ker A_s$ *is constant, then* $\ker A_s \in \mathscr{S}(X)$ *and* $\operatorname{im} A_s \in \mathscr{S}(Y)$ *are continuous families of closed subspaces (continuous in the gap norm).*

We recall the notion of semi–Fredholm pairs: Let $M,N \in \mathscr{S}(X)$. The pair M,N is called *(semi-)Fredholm* if $M+N$ is closed in X, and both of (one of) the spaces $M\cap N$ and $\dim X/(M+N)$ are (is) finite dimensional. In this case, the *index* of (M,N) is defined by

$$\operatorname{index}(M,N) \;:=\; \dim M\cap N - \dim X/(M+N) \;\in\; \mathbb{Z}\cup\{-\infty,\infty\}. \tag{5}$$

Note that by [8, Remark A.1] (see also [26, Problem 4.4.7]), $X/(M+N)$ of finite dimension implies $M+N \in \mathscr{S}(X)$.

Proposition 7 *Let X be a Hilbert space and $n \in \mathbb{N}$. Denote by $\mathscr{S}\mathscr{F}^2_{1,n}(X)$ (respectively $\mathscr{S}\mathscr{F}^2_{2,n}(X)$) the set of semi-Fredholm pairs (M,N) of closed subspaces with* $\dim M\cap N = n$ *(respectively* $\dim X/(M+N) = n$*). Then the following four natural mappings $\varphi_{k,l} : \mathscr{S}\mathscr{F}^2_{l,n}(X) \to \mathscr{S}(X)$, $k,l = 1,2$ are continuous:*

$$\varphi_{1,l}(M,N) \;:=\; M\cap N, \qquad \varphi_{2,l}(M,N) \;:=\; M+N.$$

Proof (Communicated by R. Nest) Let $(M,N) \in \mathscr{S}(X)\times\mathscr{S}(X)$. Let P_M and P_N denote the orthogonal projections of X onto M and N respectively. Then we have

$$\operatorname{im} P_M + \operatorname{im} P_N \;=\; \operatorname{im}((I-P_N)P_M) + \operatorname{im} P_N$$

and the kernel of $(I-P_N)P_M \in \mathscr{B}(\operatorname{im} P_M, \ker P_N)$ is $M\cap N$. So $M+N$ is closed if and only if $\operatorname{im}((I-P_N)P_M)$ is closed. By Lemma 13, the maps $\varphi_{k,1}$, $k=1,2$ are continuous. Recall that taking orthogonal complements is continuous. Then $\varphi_{k,2}$ is continuous by the fact that

$$\varphi_{k,2}(M,N) \;=\; \big(\varphi_{3-k,1}(M^\perp,N^\perp)\big)^\perp, \quad k=1,2.$$

A.2 Closed linear relations

This subsection discusses some general properties of closed linear relations. For additional details, see Cross [20].

Basic concepts of closed linear relations

Let X, Y be two vector spaces. A *linear relation* A between X and Y is just a linear subspace of $X \times Y$. As usual, the *domain*, the *range*, the *kernel* and the *indeterminant part* of A is defined by

$$\begin{aligned} \mathrm{dom}(A) &= \{x \in X \mid \text{ there exists } y \in Y \text{ such that } (x,y) \in A\}, \\ \mathrm{im} A &= \{y \in Y \mid \text{ there exists } x \in X \text{ such that } (x,y) \in A\}, \\ \ker A &= \{x \in X \mid (x,0) \in A\}, \\ A(0) &= \{y \in Y \mid (0,y) \in A\}, \end{aligned}$$

respectively.

Let X, Y, Z be three vector spaces. Let A, B be linear relations between X and Y, and C a linear relation between Y and Z. We define $A+B$ and CA by

$$A+B = \{(x, y+z) \in X \times Y \mid (x,y) \in A, (y,z) \in B\} \tag{6}$$
$$CA = \{(x,z) \in X \times Z \mid \exists y \in Y \text{ such that } (x,y) \in A, (y,z) \in C\}. \tag{7}$$

Definition 8 Let X, Y be two Banach spaces. A *closed linear relation* between X, Y is a closed linear subspace of $X \times Y$. We denote by $\mathscr{C}LR(X,Y) = \mathscr{S}(X \times Y)$ and $\mathscr{C}LR(X) = \mathscr{S}(X \times X)$.

Note that a linear relation A between X, Y is a graph of a linear operator if and only if $A(0) = \{0\}$. In this case we shall still denote the corresponding operator by A. After identifying an operator and its graph, we have the inclusions

$$\mathscr{B}(X,Y) \subset \mathscr{C}(X,Y) \subset \mathscr{C}LR(X,Y).$$

Let A be a linear relation between X, Y. The *inverse* A^{-1} of A is a always defined. It is the linear relation between Y, X defined by

$$A^{-1} = \{(y,x) \in Y \times X; (x,y) \in A\}. \tag{8}$$

Definition 9 Let X, Y be two Banach spaces and $A \in \mathscr{C}LR(X,Y)$.

(i) A is called *Fredholm*, if $\dim\ker A < +\infty$, $\mathrm{im} A$ is closed in Y and $\dim Y/\mathrm{im} A < +\infty$. In this case, we define the *index* of A to be

$$\mathrm{index}\,(A) = \dim\ker A - \dim Y/\mathrm{im} A. \tag{9}$$

(ii) X, Y is called *bounded invertible*, if $A^{-1} \in \mathscr{B}(Y,X)$.

Lemma 14 *(a) A is Fredholm, if and only if the pair $(A, X \times \{0\})$ is a Fredholm pair of closed subsets of $X \times Y$. In this case,* $\mathrm{index}\,(A) = \mathrm{index}\,(A, X \times \{0\})$.
(b) A is bounded invertible, if and only if $X \times X$ is the direct sum of A and $X \times \{0\}$.

Proof Our results follow from the fact that

$$\begin{aligned} A \cap (X \times \{0\}) &= \ker A \times \{0\}, \\ A + X \times \{0\} &= \{0\} \times \mathrm{im}(A) + X \times \{0\}. \end{aligned}$$

Spectral projections of closed linear relations

Definition 10 Let X be a Banach space and $A \in \mathscr{C}LR(X)$. Let ζ be a complex number. ζ is called a *regular point* of A if $A - \zeta I$ is bounded invertible. Otherwise ζ is called a *spectral point* of A. We denote the set of all spectral points of A by $\sigma(A)$ and the set of all regular points of A by $\rho(A)$. The *resolvent* of A is defined by

$$R(\zeta, A) = (A - \zeta I)^{-1}, \quad \zeta \in \rho(A). \tag{10}$$

Let X be a Banach space, and $A \in \mathscr{C}LR(X)$. Let $N \subset \mathbb{C}$ be a bounded open subset. Assume that $\sigma(A) \cap \partial N$ is a finite set. Then there exists an open subset $\widetilde{N} \subset N$ such that

$$\overline{\widetilde{N}} \subset N,\ \partial\widetilde{N} \in C^1,\ \sigma(A) \cap \widetilde{N} \ = \ \sigma(A) \cap N, \quad \text{and} \quad \sigma(A) \cap \partial\widetilde{N} \ = \ \emptyset, \tag{11}$$

and the spectral projection

$$P_N(A) \ := \ -\frac{1}{2\pi i}\int_{\partial\widetilde{N}} (A - \zeta I)^{-1} d\zeta \tag{12}$$

is well-defined and does not depend of the choice of $\widetilde{N}$. We have the following lemma (cf. [26, Theorem III.6.17]):

Lemma 15 *(a) We have*

$$P_N(A)A \subset AP_N(A) \ = \ P_N(A)AP_N(A) + \{0\} \times A(0), \tag{13}$$

where the composition is taken in the sense of (7).
(b) We have

$$P_N(A)AP_N(A) \ = \ -\frac{1}{2\pi i}\int_{\partial\widetilde{N}} \zeta(A - \zeta I)^{-1} d\zeta. \tag{14}$$

(c) If we view $P_N(A)AP_N(A)$ *as a linear relation on* $\mathrm{im}(P_N(A))$, *then we have* $P_N(A)AP_N(A) \in \mathscr{B}(\mathrm{im}(P_N(A)))$, *and*

$$\sigma(A) \cap N \ = \ \sigma\big(P_N(A)AP_N(A)\big). \tag{15}$$

Proof Let $z \in N \setminus \tilde{N}$ be a regular point. Then we have

$$P_N(A)R(z,A) = R(z,A)P_N(A) = -\frac{1}{2\pi i}\int_{\partial\widetilde{N}} (z-\zeta)^{-1}(A - \zeta I)^{-1} d\zeta.$$

Since $R(z,A)$ is bounded and $0 \neq (z-\zeta)^{-1}$ for all $\zeta \in \sigma(A) \cap \tilde{N}$, we have $\ker R(z,A) = A(0) \subset \ker(P_N(A))$. Then our results follow form the corresponding results for $R(z,A)$.

A.3 Spectral flow for closed linear relations.

Firstly we give the definition of admissible relations.

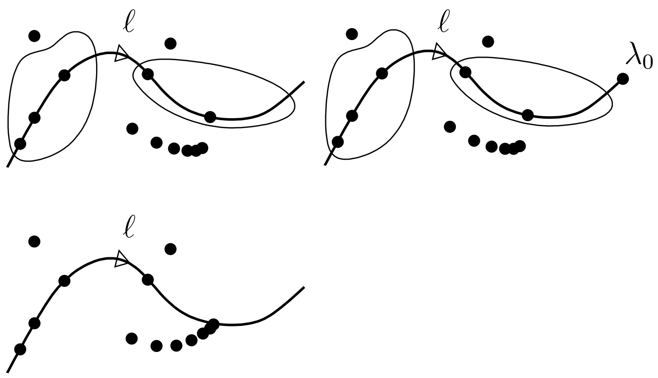

Fig. 1 Upper left: Closed linear relation with admissible spectrum with respect to ℓ. Upper right: Admissible spectrum with $\lambda_0 \in \overline{\ell} \setminus \ell$. Bottom: Non-admissible spectrum since $\sigma(A) \cap N \neq \sigma(A) \cap \ell$ and $\dim \mathrm{im}\, P_N(A) = +\infty$, each inflicting (16)(i) and (ii)

Definition 11 (Cf. Zhu [40, Definition 1.3.6], [41, Definition 2.1], and [43, Definition 2.6]). Let $\ell \subset \mathbb{C}$ be a C^1 real 1-dimensional submanifold which has no boundary and is co-oriented (i.e., with oriented normal bundle). Let X be a Banach space and $A \in \mathscr{C}LR(X)$ be a closed linear relation.
(a) We call A *admissible* with respect to ℓ, if there exists a bounded open subset N of $\mathbb{C}$ (called *test domain*) such that (see also Fig. 1)

$$\text{(i) } \sigma(A) \cap N = \sigma(A) \cap \ell \quad \text{and} \quad \text{(ii) } \dim \operatorname{im} P_N(A) < +\infty \tag{16}$$

Then $P_N(A)$ does not depend on the choice of such a test domain N. We set

$$P_\ell(A) := P_N(A) \quad \text{and} \quad \nu_\ell(A) := \dim \operatorname{im} P_N(A). \tag{17}$$

For fixed ℓ and X we shall denote the space of all ℓ-admissible closed linear relations in X by $\mathscr{A}_\ell(X)$.
(b) Let $A \in \mathscr{A}_\ell(X)$. Let $N \subset \mathbb{C}$ be open and bounded with C^1 boundary. We set $N^0 = N \cap \ell$ and assume

$$\overline{N^0} = \overline{N} \cap \ell, \sigma(A) \cap \ell \subset N, \sigma(A) \cap \partial N = \emptyset, \quad \text{and} \quad \dim \operatorname{im} P_N(A) < +\infty. \tag{18}$$

Moreover, we require that each connected component of N has connected intersection with ℓ so that the disjoint positive (negative) part $N^\pm$ of N with respect to the co-orientation of ℓ is well-defined, and we have a disjoint union $N = N^+ \cup N^0 \cup N^-$. We shall call the resulting triple $(N; N^+, N^-)$ *admissible* with respect to ℓ and A, and write $(N; N^+, N^-) \in \mathscr{A}_{\ell,A}$. See also Fig. 2

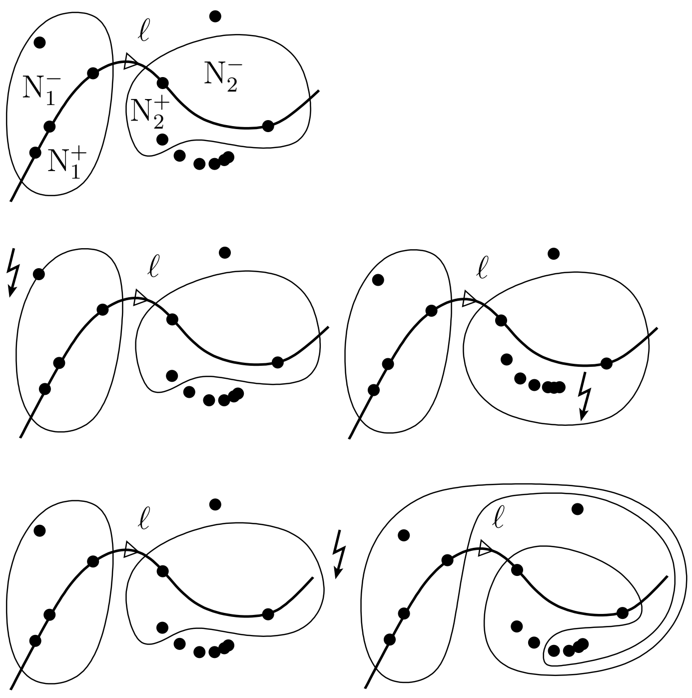

Fig. 2 Top: Admissible test domain triple (N, N^+, N^-). Middle and bottom: Non-admissible test domain triples. Middle left: $\sigma(A) \cap \partial N \neq \emptyset$. Middle right: $\dim P_N(A) = +\infty$. Bottom left: No distinct orientation. Bottom right: $N \cap \ell$ not connected while N connected

Now we are able to define spectral-continuity and the spectral flow. Our data are a co-oriented curve $\ell \subset \mathbb{C}$, a family of Banach spaces $\{X_s\}_{s\in[a,b]}$ and a family $\{A_s\}_{s\in[a,b]}$ of closed linear relations on X_s.

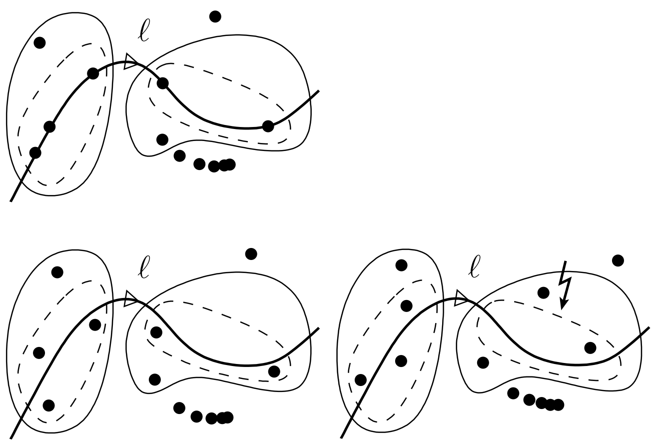

Fig. 3 The hedging of the spectra of a spectral-continuous family near ℓ at s_0 : The same test domain triple (N,N^+,N^-) (solid line) works at s_0 in the upper figure, at $s_0-\varepsilon$ in bottom left, and at $s_0+\varepsilon$ in bottom right. The sub-triple (N',N'^+,N'^-) (encircled by the broken line) will also work at s_0 and for $s_0-\varepsilon$, but only for $s_0+\varepsilon'$ with $\varepsilon'\ll\varepsilon$

Definition 12 (a) We shall call the family $\{A_s\}\in\mathscr{A}_\ell(X_s)$, $s\in[a,b]$ *spectral-continuous* near ℓ at $s_0\in[a,b]$, if there is an $\varepsilon(s_0)>0$ such that for all $\varepsilon'\in(0,\varepsilon(s_0))$ there exists a triple $(N;N^+,N^-)$ such that

$$(N;N^+,N^-)\ \in\ \mathscr{A}_{\ell,A_s}\quad \text{ for all } |s-s_0|<\varepsilon';$$

and for all triple $(N';N'^+,N'^-)\in\mathscr{A}_{\ell,A_{s_0}}$ with $\overline{N'}\subset N$, and $N'^\pm\subset N^\pm$, we have

$$(N';N'^+,N'^-)\ \in\ \mathscr{A}_{\ell,A_s}\quad \text{ for all } |s-s_0|\ll 1;$$

and $\dim\operatorname{im}P_{N'}(A_s)$ and $\dim\operatorname{im}P_{N^\pm\setminus N'^\pm}(A_s)$ do not depend on s. See also Fig. 3 and Fig. 4.

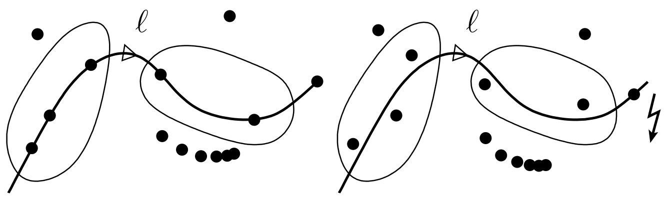

Fig. 4 A globally spectral-continuous curve of closed linear relations with admissible spectra may fail to become spectral-continuous near ℓ due to a spectral point $\lambda_0\in\overline{\ell}\setminus\ell$ for s_0 (left), which moves inward on ℓ for $s=s_0\pm\varepsilon$ (right)

We shall call the family $\{A_s\}\in\mathscr{A}_\ell(X_s)$, $s\in[a,b]$ *spectral-continuous* near ℓ, if it is spectral-continuous near ℓ at s_0 for all $s_0\in[a,b]$.

(b) Let $\{A_s\}\in\mathscr{A}_\ell$, $s\in[a,b]$ be a family of admissible operators that is spectral-continuous near ℓ. Then there exist a partition

$$a\ =\ s_0\le t_1\le s_1\le\ldots s_{n-1}\le t_n\le s_n\ =\ b \tag{19}$$

of the interval $[a,b]$, such that $s_{k-1},s_k\in(t_k-\varepsilon(t_k),t_k+\varepsilon(t_k))$, $k=1,\ldots,n$. Let $(N_k;N_k^+,N_k^-)$ be like a $(N;N^+,N^-)$ in (a) for t_k and some $\varepsilon'\in(0,\varepsilon(s_0))$ such that $s_{k-1},s_k\in(t_k-\varepsilon',t_k+\varepsilon')$, $k=1,\ldots,n$. Then we define the *spectral flow* of $\{A_s\}_{a\le s\le b}$ through ℓ by

$$\mathrm{sf}_\ell\{A_s; a\le s\le b\}$$
$$:= \sum_{k=1}^{n}\Big(\dim\operatorname{im}\big(P_{N_k^-}(A_{s_{k-1}})\big)-\dim\operatorname{im}\big(P_{N_k^-}(A_{s_k})\big)\Big). \quad (20)$$

When ℓ is a bounded open submanifold of $i\mathbb{R}$ containing 0 with co-orientation from left to right, we set

$$\mathrm{sf}\{A_s; a\le s\le b\} \;:=\; \mathrm{sf}_\ell\{A_s; a\le s\le b\}.$$

Lemma 16 *Something new.*

Note that for a family of $\{A_s\in\mathscr{A}_\ell(X_s)\}$, we always obtain a spectral-continuous family, when we are given a suitable family of transformations $T_{s,s_0}: Y_s\to Y_{s_0}$ such that the family

$$T_{s,s_0}A_sT_{s,s_0}^{-1}\in\mathscr{C}(Y_{s_0})$$

is continuously varying.

From our assumptions it follows that the spectral flow is independent of the choice of the partition (19) and admissible $(N_k;N_k^+,N_k^-)$, hence it is well-defined. From the definition it follows that the spectral flow through ℓ is path additive under catenation and homotopy invariant. For details of the proof, see [32] and [43].

We close the appendix by discussing the invariance of the spectral flow under embedding in a larger space, assuming a simple regularity condition.

Lemma 17 *Let $\{Y_s; s\in[a,b]\}$ and $\{X_s; s\in[a,b]\}$ be two families of (complex) Banach spaces with $X_s\subset Y_s$ (no density or continuity of the embeddings assumed). Let $\{A_s\in\mathscr{C}LR(Y_s); s\in[a,b]\}$ be a spectral-continuous curve near a fixed co-oriented curve $\ell\subset\mathbb{C}$. We assume that $A_s(X_s)\subset X_s$ for all s and that the curve $\{A_s|_{X_s}\in\mathscr{C}LR(Y_s); s\in[a,b]\}$ is also spectral-continuous near ℓ. Then we have*

$$\mathrm{sf}_\ell\{A_s; s\in[a,b]\} \;=\; \mathrm{sf}_\ell\{A_s|_{X_s}; s\in[a,b]\}$$

if the difference $\dim\nu_\ell(A_s)-\dim\nu_\ell(A_s|_{X_s})$, $s\in[a,b]$, is constant.

Proof We go back to the local definition of sf_ℓ and reduce to the finite-dimensional case. Denote by m the constant in our assumption. Let $s_0\in[a,b]$. Choose a triple

$$(N_1;N_1^+,N_1^-)\in\mathscr{A}_{\ell,A_{s_0}}$$

such that N_1 satisfies (16) for A_{s_0}. Then by spectral-continuity, there exists a triple $(N;N^+,N^-)$ with $\overline{N}\subset N_1$ with

$$(N;N^+,N^-)\in\mathscr{A}_{\ell,A_s}\quad\text{for } |s-s_0|\ll 1.$$

Then we have, again for $|s-s_0|\ll 1$

$$\dim\operatorname{im}P_N(A_s) \;=\; \nu_\ell(A_{s_0}) \;=\; \nu_\ell(A_s|_{X_{s_0}})+m \;=\; \dim\operatorname{im}P_N(A_s|_{X_s})+m \quad (21)$$

by spectral-continuity and our assumption. Now we consider for each $\lambda\in\mathbb{C}\cap N$ the algebraic multiplicities and find

$$\dim\ker(A_s|_{X_s}-\lambda I|_{X_s})^k\le\dim\ker(A_s-\lambda I)^k \quad (22)$$

for each $k\in\mathbb{N}$. Comparing

$$\dim\operatorname{im}P_N(A_s) \;=\; \sum_{\lambda\in\sigma(A_s)\cap N}\sum_{k\in\mathbb{N}}\dim\ker(A_s-\lambda I)^k \quad\text{and}$$
$$\dim\operatorname{im}P_N(A_s|_{X_s}) \;=\; \sum_{\lambda\in\sigma(A_s|_{X_s})\cap N}\sum_{k\in\mathbb{N}}\dim\ker(A_s|_{X_s}-\lambda I|_{X_s})^k$$

we obtain from equation (21) and the inequalities (22) that

$$\dim\ker(A_s|_{X_s}-\lambda I|_{X_s})^k \;=\; \dim\ker(A_s-\lambda I)^k$$

for each $\lambda\in N\setminus\ell$ and $k\in\mathbb{N}$. So

$$\sigma(A_s)\cap(N\setminus\ell) \;=\; \sigma(A_s|_{X_s})\cap(N\setminus\ell);$$

and the algebraic multiplicities with respect to A_s and $A_s|_{X_s}$ coincide in each point. By the definition of the spectral flow, the two spectral flows must coincide.

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