

## 'Proportions' in the Liber abbaci

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*Published in:*  
Proportions: Science - Musique - Peinture & Architecture

*Publication date:*  
2012

*Document Version*  
Early version, also known as pre-print

*Citation for published version (APA):*  
Høyrup, J. (2012). 'Proportions' in the *Liber abbaci*. In S. Rommevaux, P. Vendrix, & V. Zara (Eds.), *Proportions: Science - Musique - Peinture & Architecture* (pp. 83-100). Brepols Publishers.

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# “Proportions” in the *Liber abbaci*

Contribution to the proceedings of the meeting

Proportions

Arts – Architecture – Musique – Mathématiques – Sciences

Centre d'Études Supérieures de la Renaissance

Tours, 30 June to 4 July, 2008

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*PREPRINT*

August 2009

## Abstract

As I have argued elsewhere, Fibonacci is an early exponent of a burgeoning abacus culture, more likely to flourish in his time in other areas around the Mediterranean (including the Catalan-Provençal area) than in Italy. It is obvious, however, that the *Liber abbaci* is much more than a normal abacus book as we know it from later Italy. Indeed, analysis of the appearances of the notions of ratio and proportion in the book shows how these, not present in “proto-abacus” culture, are applied to abacus material, in principle thus seeing abacus mathematics in the perspective of “magisterial” mathematics but without changing it fundamentally. This conclusion can also be drawn regarding the modest integration of ratio and proportion in algebra. The only exception is chapter 15, section 1, whose main part explores the ancient theory of means (geometric, harmonic, their sub-contraries, etc. – without Fibonacci knowing so), showing how to find either the mean or one of the extremes from the other two numbers.

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## Preliminaries

Before taking up the substance of my topic, I shall make three preliminary remarks: one on terminology, one on notation, and one on delimitation.

Terminology first. As his contemporaries, Fibonacci speaks of a ratio/λόγος (understood as a relation between two integers, not as a single number) as *proportio/proportione*. He uses the same word where we would speak of a proportion and the Greek mathematicians of ἀναλογία, that is, an affirmation that two ratios are “the same” or “similar”. In the case of numbers being in continued proportion (εἰης ἀνάλογον), he sometimes speak of *continua proportione*, sometimes however he uses the word *proportionalitas*. An attempt to enforce a modern terminology would either divide the field in a way which does not correspond to the thought of our author, or it would force us to speak of “numbers in continued ratio” – which certainly makes sense, but is *not* modern terminology. It would also bow to the modern conceptual confusion, which uses “ratio” both in the historically proper sense, about the relation between *two* numbers, and about their quotient, a single number. I shall therefore translate *proportio* as “proportion”, etc. – while still speaking in modern ways of ratio and proportion outside direct and indirect quotations when the relation between two numbers respectively the “similitude” between two such relations is meant; the single-number “ratio” I shall refer to as the “quotient”.

Second, notation. When designating explicitly a proportion, our texts mostly say that “the first number is to the second, as the third to the fourth”,<sup>1</sup> or use some equivalent expression. For typographical convenience, I shall use the notation  $\frac{a}{b} : \frac{c}{d}$ , which should be read as representing the frame

a	c
b	d

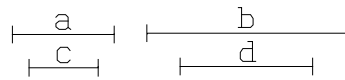
corresponding to what is found regularly in the margin in the *Liber abbaci* (p. 170 and *passim*). The two notations – as well as the line diagram used both in

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<sup>1</sup> Thus in the *Liber abbaci* (*Scritti* di Leonardo Pisano matematico del secolo decimoterzo. I. Il *Liber abbaci*, ed. Baldassare Boncompagni, Roma, Tipografia delle Scienze Matematiche e Fische, 1857, here p. 170); all further references to the *Liber abbaci* refer to the pagination of this edition.

Here as everywhere in the following, translations with no identified translator are mine.

the *Liber abbaci* (e.g., p. 395) and by Campanus<sup>2</sup>



are equally fit to serve the visualization and automation of the various operations that can be performed on the proportion:<sup>3</sup>

<p><i>e contrario:</i> <math>\frac{b}{a} : \frac{d}{c}</math></p> <p><i>permutata:</i> <math>\frac{a}{c} : \frac{b}{d}</math></p> <p><i>conjuncta:</i> <math>\frac{a+b}{b} : \frac{c+d}{d}</math></p> <p><i>disjuncta:</i> <math>\frac{a-b}{b} : \frac{c-d}{d}</math></p>	<p><i>conversa:</i> <math>\frac{a}{a+b} : \frac{c}{c+d}</math></p> <p><i>eversa:</i> <math>\frac{a}{a-b} : \frac{c}{c-d}</math></p> <p><i>aequa:</i> <math>\frac{a}{b} : \frac{a+c}{b+d}</math></p>
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and also of the equality of the products  $a \cdot d = b \cdot c$  (to which I shall refer in the following as the “product rule”). The typographically convenient notation thus involves no serious anachronism –  $a:b::c:d$ , while agreeing with the phrase “the first to the second, as the third to the fourth”, corresponds less well to the diagrams on which the medieval authors based their operational thinking. In order to distinguish, I shall write fractions (including “ratios” understood as quotients) as  $\frac{a}{b}$ . Ratios (not understood as quotients, and not constituents of a proportion) I shall denote  $a:b$ , and numbers in continued proportion will stand as  $a:b:c:\dots$

Third, delimitation. Any applied arithmetic which goes beyond the simplest accounting runs into problems of proportionality – say, of the type “for  $a$  [coins],  $b$  [units], for  $c$  [coins], how much? In Near Eastern and Greek Antiquity, this would normally be solved in an intuitively transparent way: For  $a$  [coins],  $b$  [units], for 1 [coin] therefore  $\frac{b}{a}$  [units], and for  $c$  therefore  $c \cdot \frac{b}{a}$  [units]. Some Arabic reckoners<sup>4</sup> would prefer the argument “by *nisbah* [“ratio”], for  $a$  [coins],  $b$  [units], for  $c$  therefore  $\frac{c}{a}$  as much, that is,  $(\frac{c}{a}) \cdot b$  [units]. From India, however,

<sup>2</sup> *Campanus of Novara and Euclid’s Elements*, ed. Hubert L. L. Busard. 2 vols. (Boethius, 51,1–2). Stuttgart, Franz Steiner, 2005; p. 161 and *passim*.

<sup>3</sup> This way to present them is taken from the Campanus *Elements* (see note 2), p. 171f.

<sup>4</sup> Thus Ibn Thabāt (*Die Reichtümer der Rechner (Ġunyāt al-Hussāb) von Ahmad b. Tabāt (gest. 631/1234). Die Araber – Vorläufer der Rechenkunst*, ed., trans. Ulrich Rebstock. (Beiträge zur Sprach- und Kulturgeschichte des Orients, 32). Walldorf-Hessen, Verlag für Orientkunde Dr. H. Vorndran; here p. 43–45), and al-Karajī (*Kāfi fil Hisāb (Genügendes über Arithmetik)* des Abu Bekr Muhammed ben Alhusein Alkarkhi, ed., trad. Adolph Hochheim. 3 vols. Halle, Louis Nebert, 1878; here II, p. 17].

probably via the trade routes and possibly with ultimate roots in China, Arabic merchants and after them theoretically inclined Arabic mathematicians from al-Khwārizmī onward adopted the *rule of three*, stating that  $c$  must yield  $(b \cdot c)/a$ .<sup>5</sup> Indian practical reckoners appear to have used a formulation in the style “multiply the thing [whose counterpart] you want to know by that which is not similar [to it in kind] and divide by that which is similar”. This is not the main formulation of the learned Sanskrit writers (Āryabhata, Brahmagupta, Mahāvīra, etc.), but the formulations of the latter two betray that they know it. Even in the Arabic world, it appears to have been the formulation of merchants. The theoretically trained Arabic mathematicians soon saw that the whole matter can be based on the proportion theory of *Elements* VII – if only we forget about the numbers being concrete and indeed being of two different kinds (for instance, dinars and cloth), and not abstract. None the less, many of the Arabic mathematicians betray familiarity with the traditional formulation, in spite of its conflict with the Euclidean approach (which requires ratios to be between quantities of the same kind, e.g., abstract numbers<sup>6</sup>).

In the European (that is, Italian and Ibero-Provençal) abacus environment, the rule also arrived in “non-Euclidean” interpretation (in Italy and perhaps in Provence in the traditional “non-similar/similar” formulation, in Spain (as we shall see) apparently in a different shape; even in the Christian world, however, theoretically trained writers interacting with the abacus environment, from Fibonacci to Chuquet, made use of the Euclidean formulation. This, however, I shall not discuss in any depth – not because it is not interesting but because it is a separate topic, and treated at best together with other aspects of the approach to the rule of three.

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<sup>5</sup> This, and the remains of the paragraph, builds on Jens Høyrup, “Further questions to the historiography of Arabic (but not only Arabic) mathematics from the perspective of Romance abacus mathematics”. Contribution to the “9<sup>ième</sup> Colloque Maghrébin sur l’Histoire des Mathématiques Arabes”, Tipaza, 12–13–14 mai 2007, p. 1–8. Until appearance of the proceedings available at the address <http://www.akira.ruc.dk/~jensh/Work%20in%20progress/FurtherQuestions.pdf>.

<sup>6</sup> Of course, the Euclidean approach is saved if only we use the equivalent proportion  $\frac{a}{c} : \frac{b}{d}$ . However, the sources never bother to perform this transformation.

## **Fibonacci's *Liber abbaci***

I have argued on other occasions<sup>7</sup> that Fibonacci is not the founding father of abacus culture but rather an early (towering) exponent of a culture which already flourished in his time, if not in Italy (which seems unlikely) then in Provence, Catalonia and the Maghreb and al-Andalus, perhaps even in Egypt, Syria and Byzantium, and which was connected to a culture of commercial arithmetic ranging at least as far as Iran and India; on the present occasion I shall refer to this as the “proto-abacus culture”.

That should not be taken to imply that the *Liber abbaci* is just an early abacus book. Fibonacci writes *in a mathematically educated perspective* about the kind of mathematics thriving in the environment in question; but his scope is much larger, encompassing not only what he encountered on business travels to Egypt, Syria, Constantinople, Sicily and Provence (p. 1) but also topics which almost certainly fell outside the horizon of the proto-abacus culture.<sup>8</sup> Much of his treatment of proportions (if not all of it) falls in that category.

### **Touching on proportions**

The first time numbers in proportion turn up in the *Liber abbaci* in the explanation of the algorithm for the multiplication of multi-digit numbers (p. 15). Here it is pointed out that if three numbers are proportional, then the product of the first and the third equals the product of the second by itself; and if four, then the product of the first and the fourth equals that of the second and the third; for these product rules, Fibonacci gives a generic reference to Euclid. They are combined with the observation that the “degrees” or decimal levels form an infinite continued proportion, which leads to the conclusion that multiplication of the first degree by the third gives as much as that of the second degree by itself, while the second by the third gives as much as the first by the fourth, etc.

This argument *may* have been devised by Fibonacci himself; I do not

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<sup>7</sup> See, for instance, Jens Høyrup, “Leonardo Fibonacci and *Abaco* Culture: a Proposal to Invert the Roles”. *Revue d'Histoire des Mathématiques*, XI (2005), p. 23–56.

<sup>8</sup> Though regarding the *Liber abbaci* as the archetype for abacus books, Margherita Bartolozzi & Raffaella Franci (“La teoria delle proporzioni nella matematica dell’abaco da Leonardo Pisano a Luca Pacioli”, *Bollettino di Storia delle Scienze Matematiche*, X (1990), p. 3–28, here p. 5) align it more adequately with fifteenth-century encyclopaediae like Benedetto da Firenze’s *Praticha d’arismetricha* and the anonymous MS Florence, Palatino 573.

remember having seen it in any earlier source, not even in hints.<sup>9</sup> Nice though it is, it also seems to have been a historical dead end, not to be repeated by any later writer.

A next passing reference (p. 82) to (four) numbers in proportion and to the equality of products turns up in the explanation of the decomposition of a fraction – once more with the generic reference to Euclid. This is followed closely by the presentation of the rule of three in simple and composite shape, which I shall not treat in depth.<sup>10</sup> I shall merely observe

- that Fibonacci does *not* use what was to become the standard formulation of the abacus school (the one which refers to the non-similar and the similar) – his formulations (pp. 83f) have a certain family likeness with what can be found in Arabic authors (al-Khwārizmī, al-Karajī, etc.), but their actual shape is likely to be Fibonacci’s own;
- that Fibonacci employs the rectangular frame mentioned above, leaving the position for the unknown number empty and indicating the cross-multiplication by a diagonal;
- that the treatment of the non-composite rule is argued from the product rule “which has been proved in the arithmetical [books of the *Elements*] and in the geometry”;
- that the composite rule (used in barter problems) is presented with a reference to *figura cata, scilicet sectoris* [Menelaos’ theorem] “which Ptolemy teaches in the *Almagest*”;
- that the name *proportio proportionum* is introduced (p. 131) for multiply composite ratios – wholly unconnected, of course, to Oresme’s later notion of *proportio proportionum*.

Whereas barter problems employ the rule of three “sequentially”, partnership problems use it “in parallel”; in this case (pp. 114f, 135–143), however, Fibonacci does not refer explicitly to “proportions” or proportionality – nor indeed to the rule of three itself, but since in general he has no name for that rule this is not astonishing. However, in connection with a problem about the alloying of three

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<sup>9</sup> If Fibonacci’s own invention, it could have been inspired by analogous reasoning about the sequence of algebraic powers. The parallel between the powers of the algebraic *thing* and the powers of ten was pointed out by al-Karajī [Franz Woepcke, *Extrait du Fakhri, traité d’algèbre* par Aboû Bekr Mohammed ben Alhaçan Alkarkhî; *précédé d’un mémoire sur l’algèbre indéterminé chez les Arabes*. Paris, L’Imprimerie Impériale, 1853, p. 48]; it may have been common lore among Arabic writers 200 years later.

<sup>10</sup> See, however, Bartolozzi & Franci, *op. cit.* (note 8), p. 5–7.

monies (pp. 149f), the first and the second in ratio 2:3, the second and the third in ratio 4:5, he speaks of “proportional alloying” and teaches how to harmonize these as easily composable ratios by means of multiplication. The idea of “proportional alloying” also turns up repeatedly in the following pages (but with even less theoretical effect).

### **Closer attention to ratios and proportions in “abbacus” context**

Proper interest in our topic only returns in Chapter 12, Part 2 (pp. 169–173). It starts by explaining equal, major and minor ratios, and gives the examples 3:3, 8:4, 9:3, 16:5, 4:8, 3:9 and 5:16 – providing them with names which are not in the Boethian tradition but come close to the “denomination” (although this word does not occur). For instance, 16:5 is a “triple proportion and a fifth”. It goes on with the problem of finding the number to which 6 has the same “proportion” as 3 to 5, giving first the numerical solution  $(5 \cdot 6) / 3$  and saying then that this question is stated “in our vernacular” (*ex usu nostri vulgaris*<sup>11</sup>) in the phrase “if 3 were 5, what would then 6 be?”. Similarly, it asks for the number to which 11 has the same ratio as 5 to 9, and gives it the vernacular formulation “if 5 were 9, what would 11 be?”.

This formulation is remarkable.<sup>12</sup> Only one Italian abacus treatise I know of identifies the rule of three by means of the same phrase, namely the *Columbia Algorism*<sup>13</sup> – also untypical in other respects, almost certainly dated no later than 1290<sup>14</sup> and thereby probably the earliest extant abacus text (though known only from a fourteenth-century copy). Admittedly, counterfactual questions –

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<sup>11</sup> A complete survey of the references to *modus vulgaris* and its cognates in the *Liber abbaci* shows that the genuine meaning is not the generic spoken vernacular but with one exception the simple ways of practical reckoners (the exception (p. 111) is the information that an alloy of silver and tin is called “false silver *vulgariter*”). Simple, stepwise calculation is meant in four places (pp. 115, 127, 204, 364). In the last place, the *modus vulgaris* is confronted explicitly with how one proceeds *magistraliter*.

<sup>12</sup> For full documentation, see Jens Høyrup, *Jacopo da Firenze’s Tractatus Algorismi and Early Italian Abacus Culture*. (Science Networks. Historical Studies, 34). Basel etc., Birkhäuser, 2007, p. 64–67.

<sup>13</sup> *Ein italienisches Rechenbuch aus dem 14. Jahrhundert (Columbia X 511 A13)*, ed. Kurt Vogel. (Veröffentlichungen des Deutschen Museums für die Geschichte der Wissenschaften und der Technik. Reihe C, Quellentexte und Übersetzungen, Nr. 33). München, Deutsches Museum, 1977.

<sup>14</sup> See Høyrup, *Jacopo da Firenze’s Tractatus Algorismi ...* (above, note 12), p. 31 n. 70.



and even “counterfactual calculations” in the style “if 7 were the half of 12, what would the half of 10 be?” (p. 10) – are not absent from the Italian abacus record, but they invariably turn up long after the rule of three is explained, or as secondary examples (the primary examples confronting either different currencies or goods and their monetary value). In all Ibero-Provençal treatises from before 1500 which I have inspected,<sup>15</sup> on the other hand, the rule of three is introduced first by counterfactual or abstract-number questions, “If 3 were 4, what would 5 be?” or “if  $4\frac{1}{2}$  are worth  $7\frac{2}{3}$ , what are  $13\frac{3}{4}$  worth?”. The Provençal specimens also know the formulation in terms of the non-similar and the similar, and so does Santcliment’s Catalan *Summa*.<sup>16</sup> Besides that, however, Santcliment informs us that this is spoken of “in our vernacular” (*en nostre vulgar*) by the phrase “if so much is worth so much, how much is so much worth” (*si tant val tant: que valra tant*). The same phrase (*sy tanto faze tanto, ¿qué sería tanto?*) is also used in the Castilian *Libro de arismética que es dicho algarismo*.<sup>17</sup> Wherever Fibonacci encountered the vernacular tradition he refers to, it left no appreciable traces

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<sup>15</sup> In chronological order

- the Castilian *Libro de arismética que es dicho algarismo* (in *El arte del algarismo. Un libro castellano de aritmética comercial y de ensayo de moneda del siglo XIV*. (Ms. 46 de la Real Colegiato de San Isidoro de León, ed. Betsabé Caunedo del Potro & Ricardo Córdoba de la Llave. Salamanca, Junta de Castilla y León, Consejería de Educación y Cultura, 2000);
- the “Pamiers Algorism” (partial ed. in Jacques Sesiano, “Une arithmétique médiévale en langue provençale”. *Centaurus* XXVII (1984), p. 26–75).
- the mid-fifteenth-century Franco-Provençal *Traicté de la pratique d’algorisme* (I used the transcription in Stéphane Lamassé’s unpublished dissertation, for access to which I am grateful).
- Barthélemy de Romans’ Provençal *Compendy de la pratique des nombres* (*Une arithmétique commerciale du XV<sup>e</sup> siècle. Le Compendy de la pratique des nombres de Barthélemy de Romans*, ed. Maryvonne Spiesser. (De Diversis artibus, 70) Turnhout, Brepols, 2003).
- Francesc Santcliment’s *Summa de l’art d’aritmética* (ed. Antoni Malet. Vic, Eumo Editorial, 1998);
- Francés Pellos’s *Compendion de l’abaco* (ed. Robert Lafont & Guy Tournerie. Montpellier, Édition de la Revue des Langues Romanes, 1967).

I also looked at Chuquet’s *Triparty en la science des nombres* (ed. Aristide Marre. *Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche* XIII (1880), p. 593–659, 693–814) which is not strictly Provençal but integrated in and thus a lateral witness of the Provençal tradition.

<sup>16</sup> *Op. cit.* (note 15), p. 163.

<sup>17</sup> Caunedo del Potro & Córdoba de la Llave, *op. cit.* (note 15), p. 147].

in Italy, but many in the Ibero-Provençal orbit, most clearly in its Iberian section.

Next, Fibonacci presents the counterfactual calculation that was just quoted (“if 7 were the half of 12, what would the half of 10 be?”), and another counterfactual simple question. He goes on with procedures for finding four and six integers in proportion if the first two of them are given; shows how to divide 10 into four unequal parts in proportion – namely by scaling an arbitrary proportion  $\frac{a}{b} : \frac{c}{d}$  by the factor  $\frac{10}{(a+b+c+d)}$ ; explains how to construct a continued proportion with an arbitrary number of terms (explaining the appurtenant product rules); and finally demonstrates how to find two or three numbers so that  $\frac{1}{p}n_1 = \frac{1}{q}n_2$  (and, in the case of three numbers,  $\frac{1}{r}n_2 = \frac{1}{s}n_3$ ) – in a different formulation, not used by Fibonacci but common in later Italian abacus algebra,  $\frac{n_1}{n_2} : \frac{p}{q}$  (and  $\frac{n_2}{n_3} : \frac{r}{s}$ ).

On the whole, what Fibonacci does in this chapter is thus to connect procedures and problem types belonging to the “vernacular” proto-abbacus tradition(s) he had encountered with the notion of “proportions”. The theoretical field itself is not explored in any way.

### Chapter 15 part 1: exploring the theory of means

Theoretical exploration of a kind comes in Chapter 15, Part 1 (pp. 387–397), claims to treat of “the proportions of three and four quantities, to which the solution of many questions belonging to geometry are reduced” (p. 387). Actually it deals with problems about *numbers* in proportion, and (as we shall see) its results are not used in the following “geometry”-section when they would be pertinent. These numbers are spoken of as “the first/second/third/fourth number” (or, in the case of three numbers, often “minor/middle/major”). In most cases, they are represented by letter-carrying line segments drawn in the margin – for brevity, since we are not going to follow the arguments in detail, we may designate them *P*, *Q*, *R* and (when needed) *S*. At first proportions involving three numbers are presented, afterwards (much fewer) questions involving four numbers are dealt with. By means of conjunction, disjunction, permutation etc., the given proportion is transformed in such a way that the numbers can be found from the product rules by means of addition or subtraction or, more often, *Elements* II.5–6 (II.6 being sometimes preferred even in cases where II.5 would seem the obvious choice). Strikingly, Fibonacci never refers to Euclid here, which he is otherwise fond of doing.<sup>18</sup> In the appendix I give a complete

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<sup>18</sup> Menso Folkerts, *The Development of Mathematics in Medieval Europe: The Arabs, Euclid, Regiomontanus*. (Variorum Collected Studies Series, CS811). Aldershot, Ashgate, 2006; article

list, indicating for each case the initial transformations and the strategy used to complete the solution.

As stated, Fibonacci starts by considering questions involving three numbers. In the questions (1)–(3), these are in continued proportion,  $P:Q:R$ . One of the numbers is given together with the sum of the other two. The naming of segments presupposes the alphabetic order  $a, b, c, \dots$

The sequence (4)–(38) still treats of three numbers, but now differences between the numbers are among the given magnitudes. The alphabetic order underlying naming changes to  $a, b, g, d, \dots$

(39)–(50) consider four numbers in proportion,  $\frac{P}{Q} : \frac{R}{S}$ . The underlying alphabetic order is still  $a, b, g, d, \dots$ . At first, the *e contrario* and *permutata* transformations are set out, and it is explained how any one of the numbers can be found from the three others via the product rule. Then follow problems where two of the numbers are given together with the sum of ((40)–(45)) respectively the difference between ((46)–(49)) the two others; finally, in (50), two numbers and the sum of the squares of the remaining two is given.

The most interesting group is (4)–(38). The change of alphabetic sequence seems to imply that this sequence as well as the one which follows build on (or copy from) a different source, Arabic or possibly Greek. However, since the letter  $c$  turns up in the manipulations leading to the solution in (4)–(5); since these two and the observation (6) but none of the following ones designate one of the segments by a single letter; since the continued proportion is treated again in (27)–(29); and since finally (7) is preceded by the heading *modus alius proportionis inter tres numeros*, (4)–(5) may have been inserted by Fibonacci in continuation of the topic of (1)–(3) but in emulation of the sequence which follows. The borrowed sequence should thus presumably be restricted to (7)–(38).

All of these except (26) (on which imminently) and the observations (19) and (33) deal with the 10 non-arithmetical means between two numbers discussed in ancient Greek mathematics.<sup>19</sup> More precisely, they show how to find the various means ( $Q$ ) if the extremes  $P$  and  $R$  are given, or any of the extremes if the other extreme and a mean are given. The following scheme relates Fibonacci's problems with Pappos's and Nicomachos's presentations and numbering of

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## IX.

<sup>19</sup> Cf. Thomas L. Heath, *A History of Greek Mathematics*. 2 vols. Oxford, The Clarendon Press, 1921; here vol. II, p. 85–88.

these.<sup>20</sup>

	Pappos	Nicomachos	<i>Liber abbaci</i>
$\frac{R-Q}{Q-P} : \frac{R}{R}$ (arithmet.)	P1	N1	
$\frac{R-Q}{Q-P} : \frac{R}{Q}$ or $\frac{R-Q}{Q-P} : \frac{Q}{P}$	P2	N2	27–29
$\frac{R-Q}{Q-P} : \frac{R}{P}$	P3	N3	7–9
$\frac{R-Q}{Q-P} : \frac{P}{R}$	P4	N4 (but inverted)	10–12 (inverted)
$\frac{R-Q}{Q-P} : \frac{P}{Q}$	P5	N5 (but inverted)	34–36 (inverted)
$\frac{R-Q}{Q-P} : \frac{Q}{R}$	P6	N6 (but inverted)	20–22 (inverted)
$\frac{R-P}{Q-P} : \frac{R}{P}$	absent	N7	16–18
$\frac{R-P}{R-Q} : \frac{R}{P}$	P9	N8	13–15
$\frac{R-P}{Q-P} : \frac{Q}{P}$	P10	N9	30–32
$\frac{R-P}{R-Q} : \frac{Q}{P}$	P7	N10	37–38
$\frac{R-P}{R-Q} : \frac{R}{Q}$	P8	absent	23–25

As we see, Fibonacci agrees with Nicomachos and Boethius and not with Pappos in the cases 4–6, having  $\frac{R}{P} : \frac{Q-P}{R-Q}$  instead of  $\frac{R-Q}{Q-P} : \frac{P}{R}$ , etc. However, it is not only the change of alphabetic order that seems to rule out that Fibonacci himself has produced a piece of theory inspired by Boethius. Firstly, he deals with the case P8 which is absent from Nicomachos’s list, and his order is wholly different from both Greek authors as soon as we get beyond P4=N4, the subcontrary to the harmonic mean. Secondly, where these speak of  $R-P$  directly as the difference between the extremes, Fibonacci identifies it repeatedly as the sum of the first

<sup>20</sup> Pappi Alexandrini *Collectionis quae supersunt*, ed. Friedrich Hultsch, 3 vols. Berlin, Weidmann, 1876, 1877, 1878, vol. I, p. 70–73, 84–87; Nicomachi Geraseni Pythagorei *Introductionis arithmeticae libri II*, ed. Richard Hoche. Leipzig, Teubner, 1886, p. 124–144].

Since Boethius (*De institutione arithmetica libri duo. De institutiune musica libri quinque. Accedit Geometria quae fertur Boetii*, ed. Gottfried Friedlein. Leipzig, Teubner, 1867, p. 140–169) agrees exactly with Nicomachos in all relevant respects, there is no reason to insert his translation in the table.

and the second difference. Thirdly, Fibonacci does not seem to have recognized the link to the ancient theory of *medietates* (which he is likely to have known from Boethius), nor to have seen that (27)–(29) deals with the geometric proportion which is already treated in (4)–(5) (even though he reduces (4) to (28) and points out the inverse equality in (19)). All of this confirms that Fibonacci uses an Arabic (or possibly Greek) source which was ultimately inspired by (e.g.) Nicomachos (who was well known among Arabic mathematicians) but which had gone through a thorough refashioning (involving insertion of missing cases, P8 as well as Fibonacci’s (26) and omission of the initial arithmetical mean – and, if Nicomachos is really the inspiration, by transforming the list of definitions into a sequence of problems with solutions).<sup>21</sup>

One may of course ask whether this seeming refashioning is not evidence that the link to the theory of means is spurious and the coincidence accidental. This cannot be excluded. However, the continual supposition that  $P < Q < R$  and the fact that anybody in the Greek or Arabic world who produced Fibonacci’s model can be assumed to have known the theory of means suggests that this theory was indeed the inspiration.

In (39)–(50), single-letter naming of segments and the reappearance of the letter *c* in the manipulations suggest that this sequence may come from Fibonacci’s own pen, or (less likely, I would say) from a different source.

### “Questions concerning geometry”

Chapter 15, Part 2 is claimed to deal with “questions concerning geometry”. Actually, a number of its problems have nothing to do with geometry, apart from using line diagrams for their solution; several of these – all dealing with composite gain – involve proportions.

The first of them (p. 399) is very simple. Somebody goes to one place of trade with 100 £ and earns, and afterwards earns proportionally in another place, and then has a total of 200 £. A continued proportion (represented by lettered line

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<sup>21</sup> Our medieval author is not the only one to have noticed its absence. Heath (*op. cit.* note 19, vol. II, p. 87) also sees it, and then observes that this mean is “illusory” since it only exists if the extremes coincide; for Fibonacci and his source, who have given up speaking of means, the problem is fully valid, and to be treated.

Theon of Smyrna (*Exposition des connaissances mathématiques utiles pour la lecture de Platon*, ed., trans. Jean Dupuis. Paris, Hachette, 1892, p. 175) also arrives at twelve means without specifying them completely; but he arrives at this number by adding to the 6 basic ones their subcontraries, overlooking that the arithmetical mean is its own subcontrary.

segments) shows the possession after the first travel to be  $\sqrt{(100 \cdot 200)} \approx \text{£ } 141$ , s. 8, d.  $5\frac{1}{8}$ .

The next case (p. 399) is somewhat more tricky. The initial capital is still 100 £, but after the first travel a partner invests 100 £ in the enterprise, and after the second travel the total amounts to 299 £. This gives the proportion (represented by lines)  $\frac{100}{Q} : \frac{Q+100}{299}$ . The product rule and *Elements* II.6 (still not identified) lead to the solution  $Q = 130$  £. Interchange of left and right would reduce this to case (49) above, but Fibonacci does not establish the link.

Then follows (pp. 399f) an example with three travels (beginning with 100 £ and ending with 200 £) and no extra investments, which leads to a continued proportion with four terms and thus, with reference to Euclid (namely *Elements* VII.12), a solution expressible in cube roots. This gives rise to a digression discussing numbers allowing an exact solution (24 and 81) and the notions of duplicate and triplicate proportion. From here Fibonacci goes on to the case of four travels, involving five numbers in continued proportion and a quadruplicate proportion; and to the concepts of quintuple and sextuple proportion. These are given the names “cube of squares (or square of cubes)” and “cube of cubes”; as can be seen from numerical examples, however, Fibonacci is not deceived by these names, ultimately inspired by Arabic algebraic terminology.

A final problem about composite gain (p. 401) deals with two travels with initial capital  $P$ , final total  $R$  and intermediate possession  $Q = 80$  £, with  $\frac{P}{Q} : \frac{5^2}{81}$ . Fibonacci calculates  $5 \cdot 9 = 45$  and claims without explanation that  $\frac{45}{80} : \frac{25}{P}, \frac{45^R}{80} : \frac{81^R}{80}$ . The trick is of course that  $\frac{25}{45} : \frac{45}{81}$ , while  $\frac{P}{80} : \frac{80}{R}$ ; a scaling with the factor  $\frac{45^R}{80}$  conserves the ratio between the extreme terms and adjusts the value of the middle term. Finally, Fibonacci explains it to be an equivalent problem to find two numbers  $p$  and  $q$  (namely,  $p = \sqrt{P}$ ,  $q = \sqrt{Q}$ ) so that  $\frac{1}{5}p = \frac{1}{9}q$ ,  $p \cdot q = 80$ .<sup>22</sup> This is solved via a single false position,  $p' = 5$ ,  $q' = 9$ , and subsequent scaling by the factor  $\sqrt{\frac{80}{5 \cdot 9}}$ .

The notion of “proportion” or proportionality turns up in two further places in this “geometric” section. In none of them, anything profound is meant.

First, a rule is given (p. 401) for producing “two integer roots whose squares together make the square of a number” – that is, for finding Pythagorean triples (triangles are not spoken of). The solution (that of *Elements* X, 29, lemma 1) given is to choose two square numbers or numbers having the “proportion” of squares (say,  $p^2$  and  $q^2$ ), both even or both odd. The solution given is  $p \cdot q$ ,  $(q^2 - p^2)/2$  – the third member of the triple being  $(q^2 + p^2)/2$ . This is proved by means of *Elements*

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<sup>22</sup> We recognize the structure  $\frac{1}{p}n_1 = \frac{1}{q}n_2$ , dealt with already in Chapter 12, Part 2.

## II.5 (unidentified).

Second, in the last problem of the section (pp. 405f), three numbers (say,  $a$ ,  $b$  and  $c$ ) are asked for, so that  $\frac{1}{2}a = \frac{1}{3}b$ ,  $\frac{1}{4}b = \frac{1}{5}c$ ,  $abc = a+b+c$ . Once again, this is solved by a single false position,  $a' = 8$ ,  $b' = 12$ ,  $c' = 15$ , with consecutive proportional scaling. Similarly to what he did in the last travel problem, Fibonacci goes on to discuss what to do when there are four, five and six numbers, using once again the notions of double, triple, quadruple and quintuple proportion.<sup>23</sup>

### “Proportions” and algebra

The third and final (and most famous) part of Chapter 15 (pp. 406–459) deals with “certain problems according to the method of algebra and almuchabala, that is, by proportion and restoration”.<sup>24</sup> This identification of *algebra* with “proportion” and *almuchabala* with “restoration” appears to be Fibonacci’s own invention.

Concerning “restoration” we may observe that Fibonacci knows the term from Gherardo of Cremona’s translation of al-Khwārizmī (with which he was familiar, as shown by Miura Nobuo<sup>25</sup>) and also uses it himself quite often about the cancellation of a subtractive term by addition to both sides of an equation<sup>26</sup> (alternatively he employs a mere “add”); but Gherardo will not have helped him

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<sup>23</sup> Most remarkable in this problem is presumably the use of *tetragonus* in the sense of a numerical square: everywhere else in the work this is spoken of as *quadratus*, while *tetragonus* invariably refers to a geometric square (often, (pp. 175f, 368, 408f, 421, 426f, 453)) or cube (once, (p. 403)). It is difficult not to believe Fibonacci to have used a source written in Greek without bothering to adjust its style.

<sup>24</sup> [...] *pars tertia de solutione quarundam questionum secundum modum algebre et almuchabale, scilicet ad proportionem et restaurationem.*

<sup>25</sup> MIURA Nobuo, “The Algebra in the *Liber abaci* of Leonardo Pisano”. *Historia Scientiarum* XXI (1981), p. 57–65, here p. 60.

<sup>26</sup> The “equation” as a mathematical *object* is of course *our* concept and thus strictly speaking an anachronism. Fibonacci only has the action of equating – the isolated appearance of *equatio* (p. 407) is to be understood as a corresponding verbal noun, *pace* Barnabas Hughes (*Fibonacci’ De practica geometrie*, New York, Springer, 2008, p. xxix, 361), who is seduced by Boncompagni’s mistaken punctuation (*reddigi ad equationem. Vnius* (sic) *census per diuisionem* [...] should be simply *reddigi ad equationem unius census per diuisionem* [...]).

discover that it translates *al-jabr*.<sup>27</sup> On the other hand, the term used by Gherardo to translate *al-muqābalah* and the corresponding verb *qabila* – that is, *oppositio/oppo- nere* – only occurs thrice in Fibonacci’s algebra chapter (p. 429, 436, 457), every time in the sense of confronting the two sides of an equation (in all probability the original function of the term, but not Gherardo’s normal interpretation<sup>28</sup> – and thus perhaps a coincidence).

This explains that there was space for Fibonacci’s mistaken guess – he had two slots for only one technical operation. It does not explain why he used the other slot for “proportion”, but at least this choice suggests him to have seen proportions as an important tool in the field. Why?

One hypothesis can be rejected straightaway. It has nothing to do with the proportional reduction of all coefficients when an equation is normalized. For this, Fibonacci uses *redigere* (as quoted in note 26), *reintegrare* (p. 420), or just normalizes without naming the operation; neither “proportion” nor “proportional” ever occurs in this context.<sup>29</sup>

Instead, we may observe that Fibonacci sometimes inserts pieces of reasoning based on proportion theory within algebraic or other calculations, and sometimes solves problems by means of proportion theory instead of algebra.

A relatively simple example of the first type is found in the solution of the problem, to divide 60 *denarii* first among a number (say, *r*) of men and then

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<sup>27</sup> That Fibonacci does not discover on his own should downplay Fibonacci’s Arabic skills, *pace* Barnabas Hughes, *Fibonacci’ De practica geometrie* (above, note 26), p. xix].

<sup>28</sup> There is one exception – “Gerard of Cremona’s Translation of al-Khwārizmī’s *Al-Jabr: A Critical Edition*”, ed. Barnabas B. Hughes. *Mediaeval Studies*, XLVIII (1986), p. 211–263, here p. 255.

<sup>29</sup> Also to be rejected as it stands is Barnabas Hughes suggestion (Barnabas B. Hughes, “Fibonacci, Teacher of Algebra. An Analysis of Chapter 15.3 of *Liber Abbaci*”. *Mediaeval Studies* LXVI (2004), p. 313–361, here p. 324 n. 43] that Fibonacci understood “*proportio* as a kind of operation” because “the two verbs *proportionari* and *equari* [...] are synonymous” in the Latin translation of Abū Kāmil’s algebra. Hughes overlooks that the verb *equari* is used as an editorial explanation by Jacques Sesiano (“La version latine médiévale de l’Algèbre d’Abū Kāmil”, ed. Jacques Sesiano in M. Folkerts & J. P. Hogendijk (eds), *Vestigia Mathematica. Studies in Medieval and Early Modern Mathematics in Honour of H. L. L. Busard*. Amsterdam & Atlanta, Rodopi, 1003, p. 315–452, here p. 325). What is relevant is that the fourteenth-century translator uses the verb *proportionari* in the sense of “giving/having comparable size” a single time, in agreement with possible Italian usage (of *proporzionare*) of the late Middle Ages. It is not totally excluded – though quite improbable, given that there are no other traces of this meaning in Fibonacci’s text – that Fibonacci did so too.



among  $r+2$  men, by which the share of each man decreases by  $2\frac{1}{2}$  *denarii*. Al-Khwārizmī<sup>30</sup> solves an analogous problem via (implicit) subtraction of fractions containing algebraic expressions in the denominator; Abū Kāmil<sup>31</sup> makes use of subtraction of areas within a geometric diagram; Fibonacci (p. 413) replaces this “geometric arithmetic” by operations on a proportion.

A more advanced instance of the first type deals with the gains of a complex partnership: Somebody invests 12 £, and has a certain gain after 3 months. Then somebody else invests 11 £, and after another 12 months with gain at the same monthly rate, the total gain for the two is 9 £. This is expressed in line diagrams and treated *inter alia* by operations on proportions, which in the end allow the establishment of an algebraic equation.

A simple instance of the second type is an alternative solution to the problem to find two numbers with difference 6 and quotient  $\frac{1}{3}$ . The primary solution goes via algebra: the smaller number is posited as a *thing*, the larger is thus a *thing* plus 6, etc. Alternatively, the larger is a segment  $ab$ , the smaller the partial segment  $ac$ , whence  $bc = 6$ ,  $\frac{ab}{ac} : \frac{3}{1}$ , and *disjunctim*  $\frac{bc}{ac} : \frac{2}{1}$ , etc. For somebody as familiar with proportion techniques as Fibonacci, this may indeed have been as easy as the primary solution, and for those not yet familiar with algebra it may have been easier.

Another alternative (pp. 423f), this time to an algebraic method which is mentioned but not presented, asks for a number which, when  $\frac{1}{3}$  of it and 6 are removed and the remainder multiplied by itself, yields twice the original number – in symbols,

$$(x - \frac{1}{3}x - 6)^2 = 2x .$$

In a line diagram, Fibonacci transforms this into a proportion which in symbols becomes

$$\frac{\frac{2}{3}x}{x - \frac{1}{3}x - 6} = \frac{x - \frac{1}{3}x - 6}{3} .$$

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<sup>30</sup> “Gerard of Cremona’s Translation ...” (above, note 28), p. 255; Al-Khwārizmī, *Le Commencement de l’algèbre*, ed., trans. Roshdi Rashed. (Collections Sciences dans l’histoire). Paris, Blanchard, 2007, p. 190–193.

<sup>31</sup> The *Algebra* of Abū Kāmil, *Kitāb fī al-jābr (sic) wa’l-muqābala*, in a Commentary by Mordechai Finzi. Hebrew Text, Translation, and Commentary with Special Reference to the Arabic Text, ed. Martin Levey. Madison etc, University of Wisconsin Press, 1966, p. 106; *Die Algebra, Kitāb al-Gabr wal-muqabala* des Aby Kamil Soga ibn Aslam, ed., trans. Sami Chalhoub. (Quellen und Studien über die Geschichte der Arabischen Mathematik, 7). Aleppo, University of Aleppo, Institute for the History of Arabic Science, 2004, p. 76–78, 197; “La version latine médiévale de l’Algèbre d’Abū Kāmil” (above, note 29), p. 370f.

*Disjunctim*, this allows him to apply *Elements* II.6 (unidentified once again). This time, one should have understood very little of the algebra that precedes in order to prefer the alternative. If we observe that the underlying alphabetic order is *a, b, g, d* (which it rarely is in this section) and that the problem belongs to a family which was widespread in the “supra-utilitarian” stratum of proto-abbacus arithmetic inside as well as outside algebra<sup>32</sup> we may speculate whether Fibonacci found it in a source written in Greek and presented it for the sake of completeness (which would correspond to a general practice of his).

All in all, we may conclude that “proportions” had nothing to do with algebra as Fibonacci encountered it. He writes, however, as if he thought they *should* have. He certainly has no persuasion that existing algebra should be illegitimate because it was Arabic, nor any consistent program to replace it with something legitimately belonging within the realm of Greek mathematics<sup>33</sup> – but his global view of mathematics, coloured by his understanding of the *Elements*, and his possession of a level that enabled him to merge different approaches in a not fully eclectic manner, still made him go part of the way taken eventually with greater resolve by some Renaissance writers on algebra.

This conclusion holds beyond Fibonacci’s treatment of algebra. As from algebra, the language of proportions was absent from (proto-)abbacus mathematics in general. However, Fibonacci, when writing his monumental book exactly about abbasus (mathematics), implied by using it occasionally that it *should* have a place – not replacing anything but bringing to perfection. In a non-mathematical *analogia* and with hindsight we notice that “vulgar” abbasus mathematics corresponds to *nature* in St Thomas’ famous dictum,<sup>34</sup> and proportions (and thereby “magisterial”, Greek-style mathematics) to *Grace*. Alternatively, since even St Thomas expresses himself in a veiled *analogia*,<sup>35</sup>

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<sup>32</sup> See Høyrup, *Jacopo da Firenze’s Tractatus Algorismi ...* (above, note 12), p. 131–133.

<sup>33</sup> That is, nothing like the ideal which shines through in Jordanus’s *De numeris datis* and to which Regiomontanus, Viète and others paid lip service through references to Diophantos and *analysis* – see Jens Høyrup, “A New Art in Ancient Clothes. Itineraries Chosen between Scholasticism and Baroque in Order to Make *Algebra* Appear Legitimate, and Their Impact on the Substance of the Discipline”. *Physis*, n.s., XXXV (1998), p. 11–50.

<sup>34</sup> “Grace does not abolish nature but brings it to perfection” (*Summa theologiae* I, question 1, article 1, ad 2). <http://www.corpusthomicum.org/> (accessed 29.1.2006).

<sup>35</sup> Matt. 5:17, “Think not that I am come to destroy the law, or the prophets: I am not come to destroy, but to fulfil”

abbacus mathematics corresponds to (Old Testament) Law and the Prophets, and magisterial mathematics to the Gospel – namely, once again with hindsight, to the gospel which late Renaissance mathematicians set out to implement.

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## Appendix: The problems of chapter 15 part 1

In most cases, the numbers entering the proportions are represented by letter-carrying line segments drawn in the margin – for brevity, since I do not follow the arguments in detail, I designate them  $P$ ,  $Q$ ,  $R$  and (when needed)  $S$ . For each section I indicate the initial transformations and the strategy used to complete the solution; I also indicate in superscript the page number for each page shift in the Boncompagni edition. The numbering of sections is mine; Fibonacci's headings are indicated as ---*heading*---, divisions ----- to simple paragraph divisions in the Boncompagni edition.

(1)–(3) deal with three numbers in continued proportion,  $P:Q:R$ , of which one and the sum of the other two are given. The naming of segments presupposes the alphabetic order  $a$ ,  $b$ ,  $c$ , ...:

---*Incipit pars prima*---

- (1)<sup>(387)</sup>  $P+Q = 10$ ,  $R = 9$ .  $\frac{P+Q}{Q} : \frac{Q+R}{R}$ , whence *Elements* II.6 can be applied to  $Q \cdot (Q+9) = 90$ .
- (2)  $P = 4$ ,  $Q+R = 15$ . Analogous.
- (3)  $Q = 6$ ,  $P+R = 13$ . The product rule gives  $P \cdot R = 36$ , which is transformed so as to permit use of *Elements* II.6 (direct use of II.5 seems obvious).

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(4)–(38) still treat of three numbers, but now differences between the numbers are among the given magnitudes. Now the alphabetic order underlying naming is  $a$ ,  $b$ ,  $g$ ,  $d$ , ...:

- (4)<sup>(388)</sup>  $P:Q:R$ ,  $Q-P = 2$ ,  $R = 9$ . *Disjunctim*  $\frac{R}{Q} : \frac{R-Q}{Q-P}$ . Solved by means of *Elements* II.5.
- (5)  $P:Q:R$ ,  $R-P = 5$ ,  $Q = 6$ . The product rule gives  $P \cdot R = 36$ , which allows use of *Elements* II.6.
- (6) A passage explaining that  $\frac{a}{b} : \frac{c}{d}$  entails that the squares of the numbers are also in proportion – a proportion which can then be transformed *conjunctim*, *e converso* etc. Further, that the same holds for the cubes. This is an aside, no consequence of what precedes nor a preparation for what follows immediately; when it is eventually used in (50) there is no backward reference. Fibonacci, though knowing his Euclid, is not particularly interested in corollaries or lemmas.

---*Modus alius proportionis inter tres numeros*---

- (7)<sup>(389)</sup>  $\frac{R-Q}{Q-P} : \frac{R}{P}$ ,  $Q$  unknown. This means that  $R-P$  is split into two parts having the ratio  $R:P$ , which is solved as a partnership problem.
- (8) Same proportion,  $R$  unknown. *Permutatim*  $\frac{R}{R-Q} : \frac{P}{Q-P}$ , a first-degree

problem.

(9) Same proportion,  $P$  unknown, solved similarly.

---*Modus alius proportionis inter tres numeros*---

(10)  $\frac{Q-P}{R-Q} : \frac{R}{P}$ ,  $Q$  unknown.  $\frac{(Q-P)+(R-Q)}{R-Q} : \frac{R+P}{P}$ , a first-degree problem.

(11)<sup>(390)</sup> Same proportion,  $R$  unknown. Product rule, and *Elements* II.6.

(12) Same proportion,  $P$  unknown. Analogously.

---*Modus alius proportionis in tribus numeris*---

(13)  $\frac{R}{P} : \frac{(R-Q)+(Q-P)}{R-Q}$ ,  $Q$  unknown. Since  $(R-Q)+(Q-P) = R-P$ , this is as simple first-degree problem.

(14) Same proportion,  $R$  unknown.  $\frac{R-P}{P} : \frac{Q-P}{R-Q}$ . From the product rule follows that the product of  $R-P$  and  $R-Q$  as well as their difference are known, which allows the application of *Elements* II.6.

(15)<sup>(391)</sup> Same proportion,  $P$  unknown. Product rule and *Elements* II.5.

---*Modus alius proportionis*---

(16)  $\frac{R}{P} : \frac{(R-Q)+(Q-P)}{Q-P}$ ,  $Q$  unknown.  $\frac{R}{P} : \frac{R-P}{Q-P}$ , a linear problem.

(17) Same proportion,  $R$  unknown.<sup>36</sup>  $\frac{R-P}{P} : \frac{(R-P)-(Q-P)}{Q-P}$ , whence *permutatim*  $\frac{R-P}{R-Q} : \frac{P}{Q-P}$ , a linear problem.

(18) Same proportion,  $P$  unknown. *Eversim* (although Fibonacci writes “you permute”)  $\frac{R}{R-P} : \frac{R-P}{R-Q}$ . The product rule gives  $R-P$ , whence  $P$ .

---*Incipit differentia tertia in proportione trium numerorum*---

(19) No question but the observation that if  $\frac{R}{Q} : \frac{R-Q}{Q-P}$ , then  $P$ ,  $Q$  and  $R$  are in continued proportion – namely because  $Q$  must be the same part of  $R$  as  $P$  of  $Q$ .

(20)<sup>(392)</sup>  $\frac{R}{Q} : \frac{Q-P}{R-Q}$ ,  $Q$  unknown.  $\frac{Q}{R} : \frac{R-Q}{Q-P}$ , whence  $\frac{Q+R}{R} : \frac{R-P}{Q-P}$ . The product rule and an addition allows the use of *Elements* II.6 (direct use of II.5 seems easier).

(21) Same proportion,  $R$  unknown. The product rule and *Elements* II.6 give  $R$ .

(22) Same proportion,  $P$  unknown. The product rule gives  $Q-P$ .

---*Modus proportionis in tribus numeris*---

(23)  $\frac{R}{Q} : \frac{(R-Q)+(Q-P)}{R-Q}$ ,  $Q$  unknown. *Permutatim* and *conjunctim*  $\frac{R+(R-P)}{R-P} : \frac{R}{R-Q}$ . From the product rule follows  $R-Q$ .

(24) Same proportion,  $R$  unknown. The argument is corrupt, claiming that the proportion can be transformed into  $\frac{Q}{R} : \frac{R}{Q-P}$ . The product rule and *Elements* II.5 would have led directly to a correct solution.

(25)<sup>(393)</sup> Same proportion,  $P$  unknown.  $R-P$  follows from the product rule.

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<sup>36</sup> By error, the text has *minor numerus .a.d.*, but the calculation proceeds from the premise *major numerus .a.b.*, corresponding to  $R$ .

---*Modus alius proportionibus in tribus numeris*---

- (26)  $\frac{R}{Q} : \frac{R-P}{Q-P}$ ,  $Q$  unknown. *Disjunctim*  $\frac{R-Q}{Q} : \frac{R-Q}{Q-P}$ . Since  $P$  is a number (i.e., not 0),  $R$  must equal  $Q$ . Alternatively, the proportion is transformed *permutatim* into  $\frac{R}{R-P} : \frac{Q}{Q-P}$ , and  $R$  is posited to be 8,  $P$  to be 2, from which is derived that  $Q$  must equal  $R$ . Finally it is observed that even with this transformation the numerical position is superfluous.

---*Modus alius proportionis in tribus numeris*---

- (27)  $\frac{Q}{P} : \frac{R-Q}{Q-P}$ ,  $Q$  unknown.<sup>37</sup> Instead of transforming *ex aequa*  $\frac{Q}{P} : \frac{Q+(R-Q)}{P+(Q-P)}$ , i.e.,  $\frac{Q}{P} : \frac{R}{Q}$ , Fibonacci prefers to combine transformations *permutatim* ( $\frac{R-Q}{Q} : \frac{Q-P}{P}$ ) and *conjunctim* ( $\frac{Q}{P} : \frac{R}{Q}$ , that is, producing the same outcome). From the product rule,  $Q$  is found as  $\sqrt{(PR)}$ .

- (28) Same proportion,  $R$  unknown. Fibonacci uses the transformed proportion from (27) to find  $R$  as  $Q^2/P$ .

- (29) Same proportion. It is pointed out that if  $Q$  is known (the example being  $Q = 12$ ), then either of the others can be chosen freely, the third number following (via  $\frac{Q}{P} : \frac{R}{Q}$ ) from division.

---*Modus alius proportionis in tribus numeris*---

- (30)<sup>(394)</sup>  $\frac{Q}{P} : \frac{R-P}{Q-P}$ ,  $Q$  unknown. The product rule allows application of *Elements* II.6.

- (31) Same proportion,  $R$  unknown,  $R-P = (Q \cdot [Q-P]) / P$ .

- (32) Same proportion,  $P$  unknown. *Eversim*  $\frac{Q}{Q-P} : \frac{R-P}{R-Q}$ . The product rule allows application of *Elements* II.6.

- (33) It is then asserted that if one of the numbers is known in this proportion, the others can be found. What is actually shown (and obviously meant) is that if one is known, another one can be chosen *ad libitum*, and a third determined so as to fit.

---*Modus alius proportionis in tribus numeris*---

- (34)  $\frac{Q}{P} : \frac{Q-P}{R-Q}$ ,  $Q$  unknown. *Conjunctim*  $\frac{Q+P}{P} : \frac{R-P}{R-Q}$ , which (via a trick necessitated by the line representation) allows application of *Elements* II.5.

- (35) Same proportion,  $R$  unknown.  $R-Q = \frac{P \cdot (Q-P)}{Q}$ .

- (36)<sup>(395)</sup> Same proportion,  $P$  unknown. The product rule allows application of *Elements* II.5.

- (37)  $\frac{Q}{P} : \frac{R-P}{R-Q}$ . Since *eversim*  $\frac{Q}{Q-P} : \frac{R-P}{(R-P)-(R-Q)}$ , i.e.,  $\frac{Q}{Q-P} : \frac{R-P}{Q-P}$ , this is only possible if  $Q = R-P$  – or, as Fibonacci prefers,  $P = R-Q$ . From this, any one of the numbers can be found if the other two are known.

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<sup>37</sup> The text says *ingnotus primus numerus .a.g.*, but *ag* is actually the second, that is,  $Q$ .



---*Modus ultimus proportionis in tribus numeris*---

(38) Same proportion,  $P+Q+R$  given. For three numbers  $p$ ,  $q$  and  $r$  fulfilling the condition, multiply each of them by  $\frac{P+Q+R}{p+q+r}$  (a scaling trick we have already encountered in Chapter 12, Part 2).

(39)–(50) consider four numbers in proportion,  $\frac{P}{Q} : \frac{R}{S}$ . The underlying alphabetic order is still  $a, b, g, d, \dots$

---*Incipit de proportione quattuor numerorum*---

(39) From  $\frac{P}{Q} : \frac{R}{S}$  follows  $\frac{Q}{P} : \frac{S}{R}$  and  $\frac{R}{P} : \frac{S}{Q}$ . From the product rule  $PS = QR$ , any one of the numbers can be found from the others.

(40)  $P+Q$ ,  $R$  and  $S$  known.  $\frac{P+Q}{Q} : \frac{R+S}{S}$ , whence  $Q$ .

(41)  $R+S$ ,  $P$  and  $Q$  are known. Similarly

(42)  $P+R$ ,  $Q$  and  $S$  known.  $\frac{Q+S}{S} : \frac{P+R}{R}$ , whence  $R$ .

(43)  $Q+S$ ,  $P$  and  $R$  are known. Similarly

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(44)<sup>(396)</sup>  $Q+R$ ,  $P$  and  $S$  known. The product rule allows application of *Elements* II.5.

(45) Similarly if  $P+S$ ,  $Q$  and  $R$  are known. Illustrated by an example involving *rotuli* (a weight unit) and *bizantii* and their sum.

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(46)  $P-Q$ ,  $R$  and  $S$  known.  $\frac{P-Q}{Q} : \frac{R-S}{R}$ , whence  $Q$ .

(47)  $R-S$ ,  $P$  and  $Q$  known. Similarly  $\frac{P-Q}{P-Q} : \frac{R-S}{R}$ , whence  $R$ .

(48)  $P-R$ ,  $Q$  and  $S$  known.  $\frac{P}{R} : \frac{Q}{S}$ ,  $\frac{P-R}{R} : \frac{Q-S}{S}$ , whence  $R$ .

(49)  $P-S$ ,  $Q$  and  $R$  known. The product rule allows application of *Elements* II.6.

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(50)<sup>(397)</sup>  $P^2+Q^2$ ,  $R$  and  $S$  known. Jumps directly (in a numerical example) to the proportion  $\frac{P^2+Q^2}{Q^2} : \frac{R^2+S^2}{S^2}$ .