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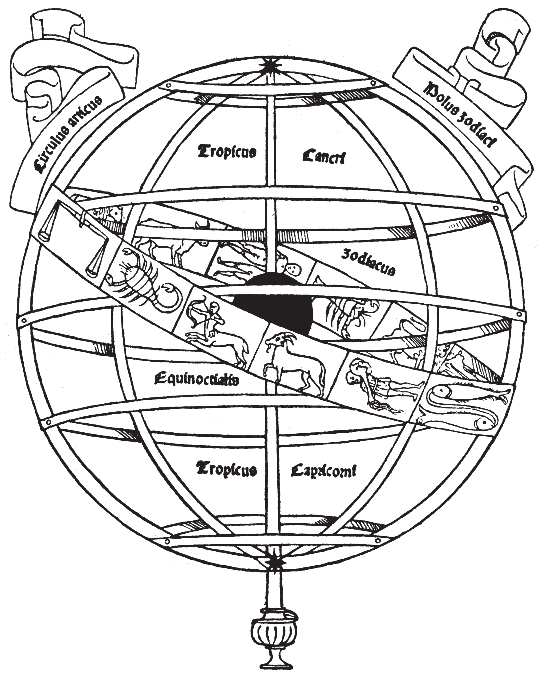

LEO S. OLSCHKI EDITORE
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"PROPORTIONS" IN AND AROUND THE ITALIAN ABBACUS TRADITION

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ABSTRACT — The language and notion of "proportions," in the senses ascribed to the term during the epoch, are traced both in ordinary abbacus books and in those extensive works that were written in the vicinity of abbacus culture by authors with erudite or humanist ambitions, such as Fibonacci's *Liber abbaci*, Benedetto da Firenze's *Trattato d'aritmetica*, and Pacioli's *Summa*. The very language turns out to have been initially absent from general abbacus culture as reflected in the ordinary books, but it slowly and modestly crept in. The authors of the extensive works addressed the topic, as indeed they had to if they wanted to relate to university and humanist mathematics; but even in their case it generally remained isolated and did not penetrate their presentation of abbacus mathematics broadly.

To Hong

1. PRELIMINARIES

Before taking up the substance of my topic, I shall make three preliminary remarks: one on terminology, one on notation, and one on delimitation. First, terminology. As other texts from the epoch, those I am going to consider speak of a *ratio/λὀγος* (understood as a relation between two integers, not as a single number) as *proportio/proportione*. Some of

* A first version of this paper was presented as a contribution to the meeting "Proportions: Arts – Architecture – Musique – Mathématiques – Sciences," at the Centre d'Études Supérieures de la Renaissance, Tours, 30 June - 4 July 2008. Being much too long for the proceedings, these are to contain instead a revised version of the part dealing with the *Liber abbaci*. That part is therefore reduced in the present paper.

them use *proportionalità* where we would speak of a proportion, and the Greek mathematicians of ἀναλογὶα, that is, an affirmation that two ratios are "the same" or "similar"; others, however, use the term *proportio/proportione* even here, or speak of the numbers involved as *proportionales*. In the case of numbers being in continued proportion (ἑξῆς ἀνάλογον), moreover, our texts speak of *numeri continui proportionales*[1]/*numeros in continua proportione*, etc. An attempt to enforce a modern terminology would either divide the field in a way that does not correspond to the thought of the authors of the period, or it would force us to speak of "numbers in continued ratio," which certainly makes sense, but is *not* modern terminology. It would also impose the modern conceptual confusion, more misleading than the medieval one, which uses "ratio" both in the historically proper sense, about the relation between *two* numbers, and when indicating their quotient. I shall therefore translate *proportio/proportione* as "proportion," etc. – while still speaking in modern ways of ratio and proportion outside direct and indirect quotations when the relation between two numbers, respectively the "similitude" between two such relations is meant; the single-number "ratio" I shall refer to as the "quotient."

Second, notation. When designating explicitly a proportion, our texts mostly say that "the first number is to the second, as the third to the fourth,"[2] or use some equivalent expression. For typographical convenience, I shall use instead the notation $\frac{a}{b} : \frac{c}{d}$, which should be read as representing the frame:

a	c
b	d

corresponding to what is found regularly in the margin in the *Liber abbaci*[3] and (according to Rodet)[4] consistently in a pre-1400 manuscript of Ibn Ezra's *Sefer ha-mispa* – whence probably more widespread.[5] The two

[1] Thus *Liber abbaci*, ed. BONCOMPAGNI, 1857, pp. 171, 399.

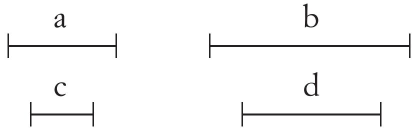
[2] Thus *Liber abbaci*, ed. BONCOMPAGNI, 1857, p. 170; as everywhere in the following, translations with no identified translator are mine.

[3] E.g., ed. BONCOMPAGNI, 1857, p. 170.

[4] As cited in SILBERBERG, 1895, p. 109.

[5] It is *not*, however, in the *Liber mahamaleth* (Paris, Bibliothèque Nationale, ms latin 7377A), even though this work makes use of the rectangular frame for other purposes.

notations, as well as the line diagram used both in the *Liber abbaci*[6] and by Campanus:[7]



are equally fit to serve the visualization and automation of the various operations that can be performed on the proportion:[8]

e contrario: $$\frac{b}{a} : \frac{d}{c}$$

permutata: $$\frac{a}{c} : \frac{b}{d}$$

conjuncta: $$\frac{a+b}{b} : \frac{c+d}{d}$$

disjuncta: $$\frac{a-b}{b} : \frac{c-d}{d}$$

conversa: $$\frac{a}{a+b} : \frac{c}{c+d}$$

eversa: $$\frac{a}{a-b} : \frac{c}{c-d}$$

aequa: $$\frac{a}{b} : \frac{a+c}{b+d}$$

and also of the equality of the products $a \cdot d = b \cdot c$ (to which I shall refer in the following as the "product rule"). The typographically convenient notation thus involves no serious anachronism – $a : b :: c : d$, while fitting the phrase "the first is to the second, as the third to the fourth," corresponds less well to the diagrams on which the medieval authors based their operational thinking. In order to distinguish, I shall write fractions (including "ratios" understood as quotients) as ${}^{a}/_{b}$. Ratios (not

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6 Ed. Boncompagni, 1857, p. 395 and *passim.*

7 Ed. Busard, 2005, p. 161 and *passim.*

8 This way to present them is taken from the Campanus *Elements* (ed. Busard, 2005, p. 171*f*).

understood as quotients, and not constituents of a proportion) I shall denote $a : b$, and numbers in continued proportion will stand as $a : b : c : \ldots$.

Third, delimitation. Any applied arithmetic that goes beyond the simplest accounting runs into problems of proportionality – say, of the type "for *a* [coins], *b* [units], for *c* [coins], how much?" In Near Eastern and Greek Antiquity, this would normally be solved in an intuitively transparent way: for *a* [coins], *b* [units], for 1 [coin] therefore $^{b}/_{a}$ [units], and for *c* therefore $c \cdot {}^{b}/_{a}$ [units]. Some Arabic reckoners[9] would prefer the argument "by *nisbah* [ratio]," for *a* [coins], *b* [units], for *c* therefore $^{c}/_{a}$ as much, that is $({}^{c}/_{a}) \cdot b$ [units]. From India, however, probably via the trade routes and possibly with ultimate roots in China, Arabic merchants and after them theoretically inclined Arabic mathematicians from al-Khwārizmī onward adopted the *rule of three*, stating that *c* must yield $(b \cdot c)/a$.[10] Indian practical reckoners appear to have used a formulation in the style "multiply the thing [whose counterpart] you want to know by that which is not similar [to it in kind] and divide by that which is similar." This is not the main formulation of the erudite Sanskrit writers (Āryabhata, Brahmagupta, Mahāvirā, etc.), but the formulations of the latter two betray that they know it. Even in the Arabic world, it appears to have been the formulation of merchants. The theoretically trained Arabic mathematicians soon saw that the whole matter can be based on proportion theory as found in *Elements* VII – if only we forget about the numbers being concrete and indeed being of two different kinds (for instance, dinars and cloth), and not abstract. Nonetheless, many of the Arabic mathematicians betray familiarity with the traditional formulation, in spite of its conflict with the Euclidean approach (which requires ratios to be between quantities of the same kind, e.g., abstract numbers).[11]

In the European (that is, Italian and Ibero-Provençal) abbacus environment, the rule also arrived in a "non-Euclidean" interpretation: in Italy and perhaps in Provence in the traditional ("non-similar/similar") formulation, in Spain (as we shall see) apparently in a different shape; even in the Christian world, however, theoretically trained writers interacting with the abbacus environment, from Fibonacci to Chuquet, made use of the Euclidean formulation. This, however, I shall not discuss

[9] Thus Ibn Thabīt (ed. and trans. REBSTOCK, 1993, pp. 43-45), and al-Karajī (ed. and trans. HOCHHEIM, 1878, II, p. 17).

[10] This, and the remains of the paragraph, builds on HØYRUP, 2007b, pp. 1-8.

[11] Of course, the Euclidean approach is saved if only we use the equivalent proportion $\frac{a}{c} : \frac{b}{d}$. However, the sources never bother to perform this transformation.

in any depth, not because it is not interesting but because it is a separate topic, to be treated at best together with other aspects of the approach to the rule of three.

2. Fibonacci's *Liber abbaci*

I have argued on other occasions[12] that Fibonacci is not the founding father of abbacus culture but rather an early (towering) exponent of a culture that already flourished in his time, if not in Italy (which seems unlikely), then in Provence, Catalonia, and the Maghreb and al-Andalus; perhaps even in Egypt, Syria, and Byzantium; and which was connected to a culture of commercial arithmetic ranging at least as far as Iran and India. On the present occasion I shall refer to this as the "proto-abbacus culture."

However, the *Liber abbaci* is not just an early abbacus book. Fibonacci writes *in a mathematically educated perspective* about the kind of mathematics thriving in the environment in question; but his scope is much larger, encompassing not only what he encountered on business travels to Egypt, Syria, Constantinople, Sicily, and Provence,[13] but also topics that almost certainly fell outside the horizon of the proto-abbacus culture.[14] At least part of his treatment of proportions falls in that category (but see the beginning of Section 3 for a sharpening of this statement).

The first time numbers in proportion turn up in the *Liber abbaci* is in the explanation of the algorithm for multiplying multi-digit numbers.[15] Combining the product rule, for which he gives an unspecific reference to Euclid, with the observation that the "degrees" or decimal levels form an infinite continued proportion, Fibonacci concludes that multiplication of the first degree by the third gives as much as that of the second degree by itself, while the second by the third gives as much as the first by the fourth, etc.

The argument could be original; I do not remember having seen it in any earlier source, not even in hints.[16] Nice though it is, it also seems to have been a historical dead end, not to be repeated by any later writer.

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12 See, for example, Høyrup, 2005.

13 Ed. Boncompagni, 1857, p. 1.

14 Bartolozzi & Franci (1990, p. 5), though regarding the *Liber abbaci* as the archetype for abbacus books, align it more adequately with fifteenth-century encyclopediae such as Benedetto da Firenze's *Praticha d'arismetricha* and the anonymous MS Florence, Palatino 573 – on both of which below.

15 Ed. Boncompagni, 1857, p. 15.

16 It may have been inspired by analogous reasoning about the sequence of algebraic powers. The

A next passing reference[17] to (four) numbers in proportion and to the product rule turns up in the explanation of the decomposition of a fraction – once more with unspecific reference to Euclid. This is followed closely by the presentation of the rule of three in simple and composite shape, which I shall not treat in depth.[18] I shall merely mention that:

– Fibonacci does *not* use what was to become the standard formulation of the abbacus school (the one that refers to the non-similar and the similar): the formulations are likely to be his own;[19]
– Fibonacci makes use of the rectangular frame mentioned above, leaving empty the position for the unknown number and indicating the cross-multiplication by a diagonal;
– the treatment of the non-composite rule is argued from the product rule "which has been proved in the arithmetical [books of the *Elements*] and in the geometry";
– the composite rule (used in barter problems) is presented with a reference to *figura cata, scilicet sectoris* (Menelaos' theorem) "which Ptolemy teaches in the *Almagest*."

Whereas barter problems employ the rule of three "sequentially," partnership problems use it "in parallel"; in this case,[20] however, Fibonacci speaks neither of "proportions" nor of proportionality – nor indeed does he refer to the rule of three itself; but since in general he has no name for that rule, this is not astonishing. However, in connection with a problem about the alloying of three monies,[21] the first and the second in ratio 2 : 3, the second and the third in ratio 4 : 5, he speaks of "proportional alloying" and teaches how to harmonize these as easily composable ratios by means of multiplication. The idea of "proportional alloying" turns up repeatedly in the following pages. Proper interest in our topic returns only in Chapter 12, Part 2.[22]

This chapter begins by explaining equal, major, and minor ratios, and gives the examples 3 : 3, 8 : 4, 9 : 3, 16 : 5, 4 : 8, 3 : 9, and 5 : 16 – providing

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parallel between the powers of the algebraic *thing* and the powers of ten is pointed out by al-Karajī (Woepcke, 1853, p. 48) and may have been common lore among Arabic writers 200 years later.

17 Cf. ed. Boncompagni, 1857, p. 82.

18 But see Bartolozzi & Franci, 1990, pp. 5-7.

19 Ed. Boncompagni, 1857, p. 83*f.*

20 Cf. *ibid.*, pp. 114*f*, 135-143.

21 *Ibid.*, p. 149*f.*

22 *Ibid.*, pp. 69-173.

them with names that are not in the Boethian tradition but come close to the "denomination" (though not using this word). For instance, 16 : 5 is a "triple proportion and a fifth." It goes on with the problem of finding the number to which 6 has the same "proportion" as 3 to 5, giving first the numerical solution $(5 \cdot 6)/3$ and saying then that this question is stated "in our vernacular" (*ex usu nostri vulgaris*)[23] in the phrase "if 3 were 5, what would then 6 be?" Similarly, it asks for the number to which 11 has the same ratio as 5 to 9, and gives it the vernacular formulation "if 5 were 9, what would 11 be?"

This formulation is remarkable.[24] Only one Italian abbacus treatise I know of identifies the rule of three by means of the same phrase, namely the *Columbia Algorism*[25] – also untypical in other respects, almost certainly dated no later than 1290[26] and thereby probably the earliest extant abbacus text (though known only from a fourteenth-century copy). Counterfactual questions – and even "counterfactual calculations" in the style "if 7 were the half of 12, what would the half of 10 be?"[27] – are certainly not absent from the Italian abbacus record, but they invariably turn up long after the rule of three is explained, or as secondary examples (the primary examples confronting either different currencies, or goods and their monetary value). In all Ibero-Provençal treatises from before 1500 that I have inspected,[28] on the other hand, the rule of three is

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23 A complete survey of the references to *modus vulgaris* and its cognates in the *Liber abbaci* shows that the genuine meaning is not the generic spoken vernacular, but with one exception the simple ways of practical reckoners (the exception, p. 111, is the information that an alloy of silver and tin is called "false silver *vulgariter*"). Simple, stepwise calculation is meant in four places (pp. 115, 127, 204, 364). In the last place, the *modus vulgaris* is confronted explicitly with how one procedes *magistraliter*.

24 Cf. full documentation in Høyrup, 2007a, pp. 64-67.

25 Ed. Vogel, 1977.

26 Høyrup, 2007, p. 31 n. 70.

27 Ed. Boncompagni, 1857, p. 10.

28 In chronological order:
– the Castilian *Libro de arismética que es dicho alguarismo* (ed. Caunedo del Potro & Córdoba de la Llave, 2000);
– the "Pamiers Algorism" (partial ed. Sesiano, 1984);
– the mid-fifteenth-century Franco-Provençal *Traicté de la praticque d'algorisme* (I used the transcription in Stéphane Lamassé's unpublished dissertation, for access to which I am grateful).
– Barthélemy de Romans' Provençal *Compendy de la praticque des nombres* (ed. Spiesser, 2003, p. 264);
– Francesc Santcliment's *Summa de l'art d'aritmètica* (ed. Malet, 1998);
– Francés Pellos's *Compendion de l'abaco* (ed. Lafont & Tournerie, 1967, pp. 132-134).
I also looked at Chuquet's *Triparty en la science des nombres* (ed. Marre, 1880), not strictly Provençal but in the Provençal tradition.

introduced first by counterfactual or abstract-number questions: "if 3 were 4, what would 5 be?" or "if $4^1/_2$ are worth $7^2/_3$, what are $13^3/_4$ worth?" All the Provençal specimens also know the formulation in terms of the non-similar and the similar, and so does Santcliment's Catalan *Summa*.[29] Besides that, however, Santcliment informs us that this is spoken of "in our vernacular" by the phrase "if so much is worth so much, how much will so much be worth?" (*si tant val tant: que valra tant?*). The same phrase (*sy tanto faze tanto, ¿qué sería tanto?*) is also used in the Castilian *Libro de arismética que es dicho alguarismo*.[30] Wherever Fibonacci encountered the vernacular tradition he refers to, it left no conspicuous traces in Italy but many in the Ibero-Provençal orbit, most clearly in its Iberian section.

Next, Fibonacci presents the counterfactual calculation that was just quoted ("if 7 were the half of 12, what would the half of 10 be?"), and another simple counterfactual question. He goes on with procedures for finding four and six integers in proportion if the first two of them are given; shows how to divide 10 into four unequal parts in proportion – namely by scaling an arbitrary proportion $\frac{a}{b}:\frac{c}{d}$ by the factor $^{10}/_{(a+b+c+d)}$; explains how to construct a continued proportion with an arbitrary number of terms (explaining again the product rules); and, finally, demonstrates how to find two or three numbers so that $^1/_p\, n_1 = {}^1/_q\, n_2$ (and, in the case of three numbers, $^1/_r\, n_2 = {}^1/_s\, n_3$) – in a different formulation, not used by Fibonacci but common in later Italian abbacus algebra, $\frac{n_1}{n_2}:\frac{p}{q}$ (and $\frac{n_2}{n_3}:\frac{r}{s}$).

On the whole, what Fibonacci does in this chapter is thus to relate to the notion of "proportions" the procedures and problem types belonging to the "vernacular" proto-abbacus tradition(s) he had encountered. The theoretical field itself is not explored in any way.

Theoretical exploration of a kind comes in Chapter 15, Part 1,[31] which claims to deal with "the proportions of three and four quantities, to which the solution of many questions belonging to geometry are reduced."[32] Actually it deals with problems about *numbers* in proportion. These numbers are spoken of as "the first/second/third/fourth number" (or, when the numbers are three numbers, "minor/middle/major"). In most cases, they are represented by letter-carrying line segments drawn in the

[29] Ed. MALET, 1998, p. 163.

[30] Ed. CAUNEDO DEL POTRO & CÓRDOBA DE LA LLAVE, 2000, p. 147.

[31] Ed. BONCOMPAGNI, 1857, pp. 387-397.

[32] *Ibid.*, p. 387.

margin; for brevity we may designate them P, Q, R and (when needed) S. At first, proportions involving three numbers are presented; afterwards, (far fewer) questions involving four numbers are dealt with. By means of conjunction, disjunction, permutation, etc., the given proportion is transformed in such a way that the numbers can be found from the product rules by means of addition or subtraction or, more often, *Elements* II.5-6 (II.6 being sometimes preferred even in cases where II.5 would seem the obvious choice). Strikingly, Fibonacci never refers to Euclid here, which he is otherwise fond of doing.[33]

We may divide into 50 sections, of which some 5 contain theorem-like observations (the delimitation is not quite sharp), and the remainder solve or show the solvability of problems.

At first, in (1)-(3) come questions about three numbers in continued proportion, $P : Q : R$. One of the numbers is given together with the sum of the other two. The naming of segments presupposes the alphabetic order *a*, *b*, *c*, ...

(4)-(38) still treat of three numbers, but now differences between the numbers are among the given magnitudes. The alphabetic order underlying naming changes to *a*, *b*, *g*, *d*, ...

In (4)-(5), the three numbers are still in continued proportion, but now one of the numbers and the difference between the two others are given.

(7)-(38) are more astonishing. They fall into groups of three, divided by separate headings by Fibonacci. To each heading corresponds one of the non-arithmetical "means" of ancient Greek mathematics[34] – geometric, harmonic, their subcontraries, etc. – and it is shown how each mean can be found from the two extremes, or one of the extremes from the other extreme and the middle. Fibonacci deals with all the means defined by Nicomachos[35] (as followed by Boethius), but also with a mean defined by Pappos[36] but left out by Nicomachos (see the scheme on page 64). However, Fibonacci does not speak of means, even though he is likely to know about them from Boethius; his order is different from those of Nicomachos and Pappos; and he does not observe that his (27)-(29) represent the geometric mean, which he has already dealt with in (4)-(5). This, together with the change of underlying alphabetic order, suggests that he has not constructed this sequence on his own under inspiration

33 Folkerts, 2006, p. ix.

34 Heath, 1921, II, pp. 85-88.

35 Ed. Hoche, 1866, pp. 124-144.

36 Hultsch, 1876, I, pp. 70-73, 84-87.

	Pappos	Nicomachos	*Liber abbaci*
$\frac{R-Q}{Q-P}:\frac{R}{R}$ (arithmet.)	P1	N1	
$\frac{R-Q}{Q-P}:\frac{R}{Q}$ or $\frac{R-Q}{Q-P}:\frac{Q}{P}$	P2	N2	27-29
$\frac{R-Q}{Q-P}:\frac{R}{P}$	P3	N3	7-9
$\frac{R-Q}{Q-P}:\frac{P}{Q}$	P4	N4 (but inverted)	10-12 (inverted)
$\frac{R-Q}{Q-P}:\frac{Q}{P}$	P5	N5 (but inverted)	34-36 (inverted)
$\frac{R-Q}{Q-P}:\frac{Q}{R}$	P6	N6 (but inverted)	20-22 (inverted)
$\frac{R-P}{Q-P}:\frac{R}{P}$	absent	N7	16-18
$\frac{R-P}{R-Q}:\frac{R}{P}$	P9	N8	13-15
$\frac{R-P}{Q-P}:\frac{Q}{P}$	P10	N9	30-32
$\frac{R-P}{R-Q}:\frac{Q}{P}$	P7	N10	37-38
$\frac{R-P}{R-Q}:\frac{R}{Q}$	P8	absent	23-25
$\frac{R}{Q}:\frac{R-P}{Q-P}$	absent	absent	26

Means dealt with by Pappos and Nikomachos and in the *Liber abbaci*.

from the ancients, but has borrowed from an Arabic or Greek treatise on the matter, which – in the interest of completeness – had also added the case (26) even though it defines no genuine mean (as pointed out by Fibonacci, the condition $\frac{R}{Q}:\frac{R-P}{Q-P}$ implies that $R=Q$).

(39)-(50) consider four numbers in proportion, $\frac{P}{Q}:\frac{R}{S}$. The underlying alphabetic order is still *a*, *b*, *g*, *d*, ... At first, the *e contrario* and *permutata* transformations are set out, and it is explained how any one of the numbers can be found from the other three via the product rule. Then follow problems where two of the numbers are given together with the sum of (40)-(45), respectively the difference between (46)-(49) the two others; finally, in (50), two numbers and the sum of the squares of the remaining two are given.

In (39)-(50) as in (4)-(5), but not in (7)-(38), some segments are labelled by a single letter, and the letter *c* is used during the manipulations. We may therefore assume that these sequences come from Fibonacci's own pen, or (less likely, I would say) from a different source than the one for (7)-(38).

Chapter 15, Part 2 is claimed to deal with "questions concerning geometry." Actually, a number of its problems have nothing to do with geometry, apart from having solutions based on line diagrams; several of these – all dealing with composite gain – involve proportions.

The first of them[37] is very simple. Somebody goes to one place of trade with £ 100 and earns, and afterwards earns proportionally in another place, and then has a total of £ 200. A continued proportion shows the possession after the first travel to be $\sqrt{(100\cdot 200)} \approx$ £ 141, s. 8, d. $5^{1}/_{8}$.

The next case[38] is somewhat more tricky. The initial capital is still £ 100, but after the first travel a partner invests £ 100 in the enterprise, and after the second travel the total amounts to £ 299. This gives the proportion (represented by lines) $\frac{100}{Q}:\frac{Q+100}{299}$. The product rule and *Elements* II.6 (still unidentified) lead to the solution $Q =$ £ 130. Interchange of left and right would reduce this to case (46) above, but Fibonacci does not establish the link.

Then follows[39] an example with three travels (£ 100 increasing to £ 200) and no extra investments, which leads to a continued proportion with four terms and thus, with reference to Euclid (namely, *Elements* VII.12), a solution expressible in cube roots. A digression follows discussing numbers

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37 Ed. Boncompagni, 1857, p. 399.

38 *Ibid.*

39 *Ibid.*, p. 399*f.*

allowing an exact solution (24 and 81), and the notions of duplicate and triplicate proportion. Fibonacci goes on to the case of four travels, involving five numbers in continued proportion and a quadruplicate proportion; and to the concepts of quintuple and sextuple proportion.

A final problem about composite gain[40] deals with two travels with initial capital P, final total R, and intermediate possession $Q = £\ 80$, with $\frac{P}{R} : \frac{5^2}{9^2}$. Fibonacci calculates $5 \cdot 9 = 45$ and claims without explanation that $\frac{45}{80} : \frac{25}{P}$, $\frac{45}{80} : \frac{81}{R}$. The trick is of course that $\frac{25}{45} : \frac{45}{81}$, while $\frac{P}{80} : \frac{80}{R}$; a scaling with the factor $^{45}/_{80}$ conserves the ratio between the extreme terms and adjusts the value of the middle term. Finally, Fibonacci explains it to be an equivalent problem to find two numbers, p and q (namely, $p = \sqrt{P}$, $q = \sqrt{Q}$) such that $^1/_5 p = {}^1/_9 q$, $p \cdot q = 80$.[41] This is solved via a single false position, $p' = 5$, $q' = 9$, and subsequent scaling by the factor $\sqrt{^{80}/_{5.9}}$.

The notion of "proportion" or proportionality turns up in two further places in this "geometric" section. In none of them is meant anything profound. First, a rule is given[42] for producing "two integer roots whose squares together make the square of a number," that is, for finding Pythagorean triples (triangles are *not* spoken of). Second, in the last problem of the section,[43] three numbers (say, a, b, and c) are asked for, so that $^1/_2 a = {}^1/_3 b$, $^1/_4 b = {}^1/_5 c$, $abc = a + b + c$. This is solved by a single false position, $a' = 8$, $b' = 12$, $c' = 15$, with consecutive proportional scaling. Similarly to what he did in the last travel problem, Fibonacci goes on to discuss what to do when there are four, five, and six numbers, using once again the notions of double, triple, quadruple, and quintuple proportion.[44]

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40 *Ibid*., p. 401.

41 We recognize the structure $^1/_p n_1 = {}^1/_q n_2 =$, dealt with already in Chapter 12, Part 2 (see above, p. 62).

42 BONCOMPAGNI, 1857, p. 401.

43 *Ibid*., p. 405*f*.

44 Most remarkable in this problem is presumably the use of *tetragonus* in the sense of a numerical square: everywhere else in the work this is spoken of as *quadratus*, while *tetragonus* invariably refers to a geometric square (often, ed. BONCOMPAGNI, 1857, pp. 175*f*, 368, 408*f*, 421, 426*f*, 453) or cube (once, ed. BONCOMPAGNI, 1857, p. 403). We can presume that Fibonacci used a source written in Greek without bothering to adjust its style.

The third and final (and most famous) part of Chapter 15[45] deals with "certain problems according to the method of algebra and *almuchabala*, that is, of proportion and restoration."[46] This identification of *algebra* with "proportion," and *almuchabala* with "restoration," is almost certainly Fibonacci's own invention.

Fibonacci knows the term "restoration" from Gherardo of Cremona's translation of al-Khwārizmī (with which he was familiar),[47] and also uses it himself quite often concerning the cancellation of a subtractive term by addition to both sides of an equation[48] (alternatively he employs a mere "add"); but Gherardo will not have helped him discover that it translates *al-jabr*.[49] On the other hand, the term used by Gherardo to translate *al-muqābalah* and the corresponding verb *qabila* – that is, *oppositio/opponere* – only occurs thrice in Fibonacci's algebra chapter,[50] each time in the sense of confronting the two sides of an equation (in all probability the original function of the term, but not Gherardo's normal interpretation).[51]

This explains that there was space for Fibonacci's mistaken guess – he had two slots for only one technical operation. It does not explain why he used the other slot for "proportion," but at least this choice suggests that he saw proportions as an important tool in the field. Why?

One hypothesis can be rejected straightaway. It has nothing to do with the proportional reduction of all coefficients when an equation is normalized. For this, Fibonacci uses *redigere*, as quoted in note 48 *reintegrare*,[52] or he performs the operation without naming it; neither "proportion" nor "proportional" ever occurs in this context.[53]

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45 Ed. Boncompagni, 1857, pp. 406-459.

46 [...] *pars tertia de solutione quarumdam questionum secundum modum algebre et almuchabale, scilicet ad proportionem et restaurationem*.

47 See Miura, 1981, p. 60.

48 The "equation" as a mathematical *object* is of course *our* concept and thus strictly speaking an anachronism. Fibonacci has only the action of equating – the isolated appearance of *equatio* (ed. Boncompagni, p. 407) is to be understood as a corresponding verbal noun, *pace* Barnabas Hughes (2008, pp. xxix, 361), who is seduced by Boncompagni's mistaken punctuation (*reddigi ad equationem. Vnius* (sic) *census per diuisionem* [...] should be simply *reddigi ad equationem unius census per diuisionem* [...]).

49 That Fibonacci does not discover it on his own should downplay Fibonacci's Arabic skills, *pace* Barnabas Hughes (2008, p. xix).

50 Ed. Boncompagni, 1857, pp. 429, 436, 457.

51 There is one exception (ed. Hughes, 1986, p. 255).

52 Ed. Boncompagni, 1857, p. 420.

53 Barnabas Hughes suggests (2004, p. 324 n. 43) that Fibonacci understood "*proportio* as a

We may observe instead that Fibonacci inserts occasional bits of reasoning based on proportion theory within algebraic or other calculations, and occasionally solves problems by means of proportion theory instead of algebra.

A simple example of the first type is found in the solution of the problem, to divide 60 *denarii* first among a number of men and then among 2 more men, for which the share of each man decreases by $2^1/_2$ *denarii*. Al-Khwārizmī[54] solves an analogous problem via (implicit) subtraction of fractions containing algebraic expressions in the denominator; Abū Kāmil[55] makes use of the subtraction of areas within a geometric diagram; Fibonacci[56] replaces this "geometric arithmetic" by means of operations on a proportion.

A more advanced instance of the first type deals with the gains of a complex partnership: Somebody invests £ 12, and has a certain gain after 3 months. Then somebody else invests £ 11, and after another 12 months with gain at the same monthly rate, the total gain for the two is £ 9. This is expressed in line diagrams and treated *inter alia* by operations on proportions, which in the end allow the establishment of an algebraic equation.

A simple instance of the second type is an alternative solution to the problem of finding two numbers with a difference of 6 and a quotient of $^1/_3$. The primary solution goes via algebra: the smaller number is posited as a *thing*, the larger is thus a *thing* plus 6, etc. Alternatively, the larger is a segment *ab*; the smaller is the partial segment *ac*, whence $bc = 6$, $\frac{ab}{ac} : \frac{3}{1}$, and *disjunctim* $\frac{3}{ac} : \frac{2}{1}$, etc. For somebody as familiar with proportion techniques as Fibonacci, this may indeed have been as easy as the primary solution, and for those not yet familiar with algebra it may have been easier.

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kind of operation" because "the two verbs *proportionari* and *equari* [...] are synonymous" in the Latin translation of Abu Kamil's algebra. Hughes overlooks the fact that the verb *equari* is used as an editorial explanation by Jacques Sesiano (1993, p. 325). What is relevant is that the fourteenth-century translator uses the verb *proportionari* in the sense of "giving/having comparable size" a single time, in agreement with possible Italian usage (of *proporzionare*) of the late Middle Ages. It is not totally excluded – though quite improbable, given that there are no other traces of this meaning in Fibonacci's text – that Fibonacci did so too. Why should he coin a semantic neologism in the heading and never use it afterwards?

54 Ed. Hughes, 1986, p. 255; ed. Rashed, 2007, pp. 190-193.

55 Ed. Levey, 1966, p. 106; ed. Chalhoub, 2004, pp. 76-78, 197; ed. Sesiano, 1993, p. 370*f.*

56 Ed. Boncompagni, 1857, p. 413.

Another alternative,[57] this time to an algebraic method that is mentioned but not presented, asks for a number that, when $^1/_3$ of it and 6 are removed and the remainder multiplied by itself, yields twice the original number; in symbols:

$$(x - {}^1/_3x - 6)^2 = 2x.$$

In a line diagram, Fibonacci transforms this into a proportion that in symbols becomes:

$$\frac{\frac{2}{3}x}{x - \frac{1}{3}x - 6} = \frac{x - \frac{1}{3}x - 6}{3}.$$

Disjunctim, this allows him to apply *Elements* II.6 (unidentified once again). This time, only a reader who had understood nothing of the algebra that precedes would be likely to prefer the alternative. If we observe that the underlying alphabetic order is *a*, *b*, *g*, *d* (which it rarely is in this section) and that the problem belongs to a family that was widespread in the "supra-utilitarian" stratum of proto-abbacus arithmetic inside as well as outside algebra,[58] one may speculate whether Fibonacci found it in a source written in Greek and presented it for the sake of completeness (which would correspond to a general practice of his).

All in all, we may conclude that "proportions" had nothing to do with algebra as Fibonacci encountered it. He writes, however, as if he thought they *should* have. Nothing led him to entertain the idea that existing algebra should be illegitimate because it was Arabic, or that he had a consistent program to replace it with something more "magisterial," legitimately belonging within the realm of Greek;[59] but his global view of mathematics, coloured by his understanding of the *Elements*, and his possession of a level that enabled him to merge different approaches in a not fully eclectic manner, still conducted him part of the way taken eventually with greater resolve by some Renaissance writers on algebra.

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57 *Ibid.*, p. 423*f.*

58 See Høyrup, 2007a, pp. 131-133.

59 That is, nothing like the ideal that shines through in Jordanus's *De numeris datis* and to which Regiomontanus, Viète, and others paid lip service through references to Diophantos and *analysis*; see Høyrup, 1998.

3. Early Abbacus Books

Examination of early Italian abbacus books reveals that Fibonacci glued proportions not only onto algebra but also more generally onto the proto-abbacus tradition, from which they were equally absent.

The *Columbia Algorism* – almost certainly the earliest extant abbacus text (cf. above before note 26) – does not speak of "proportions" a single time, not even in the sense of ratio; the rule of three, as explained, is referred to through the counterfactual "vernacular" structure: for instance, "if 25 were 12, what would 12 be?"[60]

Slightly younger[61] is a *Livero de l'abbecho secondo la oppenione de maiestro Leonardo de la chasa degli figluogle Bonaçie da Pisa* ("Abbacus book according to the opinion of master Leonardo of Pisa from the house of the Fibonacci").[62] This treatise is a mixed compilation.[63] Almost half of it (if we count lines; well over half if we count problems) has nothing at all to do with the *Liber abbaci*; the remainder is borrowed very closely from it, but often demonstrably without understanding. Apart from the contents of a final chapter containing mixed recreational problems, everything independent belongs on the basic level, the level corresponding to what should be taught in an abbacus school. What comes from Fibonacci is sophisticated, advanced – roughly speaking, adornment serving to show off (a purpose also ministered to by the reference to the famous predecessor in the title).

In the part of the text that is not borrowed from Fibonacci, the notion of "proportion" does not occur in any sense. The rule of three is presented in terms of the similar and the non-similar. The part copied from Fibonacci does borrow a number of references to the notion, translated either *propositione* (sometimes *prepositione*) or *proportione*. *Propositione* also occurs as the translation of *petitio* or *propositio* (both referring to requests or propositions that somebody give part of his possessions to somebody else). The mix-up of *propositione* and *proportione* also turns up in other abbacus texts, facilitated probably by the possibility of abbreviating both in the same way, which might of course mislead a copyist who did not

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60 Ed. Arrighi, 1989, p. 32.

61 For this revised dating, see Høyrup, 2007a, p. 31 n. 70.

62 Ed. Arrighi, 1989.

63 See Høyrup, 2005.

understand the text he copied. However, a global survey of the relevant passages suggests that the present compiler did not know that he was sometimes speaking of proportions, glaringly misunderstood as they occasionally are (see the scheme on the next page). In any case, they were not part of his own mathematical upbringing and culture as reflected in the portion of the compilation that was not copied (or miscopied) from Fibonacci.

Leaving aside for a moment Jacopo da Firenze's *Tractatus algorismi*, written in Montpellier in 1307, in which the notion of "proportion" does turn up a few times in particular contexts, I shall go on with other early abbacus treatises.

Two of these were also written in Provence: a *Liber habaci* from c. 1310, and Paolo Gherardi's *Libro di ragioni* from 1328.[64] Neither of them speaks of "proportions" at all, either under this name or as *propositioni* – with one specific kind of exception in Gherardi's book to which we shall return. The former gives the rule of three almost exactly as the *Livero de l'abbecho*, but differs from all other known abbacus writings on one singular account: all its integer numbers are written with Roman numerals, and all its fractions are spelled out in full words. Even the brief explanation of the place value system[65] does not show a single Arabic numeral. This might (but need not) reflect a style preceding the *Columbia Algorism*.

A *Libro de molte ragioni d'abaco* written around 1330 in or around Lucca by three different hands[66] (and thus, we must presume, fairly representative as a total of the linguistic habits of the local environment of the time) contains two passages of interest: for the digging of a well, the toil is said[67] to be *aproportionata* to the depth; and it is said[68] to be necessary for a certain problem solution to be valid that Florence and Lucca are either both *proportionata* as circles or both as squares. The latter request thus refers to geometric shape, considered generically as a "proportioning." In the former case it turns out in the following that the toil for each cubit is almost but not quite directly proportional to its depth, since the total

64 Both are in ARRIGHI, 1987b. Arrighi ascribes both to Gherardi, but gives no convincing reasons for which the *Liber habaci* should have come from his pen.

65 Ed. ARRIGHI, 1987b, p. 155.

66 VAN EGMOND, 1980, p. 163.

67 Ed. ARRIGHI, 1973, p. 29.

68 *Ibid.*, p. 31.

Occurrences of *proportione/propositione* in the *Livero de l'abbecho* (right column, with page numbers from ARRIGHI, 1989), and corresponding passages in the *Liber abbaci* (left column, with page numbers from BONCOMPAGNI, 1857).

270	que multiplica per 6 de proportione superius inventa	49	el quale multiplica per 6 de la prepositione de sopre trovata
131	que proportio est composita ex duabus datis proportionibus. Et cum proportio aliqua est composita ex quotcumque proportionibus; tunc proportio proportionum ipsa appellatur: que compositio qualiter fiat, lucidius demonstrabo	30	la quale proportione ène conposta da doie prepositione e proportione della propositione è chiamata perch'ella se mostra chiaramente
145	argenti uncias, que fuerint in omnibus prepositis monetis, addiscas	34	le onzie de l'argento che sonno en tutte le propositione e le monete en prende
229	positis petitionibus ipsorum [for the purchase of a horse]	69	noie devemo ponere le propositione
200	ex petitionibus ex proportionibus reliquorum hominum	78	de la petitone e de la proportone degl'altre huomene
201	ex petitionibus et ex propositionibus reliquorum	78	de la petitone e da propositione degl'altre
288	secunda aliquam datam proportionem [...] qui sunt in dicta proportione	81	secondo l'altra propositione [...] che sonno ella ditta propusitione
205	hec positio per priman regulam, hoc est per modum arborum, solui possit; tamen qualiter aliter soluatur demonstrare cupimus	87	quista propositione overo quistione se può fare per la regola del primo albero, el quale mostramo chusì
286	ut invenias proportionem, quam habent ad inuicem primum, et secundum uas	100	truova la propositione che àggiono emsieme el primo e'l sechondo vaso
133*f*	proportio uniuscuiusque numeri prime coniunctionis ad 6, qui est tertius ex numeris secunde, est composita ex duabus proportionibus quattuor reliquorum numero rum	137	la propositione de ciascuno numero de la prima congiontone a 6 el qual'è el terço numero

work for depth n is as $1 + 2 + \ldots + n$.[69] However, this numerical specification comes afterwards, the word *aproportionata* seems to stand as an explanatory everyday term, meaning loosely "corresponding to." Apart from these two passages, due to the same hand, neither "proportion" nor "proposition" (in any sense) can be found anywhere in the compilation. More or less specific notions of proportionality thus seem to have penetrated general language (perhaps coming in particular from the visual arts).[70]

Only texts in modern print allow text recognition and thus (at least fairly reliable) complete search. Regarding the extensive *Trattato di tutta l'arte dell'abacho* (written in Avignon in c. 1334[71]), existing only in manuscript form, I am therefore not able to assert that "proportion," "proportionality," and "proposition" used for "proportion" are totally absent; however, I have consulted such passages in the earliest manuscript[72] where the concepts could be expected to turn up if they belonged to the standard vocabulary of the author, without finding any of them. All in all it seems a reasonable conclusion that the relation of early abbacus culture with proportions and proportionality was like that of Molière's Monsieur Jourdain with prose: he had spoken it for forty years without knowing anything about it. In other words: *We* may find reasoning based on proportions, but this observation of ours does not correspond to the conceptual world of the abbacus teachers.

It seems reasonable to infer that Fibonacci glued proportions not only onto algebra, but also onto the whole of proto-abbacus mathematics.

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69 The toil is thus supposed to be proportional to the depth of the bottom of the stratum, not to its average depth.

70 The family of words derived from "proportion" has one representative in Dante's *Divina Commedia*, namely a reference (XXX.56) to the *proporzione* of a giant; and one in Boccaccio's *Decameron* (sesta giornata, novella sesta), a reference to the duly *proporzionati* faces produced by a painter. Both have to do with geometric shape, the latter with being well shaped.

Not clearly linked to shape and aesthetic proportionality, however, is the observation in Dante's *Convivio* IV that the human intellect is *improporzionalmente* surpassed by the divine intellect; and likewise the one in his *Vita nova* XXV that "rhyme" in the vernacular is much the same as "verse" in Latin, with the added proviso *secondo alcuna proporzione* – a *mutatis mutandis* with quantitative connotations (I used the electronic versions of the texts on http://www.liberliber.it).

71 As shown by Cassinet, 2001.

72 Florence, Bibl. Naz. Centr., fond. prin. II,IX.57, the author's draft autograph.

4. Jacopo da Firenze and Early Abbacus Algebra

Three manuscripts exist that claim in identical colophons to contain Jacopo's *Tractatus algorismi*, written in Montpellier in September 1307: Milan, Trivulziana MS 90, dated by watermarks to c. 1410; Florence, Riccardiana MS 2236, undated; and Vatican, Vat. Lat. 4826, dated by watermarks to c. 1450.[73] Editions of all three are in my book of 2007.[74] The Vatican manuscript contains an algebra, a chapter with problems about wages in growing continued proportion and a final collection of mixed problems that is absent from the others. As I have argued,[75] the Vatican manuscript is a faithful copy of a shared archetype at least antedating 1328 considerably (and thus likely to be Jacopo's original), whereas the other manuscripts, very close to each other, represent an abridgment adapted to the needs of the abbacus school.[76]

Let us therefore look first at the Vatican manuscript. Its algebra contains rules for all "cases" (simplified equation types) up to the fourth degree that are either homogeneous or reducible to second-degree equations by division, and one of the three possible biquadratic equations. For the six equations of the first and of the second degree, one or several examples are given (ten in total).

The *theory* of proportions is not used here, not even its simplest level.[77] However, the problem statements are of some interest. Five are sham commercial problems,[78] and two number problems belong to classical

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73 For these datings, see Van Egmond, 1980, pp. 225, 148, 166. Van Egmond gives the date 1307 for the Florence manuscript, but this is merely the date stated in the colophon, common to all three manuscripts. Since this manuscript is close to the Trivulziana 90 (see imminently) but with more errors, a date later in the fifteenth century is plausible.

74 Høyrup, 2007a. The edition of the Riccardiana manuscript (as implicitly included in the critical edition with the Trivulziana manuscript) is a re-edition of Annalisa Simi's transcription (1995).

75 Høyrup, 2007a, pp. 5-25.

76 In a brief inserted note in Van Egmond (2008, p. 313), it is claimed that the Vatican algebra belongs to a family that Van Egmond names after Benedetto da Firenze. He takes it for granted that the undated Florence manuscript is actually from 1307, refers to the Milan manuscript as "several later copies of it," and overlooks the fact that the verbatim repetition of the Vatican text in the *Trattato dell'alcibra amuchabile* from c. 1365 (with one improvement showing the model of the Vatican algebra to be earlier) excludes any date after 1365. His claim can be safely disregarded – as can much of his construction of "families," based exclusively on the appearance and order of equation *types*, with no regard for formulations, choice of examples, or incipient symbolism, and almost none for terminology.

77 Yet it *could* have served. In one problem, the partnership serves instead to establish the equation; in another one dealing with composite interest, the rule of three is used.

78 One of these (ed. Høyrup, 2007a, p. 314*f*) deals with composite gain; it does not coincide with any of those found in the *Liber abaci*, but it also leads to a problem of the second degree. In

types already found in al-Khwārizmī's *Algebra*. Two, however, have a dress that appears not to be known from any earlier source:[79]

> [...] find me two numbers that are in proportion [*propositione*] as is 2 of 3. And when each (of them) is multiplied by itself, and one multiplication is detracted from the other, 20 remains [...].
>
> Find me 2 numbers that are in proportion [*propositione*] as is 4 of 9. And when one is multiplied against the other, it makes as much as when they are joined together.

At first, these may look intricate, but at slightly closer inspection they are nothing but more complicated ways to ask for "a number that, when multiplied by itself and by 5, gives 20," and "a number that, when multiplied by itself and by 36, gives as much as when it is multiplied by 13."

As we shall see presently, later writers use the same principle to show off cheaply, but overall they mostly use the formulation "the first is such a part of the second as [say] 2 is of 3." Jacopo is thus not likely to have invented the mathematical principle, but the explicit use of the notion of proportion (expressed as *propositione*) *could* a priori have been his idea; see however note 84.

However that may be, the "proportion" concept turns up again slightly later, in a sequence of problems about the manager of a warehouse (a *fondaco*, written *fondicho*, etc.) whose wages are supposed to increase in continued proportion. The statements run as follows:

> Somebody stays in a warehouse 3 years, and in the first and third year together he gets in salary 20 *fiorini*. The second year he gets 8 *fiorini*. I want to know what he received precisely the first year and the third year, each one by itself. Do thus, and let this always be in your mind, that the second year multiplied by itself will make as much as the first in the third. [...]
>
> Somebody stays in a warehouse for 4 years, and in the first year he got 15 gold *fiorini*. The fourth he got 60 *fiorini*. I want to know how much he got the second year

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Jacopo's variant, the gain in the first travel (£ 12) and the total possession after the second (£ 54) are known. The problem of course involves a continued proportion, $\frac{C+12}{C} : \frac{54}{C+12}$ (C being the initial capital), but Jacopo deals with it by means of the rule of three – identified only as *la regola* – integrated in algebraic reasoning.

79 Ed. Høyrup, 2007a, pp. 307, 309. It *may* be related, but then only distantly, to Fibonacci's repeated reference to numbers for which ${}^1/_p n_1 = {}^1/_q n_2$ (two examples above, the paragraphs after note 30 and before 44).

and the third at that same rate. Do thus, that you divide that which he got in the fourth year in that which he got in the first year. And you will say that what results from it is cube root. [...]

Somebody stays in a warehouse for 4 years. And in the first year and the fourth together he got 90 gold *fiorini*. And in the second year and the third together he got 60 gold *fiorini*. I want to know what resulted for him, each one by itself. And let them be in proportion and let the first be such part of the second as the second of the third, and as the third of the fourth. And let it always stay in your mind this, that to multiply the first year in the fourth makes as much as the second year in the third. And it makes as much to divide the fourth year in the second as the third year in the first. [...] And 40 *fiorini* he got the third year. And it is done, and you see well clearly that each of these numbers are in proportion. And such part is the first of the second as the second of the third, and as the third of the fourth: each is the half. [...].

Somebody stays in a warehouse for 4 years. And in the first year and the third together he got gold *fiorini* 20. And in the second and the fourth year he got gold *fiorini* 30. I want to know what was due to him the first year and the second and the third and the fourth. And that the first be such part of the second as the third is of the fourth. [...].

As we see, the geometric proportion is taken for granted, as belonging tacitly to the dress. As soon as the procedure is explained, however, the necessary fundaments of proportion theory turn up, and in the third and fourth problem we even find the word (in the form *propositione*)[80] together with the alternative formulation "to be such a part as."

The third problem goes beyond what can be found in the *Elements*, though it is based on knowledge that had been current in Arabic scientific mathematics since al-Karajī. If a, b, c, and d designate the respective yearly wages, the first step of the solution is to state that

$$a \cdot d = b \cdot c = \frac{(b+c)^3}{3(b+c)+(a+d)}$$

This certainly goes beyond Jacopo's mathematical competence. *He* cannot have invented the problems. On his own he *may* have had the idea to introduce the term *propositione* – though no scholar, he was not quite without scholarly pretensions, and his five-line colophon is in Latin. On

80 It should be noted that something that is proposed is spoken of as a *proponimento* by Jacopo (ed. Høyrup, 2007a, pp. 250, 425).

the whole, however, the appearances of the word in the algebra and in this quasi-algebraic chapter are more likely to have been borrowed together with the rest of that text: it turns up nowhere else in the treatise, and seems well integrated when appearing in the *fondaco*-problems. His pretensions may then have caused him not to eliminate it.[81]

Evidently, the references to proportions in the Vatican manuscript have no counterparts in the Florence and Milan manuscripts, from which the very chapters where they should turn up are missing. Nonetheless, the word appears a single time,[82] namely in the counterfactual calculation: "if 5 times 5 made 26, say me how much 7 times 7 would make in that same proportion (*in quela medesima proportione)*," that is, in the sense of "rate"; the result is then stated with the words: "we shall say that 7 times 7 makes 50 and $^{24}/_{25}$ at this same rate (*diremo che 7 via 7 facia 50 et* $^{24}/_{25}$ *a quela medesima rasone*)." Further on, *rasone* is used in a counterfactual calculation that follows immediately, and in nine places where rates are spoken of. In the Vatican manuscript, the first counterfactual question[83] has *ragione* where the Milan and Florence manuscripts have *proportione*. This single appearance of the word is probably a secondary modification reflecting a general tendency in the later fourteenth century to absorb bits of the terminology of university mathematics.

A number of abbacus texts written between 1307 and 1345 (all mentioned above) contain a smaller or larger amount of algebra:

– Paolo Gherardi's *Libro di ragioni* from 1328;
– the *Libro de molte ragioni d'abaco* from c. 1330;
– the *Trattato di tutta l'arte dell'abacho* from c. 1334.

Gherardi has a systematic presentation of algebraic cases: all cases up to the third degree treated by Jacopo; four more cases of the third degree that cannot be reduced to quadratic equations (the solutions for which are therefore false, produced by superficial imitation of second-degree solutions); and the case "cubes equal to the square root of a number." Ten of these are illustrated by problems of the type asking for numbers in a given ratio (invariably, when more than two numbers are involved, as

81 Given the spelling *propositione* and the general conscientious precision of the manuscript we possess, the term is not likely to have been inserted in the text during the transmission process.

82 Ed. Høyrup, 2007a, p. 420.

83 *Ibid.*, p. 238.

n : *m* and as *m* : *p*, etc., avoiding thus the need for composing ratios); but in seven of them the formulation of the matter is "such part [...] as *m* is of *n*." Only three ask for "3 numbers that are in position [*positione*] together, that is, the first of the second as 2 of 3, and the second of the third, as 3 of 4."[84]

In the *Libro de molte ragioni d'abaco* and the *Trattato di tutta l'arte dell'abacho*, problems with the "such part" formulation are found, but never the "proportion" formulation.

In 1344, a certain Dardi of Pisa wrote the first treatise in the abbacus tradition dedicated exclusively to algebra.[85] This work contains several hundred problems, a large part of which deal with two or three numbers in a given ratio. Mostly these use the formula "such part [. . .] as *m* is of *n*." In one case,[86] however, a two-number problem asks for "two proportional numbers in continued proportion [*proportionali in continua proportione*] so that the first is such a part of the second as 4 is of 5." More meaningful is the question[87] for "three numbers in continued proportion [*in continua proportione*], that is, that the first is of the second as the second of the third, and be such a proportion as 2 of 3" – but we notice that the final words of the question use "proportion" as a synonym for "part."

In three later problems of the same type,[88] the Arizona manuscript replaces the information about the continued proportion by the phrase "that one is such a part of the other as [...]," apparently meant as "each [...] of the following one." The Vatican and Siena manuscripts have the same construction in no. 75, but state "that the first is such a part of the second as [...]" in no. 69, saying nothing about the ratio between the second and the third. So does the Vatican manuscript in no. 67, whereas the Siena manuscript adds "and that they are in continued proportion." It

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84 Ed. ARRIGHI, 1987b, pp. 102, 106, 107. But still three, while one example that is shared with Jacopo has the alternative formulation. This decreases the likelihood that Jacopo should be the one who introduced the proportion formulation (and the respective *propositione*/*positione* suggests perhaps shared dependency on a source or environment where *proportione* was reinterpreted as one or the other, either because numbers may be *positioned* in ratio or *posited* (namely, as 2 *things* and 3 *things* if their ratio is 2:3), or because a specific ratio is *proposed* (but cf. note 80).

85 I have consulted Vatican, Chigi M.VIII.170 from c. 1390; Franci (2001), an edition of Siena, I.VII.17 (c. 1470); and Van Egmond's personal transcription of the Arizona manuscript written in 1429 (for access to which I am grateful).

86 Ed. FRANCI, 2001, p. 89, similarly the manuscripts; counted as no. 10 by Dardi.

87 Ed. FRANCI, 2001, p. 139, similarly the manuscripts; Dardi's no. 64.

88 Dardi's nos. 67, 69, and 75.

seems likely that the Arizona manuscript corresponds to the original on this point, and that Dardi has thus explained the notion of continued proportion in no. 64 (after having used it wrongly in no. 10), and afterwards just uses "one [...] the other" as a way to indicate a repeated ratio; Siena and Vatican at first overlook this finesse, but in no. 69 Siena sees that information is then missing; in no. 75, both copyists have discovered it.

Dardi, like Jacopo, has scholarly pretensions (and much greater mathematical competence and ambitions, but that is immaterial in this connection): his preface explains[89] the four Aristotelian causes (*rispetti*) of his book, in the best scholastic manner. He *may* therefore have adopted a term from Latin university mathematics, without having much use for it (and, as we see in no. 10 and no. 64, without being quite sure of its use). His treatise is thus yet another piece of evidence that *proportione* and *proportionalità* did not yet belong to the standard terminology of the abbacus ambience.

Before c. 1340, a master Biagio known later as *il Vecchio* (the Old), wrote an abbacus treatise that has been lost, but from which a collection of algebraic problems was copied by Benedetto da Firenze for his encyclopedia (see note 14). This collection confirms the picture, and adds some shades (with the proviso that we cannot be quite sure Benedetto did not change the precise wording of his model).

Firstly, we find again a large number of problems asking for numbers or quantities in a given ratio: 19 in all.[90] Only the last of them[91] asks for "2 numbers, or 2 quantities, which are in proportion as 5 is to 7, that is, that the first quantity is to the second as 5 is to 7"; all the others ask either for quantities (10 of them) or numbers (8 of them), and all use the formula "such part [...] as *m* is of *n*."[92] We may speculate that the first 18 occurrences are borrowed material, and the last one Biagio's own construction, in which he shows the applicability of the proportion concepts to this problem type (and points to the equivalence of number- and quantity-formulations).

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89 Ed. FRANCI, 2001, p. 37.

90 One of them (ed. PIERACCINI, 1983, p. 18*f*) asks for three numbers in ratios 2 : 3 and 2 : 5, but this does not lead to a presentation or investigation of the composition of ratios. Biagio simply posits the numbers to be 2 *things*, 3 *things*, and $7\frac{1}{2}$ *things* without explanation. All others, as usual, have the ratios nicely fitting together.

91 Ed. PIERACCINI, 1983, p. 126.

92 The homonymy should not mislead us into believing that the "quantities" are continuous magnitudes – lengths, areas, volumes, durations, weights – as they would have been in contemporary Aristotelian university discourse. None of the authors I treat before Pacioli uses the term in this way.

This is not the first time Biagio refers to "proportions." In a problem about a loan with compound interest over three years,[93] he introduces the notion of a continued proportion and uses the product rule to establish the equation. Later on, in Jacopo's fourth *fondaco* problem (still told about the manager of a *fondaco*, with the data 40 and 60),[94] he first shows that the product rule $ad = bc$ gives a tautology, and then that the rule $ac = b^2$ yields an equation.[95] In an indeterminate problem about three monies with unknown metal content,[96] he postulates that the quantities are in continued proportion with a ratio of 2 : 1, and thereby gets a single determinate equation. Finally, in a more intricate problem about composite gain[97] – given difference between the interest rates and the given ratio between the total interests of the first and the second year – this ratio is at first defined as being "as 2 to 3," but when it is used later we are told that "the proportion of the interest of the first year and that of the second is as 2 to 3." The last instance (perhaps also the second-last) sounds as if the idiom of proportions felt natural for Biagio; the formulations of the final problem about "2 numbers, or 2 quantities" may then indicate that he was aware that his public was less familiar with it. However that may be, 7 occurrences of the words *proportione* and *proportionalità* in a text of some 30000 words must be characterized as a modest intrusion in Biagio's language.

A final algebraic treatise, written after our limit 1345, but throwing light on the early epoch, is a *Trattato dell'alcibra amuchabile* from c. 1365.[98] It consists of three parts:

– rules for calculation with signs, square roots, and binomials consisting of number and square root;
– a list of algebraic "cases," provided in part with examples; and
– a collection of problems.

Only the second of these concerns us at present.[99] It contains all of Jacopo's cases including his examples almost verbatim. This segment of the second part is almost certainly derived from Jacopo's text, and it is

93 Ed. PIERACCINI, 1983, pp. 67-69.
94 *Ibid.*, pp. 89-91.
95 Jacopo, in contrast, had only provided a rule for determining the solution.
96 Ed. PIERACCINI, 1983, p. 109*f.*
97 *Ibid.*, pp. 119-121.
98 Ed. SIMI, 1994.
99 Cf. HØYRUP, 2007a, pp. 160, 163.

therefore no wonder that the examples with numbers in a given ratio speak of "proportion," as does Jacopo. More interesting is that it also presents cases and examples that are in Gherardi's algebra but not in Jacopo's, moreover in a version that appears to antedate Gherardi's – seven examples in total, all constructed around numbers in given ratio(s). Four of these ask for numbers *in proporzione*, only three use the "such part" formulation. Of Gherardi's counterparts, 6 are of the latter type, and only one asks for numbers *in positione*; this latter question corresponds to a "proportion"-formulation in the *Trattato*. It thus looks (but the statistics are not sufficient to allow any certainty) that Gherardi had a tendency to use "such part" even when his source had (we may presume) *propositione* or *positione*; this augments the likelihood that Jacopo did not introduce the "proportion" language on his own, but took it over from his source.

5. Antonio de' Mazzinghi

Antonio de' Mazzinghi (c. 1355 to 1385-86)[100] is praised highly for his algebraic competence in three encyclopedias from the mid-fifteenth century,[101] which are also our only sources for his mathematics. The largest extract is his *Fioretti*.[102] This is an outstanding text, which fully confirms the praises heaped upon him. That is not what concerns us here, but it is good to know as a background to what follows.

The *Fioretti* do not contain a single problem of the kind asking for numbers or quantities in a given ratio. We may guess that Antonio found it below his mathematical dignity to stoop to the use of such cheap tricks; alternatively, he may not have found them fitting for a collection of "little flowers."

Two problems have a formal similarity with the cheap type,[103] asking indeed for numbers in ratio; however, this ratio is not given numerically, but as that between two other numbers fulfilling algebraic conditions. Written in letter symbols, the respective structures are:

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100 See Ulivi, 1996, pp. 109-115.

101 Benedetto's *Praticha d'arismetricha*, see above, note 14; Vatican, Ottobon. lat. 3307; and Florence, Bibl. Naz., Palatino 573.

102 Ed. Arrighi, 1967a.

103 *Ibid.*, pp. 46-51.

$$ab = (a-b)^2,\ \frac{c}{d} : \frac{a}{b},\ 19 = c+d,\ cd = c^2+d^2$$

and

$$a^2+b^2 = 60,\ \frac{c}{d} : \frac{a}{b},\ cd = 10,\ c^2+d = ab$$

The first is the one where Antonio famously has to invent a trick that enables him to calculate with two unknowns.[104]

In problems about compound interest, Antonio points out with greater clarity than any predecessor[105] that interest *a chapo d'anno* ("[making up accounts] at the end of year," that is, compound interest) "proceeds in continued proportionality," not only with correct calculations up to five years based explicitly upon the product rules (thus no longer the rule of three), but also with the finding of equivalent rates of interest if accounts are prepared every 8 or 9 months. Belonging to the same field of theoretical interest is the problem of finding a five-term continued proportion beginning with 16 and ending with 81.[106]

A large number of problems ask for three or four numbers in continued proportion that fulfil other algebraic conditions of the first or second degree. In symbolic abbreviation and with the numeration of Arrighi's edition (taken from the manuscript) they are:

#1 $19 = a+b+c,\ a\cdot(b+c)+b\cdot(c+a)+c\cdot(a+b) = 228$

#2 $a\cdot(b+c)+b\cdot(c+a)+c\cdot(a+b) = 888,\ a^2+b^2+c^2 = 481$

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104 If for example 10 has to be divided into two parts, it was often found convenient to take these as 5 – *thing* and 5 + *thing*. As we have seen, an unspecified number is regularly also spoken of as a "quantity." Antonio combines the two ideas, taking *a* to be "a *thing* minus a *quantity*," *b* to be "a *thing* plus a *quantity*." The intellectual jump involved in this seems to have gone almost unnoticed at the time and to have inspired little imitation, perhaps because the use of habitual words made Antonio's readers (including perhaps Benedetto, but see imminently) overlook the fact that something (potentially) important had occurred – exactly as had happened to Fibonacci's similar trick (using *res* and *causa* for the two unknowns) in the *Flos* (ed. BONCOMPAGNI, 1862, p. 236).

Benedetto does use *quantity* as an algebraic unknown in his *Tractato d'abbaco* (ed. ARRIGHI, 1974, pp. 153, 168, 181; Arrighi ascribes the treatise to Pier Maria Calandri), namely in solutions by means of *modo retto/repto/recto*, first-degree algebra designated *regula recta* by Fibonacci, who calls the unknown *res* (ed. BONCOMPAGNI, 1857, pp. 191 and *passim*).

The *quantità* also turns up in this function in Benedetto's *Trattato de praticha d'arismetrica* – in the manuscript Siena, Biblioteca Comunale degli Intronati, L.IV.21 (Benedetto's original, cf. below, note 127), fol. 263v, even together with a second unknown *borsa*, [the unknown contents of] a purse. This casual naming might perhaps be taken to suggest that Benedetto had assimilated the possibility of two unknowns to such a degree that he could implement it on the rare occasions where he would need it without even thinking of it as a borrowing from Antonio.

105 Ed. ARRIGHI, 1967a, p. 36.

106 *Ibid.*, p. 69.

#3 $9^1/_2 = a + b + c,\ a^2 + b^2 + c^2 = 33^1/_4$
#4 $19 = a + b + c,\ 3a + 4b + 5c = 81$
#5 $a + c = 21,\ b + d = 39$
#8 $a \cdot b \cdot c \cdot d = 2916,\ a + b = 17^1/_2$
#23 $a + b + c = 14,\ a \cdot b \cdot c = 64$
#25 $a^2 + b^2 + c^2 = 84,\ {}^{20}/_a + {}^{20}/_b + {}^{20}/_c = 125$
#26 $10 = a + b + c,\ 3a + 4b = 5c$
#29[107] $c - a = 50,\ d - b = 80$

Repeatedly, as can be expected, the solutions make use of the product rules. A couple of times, however, Antonio appeals to more advanced properties of proportions or continued proportions. In #25, he finds it "rather clear and obvious" (*è cosa assai chiara e manifesta*) that if a, b, and c are in continued proportion, then the same can be said about ${}^{20}/_a$, ${}^{20}/_b$, and ${}^{20}/_c$.[108] In #29, the *disjuncta* mode is described and used.[109]

The genre as such was not new, neither in general nor to abbacus mathematics. Abū Kāmil[110] has a problem about numbers in continued proportion with the structure:

$$10 = a + b + c,\ a^2 + b^2 = c^2$$

and there are two examples in the third part of the above-mentioned *Trattato dell'alcibra amuchabile*[111] (in symbolic abbreviation):

$$10 = a + b + c,\ a \cdot b = 4,\ b \cdot c = 8$$

$$10 = a + b + c,\ a^2 + b^2 + c^2 = 40$$

The former is overdetermined and impossible, and the solution proposed is wrong. The latter, we observe, has the structure of Antonio's #3. Antonio's

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[107] Actually, #29 starts with a problem $10 = a + b,\ a^2 + b^2 + \sqrt{a} + \sqrt{b} = 86$, and a position $a = 5 - t,\ b = 5 + t$. When this has been reduced to

$$\sqrt{5 - t} + \sqrt{5 + t} = 36 - 2t^2$$

Antonio comments: "I do not like it, and therefore I do not complete it" – and goes on with the problem about three numbers in continued proportion.

[108] Ed. ARRIGHI, 1967a, p. 54.

[109] *Ibid.*, p. 63.

[110] Ed. and trans. LEVEY, 1966, p. 186; SESIANO, 1993, p. 405; CHALHOUB, 2004, p. 148.

[111] Ed. SIMI, 1994, p. 39.

#5, on its part, has the same structure as Jacopo's fourth *fondaco* problem, the one that was also solved by Biagio. Antonio solves it in the same algebraic way as Biagio, omitting however Biagio's pedagogical blind alley. I know of no evidence allowing one to determine whether the number genre as found in the *Trattato dell'alcibra amuchabile* has the same ultimate origin as the *fondaco* genre, or whether Antonio fused the two.

What seems fairly certain is that the present type of number problems has no strong links to Fibonacci's division of 10 into four unequal parts in proportion;[112] Antonio certainly knew and appreciated Fibonacci,[113] but nothing suggests the same for the compiler of the *Trattato dell'alcibra amuchabile* or his source. Moreover, Fibonacci speaks of *any*, not a continued proportion (and uses the example $\frac{3}{7} : \frac{6}{14}$); afterwards he shows how to construct a sequence of any length of numbers in continued proportion, but now without constraint on their sum.

Towards the end of the *Fioretti* comes a section *Mirabile dictum*,[114] showing how to divide a number (say, N)[115] into parts (say, a, b, c, d, and e) in such a way that:

$$^{N}/_{a} + ^{N}/_{b} + ^{N}/_{c} + ^{N}/_{d} + ^{N}/_{e} = N$$

This section was analyzed by Margherita Bartolozzi and Raffaella Franci.[116] Since it was certainly due to Antonio himself and, as far as I know, had little further impact on the abbacus tradition (apart from being copied by an obviously impressed Benedetto and being used by Pacioli, for which see below), I shall not discuss it any further.

Antonio was familiar with Book 15, Part 1 of the *Liber abbaci*: Palatino 573 from c. 1470 quotes his *gran trattato* for presupposing "that the proportions from the first part of the 15th chapter [of the *Liber abbaci*] be clear to you."[117] But this familiarity left no trace in the *Fioretti*. Whether Antonio thought of the connection to the Boethian means cannot be decided with certainty, but since those who read him have not noticed it, it is unlikely.

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112 Ed. Boncompagni, 1857, p. 170.

113 Quotation in Ottobon. lat. 3307, ed. Arrighi, 2004/1968, p. 221.

114 Ed. Arrighi, 1967a, pp. 81-87.

115 Antonio builds his solution up around the core $(\sqrt{6} - \sqrt{2}, \sqrt{6} + \sqrt{2})$, but then states that any pair of binomials $\sqrt{b} \pm \sqrt{a}$ serves if $b : a$ is a multiple. Unfortunately, as pointed out by Bartolozzi & Franci (1990, p. 10 n. 16), Antonio's condition is insufficient.

116 Bartolozzi & Franci, 1990, p. 10*f.*

117 Arrighi, 2004/1967, p. 190.

6. Late Fourteenth Century Otherwise

Like Fibonacci, Antonio knew theoretical mathematics well enough to adopt it creatively into his abbacus heritage. He was exceptional, and in consequence an exception; to what extent would his near-contemporaries make use of the notion of "proportion," in any of its possible senses? Two examples will have to suffice.

The first is Giovanni de' Danti d'Arezzo's *Tractato de algorisimo* from 1370.[118] This is a decent but not sophisticated abbacus book, containing no systematic presentation of algebra but a short passage about the arithmetic of square roots and a few algebraic problems.[119] Giovanni's remoteness from any scholarly mathematical environment is illustrated by his explanation of the existence of surds: God does not want that anything but Himself be perfect.[120]

The word "proportion" (in any of the spellings we have encountered) is as absent from this treatise as from the non-Fibonacci parts of the *Livero*, from the *Liber habaci*, and from the non-algebraic parts of Jacopo's *Tractatus* and Gherardi's *Libro di ragioni*. *Propositione* is found often, but it means "a question that is proposed."

There are seven problems asking for numbers in a given ratio; all three use the "such part" formulation. For once, the same formulation is also found in a non-algebraic business problem:[121] a loan, on which the interest in the first year is such a part of that in the second year as 3 is of 4. Since nothing similar is found in treatises from the first half of the century, this type is likely to be an offset from the analogous pure-number problems.

The second is a *Trattato d'algibra*, constituting the last fifth of a larger abbacus treatise (Florence, Bibl. Naz. Centr., fond. prin. II,V.152), according to internal evidence written in the 1390s – thus after Antonio's death, but apparently in the tradition after Biagio (or his source). Only the algebra has been published.[122] My discussion is restricted to this published part; I have not seen the manuscript.

In this algebra, the word *proporzione* turns up in two different contexts: in the theoretical introduction, and in the problems.

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118 Ed. Arrighi, 1987a.

119 *Ibid.*, pp. 52-57, 65-69.

120 *Ibid.*, p. 53.

121 *Ibid.*, p. 34.

122 Ed. Franci & Pancanti, 1988.

The theoretical introduction is an investigation of the sequence of algebraic powers.[123] This introduction is both interesting and puzzling.[124] What concerns us here, however, is merely that the sequence of powers is seen to be in continued proportion, which is used to show that *censo* times *censo* is the same as *thing* times *cube*, and *censo* times *cube* as much as *thing* times *censo di censo*.[125]

Proportions also turn up in four problem types: in problems about compound interest over three or four years (very similar to what Biagio does); in the *fondaco* problem also solved by Biagio;[126] in two problems about the division of ten in three parts in continued proportion (one analogous to Antonio's #3 and the second problem cited from the *Trattato dell'alcibra amuchabile*):

$$10 = a + b + c,\ a^2 + b^2 + c^2 = 70;$$

and one similar to Antonio's #4:

$$10 = a + b + c,\ 3a + 4b + 5c = 35;$$

and finally in a problem about three quantities of money (in the sense of coinable metal) in continued proportion, structurally identical with Antonio's #4. There are also numerous problems about two, three, or four numbers in given ratio, all in "such part" formulation. In an inverse variant of the well-digging problem from the *Libro de molte ragioni d'abaco*, nothing is said about the toil being in correspondence (*aproportionata* or otherwise) with the depth; we only get the solution by means of triangular numbers.

This treatise is mathematically very sophisticated. Nonetheless, as we see, the use of proportion theory (or the very recourse to its terminology) expands only slightly beyond what was known at least since Biagio: proportions fully displace the rule of three in problems about compound interest, and they enter the explanation of the sequence of algebraic powers.

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123 *Ibid.*, pp. 3-6.

124 See HØYRUP, 2008, pp. 30-32.

125 According to Palatino 573 (ARRIGHI, 2004/1967, p. 191), Antonio appears to have made the same observation in his *gran trattato*.

126 The solution of this *fondaco* problem (FRANCI & PANCANTI, 1988, pp. 80-82) runs almost exactly as Biagio's, but there is one telling difference. Biagio takes the wage of the first year to be two *things*, whereas the present author chooses 2 *censi*. He does not know, however, that this word translates the Arabic *mal*, not only (namely in algebra) the square of the *thing* but also an amount of money (a capital, a dowry, etc.). In the end he therefore feels obliged to find the *thing* from the *censo* – only to square it again. This implies that the (direct or indirect) source cannot be Biagio's text; it must be traced to a context in which the original meaning of the *censo*/*mal* was still alive.

7. The Mid-Fifteenth-Century *Abbacus Encyclopediae*

Around 1460, three extensive works of encyclopedic character were produced in the Florentine abbacus environment, already listed together in note 10:

– Benedetto da Firenze's *Praticha d'arismetricha* (existing in many copies);[127]
– Palatino 573;[128]
– Ottobon. lat. 3307.[129]

All of them contain material that was foreign to the abbacus tradition, and extensive extracts from the writings of (mostly) *named* abbacus predecessors (quite unusual in the abbacus environment). They are indeed our only source for Biagio's and Antonio's mathematics, but also contain long translated extracts from the *Liber abbaci*. They *may* possibly have had a model in Antonio's lost *gran trattato* (see above).

Benedetto's work is divided into sixteen books. Three of these have to do specifically with proportions, and build for this on "foreign" material. Book II about "the nature and properties of numbers" (*la natura e propietà de' numeri*) is a presentation of speculative arithmetic in the Boethian tradition. It also offers an exposition and explanation of the way ratios were labelled in this tradition:[130] multiple, submultiple, superparticular, superpartiens, sesquialtera, sesquitertia, etc. But it does not use the word *proportione* (or *ragione*), but speaks of "a number that is referred to another number" (*numero che è riferito ad altro numero*). Book V is stated to deal with "the nature of numbers and proportional quantities" (*la nature de' numeri e quantità proportionali*).[131] Its first part builds on the Campanus version of *Elements* V-IX and on Campanus's *De proportione et proportionalitate* regarding the composition of ratios; the second part

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127 Described in Arrighi, 2004/1965; see Van Egmond, 1980, p. 356. Inspection of the marginal computations in the manuscript Siena, Biblioteca Comunale degli Intronati, L.IV.21, shows that some of them (for instance the complicated problem with the unknowns *quantità* and *borsa* mentioned in note 104) were written first, and the main text in whatever space was left over on the page. This shows that this manuscript was Benedetto's working copy, and that the date given in the colophon (1463) is that of the manuscript. Fortunately, all partial editions of Benedetto's text are based on this manuscript. Similar marginal computations in the other two manuscripts show that even they are their authors' working copies.

128 Described in Arrighi, 2004/1967.

129 Described in Arrighi, 2004/1968.

130 Ed. Arrighi, 1967b, p. 324*f.*

131 See Bartolozzi & Franci, 1990, pp. 12-14.

concerns metrological conversions. The first part of Book XI presents *Elements* II (including the division into extreme and mean ratio); the second part is a translation of Book XV, Part I of the *Liber abbaci*.

How much does this general mathematical erudition influence what Benedetto did *within* abbacus mathematics? We may look at his own algebra, contained in Book XIII.

Firstly, it is said more clearly[132] than in the above-mentioned Florence manuscript (Bibl. Naz. Centr., fond. prin. II,V.152) that the "algebraic powers" (*termini dell'algebra*) are in geometric proportion; on the other hand, Benedetto does not *use* this observation for anything. The associative law for multiplication may have been too obvious for him, for which he sees no need to do so; alternatively, this should be said about Antonio, whom Benedetto seems to follow.[133]

Secondly, the word *proportione* turns up once[134] inside one of the many problems about numbers given in ratio, which however are all defined in terms of "such part."

Palatino 573 was written by a former student of one Domenico d'Agostino *vaiaio* ("the tanner" or "the furrier" – whether this profession was indeed his or that of an ancestor is not to be determined). Its author says that he uses Benedetto's homonymous treatise, written "already some time ago" (for which reason the present manuscript should probably be dated around 1470)[135] as a model, adding and removing matters as needed.[136] It falls into eleven parts, subdivided into chapters. I shall discuss the relevant aspects.[137]

Chapter II.8,[138] about "the way to express as part, and, first, the definition," starts by quoting Boethius's, Euclid's and Jordanus's definitions of a ratio (*proportione*) as a relation between two numbers or quantities, and goes on with the traditional names (*doppia*, *sexquialtera*, etc.). This is not unproblematic, for according to the definition a ratio is *not* a (possibly broken) number, as is the "part" he

132 Ed. SALOMONE, 1982, p. 20.

133 Cf. note 125.

134 Ed. SALOMONE, 1982, p. 40.

135 Van Egmond (1980, p. 124) dates it on the basis of watermarks to c. 1460, but 1470 is still compatible with these.

136 Ed. ARRIGHI, 2004/1967, p. 168.

137 On the basis of the extracts in ARRIGHI, 2004/1967, pp. 168-194 (introductions to parts and chapters) and the description and quotations in BARTOLOZZI & FRANCI, 1988, pp. 14-16.

138 ARRIGHI, 2004/1967, p. 176; BARTOLOZZI & FRANCI, 1990, p. 15.

wishes to express. The author glosses over the difficulty by regarding it merely as a question of language (thus reminiscent of certain discussions within the contemporary historiography of mathematics): "we in the schools do not use such terms [*vocaboli*] but say instead [...] that 8 is $^2/_3$ of 12, and that 12 is $^3/_2$ of 8." The author also points to the necessity that the two magnitudes in a ratio be of the same kind, without noticing that this should create difficulties when, later, the concept is used to explain the rule of three. This is symptomatic of the whole project (as shared with Benedetto): abbacus mathematics is put into the framework of scholarly (in Fibonacci's word, "magisterial") mathematics, but the author reinterprets concepts as needed, and does not care much about the contradictions that may arise.

Part III[139] is similar in its aim. The introduction announces that its first chapter will deal with "the four proportional quantities or numbers, in the vernacular called the rule of three things." The chapter itself starts by defining the *proportionalità* as the equality of two ratios and explaining that such *proportionalità* may be continued or not continued. It goes on to present the product rule and uses it to determine one term in a continued proportion from the two others. Chapter III.2 applies the rule to commercial examples, and ends by saying that

> it is true that many who want to show this rule have said that one multiplies the quantity that one wants to know by the one which is not similar, and divides in the other quantity. And they actually say the truth. Because when you multiply a quantity by another one which is not similar, it is as multiplying the first by the fourth or the second by the third.

Nothing, as we see, is said about the impossibility of speaking properly of ratios between dissimilar quantities; the author obviously thinks of nothing but the measuring of numbers in the already established units of the statement of the problem.

Chapter III.3 translates the second part of Chapter XII of the *Liber abbaci* (discussed above).

Chapter V.3[140] introduces problems about numbers in given ratio by giving once more the names of ratios. This time, however, it identifies ratios with numbers in the abbacus manner ("5 to 16 are $^5/_{16}$ because from 5 divided by 16 comes $^5/_{16}$").

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139 Arrighi, 2004/1967, pp. 176-178; Bartolozzi & Franci, 1990, p. 15*f.*

140 Arrighi, 2004/1967, p. 181.

Part IX[141] translates Chapter XV, Part 1 of the *Liber abbaci*. The introduction refers to objections against the relevance of this topic for algebra ("many strain themselves to prove [...]"). In defense of this relevance it cites Paolo dell'Abbaco's (otherwise unknown) *trattato delle quantità chontinue*, Antonio's *gran trattato*, and the oral injunctions of his own teacher, the *vaiaio*. No argument beyond these appeals to authority is offered (nor could there probably be any). The link to Boethius's presentation of the ten means goes once more unnoticed.

Part X[142] is dedicated to algebra. The introduction states that in order to restore the almost lost *Maumetto arabicho* (that is, al-Khwārizmī), the presentation will be based on him.[143] Chapter 1 starts by quoting Fibonacci for the explanation that *algebra almuchabale* means "restoration and opposition, because the parts are opposed to each other, as you will see in the examples" – a misquotation, as we know, caused perhaps by al-Khwārizmī's text, perhaps (and rather) by earlier abbacus writers.[144] It then continues with al-Khwārizmī's text in Gherardo's translation.

After this pious homage to tradition in humanist style, the author feels the need to be modern, and starts by explaining the algebraic powers as a continued proportion, following (as he says) Antonio's *trattato* (probably the *gran trattato* of which he has already spoken).

A large number of problems follow, in part taken from named predecessors.

Ottobon. lat. 3307 was written by another former student of the *vaiaio*, perhaps slightly before 1463 (at least there is no evidence whatsoever that its author knew about Benedetto's *Trattato*). It is divided similarly to Palatino 573. Beyond the chapter headings,[145] I used a photocopy from a microfilm of the final algebraic part, which already allows me to say (I shall omit the documentation) that this third encyclopedia is of a markedly lower quality than the other two, but not to derive much of relevance for the topic of proportions.

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141 ARRIGHI, 2004/1967, p. 190*f*; BARTOLOZZI & FRANCI, 1990, p. 16.

142 ARRIGHI, 2004/1967, pp. 191-194.

143 Similarly, Benedetto (ed. SALOMONE, 1982, p. 20) presents not Fibonacci's, but al-Khwārizmī's geometric proofs "because more ancient."

144 A third possibility is Guglielmo de Lunis's lost translation (probably a revised version) of al-Khwārizmī's algebra, whose introduction (quoted for instance by Benedetto; ed. SALOMONE, 1982, p. 1*f*) has this interpretation of opposition; cf. also above, text before note 51.

145 As quoted in ARRIGHI, 2004/1968.

Maybe, however, there is not much of relevance at all: the algebraic powers (for which this author has no general term like Benedetto's *termini dell'algebra*) are *not* explained to be in continued proportion; further, the 33 folios of problems contain 2 questions (fols. 336r, 342v) about 4, respectively 3, numbers in continued proportion, and none about numbers in a given ratio.

We should be aware that these three encyclopediae all come from a specific strand in the abbacus tradition, centred upon a specific group of Florentine abbacus schools and tracing its origins back to Antonio, Paolo dell'Abbaco, and Biagio. Its interest in precursors – outside of those belonging to the strand itself, al-Khwārizmī and Fibonacci – was not generally shared. All the more telling is it that their copying of Fibonacci's sections on proportions did not really influence what they themselves did in abbacus style, apart from the formulation of the rule of three as a rule of four quantities in proportion. The interests and orientation of abbacus mathematics proper, we may perhaps conclude, left no space for that.

8. Luca Pacioli

Even Luca Pacioli's *Summa de Arithmetica Geometria Proportioni et Proportionalita* (1494; 1523),[146] divided into nine *distinctiones*,[147] might be characterized as a kind of encyclopedia, *not* belonging to the strand discussed in the previous section. A symptom of this different affiliation is that Pacioli feels no need to give specific references for his borrowings – Pacioli instead has a general acknowledgment in the initial unfoliated *Sommario* that most of his volume has been taken from Euclid, Boethius, Fibonacci, Jordanus, Blasius of Parma, Sacrobosco, and Prosdocimo de' Beldomandi. Actually, most of it probably comes from less prestigious, anonymous abbacus sources – whatever the citation strategy in the encyclopedias, it was always a strategy.

Pacioli's title might lead us to predict that ratios and proportions play a greater role and are more integrated than in the preceding manuscript sources. The list of his confessed sources could make us expect the same.

146 In general, the second edition is faithful to the first, word for word and line for line. There are a few corrections (and different misprints), but the main difference is that a number of abbreviations are expanded in the second edition.

147 I consider only the first, arithmetical part (1-224) and disregard the geometry (76 fols., 8 *distinctiones*).

Actually, "proportions and proportionalities" are mainly dealt with in the Sixth Distinction (at great length, fols. 67v-98v), but they do serve elsewhere.[148]

The topic is taken up for the first time in connection with the rule of three.[149] Initially, this rule is presented in terms of the similar and dissimilar, and in a different but equally "a-theoretical" way (fol. 57r).[150] After some examples, however, comes an explanation (fol. 57v) *unde regula predicta procedat* (where the said rule comes from), referring to *Elements* V. Actually, this explanation goes beyond what is needed for the purpose, as Pacioli points out, namely because he wants the reader to "better understand the fundaments" of the rule given.

He begins by stating that the force of the rule of three "proceeds from the mutual proportionality of quantities, be they continuous or be they discrete, that is, be they numbers or be they measures, and be the proportionalities continued or not continued [*incontinua*]," with three, respectively four, terms. After pointing out that "in all the calculations of the trading and business world, 4 numbers always occur, of which 3 are always known and the fourth unknown" and repeating the need for three, respectively four, terms, Pacioli makes a more interesting point: that in continued proportions all terms must be of the same nature because all (except the extremes) stand both as antecedent and as consequent, while the non-continued proportions require only pairwise identical nature. His explanation of these natures comes from the Aristotelian tradition – they are numbers, lines, surfaces, and bodies. He then goes on with the product rule, with explanation of a schematic representation, and with examples of what to do if any one of the four terms is unknown.

9. The Sixth Distinction

The Sixth Distinction is described by Margherita Bartolozzi and Raffaella Franci,[151] for which reason I shall elaborate only on such things

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148 I got access to Fenny Smith's work on *Proportion in the 'Summa de arithmetica, geometria, proportione et proportionalità' of Luca Pacioli* (1998) only after having finished the work. There is some overlap, but on the whole rather little, not least because our approaches to the text, as well as our emphasis, differ.

149 Continued proportions are mentioned but not really treated on fol. 37v, during the treatment of progressions.

150 The second edition (1523) has this folio number misprinted as 64.

151 Bartolozzi & Franci, 1990, pp. 17-27.

as go beyond their work. It is subdivided into six *tractati*, the first of which (fols. 67v-72v) deals with "proportions," argued initially to be necessary not only by means of a list of glorious *names* – Euclid, the Stoics, the Platonists, the Peripatetics, Boethius, Jordanus, etc. – but also with reference to the prestigious uses of the concept: in Archimedes' *De mensura circuli*,[152] in law, in medicine (namely in composite drugs and the determination of diets), in mechanical artifices, in the painter's mixing of colours and in the canonical proportions of the human body,[153] in rhetoric, in architecture, in carpentry, in music, etc. Only afterwards (fol. 69r) come the "definitions of the various proportions," said to be preceded in Euclid by the definition of parts "as we did in the definition of fractions" (namely fol. 48r). A ratio can, with Plato and Boethius, be determined for any two magnitudes of the same kind, but not, as "in a certain abuse of common speech," between the sharpness of a voice and that of a knife.

After this very general "definition" comes the subdivision into geometrical, arithmetical, and harmonic proportion, the first of which is supposed at this point to be applicable to continuous quantities only, the second to discrete as well as continuous quantities,[154] the third to sound and song. As can also be seen from the examples, Pacioli primarily links the three types to the disciplines from which they take their name, even though he does explain later on that the arithmetical proportion has to do with "excesses or differences," and says on fol. 75v that this linking is what "certain blunt minds" (*alcuni roçi*) think. The fact that a harmonic proportion has to involve three terms leads to a digression (provided one can distinguish digressions from the rest in Pacioli's almost Borgesian prose) about the careless habit of using *proportione* where the precise word would be *proportionalità*, and about the subdivision of such proportions into continued and discontinued.[155]

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152 It may be taken note of that Pacioli follows the abbacus tradition and not the Archimedean treatise in the value of these ratios; abbacus geometry had always taken the quotient between perimeter and diameter to be exactly $^{22}/_{7}$ (not 22 : 7, since it was not interested in ratios), and that between the areas and the circumscript square to be $^{11}/_{14}$. Pacioli's reading of Archimedes' treatise must have been (at best) superficial.

153 In this connection there is a laudatory reference to *maestro Pietro de li Franceschi nostro conterraneo del Borgo san Sepolchro* and his *De prospectiva pingendi*.

154 Pacioli even gives an argument for this claim: discrete quantities can only enter into rational ratios, continuous ones indifferently into rational and irrational ratios. Everywhere else Pacioli evidently operates with "geometric proportions," that is, ratios, between numbers. *Interdum dormit Homerus* – and lesser spirits too.

155 Pacioli's term has changed since fol. 57v, now he uses *discontinua* instead of *incontinua*.

The presumed observation that only geometrical proportions can be between rational as well as irrational quantities (*viz.*, because arithmetic *as a discipline* considers only rational magnitudes) leads to a discussion of commensurability and incommensurability, with a reference to *Elements* X. Since Pacioli's example is the diagonal in a square with side 10, this is superfluous sophistication, and in fact he goes on with an (unreferenced) borrowing from the scholastic theory of ratios, namely when speaking of the ratio diameter:side as *meççadoppia*, (half of double), explaining (with reference to *Elements* VII-VIII) that this ratio duplicated is the double ratio.

There follows a long presentation (fols. 71r-72v) of the Boethian subdivisions of the category of "rational proportions" – now obviously only geometrical, but that goes unmentioned: equal, major, minor, multiplex, simple and multiplex, superparticular and superpartient, submultiplex, etc. In the end comes the wonderful admission that "these terms which serve to denominate these many kinds of proportions serve (for you, practitioner [*a te pratico*]) no other purpose than speaking solemnly [*proferire*] about the species you have found." Fol. 82r presents the Boethian categories in a scheme.

The second treatise (fols. 72v-76r) takes up *proportionalità* for good, defining these (with *Elements* V) as similitude of ratios.[156] Once more we get the geometrical, arithmetical, and harmonic proportions, this time with numerical examples for the former two,[157] the harmonic proportion being left explicitly aside. Those still considered may be continued or discontinued; the need for similarity of kind is repeated. For continued proportions of both remaining types, the necessity of equal kind for all members is pointed out.

After a discussion of *disproportionality* (fol. 74r) – the first "proportion" being either larger or smaller than the second – Pacioli deals with the six ways to come to grips with proportions (*de sex specibus sive modis arguendi proportionalitatum*), repeating Campanus's seven modes (see page 57) and coming down to six by conflating *e contrario* with *e conversa*. Pacioli says afterwards (fol. 75v) that he now deals with geometric proportions only, at which point he also asks how much of it holds for arithmetical proportions. He shows the *permutatim* mode to be valid, and

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156 Bartolozzi & Franci (1990, p. 19) reproach Pacioli for the fact that this similitude is meaningless without its definition via equimultiples, forgetting that this is not only absent from Pacioli's *Summa* but also from the Campanus *Elements*.

157 For geometric proportionality, the example is that 6 is to 3 as 4 is to 2 (no problem with numbers); cf. note 54 and the preceding text.

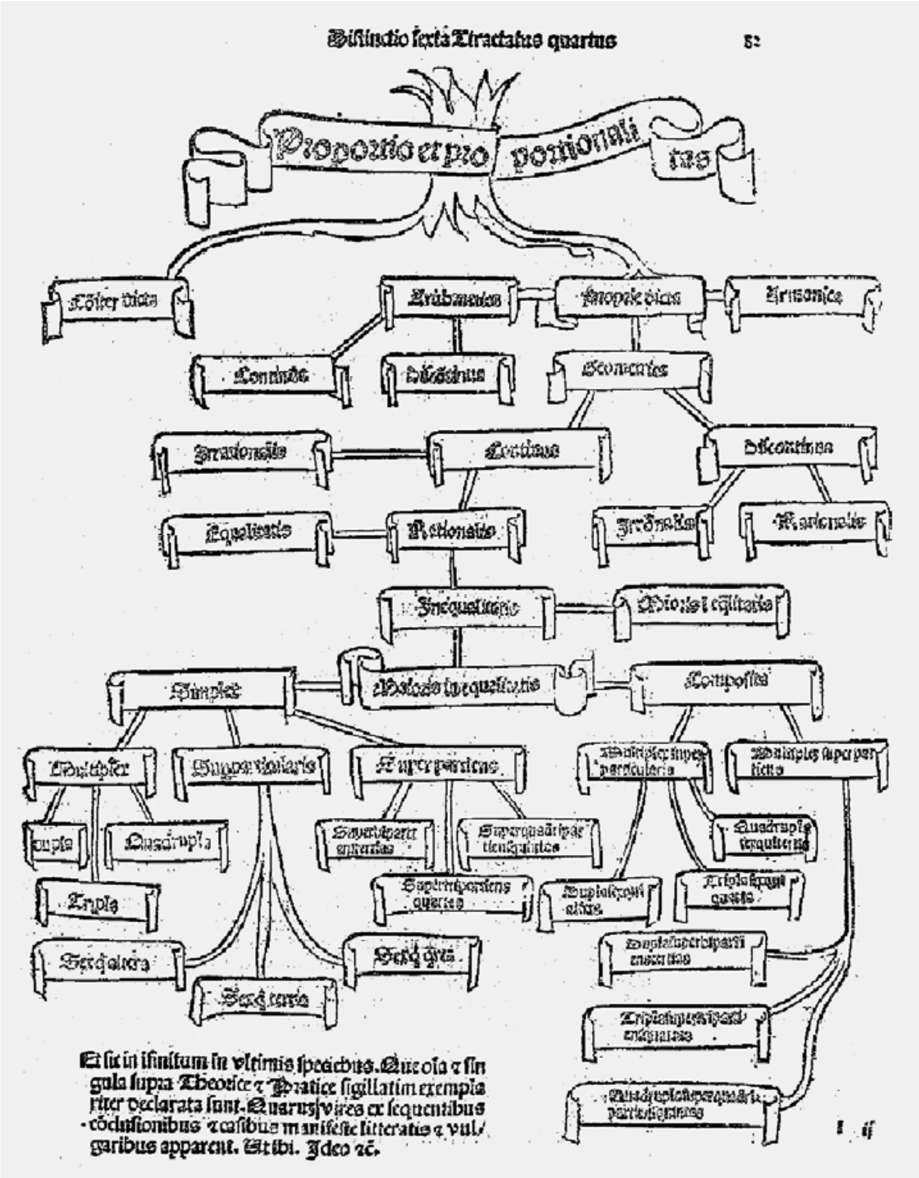

Fig. 1 – Pacioli's schematic presentation of the Boethian categories (1494, fol. 82r).

then generalizes that it should hold for all – obviously without calculating, since only the *ex aequa* mode is true.

The third treatise (fols. 76r-80r) begins by explaining the *denominations* of ratios (not to be confused with the Boethian *names*), the number resulting from the division of one term by the other, in agreement with the terminology of Jordanus and Campanus in their treatises on proportions.[158] After the dismissive remark about the utility of the Boethian terminology in the first treatise (followed up here with references to *phylosophi*), Pacioli thus chooses not to do as in Palatino 573, which identifies ratios with numbers (or replaces them with numbers); what he does is equivalent, but in the guise of established theory. The cost (which the loquacious Pacioli may not have seen as a cost) is that what Palatino 573 does in a couple of lines now needs two dense pages (fols. 76r-77r) to be explained.

Procured with the denomination concept and in agreement with the Campanus *Elements* VII,[159] Pacioli can return (fol. 77r-77v) to the question of whether one ratio (among numbers, which he does not say) is equal to, greater than, or smaller than another one. He uses the occasion to show how this can be done also for the Boethian names, translating them into denominating numbers. He can also take up the composition of ratios (fols. 77v-78r), "without comparison much more difficult" than the operations on integers, fractions, and roots, and (once more) necessary for instance for the physician in his preparation of composite drugs.

First he deals with continued proportions (fol. 78r), where we see that Pacioli spontaneously tends to forget the distinction between the ratio and its denomination: in order to find the ratio between the first and the third term "it is sufficient to multiply [that between the first and the second term] by itself, or its denomination by itself, and it will make the denomination of the proportion between the first and the third." The same tendency underlies an explanatory observation on Campanus, "by duplicated [ratio] Campanus understands (as true is) multiplied by itself" and as well as in the corresponding reference to the "multiplication of the double [ratio] by itself" and in the general claim that "as multiplying a proportion by itself makes a third proportion, thus to multiply the denomination of the said proportion by itself will make the denomination of that third proportion." The composition of unequal ratios arrives

158 Ed. BUSARD, 1971, pp. 205, 213; ed. BUSARD, 2005, p. 230.
159 Ed. BUSARD, 2005, p. 230.

briefly on fol. 79v. Now Pacioli is more faithful to his theoretical base, and speaks of "joining" the ratios 2 : 1, 6 : 2, and 24 : 6; the way is of course to multiply the denominations 2, 3, and 4.

Next follows (fols. 79v-80r) the problem of how to divide a given ratio into several ratios from which it is composed.[160] It is correctly said, and demonstrated by examples, that this can be done in many ways, by the insertion of intermediate terms ad libitum.[161] Finally (fol. 80r) Pacioli teaches how to determine one term of a ratio if the denomination and the other term is known (or the other chosen freely, if none is fixed).

The fourth treatise (fols. 80r-81v) is an attempted *Algorismus proportionum*, based once again on Witelo. It teaches how to add (that is, compose) and subtract ratios and how to "multiply" and "divide" ratios. "Multiplication," however, is simply the composition of several not necessarily equal ratios, and "division" is the splitting of a ratio into several not necessarily equal ratios (the examples for both use unequal ratios). Oresme's work is clearly no inspiration.

The fifth treatise (fols. 832v-84r) examines what happens to ratios and arithmetical proportions if they are changed in various ways (examples in the margin combine denominations and Boethian names).[162] Namely that:

– a ratio increases if the major term is augmented (all ratios are supposed to have the major term first) or the minor diminished;
– the same happens if to both terms something is added, to the major something larger than itself, to the minor something smaller than itself;
– if between the extreme terms of a ratio one or several others are inserted, then any ratio between any two intermediate terms or between one extreme and an intermediate term is smaller than the original ratio;
– the increase of both major terms or both minor terms or the decrease or increase by the same amount of all four terms in an arithmetical proportion conserves the proportionality;
– a ratio does not change if both terms increase *geometrice* – further explained as increase by the same part;

160 The title asks for several equal ratios, but that does not correspond to what actually follows. In Pacioli, 1523, p. 79v, the word "equal" has justly been removed – the publisher Paganinus de Paganino may have had the assistance of somebody who understood the matter, or based himself on a copy with corrections inserted. The latter seems plausible; in the first line of fol. 81v, an erroneous $^2/_3$ is not corrected into $1^2/_3$.

161 Pacioli refers for this to Witelo's *Perspectiva*, which he has already said on fol. 79r to have consulted years ago.

162 Bartolozzi & Franci (1990, p. 23) offer a translation into modern mathematical symbols.

– a ratio diminishes if to both terms the same absolute (*arithmetice*) amount is added, and it increases if the same absolute amount is subtracted from both;
– if both terms of a ratio are increased geometrically, then their "arithmetical proportion" (that is, their difference) increases;
– if both are diminished geometrically, then their "arithmetical proportion" decreases;
– if both terms of a ratio are equal, arithmetically increasing or decreasing both equally is the same as geometrically increasing or decreasing them equally, and their ratio is conserved.

Three corollaries follow that are related to the Peripatetic theory of motion.

According to its title, the sixth treatise (fols. 84r-98v) deals with the "seven marvels [*mirabiles*] from the proportions between two quantities." Actually, it *begins* with seven "marvels" involving two quantities and then considers others that concern three or more. The first marvel is that

any two quantities you want in any proportion joined together, and then the sum divided by each of the said quantities; the results then joined together, and then the sum of the said results equally divided by each of the said results; and again these latter two results joined together, will always be the sum of the first two results, and it never fails.

In symbols:[163]

$$\frac{\frac{a+b}{a}+\frac{a+b}{b}}{\frac{a+b}{a}}+\frac{\frac{a+b}{a}+\frac{a+b}{b}}{\frac{a+b}{b}}=\frac{a+b}{a}+\frac{a+b}{b} \qquad (1)$$

I shall not go through all seven marvels (all are rendered in symbols by Bartolozzi and Franci),[164] but two are noteworthy, in symbols respectively:[165]

163 The fraction lines stand for the operation that Pacioli speaks of as "division"; *denom* in (5) stands for "denomination of" the ensuing "proportion."

164 BARTOLOZZI & FRANCI, 1990, pp. 23-24. There is a (mathematical as well as translational) error in the fourth, which should be:

$$\frac{\frac{a+b}{a}+\frac{a+b}{b}}{\frac{a+b}{a}}=\frac{a+b}{b} \text{ and } \frac{\frac{a+b}{a}+\frac{a+b}{b}}{\frac{a+b}{b}}=\frac{a+b}{a} \qquad (4)$$

$$\frac{a+b}{a} \times \frac{a+b}{b} = \frac{a+b}{a} + \frac{a+b}{b} \tag{3}$$

and

$$\frac{a+b}{a} + \frac{a+b}{b} = 2\,denom(a:b) + denom(b:a) \tag{5}$$

The marvels seem to regard the connection between problems about the splitting of 10 into two parts, *a* and *b*, where ${}^{a}/_{b} + {}^{b}/_{a}$ respectively ${}^{10}/_{a} + {}^{10}/_{b}$ is given. Such problems are known since the beginning of the algebra tradition,[166] and they had also been taken up by Jordanus in *De numeris datis*.[167] Even though I do not remember having seen Pacioli's rules in earlier sources, I therefore suspect him of having borrowed at least some of them.

After the seven marvels, as mentioned, others follow (fol. 85r-v) regarding three, four, or five numbers in continued proportion, the first of which is that if three numbers are in continued proportion, then the division of their sum by the single numbers produces another continued proportion. This was (for an arbitrary dividend) what Antonio considered "rather clear and obvious" (page 83) and in fact a useful theorem for certain problems about the splitting of a number into a sum of numbers in continuous proportion. We may take it for granted that Pacioli took it from the tradition; perhaps indeed directly or indirectly from Antonio, since he goes on (fols. 85v-86r) to apply the rules to binomials in the way Antonio had done in his "*Mirabile dictum*" (page 84). As pointed out by Bartolozzi and Franci,[168] Pacioli generalizes Antonio's method further than Antonio himself had done (asking only for a rational ratio $b:a$) without controlling and errs (or so it seems – the text is not fully clear as to how many conditions Pacioli wants to fulfil).

The authors have overlooked that the equality between the first and second results are said to be *e converso*. As Pacioli points out, the first marvel follows from this, as do the second versions of (3) and (5), not given by Bartolozzi & Franci, in which the right-hand sides of (4) are replaced by the left-hand sides.

165 In both cases Pacioli also points out that the rules hold for the "second results" as well.

166 See Rosen, 1831, pp. 44-46; Rashed, 2007, pp. 167-165 (al-Khwārizmī); Levey, 1966, pp. 94-102, cf. 132-140; Sesiano, 1993, pp. 365-369, cf. 382-388; Chalhoub, 2004, pp. 58-65, 103-109 (Abū Kāmil); and Woepcke, 1853, p. 91*f* (al-Karajī). All three give general rules for the behaviour of the quotients, e.g., ${}^{a}/_{b} \cdot {}^{b}/_{a} = 1$ and $({}^{a}/_{b} + {}^{b}/_{a}) \cdot ab = a^2 + b^2$.

167 I.20 and I.21a, ed. Hughes, 1981, p. 64.

168 Bartolozzi & Franci, 1990, p. 24.

Next (fols. 86v-87v) come a number of rules about three, four, or more numbers in continued or (occasionally) non-continued proportion. Most, as Pacioli states, follow from *Elements* VI.15-16 and VII.20 (our VI.16-17 and VII.19 – the product rule for three or four segments or numbers in proportion/continued proportion): how, if two (or, in an overdetermined case, three) neighbouring quantities in a continued proportion are known, to find the remaining one(s).

Slightly more intricate are the cases where the first and the last of four or five quantities in continued proportion are known. In the case of four quantities, this coincides mathematically with Jacopo's second *fondaco* problem, but whereas Jacopo merely prescribes the extraction of the cube root of the ratio between the fourth and the first quantity without explaining why, Pacioli uses algebra, without which he finds it difficult to solve the problem. In the case of five quantities, the middle quantity is found first from the product rule.[169]

Between these two cases, Pacioli gives the abstract analogue of Jacopo's third *fondaco* problem. Without explanation Pacioli gives the same rule as Jacopo (page 76); he certainly does not know how it comes about (if so, the algebraic solution of the preceding problem shows that he would have explained it). However, the last step of his procedure (how to find two numbers from their sum and their product) suggest that Jacopo is not his direct or indirect source: it contains a hint of an underlying geometric procedure (a reference to an operation with two *different* halves of a quantity) that is absent from Jacopo's text, and which neither Pacioli nor any intermediate abbacus writer is likely to have introduced on his own.[170]

Pacioli now (fol. 88r) supplies a number of "keys," likened (nothing less!) to the two spiritual keys of gold and silver by which "in our Catholic

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[169] In generic terms, Pacioli says that the same method can be used for "6, 7, 8, etc." terms, but he abstains (maybe wisely) from implementing this insight – in fol. 182r he speaks of the sixth root as the "cube root of the cube root," and of the seventh root as the "root of the root of the cube root." Possibly, these composite expressions indicate that Pacioli believed they could be found by stepwise calculation. This is not quite certain, however: as we have seen above, Fibonacci speaks in the *Liber abbaci* (ed. BONCOMPAGNI, 1857, p. 400) of the quintupled proportion as the "cube of the square, or square of the cube" and of the sextupled ratio as the "cube of the cube," but his numerical examples (32 : 1 as the quintuple of 2 : 1, 729 : 1 as the sextuple of 3 : 1) show he was not misled.

[170] When solving in the second part of the *Summa* the corresponding geometric problem, Pacioli (1494, II, fol. 18r) merely refers to the contents of *Elements* II.5, as does his ultimate source (Fibonacci's *Pratica geometrie*, ed. BONCOMPAGNI, 1862, p. 63). Similarly also (with explicit citation of Euclid's proposition) in the arithmetical part, fol. 93v.

Militant Church the first shepherd Saint Peter" opens and closes the doors of Paradise and Hell for us. The keys – fifteen in total – are theorems (not so labelled by Pacioli), in part near or full repetitions of what he has already explained before or easy corollaries of familiar matter, in part new to the book and not easily guessed without symbolic manipulation. All are illustrated by numerical examples. I list them in symbolic translation, indicating the beginning of new pages:

(1)(88r) If $a : b : c : d$, then $\dfrac{b+c}{a+b+c+d} : \dfrac{b}{a+c}$

(2) If $a : b : c : d$, then $\dfrac{a+b}{c+d} : \dfrac{a}{c}$

(3) If $a : b : c : d$, then $\dfrac{a+c}{b+d} : \dfrac{a}{b}$

(4) If $a : b : c : d$ and $S = a + b + c + d$, then ${}^{S}/_{a} : {}^{S}/_{b} : {}^{S}/_{c} : {}^{S}/_{d}$; with three members, this was the first three-number "marvel" on fol. 85r.

(5) If $\dfrac{a}{b} : \dfrac{c}{d}$, then $ad = bc$; the product rule, amply used before.

(6)(88v) If $\dfrac{a}{b} : \dfrac{c}{d}$ and if $c^2 + d^2 = a \cdot b$, then $\sqrt{(a^2 + b^2)(cd)}$ has the same value. Actually, given only the proportion, $(a^2+b^2) \cdot cd = ab \cdot (c^2+d^2)$.

(7) If $\dfrac{a}{b} : \dfrac{c}{d}$, then $([a \cdot b] \cdot c) \cdot d = (a \cdot d) \cdot (b \cdot c)$; evidently, this does not depend on the proportionality.

(8) If $a : b : c : d$, then $(a+b+c+d)^2 = \mathrm{a} \cdot (b+c+d) + b \cdot$ $\cdot (a+c+d) + d \cdot (a+b+c) + c \cdot (a+b+d) + a^2 + b^2 +$ $+ c^2 + d^2$; this time, Pacioli himself points out that the rule does not depend on the proportionality.

(9) If $a : b : c$, then $(a \cdot b) \cdot c = b^3$.

(10) If $a : b : c$, and if, for some quantity Q, ${}^{Q}/_{a} + {}^{Q}/_{b} + {}^{Q}/_{c} =$ $= a + b + c$, then $b = \sqrt{Q}$.

(11)(89r) If $a:b:c$, then ${}^{(a\cdot b)\cdot c}/_{a} = b\cdot c$, ${}^{(a\cdot b)\cdot c}/_{b} = a\cdot c$, ${}^{(a\cdot b)\cdot c}/_{c} = a\cdot b$, and ${}^{(a\cdot b)\cdot c}/_{a\cdot b} = c$, ${}^{(a\cdot b)\cdot c}/_{a\cdot c} = b$, ${}^{(a\cdot b)\cdot c}/_{b\cdot c} = a$; Pacioli points out that this does not depend on the proportionality.

(12) If $a:b:c$ and further $\frac{a}{b}:\frac{p}{q}$, then $p\cdot(b+c) = q\cdot(a+b)$.

(13) If $a:b:c$, then $2\cdot(a\cdot c+b\cdot[a+c]) = a(b+c)+\mathrm{b}(a+c)+$ $+c(a+b)$. With references to *Elements* II.2 and the formulations "in other words" in *Elements* VI and IX, Pacioli points out that this does not depend on the proportionality.

(14)(89v) If $a:b:c$, then ${}^{a(b+c)+b(a+c)+c(a+b)}/_{2\cdot(a+b+c)} = b$.

(15) If $a:b:c$, then $\frac{a^2}{b^2}:\frac{a}{c}$.

This is the last "key." Under the heading "to find mean proportionals between two quantities," two sophisticated counterfactual calculations follow (fol. 89v), which I guess are Pacioli's own invention: If 2 is the arithmetical respectively geometric mean between 5 and 11, what is then the corresponding mean between 7 and 13? In both cases, the true means between 5 and 11 and between 7 and 13 are found (8 and 10 respectively $\sqrt{55}$ and $\sqrt{91}$), and the rule of three is applied. In the arithmetical case, a proof is performed, consisting in a corresponding proportional change of the limits, after which the true means between these limits are shown to coincide with what was found before; in the geometrical case, a similar proof is sketched but not performed.

The "second case" under the same heading is a traditional question: "three is (too) little and 4 is (too) much." The "just or due" amount is said to be $\sqrt{12}$, the geometric mean; this – not the arithmetical mean – is then stated to be what is used in all commercial matters (*in omnibus mercantiis*). Primarily, this is probably extrapolated from the observation that the rule of three is based on geometric *proportionality*. But Pacioli may also think of the use of the geometric mean in certain mathematical *problems* in commercial disguise.

In any case, such a problem, about three pearls, follows as the "third case." The first pearl weighs 1 carat and is worth 200 *ducati*, the second weighs 2 carats and is worth 1000 *ducati*, the third weighs 3 carats. What is its just price?

Pacioli posits a fourth pearl that weighs 4 carats. To the weights 1 : 2 : 4 in continued proportion must correspond prices in continued proportion, i.e., 200 : 1000 : 5000. Therefore the price of the 4-carat pearl must be 5000 *ducati*. Three carats being the (arithmetical) mean between 2 and 4, the price of the 3-carat pearl must be $\sqrt{(1000 \cdot 5000)}$.

A fourth case is also about justice. The Holy Father Innocent VIII orders that 10000 *ducati* be distributed justly among the citizens of Perugia for service rendered. This gives rise to a long discourse (more than 500 words) about Aristotle's two kinds of justice from the [Nicomachean] *Ethics* V. 2-5:[171] "commutative,"[172] applicable to commercial exchange, and distributive. Both, according to Pacioli, "can, broadly speaking, be understood in two ways, geometrically and arithmetically, though, strictly and properly speaking, the maximal distributive sort can only be geometrical."[173] After the digression into ethical theory, it is then explained that the money is justly distributed if done in geometric proportion to the "quality" (*bontà*) of each.

The sixth distinction ends (fols. 90v-98r) with 35 problems[174] and an epilogue (fol. 98r-v). The final two have nothing to do with proportions – #34 is "Bachet's weight problem," and #35 belongs to the same family; parallels in the wording suggest that they are borrowed from the *Liber abbaci*.[175] In all the others, "proportions" play a role.

First come 23 problems about three numbers in continued proportion. In seven of them, a number (19, 19, 14, 10, "a number,"[176] 10, 10) is split into such constituents; towards the end of the sequence, four are presented as dealing with economic life.[177] In #1-6, specified "keys" are used as first steps in the procedure, which in these and the other cases often makes use of algebra or (in #5, #6, and #18) of *Elements* II.[178]

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171 Barnes, 1984, II, pp. 1784-1789.

172 Nowadays normally translated "rectificatory," but Pacioli follows his fellow friar Thomas Aquinas (*Summa theologiae* 1ª q. 21 a. 1 s 1 co; see *Corpus thomisticum*), whom he cites.

173 This point comes from Aristotle, whose Chapter 3 also contains a discourse on proportion theory. Mathematical proportions (represented by lines and letters) are used further in Chapters 4 and 5.

174 Pacioli also counts until 35, but has two #18, skips #19 and #28, and has two #29.

175 Ed. Boncompagni, 1857, p. 297*f.*

176 This problem (#15) is indeterminate. Afterwards, the number is chosen to be 10, whereby it is made determinate.

177 #18bis deals with a gambler's gains, where the product rule is explained once again, suggesting perhaps the text to be borrowed (but Pacioli is too fond of repeating to make the inference certain); #21, which "was proposed to me in Florence in 1480, the 22nd of June," deals with a purchase of saffron, cinnamon, and mastic, and #22-23 with alloys.

178 Algebra is thus *used* by Pacioli well before he presents it systematically. Often, this algebra is quite complex. In #4, for instance, Pacioli has to operate with two unknowns in the same way as Antonio, i.e., with "a *thing* less a quantity" and "a *thing* plus a quantity." The problem in which this

Next follows a sequence of ten problems about four magnitudes in continued proportion, none of them in concrete form. Once again, the first ones make use of specified "keys" (#24-27, but also #31-32). Most interesting are probably #31-33: #31 and #33 are pure-number versions of Jacopo's third and fourth *fondaco* problems; #32 of a similar problem where the sums of the wages for the first two and for the last two years are given. In #31, key (1) is used to reduce the problem; then the second number is taken as the *thing* and found by second-degree algebra to be $12^1/_2 + \sqrt{7^{37}/_{84}}$ – at which point Pacioli cautiously leaves it to the reader to continue.[179] Since his present method does not lead easily[180] to the formula used by Jacopo and by Pacioli in the first presentation of the abstract problem just before the "keys" (above, page 45), Pacioli appears not to have noticed the connection.

The use of the "keys" in problem reductions leaves little doubt that these *new* theorems about the behaviour of proportions were created as tools for the solution of problems – but apparently only problems formulated in terms of proportions or proportionality, for whose initial reduction they served. Pacioli's way of adding observations about (8), (11), and (13) strongly suggests that the basic set was not his own. It is likely to have been created during the fifteenth century and *seems to reflect a more intimate integration between algebra and proportions than other sources would make us expect*.

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is done is not the same as the one where Antonio introduces it in the *Fioretti*, or as the one from the *Flos* where Fibonacci employs it; cf. note 104.

179 The solution is correct, but corresponds to a decreasing sequence, which is certainly not what Pacioli intended; in order to have an increasing sequence, he should have chosen the other root of the equation, $12^1/_2 - \sqrt{7^{37}/_{84}}$. Since Pacioli did not discover it, he cannot have finished the calculations. When applying later the same method to an analogous wage problem with rational solutions, Pacioli makes the complete calculation and chooses the correct solution – see presently.

180 Of course it *can* lead to it, but only if one is able to express the double root (the second and the third number, respectively) as

$$\frac{P}{2} \pm \sqrt{\frac{P^2}{4} - \frac{P^3}{3P+Q}}$$

(*P* being the sum of the second and the third number, *Q* that of the first and the fourth). The product of these is indeed

$$\frac{P^3}{3P+Q}$$

as required; but this would have been far too complicated for Pacioli.

10. Further "Proportions" in Pacioli's *Summa*

Proportionality turns up in (at least) three other contexts in the *Summa*: in the general presentation of algebra, and in two sets of problems.

Fol. 143r lists a sequence of 30 algebraic powers (*dignità*) in two different terminologies and observes that the reader may go on *proceeding proportionally* "as long as you want." On fol. 145v, the same insight (which as we know was not new) is hinted at in the statement that all solvable cases are *proportionati* to the six basic cases.[181] It becomes more explicit (and somewhat more innovative) on fols. 149v-150r, after a short list of select possible and impossible cases. In order to find out to which basic case a given equation reduces, one shall locate the *dignità* in the ordered sequence and reduce[182] geometrically equal to the lowest possible degree by counting downwards. However, if the *intervalli* between the three powers in a three-term equation (the only equations Pacioli considers) are not equal, it has "so far not been possible to form general rules because of their disproportionality."

On fols. 186r-187v, a number of problems deal with gain (occasionally loss) "at the same rate" in two or more travels. Mostly, the proportionality leads to the application of the rule of three, but once, in an alternative ("and more beautiful" [*pulchrius*]) solution (fol. 186r), proportionality is mentioned explicitly, and the product rule applied. This evidently gives the same calculations as the rule of three; the aesthetic advantage is solely in the use of "magisterial" terminology – "speaking solemnly" in Pacioli's earlier words.

Finally, fol. 194r presents six problems about the wages of a servant, in two of which the wage is supposed to increase "at the same rate" each year (four years in total). In the first of them the wage of the first year is 10, and that of the last year is 60; apart from a change of the wage of the first year, this coincides with Jacopo's second *fondaco* problem. This has to be done "according to what I showed you in the proportions, and I shall say no more, except that there are four proportional numbers, and the first is 10, and the last is 60. I ask for the means" – which are then stated to be $\sqrt[3]{36000}$ and $\sqrt[3]{336000}$.

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[181] This, certainly, is not true *stricto sensu* if we consider as solvable, e.g., the case "*cubes* equal to number," but Pacioli's target is the proliferation of false solutions to non-homogeneous higher-degree equations.

[182] The verb is *schizzare*, which mostly refers to the reduction of a fraction through the division of numerator and denominator by the same divisor.

The second coincides with Jacopo's third *fondaco* problem, even in its choice of parameters. For the solution, Pacioli refers to the "first key" and to what he has already taught. This time, the numbers are convenient, and Pacioli makes the complete calculation, finding the second number to be $30 - \sqrt{100}$ and the third to be $\sqrt{100 + 30}$ (an order which suggests that he has not used the double solution, but subtracted from 60).

There may be other scattered references to the concept of proportionality in the work. All in all, however, "proportions and proportionalities" are mainly treated in the sixth distinction, which is indeed extensive and profound enough to justify the appearance of the terms in the title of the work; to this same distinction are also moved traditional abbacus problem types about numbers in proportion, abstract as well as in commercial dress. Outside the sixth distinction, "proportions" play as modest a role as in most abbacus treatises.

11. Summing up

In this way, Pacioli's *opus magnum* suggests the general summary we may draw up. Abbacus mathematics, based on practical arithmetic, was always centred around problems of simple (direct or inverse) proportionality; to this was added a strand of algebraic thought with great prestige due both to its efficiency and to its character of "theoretical level of practical arithmetic."[183] Initially, neither the language nor the theory of proportions had anything to do with either; gradually, but hardly to a larger extent than its penetration in daily discourse, the language of proportion would pop up. Problems might also be formulated in terms of quantities or numbers in proportion. *Theory* beyond the product rules remained outside.

To this, only writers with "magisterial" pretensions – Fibonacci, Antonio, Benedetto, the author of Palatino 573, Pacioli – and the shadowy inventor of Pacioli's "keys" constitute exceptions. Apart from

183 Two quotations may suffice to illustrate this prestige. Palatino 573 (ed. Arrighi, 2004/1967, p. 191) opens the part on algebra (as we recall, this encyclopedia is divided into eleven "parts") with the words "every part would be in vain if this [part] was left out; because [...] this is the one that gives solution to all cases." Pacioli (1494, p. 144r) observes in the corresponding place that we have now "arrived with the help of God to the much desired place: that is, to the mother of all the cases popularly called the *regola della cosa* or *Arte magiore*, that is, *pratica speculativa*, otherwise called *algebra & almucabala* in the Arabic or Chaldean tongue." The words *pratica speculativa* (used again about algebra in Pacioli's *Divina proportione*, I, cap. iv, 1509, p. 3v) mean exactly "theoretical [level of] practical arithmetic."

the inventor of the keys, who to some extent made *new* theory, what they offer in terms of theory and technicalities beyond the product rules are isolated chapters, in some cases just copied from Fibonacci (and never understanding more about these than Fibonacci himself did). They are there more as a result of pious duty than of mathematical necessity.

As regards the Boethian terminology for ratios, the situation is even more blatant. Since late Carolingian times, this categorization had been the almost sacred core of the mathematics of Latin schools and universities. Benedetto and Palatino 573 introduce them, but the latter dismisses them immediately, and the former makes no use of them. Only Pacioli employs this "solemn speech" rather consistently when explaining the composition of ratios in Distinction 6, 4th Treatise. The general tendency is to speak "at best" (as judged from the perspective of the schools) of the denominations of ratios, "at worst" to come close to identifying the ratio with the quotient (as when Pacioli identifies the composition and multiplication of ratios).

All in all, abbacus mathematics is much more modern on this account (and on several others) than scholarly mathematics of its epoch. As scholars eventually digested the abbacus heritage, they took over norms, and not just technical algebraic knowledge.

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